

Slowly varying asymptotics for signed stochastic difference equations

Dmitry Korshunov

March 8, 2021

Abstract

For a stochastic difference equation $D_n = A_n D_{n-1} + B_n$ which stabilises upon time we study tail distribution asymptotics for D_n under the assumption that the distribution of $\log(1 + |A_1| + |B_1|)$ is heavy-tailed, that is, all its positive exponential moments are infinite. The aim of the present paper is three-fold. Firstly, we identify the asymptotic behaviour not only of the stationary tail distribution but also of D_n . Secondly, we solve the problem in the general setting when A takes both positive and negative values. Thirdly, we get rid of auxiliary conditions like finiteness of higher moments introduced in the literature before.

MSC: 60H25; 60J10

1 Introduction

Let (A, B) be a random vector in \mathbb{R}^2 such that $\mathbb{E} \log |A| = -a < 0$. Let (A_k, B_k) , $k \in \mathbb{Z}$, be independent copies of (A, B) . Consider the following stochastic difference equation

$$\begin{aligned} D_n &= A_n D_{n-1} + B_n \\ &= \Pi_1^n D_0 + \sum_{k=1}^n \Pi_{k+1}^n B_k, \quad n \geq 1, \end{aligned} \tag{1}$$

where D_0 is independent of (A_k, B_k) 's, $\Pi_k^n := A_k \cdot \dots \cdot A_n$ for $k \leq n$ and $\Pi_{n+1}^n = 1$. The process D_n clearly constitutes a Markov chain and satisfies the following equality in distribution

$$D_n \stackrel{st}{=} \Pi_{-n}^{-1} D_0 + \sum_{k=-n}^{-1} \Pi_{k+1}^{-1} B_k.$$

If $a < \infty$ then, by the strong law of large numbers applied to the logarithm of $|\Pi|$, with probability 1, $e^{-2an} \leq \Pi_1^n \leq e^{-an/2}$ ultimately in n , hence the process D_n , $n \geq 1$, is stochastically bounded if and only if $\mathbb{E} \log(1 + |B|) < \infty$. If $\mathbb{P}\{A = 0\} > 0$ which implies $a = \infty$, then the process D_n is always stochastically bounded. In both cases, the Markov chain D_n is stable, its stationary distribution is given by the following random series

$$D_\infty := \sum_{k=-\infty}^{-1} \Pi_{k+1}^{-1} B_k \stackrel{st}{=} \sum_{k=1}^{\infty} \Pi_1^{k-1} B_k$$

and D_n weakly converges to the stationary distribution as $n \rightarrow \infty$; in the context of financial mathematics such random variables are called stochastic perpetuities. Stability results for D_n are dealt with in [17], see also [2]; the case where $\mathbb{E} \log |A|$ is not necessarily finite is treated in [9].

Both perpetuities and stochastic difference equations have many important applications, among them life insurance and finance, nuclear technology, sociology, random walks and branching processes in random environments, extreme-value analysis, one-dimensional ARCH processes, etc. For particularities, we refer the reader to, for instance, Embrechts and Goldie [5], Rachev and Samorodnitsky [15] and Vervaat [17] for a comprehensive survey of the literature.

If $A \geq 0$ and $\mathbb{P}\{A > 1\} > 0$, then $\mathbb{E}A^\gamma \rightarrow \infty$ as $\gamma \rightarrow \infty$, so $\mathbb{E}A^\beta > 1$ for some $\beta < \infty$. If in addition $B \geq 0$, then it follows from the stationary version of the recursion (1) that $\mathbb{E}D_\infty^\beta \geq \mathbb{E}D_\infty^\beta \mathbb{E}A^\beta$ which implies that $\mathbb{E}D_\infty^\beta = \infty$, in other words, with necessity, not all moments of D_∞ are finite; see [8] for a similar conclusion for signed A and B . It was proven in the seminal paper by Kesten [11, Theorem 5], see also [7], that if $\mathbb{E}|A|^\beta = 1$ for some $\beta > 0$, then a power tail asymptotics for the stationary distribution holds, $\mathbb{P}\{|D_\infty| > x\} \sim c/x^\beta$ as $x \rightarrow \infty$, for some $c > 0$.

The problem we address in this paper is about the tail asymptotic behaviour of D_n and of its stationary version D_∞ in the case where the distribution of $\log |A|$ is heavy-tailed, that is, all positive exponential moments of $\log |A|$ are infinite, in other words, $\mathbb{E}|A|^\gamma = \infty$ for all $\gamma > 0$. It can only happen if the random variable $|A|$ has right unbounded support.

The only result in that direction we are aware of is that by Dyszewski [4] where in the context of iterated random functions it is proven that the stationary tail distribution is asymptotically equivalent to

$$\frac{1}{a} \int_x^\infty \mathbb{P}\{\log C > y\} dy \quad \text{as } x \rightarrow \infty,$$

where $C := \max(A, B)$, provided $A, B \geq 0$, the integrated tail distribution of $\log C$ is subexponential and under additional moment condition that $\mathbb{E} \log^\gamma C < \infty$

for some $\gamma > 1$. In the case of a signed B , only lower and upper asymptotic bounds are derived in [4]. An alternative approach to lower and upper bounds for the tail distribution of D_∞ is developed in [3] in the case of positive A and B .

The aim of the present paper is three-fold. Firstly, we identify the asymptotic behaviour not only of the stationary tail distribution but also for D_n in the heavy-tailed case. Secondly, we solve the problem in the general setting when A takes both positive and negative values. Thirdly, we get rid of auxiliary conditions like finiteness of higher moments.

Our approach to the problem is based on reduction of D_n – roughly speaking by taking the logarithm of it – to an asymptotically homogeneous in space Markov chain with heavy-tailed jumps and on further analysis of such chains. Namely, we define a Markov chain X_n on \mathbb{R} as follows

$$X_n := \begin{cases} \log(1 + D_n) & \text{if } D_n \geq 0, \\ -\log(1 + |D_n|) & \text{if } D_n < 0, \end{cases} \quad (2)$$

hence the distribution tail of D_n may be computed as

$$\mathbb{P}\{D_n > x\} = \mathbb{P}\{X_n > \log(1 + x)\} \quad \text{for } x > 0. \quad (3)$$

At any state $x \geq 0$, the jump of the Markov chain X_n is a random variable distributed as

$$\xi(x) = \begin{cases} \log(1 + A(e^x - 1) + B) - x & \text{if } A(e^x - 1) + B \geq 0, \\ -\log(1 + |A(e^x - 1) + B|) - x & \text{if } A(e^x - 1) + B < 0, \end{cases} \quad (4)$$

and at any state $x \leq 0$,

$$\xi(x) = \begin{cases} \log(1 + A(1 - e^{-x}) + B) - x & \text{if } A(1 - e^{-x}) + B \geq 0, \\ -\log(1 + |A(1 - e^{-x}) + B|) - x & \text{if } A(1 - e^{-x}) + B < 0. \end{cases} \quad (5)$$

Also define a sequence of independent random fields $\xi_n(x)$, $x \in \mathbb{R}$, which are independent copies of $\xi(x)$. Then the recursion (1) may be rewritten as

$$X_{n+1} = X_n + \xi_n(X_n).$$

The Markov chain X_n is *asymptotically homogeneous in space*, that is, the distribution of its jump $\xi(x)$ weakly converges to that of $\xi := \log A$ as $x \rightarrow \infty$; it is particularly emphasised in [7, Section 2]. Let us underline that, in general, $\log(A + (1 - A + B)e^{-x})$ may not converge to ξ as $x \rightarrow \infty$ in total variation norm.

