ON THE CORRESPONDENCE BETWEEN GRADED-LOCAL CONFORMAL NETS AND VERTEX OPERATOR SUPERALGEBRAS WITH APPLICATIONS

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Abstract

This thesis deals with two different mathematical axiomatisations of chiral Conformal Field Theory (CFT). On the one hand, there is the axiomatisation via Haag-Kastler axioms, in an operator algebraic setting, dealing with families of von Neumann algebras, also known as graded-local conformal net theory. On the other hand, there is the axiomatisation via vertex operator superalgebras (VOSAs).

In [CKLW18], the authors establish a correspondence between these two settings, taking into account Bose theories only, that is, local conformal net theory and vertex operator algebra (VOA) theory respectively.

In this thesis, we extend the correspondence given in [CKLW18] to the case of graded-local conformal nets and VOSAs, in order to include Fermi theories too. Furthermore, we present completely new results about theories with non-trivial grading.

As an application of the correspondence between local conformal nets and VOAs, we classify all the subtheories of specific concrete models, known as even rank-one lattice chiral CFTs.

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Segni, 5th July 2021. *Tiziano Gaudio* [1].

[1] Un tifoso giallorosso. Daje Roma sempre! - *(An A.S. Roma supporter. Always come on Roma!)*

Contents

Introduction

It is possible to approach *chiral Conformal Field Theory*, briefly *chiral CFT*, from different ways of mathematics and the relationships between those several approaches have been subjected to extensive studies. Indeed, the relation between the two, that is the analytic and the algebraic way, constitutes the main topic of this thesis. Before going further with the presentation of it, it is worthwhile to report some facts to introduce and correctly place chiral CFT in the wide world of mathematics. However, it is not our intention to give a full account of the genesis of chiral CFT. Thus, we redirect more curious readers to the specific references, which we give throughout the exposition. It is in no way our aim to give a complete list of references and we apologise if we miss some of the major ones.

A brief historical background

Quantum Field Theory (QFT) brings together three different branches of physics: Classical Field Theory, Special Relativity and Quantum Mechanics. The main goal of QFT is to provide physical models for the description of systems of subatomic particles (such as, photons, electrons, quarks and so on) and their interactions. Therefore, QFT constitutes the ground for modern elementary particle physics and can be successfully used in other areas of physics, such as nuclear and atomic physics, condensed matter physics and astrophysics. Moreover, the Standard Model, the physical theory which unifies three of the four known fundamental forces of Nature (strong, weak and electromagnetic interactions, but not gravitational one), is described as a QFT. What is required of QFT to be efficient is having a predictive feature to describe particle scatterings. The branch of QFT studying particle scatterings is known as perturbative QFT and its major tool is renormalization. Naively speaking, renormalization can be understood as those "technical procedures" used by quantum field theorists to solve a perturbative model and thus to give effective predictions on a real world interacting system. Both a blessing and a curse, renormalization is a sturdy bridge between experimental feedbacks and theoretical predictions, but it lacks a rigorous mathematical playground. At this point, the pragmatic reader may ask whether humans really need a rigorous mathematical playground for QFT or it is just a mathematician's whim. The answer is we definetely need it. It should be sufficient to think that the Standard Model, which is the most important theory expressible through QFT, lacks completeness, describing just a little part of our real world. Even a unification of the Standard Model and General Relativity must pass through a deeper understanding of QFT. Mathematics must be the vehicle to reach that goal. For an introduction to QFT and related topics, the reader may consult [Wei05I, Wei05II, Wei05III], see also [Fol08] or the physics textbooks [PS95] and [Ste94].

The search for a rigorous mathematical framework for QFT brought to the foundation of a parallel area of mathematical physics, blessing the birth of two different axiomatisations of the theory: *Wightman QFT* in the '50s, see [SW64] and *Algebraic Quantum Field Theory (AQFT)* later in the '60s, see [Haag96] and [Ara10]. Wightman QFT focuses on the mathematical definition of *quantum fields*, whereas AQFT on the *observable quantities* or just *observables* of the underlying physical system, such as position and momentum.

A question arises naturally: physically speaking, what is a quantum field? The concept of field was introduced to implement Faraday's idea of *locality*, proposed around 1830, which substituted the concept of forces acting at distance of Classical Mechanics. In contrast, the locality principle affirms that physical effects propagate from point to neighbouring point, so that all points of the space participate to the physical process. Then, see the beginning of [Haag96, Section I.2], fields are "the vehicles by which this idea is put to work [...]". A classical example is the electrostatic field generated by a stationary electric charge, where to every point of the space is associated a three-dimensional vector. Therefore, a quantum field can be understood as a way to implement the locality principle in the quantum mechanical context, reconciling Classical Field Theory and Quantum Mechanics.

Now, let us have a naive look at these two axiomatisations of QFT, just to have a generic idea of the physical interpretation of the mathematics as presented.

Mathematically, a quantum field in the Wightman framework is an operator-valued distribution, that is, a linear function Φ from the Schwartz space $\mathcal{S}$ of rapidly decreasing functions on a d-dimensional ($d \geq 2$) Minkowski spacetime $\mathcal{M}$ to the space of (not necessarily bounded) operators on a given Hilbert space $\mathcal{H}$. Note that the definition of *Wightman fields* just given is a precise quantum version of a classical field, which is usually regarded as a scalar/vector/tensor-valued linear map on the d-dimensional Minkowski spacetime as for example, the electromagnetic field is. On the one hand, the Minkowski spacetime, where physical systems live, is a heritage of Special Relativity. On the other hand, the Hilbert space $\mathcal{H}$ realises the probabilistic interpretation of the theory, coming from Quantum Mechanics. Indeed, a vector ψ in $\mathcal{H}$ represents a possible configuration or *state* of the system. Furthermore, we want to include in our theory a state Ω representing the *quantum vacuum*, that is, when the system is in the configuration with lowest possible energy. Hence, from the creation-annihilation point of view, an application of the *smeared field* $\Phi(f)$ for a function $f \in \mathcal{S}$ to the vacuum Ω (so the vacuum state must be in the domain of all the smeared fields) produces a state of the system $\Phi(f)\Omega$ with one particle alone in the world; whereas an application of its adjoint $\Phi(f)^*$ to a generic state lower the particle number by one, creating an antiparticle when applied to the vacuum state. Note that the quantum vacuum is the state with no particles by this interpretation. Moreover, it is usually required that polynomials in the smeared fields and their adjoints generate a dense subset of $\mathcal{H}$ whenever applied to the vacuum state, that is, the vacuum is a cyclic state. Insisting on the physical interpretation, the locality principle, proper of a field theory, is fused with the relativistic *Einstein's causality principle*, that is, no physical effect can propagate faster than light. This fusion is realised by asking that the smeared fields commute or anticommute whenever the supports of their schwartzian test functions are contained in space-like [2] separated regions of $\mathcal{M}$. Commuting or anticommuting depends on the fact of life that there exist (as far as we known) only two kinds of quantum fields as well as of subatomic particles: *bosons* and *fermions* respectively. The first obey the *Bose-Einstein statistic*, that is, they are unaffected by the exchange of two of them; whereas the second are antisymmetric under the exchange of two of them, which is the so called *Fermi-Dirac statistic*.

The *Haag-Kastler axioms* constitute the mathematical framework for AQFT. Observables in the Haag-Kastler setting are self-adjoint elements of an isotonous (inclusion preserving) net of concrete von Neumann algebras, which is the main character of the theory. In complete analogy with Wightman setting, this net is a linear map $\mathcal{M} \supset \mathcal{O} \mapsto \mathcal{A}(\mathcal{O}) \subset B(\mathcal{H})$, where $\mathcal{O}$ is a compact closure open region of a d-dimensional Minkowski spacetime and $B(\mathcal{H})$ is the C*-algebra of bounded operators acting on a given Hilbert space $\mathcal{H}$. Also in this case, we require the existence of a vacuum vector Ω in $\mathcal{H}$, which is cyclic for the net of von Neumann algebras. The physical interpretation of $\mathcal{M}$, $\mathcal{H}$ and Ω is the same as for the Wightman framework. Similarly, the locality principle is implemented requiring that operators contained in von Neumann algebras coming from causally disjoint regions, commute or anticommute depending on their statistics. The algebraic character of the Haag-Kastler approach can be further emphasised considering the isotonous net of concrete von Neumann algebras as deriving from a GNS representation of an isotonous net of abstract C*-algebras, induced by a linear functional giving the vacuum vector Ω and the Hilbert space $\mathcal{H}$ of the theory [3], see [FR20] for a brief introduction in this spirit.

Another fundamental concept for a physical theory, which has not been introduced until now, is *symmetry*. The choice of required symmetries which a given theory must satisfy, is due to several considerations: they can be chosen by a purely theoretical argument or suggested by experimental settings. The usual real world situation described by QFT is a subatomic

[2]Recall that for every *point-event* x of $\mathcal{M}$, there are three distinguished regions as a consequence of the Einstein's causality principle: the *forward cone* V^+, containing all the events which can be causally influenced by x; the *backward cone* V^-, containing all the events which can have influenced x; the *space-like region*, which is the complement of the two cones and consists of all events having no causal relation with x. V^+ and V^- are usually called the *future* and the *past* of x, whereas the boundary of V^+ is constituted by all events which can be reached by *light signals* only (as the fastest possible ones) from x.

[3]This cut between the algebraic data and its concrete realisation becomes relevant when one lacks a privileged vacuum representation for the observables. This is what usually happens in perturbative AQFT and AQFT on curved spacetimes, see e.g. [Rej16] and [FR16] respectively.

particle experiment implemented in a laboratory displaced somewhere on the Earth or on a spaceship crossing the Universe. Then, all possible displacements of our laboratory, limitedly to the prescriptions of Special Relativity, are given by the isometries of the Minkowski spacetime, which constitute the *Poincaré group*. E. P. Wigner was the first to introduce Poincaré symmetries in terms of projective unitary representations acting on the Hilbert space $\mathcal{H}$ of a given QFT, see [Wig59]. On this line, both the QFT axiomatisations above provide that a projective unitary representation U of the Poincaré group is implemented on the Hilbert space $\mathcal{H}$ of the respective theories. This fact reflects the fascinating as well as obvious concept that the possible states of the system are limited by the symmetries of the system itself. Moreover, the vacuum vector Ω is characterised as the unique, up to a constant factor, invariant vector with respect to U. This makes explicit that the physical quantum vacuum remains the same for all the observers jointed by any Poincaré transformation. In application of the *covariance principle*, U must act covariantly on Wightman fields and on von Neumann algebras respectively. This implies that the physical laws are preserved when we change the frame or reference by an isometry of the Minkowski spacetime.

Isometries of the Poincaré group are those transformations which preserve both the causal structure and distances of the Minkowski spacetime. However, if we have to analyse a system which shows additional symmetries, such as scaling invariance, then it makes sense to not force the theory to be distance-preserving. This is what happens in several situations, such as in statistical critical phenomena (phase transitions), massless field theories or (perturbative) String Theory. The enlarged group of symmetries of the spacetime, which allows all those transformations preserving the casual structure only, is called the *conformal group*. Accordingly, a quantum (or classical) field theory showing conformal symmetries is called a *Conformal Field Theory*. It turns out that the conformal group can be described by a finite number of independent parameters (even if they are increased compare to the Poincaré group, e.g. see [BGL93, Section 1]), whenever the spacetime dimension is strictly bigger then two. Contrarily, for two-dimensional spacetimes, the conformal group is very rich. Indeed, there is an infinite number of parameters because every locally analytic function can describe a well-defined local conformal transformation. This richness is fundamental in the resolution of models presenting conformal symmetries. Indeed, the more symmetries are shown by the system, the more restrictions can be imposed to completely describe it. (See [DMS97, Chapter 1] and also [Wei05, Chapter 1.2]). Even more, in several two-dimensional CFT models, quantum fields can be decomposed into two components "living" in the left and right lightrays, called *chiral parts*. Under certain conditions, it is convenient to compactify a lightray to the unit circle S^1, so that the conformal group for every chiral part is represented by (the universal cover $\mathrm{Diff}^+(S^1)^{(\infty)}$ of) the Lie group $\mathrm{Diff}^+(S^1)$ of all *orientation-preserving diffeomorphisms* of S^1. Therefore, a *chiral CFT* is, naively speaking, a QFT living in the unit circle S^1 with symmetries given by the universal cover $\mathrm{Diff}^+(S^1)^{(\infty)}$ of the conformal group $\mathrm{Diff}^+(S^1)$.

The AQFT approach via Haag-Kastler axioms to chiral CFT falls under the name of *graded-local conformal net theory*. Here, the net [4] of von Neumann algebras $\mathcal{A}$ is given by an isotonous map from the set of *intervals* (that is, non-empty non-dense connected open subsets) of S^1 to some subsets of $B(\mathcal{H})$. The other axioms remain the same, apart from some prescriptions: causally disjoint regions are disjoint intervals; whereas the covariance principle is implemented through a projective unitary representation U of $\mathrm{Diff}^+(S^1)^{(\infty)}$ on $\mathcal{H}$. The specific case where Bose fields only are considered, is called just *(local) conformal net theory*. We refer the reader to [DMS97] and [FST89] as classical references for two-dimensional CFT; [GF93] and [CKL08] for graded-local conformal net theory.

If graded-local conformal nets are the analytic way to chiral CFT, *vertex operator superalgebra (VOSA) theory*, see [Kac01, FHL93, FLM88, LL04], represents the algebraic one. While VOSAs pop up in several areas of mathematics, their axiomatic setting appears as an attempt of reducing Wightman theory to a completely algebraic one, taking away the functional analytic part from it. Indeed, the data of a VOSA is a quadruple (V, Ω, Y, ν) with the following interpretations. V is a $\mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z}$-graded complex vector space, playing the role of the finite-energy part of the Hilbert space of states, where the grading distinguishes between bosons and fermions with parity $\overline{0}$ and $\overline{1}$ respectively. Ω is an even vector of V, called the *vacuum vector*. Y is called the *state-field correspondence* and constitutes of a linear map, associating to every vector a

[4] Note that the set of intervals of the unit circle S^1 is not a directed set and thus, strictly speaking, $\mathcal{A}$ is not a proper net. Anyway, we keep this terminology because it is commonly used.

in V a formal doubly-infinite Laurent series $Y(a,z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1}$, where every $a_{(n)}$ is a parity-preserving endomorphism of V and z is a formal variable. As the name suggests, a *field* $Y(a,z)$ is an algebraic prototype of a Wightman field attached to the state of the system a in the spirit of field-particle duality, interpreting z as insisting on the complex plane $\mathbb{C}$ and thus on S^1 too. Therefore, it is natural to require a suitable *locality axiom* for the set of fields. Note that, differently from the Wightman setting, the set of fields attached to a given VOSA constitutes the whole Borchers class of the theory and not only some its representatives, see [Haag96, Section II.5.5] for details. Finally, the conformal symmetries are realised through the even vector ν in V, called the *conformal vector*. The field $Y(\nu,z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$ associated to ν, gives the generators $\{L_n \mid n \in \mathbb{Z}\}$ of a *Virasoro algebra*. This is an infinite-dimensional Lie algebra, which is naturally associated to the Lie group $\mathrm{Diff}^+(S^1)^{(\infty)}$. When the data of a conformal vector is given, fields are called with the evocative name of *vertex operators*. To have a blend of analysis, a VOSA is usually endowed with an *invariant scalar product* $(\cdot|\cdot)$, where "invariant" stands naively for the property to give a formal adjoint to every vertex operator with respect to this scalar product. Therefore, the quintuple $(V,\Omega,Y,\nu,(\cdot|\cdot))$ is called *unitary VOSA*, see [DL14], [AL17] and [CKLW18, Chapter 5], and constitutes the basic algebraic data of a Wightman theory for a chiral CFT. As for the Haag-Kastler case, we can consider Bose theory only reducing the $\mathbb{Z}_2$-grading to a trivial one, so that (V,Ω,Y,ν) is simply called a *vertex operator algebra (VOA)* [5].

It goes without saying that complex and functional analysis as well as operator algebra theory play a crucial role in the development of mathematical methods for QFT and quantum physics in general, see e.g. [BR02, BR02b]. Results from AQFT have been bringing contributions to the operator algebraic environment in return. As an example, Q-systems, first outlined in [Lon94], are a fantastic tool which link the tensor category theory with finite-index subfactors of type III von Neumann algebras, extending the Jones' index theory for type II subfactors [Jon83], see [BKLR15]. Q-systems appear in the representation theory of Haag-Kastler nets of von Neumann algebras, where the so called Doplicher-Haag-Roberts (DHR) analysis [DHR69I, DHR69II, DHR71, DHR74] and the Doplicher-Roberts reconstruction theorem [DR89] are notable examples of link between C*-tensor categories and group representation theory, see also [Haag96, Chapter IV] and [Ara10, Chapter 6], [FR20, Section 8]. Moreover, the representation theory of conformal nets can be expressed via the KK-theory formalism, see [CCH13, CCHW13]. Fermi theories, which are central in this thesis, naturally introduce *Supersymmetry*, which noncommutative geometry applies to, see e.g. [CHL15, CHKLX15, CH17]. Similarly, VOSA theory is a bridge linking several areas of mathematics. Indeed, its axiomatic form appeared for the first time in [Bor86], where R. E. Borcherds used it to study the representation theory of Kac-Moody algebras, a kind of infinite-dimensional Lie algebras [Kac95]. He also recognised that the module constructed in [FLM84] by Frenkel, Lepowsky and Meurman is actually a V(O)A. This discover unearths the marvellous link, known as "Monstrous Moonshine", between infinite-dimensional Lie algebras, VOAs, modular function theory and finite group theory. Moreover, the Moonshine VOA is constructed from the Leech lattice and more in general, positive-definite even unimodular lattices (see e.g. [CS99, Chapter 2]) are used to classify holomorphic VOAs, which are of interest for CFT too, see [EMS20] and references therein. Besides, note that the term "vertex operator" derives from the dawn of String Theory: Fubini and Veneziano used vertex operators in [FV70] to describe particle scattering at some vertexes of the underlying base space. For a brief historical review of VOAs, see the Introduction of [FLM88]. For the sake of completeness and as a proof of the wide links of chiral CFT with several areas of mathemtics, we mention that there is a third approach to it with a marked geometrical character, called Segal CFT, which is not treated by this work, see [Hen18] and the series of papers [Ten17, Ten19, Ten19b]. An interesting book of T. Gannon [Gan06] collects together most of the areas introduced above. Similarly to AQFT, tensor categories play an important role in the representation theory of VOSA, see [Hua08], with notable recent developments [Gui19I, Gui19II, Gui20, Gui21], where, among other things, a Q-system theory for unitary VOA is established.

[5]Some authors prefer to use the nomenclature VOA theory for VOSA theory too, as in [Kac01]. We prefer to keep such difference, considering the nature and the content of this thesis.

Main results of this thesis

Turning back to the Haag-Kastler and Wightman axiomatisations of QFT, the reader may ask whether they are equivalent. Unfortunately, this is still an open problem, despite much has been already done in different settings, see [BY90, BY92, BZ63, Bos05, DF77, DSW86, FH81, FJ96, Jor96, GJ87]. As far as chiral CFT is concerned, the existence of a sort of equivalence between graded-local conformal nets and unitary VOSAs is supported by several cases where the two theories touch each other. There are different models of graded-local conformal nets constructed from unitary VOSAs as well as tools and methods developed for graded-local conformal nets transposed to unitary VOSAs and vice versa. Some examples are: the conformal net constructed from the Moonshine VOA [KL06], where methods from framed VOAs [DGH98] are used; the graded-local conformal nets constructed in [Kaw10] arise from the VOSAs whose automorphism groups are the Conway's largest sporadic group and the Rudvalis's one, constructed by J. F. Duncan in [Dun07] and [Dun08] respectively; the conformal nets and the VOAs associated with lattices [DX06] and [DG98]; the methods of mirror extensions [Xu07] and [DJX14]; the classification of theories with central charge $c < 1$ [KL04] and [DL15], see also [Gui21, Corollary 3.33]. Indeed, a great answer to this equivalence problem is given by [CKLW18], where the authors establish a correspondence between conformal nets and unitary VOAs under very natural assumptions. Specifically, they construct a unique, up to isomorphism, irreducible conformal net from a simple unitary VOA [6], provided the latter satisfies a certain analytic assumption called *strong locality*. Conversely, using the Tomita-Takesaki modular theory (see [BR02, Chapter 2.5]), they construct a kind of "pre-"vertex operators from a given irreducible conformal net, called *FJ vertex operators* [7]. Then, under a specific "growth" condition called *energy boundedness*, the FJ vertex operators are actual vertex operators generating a simple strongly local unitary VOA such that, the associated irreducible conformal net is the one we start with.