Asymptotically homogeneous in space Markov chains are studied in detail in [1, 13] from the point of view of their asymptotic tail behaviour in subexponential case. However, that results for general asymptotically homogeneous in space

Markov chains are not directly applicable to stochastic difference equations as it is formally assumed in [1, Theorem 3] that the distribution of a Markov chain X_n converges to the invariant distribution in the total variation norm which is not always true for stochastic difference equations. Secondly, stochastic difference equations possess some specific properties that allow us to find tail asymptotics in a simpler way than it is done in [1, Theorem 3] or in [4, Theorem 3.1]; we explore that below however our approach still follows some ideas of the proof for Markov chains in [1].

Let us recall some relevant classes of distributions needed in our analysis of the heavy-tailed case.

Definition 1. A distribution H with right unbounded support is called *long-tailed*, $H \in \mathcal{L}$, if, for each fixed y , $\overline{H}(x+y) \sim \overline{H}(x)$ as $x \rightarrow \infty$; hereinafter $\overline{H}(x) = H(x, \infty)$ is the tail of H .

A random variable A has slowly varying at infinity distribution if and only if the distribution of $\xi := \log(A^+)$ is long-tailed.

Definition 2. A distribution H on \mathbb{R}^+ with unbounded support is called *subexponential*, $H \in \mathcal{S}$, if $\overline{H} * \overline{H}(x) \sim 2\overline{H}(x)$ as $x \rightarrow \infty$. Equivalently, $\mathbb{P}\{\zeta_1 + \zeta_2 > x\} \sim 2\mathbb{P}\{\zeta_1 > x\}$, where random variables ζ_1 and ζ_2 are independent with common distribution H . A distribution H of a random variable ζ on \mathbb{R} with right-unbounded support is called *subexponential* if the distribution of ζ^+ is so.

As well-known (see, e.g. [6, Lemma 3.2]) the subexponentiality of H on \mathbb{R}^+ implies long-tailedness of H . In particular, if the distribution of a random variable $\zeta \geq 0$ is subexponential then ζ is heavy-tailed.

For a distribution H with finite mean, we define the *integrated tail distribution* H_I generated by H as follows:

$$\overline{H}_I(x) := \min\left(1, \int_x^\infty \overline{H}(y)dy\right).$$

Definition 3. A distribution H on \mathbb{R}^+ with unbounded support and finite mean is called *strong subexponential*, $H \in \mathcal{S}^*$, if

$$\int_0^x \overline{H}(x-y)\overline{H}(y)dy \sim 2m\overline{H}(x) \quad \text{as } x \rightarrow \infty,$$

where m is the mean value of H . It is known that if $H \in \mathcal{S}^*$ then both H and H_I are subexponential distributions, see e.g. [6, Theorem 3.27].

In what follows we use the following notation for distributions: we denote

- (i) the distribution of $\log(1 + |A| + |B|)$ by H ;
- (ii) the distribution of $\log(1 + |A|)$ by F ;
- (iii) the distribution of $\log(1 + |B|)$ by G ;

- (iv) the distribution of $\log(1 + B^+)$ by G^+ ;
- (v) the distribution of $\log(1 + B^-)$ by G^- .

The paper is organised as follows. In Sections 2, 4 and 5 we assume that $\log |A|$ has finite negative mean and successively investigate three different cases in the order of increasing difficulty: (i) both A and B are positive, see Theorem 1; (ii) A is positive and B is a signed random variable, see Theorem 6; (iii) both A and B are signed, see Theorem 7. In the case (i) we also explain in Theorem 4 the most probable way by which large deviations of D_n can occur – it is a version of the principle of a single big jump playing the key role in the theory of subexponential distributions. The aim of Section 3 is to explain what happens if the distribution of A has an atom at zero; in that case the tail asymptotics of D_n is essentially different from what we observe if A has no atom at zero.

2 Positive stochastic difference equation

In this section we consider a positive D_n , so $A > 0$, $B \geq 0$ – we exclude the case where A has an atom at zero as then the tail asymptotics of D_n are essentially different, see the next section. Then the Markov chain $X_n := \log(1 + D_n)$ is positive too. As above, we denote $\xi := \log A$ and the distribution of the random variable $\log(1 + A + B)$ by H .

Theorem 1. *Suppose that $A > 0$, $B \geq 0$, $\mathbb{E}\xi = -a \in (-\infty, 0)$ and $\mathbb{E}\log(1 + B) < \infty$, so that D_n is positive recurrent.*

If the integrated tail distribution H_I is long-tailed, then

$$\mathbb{P}\{D_\infty > x\} \geq (a^{-1} + o(1))\overline{H_I}(\log x) \quad \text{as } x \rightarrow \infty. \quad (6)$$

If, in addition, the distribution H is long-tailed itself, then

$$\mathbb{P}\{D_n > x\} \geq \frac{1+o(1)}{a} \int_{\log x}^{\log x+na} \overline{H}(y)dy \quad \text{as } x \rightarrow \infty \text{ uniformly for all } n \geq 1. \quad (7)$$

If the integrated tail distribution H_I is subexponential then

$$\mathbb{P}\{D_\infty > x\} \sim a^{-1}\overline{H_I}(\log x) \quad \text{as } x \rightarrow \infty. \quad (8)$$

If moreover the distribution H is strong subexponential then

$$\mathbb{P}\{D_n > x\} \sim \frac{1}{a} \int_{\log x}^{\log x+na} \overline{H}(y)dy \quad \text{as } x \rightarrow \infty \text{ uniformly for all } n \geq 1. \quad (9)$$

The main contribution of Theorem 1 is (9) that states uniform asymptotic behaviour for all $n \geq 1$. It is much stronger than a rather simple conclusion that (9) holds for a fixed n demonstrated by Dyszewski in [4, Theorem 3.3] by induction argument that clearly does not work for the tail asymptotics for the entire range of $n \geq 1$.

In [4], a sufficient condition for the asymptotics (8) is formulated in terms of the distribution of $\log \max(A, B)$ instead of H . Let us show that these two approaches are equivalent. Indeed, for any two positive random variables A and B , since

$$\begin{aligned} \max(\log(1 + A), \log(1 + B)) &\leq \log(1 + A + B) \\ &\leq \log(1 + 2 \max(A, B)) \\ &< \log 2 + \max(\log(1 + A), \log(1 + B)), \end{aligned}$$

it follows that

(i) the distribution H is long-tailed/subexponential/strong subexponential if and only if the distribution of $\max(\log(1 + A), \log(1 + B))$ is long-tailed/subexponential/strong subexponential respectively;

(ii) the distribution H_I is subexponential if and only if the integrated tail distribution of $\max(\log(1 + A), \log(1 + B))$ is so.

Denote the distribution of $\log(1 + A)$ by F and that of $\log(1 + B)$ by G . In the next result we discuss some sufficient conditions for subexponentiality and related properties of H .

Lemma 2. *Let A and B be any two positive random variables such that either of the following two conditions holds:*

- (i) *the distribution H of $\log(1 + A + B)$ is long-tailed or*
- (ii) *the random variables A and B are independent.*

Then if the distribution $(F + G)/2$ is subexponential or strong subexponential, then the distribution H is subexponential or strong subexponential respectively.

If the integrated tail distribution $(F_I + G_I)/2$ is subexponential, then H_I is subexponential too.

Proof. First assume that (i) holds. On the one hand,

$$\begin{aligned} \overline{H}(x) &= \mathbb{P}\{\log(1 + A + B) > x\} \\ &\geq \frac{\mathbb{P}\{\log(1 + A) > x\} + \mathbb{P}\{\log(1 + B) > x\}}{2} \\ &= (\overline{F}(x) + \overline{G}(x))/2 \end{aligned} \tag{10}$$

and thus, for all sufficiently large x ,

$$\overline{H_I}(x) \geq (\overline{F_I}(x) + \overline{G_I}(x))/3. \tag{11}$$

On the other hand,

$$\begin{aligned}\bar{H}(x) &\leq \mathbb{P}\{\log(1+2A) > x\} + \mathbb{P}\{\log(1+2B) > x\} \\ &\leq \bar{F}(x - \log 2) + \bar{G}(x - \log 2).\end{aligned}\quad (12)$$

If $(F + G)/2$ is subexponential then it is long-tailed and hence

$$\bar{H}(x) \leq (1 + o(1))(\bar{F}(x) + \bar{G}(x)) \quad \text{as } x \rightarrow \infty. \quad (13)$$

If $(F_I + G_I)/2$ is subexponential then similarly

$$\bar{H}_I(x) \leq (1 + o(1))(\bar{F}_I(x) + \bar{G}_I(x)) \quad \text{as } x \rightarrow \infty. \quad (14)$$

The two bounds (13) and (10) in the case of long-tailed H allow us to apply Theorem 3.11 or 3.25 from [6] and to conclude subexponentiality or strong subexponentiality of H respectively provided $(F + G)/2$ is so.

The two bounds (14) and (11) in the case of long-tailed H_I allow us to apply Theorem 3.11 from [6] and to conclude subexponentiality of H_I provided $(F_I + G_I)/2$ is so.

Now let us consider the case where A and B are independent which yields the following improvement on the lower bound (10). For all $x > 0$,

$$\begin{aligned}\bar{H}(x) &\geq \mathbb{P}\{\log(1+A) > x\} + \mathbb{P}\{\log(1+A) \leq x\}\mathbb{P}\{\log(1+B) > x\} \\ &= \bar{F}(x) + F(x)\bar{G}(x) \\ &\sim \bar{F}(x) + \bar{G}(x) \quad \text{as } x \rightarrow \infty.\end{aligned}$$

Therefore, H inherits the tail properties of the distribution $(F + G)/2$, and H_I the tail properties of $(F_I + G_I)/2$. \square

Proof of Theorem 1. At any state $x \geq 0$, the Markov chain X_n has jump

$$\begin{aligned}\xi(x) &= \log(1 + A(e^x - 1) + B) - x \\ &= \log(A + e^{-x}(1 - A + B)) \\ &\geq \log(A - e^{-x}A),\end{aligned}$$

as $B \geq 0$. Fix an $\varepsilon > 0$. Choose x_0 sufficiently large such that $\log(1 - e^{-x_0}) \geq -\varepsilon/2$. Then the family of jumps $\xi(x)$, $x \geq x_0$, possesses an integrable minorant

$$\begin{aligned}\xi(x) &\geq \xi + \log(1 - e^{-x_0}) \\ &\geq \xi - \varepsilon/2 =: \eta.\end{aligned}\quad (15)$$

On the other hand, since $A > 0$ and $B \geq 0$, the family of jumps $\xi(x)$, $x \geq x_0$, possesses an integrable majorant $\zeta(x_0) := \log(A + e^{-x_0}(1+B))$. For a sufficiently large x_0 ,

$$\mathbb{E} \log(A + e^{-x_0}(1+B)) \leq \mathbb{E} \xi + \varepsilon, \quad (16)$$

owing to the dominated convergence theorem which applies because firstly $\log(A + e^{-x_0}(1+B)) \rightarrow \log A = \xi$ a.s. as $x_0 \rightarrow \infty$ and secondly, by the concavity of the function $\log(1+z)$,

$$\begin{aligned} \log(A + e^{-x_0}(1+B)) &< \log(1 + A + e^{-x_0}(1+B)) \\ &\leq \log(1+A) + \log(1 + e^{-x_0}(1+B)), \end{aligned}$$

which is integrable by the finiteness of $\mathbb{E} \xi$ and $\mathbb{E} \log(1+B)$.

Let us first prove the lower bound (6) following the single big jump technique known from the theory of subexponential distributions. Since D_n is assumed convergent, the associated Markov chain X_n is stable, so there exists a $c > 2$ such that

$$\mathbb{P}\{X_n \in (1/c, c]\} \geq 1 - \varepsilon \quad \text{for all } n \geq 0.$$

Let us consider an event

$$\Omega(k, n, c) := \{\eta_{k+1} + \dots + \eta_{k+j} \geq -c - n(a + \varepsilon) \text{ for all } j \leq n\}, \quad (17)$$

where η_k are independent copies of η defined in (15). By the strong law of large numbers, there exists a sufficiently large c such that

$$\mathbb{P}\{\Omega(k, n, c)\} \geq 1 - \varepsilon \quad \text{for all } k \text{ and } n. \quad (18)$$

It follows from (15) that any of the events

$$\{X_{k-1} \leq c, X_k > x + c + (n-k)(a + \varepsilon), \Omega(k, n-k, c)\} \quad (19)$$

implies $X_n > x$ and they are pairwise disjoint. Therefore, by the Markov property and (18),

$$\begin{aligned} &\mathbb{P}\{X_n > x\} \\ &\geq \sum_{k=1}^n \mathbb{P}\{X_{k-1} \leq c, X_k > x + c + (n-k)(a + \varepsilon)\} \mathbb{P}\{\Omega(k, n-k, c)\} \\ &\geq (1 - \varepsilon) \sum_{k=1}^n \mathbb{P}\{X_{k-1} \in (1/c, c], X_k > x + c + (n-k)(a + \varepsilon)\}. \end{aligned}$$

The k th probability on the right hand side equals

$$\begin{aligned} & \int_{1/c}^c \mathbb{P}\{X_{k-1} \in dy\} \mathbb{P}\{y + \xi(y) > x + c + (n-k)(a + \varepsilon)\} \\ &= \int_{1/c}^c \mathbb{P}\{X_{k-1} \in dy\} \mathbb{P}\{\log(1 + A(e^y - 1) + B) > x + c + (n-k)(a + \varepsilon)\}. \end{aligned}$$