Therefore, one of the main tasks of ***this thesis*** is to extend the correspondence [CKLW18] to the case of graded-local conformal nets and unitary VOSAs and indeed we have the following:

Theorem 1. *A simple unitary VOSA V satisfying a suitable analytic condition, called strong graded locality, gives rise to a unique, up to isomorphism, irreducible graded-local conformal net $\mathcal{A}_V$. Conversely, an irreducible graded-local conformal net $\mathcal{A}$ gives rise to some FJ vertex operators, which determine a simple strongly graded-local unitary VOSA V such that $\mathcal{A}_V = \mathcal{A}$, provided that these FJ vertex operators satisfy a certain set of conditions.*

The first part of Theorem 1 is proved in Chapter 3. The rough idea is to construct a kind of Wightman fields from vertex operators of a simple strongly graded-local unitary VOSA, that is, operator-valued distributions from some test function spaces on S^1 to $B(\mathcal{H})$, where $\mathcal{H}$ is the Hilbert space completion of V with respect to its invariant scalar product. These objects are called *smeared vertex operators* and the von Neumann algebras they generate, constitute the irreducible graded-local conformal net $\mathcal{A}_V$. Here, the strong graded locality enters in two occasions. First, it assures the necessary analytic conditions to well-define the smeared vertex operators, requiring that the VOSA V is *energy-bounded*. Second, it assumes that $\mathcal{A}_V$ is graded-local, a central property which we cannot derive automatically from the locality of the vertex operators. This is due to a well-known results of E. Nelson [Nel59, Section 10], see also [RS80, Section VIII.5], which can be summarised as follows: there exist two self-adjoint operators on a separable Hilbert space, commuting in a common core, whose generated von Neumann algebras do not. In Chapter 3, we also prove that the group of unitary automorphisms of a simple strongly graded-local unitary VOSA V equals the group of automorphisms of the associated irreducible graded-local conformal net $\mathcal{A}_V$.

The second part of Theorem 1 is proved in Chapter 7.1. As announced, we construct some *FJ vertex operators* from an irreducible graded-local conformal net $\mathcal{A}$, using the Tomita-Takesaki modular theory. If the FJ vertex operators satisfy a suitable *energy bound condition*, similar to the one introduced just above, then they turn out to be ordinary vertex operators of a simple strongly graded-local unitary VOSA V such that $\mathcal{A}_V = \mathcal{A}$. At this point, should not be difficult to believe that the correspondence established by Theorem 1 is an extension of the one given by [CKLW18]. In Chapter 7.2, we also show that the correspondence $V \mapsto \mathcal{A}_V$ of Theorem 1 can be viewed as an isomorphism of categories.

[6] As we will see also in this thesis, simplicity and irreducibility are necessary and natural conditions to deal with.

[7] The prefix "FJ" is for [FJ96], which inspires that construction.

We conjecture that the strong graded-locality condition as well as the energy bound can be dropped, so that the correspondence $V \mapsto \mathcal{A}_V$ of Theorem 1 becomes a correspondence between simple unitary VOSAs and irreducible graded-local conformal nets. This is a very challenging problem, which would represent another great step forward in the understanding of the connection between VOSAs and graded-local conformal nets.

We also generalise all the results in [CKLW18] to the case of the correspondence given by Theorem 1. Among these, we highlight the two main theorems of Chapter 5, which are of great relevance. Indeed, Theorem 5.1.1 states that:

Theorem 2. *There is a one-to-one correspondence between unitary vertex subalgebras of a simple strongly graded-local unitary VOSA V and covariant subnets of the associated irreducible graded-local conformal net $\mathcal{A}_V$.*

As a consequence, we have that the coset construction as well as the fixed point one agree under the correspondence established in Theorem 1. Instead, Theorem 5.2.1 shows that:

Theorem 3. *Let $\mathfrak{F}$ be a subset of quasi-primary generating elements of a simple energy-bounded unitary VOSA V. If $\mathfrak{F}$ satisfies the strong graded locality, then V is strongly graded-local. Moreover, $\mathfrak{F}$ gives rise to an irreducible graded-local conformal net which is exactly $\mathcal{A}_V$.*

This last result has several consequences. First, we prove that simple unitary VOSAs generated by elements of conformal weight $\frac{1}{2}$ and 1 together with quasi-primary PCT-invariant Virasoro vectors are automatically strongly graded-local. Second, we have that graded tensor product constructions agree under the correspondence established by Theorem 1. Then, the great importance of those results is manifest in Chapter 6, where they are massively used for the construction of examples.

We also give results which do not have a local counterpart. An interesting example of this fact is in Section 2.7. Here, we show that a simple VOSA V is unitary if and only if its even part $V_{\overline{0}}$ is unitary and its odd part $V_{\overline{1}}$ is unitary as $V_{\overline{0}}$-module. And, in Section 6.2, we present the well-known *superconformal theories* in both VOSA and graded-local conformal net settings. Then, we show that also superconformal structures are preserved under the correspondence of Theorem 1. Furthermore, we give a very interesting characterisation of the automorphisms of V preserving the superconformal structure.

Finally, we give an application of the correspondence between VOAs and local conformal nets. In Section 6.3, we present a classification result for specific chiral CFT models, known as *even rank-one lattice chiral CFT models*. Specifically, relaying on a previous work of C. Dong and R. L. Griess [DG98], we use unitary VOA methods only (so no strong locality results are involved) to give a complete classification of unitary vertex subalgebras of even rank-one lattice type VOAs $V_{L_{2N}}$ for $N \in \mathbb{Z}_{>0}$ as defined in e.g. [DG98], see also [Don93], [Kac01, Section 5.4], [LL04, Section 6.4 and Section 6.5] for the more general case. $V_{L_{2N}}$ are simple strongly local unitary VOAs and thus, as a conseuquence of [CKLW18, Theorem 7.5] or Theorem 2, we automatically have a complete classification of covariant subnets of the corresponding irreducible conformal nets $\mathcal{A}_{\mathrm{U}(1)_{2N}}$ for $N \in \mathbb{Z}_{>0}$. We call the latter U(1)*-current extension nets* because they can be constructed, see [BMT88], by extensions of the U(1)*-current net*, which realises the *free boson chiral CFT*, see also [DX06], [Xu05].

This classification result is interesting from several points of view. First, it shows how powerful the theory developed above can be as it allows to transfer results from one setting to the other one with just a little effort. Second, it is one of the few classification results for covariant subnets of concrete models. Indeed, apart from the trivial subnet, it is known that: the Virasoro net has no other covariant subnets, see [Car98]; any other covariant subnet of $\mathcal{A}_{\mathrm{SU}(2)_1}$, which is the conformal net constructed from the loop group SU(2) at level 1, is a fixed point subnet for a closed subgroup of the automorphism group SO(3) of $\mathcal{A}_{\mathrm{SU}(2)_1}$, see [Car99]. These models are known to be strongly local, see [CKLW18, Example 8.4 and Example 8.9] and indeed, we have corresponding classification results in the VOA setting too, see [CKLW18, Theorem 8.5 and Theorem 8.13] respectively. Furthermore, conformal nets with central charge $c < 1$ are completely classified in [KL04] and so are their possible covariant subnets. Also this result has a VOA counterpart in [DL15] and [Gui21, Corollary 3.33], which gives a complete classification of simple unitary VOAs with central charge $c < 1$. This last remark brings us to the third point which makes our classification result so interesting. Indeed, it is commonly believed that the even rank-one lattice chiral CFT models, together with their fixed point subtheories, exhaust all possible chiral CFTs with central charge $c = 1$, see [DVVV89, Gin88, Kir89]. In [Xu05], the author proved that any conformal net with $c = 1$, satisfying an additional assumption called "Spectrum Condition", is realised in that way. From this point of view, our classification result is a step forward in that direction, proving that no new chiral CFT models with $c = 1$ appear as subtheories of rank-one lattice chiral CFT models.

To have a general idea of our classification results, let $\mathbb{Z}_k$, $\mathbb{T}$ and $D_\infty := \mathbb{T} \rtimes \mathbb{Z}_2$ be the *cyclic group of order* $k \in \mathbb{Z}_{>0}$, the *one-dimensional torus* and the *infinite dihedral group* respectively. Then, for all $N \in \mathbb{Z}_{>0}$, these groups embed as unitary automorphisms of the VOSA $V_{L_{2N}}$. Moreover, it is well-known that V_{L_2} is unitarily equivalent to the unitary affine VOA $V_{\mathfrak{sl}(2,\mathbb{C})_1}$, constructed from the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ at level one, whose unitary automorphism group is isomorphic to SO(3), see e.g. [CKLW18, Example 8.8 and Corollary 8.12]. For any closed group H of unitary automorphisms of $V_{L_{2N}}$ for every $N \in \mathbb{Z}_{>0}$, denote by $V_{L_{2N}}^H$ the corresponding fixed point subalgebra. Then, our classification theorem in the VOA setting says that:

Theorem 4. *The non-trivial unitary subalgebras W of the even rank-one lattice type VOAs $V_{L_{2N}}$ are classified as follows. Apart from the Virasoro subalgebra $L(1,0)$, we have:*

(i) If $N = k^2$ for some positive integer k, then after the identification of $V_{L_{2N}}$ with $V_{L_2}^{\mathbb{Z}_k}$, $W = V_{L_2}^H$ for some closed subgroup $H \subseteq SO(3)$ containing $\mathbb{Z}_k$.

(ii) If $N > 2$ is not the square of any positive integer, then $W = V_{L_{2N}}^H$ for some closed subgroup $H \subseteq D_\infty$.

(iii) If $N = 2$, then either $W = V_{L_4}^H$ for some closed subgroup $H \subseteq D_\infty$ or W is in the continuous series of unitary subalgebras $\{W_t \mid t \in \mathbb{T}\}$ isomorphic to the Virasoro VOA $L(\frac{1}{2},0)$.

Hence, the classification theorem in the conformal net setting, see Theorem 6.3.26, is stated in a similar form as a direct consequence of [CKLW18, Theorem 7.5] or its super counterpart Theorem 2. Note that the conformal net theory arisen from V_{L_2} is exactly the conformal net $\mathcal{A}_{\mathrm{SU}(2)_1}$ presented above.

Generally, the classification of subtheories of concrete chiral CFT models is not an easy task because requires several information about the model under investigation. This is the reason why a classification theorem for other lattice chiral CFT models, such as the higher-dimensional or the graded-local ones, is a very big challenge. In the light of the new results about the correspondence between graded-local conformal nets and VOSAs, we hope that methods and tools used in the proof of Theorem 4, can help for a classification theorem for subtheories of odd rank-one lattice chiral CFT models. Anyway, this is out of the scope of the present thesis.

Outline of the contents

The material follows mostly the two papers [CGH19] and [CGH21] of the author with S. Carpi and R. Hillier. In particular, [CGH19], which is published as *Editor's Pick* in *Journal of Mathematical Physics*, is related to Section 6.3; whereas the rest of the thesis is based on [CGH21], which is still in preparation. Additionally, this thesis provides details in various proofs, gives some secondary or slightly general results and some introductory parts, which are definitely useful for the overall comprehension of the work. We highlight that, apart from Section 6.3, the spirit and most of the results and proofs are close to what is done in [CKLW18] due to its nature of extension of an existing theory. Nevertheless, not all the arguments used in [CKLW18] can be adapted to extend the theory, so that completely new procedures are needed in some cases. A clear example of this fact are the results in Chapter 4, which are proved with technical tools completely different to the ones used in [CKLW18, Appendix B], which can be considered as its "non-super" counterpart. Note that Chapter 4 is vital for the proof of Theorem 3 and the proof of the second part of Theorem 1. Similar discussion holds for the construction of the irreducible graded-local conformal net $\mathcal{A}_V$ in the first part of Theorem 1, where we develop a strategy which must take into account the presence of odd elements, present in the super environment only. Furthermore, we highlight again that some of the results have no local counterpart, such as the unitarity of VOSAs in terms of their even and odd part in Section 2.7, the superconformal theories as well as the graded-local and superconformal models given in Chapter 6. We give a detailed treatment of some classical local and graded-local models there, also proving the well-known unitarity of certain VOSAs relying on the theory developed in Section 2.5. This gives a different approach to unitarity of those VOSAs from the one used in [AL17]. We give a brief outline about the main differences with respect to the local case at the beginning of every chapter. Now, let us have a general look to the content of this thesis.

Chapter 1 is an introduction to basic theory of $\mathrm{Diff}^+(S^1)^{(\infty)}$ and of graded-local conformal nets, which are necessary for the development of the work. Moreover, we settle down some useful notations, used throughout the thesis, in Section 1.1. The role of *Chapter 2* is twofold. On the one hand, we introduce unitary VOSAs and its module theory. On the other hand, we present new results about the uniqueness of the unitary structure of VOSAs, the characterisation of the automorphism group of unitary VOSAs and of the structure of unitary vertex subalgebras. Furthermore, we give a PCT theorem for VOSAs and we characterise the unitary structure of a $\mathbb{Z}_2$-graded simple current extension of a VOSA. This last result allows us to characterise the unitarity of a VOSA in terms of the unitarity of its even and odd parts. In *Chapter 3*, we construct the *smeared vertex operators* associated to an *energy bounded* unitary VOSA. Then, we introduce the notion of *strong graded-locality*, which allows us to construct the irreducible graded-local conformal nets $\mathcal{A}_V$ associated to a simple unitary VOSA V. Hence, we prove that the group of unitary automorphisms of V coincides with the group of automorphisms of $\mathcal{A}_V$. Differently from [CGH21], we give an outline of the graded-local conformal nets which one can naturally construct on the infinite and double cover of S^1. In *Chapter 4*, we give a Bisognano-Wichmann property for smeared vertex operators, which will be fundamental for later results. In *Chapter 5*, we prove the one-to-one correspondence between unitary subalgebras of a simple strongly graded-local unitary VOSA and covariant subnets of the associated irreducible graded-local conformal net. Moreover, we give an important criteria to prove strong graded-locality from a particular subset of generators. As a consequence of those results, we are able to prove that the correspondence "preserves" general constructions as fixed point ones, cosets and graded tensor products. Furthermore, simple unitary VOSAs generated by quasi-primary PCT-invariant Virasoro vectors and vectors of conformal weight $\frac{1}{2}$ and 1 are automatically strongly graded-local. *Chapter 6* shows how to systematically construct most of the well-known models of graded-local conformal nets from VOSAs, using the whole power of the theory just developed. We also introduce the superconformal structures in both settings, showing that they are preserved by the correspondence. Finally, we construct the rank-one lattice models and we illustrate the classification result of [CGH19]. *Chapter 7.1* is the backwards part of the theory. We first prove that the construction in Chapter 3 produces different irreducible graded-local conformal nets from different simple strongly graded-local unitary VOSAs. Second, we prove again that result, but this time developing the theory of FJ vertex operators. The introduction of the FJ vertex operators allows also us to prove under which sufficient assumptions an irreducible graded-local conformal net actually comes from a simple strongly graded-local unitary VOSA. We close merging the correspondence just developed in a categorical environment in *Chapter* 7.2. Finally, *Appendix A* shows the equivalence between Wightman and VOSA locality under the assumption of energy boundedness, which is used in several occasions throughout the thesis.

Chapter 1

Graded-local conformal nets

Section 1.1 is used to settle down the notation and recall some well-known facts about L^2-spaces, which we will use throughout the thesis. Then, we recall the main facts about the orientation-preserving diffeomorphisms of the circle in Section 1.2 and about graded-local conformal nets on S^1 and its covers $\mathbb{R}$ and $S^{1(2)}$ in Section 1.3 and Section 1.4 respectively, redirecting the reader to the main references when necessary.

Comparing with the local case... The following is a collection of well-known facts about graded-local conformal nets mostly from [CKL08] and thus there are no new results. We just present proofs of a couple of well-known facts about subnets and isomorphisms in the graded-local setting, which, to the author's knowledge, are not in the literature.

1.1 Notation for function spaces

We use $C^\infty(\mathbb{R})$ and $C^\infty(\mathbb{R},\mathbb{R})$ to denote the spaces of infinitely differentiable complex- and real-valued functions on $\mathbb{R}$ respectively. Moreover, we use $C_c^\infty(\mathbb{R})$ and $C_c^\infty(\mathbb{R},\mathbb{R})$ to denote the respective subspaces of functions with compact support.

We use $\sim_{\text{a.e.}}$ for the relation of equality almost everywhere for functions and $\{\cdot\}_{\sim_{\text{a.e.}}}$ for a quotient set by this relation. Moreover, with an abuse of notation, we denote the a.e.-class of a function with the function itself.

Square-integrable functions on $[-\pi,\pi]$. Let

$$L^2([-\pi,\pi]) := \Big\{ f : [-\pi,\pi] \to \mathbb{C} \mid \|f\|_{2,\pi} < +\infty \Big\}_{\sim_{\text{a.e.}}} \tag{1.1}$$

$$\|f\|_{2,\pi}^2 := \frac{1}{2\pi}\int_{-\pi}^{\pi} \big|f(x)\big|^2 \,\mathrm{d}x\,. \tag{1.2}$$

It is known that $\{e^{-inx} \mid n\in\mathbb{Z}\}$ and $\{e^{-inx} \mid n\in\mathbb{Z}-\frac{1}{2}\}$ are orthonormal bases of $L^2([-\pi,\pi])$ and that for all $f\in L^2([-\pi,\pi])$,

$$f(x) = \sum_{n\in\mathbb{Z}} \widehat{f}_n e^{inx} = \sum_{n\in\mathbb{Z}-\frac{1}{2}} \widehat{f}_n e^{inx}\,, \qquad \widehat{f}_n := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,\mathrm{d}x \quad \forall n\in\frac{1}{2}\mathbb{Z}\,. \tag{1.3}$$

Moreover, we have **Parseval's Theorem**: for all $f,g\in L^2([-\pi,\pi])$,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{g(x)}\,\mathrm{d}x = \sum_{n\in\mathbb{Z}} \widehat{f}_n\overline{\widehat{g}_n} = \sum_{n\in\mathbb{Z}-\frac{1}{2}} \widehat{f}_n\overline{\widehat{g}_n}\,. \tag{1.4}$$

An operation on functions in $L^2([-\pi,\pi])$, which we use is the **convolution product** $*$. It is defined by

$$(f*g)(x) := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)g(x-y)\,\mathrm{d}y \qquad \forall x\in[-\pi,\pi] \tag{1.5}$$

where we have eventually extended the (class of) $f,g\in L^2([-\pi,\pi])$ to the whole real axis by 2π-periodicity. The important fact here is that $f*g\in L^2([-\pi,\pi])$ and the following formula holds

$$\widehat{(f*g)}_n = 2\pi \widehat{f}_n \widehat{g}_n \qquad \forall n \in \frac{1}{2}\mathbb{Z}\,. \tag{1.6}$$

We write $C^\infty([-\pi,\pi])$ for the vector space of infinitely differentiable complex-valued functions on $[-\pi,\pi]$ and $C_c^\infty((-\pi,\pi))$ for the vector subspace of the $C^\infty([-\pi,\pi])$-functions with compact support in $(-\pi,\pi)$.

Square-integrable functions on S^1. In our application, we are interested in working with functions on the **unit circle**, that is the set of complex numbers with unit norm:

$$S^1 := \left\{ z \in \mathbb{C} \mid |z| = 1 \right\} = \left\{ e^{ix} \mid x \in (-\pi,\pi] \right\} \subset \mathbb{C} \cong \mathbb{R}^2\,, \tag{1.7}$$

where we identify every $z \in S^1$ with e^{ix} for the unique $x \in (-\pi,\pi]$ when necessary. Then, we give the usual subspace topology to S^1. Let

$$L^2(S^1) := \left\{ f : S^1 \to \mathbb{C} \mid \|f\|_{S^1} < +\infty \right\}_{\sim_{\text{a.e.}}} \tag{1.8}$$

where

$$\|f\|^2_{S^1} := \oint_{S^1} \left|f(z)\right|^2 \frac{\mathrm{d}z}{2\pi i z} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|f(e^{ix})\right|^2 \mathrm{d}x\,. \tag{1.9}$$

At this stage, it is convenient to identify $L^2(S^1)$ with $L^2([-\pi,\pi])$ through the isometric isomorphism

$$L^2(S^1) \ni f \longmapsto \widetilde{f} \in L^2([-\pi,\pi])\,, \quad \widetilde{f}(x) := f(e^{ix}) \quad \forall x \in [-\pi,\pi]\,. \tag{1.10}$$

Then, it is clear that $\left\{ z^{-n} \mid n \in \mathbb{Z} \right\}$ and $\left\{ z^{-n} \mid n \in \mathbb{Z} - \frac{1}{2} \right\}$ are orthonormal basis for $L^2(S^1)$ and thus every function $f \in L^2(S^1)$ can be written as a convergent series of its Fourier coefficients so defined

$$\widehat{f}_n := \oint_{S^1} f(z) z^{-n} \frac{\mathrm{d}z}{2\pi i z} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{ix}) e^{-inx}\,\mathrm{d}x = \widehat{\widetilde{f}}_n \qquad \forall n \in \frac{1}{2}\mathbb{Z}\,. \tag{1.11}$$

Furthermore, we still have Parseval's Theorem: for all $f, g \in L^2(S^1)$,

$$\oint_{S^1} f(z)\overline{g(z)}\,\frac{\mathrm{d}z}{2\pi i z} = \sum_{n\in\mathbb{Z}} \widehat{f}_n \overline{\widehat{g}_n} = \sum_{n\in\mathbb{Z}-\frac{1}{2}} \widehat{f}_n \overline{\widehat{g}_n}\,. \tag{1.12}$$

Also in this case, we define the vector space of infinitely differentiable complex-valued functions:

$$C^\infty(S^1) := \left\{ f : S^1 \to \mathbb{C} \mid \widetilde{f} \in C^\infty([-\pi,\pi]),\ \widetilde{f}^{(s)}(-\pi) = \widetilde{f}^{(s)}(\pi)\ \forall s \in \mathbb{Z}_{\geq 0} \right\}\,, \tag{1.13}$$

where $\widetilde{f}(x) := f(e^{ix})$ for all $x \in [-\pi,\pi]$. Note that the map in (1.10) (allowing x going through the whole real axis) gives an isomorphism between $C^\infty(S^1)$ and the vector space of infinitely differentiable 2π-periodic complex-valued functions on $\mathbb{R}$. Thus, we have that $f \in C^\infty(S^1)$ if and only if (the extension to the whole real axis of) $\widetilde{f}$ is 2π-periodic and in $C^\infty(\mathbb{R})$.