For all $y > 1/c$,

$$\begin{aligned} \log(1 + A(e^y - 1) + B) &\geq \log(1 + A(e^{1/c} - 1) + B) \\ &\geq \log(1 + A + B) + \log(e^{1/c} - 1), \end{aligned}$$

because $e^{1/c} - 1 < \sqrt{e} - 1 < 1$. Therefore, the value of the last integral is not less than

$$\mathbb{P}\{X_{k-1} \in (1/c, c]\} \mathbb{P}\{\log(1 + A + B) > x + c_1 + (n-k)(a + \varepsilon)\},$$

where $c_1 := c - \log(e^{1/c} - 1)$. Hence, due to the choice of c ,

$$\mathbb{P}\{X_n > x\} \geq (1 - \varepsilon)^2 \sum_{k=1}^n \bar{H}(x + c_1 + (n-k)(a + \varepsilon)).$$

Since the tail is a decreasing function, the last sum is not less than

$$\frac{1}{a + \varepsilon} \int_0^{n(a + \varepsilon)} \bar{H}(x + c_1 + y) dy. \quad (20)$$

Letting $n \rightarrow \infty$ we obtain that the tail at point x of the stationary distribution of the Markov chain X is not less than

$$\begin{aligned} \frac{(1 - \varepsilon)^2}{a + \varepsilon} \int_0^\infty \bar{H}(x + c_1 + y) dy &= \frac{(1 - \varepsilon)^2}{a + \varepsilon} \bar{H}_I(x + c_1) \\ &\sim \frac{(1 - \varepsilon)^2}{a + \varepsilon} \bar{H}_I(x) \quad \text{as } x \rightarrow \infty, \end{aligned}$$

due to the long-tailedness of the integrated tail distribution H_I . Summarising altogether we deduce that, for every fixed $\varepsilon > 0$,

$$\liminf_{x \rightarrow \infty} \frac{\mathbb{P}\{D_\infty > x\}}{\bar{H}_I(\log x)} \geq \frac{(1 - \varepsilon)^2}{a + \varepsilon},$$

which implies the lower bound (6) due to the arbitrary choice of $\varepsilon > 0$.

If the distribution H is long-tailed itself, then the integral in (20) is asymptotically equivalent to the integral

$$\int_x^{x+n(a+\varepsilon)} \overline{H}(y) dy \quad \text{as } x \rightarrow \infty \text{ uniformly for all } n \geq 1,$$

which implies the second lower bound (7).

Now let us turn to the asymptotic upper bound under the assumption that the integrated tail distribution H_I is subexponential. Fix an $\varepsilon \in (0, a)$. Let x_0 be defined as in (16), so $\mathbb{E}\zeta(x_0) \leq -a + \varepsilon$. Let J be the distribution of $\zeta(x_0)$. Since

$$\log(1 + A + B) - x_0 \leq \zeta(x_0) \leq \log(1 + A + B),$$

we have $\overline{H}(x + x_0) \leq \overline{J}(x) \leq \overline{H}(x)$. Then subexponentiality of H_I yields subexponentiality of the integrated tail distribution J_I and $\overline{J}_I(x) \sim \overline{H}_I(x)$ as $x \rightarrow \infty$.

By the construction of $\zeta(x_0)$,

$$x + \xi(x) \leq y + \zeta(x_0) \quad \text{for all } y \geq x \geq x_0. \quad (21)$$

Also, by the positivity of A ,

$$\begin{aligned} x + \xi(x) &= \log(1 + A(e^x - 1) + B) \\ &\leq \log(1 + A(e^{x_0} - 1) + B) \\ &= x_0 + \xi(x_0) \leq x_0 + \zeta(x_0) \quad \text{for all } x \leq x_0. \end{aligned} \quad (22)$$

Consider a random walk Z_n delayed at the origin with jumps $\zeta(x_0)$:

$$Z_0 := 0, \quad Z_n := (Z_{n-1} + \zeta_n(x_0))^+,$$

where $\zeta_n(x_0)$ are independent copies of $\zeta(x_0)$. The upper bounds (21) and (22) yield that the two chains X_n and Z_n can be constructed on a common probability space in such a way that, with probability 1,

$$X_n \leq x_0 + Z_n \quad \text{for all } n, \quad (23)$$

so X_n is dominated by a random walk on $[x_0, \infty)$ delayed at point x_0 . Since the integrated tail distribution J_I is subexponential, the tail of the invariant measure of the chain Z_n is asymptotically equivalent to $\overline{J}_I(x)/(a - \varepsilon) \sim \overline{H}_I(x)/(a - \varepsilon)$ as $x \rightarrow \infty$, see, for example, [6, Theorem 5.2]. Thus, the tail of the invariant measure of X_n is asymptotically not greater than $\overline{H}_I(x - x_0)/(a - \varepsilon)$ which is equivalent to $\overline{H}_I(x)/(a - \varepsilon)$, since H_I is long-tailed by subexponentiality. Hence,

$$\limsup_{x \rightarrow \infty} \frac{\mathbb{P}\{D_\infty > x\}}{\overline{H}_I(\log x)} \leq \frac{1}{a - \varepsilon}.$$

Due to arbitrary choice of $\varepsilon > 0$ and the lower bound proven above this completes the proof of the first asymptotics (8).

The same arguments with the same majorant (23) allow us to conclude the finite time horizon asymptotics for D_∞ if we apply Theorem 5.3 from [6] instead of Theorem 5.2. \square

Theorem 1 makes it possible to identify a moment of time after which the tail distribution of D_n is equivalent to that of D_∞ , in some particular strong subexponential cases.

Corollary 3. *Suppose that $\mathbb{E} \log A = -a < 0$, $B > 0$ and $\mathbb{E} \log(1 + B) < \infty$.*

If the distribution H of $\log(1 + A + B)$ is regularly varying at infinity with index $\alpha < -1$, then $\mathbb{P}\{D_n > x\} \sim \mathbb{P}\{D_\infty > x\}$ as $n, x \rightarrow \infty$ if and only if $n/\log x \rightarrow \infty$.

If $\bar{H}(x) \sim e^{-x^\beta}$ for some $\beta \in (0, 1)$, then $\mathbb{P}\{D_n > x\} \sim \mathbb{P}\{D_\infty > x\}$ as $n, x \rightarrow \infty$ if and only if $n/\log^{1-\beta} x \rightarrow \infty$.

We conclude this section by a version of the principle of a single big jump for D_n . For any $c > 1$ and $\varepsilon > 0$ consider events

$$\Omega_k := \{1/c < X_{k-1} \leq c, X_k > \log x + c + (n-k)(a+\varepsilon), \\ |X_{k+j} - X_k + aj| \leq c + j\varepsilon \text{ for all } j \leq n-k\}$$

or, in terms of D_n ,

$$\Omega_k^D := \{1/c < D_{k-1} \leq c, A_k/c + B_k > xe^{c+(n-k)(a+\varepsilon)}, \\ e^{-c-j(a+\varepsilon)} \leq D_{k+j}/D_k \leq e^{c-j(a-\varepsilon)} \text{ for all } j \leq n-k\}.$$

Roughly speaking, it describes a trajectory such that, for large x , the D_{k-1} is neither too far away from zero nor too close, then a single big jump occurs, both A_k and B_k may contribute to that big jump, and then the logarithm of D_{k+j} , $j \leq n-k$, moves down according to the strong law of large numbers with drift $-a$. As stated in the next theorem, the union of all these events describes more precisely than the lower bound of Theorem 1 the most probable way by which large deviations of D_n do occur.

Theorem 4. *Let the distribution H of $\log(1 + A + B)$ be strong subexponential. Then, for any fixed $\varepsilon > 0$,*

$$\lim_{c \rightarrow \infty} \liminf_{x \rightarrow \infty} \inf_{n \geq 1} \mathbb{P}\{\cup_{k=0}^{n-1} \Omega_k \mid D_n > x\} = 1.$$

Proof. The events $\Omega(k)$, $k \leq n$, are pairwise disjoint and any of them implies $\{X_n > \log x\}$. Then similar arguments as in the proof of lower bound in Theorem 1 apply. \square

3 Impact of atom at zero

In this section we demonstrate what happens if the distribution of A has an atom at zero. It turns out that then the tail asymptotics of D_n are essentially different – they are proportional to the tail of H which is lighter than given by the integrated tail distribution H_I in the case where $A > 0$ – because the chain satisfies Doeblin’s condition, see e.g. [14, Ch. 16]. As above, we denote by H the distribution of the random variable $\log(1 + A + B)$. For simplicity, we assume that $B > 0$.

Theorem 5. *Suppose that $A \geq 0$, $B > 0$ and $p_0 := \mathbb{P}\{A = 0\} \in (0, 1)$. If the distribution H is long-tailed and $D_0 > 0$, then*

$$\mathbb{P}\{D_n > x\} \geq \left(\frac{1 - (1 - p_0)^n}{p_0} + o(1) \right) \overline{H}(\log x) \quad (24)$$

as $x \rightarrow \infty$ uniformly for all $n \geq 1$. In particular,

$$\mathbb{P}\{D_\infty > x\} \geq (p_0^{-1} + o(1)) \overline{H}(\log x) \quad \text{as } x \rightarrow \infty. \quad (25)$$

If the distribution H is subexponential, $D_0 > 0$ and $\{D_0 > x\} = o(\overline{H}(x))$ then

$$\mathbb{P}\{D_n > x\} \sim \frac{1 - (1 - p_0)^n}{p_0} \overline{H}(\log x) \quad (26)$$

as $x \rightarrow \infty$ uniformly for all $n \geq 1$. In particular,

$$\mathbb{P}\{D_\infty > x\} \sim p_0^{-1} \overline{H}(\log x) \quad \text{as } x \rightarrow \infty. \quad (27)$$

Proof. Let H_0 be the distribution of $\log(1 + A + B)$ conditioned on $A > 0$ and G_0 be the distribution of $\log(1 + B)$ conditioned on $A = 0$, then $H = p_0 G_0 + (1 - p_0) H_0$.

Let us decompose the event $X_n > x$ according to the last zero value of A_k , which gives equality

$$\begin{aligned} \mathbb{P}\{X_n > x\} &= \mathbb{P}\{A_1, \dots, A_n > 0, X_n > x\} \\ &\quad + \sum_{k=1}^n \mathbb{P}\{A_k = 0, A_{k+1} > 0, \dots, A_n > 0, X_n > x\} \\ &= (1 - p_0)^n \mathbb{P}\{X_n > x \mid A_1, \dots, A_n > 0\} \\ &\quad + p_0 \sum_{k=1}^n (1 - p_0)^{n-k} \mathbb{P}\{X_n > x \mid A_k = 0, A_{k+1}, \dots, A_n > 0\} \\ &= (1 - p_0)^n \mathbb{P}\{X_n > x \mid A_1, \dots, A_n > 0\} \\ &\quad + p_0 \sum_{k=0}^{n-1} (1 - p_0)^k \mathbb{P}\{X_{k+1} > x \mid A_1 = 0, A_2, \dots, A_{k+1} > 0\}, \end{aligned} \quad (28)$$

by the Markov property. In particular, the sum from 0 to $n - 1$ on the right hand side is increasing as n grows as all terms are positive. For that reason, for the lower bounds for $\mathbb{P}\{D_n > x\}$ it suffices to prove by induction that, for any fixed $k \geq 0$ and $\gamma > 0$, there exists a $c < \infty$ such that

$$\begin{aligned} \mathbb{P}\{X_{k+1} > x \mid A_1 = 0, A_2, \dots, A_{k+1} > 0\} \\ \geq (1 - \gamma)(\overline{G}_0(x + c) + k\overline{H}_0(x + c)), \end{aligned} \quad (29)$$

$$\mathbb{P}\{X_{k+1} > x \mid A_1, \dots, A_{k+1} > 0\} \geq (1 - \gamma)(k + 1)\overline{H}_0(x + c) \quad (30)$$

for all sufficiently large x , because then

$$\begin{aligned} \mathbb{P}\{X_n > x\} &\geq (1 - \gamma) \left((1 - p_0)^n n \overline{H}_0(x + c) \right. \\ &\quad \left. + p_0 \sum_{k=0}^{n-1} (1 - p_0)^k (\overline{G}_0(x + c) + k\overline{H}_0(x + c)) \right) \\ &= (1 - \gamma) \left((1 - (1 - p_0)^n) \left(\overline{G}_0(x + c) + \frac{1 - p_0}{p_0} \overline{H}_0(x + c) \right) \right) \\ &= (1 - \gamma) \frac{1 - (1 - p_0)^n}{p_0} \overline{H}(x + c), \end{aligned}$$

with further application of long-tailedness of H .

To prove (29), first let us note that the induction basis $k = 0$ is immediate, since the distribution of X_1 conditioned on $A_1 = 0$ is G_0 . Now let us assume that (29) is true for some k . Denote

$$G_k(dy) := \mathbb{P}\{X_{k+1} \in dy \mid A_1 = 0, A_2, \dots, A_{k+1} > 0\}, \quad k \geq 0,$$

which is a distribution on $(0, \infty)$. Then

$$\begin{aligned} \overline{G}_{k+1}(x) &= \int_0^\infty \mathbb{P}\{\log(1 + A(e^y - 1) + B) > x \mid A > 0\} G_k(dy) \\ &\geq \int_\varepsilon^{1/\varepsilon} \mathbb{P}\{\log(1 + A\delta + B) > x \mid A > 0\} G_k(dy) \\ &\quad + \int_{x+1/\varepsilon}^\infty \mathbb{P}\{\log(A(e^y - 1)) > x \mid A > 0\} G_k(dy) \\ &=: I_1 + I_2, \end{aligned}$$

for any $\varepsilon \in (0, 1/2]$ where $\delta = e^\varepsilon - 1 < \sqrt{e} - 1 < 1$. Let us observe that then

$$\begin{aligned} \mathbb{P}\{\log(1 + A\delta + B) > x \mid A > 0\} &= \mathbb{P}\{\log(1/\delta + A + B/\delta) > x - \log \delta \mid A > 0\} \\ &\geq \overline{H}_0(x - \log \delta). \end{aligned}$$

Therefore,

$$I_1 \geq \overline{H}_0(x - \log \delta) G_k(\varepsilon, 1/\varepsilon].$$

The second integral may be bounded below as follows:

$$\begin{aligned} I_2 &\geq \mathbb{P}\{\log(A(e^{x+1/\varepsilon} - 1)) > x \mid A > 0\} \overline{G}_k(x + 1/\varepsilon) \\ &\geq \mathbb{P}\{\log(Ae^{x+1/2\varepsilon}) > x \mid A > 0\} \overline{G}_k(x + 1/\varepsilon) \\ &= \mathbb{P}\{A > e^{-1/2\varepsilon} \mid A > 0\} \overline{G}_k(x + 1/\varepsilon), \end{aligned}$$

for all sufficiently large x . Letting $\varepsilon \rightarrow 0$ we obtain that, for any fixed $\gamma > 0$, there exists a $c < \infty$ such that the following lower bound holds

$$\overline{G}_{k+1}(x) \geq (1 - \gamma)(\overline{H}_0(x + c) + \overline{G}_k(x + c))$$

for all sufficiently large x , which implies the induction step.