Consider now the function $\chi(z) := e^{i\frac{x}{2}}$ for all $z = e^{ix}$ with $x \in (-\pi,\pi]$. Then, $C^\infty(S^1)$ is isomorphic to $C^\infty_\chi(S^1) := \chi C^\infty(S^1)$ through the map $f \mapsto \chi f$. Moreover, their intersection contains $C_c^\infty(S^1\backslash\{-1\})$, that is the subspace of $C^\infty(S^1)$ of infinitely differentiable complex-valued functions on S^1 with compact support contained in $S^1\backslash\{-1\}$. The former is isomorphic to $C_c^\infty((-\pi,\pi))$ through the map (1.10).

Every function in $C^\infty(S^1)$ defines a proper a.e.-class and thus $C^\infty(S^1)$ naturally embeds in $L^2(S^1)$ as well as $C^\infty_\chi(S^1)$. Therefore, for every $f \in C^\infty(S^1)$ and for every $g \in C^\infty_\chi(S^1)$, $(\widehat{f}_n)_{n\in\mathbb{Z}}$ and $(\widehat{g}_n)_{n\in\mathbb{Z}-\frac{1}{2}}$ are rapidly decaying sequences. As a matter of fact, if $g = \chi h$ with $h \in C^\infty(S^1)$, then $\widehat{g}_n = \widehat{h}_{\frac{2n-1}{2}}$ for all $n \in \mathbb{Z} - \frac{1}{2}$, which gives us the rapid decay because $(\widehat{h}_m)_{m\in\mathbb{Z}}$ is.

Remark 1.1.1. We have a natural definition of derivative for a function $f \in C^\infty(S^1)$ induced by $\widetilde{f} \in C^\infty([-\pi,\pi])$. As a matter of fact, we write $f' \in C^\infty(S^1)$ as

$$f'(z) := \frac{\mathrm{d}\widetilde{f}}{\mathrm{d}x}(x) \qquad \forall z = e^{ix} \in S^1\,. \tag{1.14}$$

Similarly, for every $g \in C^\infty_\chi(S^1)$, we denote $g' \in C^\infty_\chi(S^1)$ by

$$g'(z) := \frac{i}{2}g(z) + \chi(z)h'(z) \qquad \forall z \in S^1 . \tag{1.15}$$

where we have considered $g = \chi h$ with $h \in C^\infty(S^1)$.

We can make $C^\infty(S^1)$ and $C^\infty_\chi(S^1)$ Fréchet spaces using the family of seminorms:

$$\left\{ \|\cdot\|_s \mid s \in \mathbb{Z}_{\geq 0} \right\} , \qquad \|f\|_s := \sup_{z \in S^1} \left| f^{(s)}(z) \right| \qquad \forall s \in \mathbb{Z}_{\geq 0} . \tag{1.16}$$

Then, the Fréchet topology on $C^\infty_\chi(S^1)$ is the one induced by the one of $C^\infty(S^1)$.

Remark 1.1.2. Recall that for any $f \in C^\infty(S^1)$, $n\widehat{f}_n = -i\widehat{f'}_n$ for all $n \in \mathbb{Z}$ and thus the seminorms (1.16) can be equivalently defined as

$$\|f\|_s = \sum_{n\in\mathbb{Z}} (1 + |n|)^s \widehat{f}_n \qquad \forall f \in C^\infty(S^1) \;\; \forall s \in \mathbb{Z}_{\geq 0} . \tag{1.17}$$

Similarly, it is not hard to show that for all $g \in C^\infty_\chi(S^1)$, $n\widehat{g}_n = -i\widehat{g'}_n$ for all $n \in \mathbb{Z} - \frac{1}{2}$. Indeed, if we write $g = \chi h$ with $h \in C^\infty(S^1)$, then

$$\begin{aligned} n\widehat{g}_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} h(e^{ix})\chi(e^{ix})ne^{-inx}\,\mathrm{d}x \\ &= -i\widehat{(\chi h')}_n + \frac{1}{2}\widehat{g}_n = -i\widehat{g'}_n \qquad \forall n \in \mathbb{Z} - \frac{1}{2} , \end{aligned}$$

Therefore, we can similarly define the Fréchet topology on $C^\infty_\chi(S^1)$ as the one induced by the seminorms

$$\|g\|_s := \sup_{z \in S^1} \left| g^{(s)}(z) \right| \qquad \forall g \in C^\infty_\chi(S^1) \;\; \forall s \in \mathbb{Z}_{\geq 0} \tag{1.18}$$

or equivalently

$$\|g\|_s = \sum_{n\in\mathbb{Z}-\frac{1}{2}} (1 + |n|)^s \widehat{g}_n \qquad \forall g \in C^\infty_\chi(S^1) \;\; \forall s \in \mathbb{Z}_{\geq 0} . \tag{1.19}$$

Infinitely differentiable functions on the double cover of the circle. It is useful to look at $C^\infty_\chi(S^1)$ in a different way. Consider the double cover $S^{1(2)}$ of the circle S^1, which we can identify as

$$S^{1(2)} \equiv \left\{ e^{i\frac{x}{2}} \in \mathbb{C} \mid x \in (-2\pi, 2\pi] \right\} . \tag{1.20}$$

Denote by $C^\infty([-2\pi, 2\pi])$ the vector space of infinitely differentiable complex-valued functions on $[-2\pi, 2\pi]$ and define the vector space of infinitely differentiable complex-valued functions on $S^{1(2)}$ as

$$C^\infty(S^{1(2)}) := \left\{ f : [-2\pi, 2\pi] \to \mathbb{C} \mid f \in C^\infty([-2\pi, 2\pi]),\;\; f^{(s)}(-2\pi) = f^{(s)}(2\pi) \;\forall s \in \mathbb{Z}_{\geq 0} \right\} .$$

A linear basis for $C^\infty(S^{1(2)})$ is given by

$$\left\{ e^{-inx} \mid n \in \mathbb{Z} \right\} \cup \left\{ e^{-inx} \mid n \in \mathbb{Z} - \frac{1}{2} \right\} ,$$

where $\left\{ e^{-inx} \mid n \in \mathbb{Z} \right\}$ generates the subspace of 2π-periodic functions, which we call $C^\infty_+(S^{1(2)})$, whereas $\left\{ e^{-inx} \mid n \in \mathbb{Z} - \frac{1}{2} \right\}$ generates the one of 2π-anti-periodic functions, which we denote by $C^\infty_-(S^{1(2)})$. Then, it is clear that $C^\infty(S^1)$ is identified with $C^\infty_+(S^{1(2)})$, whereas $C^\infty_\chi(S^1)$ can be identified with $C^\infty_-(S^{1(2)})$. Then, we have the following decomposition in vector subspaces

$$C^\infty(S^{1(2)}) = C^\infty_+(S^{1(2)}) \oplus C^\infty_-(S^{1(2)}) \cong C^\infty(S^1) \oplus C^\infty_\chi(S^1) . \tag{1.21}$$

Fourier transform. For every $f \in L^2(\mathbb{R})$ (equipped as usual with Lebesgue measure) we define its Fourier transform

$$\widehat{f}(p) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x)e^{-ipx}\,\mathrm{d}x \quad \forall p \in \mathbb{R} . \tag{1.22}$$

Recall one consequence of Plancherel's Theorem ([RS75, Theorem IX.6]) is that for all $f, g \in L^2(\mathbb{R})$ we have

$$\int_{-\infty}^{+\infty} f(x)\overline{g(x)}\,\mathrm{d}x = \int_{-\infty}^{+\infty} \widehat{f}(p)\overline{\widehat{g}(p)}\,\mathrm{d}p\,. \tag{1.23}$$

Convention 1.1.3. Throughout the thesis, we use the symbols $C^\infty_\star(\cdot,\mathbb{R})$ and $L^2(\cdot,\mathbb{R})$ for the subsets of real-valued $C^\infty_\star(\cdot)$- and $L^2(\cdot)$-functions respectively.

1.2 The orientation-preserving diffeomorphisms of S^1

A **smooth real vector field** on S^1 is denoted by $f\frac{\mathrm{d}}{\mathrm{d}x}$ with $f \in C^\infty(S^1,\mathbb{R})$, see Section 1.1 for notation. These form a topological vector space $\mathrm{Vect}(S^1)$ with the usual C^∞ Fréchet topology. $\mathrm{Vect}(S^1)$ can be used to model $\mathrm{Diff}^+(S^1)$, which is the infinite dimensional Lie group of **orientation-preserving diffeomorphisms of** S^1. Note also that $\mathrm{Vect}(S^1)$ turns out to be the Lie algebra associated to $\mathrm{Diff}^+(S^1)$ with the Lie bracket given by the negative of the usual bracket of vector fields, that is

$$[g\frac{\mathrm{d}}{\mathrm{d}x}, f\frac{\mathrm{d}}{dx}] = \big(g'f - f'g\big)\,\frac{\mathrm{d}}{\mathrm{d}x} \qquad \forall g, f \in C^\infty(S^1,\mathbb{R})\,. \tag{1.24}$$

Then, we denote by $\exp : \mathrm{Vect}(S^1) \to \mathrm{Diff}^+(S^1)$ the related exponential map and with $\exp(tf)$ the one-parameter subgroup of $\mathrm{Diff}^+(S^1)$ generated by the smooth vector field $f\frac{d}{dx}$ with $f \in C^\infty(S^1,\mathbb{R})$ and t going through $\mathbb{R}$. We redirect the reader to [Mil84, Section 6] for a detailed treatment. $\mathrm{Diff}^+(S^1)$ is connected, but not simply connected, see [Mil84, Section 10], [Ham82, Example 4.2.6] and [Lok94, Section 1.1]. Therefore, we denote by $\mathrm{Diff}^+(S^1)^{(n)}$ for all $n \in \mathbb{Z}_{>0}$, the n-cover of $\mathrm{Diff}^+(S^1)$ and with $\mathrm{Diff}^+(S^1)^{(\infty)}$ its universal cover. We can identify every $\gamma \in \mathrm{Diff}^+(S^1)^{(\infty)}$ with a unique function ϕ_γ in the closed subgroup of diffeomorphisms of $\mathbb{R}$ defined by

$$\mathrm{Diff}^+(S^1)^{(\infty)} \equiv \left\{\phi \in C^\infty(\mathbb{R},\mathbb{R}) \mid \phi(x+2\pi) = \phi(x) + 2\pi\,,\ \phi'(x) > 0\ \forall x \in \mathbb{R}\right\}\,. \tag{1.25}$$

Under the above identification, the centre of $\mathrm{Diff}^+(S^1)^{(\infty)}$ is given by

$$\{\phi \in \mathrm{Diff}^+(S^1)^{(\infty)} \mid \phi(x) = x + 2\pi k\ \ \forall x \in \mathbb{R}\,,\ \ k \in \mathbb{Z}\} \cong 2\pi\mathbb{Z}\,. \tag{1.26}$$

Accordingly, for every $n \in \mathbb{Z}_{>0}$, we have the following identifications

$$\mathrm{Diff}^+(S^1)^{(n)} \equiv {\mathrm{Diff}^+(S^1)^{(\infty)}}\big/{2n\pi\mathbb{Z}} \qquad \forall n \in \mathbb{Z}_{>0}\,. \tag{1.27}$$

For every $n \in \mathbb{Z}_{>0}$ and every $\gamma \in \mathrm{Diff}^+(S^1)^{(n)}$, we denote by $\phi_\gamma \in \mathrm{Diff}^+(S^1)^{(\infty)}$ a representative of γ in its $2n\pi\mathbb{Z}$-class of diffeomorphisms (which is then given by $\{\phi_\gamma + 2n\pi k \mid k \in \mathbb{Z}\}$). Moreover, note that for all $\gamma, \gamma_1, \gamma_2 \in \mathrm{Diff}^+(S^1)^{(n)}$, $\phi_{\gamma_1\gamma_2}$ and $\phi_{\gamma_1} \circ \phi_{\gamma_2}$ are in the same class of diffeomorphisms as well as ϕ_γ^{-1} and $\phi_{\gamma^{-1}}$. For all $n \in \mathbb{Z}_{>0} \cup \{\infty\}$ and every $\gamma \in \mathrm{Diff}^+(S^1)^{(n)}$, we use $\dot\gamma$ to denote the image of γ under the covering map $p : \mathrm{Diff}^+(S^1)^{(\infty)} \mapsto \mathrm{Diff}^+(S^1)$. Accordingly, we use $\exp^{(n)}(tf)$ to denote the lift of $\exp(tf)$ to $\mathrm{Diff}^+(S^1)^{(n)}$.

Remark 1.2.1. By the results of Epstein, Herman and Thurston [Eps70, Her71, Thu74], we know that $\mathrm{Diff}^+(S^1)$ is a simple group. It follows that every element of $\mathrm{Diff}^+(S^1)$ is a finite product of exponentials of vector fields $\exp(tf)$, where $f \in C^\infty(S^1,\mathbb{R})$ with $\mathrm{supp} f \subset I$ for some interval $I \subset S^1$, see [Mil84, Remark 1.7] and [CKLW18, Remark 3.1]. It also follows that every element of $\mathrm{Diff}^+(S^1)^{(2)}$ is a finite product of exponentials of vector fields $\exp^{(2)}(tf)$ with f as above by the proof of [CKL08, Proposition 38].

Note that if we identify the double cover $S^{1(2)}$ of the circle S^1 as the set

$$S^{1(2)} \equiv \left\{e^{i\frac{x}{2}} \in \mathbb{C} \mid x \in (-2\pi, 2\pi]\right\}, \tag{1.28}$$

then every $\gamma \in \mathrm{Diff}^+(S^1)^{(2)}$ acts naturally on it by $\gamma(e^{i\frac{x}{2}}) = e^{i\frac{\phi_\gamma(x)}{2}}$, where ϕ_γ is one of the representatives in its $4\pi\mathbb{Z}$-class of diffeomorphisms given by (1.27). Note that the former action does not depend on the choice of the representative ϕ_γ.

It is useful to introduce the **Virasoro algebra** $\mathfrak{Vir}$, see [KR87, Lecture 1]. Consider the complexification $\mathrm{Vect}_{\mathbb{C}}(S^1)$ of $\mathrm{Vect}(S^1)$ and its infinite dimensional Lie subalgebra spanned by generators $l_n := -ie^{inx}\frac{d}{dx}$ for all $n \in \mathbb{Z}$, known as the **complex Witt algebra** $\mathfrak{Witt}$. These generators satisfies the commutation relations:

$$[l_n, l_m] = (n-m)l_{n+m} \qquad \forall n, m \in \mathbb{Z} \tag{1.29}$$

where the bracket above is the natural extension of the one in (1.24) to $\mathrm{Vect}_{\mathbb{C}}(S^1)$. The Virasoro algebra is defined as the nontrivial central extension $\mathfrak{Witt} \oplus \mathbb{C}k$ with commutation relations:

$$[l_n, l_m] = (n-m)l_{n+m} + \frac{n^3-n}{12}\delta_{n,-m}k\,, \quad [l_n, k] = 0 \qquad \forall n, m \in \mathbb{Z}\,. \tag{1.30}$$

Note that the Lie subalgebra of $\mathfrak{Witt}$ generated by l_n with $n \in \{-1, 0, 1\}$ is isomorphic to the finite Lie algebra $\mathfrak{sl}(2, \mathbb{C})$. Moreover, the smooth real vector fields il_0, $\frac{i(l_1+l_{-1})}{2}$ and $\frac{l_1-l_{-1}}{2}$ generate a Lie subalgebra of $\mathrm{Vect}(S^1)$ isomorphic to $\mathfrak{sl}(2, \mathbb{R}) \cong \mathrm{PSU}(1, 1) \cong \mathrm{PSL}(2, \mathbb{R})$, which exponentiates to the three-dimensional Lie subgroup $\mathrm{Möb}(S^1)$ of $\mathrm{Diff}^+(S^1)$ of **Möbius transformations** of S^1.

We set the following Möbius transformations of S^1:

$$r(t) := \exp(itl_0)\,, \quad \delta(t) := \exp\left(t\frac{l_1 - l_{-1}}{2}\right)\,, \quad w(t) := \exp\left(\frac{it}{2}\left[l_0 + \frac{l_1 + l_{-1}}{2}\right]\right) \tag{1.31}$$

which are known as the **rotation**, **dilation** and **translation subgroups** of $\mathrm{Möb}(S^1)$ respectively. Their explicit actions on S^1 are respectively given by

$$r(t)z = e^{it}z \qquad \forall t \in \mathbb{R}\ \forall z \in S^1 \tag{1.32}$$

$$\delta(t)z = \frac{z\cosh(t/2) - \sinh(t/2)}{-z\sinh(t/2) + \cosh(t/2)} \qquad \forall t \in \mathbb{R}\ \forall z \in S^1 \tag{1.33}$$

$$w(t)z = \frac{z(4+it)+it}{-zit+(4-it)} \qquad \forall t \in \mathbb{R}\ \forall z \in S^1\,. \tag{1.34}$$

If the action (1.32) explains why $r(t)$ is called the rotation subgroup, note that under the **Cayley transform**, that is the diffeomorphism $C : S^1\backslash\{-1\} \to \mathbb{R}$ defined by

$$C(z) := 2i\frac{1-z}{1+z}\,, \qquad C^{-1}(x) = \frac{1+\frac{i}{2}x}{1-\frac{i}{2}x}\,,$$

the actions (1.33) and (1.34) send $x \in \mathbb{R}$ to $e^t x$ and $x + t$ for all $t \in \mathbb{R}$ respectively, so justifying the remaining nomenclature given above.

We denote by $\mathrm{Möb}(S^1)^{(n)}$ the corresponding n-cover contained in $\mathrm{Diff}^+(S^1)^{(n)}$ for all $n \in \mathbb{Z}_{>0} \cup \{\infty\}$. In particular, $\mathrm{Möb}(S^1)^{(2)} = \mathrm{SU}(1, 1) \cong \mathrm{SL}(2, \mathbb{R})$. Recall that the covering map p restricts to the one of $\mathrm{Möb}(S^1)^{(\infty)}$ on $\mathrm{Möb}(S^1)$. Moreover, we have the identifications corresponding to (1.27) for every $\mathrm{Möb}(S^1)^{(n)}$, namely

$$\mathrm{Möb}(S^1)^{(n)} \equiv \mathrm{Möb}(S^1)^{(\infty)}/2n\pi\mathbb{Z} \qquad \forall n \in \mathbb{Z}_{>0}\,. \tag{1.35}$$

For every $n \in \mathbb{Z}_{>0} \cup \{\infty\}$, we use $r^{(n)}(t)$, $\delta^{(n)}(t)$ and $w^{(n)}(t)$ for the lifts of the corresponding subgroups given by the exponential map.

Of fundamental interest for our analysis is the representation theory of $\mathrm{Diff}^+(S^1)^{(\infty)}$ and $\mathrm{Möb}(S^1)^{(\infty)}$. We do not report here a treatment of such theory, which can be found in a resumed form in [CKLW18, Section 3.2]. We present some key facts which can be useful to recall before proceeding further.

For a topological group $\mathscr{G}$, a **strongly continuous projective unitary representation** U on an Hilbert space $\mathcal{H}$ is a strongly continuous homomorphism from $\mathscr{G}$ to the quotient $\mathrm{U}(\mathcal{H})/\mathbb{T}$ of the group of unitary operators on $\mathcal{H}$ by the circle subgroup $\mathbb{T}$ of operators with unit norm. A strongly continuous projective unitary representation U of $\mathrm{Diff}^+(S^1)^{(\infty)}$ restricts to one of the subgroup $\mathrm{Möb}(S^1)^{(\infty)}$, which can always be lift to a strongly continuous unitary representation $\widetilde{U}$ of $\mathrm{Möb}(S^1)^{(\infty)}$, see [Bar54]. U is said to be **positive-energy** if $\widetilde{U}$ is, that is, if the infinitesimal self-adjoint generator L_0 of the strongly continuous one-parameter group $t \mapsto e^{itL_0} := \widetilde{U}(r^{(\infty)}(t))$ is a positive operator on $\mathcal{H}$, namely, it has a non-negative spectrum. Note that U factors through a positive-energy strongly continuous projective unitary representation of $\mathrm{Diff}^+(S^1)^{(2)}$ whenever $e^{i4\pi L_0} = 1_{\mathcal{H}}$. In that case, it restricts also to a positive-energy strongly continuous projective unitary representation of $\mathrm{Möb}(S^1)^{(2)}$.

A **positive-energy unitary representation** of the Virasoro algebra $\mathfrak{Vir}$ is a Lie algebra representation π of $\mathfrak{Vir}$ on a complex vector space V equipped with a scalar product $(\cdot|\cdot)$ such that:

- π is **unitary**, that is, $(a|L_n b) = (L_{-n}a|b)$ for all $n \in \mathbb{Z}_{\geq 0}$ and all $a, b \in V$;
- π has **positive energy**, which means that L_0 is diagonalizable on V with non-negative eigenvalues, namely,

$$V = \bigoplus_{\lambda \in \mathbb{R}_{\geq 0}} V_\lambda \,, \quad V_\lambda := \mathrm{Ker}(L_0 - \lambda 1_V)\,; \tag{1.36}$$

- π has a **central charge** $c \in \mathbb{C}$, that is, $K = c1_V$;

where $K := \pi(k)$ and $L_n := \pi(l_n)$ for all $n \in \mathbb{Z}$.

Finally, we remark that there is a well-known correspondence between positive-energy strongly continuous projective unitary representations of $\mathrm{Diff}^+(S^1)^{(\infty)}$ and positive-energy unitary representations of $\mathfrak{Vir}$, which is summarised in [CKLW18, Theorem 3.4]:

Theorem 1.2.2. *Every positive-energy unitary representation π of the Virasoro algebra $\mathfrak{Vir}$ on a complex inner product space V integrates to a unique positive-energy strongly continuous projective unitary representation U_π of $\mathrm{Diff}^+(S^1)^{(\infty)}$ on the Hilbert space completion $\mathcal{H}$ of V. Furthermore, every positive-energy strongly continuous projective unitary representation of $\mathrm{Diff}^+(S^1)^{(\infty)}$ on a Hilbert space $\mathcal{H}$ arises in this way, whenever the subspace $\mathcal{H}^{\mathrm{fin}} := \bigoplus_{\alpha \in \mathbb{R}_{\geq 0}} \mathrm{Ker}(L_0 - \alpha 1_{\mathcal{H}})$ is dense in $\mathcal{H}$. The map $\pi \mapsto U_\pi$ becomes one-to-one when restricting to representations π on the complex inner product spaces V, whose Hilbert space completion $\mathcal{H}$ satisfies $\mathcal{H}^{\mathrm{fin}} = V$. These are exactly those complex inner product spaces V such that $V_\lambda := \mathrm{Ker}(L_0 - \lambda 1_V) \subseteq V$ is complete (that is, it is a Hilbert space) for all $\lambda \in \mathbb{R}_{\geq 0}$. Finally, U_π is irreducible if and only if π is irreducible, i.e. if and only if the corresponding $\mathfrak{Vir}$-module is $L(c, 0)$ for some $c \geq 1$ or $c_m = 1 - \frac{6}{m(m+1)}$ with integers $m \geq 2$.*

The proofs of the facts stated by Theorem 1.2.2 are essentially given by [GW85] and [Tol99, Theorem 5.2.1, Proposition 5.2.4] for the integrability statement and by [Lok94, Chapter 1] and [Car04, Appendix A] for the remaining part, see Example 6.1.1 for the definition of the $\mathfrak{Vir}$-modules $L(c, 0)$; see also [NS15] for related results.

1.3 Graded-local conformal nets on S^1

We recall some preliminaries about graded-local conformal nets from [CKL08, Section 2], [CHL15, Section 2] and references therein.

Let $\mathcal{J}$ be the set of **intervals** of the unit circle, that is, the open, connected, non-empty and non-dense subsets of S^1. For every interval $I \in \mathcal{J}$, we set $I' := S^1 \backslash \overline{I}$, that is the complement of I in S^1. Set

$$\mathrm{Diff}(I) := \left\{ \gamma \in \mathrm{Diff}^+(S^1) \mid \gamma(z) = z \ \forall z \in I' \right\} \qquad \forall I \in \mathcal{J}\,. \tag{1.37}$$

Accordingly to the notation of Section 1.2, for all $I \in \mathcal{J}$ and all $n \in \mathbb{Z}_{>0} \cup \{\infty\}$, $\mathrm{Diff}(I)^{(n)}$ denotes the connected component to the identity of the pre-image of $\mathrm{Diff}(I)$ in $\mathrm{Diff}^+(S^1)^{(n)}$ under the covering map p.

A **graded-local Möbius covariant net on** S^1 is a family of von Neumann algebras $\mathcal{A} := (\mathcal{A}(I))_{I \in \mathcal{J}}$ on a fixed infinite-dimensional separable Hilbert space $\mathcal{H}$, also called **net** of von Neumann algebras, with the following properties:

(A) (Isotony). For all $I_1, I_2 \in \mathcal{J}$ such that $I_1 \subseteq I_2$, then $\mathcal{A}(I_1) \subseteq \mathcal{A}(I_2)$.

(B) **(Möbius covariance)**. There exists a strongly continuous unitary representation U of $\mathrm{M\ddot{o}b}(S^1)^{(\infty)}$ on $\mathcal{H}$ such that

$$U(\gamma)\mathcal{A}(I)U(\gamma)^{-1} = \mathcal{A}(\dot{\gamma}I) \qquad \forall \gamma \in \mathrm{M\ddot{o}b}(S^1)^{(\infty)},\ \forall I \in \mathcal{J}\,. \tag{1.38}$$

(C) **(Positivity of the energy)**. The infinitesimal generator L_0 of the (universal cover of) rotation one-parameter subgroup $t \mapsto U(r^{(\infty)}(t))$, called **conformal Hamiltonian**, is a positive operator on $\mathcal{H}$. This means that U is a positive energy representation.

(D) **(Existence of the vacuum)**. There exists a U-invariant unit vector $\Omega \in \mathcal{H}$, called **vacuum vector**, which is also cyclic for the von Neumann algebra $\mathcal{A}(S^1) := \bigvee_{I\in\mathcal{J}} \mathcal{A}(I)$.

(E) **(Graded locality)**. There exists a self-adjoint unitary operator Γ on $\mathcal{H}$, called **grading unitary**, such that $\Gamma\Omega = \Omega$ and

$$\Gamma\mathcal{A}(I)\Gamma = \mathcal{A}(I)\,, \quad \mathcal{A}(I') \subseteq Z\mathcal{A}(I)'Z^* \qquad \forall I \in \mathcal{J} \tag{1.39}$$

with

$$Z := \frac{1_{\mathcal{H}} - i\Gamma}{1-i}\,. \tag{1.40}$$

Note that every operator $A \in \mathcal{A}(I)$ with $I \in \mathcal{J}$ can be written as the sum of the two operators

$$A_0 := \frac{A + \Gamma A \Gamma}{2}\,, \qquad A_1 := \frac{A - \Gamma A\Gamma}{2} \tag{1.41}$$

which have the property that $\Gamma A_k \Gamma = (-1)^k A_k$ for all $k \in \{0,1\}$. Then, we call an operator A **homogeneous** with **degree** ∂A equal to either 0 or 1 if $\Gamma A \Gamma$ is equal to either A or $-A$ respectively. Contextually, we say A is either a **Bose** or a **Fermi** element respectively. With this notation, graded-locality (1.39) is equivalent to saying that $[\mathcal{A}(I_1), \mathcal{A}(I_2)] = 0$ whenever $I_1 \subseteq I_2'$, where the graded-commutator here is defined through homogeneous elements A and B by

$$[A,B] := AB - (-1)^{\partial A \partial B} BA$$

and thus extended to arbitrary elements by linearity.

A graded-local Möbius covariant net $\mathcal{A}$ is said to be **irreducible** if $\mathcal{A}(S^1) = B(\mathcal{H})$. This is equivalent to say that Ω in axiom **(D)** above is the unique vacuum vector up to a multiplication for a complex number and equivalently that all local algebras $\mathcal{A}(I)$ are type III_1 factors (provided that $\mathcal{A}(I) \neq \mathbb{C}\Omega$ for all $I \in \mathcal{J}$), see [CKL08, Proposition 7] and references therein.

As well-explained in [CKL08, Section 2.2], the axioms **(A)-(E)** above have important consequences. We rewrite here below the ones which we will use more often throughout the thesis:

- **(Reeh-Schlieder theorem)**. See [CKL08, Theorem 1]. The vacuum Ω is cyclic and separating for every von Neumann algebra $\mathcal{A}(I)$ with $I \in \mathcal{J}$.
- **(Bisognano-Wichmann property)**. See [CKL08, Theorem 2]. For every $I \in \mathcal{J}$, let $\delta_I(t)$ with $t \in \mathbb{R}$ be the one-parameter subgroup of $\mathrm{M\ddot{o}b}(S^1)$ of **dilations associated to** I, defined as follows, see [Lon08b, Section 1.1] or [BGL93, Eq. (1.6)]. Let $S^1_+ \in \mathcal{J}$ be the upper semi-circle and let $\gamma_I \in \mathrm{M\ddot{o}b}(S^1)$ be such that $\gamma_I S^1_+ = I$. Then, $\delta_I(t) := \gamma_I \delta(t) \gamma_I^{-1}$ for all $t \in \mathbb{R}$. Note that δ_I is independent on the choice of γ_I. Indeed, $\delta(t)S^1_+ = S^1_+$ for all $t \in \mathbb{R}$ and if $\gamma \in \mathrm{M\ddot{o}b}(S^1)$ is such that $\gamma S^1_+ = S^1_+$, then $\gamma = \delta(\lambda)$ for some $\lambda \in \mathbb{R}$. It follows that if $\gamma_1, \gamma_2 \in \mathrm{M\ddot{o}b}(S^1)$ are such that $\gamma_i S^1_+ = I$ for all $i \in \{1,2\}$, then $\gamma_2^{-1}\gamma_1\delta(t)\gamma_1^{-1}\gamma_2 = \delta(t)$ for all $t \in \mathbb{R}$. This clearly implies that δ_I is well-defined. Moreover, $\delta_{S^1_+}(t) = \delta(t)$ and $\gamma\delta_I(t)\gamma^{-1} = \delta_{\gamma I}(t)$ for all $t \in \mathbb{R}$ and all $\gamma \in \mathrm{M\ddot{o}b}(S^1)$. For every $n \in \mathbb{Z}_{>0} \cup \{\infty\}$, denote by $\delta_I^{(n)}(t)$ the corresponding lift to $\mathrm{M\ddot{o}b}(S^1)^{(n)}$. Use also $j_I : S^1 \to S^1$ to denote the **reflection associated to** I, mapping I onto I', see [Lon08b, Section 1.6.2] or [BGL93]. Let Δ_I and J_I be the modular operator and the modular conjugation associated to the couple $(\mathcal{A}(I), \Omega)$ from the Tomita-Takesaki modular theory for von Neumann algebras, see [BR02, Chapter 2.5]. Then,

$$U(\delta_I^{(\infty)}(-2\pi t)) = \Delta_I^{it} \qquad \forall I \in \mathcal{J}\ \ \forall t \in \mathbb{R}\,. \tag{1.42}$$

Moreover, U extends uniquely to an anti-unitary representation of $\mathrm{Möb}(S^1)^{(\infty)} \ltimes \mathbb{Z}_2$ given by

$$U(j_I) = ZJ_I \qquad \forall I \in \mathcal{J} \tag{1.43}$$

and acting covariantly on $\mathcal{A}$, that is,

$$U(\gamma)\mathcal{A}(I)U(\gamma)^* = \mathcal{A}(\dot{\gamma}I) \qquad \forall\gamma \in \mathrm{Möb}(S^1)^{(\infty)} \ltimes \mathbb{Z}_2 \;\; \forall I \in \mathcal{J}\,. \tag{1.44}$$

- **(Uniqueness of the Möbius representation)**. See [CKL08, Corollary 3]. The strongly continuous unitary representation U of $\mathrm{Möb}(S^1)^{(\infty)}$ is unique.

- **(Twisted Haag duality)**. See [CKL08, Theorem 5].

$$\mathcal{A}(I') = Z\mathcal{A}(I)'Z^* = Z^*\mathcal{A}(I)'Z \qquad \forall I \in \mathcal{J}\,. \tag{1.45}$$

- **(Vacuum Spin-Statistics theorem)**. See [CKL08, Proposition 8 and Corollary 9]. The grading unitary Γ is unique and

$$U(r^{(\infty)}(2\pi)) = e^{i2\pi L_0} = \Gamma\,. \tag{1.46}$$

- **(External continuity)**. By Möbius covariance, see the proof of [CKL08, Corollary 4] and [Lon08b, p. 48], it holds that

$$\mathcal{A}(I) = \bigcap_{\mathcal{J}\ni J\supset \overline{I}} \mathcal{A}(J) \qquad \forall I \in \mathcal{J}\,. \tag{1.47}$$

A **graded-local conformal net on** S^1 is a graded-local Möbius covariant net on S^1 with the following additional property:

(F) **(Diffeomorphism covariance)**. There exists a strongly continuous projective unitary representation U^{ext} of $\mathrm{Diff}^+(S^1)^{(\infty)}$, which extends U and such that

$$U^{\mathrm{ext}}(\gamma)\mathcal{A}(I)U^{\mathrm{ext}}(\gamma)^{-1} = \mathcal{A}(\dot{\gamma}I) \qquad \forall\gamma \in \mathrm{Diff}^+(S^1)^{(\infty)} \;\; \forall I \in \mathcal{J}\,; \tag{1.48}$$

$$U^{\mathrm{ext}}(\gamma)AU^{\mathrm{ext}}(\gamma)^{-1} = A \qquad \forall\gamma \in \mathrm{Diff}(I)^{(\infty)} \;\; \forall A \in \mathcal{A}(I') \;\; \forall I \in \mathcal{J}\,. \tag{1.49}$$

With an abuse of notation, we use the same symbol U to denote U^{ext}.

Note that the Vacuum Spin-Statistics theorem above implies that the representation U of $\mathrm{Möb}(S^1)^{(\infty)}$ and consequently its extension to $\mathrm{Diff}^+(S^1)^{(\infty)}$ factor through representations of $\mathrm{Möb}(S^1)^{(2)}$ and $\mathrm{Diff}^+(S^1)^{(2)}$ respectively, which we still denote by U. Then, we have the following uniqueness result:

- **(Uniqueness of the diffeomorphism representation)**. See [CKL08, Corollary 11]. The strongly continuous projective unitary representation U of $\mathrm{Diff}^+(S^1)^{(2)}$ (and thus as representation of $\mathrm{Diff}^+(S^1)^{(\infty)}$ too) is unique (up to a projective phase).

A graded-local Möbius covariant (or conformal) net is simply called **Möbius covariant (or conformal) net** if $\Gamma = 1_{\mathcal{H}}$.

Now, we introduce some basics about covariant subnets of a given graded-local Möbius covariant (or conformal) net $\mathcal{A}$ on S^1. A **Möbius covariant subnet** $\mathcal{B}$ of $\mathcal{A}$ is a family of von Neumann algebras $(\mathcal{B}(I))_{I\in\mathcal{J}}$ acting on $\mathcal{H}$ with the following properties:

- $\mathcal{B}(I) \subseteq \mathcal{A}(I)$ for all $I \in \mathcal{J}$;
- if $I_1 \subseteq I_2$ are intervals in $\mathcal{J}$, then $\mathcal{B}(I_1) \subseteq \mathcal{B}(I_2)$;
- $U(\gamma)\mathcal{B}(I)U(\gamma) = \mathcal{B}(\dot{\gamma}I)$ for all $\gamma \in \mathrm{Möb}(S^1)^{(\infty)}$ and all $I \in \mathcal{J}$.

We use $\mathcal{B} \subseteq \mathcal{A}$ to say that $\mathcal{B}$ is a Möbius covariant subnet of $\mathcal{A}$. Define

$$\mathcal{B}(S^1) := \bigvee_{\substack{I\in\mathcal{J}\\ I\subset S^1}} \mathcal{B}(I)\,, \qquad \mathcal{H}_{\mathcal{B}} := \overline{\mathcal{B}(S^1)\Omega} \subseteq \mathcal{H}$$

and denote by $e_{\mathcal{B}} \in \mathcal{B}(S^1)' \cap U(\mathrm{Möb}(S^1)^{(\infty)})'$ the projection on the Hilbert space $\mathcal{H}_{\mathcal{B}}$. Then, we have:

Proposition 1.3.1. *Let $\mathcal{B}$ be a Möbius covariant subnet of a graded-local Möbius covariant net $\mathcal{A}$. Then, the restriction maps*

$$\mathcal{J} \ni I \mapsto \mathcal{B}(I)_{e_\mathcal{B}} := \mathcal{B}(I)|_{\mathcal{H}_\mathcal{B}}\,, \qquad \text{Möb}(S^1)^{(\infty)} \ni \gamma \mapsto U(\gamma)|_{\mathcal{H}_\mathcal{B}}$$

define a graded-local Möbius covariant net $\mathcal{B}_{e_\mathcal{B}}$ acting on $\mathcal{H}_\mathcal{B}$, which is irreducible if $\mathcal{A}$ is.

If $\mathcal{A}$ is conformal, then $\mathcal{B}$ is diffeomorphism covariant, that is, $U(\gamma)\mathcal{B}(I)U(\gamma)^* = \mathcal{B}(\dot\gamma I)$ for all $\gamma \in \text{Diff}^+(S^1)^{(\infty)}$ and all $I \in \mathcal{J}$. Moreover, there exists an extension of $U(\cdot)|_{\mathcal{H}_\mathcal{B}}$ to $\text{Diff}^+(S^1)^{(\infty)}$ on $\mathcal{H}_\mathcal{B}$, which makes $\mathcal{B}_{e_\mathcal{B}}$ conformal.

Remark 1.3.2. Note that $\mathcal{H}_\mathcal{B} = \mathcal{H}$ if and only if $\mathcal{B}(I) = \mathcal{A}(I)$ for all $I \in \mathcal{J}$, see [Ten19, Proposition 2.30]. This means that Ω is generally not cyclic for $\mathcal{B}(S^1)$ and thus $\mathcal{B}$ does not determine a graded-local Möbius covariant net on $\mathcal{H}$.

Proof. If $\mathcal{A}$ is a graded-local Möbius covariant net, then properties **(A)-(D)** are clearly satisfied by $\mathcal{B}_{e_\mathcal{B}}$ and $U(\cdot)|_{\mathcal{H}_\mathcal{B}}$ on the Hilbert space $\mathcal{H}_\mathcal{B}$ with vacuum Ω. It remains to prove the graded-locality **(D)** of $\mathcal{B}_{e_\mathcal{B}}$. By the Vacuum Spin-Statistics theorem and the Möbius covariance of $\mathcal{B}$, it is clear that $\Gamma\mathcal{B}_{e_\mathcal{B}}(I)\Gamma = \mathcal{B}_{e_\mathcal{B}}(I)$ for all $I \in \mathcal{J}$. Moreover, by the graded-locality of $\mathcal{A}$, we get that

$$\mathcal{B}(I') \subseteq \mathcal{A}(I') \subseteq Z\mathcal{A}(I)'Z^* \subseteq Z\mathcal{B}(I)'Z^* \qquad \forall I \in \mathcal{J}\,,$$

which implies the graded-locality of $B_{e_\mathcal{B}}$. Suppose further that $\mathcal{A}$ is irreducible and that $\Omega_1 \in \mathcal{H}_\mathcal{B}$ is a cyclic $U_\mathcal{B}$-invariant vector for $\mathcal{B}_{e_\mathcal{B}}$ on $\mathcal{H}_\mathcal{B}$. This implies that $\mathcal{H}_\mathcal{B} \subseteq \overline{\mathcal{A}(S^1)\Omega_1}$ and thus $\Omega \in \overline{\mathcal{A}(S^1)\Omega_1}$. Hence,

$$A\Omega \in A\overline{\mathcal{A}(S^1)\Omega_1} = \overline{\mathcal{A}(S^1)\Omega_1} \qquad \forall A \in \mathcal{A}(S^1)\,,$$

which implies that Ω_1 is cyclic for $\mathcal{A}(S^1)$ on $\mathcal{H}$. But, Ω_1 is also U-invariant and thus $\Omega_1 = \Omega$ by the irreducibility of $\mathcal{A}$. To sum up, we have proved that $\mathcal{B}_{e_\mathcal{B}}$ is a Möbius covariant net on $\mathcal{H}_\mathcal{B}$, which is irreducible if $\mathcal{A}$ is.