The second lower bound, (30), follows by similar arguments provided $D_0 > 0$.

Let us now proceed with a matching upper bound under the assumption that H is a subexponential distribution. Since $A, B \geq 0$,

$$\xi(x) = \log(A + e^{-x}(1 - A + B)) \quad (31)$$

$$\leq \log(1 + A + B) \quad \text{for all } x > 0. \quad (32)$$

Let η and ζ be random variables with the following tail distributions

$$\begin{aligned} \mathbb{P}\{\eta > x\} &= \min\left(1, \frac{\mathbb{P}\{\log(1 + A + B) > x\}}{\mathbb{P}\{A = 0\}}\right), \\ \mathbb{P}\{\zeta > x\} &= \min\left(1, \frac{\mathbb{P}\{\log(1 + A + B) > x\}}{\mathbb{P}\{A > 0\}}\right), \quad x > 0. \end{aligned}$$

Both are subexponential random variables provided $\log(1 + A + B)$ is so, see e.g. [6, Corollary 3.13]. It follows from (31) that, for all $x > 0$,

$$\begin{aligned} \mathbb{P}\{\xi(x) > y \mid A = 0\} &\leq \mathbb{P}\{\eta > y\}, \\ \mathbb{P}\{\xi(x) > y \mid A > 0\} &\leq \mathbb{P}\{\zeta > y\}, \end{aligned}$$

which implies that

$$\mathbb{P}\{X_{k+1} > x \mid A_1 = 0, A_2, \dots, A_{k+1} > 0\} \leq \mathbb{P}\{\eta + \zeta_1 + \dots + \zeta_k > x\},$$

where ζ_i 's are independent copies of ζ independent of η . Then standard technique based on Kesten's bound for convolutions of subexponential distributions, see e.g.

Theorem 3.39 in [6], allows us to deduce from (28) that, for any fixed $\gamma > 0$,

$$\begin{aligned} \mathbb{P}\{X_n > x\} &\leq (1 + \gamma) \left((1 - p_0)^n \bar{G}_0(x) + p_0 \sum_{k=0}^{n-1} (1 - p_0)^k (\bar{G}_0(x) + k \bar{H}_0(x)) \right) \end{aligned}$$

for all $n \geq 1$ and sufficiently large x . Therefore,

$$\mathbb{P}\{X_n > x\} \leq (1 + \gamma) \frac{1 - (1 - p_0)^n}{p_0} \bar{H}(x),$$

which together with the lower bound proves (26). \square

4 The case of positive A and signed B

In this section we consider the case where D_n takes both positive and negative values because of signed B , while A is still assumed positive in this section, $A > 0$. The Markov chain X_n is defined as in (2).

As B is no longer assumed positive, it makes the tail behaviour of D quite different if no further assumptions are made on dependency between A and B . For example, in the extreme case where $B = -cA$ for some $c > 0$, so $D_{n+1} = A_{n+1}(D_n - c)$, we have that D_n is eventually negative if stable, hence $D_\infty < 0$ with probability 1.

More generally, if $B = A\eta$ where η is independent of A and takes values of both signs, then we conclude similar to (6) that, as $x \rightarrow \infty$,

$$\mathbb{P}\{D_\infty > x\} \geq \left(\frac{1}{a} \int_{\mathbb{R}} \mathbb{P}\{\eta > -c\} \mathbb{P}\{D_\infty \in dc\} + o(1) \right) \bar{F}_I(\log x),$$

provided the distribution F_I is long-tailed. However, the technique used in Section 2 for proving the matching upper bound does not work in such cases as the Lindley majorant returns the coefficient a^{-1} which is greater than that in the lower bound above. For that reason we restrict further considerations to the case where A and B are independent.

Theorem 6. *Suppose that $A > 0$, A and B are independent, $\mathbb{E}\xi = -a \in (-\infty, 0)$ and $\mathbb{E} \log(1 + |B|) < \infty$.*

If the integrated tail distributions F_I and G_I^+ are long-tailed, then

$$\mathbb{P}\{D_\infty > x\} \geq (a^{-1} + o(1)) \left(\mathbb{P}\{D_\infty > 0\} \bar{F}_I(\log x) + \bar{G}_I^+(\log x) \right) \text{ as } x \rightarrow \infty. \quad (33)$$

If, in addition, the distributions F and G^+ are long-tailed itself, then, as $x, n \rightarrow \infty$,

$$\begin{aligned} & \mathbb{P}\{D_n > x\} \\ & \geq \frac{1+o(1)}{a} \left(\mathbb{P}\{D_\infty > 0\} \int_{\log x}^{\log x+na} \overline{F}(y) dy + \int_{\log x}^{\log x+na} \overline{G^+}(y) dy \right). \end{aligned} \quad (34)$$

If $\mathbb{P}\{D_\infty = 0\} = 0$, the integrated tail distributions F_I , G_I^+ and G_I^- are long-tailed, $\overline{G_I^-}(z) = O(\overline{F_I}(z) + \overline{G_I^+}(z))$ and H_I is subexponential then

$$\mathbb{P}\{D_\infty > x\} \sim a^{-1} \left(\mathbb{P}\{D_\infty > 0\} \overline{F_I}(\log x) + \overline{G_I^+}(\log x) \right) \text{ as } x \rightarrow \infty. \quad (35)$$

If moreover the distributions F , G^+ and G^- are long-tailed, $\overline{G^-}(z) = O(\overline{F}(z) + \overline{G^+}(z))$ and H is strong subexponential then, as $x, n \rightarrow \infty$,

$$\mathbb{P}\{D_n > x\} \sim \frac{1}{a} \left(\mathbb{P}\{D_\infty > 0\} \int_{\log x}^{\log x+na} \overline{F}(y) dy + \int_{\log x}^{\log x+na} \overline{G^+}(y) dy \right). \quad (36)$$

Proof. Fix an $\varepsilon > 0$. As follows from (4), for $x \geq 0$,

$$\xi(x) \geq \begin{cases} \log(A(1 - e^{-x}) - e^{-x}B^-) & \text{if } A(e^x - 1) + B \geq 0, \\ -\log(1 + A + |B|) & \text{if } A(e^x - 1) + B < 0, \end{cases}$$

where the second line follows due to $A > 0$. The minorant on the right hand side is stochastically increasing as x grows, therefore, there exists a sufficiently large x_0 and a random variable η such that

$$\xi(x) \geq \eta \quad \text{for all } x \geq x_0 \quad \text{and} \quad \mathbb{E}\eta > -a - \varepsilon/2. \quad (37)$$