For the second part, suppose that $\mathcal{A}$ is a graded-local conformal net. To prove that $\mathcal{B}_{e_\mathcal{B}}$ has a conformal structure, we can adapt [Wei05, Corollary 6.2.13], paying attention to the fact that there it is used [Car04, Proposition 3.7], which can be adapted to the graded-local case as done in the proof of Theorem 3.2.17. Finally, we can obtain the diffeomorphism covariance of $\mathcal{B}$ by adapting [Wei05, Proposition 6.2.28]. The proof of [Wei05, Proposition 6.2.28] uses different results coming mostly from [Wei05, Sections 6.4, 6.5 and 6.6]. [Wei05, Sections 6.4] brings to the proof of [Wei05, Corollary 6.2.13], which can be adapted as discussed above. [Wei05, Section 6.5] deals with the representation theory of Virasoro nets (see e.g. Example 6.1.1 and references therein for the construction of these models), which then has a general validity. The final part of the proof is in [Wei05, Section 6.6], which can be easily adapted. Here, note that in the proof of [Wei05, Proposition 6.2.23], results from [Kös02] are used. Anyway, these are expressed in a wider generality for von Neumann algebras and thus they can be used in the graded-local setting too, eventually using the version of [Kös02] (see the discussion just after the proof of [Kös02, Theorem 5]) given by [Kös02v2]. □

Thanks to the above result, we will denote the net $\mathcal{B}_{e_\mathcal{B}}$ by just $\mathcal{B}$, specifying if it acts on $\mathcal{H}_\mathcal{B}$ or $\mathcal{H}$ respectively when confusion can arise. Moreover, we can refer to a Möbius covariant subnet simply as a **covariant subnet**.

A first example of covariant subnet is the **trivial subnet**, which is defined by $\mathcal{B}(I) := \mathbb{C}1_{\mathbb{C}\Omega}$ for all $I \in \mathcal{J}$ with $\Omega := 1 \in \mathbb{C}$. In the following, we present some constructions which help in the production of examples of covariant subnets.

Let $(\mathcal{A}_1, \mathcal{H}_1, \Omega_1, U_1)$ and $(\mathcal{A}_2, \mathcal{H}_2, \Omega_2, U_2)$ be two graded-local Möbius covariant nets. We say that $\mathcal{A}_1$ and $\mathcal{A}_2$ are **isomorphic** or **unitarily equivalent** if there exists a unitary operator $\phi : \mathcal{H}_1 \to \mathcal{H}_2$ such that $\phi(\Omega_1) = \Omega_2$ and $\phi\mathcal{A}_1(I)\phi^{-1} = \mathcal{A}_2(I)$ for all $I \in \mathcal{J}$. This implies that

$$\phi U_1(\gamma)\phi^{-1} = U_2(\gamma) \qquad \forall \gamma \in \text{Möb}(S^1)^{(\infty)} \tag{1.50}$$

thanks to the uniqueness of the representations U_i, see p. 16, which are also completely determined by their respective vacuum vectors. Therefore, we define the **automorphism group** $\text{Aut}(\mathcal{A})$ of the graded-local Möbius covariant net $\mathcal{A}$ as

$$\text{Aut}(\mathcal{A}) := \Big\{\phi \in U(\mathcal{H}) \mid \phi(\Omega) = \Omega\,,\ \ \phi\mathcal{A}(I)\phi^{-1} = \mathcal{A}(I)\ \ \forall I \in \mathcal{J}\Big\}\,. \tag{1.51}$$

Then, equation (1.50) implies that that every $\phi \in \mathrm{Aut}(\mathcal{A})$ commutes with $U(\gamma)$ for all $\gamma \in \mathrm{M\ddot{o}b}(S^1)^{(\infty)}$. Note also that $\mathrm{Aut}(\mathcal{A})$ equipped with the topology induced by the strong one of $B(\mathcal{H})$ is a topological group. Furthermore, we have the following desirable result:

Proposition 1.3.3. *Let ϕ realise an isomorphism between two graded-local Möbius covariant nets $\mathcal{A}_1$ and $\mathcal{A}_2$. If $\mathcal{A}_1$ and $\mathcal{A}_2$ are also diffeomorphism covariant, then*

$$\phi U_1(\gamma)\phi^{-1} = U_2(\gamma) \qquad \forall \gamma \in \mathrm{Diff}^+(S^1)^{(\infty)} .$$

As a consequence, if $\phi \in \mathrm{Aut}(\mathcal{A})$ for a graded-local conformal net $\mathcal{A}$, then ϕ commutes with the conformal symmetries $U(\gamma)$ for all $\gamma \in \mathrm{Diff}^+(S^1)^{(\infty)}$.

Proof. If $\mathcal{A}_1$ and $\mathcal{A}_2$ had been local Möbius covariant nets, then a trivial adaptation of the proof of [CW05, Corollary 5.8] would have proved the statement. In the graded-local case, we have some complications, which we deal with as follows.

Recall that thanks to the Vacuum Spin-Statistics theorem, see (1.46), the representation U of $\mathrm{Diff}^+(S^1)^{(\infty)}$ factors through a representation of $\mathrm{Diff}^+(S^1)^{(2)}$, which we still denote by U. By the uniqueness of the representation U of $\mathrm{Diff}^+(S^1)^{(2)}$, see p. 16, we have that

$$\phi U_1(\gamma)\phi^{-1} = \alpha(\gamma)U_2(\gamma) \qquad \forall \gamma \in \mathrm{Diff}^+(S^1)^{(2)}$$

where $\alpha(\gamma) := \phi U_1(\gamma)\phi^{-1}U_2(\gamma)^{-1}$ for all $\gamma \in \mathrm{Diff}^+(S^1)^{(2)}$. It is not difficult to check that α is a character of $\mathrm{Diff}^+(S^1)^{(2)}$, which is also continuous thanks to the continuity of U. Therefore, we have to prove that α is the trivial character, that is $\alpha(\gamma) = 1$ for all $\gamma \in \mathrm{Diff}^+(S^1)^{(2)}$. To this aim, let H be the kernel of the character α. Then, $p(H)$, that is the projection of H under the covering map $p : \mathrm{Diff}^+(S^1)^{(2)} \to \mathrm{Diff}^+(S^1)$, is a non-trivial normal subgroup of $\mathrm{Diff}^+(S^1)$. It follows that $\mathrm{Diff}^+(S^1) = p(H)$ because $\mathrm{Diff}^+(S^1)$ is a simple group, see Remark 1.2.1. Hence, $\mathrm{Diff}^+(S^1)^{(2)} = H \cup Hz$, where $z := r^{(2)}(2\pi)$ is the rotation by 2ϕ in $\mathrm{Diff}^+(S^1)^{(2)}$, see (1.31) for the notation. Suppose by contradiction that $z \notin H$. Note that H is closed by the continuity of α and thus Hz is closed too. This means that $\mathrm{Diff}^+(S^1)^{(2)}$ is the union of two disjoint closed subsets, which contradicts the connectedness of $\mathrm{Diff}^+(S^1)^{(\infty)}$. As a consequence, $z \in H$ and $\mathrm{Diff}^+(S^1)^{(2)} = H$, that is $\alpha(\gamma) = 1$ for all $\gamma \in \mathrm{Diff}^+(S^1)^{(2)}$. Then, the remaining statement is a clear consequence of what proved. □

Example 1.3.4. Let $\mathcal{A}$ be a graded-local conformal net and G be a compact subgroup of $\mathrm{Aut}(\mathcal{A})$. Then, we define the **fixed point subnet** $\mathcal{A}^G$ with respect to G as

$$\mathcal{A}^G(I) := \Big\{ A \in \mathcal{A}(I) \mid gAg^{-1} = A \ \forall g \in G \Big\} \quad \forall I \in \mathcal{J} .$$

It is then easy to check that $\mathcal{A}^G$ is actually a graded-local conformal net because every automorphism g is in $U(\mathrm{Diff}^+(S^1)^{(\infty)})'$ as we have explained above. If G is finite, we call $\mathcal{A}^G$ an **orbifold subnet**. $\mathcal{A}^\Gamma := \mathcal{A}^{\{1_\mathcal{H}, \Gamma\}}$, the set of Bose elements of $\mathcal{A}$ is a conformal net, usually called **the Bose subnet** of $\mathcal{A}$.

Now, we introduce two ways to produce new examples of graded-local conformal nets and covariant subnets from given ones.

Example 1.3.5. The **graded tensor product** of two graded-local conformal nets $\mathcal{A}_1$ and $\mathcal{A}_2$, see [CKL08, Section 2.6] for details, is the isotone map of von Neumann algebras $\mathcal{J} \ni I \mapsto (\mathcal{A}_1\hat{\otimes}\mathcal{A}_2)(I)$, acting on the tensor product Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$ with grading unitary $\Gamma_1 \otimes \Gamma_2$ and vacuum vector $\Omega_1 \otimes \Omega_2$, defined by the following tensor product:

$$\mathcal{A}_1(I)\hat{\otimes}\mathcal{A}_2(I) := \Big\{ A_1 \otimes A_2 \,,\ B_1 \otimes 1_{\mathcal{H}_2} \,,\ \Gamma_1 \otimes B_2 \mid {}^{A_1, B_1 \in \mathcal{A}_1(I)\,,\ A_2, B_2 \in \mathcal{A}_2(I)}_{\partial A_i = 0\,,\ \partial B_i = 1} \Big\}'' . \tag{1.52}$$

The definition above moved from the necessity that the Fermi elements on the right of the tensor product must anticommute with the ones on the left and vice versa. For a graded-local conformal net $\mathcal{A}$ and for every $I \in \mathcal{J}$, let us denote by $\mathcal{A}_f(I)$ the **Fermi subspace** of $\mathcal{A}(I)$, that is, the subspace given by the Fermi elements. Accordingly, $\mathcal{A}_1(I)\hat{\otimes}\mathcal{A}_2(I)$ is the vector space direct sum

$$\underbrace{\mathcal{A}_1^{\Gamma_1}(I) \otimes \mathcal{A}_2^{\Gamma_2}(I) \oplus \mathcal{A}_{1f}(I)\Gamma_1 \otimes \mathcal{A}_{2f}(I)}_{\text{Bosons}} \oplus \underbrace{\mathcal{A}_1^{\Gamma_1}(I)\Gamma_1 \otimes \mathcal{A}_{2f}(I) \oplus \mathcal{A}_{1f}(I) \otimes \mathcal{A}_2^{\Gamma_2}(I)}_{\text{Fermions}} . \tag{1.53}$$

Moreover, we have the following isomorphisms of von Neumann algebras

$$\begin{aligned} \hat{\mathcal{A}}_1(I) &:= \mathcal{A}_1(I) \otimes 1_{\mathcal{H}_2} \cong \mathcal{A}_1(I) \\ \hat{\mathcal{A}}_2(I) &:= (1_{\mathcal{H}_1} \otimes \mathcal{A}_2^{\Gamma_2}(I)) \vee (\Gamma_1 \otimes \mathcal{A}_{2f}(I)) \cong 1_{\mathcal{H}_1} \otimes \mathcal{A}_2(I) \cong \mathcal{A}_2(I) \end{aligned} \tag{1.54}$$

for all $I \in \mathcal{J}$. Furthermore, we have that

$$\mathcal{A}_1(I) \hat{\otimes} \mathcal{A}_2(I) = \hat{\mathcal{A}}_1(I) \vee \hat{\mathcal{A}}_2(I), \quad [\hat{\mathcal{A}}_1(I), \hat{\mathcal{A}}_2(I)] = 0 \qquad \forall I \in \mathcal{J}. \tag{1.55}$$

Example 1.3.6. The **coset subnet** $\mathcal{B}^c$ of a covariant subnet $\mathcal{B}$ of a graded-local conformal net $\mathcal{A}$ is the covariant subnet of $\mathcal{A}$ defined by the graded-local von Neumann algebras

$$\mathcal{B}^c(I) := \left\{ A \in \mathcal{A}(I) \mid [A, B] = 0 \ \forall B \in \mathcal{B}(S^1) \right\} \qquad \forall I \in \mathcal{J}. \tag{1.56}$$

1.4 Graded-local conformal nets on $\mathbb{R}$ and $S^{1(2)}$

We are interested in working with covariant nets on $\mathbb{R}$, which we introduce following [CKL08, Section 3.1]. Let $\mathcal{J}^{\mathbb{R}}$ be the set of bounded open intervals of $\mathbb{R}$, which we can identify through the Cayley transform (see Remark 4.0.2) with the subset of $\mathcal{J}$ given by the intervals of S^1 which do not contain -1. Then, a **graded-local conformal net on** $\mathbb{R}$ is a family of von Neumann algebras $\mathcal{A} := (\mathcal{A}(I))_{I \in \mathcal{J}^{\mathbb{R}}}$ on a fixed infinite-dimensional separable Hilbert space $\mathcal{H}$, also called **net** of von Neumann algebras, with the following properties:

(Ar) **(Isotony)**. For all $I_1, I_2 \in \mathcal{J}^{\mathbb{R}}$ such that $I_1 \subseteq I_2$, then $\mathcal{A}(I_1) \subseteq \mathcal{A}(I_2)$.

(Br) **(Möbius covariance)**.

$$U(\gamma)\mathcal{A}(I)U(\gamma)^{-1} = \mathcal{A}(\dot{\gamma}I) \qquad \forall \gamma \in \text{Möb}(S^1)_I^{(\infty)} \ \forall I \in \mathcal{J}^{\mathbb{R}} \tag{1.57}$$

where for all $I \in \mathcal{J}^{\mathbb{R}}$, $\text{Möb}(S^1)_I^{(\infty)}$ is the connected component to the identity in $\text{Möb}(S^1)^{(\infty)}$ of the open subset $\left\{ \gamma \in \text{Möb}(S^1)^{(\infty)} \mid \dot{\gamma}I \in \mathcal{J}^{\mathbb{R}} \right\}$.

(Cr) **(Positivity of the energy)**. The infinitesimal generator L_0 of the (universal cover of) rotation one-parameter subgroup $t \mapsto U(r^{(\infty)}(t))$, called **conformal Hamiltonian**, is a positive operator on $\mathcal{H}$. This means that U is a positive energy representation.

(Dr) **(Existence of the vacuum)**. There exists a U-invariant unit vector $\Omega \in \mathcal{H}$, called **vacuum vector**, which is also cyclic for the von Neumann algebra $\bigvee_{I \in \mathcal{J}^{\mathbb{R}}} \mathcal{A}(I)$.

(Er) **(Graded locality)**. There exists a self-adjoint unitary operator Γ on $\mathcal{H}$, called **grading unitary**, such that $\Gamma\Omega = \Omega$ and

$$\Gamma\mathcal{A}(I)\Gamma = \mathcal{A}(I) \quad \forall I \in \mathcal{J}^{\mathbb{R}}, \qquad \mathcal{A}(I_1) \subseteq Z\mathcal{A}(I_2)'Z^*$$

whenever I_1 and I_2 are disjoint intervals in $\mathcal{J}^{\mathbb{R}}$, with $Z := \frac{1_{\mathcal{H}} - i\Gamma}{1-i}$.

(Fr) **(Diffeomorphism covariance)**. There exists a strongly continuous projective unitary representation U^{ext} of $\text{Diff}^+(S^1)^{(\infty)}$, which extends U and such that

$$\begin{aligned} U^{\text{ext}}(\gamma)\mathcal{A}(I)U^{\text{ext}}(\gamma)^{-1} &= \mathcal{A}(\dot{\gamma}I) \qquad \forall \gamma \in \text{Diff}^+(S^1)_I^{(\infty)} \ \forall I \in \mathcal{J}^{\mathbb{R}} \\ U^{\text{ext}}(\gamma)AU^{\text{ext}}(\gamma)^{-1} &= A \qquad \forall \gamma \in \text{Diff}(I)_I^{(\infty)} \ \forall A \in \mathcal{A}(I_1) \end{aligned}$$

for all disjoint $I, I_1 \in \mathcal{J}^{\mathbb{R}}$, where $\text{Diff}^+(X)_I^{(\infty)}$, for X equal to S^1 or I, denotes that we are considering only those diffeomorphisms γ in the connected component to the identity of the open subset $\left\{ \gamma \in \text{Diff}^+(X)^{(\infty)} \mid \dot{\gamma}I \in \mathcal{J}^{\mathbb{R}} \right\}$. With an abuse of notation, we use the same symbol U to denote U^{ext}.

As explained in [CKL08, Section 3.2], we can define a net on a cover of S^1. Specifically, we are interested in nets on the double cover $S^{1(2)}$. As usual, we define the set $\mathcal{J}^{(2)}$ of **intervals** of $S^{1(2)}$ as the set of connected subsets of $S^{1(2)}$ whose projections onto S^1 are intervals of $\mathcal{J}$. For any $I \in \mathcal{J}^{(2)}$, let $I' := S^{1(2)} \backslash \overline{I}$ be the complement of I in $S^{1(2)}$ and denote by pI its projection onto S^1. Then, a **graded-local conformal net on** $S^{1(2)}$ is a family of von Neumann algebras $\mathcal{A} := (\mathcal{A}(I))_{I \in \mathcal{J}^{(2)}}$ such that:

(A2) **(Isotony)**. For all $I_1, I_2 \in \mathcal{J}^{(2)}$ such that $I_1 \subseteq I_2$, then $\mathcal{A}(I_1) \subseteq \mathcal{A}(I_2)$.

(B2) **(Möbius covariance)**. There exists a strongly continuous unitary representation U of $\text{Möb}(S^1)^{(\infty)}$ on $\mathcal{H}$ such that

$$U(\gamma)\mathcal{A}(I)U(\gamma)^{-1} = \mathcal{A}(\dot{\gamma}I) \qquad \forall\gamma \in \text{Möb}(S^1)^{(\infty)} \ \forall I \in \mathcal{J}^{(2)}$$

where $\dot{\gamma}$ is the projection of γ on $\text{Möb}(S^1)^{(2)}$.

(C2) **(Positivity of the energy)**. The infinitesimal generator L_0 of the (universal cover of) rotation one-parameter subgroup $t \mapsto U(r^{(\infty)}(t))$, called **conformal Hamiltonian**, is a positive operator on $\mathcal{H}$. This means that U is a positive energy representation.

(D2) **(Existence of the vacuum)**. There exists a U-invariant unit vector $\Omega \in \mathcal{H}$, called **vacuum vector**, which is also cyclic for the von Neumann algebra $\bigvee_{I\in\mathcal{J}^{(2)}} \mathcal{A}(I)$.

(E2) **(Graded locality)**. There exists a self-adjoint unitary operator Γ on $\mathcal{H}$, called **grading unitary**, such that $\Gamma\Omega = \Omega$ and

$$\Gamma\mathcal{A}(I)\Gamma = \mathcal{A}(I)\,, \quad \mathcal{A}(I') \subseteq Z\mathcal{A}(I)'Z^* \qquad \forall I \in \mathcal{J}^{(2)}$$

with $Z := \frac{1_{\mathcal{H}} - i\Gamma}{1-i}$.

(F2) **(Diffeomorphism covariance)**. There exists a strongly continuous projective unitary representation U^{ext} of $\text{Diff}^+(S^1)^{(\infty)}$, which extends U and such that

$$U^{\text{ext}}(\gamma)\mathcal{A}(I)U^{\text{ext}}(\gamma)^{-1} = \mathcal{A}(\dot{\gamma}I) \quad \forall\gamma \in \text{Diff}^+(S^1)^{(\infty)} \ \forall I \in \mathcal{J}^{(2)}$$
$$U^{\text{ext}}(\gamma)AU^{\text{ext}}(\gamma)^{-1} = A \qquad \forall\gamma \in \text{Diff}(pI)^{(\infty)} \ \forall A \in \mathcal{A}(I') \ \forall I \in \mathcal{J}^{(2)}$$

where $\dot{\gamma}$ is the projection of γ on $\text{Diff}^+(S^1)^{(2)}$. With an abuse of notation, we use the same symbol U to denote U^{ext}.

Let $\mathcal{A}$ be a graded-local conformal net on S^1 and U be the corresponding representation of $\text{Diff}^+(S^1)^{(\infty)}$. As it is well explained in [CKL08, Section 3.2], we can associate a graded-local conformal net on $\mathbb{R}$ to $\mathcal{A}$, which we denote by $\mathcal{A}^{\mathbb{R}}$. This is defined as the restriction of the net $\mathcal{A}$ to $\mathcal{J}^{\mathbb{R}}$, namely $\mathcal{A}^{\mathbb{R}}(I) := \mathcal{A}(I)$ for all $I \in \mathcal{J}^{\mathbb{R}}$. Moreover, the Möbius and diffeomorphism covariance are obtained restricting the transformations of $\text{Möb}(S^1)^{(\infty)}$ and $\text{Diff}^+(S^1)^{(\infty)}$ as explained in **(Br)** and **(Fr)** respectively. Extend the net $\mathcal{A}^{\mathbb{R}}$ to the net $\mathcal{A}^{\mathbb{R}}_{\text{ext}}$ by

$$\mathcal{A}^{\mathbb{R}}_{\text{ext}}(I) := U(\gamma)\mathcal{A}^{\mathbb{R}}(I_1)U(\gamma)^* \qquad \forall I \in \mathcal{J}$$

for any $I_1 \in \mathcal{J}^{\mathbb{R}}$ and any $\gamma \in \text{Möb}(S^1)^{(\infty)}$ such that $\gamma I_1 = I$. Then, it is even true that $\mathcal{A}^{\mathbb{R}}_{\text{ext}} = \mathcal{A}$. Following what is explained in [CKL08, Section 3.2], thanks to the Vacuum Spin-Statistics theorem and [CKL08, Corollary 17], we can **promote** $\mathcal{A}^{\mathbb{R}}$ to a graded-local conformal net on $S^{1(2)}$, which is denoted by $\mathcal{A}^{(2)}$. Then, it holds that $\mathcal{A}^{(2)}(I) = \mathcal{A}(pI)$ for all $I \in \mathcal{J}^{(2)}$.

Chapter 2

Vertex superalgebras and their further structures

On the one hand, we introduce the basic theory of unitary vertex operator superalgebras (VOSAs) and their modules in Section 2.1 and Section 2.2. On the other hand, we generalise most of the results of [CKLW18, Section 4 and Section 5] from Section 2.3 to Section 2.6. Moreover, we study the unitarity of $\mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z}$-graded simple current extension of a VOSA in Section 2.7, so characterising the unitarity of a VOSA in terms of its even and odd parts.

Comparing with the local case... The main difficulty in generalising those results is due to the $\mathbb{Z}_2$ gradation of vertex superalgebras. Indeed, this reflects the definition of the unitary structure of a VOSA, which now involves the operator Z_V and powers of the imaginary unit i, see definitions (2.32) and (2.33). Of course, this makes the calculations harsher, but they can be dealt with by making local changes. Apart from these technical issues, the results considered generalise from the local case as common sense would suggest. Finally, Corollary 2.7.3 in Section 2.7 is a result particular to the "super" environment as relating the unitarity of a VOSA in terms of its even and odd parts.

2.1 Unitary vertex operator superalgebras

For the sake of consistency and to help the understanding of the vertex superalgebra axioms, we give some preliminary notions about the calculus of formal distributions. Later, we introduce unitary VOSAs and correlated objects. We mostly follow [Kac01] for the general settings about formal distributions and VOSAs, whereas [AL17] and [Ten19b, Section 2.1] for the unitary part. Note that our notation follows the one in [CKLW18, Sections 4 and 5] as closely as possible.

We define a **formal distribution** in the formal variables $z, w, \dots$ with values in a vector space U over $\mathbb{C}$ as a formal series of the form:

$$a(z, w, \dots) := \sum_{m,n,\dots \in \mathbb{Z}} a_{m,n,\dots} z^m w^n \cdots \tag{2.1}$$

where $a_{m,n,\dots}$ are elements of U. Such formal distributions form a vector space, which is usually denoted by $U[[z, z^{-1}, w, w^{-1}, \dots]]$. We remark that the multiplication of two formal distributions is defined if and only if the coefficient of every monomial in the formal variables is a finite sum. Consequently, we can always multiply a formal distribution by a Laurent polynomial, that is, a formal series with a finite non-zero number of coefficients corresponding to negative powers of the formal variables. This brings us to the following consideration: if we define the **residue** of a formal distribution $a(z)$ as $\operatorname{Res}_z a(z) := a_{-1}$, then $\langle a, \varphi \rangle := \operatorname{Res}_z a(z)\varphi(z)$ defines a pairing, which takes values in U, between $U[[z, z^{-1}]]$ and the algebra of Laurent polynomials in the formal variable z with coefficient in $\mathbb{C}$. In other words, these Laurent polynomials play the role of "formal test functions" for the formal distributions in $U[[z, z^{-1}]]$. If $\partial_\cdot$ is the formal partial derivation in any formal variable, we have the following well-known formulae:

$$\operatorname{Res}_z \partial_z a(z) = 0\,, \qquad \operatorname{Res}_z \partial_z a(z) b(z) = -\operatorname{Res}_z \partial_z b(z) a(z) \tag{2.2}$$

whenever the products of formal distributions above is defined. Note that the second formula above is the usual integration by parts. In the following, we use the notation: $\partial^{(j)} := \frac{\partial^j}{j!}$ for all $j \in \mathbb{Z}_{\geq 0}$.

Of particular interest is the so called **formal delta-function** $\delta(z-w)$, which is the following formal distribution in the variable z and w with coefficients in $\mathbb{C}$:

$$\delta(z-w) \equiv \delta(z,w) := z^{-1} \sum_{n\in\mathbb{Z}} \left(\frac{w}{z}\right)^n . \tag{2.3}$$

To stay true to definition [Kac01, Eq. (2.1.3)], we use the notation $\delta(z-w)$, despite δ is a function of z and w, not of $z-w$; in any case, the two notations can cohabit without making confusion and we will use one rather than the other one depending on which is more appropriate for the context. By [Kac01, Eq. 2.1.5a], we have the formula:

$$\partial_w^{(j)} \delta(z-w) = \sum_{n\in\mathbb{Z}} \binom{n}{j} z^{-n-1} w^{n-j} \qquad \forall j \in \mathbb{Z}_{\geq 0} . \tag{2.4}$$

It is then interesting to note the following properties of the formal delta-function given by [Kac01, Proposition 2.1]:

$$\begin{aligned}
\mathrm{Res}_z a(z)\delta(z-w) &= a(w) \qquad \forall a \in U[[z,z^{-1}]] && (2.5)\\
\delta(z-w) &= \delta(w-z) && (2.6)\\
\partial_z \delta(z-w) &= -\partial_w \delta(z-w) && (2.7)\\
(z-w)\partial_w^{(j+1)}\delta(z-w) &= \partial_w^{(j)}\delta(z-w) \qquad \forall j \in \mathbb{Z}_{>0} && (2.8)\\
(z-w)^{j+1}\partial_w^{(j)}\delta(z-w) &= 0 \qquad \forall j \in \mathbb{Z}_{>0} . && (2.9)
\end{aligned}$$

A definitely fundamental definition is the locality for formal distributions: a formal distribution $a(z,w)$ is said to be **local** if

$$(z-w)^N a(z,w) = 0 \tag{2.10}$$

for any positive number N big enough. A characterisation of locality is that the formal distribution $a(z,w)$ has what is called the **OPE expansion**:

$$a(z,w) = \sum_{j=0}^{N-1} c^j(w)\partial_w^{(j)}\delta(z-w) , \qquad c^j(w) := \mathrm{Res}_z\, a(z,w)(z-w)^j . \tag{2.11}$$

where the c^js are called **OPE coefficients**, see [Kac01, Definition 2.2]. To specify the above definition to the vertex superalgebra case, we suppose that U carries a further structure of $\mathbb{Z}_2$-graded associative algebra, that is, $U := U_{\bar{0}} \oplus U_{\bar{1}}$ and $U_j U_k \subseteq U_{j+k}$ for all $j,k \in \mathbb{Z}_2$. In that case, U is usually called associative superalgebra. We call **even** the elements of $U_{\bar{0}}$ and **odd** the elements of $U_{\bar{1}}$. Accordingly, we denote the **parity** of $a \in U_j$ by $p(a) = j$ for any $j \in \mathbb{Z}_2$. We can define a **bracket** by

$$[a,b] = ab - (-1)^{p(a)p(b)} ba \tag{2.12}$$

for all $a,b \in U$ with given parity. Hereafter, with an abuse of notation, we use $(-1)^{p(a)p(b)}$ as in (2.12) to denote $(-1)^{p_a p_b}$, where $p_a, p_b \in \{0,1\}$ are representatives of the remainder class of $p(a)$ and $p(b)$ in $\mathbb{Z}_2$ respectively. A classical example, widely used in the vertex superalgebra setting, is provided by the endomorphism algebra $\mathrm{End}(V)$ of a vector superspace V, where the parity is given by

$$\mathrm{End}(V)_j := \{a \in \mathrm{End}(V) \mid aV_k \subseteq V_{j+k}\}$$

and bracket defined as above. Therefore, it makes sense to define locality for formal distributions with values in a Lie superalgebra $\mathfrak{g}$ (see [Kac77]): two formal distributions $a(z)$ and $b(z)$, taking values in $\mathfrak{g}$, are said to be **mutually local** or just **local** if $[a(z),b(w)]$ in $\mathfrak{g}[[z,z^{-1},w,w^{-1}]]$ is local in the meaning given just above, that is,

$$(z-w)^N [a(z),b(w)] = 0 \qquad N \gg 0 . \tag{2.13}$$

Finally, we highlight that in the vertex superalgebra theory, it is widely used to write a formal distribution $a(z)$ with the following slightly different notation:

$$a(z) = \sum_{n\in\mathbb{Z}} a_n z^n = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1} \tag{2.14}$$

and similarly whenever more then one formal variable is involved.

Now, we are in the conditions to give the definition of a **vertex superalgebra** in details, that is, a quadruple (V, Ω, T, Y), where:

- V is a **complex vector superspace**, namely a complex vector space equipped with an automorphism Γ_V such that $\Gamma_V^2 = 1_V$ and V decomposes as the direct sum of the following vector subspaces

$$V_{\overline{0}} := \left\{a \in V \mid \Gamma_V a = a\right\} \tag{2.15}$$
$$V_{\overline{1}} := \left\{a \in V \mid \Gamma_V a = -a\right\}, \tag{2.16}$$

  where $p \in \{\overline{0}, \overline{1}\}$ denotes an element of the cyclic group of order two $\mathbb{Z}_2$. Hence, we say an element $a \in V_{\overline{0}}$ is **even**, whereas $a \in V_{\overline{1}}$ is **odd**. Moreover, we say that $a \in V$ has **parity** $p(a) \in \left\{\overline{0}, \overline{1}\right\}$ if $a \in V_{p(a)}$.

- $\Omega \in V_{\overline{0}}$ is called the **vacuum vector** of V.

- T is an even endomorphism of V called the **infinitesimal translation operator**.

- Y is a vector space linear map, called the **state-field correspondence**, defined by

$$V \ni a \longmapsto Y(a,z) := \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1} \in (\mathrm{End}(V))[[z, z^{-1}]]\,. \tag{2.17}$$

  Furthermore, Y must satisfy the following properties:

  - **(Parity-preserving)**. $a_{(n)} V_p \subseteq V_{p+p(a)}$ for all $a \in V_{p(a)}$, all $n \in \mathbb{Z}$ and all $p \in \{\overline{0}, \overline{1}\}$.
  - **(Field)**. For all $a, b \in V$, there exists a non-negative integer N such that $a_{(n)} b = 0$ for all $n \geq N$. Then, we call $Y(a,z)$ a **field**.
  - **(Translation covariance)**. $[T, Y(a,z)] = \frac{\mathrm{d}}{\mathrm{d}z} Y(a,z)$ for all $a \in V$.
  - **(Vacuum)**. $Y(\Omega, z) = 1_V$, $T\Omega = 0$ and $a_{(-1)}\Omega = a$ for all $a \in V$.
  - **(Locality)**. For all $a, b \in V$ with given parities, there exists a positive integer M such that

$$(z-w)^N [Y(a,z), Y(b,w)] = 0 \tag{2.18}$$

    for all $N \geq M$, where

$$[Y(a,z), Y(b,w)] := Y(a,z)Y(b,w) - (-1)^{p(a)p(b)} Y(b,w)Y(a,z) \tag{2.19}$$

    is the **graded commutator** of the fields $Y(a,z)$ and $Y(b,w)$. Hereafter, with an abuse of notation, we use $(-1)^{p(a)p(b)}$ as in (2.19) to denote $(-1)^{p_a p_b}$, where $p_a, p_b \in \{0, 1\}$ are representatives of the remainder class of $p(a)$ and $p(b)$ in $\mathbb{Z}_2$ respectively.

  Every $Y(a,z)$ with the properties above is called a **vertex operator**.

An important consequence of the above axioms, see [Kac01, Section 4.8], is the **Borcherds identity** (or **Jacobi identity**):

$$\begin{aligned}\sum_{j=0}^{\infty} \binom{m}{j} \left(a_{(n+j)} b\right)_{(m+k-j)} c = \\ \sum_{j=0}^{\infty} (-1)^j \binom{n}{j} \left(a_{(m+n-j)} b_{(k+j)} - (-1)^{p(a)p(b)} (-1)^n b_{(n+k-j)} a_{(m+j)}\right) c\end{aligned} \tag{2.20}$$

for all $a,b\in V$ with given parities, all $c\in V$ and all $m,n,k\in\mathbb{Z}$. Putting $n=0$ in the Borcherds identity above, we get the famous **Borcherds commutator formula** for two vectors $a,b\in V$ with given parities, see [Kac01, Eq. (4.6.3)]:

$$[a_{(m)},b_{(k)}]c=\sum_{j=0}^{\infty}\binom{m}{j}\left(a_{(j)}b\right)_{(m+k-j)}c\qquad\forall c\in V\ \forall m,k\in\mathbb{Z}\,. \tag{2.21}$$

Choosing instead $m=0$, we get the **Borcherds associative formula**:

$$(a_{(n)}b)_{(k)}c=\sum_{j=0}^{\infty}(-1)^j\binom{n}{j}\left(a_{(n-j)}b_{(k+j)}-(-1)^{p(a)p(b)}(-1)^n b_{(n+k-j)}a_{(j)}\right)c\,. \tag{2.22}$$

for all $a,b\in V$ with given parities, all $c\in V$ and all $n,k\in\mathbb{Z}$.

A further property which we will use considerably is the **skew-symmetry**, see [Kac01, Eq. (4.2.1)]: for any $a,b\in V$ with given parities,

$$Y(a,z)b=(-1)^{p(a)p(b)}e^{zL_{-1}}Y(b,-z)a\,. \tag{2.23}$$

An even element ν of a vertex superalgebra (V,Ω,T,Y) is called a **Virasoro vector** of **central charge** $c\in\mathbb{C}$ if the following commutation relations hold

$$[L_n,L_m]=(n-m)L_{n+m}+\frac{c(n^3-n)}{12}\delta_{n,-m}1_V \tag{2.24}$$

for all $n,m\in\mathbb{Z}$ where $L_n:=\nu_{(n+1)}$. $Y(\nu,z)$ is called a **Virasoro field**. If L_0 is diagonalizable on V and $L_{-1}=T$, then ν is called a **conformal vector**, L_0 a **conformal Hamiltonian** and $Y(\nu,z)$ an **energy-momentum field**. A vertex superalgebra with a fix conformal vector is called a **conformal vertex superalgebra**. Then, we define a **vertex operator superalgebra (VOSA)** as a conformal vertex superalgebra (V,Ω,T,Y,ν) with the additional properties:

- $V_n:=\mathrm{Ker}(L_0-n1_V)=0$ if $n\notin\frac{1}{2}\mathbb{Z}$ with $V_{\overline{0}}=\bigoplus_{n\in\mathbb{Z}}V_n$ and $V_{\overline{1}}=\bigoplus_{n\in\mathbb{Z}+\frac{1}{2}}V_n$.
- $\dim V_n<+\infty$ and there exists an integer $N\leq 0$ such that $V_n=0$ for all $n<N$.

With an abuse of notation, we will often use V in place of the quintuple (V,Ω,T,Y,ν).

Every non-zero element $a\in V_n$ with $n\in\frac{1}{2}\mathbb{Z}$ is called **homogeneous** with **conformal weight** $d_a:=n$. Accordingly, we rewrite

$$Y(a,z)=\sum_{n\in\mathbb{Z}-d_a}a_nz^{-n-d_a}\,,\quad a_n:=a_{(n+d_a-1)} \tag{2.25}$$

for all homogeneous $a\in V$. For our convenience, if $d_a\in\mathbb{Z}$, we set $a_n:=0$ for all $n\in\mathbb{Z}-\frac{1}{2}$; whereas if $d_a\in\mathbb{Z}-\frac{1}{2}$, we set $a_n:=0$ for all $n\in\mathbb{Z}$. Furthermore, $a\in V$ homogeneous is called **primary** if $L_na=0$ for all $n\in\mathbb{Z}_{>0}$, whereas it is called **quasi-primary** if just $L_1a=0$. It is not difficult to prove that Ω is a primary vector in V_0 (see [CKLW18, Remark 4.1]), whereas ν is a quasi-primary vector in V_2 which cannot be primary if $c\neq 0$ (see [Kac01, Theorem 4.10]). From [Kac01, Sections 4.9, 4.10] we have the following useful commutation relations:

$$[L_0,a_n]=-na_n \tag{2.26}$$
$$[L_{-1},a_n]=(-n-d_a+1)a_{n-1} \tag{2.27}$$
$$[L_1,a_n]=(-n+d_a-1)a_{n+1}+(L_1a)_{n+1} \tag{2.28}$$

for all homogeneous $a\in V$ and all $n\in\frac{1}{2}\mathbb{Z}$.

Now, we are ready to define a unitary structure on a VOSA. First of all, we need to define **(anti)linear homomorphisms** between VOSAs $(V,\Omega_V,T_V,Y_V,\nu_V)$ and $(W,\Omega_W,T_W,Y_W,\nu_W)$ as a(n) (anti)linear complex vector space map ϕ between V and W with the additional conditions: $\phi(a_{(n)}b)=\phi(a)_{(n)}\phi(b)$ for all $a,b\in V$ (ϕ respects the (n)-product), $\phi(\Omega_V)=\Omega_W$ and $\phi(\nu_V)=\nu_W$. The last condition implies that ϕ commutes with every L_n for $n\in\mathbb{Z}$ and thus it is actually parity-preserving. We also define **isomorphisms** and **automorphisms** of VOSAs in the obvious way. We denote by $\mathrm{Aut}(V)$ the group of automorphisms of V. Note that $\mathrm{Aut}(V)$ is a subset of the group $\prod_{n\in\frac{1}{2}\mathbb{Z}}\mathrm{GL}(V_n)$ of grading-preserving vector space automorphisms of V. Therefore,

$\mathrm{Aut}(V)$ can be turned into a metrizable topological group if it is equipped with the relative topology induced by the product topology of $\prod_{n\in\frac{1}{2}\mathbb{Z}}\mathrm{GL}(V_n)$. We say that an automorphism ϕ is an **involution** if $\phi^2=1$. Note that Γ_V is a linear involution of V. Moreover, due to the parity preservation, we deduce that Γ_V commutes with every (anti)linear automorphism of V. Furthermore, we define the following operator on V:

$$Z_V := \frac{1_V - i\Gamma_V}{1-i}\,. \tag{2.29}$$

Note that Z_V is a linear automorphism of the vector space V which preserves the vacuum and the conformal vectors, but it is not an automorphism of the VOSA because it does not respect the (n)-product. Indeed, we have that

$$\begin{aligned} Z_V(a_{(n)}b) &= \begin{cases} a_{(n)}b & \text{if } p(a)=p(b)=\overline{0} \\ ia_{(n)}b & \text{if } p(a)=\overline{1}+p(b) \\ a_{(n)}b & \text{if } p(a)=p(b)=\overline{1} \end{cases}, \\ (Z_Va)_{(n)}(Z_Vb) &= \begin{cases} a_{(n)}b & \text{if } p(a)=p(b)=\overline{0} \\ ia_{(n)}b & \text{if } p(a)=\overline{1}+p(b) \\ -a_{(n)}b & \text{if } p(a)=p(b)=\overline{1} \end{cases} \end{aligned} \tag{2.30}$$

for all $a,b\in V$ with given parities and all $n\in\mathbb{Z}$. Hence, (2.30) can be rewritten in the useful formula

$$Z_V(a_{(n)}b) = (-1)^{p(a)p(b)}(Z_Va_{(n)})(Z_Vb) \tag{2.31}$$

for any $a,b\in V$ with given parity and all $n\in\mathbb{Z}$.