As in the last proof, we start with the lower bound (33) following the single big jump technique. Since D_n is assumed to be convergent, the associated Markov chain X_n is stable, so there exist n_0 and $c > 2$ such that

$$\begin{aligned} \mathbb{P}\{X_n \in (1/c, c]\} & \geq (1 - \varepsilon) \mathbb{P}\{D_\infty > 0\} \quad \text{for all } n \geq n_0, \\ \mathbb{P}\{|X_n| \leq c\} & \geq 1 - \varepsilon \quad \text{for all } n, \end{aligned}$$

and also $\mathbb{P}\{A \leq c\} \geq 1 - \varepsilon$, $\mathbb{P}\{|B| \leq c\} \geq 1 - \varepsilon$. For all k, n and c , let us consider the events $\Omega(k, n, c)$ defined in (17) and satisfying (18). It follows from (37) that any of the events (19) implies $X_n > x$ and they are pairwise disjoint. Therefore,

by the Markov property and (18),

$$\begin{aligned}
& \mathbb{P}\{X_n > x\} \\
& \geq \sum_{k=1}^n \mathbb{P}\{X_{k-1} \leq c, X_k > x + c + (n-k)(a + \varepsilon)\} \mathbb{P}\{\Omega(k, n-k, c)\} \\
& \geq (1 - \varepsilon) \sum_{k=1}^n \mathbb{P}\{X_{k-1} \leq c, X_k > x + c + (n-k)(a + \varepsilon)\}, \tag{38}
\end{aligned}$$

The k th term of the sum is not less than

$$\begin{aligned}
& \left(\int_{-c}^0 + \int_0^c \right) \mathbb{P}\{X_{k-1} \in dy\} \mathbb{P}\{y + \xi(y) > z_{n-k}\} \\
& = \int_{-c}^0 \mathbb{P}\{X_{k-1} \in dy\} \mathbb{P}\{\log(1 + A(1 - e^{-y}) + B) > z_{n-k}\} \\
& \quad + \int_0^c \mathbb{P}\{X_{k-1} \in dy\} \mathbb{P}\{\log(1 + A(e^y - 1) + B) > z_{n-k}\} \\
& =: I_1 + I_2,
\end{aligned}$$

where $z_k = x + c + k(a + \varepsilon)$. For all $y \in [-c, 0]$ and $z > 0$, owing to the condition $A > 0$ and independence of A and B

$$\begin{aligned}
\mathbb{P}\{\log(1 + A(1 - e^{-y}) + B) > z\} & \geq \mathbb{P}\{\log(1 - Ae^c + B) > z\} \\
& \geq \mathbb{P}\{A \leq c\} \mathbb{P}\{\log(1 - ce^c + B) > z\} \\
& \geq \mathbb{P}\{A \leq c\} \overline{G^+}(z + 1)
\end{aligned}$$

for all sufficiently large z which yields that

$$\begin{aligned}
I_1 & \geq \mathbb{P}\{A \leq c\} \mathbb{P}\{X_{k-1} \in [-c, 0]\} \overline{G^+}(z_{n-k} + 1) \\
& \geq (1 - \varepsilon) \mathbb{P}\{X_{k-1} \in [-c, 0]\} \overline{G^+}(z_{n-k} + 1), \tag{39}
\end{aligned}$$

due to the choice of c . For all $y > 0$,

$$\begin{aligned}
& \mathbb{P}\{\log(1 + A(e^y - 1) + B) > z\} \\
& \geq \mathbb{P}\{|B| \leq c\} \mathbb{P}\{\log(1 + A(e^y - 1) - c) > z\} + \mathbb{P}\{\log(1 + B) > z\},
\end{aligned}$$

which yields that

$$\begin{aligned}
I_2 & \geq \mathbb{P}\{|B| \leq c\} \int_{1/c}^c \mathbb{P}\{\log(1 + A(e^y - 1) - c) > z_{n-k}\} \mathbb{P}\{X_{k-1} \in dy\} \\
& \quad + \overline{G^+}(z_{n-k}) \mathbb{P}\{X_{k-1} \in (0, c]\} \\
& \geq (1 - \varepsilon) \mathbb{P}\{\log(1 + A(e^{1/c} - 1) - c) > z_{n-k}\} \mathbb{P}\{X_{k-1} \in (1/c, c]\} \\
& \quad + \overline{G^+}(z_{n-k}) \mathbb{P}\{X_{k-1} \in (0, c]\}.
\end{aligned}$$

Therefore, by the choice of c , for all sufficiently large x and $k > n_0$,

$$I_2 \geq (1 - \varepsilon)^2 \mathbb{P}\{D_\infty > 0\} \overline{F}(z_{n-k} + 1) + \overline{G}^+(z_{n-k}) \mathbb{P}\{X_{k-1} \in (0, c]\}. \quad (40)$$

Substituting (39) and (40) into (38) we deduce that

$$\begin{aligned} \mathbb{P}\{X_n > x\} &\geq (1 - \varepsilon)^2 \sum_{k=n_0+1}^n \left(\mathbb{P}\{D_\infty > 0\} \overline{F}(x + c + 1 + (n - k)(a + \varepsilon)) \right. \\ &\quad \left. + \overline{G}^+(x + c + 1 + (n - k)(a + \varepsilon)) \right) \end{aligned}$$

Since the tail is a non-increasing function, the last sum is not less than

$$\frac{1}{a + \varepsilon} \int_0^{(n-n_0-1)(a+\varepsilon)} \left(\mathbb{P}\{D_\infty > 0\} \overline{F}(x + c + 1 + y) + \overline{G}^+(x + c + 1 + y) \right) dy. \quad (41)$$

Letting $n \rightarrow \infty$ we obtain that the tail at point x of the stationary distribution of the Markov chain X is not less than

$$\begin{aligned} &\frac{(1 - \varepsilon)^2}{a + \varepsilon} \int_0^\infty \left(\mathbb{P}\{D_\infty > 0\} \overline{F}(x + c + 1 + y) + \overline{G}^+(x + c + 1 + y) \right) dy \\ &= \frac{(1 - \varepsilon)^2}{a + \varepsilon} \left(\mathbb{P}\{D_\infty > 0\} \overline{F}_I(x + c + 1) + \overline{G}_I^+(x + c + 1) \right) \quad (42) \\ &\sim \frac{(1 - \varepsilon)^2}{a + \varepsilon} \left(\mathbb{P}\{D_\infty > 0\} \overline{F}_I(x) + \overline{G}_I^+(x) \right) \quad \text{as } x \rightarrow \infty, \end{aligned}$$

due to the long-tailedness of the integrated tail distributions F_I and G_I^+ . Summarising altogether we deduce that, for every fixed $\varepsilon > 0$,

$$\liminf_{x \rightarrow \infty} \frac{\mathbb{P}\{D_\infty > x\}}{\mathbb{P}\{D_\infty > 0\} \overline{F}_I(\log x) + \overline{G}_I^+(\log x)} \geq \frac{(1 - \varepsilon)^2}{a + \varepsilon},$$

which implies the lower bound (33) due to the arbitrary choice of $\varepsilon > 0$.