A **unitary vertex operator superalgebra** is a VOSA V equipped with an antilinear automorphism θ, which we call the **PCT operator**, associated with a scalar product (i.e., a positive-definite Hermitian form, linear in the second variable) $(\cdot|\cdot)$ on V such that:

- θ is an **involution**, that is, $\theta^2=1_V$ (see Remark 2.1.1 below);
- $(\cdot|\cdot)$ is **normalized**, which means that $(\Omega|\Omega)=1$;
- $(\cdot|\cdot)$ is **invariant**, that is, (see [AL17, Section 2.1])

$$(Y(\theta(a),z)b|c) = (b|Y(e^{zL_1}(-1)^{2L_0^2+L_0}z^{-2L_0}a,z^{-1})c) \qquad \forall a,b,c\in V \tag{2.32}$$

or equivalently (see [Ten19b, Definition 2.3])

$$(Y(\theta(a),z)b|c) = (b|Y(e^{zL_1}(iz^{-1})^{2L_0}Z_Va,z^{-1})c) \qquad \forall a,b,c\in V\,, \tag{2.33}$$

where (2.32) and (2.33) are to be understood as equalities between doubly-infinite formal Laurent series in the formal variable z (which is not affected by the complex conjugation) with coefficients in $\mathbb{C}$. In particular, for $a\in V$ homogeneous, we use the following definitions:

$$(-1)^{2L_0^2+L_0}a := (-1)^{2d_a^2+d_a}a \tag{2.34}$$

$$e^{zL_1}a := \sum_{l\in\mathbb{Z}_{\geq 0}}\frac{z^l}{l!}L_1^la \tag{2.35}$$

$$(iz^{-1})^{2L_0}a := (iz^{-1})^{2d_a}a\,. \tag{2.36}$$

Note also that

$$(-1)^{2d^2+d} = \begin{cases} (-1)^d & \text{if } d\in\mathbb{Z} \\ (-1)^{d+\frac{1}{2}} & \text{if } d\in\mathbb{Z}-\frac{1}{2} \end{cases}. \tag{2.37}$$

Then, the equivalence of (2.32) and (2.33) is clear from $i^{2L_0}Z_V=(-1)^{2L_0^2+L_0}$.

Remark 2.1.1. In the references [AL17, Section 2.1] and [Ten19b, Definition 2.3] given above for the definition of unitary VOSAs, it is required that the PCT operator θ is an involution. For our convenience, we have kept this axiom in the definition given above. Nevertheless, we point out that the involution axiom is not necessary because we will prove that it is actually a consequence of the other ones in (i) of Proposition 2.4.1.

If a (unitary) vertex (operator) superalgebra V has $V_{\overline{1}} = \{0\}$, then it is a **(unitary) vertex (operator) algebra** as defined in [CKLW18].

Remark 2.1.2. Let V be a unitary VOSA. The real subspace

$$V_{\mathbb{R}} := \{a \in V \mid \theta(a) = a\} \tag{2.38}$$

contains the vacuum and the conformal vectors because they are θ-invariant. Moreover, $V_{\mathbb{R}}$ inherits the structure of a **real** VOSA from the one of V. Note also that V is the vector space complexification of $V_{\mathbb{R}}$ because $V = V_{\mathbb{R}} + iV_{\mathbb{R}}$ and $V_{\mathbb{R}} \cap iV_{\mathbb{R}} = \{0\}$. Then, $V_{\mathbb{R}}$ is known as a **real form** for V, see [CKLW18, Remark 5.4] and references therein. Restricting the scalar product on V to $V_{\mathbb{R}}$, we obtain a positive-definite normalized real-valued $\mathbb{R}$-bilinear form with the invariant properties (2.32) and (2.33). Conversely, if $\widetilde{V}$ is a real VOSA with a positive-definite normalized real-valued invariant (in the meaning given just above) $\mathbb{R}$-bilinear form $(\cdot\,,\cdot)$, then its complexification $\widetilde{V}_{\mathbb{C}}$ is a VOSA with an invariant scalar product which extends $(\cdot\,,\cdot)$. Furthermore, $(\widetilde{V}_{\mathbb{C}})_{\mathbb{R}} = \widetilde{V}$.

We say that an element ϕ in $\mathrm{Aut}(V)$ is a **unitary automorphism** if it preserves the scalar product, that is $(\phi(a)|\phi(b)) = (a|b)$ for all $a, b \in V$. We denote such subgroup of $\mathrm{Aut}(V)$ by $\mathrm{Aut}_{(\cdot|\cdot)}(V)$. Similarly, we can define a **unitary isomorphism** between two generic unitary VOSAs.

In the following, we introduce some useful relations. Let $(V, (\cdot|\cdot))$ be a unitary VOSA and let $\mathcal{H}_{(V,(\cdot|\cdot))}$ be the Hilbert space completion of V with respect to $(\cdot|\cdot)$. Consider Γ and Z, the extensions to $\mathcal{H}_{(V,(\cdot|\cdot))}$ of the operators Γ_V and Z_V respectively. Then it is easy to verify that $\Gamma = \Gamma^{-1} = \Gamma^*$ and consequently

$$Z^* = Z^{-1} = \frac{1_V + i\Gamma}{1+i}\,. \tag{2.39}$$

Moreover, due to the antilinearity of θ, we have that

$$Z\theta Z = \theta\,. \tag{2.40}$$

If $a \in V$ is homogeneous of conformal weight d_a, then $L_1^l a$ is still homogeneous of conformal weight $d_a - l$ for all $l \in \mathbb{Z}_{\geq 0}$ thanks to (2.26). Then, by replacing a with $\theta(a)$ in (2.32) and (2.33) and remembering that Z and θ commute with every L_n for all $n \in \mathbb{Z}$, we can calculate the following equation

$$(a_n b|c) = (-1)^{2d_a^2+d_a} \sum_{l\in\mathbb{Z}_{\geq 0}} \frac{(b|(\theta L_1^l a)_{-n} c)}{l!} = i^{2d_a} \sum_{l\in\mathbb{Z}_{\geq 0}} \frac{(b|(Z\theta L_1^l a)_{-n} c)}{l!} \tag{2.41}$$

for all $b, c \in V$, all homogeneous $a \in V$ and all $n \in \frac{1}{2}\mathbb{Z}$. In particular, if a is quasi-primary, we have

$$(a_n b|c) = (-1)^{2d_a^2+d_a}(b|(\theta a)_{-n} c) = i^{2d_a}(b|(Z\theta a)_{-n} c) = i^{2d_a}(b|(\theta Z^* a)_{-n} c) \tag{2.42}$$

$$(b|a_n c) = (-1)^{2d_a^2+d_a}((\theta a)_{-n} b|c) = i^{-2d_a}((\theta Z^* a)_{-n} b|c) = i^{-2d_a}((Z\theta a)_{-n} b|c)\,. \tag{2.43}$$

Equation (2.42) says that

$$(L_n a|b) = (a|L_{-n} b) \qquad \forall a, b \in V \;\; \forall n \in \mathbb{Z} \tag{2.44}$$

which also implies, considering the case $n = 0$, that $(V_l|V_k) = 0$ whenever $l \neq k$. In this context, we say that a field $Y(a, z)$, associated to a quasi-primary element $a \in V$, is **Hermitian** (with respect to $(\cdot|\cdot)$) if $(a_n b|c) = (b|a_{-n} c)$ for all $n \in \frac{1}{2}\mathbb{Z}$ and all $b, c \in V$.

It is important to introduce here the notion of **vertex subalgebra** of a vertex superalgebra V. This is given by a vector subspace W of V containing the vacuum vector Ω and such that $a_{(n)} W \subseteq W$ for all $a \in W$ and all $n \in \mathbb{Z}$. By the vertex superalgebra axioms, it follows that $Ta = a_{(-2)}\Omega$ for all $a \in V$. Accordingly, W is automatically T-invariant and therefore $(W, \Omega, T|_W, Y|_W)$ is a vertex superalgebra. It is then clear that $\mathbb{C}\Omega$ and $V_{\overline{0}}$ are vertex subalgebras of V, which are also vertex algebras. Given a subspace $\mathfrak{F}$ of V, we define $W(\mathfrak{F})$ as the smallest vertex subalgebra of V containing $\mathfrak{F}$ and we say that $W(\mathfrak{F})$ is **generated by** $\mathfrak{F}$.

Moreover, we can define an **ideal** of a vertex superalgebra V as a T-invariant vector subspace $\mathscr{J}$ such that $a_{(n)}\mathscr{J} \subseteq \mathscr{J}$ for all $a \in V$ and all $n \in \mathbb{Z}$. From [Kac01, Eq. (4.3.1)], we have that $a_{(n)}V \subseteq \mathscr{J}$ for all $a \in \mathscr{J}$ and all $n \in \mathbb{Z}$. As usual, we say that a vertex superalgebra V is **simple** if the only ideals which it contains are $\{0\}$ and V itself.

Finally, we say a vertex operator algebra V is of **CFT type** if the corresponding conformal vector is of CFT type: $V_n = 0$ for all $n \notin \frac{1}{2}\mathbb{Z}_{\geq 0}$ and $V_0 = \mathbb{C}\Omega$.

2.2 Modules of vertex operator superalgebras

We introduce the representation theory for VOSAs, proving some classical results. Besides the general interest, the statements here contained are fundamental to obtain some characterisations of the unitary structure of a VOSAs in the following sections.

Working Hypothesis 2.2.1. Throughout the present section V will be a VOSA.

A V**-module** is a $\mathbb{Z}_2$-graded complex vector space $M = M_{\overline{0}} \oplus M_{\overline{1}}$ equipped with the following structure:

- a vector space linear map Y^M defined by

$$V \ni a \longmapsto Y^M(a,z) := \sum_{n\in\mathbb{Z}} a^M_{(n)} z^{-n-1} \in (\mathrm{End}(M))[[z,z^{-1}]] \tag{2.45}$$

  such that:

  - **(Parity-preserving).** $a^M_{(n)}M_p \subseteq M_{p+p(a)}$ for all $a \in V_{p(a)}$ and all $n \in \mathbb{Z}$.
  - **(Field).** For all $a \in V$ and all $b \in M$, there exists a non-negative integer N such that $a^M_{(n)}b = 0$ for all $n \geq N$.
  - **(Translation covariance).** $\frac{\mathrm{d}}{\mathrm{d}z}Y^M(a,z) = Y^M(L_{-1}a,z)$ for all $a \in V$.
  - **(Vacuum).** $Y^M(\Omega,z) = 1_M$.
  - **(Borcherds identity).** For every $a, b \in V$ with given parities, all $c \in M$ and all $m,n,k \in \mathbb{Z}$, we have that

$$\begin{aligned}\sum_{j=0}^{\infty}\binom{m}{j}\left(a_{(n+j)}b\right)^M_{(m+k-j)}c = \\ \sum_{j=0}^{\infty}(-1)^j\binom{n}{j}\left(a^M_{(m+n-j)}b^M_{(k+j)} - (-1)^{p(a)p(b)}(-1)^n b^M_{(n+k-j)}a^M_{(m+j)}\right)c\,.\end{aligned} \tag{2.46}$$

- Let $Y^M(\nu,z) = \sum_{n\in\mathbb{Z}} L^M_n z^{-n-2}$, then M has a grading compatible with the parity given by the eigenspaces of L^M_0:

$$M = \bigoplus_{n\in\mathbb{C}} M_n\,, \quad M_p = \bigoplus_{n\in\mathbb{C}} M_p \cap M_n \quad \forall p \in \{\overline{0},\overline{1}\}\,, \quad M_n := \mathrm{Ker}(L^M_0 - n1_M) \quad \forall n \in \mathbb{C}$$

  with $\dim M_n < +\infty$ for all $n \in \mathbb{C}$ and $M_n \neq \{0\}$ at most for countably many n.

Note that the translation covariance and the Borcherds identity imply that

$$[L^M_{-1}, Y^M(a,z)] = \frac{\mathrm{d}}{\mathrm{d}z}Y^M(a,z) = Y^M(L_{-1}a,z) \qquad \forall a \in V\,. \tag{2.47}$$

Moreover, if $c \in \mathbb{C}$ is the central charge of V, then the following Virasoro algebra commutation relations hold (see [Li96, Remark 2.3.2], cf. [LL04, Proposition 4.1.5]):

$$[L^M_n, L^M_m] = (n-m)L^M_{n+m} + \frac{c(n^3-n)}{12}\delta_{n,-m}1_M \qquad \forall n,m \in \mathbb{Z}\,. \tag{2.48}$$

A **submodule** S of a V-module M is a vector subspace of M invariant for the action of V. M is called **irreducible** if there are no non-trivial submodules.

We define a V**-module homomorphism** f between M^1 and M^2 as an even vector superspace homomorphism from M^1 to M^2 such that $f(Y^{M^1}(a,z)b) = Y^{M^2}(a,z)f(b)$ for all $a \in V$ and all $b \in M^1$. Further definitions of V**-module iso/endo/ automorphisms** are given in the obvious way.

A vector b in a V-module M is called **vacuum-like** if $a^M_{(n)}b = 0$ for all $n \in \mathbb{Z}_{\geq 0}$. Then, we have that:

Lemma 2.2.2. *Let V be a VOSA. Then, a vector b of a V-module M is vacuum-like if and only if $L^M_{-1}b = 0$.*

Proof. We rewrite the argument of the proof of [Li94, Proposition 3.3 (a)], which works well in our case too. The "only if" part follows by definition of vacuum-like vector because $L^M_{-1} = \nu^M_{(0)}$. Conversely, suppose that $L^M_{-1}b = 0$ and let a be an arbitrary element of V. By the field axiom in the definition of V-module, there exists an $N \geq -1$ such that $a^M_{(n)}b = 0$ for all $n > N$, but $a^M_{(N)}b \neq 0$. By translation covariance (2.47), we have that

$$-(N+1)a^M_{(N)}b = [L_{-1}, a_{(N+1)}]b = 0$$

which necessarily implies that $N = -1$, that is, b is a vacuum-like vector of M. □

The basic example of a V-module is V itself by setting $Y^V := Y$, this is called the **adjoint module**. Furthermore, note that the VOSA V is simple if and only if its adjoint module is irreducible. A second example is represented by the **contragredient module** M' of a given V-module M, which is the graded dual vector space

$$M' := \bigoplus_{n\in\mathbb{C}} (M_{\overline{0}} \cap M_n)^* \oplus \bigoplus_{n\in\mathbb{C}} (M_{\overline{1}} \cap M_n)^* \tag{2.49}$$

with the state-field correspondence $Y^{M'}$ defined by the formula

$$\langle Y^{M'}(a,z)b', c\rangle = \langle b', Y^M(e^{zL_1}(-1)^{2L_0^2+L_0}z^{-2L_0}a, z^{-1})c\rangle \ \ \forall a \in V \ \ \forall c \in M \ \ \forall b' \in M' , \tag{2.50}$$

where $\langle\cdot,\cdot\rangle$ is the natural pairing between M and M'. $Y^{M'}(a,z)$ is called the **adjoint vertex operator** of $Y^M(a,z)$. Then, we can prove the following.

Proposition 2.2.3. *For a given VOSA V and a V-module M, we have that $(M', Y^{M'})$ is a V-module too.*

Proof. The following proof is based on the proof of [FHL93, Theorem 5.2.1] as suggested for [Yam14, Lemma 2].

From the definition of contragredient module (2.49), it follows that every adjoint vertex operator $Y^{M'}(a,z)$ is actually a parity-preserving field with the related vacuum property. Moreover, by the defining formula (2.50), we check that

$$\langle L^{M'}_n b', c\rangle = \langle b', L^M_{-n}c\rangle \qquad \forall n \in \mathbb{Z} \ \forall c \in M \ \ \forall b' \in M' , \tag{2.51}$$

which implies all the required properties for $Y^M(\nu, z)$. Hence, it remains to prove that the translation covariance and the Borcherds identity for $Y^{M'}$ hold.

As for the former one, we need to prove that $\frac{\mathrm{d}}{\mathrm{d}z}Y^{M'}(a,z) = Y^{M'}(L_{-1}a, z)$ for all $a \in V$. Proceeding as in [FHL93, Eq. (5.2.22)], that is, using the invariance (2.50) and the chain rule, we get that

$$\begin{aligned}\langle \frac{\mathrm{d}}{\mathrm{d}z}Y^{M'}(a,z)b', c\rangle &= \frac{\mathrm{d}}{\mathrm{d}z}\langle Y^{M'}(a,z)b', c\rangle \\ &= \langle b', \frac{\mathrm{d}}{\mathrm{d}z}Y^M(e^{zL_1}(-1)^{2L_0^2+L_0}z^{-2L_0}a, z^{-1})c\rangle \\ &= \langle b', Y^M(\frac{\mathrm{d}}{\mathrm{d}z}e^{zL_1}(-1)^{2L_0^2+L_0}z^{-2L_0}a, z^{-1})c\rangle + \langle b', \frac{\mathrm{d}}{\mathrm{d}z}Y^M(\cdot, z^{-1})|_{a_1}c\rangle\end{aligned} \tag{2.52}$$

where $a_1 = e^{zL_1}(-1)^{2L_0^2+L_0}z^{-2L_0}a$. On the one hand, an easy calculation gives us that

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(e^{zL_1}(-1)^{2L_0^2+L_0}z^{-2L_0}\right) = L_1e^{zL_1}(-1)^{2L_0^2+L_0}z^{-2L_0} - 2e^{zL_1}(-1)^{2L_0^2+L_0}L_0z^{-2L_0-1} . \tag{2.53}$$

On the other hand, if a is homogeneous, then

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}z}Y^M(\cdot,z^{-1})|_{a_1} &= -z^{-2}\frac{\mathrm{d}}{\mathrm{d}z^{-1}}Y^M(\cdot,z^{-1})|_{a_1}\\
&= -z^{-2}Y^M(L_{-1}e^{zL_1}(-1)^{2L_0^2+L_0}z^{-2L_0}a,z^{-1})\\
&= -z^{-2}Y^M([e^{zL_1}L_{-1}-2ze^{zL_1}L_0+z^2L_1e^{zL_1}](-1)^{2L_0^2+L_0}z^{-2L_0}a,z^{-1})\\
&= Y^M(e^{zL_1}(-1)^{2d_a^2+d_a+1}z^{-2d_a-2}L_{-1}a,z^{-1})\\
&\qquad + 2z^{-1}Y^M(e^{zL_1}L_0(-1)^{2d_a^2+d_a}z^{-2d_a}a,z^{-1})\\
&\qquad - Y^M(L_1e^{zL_1}(-1)^{2d_a^2+d_a}z^{-2d_a}a,z^{-1})\\
&= Y^M(e^{zL_1}(-1)^{2L_0^2+L_0}z^{-2L_0}L_{-1}a,z^{-1})\\
&\qquad + 2z^{-1}Y^M(e^{zL_1}(-1)^{2L_0^2+L_0}L_0z^{-2L_0}a,z^{-1})\\
&\qquad - Y^M(L_1e^{zL_1}(-1)^{2L_0^2+L_0}z^{-2L_0}a,z^{-1})
\end{aligned}\tag{2.54}$$

where we have used [FHL93, Eq. (5.2.14)] for the third equality and the fact that $L_{-1}a$ is in V_{d_a-1} for the last one. Thus, putting (2.53) and (2.54) in (2.52), we get the translation covariance property for $Y^{M'}$ through a straightforward calculation.