If the distributions F and G^+ are long-tailed itself, then the integral in (41) is asymptotically equivalent to the integral

$$\int_x^{x+n(a+\varepsilon)} \left(\mathbb{P}\{D_\infty > 0\} \overline{F}(y) + \overline{G}^+(y) \right) dy \quad \text{as } x, n \rightarrow \infty,$$

and the second lower bound (34) follows too.

To prove matching upper bounds let us first observe that

$$|D_{n+1}| \leq A_n |D_n| + |B_n| \quad \text{for all } n, \quad (43)$$

where the right hand side is increasing in D_n . Hence, $|D_n| \leq \tilde{D}_n$, where \tilde{D}_n is a positive stochastic difference recursion,

$$\tilde{D}_{n+1} = A_n \tilde{D}_n + |B_n|.$$

Since H_I is subexponential, Theorem 1 applies to \tilde{D}_n , so

$$\mathbb{P}\{\tilde{D}_\infty > x\} \sim a^{-1} \overline{H_I}(\log x) \quad \text{as } x \rightarrow \infty,$$

and hence

$$\mathbb{P}\{|D_\infty| > x\} \leq (a^{-1} + o(1)) \overline{H_I}(\log x) \quad \text{as } x \rightarrow \infty,$$

It follows from (12) that

$$\overline{H}(x) \leq \mathbb{P}\{\log(1+A) > x-1\} + \mathbb{P}\{\log(1+|B|) > x-1\}.$$

Integrating the last inequality we get an upper bound

$$\begin{aligned} \overline{H_I}(x) &\leq \overline{F_I}(x-1) + \overline{G_I^-}(x-1) + \overline{G_I^+}(x-1) \\ &\sim \overline{F_I}(x) + \overline{G_I^-}(x) + \overline{G_I^+}(x) \quad \text{as } x \rightarrow \infty, \end{aligned} \quad (44)$$

because all three distributions, F_I , G_I^- and G_I^+ are assumed long-tailed. Hence the following upper bound holds for the tail of $|D_\infty|$, as $x \rightarrow \infty$:

$$\mathbb{P}\{|D_\infty| > x\} \leq (a^{-1} + o(1)) (\overline{F_I}(\log x) + \overline{G_I^-}(\log x) + \overline{G_I^+}(\log x)). \quad (45)$$

The long-tailedness of F_I and G_I^- similarly to (33) implies that

$$\mathbb{P}\{D_\infty < -x\} \geq (a^{-1} + o(1)) (\mathbb{P}\{D_\infty < 0\} \overline{F_I}(\log x) + \overline{G_I^-}(\log x)),$$

and the two lower bounds together imply that, as $x \rightarrow \infty$,

$$\mathbb{P}\{|D_\infty| > x\} \geq (a^{-1} + o(1)) (\overline{F_I}(\log x) + \overline{G_I^+}(\log x) + \overline{G_I^-}(\log x)),$$

because $\mathbb{P}\{D_\infty = 0\} = 0$. Together with the upper bound (45) it yields that

$$\mathbb{P}\{D_\infty > x\} = a^{-1} (\mathbb{P}\{D_\infty > 0\} \overline{F_I}(\log x) + \overline{G_I^+}(\log x)) + o(\overline{H_I}(\log x)),$$

and the first asymptotics (35) follows by the condition $\overline{G_I^-}(z) = O(\overline{F_I}(z) + \overline{G_I^+}(z))$.

The second asymptotics (36) follows along similar arguments. \square

5 Balance of negative and positive tails in the case of signed

A

In this section we turn to the general case where D_n takes both positive and negative values, with A taking values of both signs. Denote $\xi := \log |A|$ and the distribution of $\log(1 + |A|)$ by F . Recall that the distribution of $\log(1 + |B|)$ is denoted by G and the distribution of $\log(1 + |A| + |B|)$ by H .

The Markov chain X_n is defined as above in (2).

Theorem 7. *Suppose that $\mathbb{P}\{D_\infty = 0\} = 0$,*

$$0 < \mathbb{P}\{A > 0\} < 1, \quad (46)$$

A and B are independent, $\mathbb{E}\xi = -a \in (-\infty, 0)$ and $\mathbb{E} \log(1 + |B|) < \infty$.

If the integrated tail distribution H_I is long-tailed, then

$$\mathbb{P}\{D_\infty > x\} \geq (1/2a + o(1))\overline{H}_I(\log x) \quad \text{as } x \rightarrow \infty. \quad (47)$$

If, in addition, the distribution H is long-tailed itself, then

$$\mathbb{P}\{D_n > x\} \geq \frac{1 + o(1)}{2a} \int_{\log x}^{\log x + na} \overline{H}(y) dy \quad \text{as } n, x \rightarrow \infty. \quad (48)$$

If the integrated tail distribution H_I is subexponential then

$$\mathbb{P}\{D_\infty > x\} \sim \frac{1}{2a} \overline{H}_I(\log x) \quad \text{as } x \rightarrow \infty. \quad (49)$$

If moreover the distribution H is strong subexponential then

$$\mathbb{P}\{D_n > x\} \sim \frac{1}{2a} \int_{\log x}^{\log x + na} \overline{H}(y) dy \quad \text{as } n, x \rightarrow \infty. \quad (50)$$

Proof. The same arguments based on the single big jump technique used in the last section for proving (42) show that, for any fixed $\varepsilon > 0$, there exists a $c < \infty$ such that

$$\mathbb{P}\{|X_\infty| > x\} \geq \frac{1 - \varepsilon}{a} (\mathbb{P}\{D_\infty \neq 0\} \overline{F}_I(x + c + 1) + \overline{G}_I(x + c + 1))$$

for all sufficiently large x . Similar to (44),

$$\overline{H}_I(x) \leq \overline{F}_I(x - 1) + \overline{G}_I(x - 1)$$

for all sufficiently large x , which together with the condition $\mathbb{P}\{D_\infty = 0\} = 0$ implies that

$$\begin{aligned}\mathbb{P}\{|X_\infty| > x\} &\geq \frac{1-\varepsilon}{a}\overline{H_I}(x+c+2) \\ &\sim \frac{1-\varepsilon}{a}\overline{H_I}(x) \quad \text{as } x \rightarrow \infty,\end{aligned}$$

due to the long-tailness of the distribution H_I . Therefore,

$$\mathbb{P}\{|X_\infty| > x\} \geq (a^{-1} + o(1))\overline{H_I}(x) \quad \text{as } x \rightarrow \infty. \quad (51)$$

At any time epoch n large absolute value of X_n changes its sign with asymptotic (as $x \rightarrow \infty$) probability $p^- = \mathbb{P}\{A < 0\}$ and keeps its sign with asymptotic probability $p^+ = \mathbb{P}\{A > 0\}$, so sign change may be asymptotically described as a Markov chain with transition probability matrix

$$\begin{pmatrix} p^+ & p^- \\ p^- & p^+ \end{pmatrix},$$

whose asymptotic distribution is $(1/2, 1/2)$, owing to the condition (46). For that reason, the probability of a large positive value of X_n is approximately at least one half of the right hand side of (51), and the proof of (47) is complete. The proof of (48) follows the same lines.

To prove the upper bound (49), similar to (43) we first note that

$$|D_{n+1}| \leq |A_n||D_n| + |B_n| \quad \text{for all } n,$$

which allows to conclude the proof as it was done in the last section. \square

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