Now, we prove the Borcherds identity (2.46) for M'. We use the formal calculus for the δ-function as presented in Section 2.1; just note that the δ-function used in the proof of [FHL93, Theorem 5.2.1] is defined differently from our one, see [FHL93, Section 2.1]. We note that the Borcherds identity for every V-module N is equivalent to

$$\begin{aligned}
\delta(z_2,z_1-z_0)Y^N(Y(a,z_0)b,z_2)c' &= \delta(z_0,z_1-z_2)Y^N(a,z_1)Y^N(b,z_2)c'\\
&\quad + (-1)^{p(a)p(b)}\delta(-z_0,z_2-z_1)Y^N(b,z_2)Y^N(a,z_1)c'
\end{aligned}\tag{2.55}$$

for all homogeneous $a,b\in V$ and all $c'\in N$. M is a V-module and then we already know that the Borcherds identity (2.55) holds for the choice $N=M$. Replacing $z_1\to z_1^{-1}$, $z_2\to z_2^{-1}$, $z_0\to\frac{-z_0}{z_1z_2}$, $a\to e^{z_1L_1}(-1)^{2L_0^2+L_0}z_1^{-2L_0}a$ and $b\to e^{z_2L_1}(-1)^{2L_0^2+L_0}z_2^{-2L_0}b$ in (2.55), we obtain

$$\begin{aligned}
&\langle c',\delta(z_2^{-1},z_1^{-1}+\frac{z_0}{z_1z_2})Y^M(Y(e^{z_1L_1}(-1)^{2L_0^2+L_0}z_1^{-2L_0}a,\frac{-z_0}{z_1z_2})e^{z_2L_1}(-1)^{2L_0^2+L_0}z_2^{-2L_0}b,z_2^{-1})d\rangle\\
&= (-1)^{p(a)p(b)}\langle c',\delta(\frac{z_0}{z_1z_2},z_2^{-1}-z_1^{-1})Y^M(e^{z_2L_1}(-1)^{2L_0^2+L_0}z_2^{-2L_0}b,z_2^{-1})\cdot\\
&\qquad\qquad\cdot Y^M(e^{z_1L_1}(-1)^{2L_0^2+L_0}z_1^{-2L_0}a,z_1^{-1})d\rangle\\
&\quad +\langle c',\delta(\frac{-z_0}{z_1z_2},z_1^{-1}-z_2^{-1})Y^M(e^{z_1L_1}(-1)^{2L_0^2+L_0}z_1^{-2L_0}a,z_1^{-1})\cdot\\
&\qquad\qquad\cdot Y^M(e^{z_2L_1}(-1)^{2L_0^2+L_0}z_2^{-2L_0}b,z_2^{-1})d\rangle
\end{aligned}\tag{2.56}$$

for all homogeneous $a,b\in V$, all $d\in M$ and all $c'\in M'$. Through some standard calculations, we rewrite (2.56) as

$$\begin{aligned}
&\langle c',\delta(z_1,z_2+z_0)Y^M(Y(e^{z_1L_1}(-1)^{2L_0^2+L_0}z_1^{-2L_0}a,\frac{-z_0}{z_1z_2})e^{z_2L_1}(-1)^{2L_0^2+L_0}z_2^{-2L_0}b,z_2^{-1})d\rangle\\
&= (-1)^{p(a)p(b)}\langle c',\delta(z_0,z_1-z_2)Y^M(e^{z_2L_1}(-1)^{2L_0^2+L_0}z_2^{-2L_0}b,z_2^{-1})\cdot\\
&\qquad\qquad\cdot Y^M(e^{z_1L_1}(-1)^{2L_0^2+L_0}z_1^{-2L_0}a,z_1^{-1})d\rangle\\
&\quad +\langle c',\delta(-z_0,z_2-z_1)Y^M(e^{z_1L_1}(-1)^{2L_0^2+L_0}z_1^{-2L_0}a,z_1^{-1})\cdot\\
&\qquad\qquad\cdot Y^M(e^{z_2L_1}(-1)^{2L_0^2+L_0}z_2^{-2L_0}b,z_2^{-1})d\rangle
\end{aligned}\tag{2.57}$$

for all homogeneous $a,b\in V$, all $d\in M$ and all $c'\in M'$. Therefore, our aim is to prove the formal equality:

$$\begin{aligned}
&(-1)^{p(a)p(b)}\delta(z_1,z_2+z_0)Y^M(e^{z_2L_1}(-1)^{2L_0^2+L_0}z_2^{-2L_0}Y(a,z_0)b,z_2^{-1})\\
&= \delta(z_1,z_2+z_0)Y^M(Y(e^{z_1L_1}(-1)^{2L_0^2+L_0}z_1^{-2L_0}a,\frac{-z_0}{z_1z_2})e^{z_2L_1}(-1)^{2L_0^2+L_0}z_2^{-2L_0}b,z_2^{-1})\,.
\end{aligned}\tag{2.58}$$

By (2.5), taking the residue in z_1 of both sides of (2.58), we equivalently get that

$$\begin{aligned}&(-1)^{p(a)p(b)}Y^M(e^{z_2L_1}(-1)^{2L_0^2+L_0}z_2^{-2L_0}Y(a,z_0)b,z_2^{-1})\\&=Y^M(Y(e^{(z_2+z_0)L_1}(-1)^{2L_0^2+L_0}(z_2+z_0)^{-2L_0}a,\frac{-z_0}{(z_2+z_0)z_2})e^{z_2L_1}(-1)^{2L_0^2+L_0}z_2^{-2L_0}b,z_2^{-1})\end{aligned}$$

for all homogeneous $a,b\in V$. It is sufficient to prove that

$$\begin{aligned}&e^{z_2L_1}(-1)^{2L_0^2+L_0}z_2^{-2L_0}Y(a,z_0)b\\&=(-1)^{p(a)p(b)}Y(e^{(z_2+z_0)L_1}(-1)^{2L_0^2+L_0}(z_2+z_0)^{-2L_0}a,\frac{-z_0}{(z_2+z_0)z_2})e^{z_2L_1}(-1)^{2L_0^2+L_0}z_2^{-2L_0}b\end{aligned} \tag{2.59}$$

for all homogeneous $a,b\in V$. Indeed, we have the following equalities for the left hand side of (2.59):

$$\begin{aligned}&e^{z_2L_1}(-1)^{2L_0^2+L_0}z_2^{-2L_0}Y(a,z_0)b\\&=e^{z_2L_1}(iz_2^{-1})^{2L_0}ZY(a,z_0)b\\&=e^{z_2L_1}(iz_2^{-1})^{2L_0}(-1)^{p(a)p(b)}Y(Za,z_0)Zb\\&=e^{z_2L_1}(-1)^{p(a)p(b)}Y((iz_2^{-1})^{2L_0}Za,-z_2^{-2}z_0)(iz_2^{-1})^{2L_0}Zb\\&=(-1)^{p(a)p(b)}e^{z_2L_1}Y((-1)^{2L_0^2+L_0}z^{-2L_0}a,-z_2^{-2}z_0))(-1)^{2L_0^2+L_0}z_2^{-2L_0}b\\&=(-1)^{p(a)p(b)}Y(e^{(z_2+z_0)L_1}(-1)^{2L_0^2+L_0}(z_2+z_0)^{-2L_0}a,\frac{-z_0}{z_2(z_2+z_0)})e^{z_2L_1}(-1)^{2L_0^2+L_0}z^{-2L_0}b\end{aligned}$$

for all homogeneous $a,b\in V$, where we have used [Kac01, Eq.s (4.9.16), (4.9.17)] for the third and last equalities respectively. This prove (2.58). Putting (2.58) into (2.57) and using the definition of invariance (2.50), we obtain the Borcherds identity for M' as stated in (2.55), concluding the proof. □

We end this section defining what a unitary structure on a V-module is and presenting an important example.

Definition 2.2.4. Let V be a unitary VOSA and M be a V-module. M is called **unitary** (or **unitarizable**) if there exists a scalar product $(\cdot|\cdot)_M$ on M with the invariant property:

$$(Y^M(\theta(a),z)b|c)_M=(b|Y^M(e^{zL_1}(-1)^{2L_0^2+L_0}z^{-2L_0}a,z^{-1})c)_M\quad\forall a\in V\ \ \forall b,c\in M\,. \tag{2.60}$$

Given a module M on a unitary VOSA V, we define the **conjugate module** $\overline{M}$ of M as the complex vector superspace constituted by M itself as a set and the scalar multiplication by $\lambda\in\mathbb{C}$ realised via the usual scalar multiplication on M by its complex conjugate $\overline{\lambda}$. Vertex operators on $\overline{M}$ are defined by $Y^{\overline{M}}(a,z):=Y^M(\theta(a),z)$ for all $a\in V$, where θ is the PCT operator of V. It is not difficult to prove that $\overline{M}$ so defined is a V-module.

2.3 Invariant bilinear forms

We give the definition of invariant bilinear form for a VOSA and we prove the fundamental Proposition 2.3.3, using the notion of contragredient module previously introduced in Section 2.2. We use this proposition to characterise the simplicity of a unitary VOSA and to state further results in the forthcoming sections.

Definition 2.3.1. Let V be a VOSA. An **invariant bilinear form** on V is a bilinear form $(\cdot\,,\cdot)$ on V such that

$$(Y(a,z)b,c)=(b,Y(e^{zL_1}(-1)^{2L_0^2+L_0}z^{-2L_0}a,z^{-1})c)\qquad\forall a,b,c\in V\,. \tag{2.61}$$

Furthermore, we say that $(\cdot\,,\cdot)$ is **normalized** if $(\Omega,\Omega)=1$.

Remark 2.3.2. Note that if V is a unitary VOSA with scalar product $(\cdot|\cdot)$ and PCT operator θ, then $(\cdot\,,\cdot):=(\theta(\cdot)|\cdot)$ is a non-degenerate normalized invariant bilinear form on V.

From the definition, we have that for all homogeneous $a \in V$,

$$(a_n b, c) = (-1)^{2d_a^2+d_a} \sum_{l \in \mathbb{Z}_{\geq 0}} \frac{(b, (L_1^l a)_{-n} c)}{l!} \qquad \forall n \in \frac{1}{2}\mathbb{Z} \ \forall b, c \in V\,. \tag{2.62}$$

In particular,

$$(L_n a, b) = (a, L_{-n} b) \qquad \forall n \in \mathbb{Z} \ \forall a, b \in V\,, \tag{2.63}$$

which implies that $(V_l, V_k) = 0$ whenever $l \neq k$. Moreover, using the straightforward commutation identities (verified on homogeneous elements first and extended then by linearity, cf. [FHL93, Eq. (5.3.1)])

$$\begin{aligned} z^{-2L_0} e^{z^{-1}L_1} &= e^{zL_1} z^{-2L_0} \\ (-1)^{2L_0^2+L_0} e^{-zL_1} &= e^{zL_1} (-1)^{2L_0+L_0} \end{aligned} \tag{2.64}$$

we prove the **inverse invariance property** for an invariant bilinear form $(\cdot\,,\cdot)$ on V:

$$(c, Y(a,z)b) = (Y(e^{zL_1}(-1)^{2L_0^2+L_0} z^{-2L_0} a, z^{-1})c, b) \qquad \forall a, b, c \in V\,. \tag{2.65}$$

Now, we use the contragredient module of the adjoint module, which we denote by (V', Y') instead of using $(V', Y^{V'})$, to prove Proposition 2.3.3 below. Accordingly, we will also use the notation $a'_{(n)}$ to denote an endomorphism $a^{V'}_{(n)}$ coming from a generic element a in V.

Proposition 2.3.3. *Let V be a VOSA. Then:*

(i) Every invariant bilinear form on V is symmetric, that is,

$$(a, b) = (b, a) \qquad \forall a, b \in V\,.$$

(ii) The map $(\cdot\,,\cdot) \mapsto (\Omega, \cdot)|_{V_0}$ realises a linear isomorphism from the space of invariant bilinear forms on V to $\mathrm{Hom}_{\mathbb{C}}(V_0/L_1V_1, \mathbb{C}) = (V_0/L_1V_1)^*$.

(iii) If V is simple, then every non-zero invariant bilinear form on V is non-degenerate. Furthermore, if $(\cdot\,,\cdot)$ is a non-zero invariant bilinear form on V, then every other invariant bilinear form on V is given by $\alpha(\cdot\,,\cdot)$ where $\alpha \in \mathbb{C}$.

(iv) If V has a non-degenerate invariant bilinear form and $V_0 = \mathbb{C}\Omega$, then V is simple.

Proof. The statements (iii) and (iv) are proved as [CKLW18, Proposition (iii) and (iv)] respectively. (i) and (ii) are [Yam14, Proposition 1], but the author does not give detailed proofs. We unsuccessfully tried to find such proofs in the literature and thus, for the reader's benefit, we present one here below.

We can adapt [FHL93, Proposition 5.3.6] to prove (i). Indeed, thanks to (2.63), we can restrict to consider only $a, b \in V_d$ for some $d \in \frac{1}{2}\mathbb{Z}$. Then, we have

$$\begin{aligned} (a,b) &= (e^{zL_{-1}}a, b) = (Y(a,z)\Omega, b) \\ &= (\Omega, Y(e^{zL_1}(-1)^{2L_0^2+L_0} z^{-2L_0} a, z^{-1})b) \\ &= (-1)^{p(a)^2}(-1)^{2d^2+d} z^{-2d} (\Omega, e^{z^{-1}L_{-1}} Y(b, -z^{-1}) e^{zL_1} a) \\ &= (-1)^{p(a)^2}(-1)^{2d^2+d} z^{-2d} (e^{zL_{-1}} Y(e^{-z^{-1}L_1}(-1)^{2L_0^2+L_0}(-1)^{-2L_0} z^{2L_0} b, -z) e^{z^{-1}L_1}\Omega, a) \\ &= (-1)^{p(a)^2}(-1)^{2d}(e^{zL_{-1}} Y(e^{-z^{-1}L_1} b, -z)\Omega, a) \\ &= (-1)^{p(a)^2}(-1)^{2d}(e^{-zL_{-1}} e^{zL_{-1}} Y(\Omega, z) e^{-z^{-1}L_1} b, a) \\ &= (-1)^{p(a)^2}(-1)^{2d}(e^{-z^{-1}L_1} b, a) \\ &= (b, a) \end{aligned}$$

where we have used: (2.63) for the first equality; [Kac01, Eq. (4.1.2)] for the second one; the invariance of the bilinear form (2.61) for the third one; skew-symmetry (2.23) for the fourth and for the seventh one; the inverse invariant formula (2.65) together with (2.63) for the fifth one; the properties of the vacuum vector and some standard calculations for the sixth one; again (2.63) and that $(-1)^{p(a)^2}(-1)^{2d} = 1$ for the last one.

The proof of (ii) is an adaptation of [Li94, Theorem 3.1] (cf. also [Roi04, Theorem 1]) and therefore it requires some extra work. Note that thanks to the invariance property (2.62) and the orthogonality (2.63), an invariant bilinear form $(\cdot\,,\cdot)$ on V is completely determined by the values of the linear functional $f_{(\Omega,\cdot)}$ on V_0 defined by

$$f_{(\Omega,\cdot)}(a) := (\Omega, a) \qquad \forall a \in V_0\,. \tag{2.66}$$

Moreover, by (2.63), we have that

$$f_{(\Omega,\cdot)}(L_1V_1) = (\Omega, L_1V_1) = (L_{-1}\Omega, V_1) = 0$$

and therefore $f_{(\Omega,\cdot)}$ induces a linear functional on V_0/L_1V_1, which we denote by the same symbol. Conversely, we want to define an invariant bilinear form starting from a linear functional $f \in (V_0/L_1V_1)^*$, which can be considered as an element of the contragredient module V'. We define a bilinear form $(\cdot\,,\cdot)_f$ through the formula

$$(a, b)_f := \langle a'_{(-1)}f, b\rangle \qquad \forall a, b \in V \tag{2.67}$$

where $\langle\cdot,\cdot\rangle$ is the natural pairing between V' and V. To prove invariance of $(\cdot\,,\cdot)_f$, we first need to prove that f is a vacuum-like vector. By Lemma 2.2.2, f is a vacuum-like vector if and only if

$$0 = \langle L'_{-1}f, b\rangle = \langle f, L_1b\rangle = f(L_1b) \qquad \forall b \in V\,.$$

By definition, the pairing $\langle\cdot,\cdot\rangle$ satisfies $\langle V_k^*, V_h\rangle = 0$ whenever $h \neq k$. Because f is an element of $(V_0/L_1V_1)^*$, we must have that f is vacuum-like if and only if $f(L_1b) = 0$ for all $b \in V_1$ and $0 = \nu'_{(1)}f = L'_0f$, which is actually the case. We can now prove the invariance (2.61) for $(\cdot\,,\cdot)_f$:

$$\begin{aligned}
(a_{(n)}b, c)_f &= \langle (a_{(n)}b)'_{(-1)}f, c\rangle \\
&= \langle \sum_{j\in\mathbb{Z}_{\geq 0}} \binom{n}{j} a'_{(n-j)}b'_{(j-1)}f - (-1)^{p(a)p(b)} \sum_{j\in\mathbb{Z}_{\geq 0}} (-1)^{j+n}\binom{n}{j} b'_{(n-j-1)}a'_{(j)}f, c\rangle \\
&= \langle a'_{(n)}b'_{(-1)}f, c\rangle \\
&= (-1)^{d_a^2+d_a} \sum_{l\in\mathbb{Z}_{\geq 0}} \frac{\langle b'_{(-1)}f, (L_1^l a)_{(-n)}c\rangle}{l!} \\
&= (-1)^{d_a^2+d_a} \sum_{l\in\mathbb{Z}_{\geq 0}} \frac{(b, (L_1^l a)_{(-n)}c)_f}{l!}
\end{aligned}$$

for all homogeneous $a \in V$, all $b, c \in V$ and all $n \in \mathbb{Z}$, where we have used Borcherds identity (2.46) with $m = 0$ and $k = -1$ for the second equality thanks to Proposition 2.2.3; the definition of vacuum-like vector for the third one; the invariance of the contragredient module (2.50) for the fourth one. Therefore, $(\cdot\,,\cdot)_f$ is invariant by (2.62). Finally, it is easy to verify that the two maps raised from (2.66) and (2.67) are one the inverse of the other one, concluding the proof. □

Remark 2.3.4. Let V be a VOSA. Suppose that $V_0 = \mathbb{C}\Omega$, then V has a non-zero invariant bilinear form if and only if $L_1V_1 = \{0\}$ as a consequences of (ii) of Proposition 2.3.3. Moreover, in the latter case, there is exactly one non-zero invariant bilinear form which is also normalized. Suppose instead that V is simple, then there is at most one such a normalized invariant bilinear form by (iii) of Proposition 2.3.3.

Remark 2.3.5. Checking the proofs given above and as it is remarked in [Yam14, Remark 1], the factor $(-1)^{2L_0^2+L_0}$ which we chose for the definitions of invariance (2.61) and (2.50) is necessary for V' to indeed be a V-module. Nevertheless, we can find in the literature different definitions of invariance, as for example in [Sch98, Section 2.4] (there the author deals with real vertex superalgebra, but the treatment can be extended to the complex case, see [Sch98, p. 377]). One can prove that there is a one-to-one correspondence between these two different definitions. Indeed, let $(\cdot\,,\cdot)$ be an invariant bilinear form as in (2.61) and let $(\cdot\,,\cdot)_S$ be an invariant one as in [Sch98, Eq. (2.20)] with $\lambda = -i$. Thanks to (2.31), it is not difficult to prove that $(Z\cdot,\cdot)_S$ is an invariant bilinear form in the meaning of (2.61), whereas that $(Z^*\cdot,\cdot)$ is an invariant bilinear form in the sense of [Sch98, Eq. (2.20)].

A first view on the link between simplicity and CFT type condition for VOSAs is given by Proposition 2.3.3 and Remark 2.3.4 in presence of an invariant bilinear form. More generally, simplicity turns out to be an important feature to develop interesting results in what follows. In this light, it is useful to have the following characterisation of simplicity:

Proposition 2.3.6. *Let V be a unitary VOSA. Then, the following are equivalent:*

(i) V is simple;

(ii) $V_0 = \mathbb{C}\Omega$;

(iii) V is of CFT type.

Proof. (i) implies (ii) is a straightforward adaptation of the proof of [CKLW18, Proposition 5.3]. Note that $(\theta(\cdot)|\cdot)$ realises a non-degenerate invariant bilinear form of V and thus the claim (ii) implies (i) is (iv) of Proposition 2.3.3. (iii) implies (ii) is trivial by definition. So the only remaining implication is (ii) implies (iii). Let $n \in \frac{1}{2}\mathbb{Z}$ be such that $V_n \neq \{0\}$ and $V_N = \{0\}$ for all $N < n$. Pick $a \in V_n$, then using the Virasoro commutation relations (2.24), we have that

$$2n(a|a) = (a|2L_0a) = (a|[L_1, L_{-1}]a) = (a|L_1L_{-1}a) = (L_{-1}a|L_{-1}a) \geq 0$$

which implies that $n \geq 0$ because $(a|a) \geq 0$ too. Therefore, V is of CFT type. □

2.4 Uniqueness results and the automorphism group

In the present section, we investigate conditions which ensure the uniqueness of the unitary structure introduced in Section 2.1. For this reason, we deal with invariant bilinear forms on VOSAs, see Section 2.3, recovering the possible scalar products from them. We further describe the automorphism group of (unitary) VOSAs.

Thanks to Proposition 2.3.3, we can prove the following result:

Proposition 2.4.1. *Let V be a VOSA with a normalized scalar product $(\cdot|\cdot)$, which is invariant with respect to an antilinear automorphism θ. Then, we have that:*

(i) θ is an involution and thus V is unitary. Moreover, θ is antiunitary, that is, $(\theta(a)|\theta(b)) = (b|a)$ for all $a, b \in V$.

(ii) With V unitary, θ is the unique PCT operator associated with $(\cdot|\cdot)$. Moreover, if V is simple, then $(\cdot|\cdot)$ is the unique normalized invariant scalar product associated with θ.

Proof. We can prove (i) and the first part of (ii) as in [CKLW18, Proposition 5.1]. Regarding the second part of (ii), $(\theta(\cdot)|\cdot)$ defines a non-degenerate normalized invariant bilinear form on V. According to (iii) of Proposition 2.3.3, if V is simple, then $(\theta(\cdot)|\cdot)$ must be unique, which implies the remaining part of the proposition. □

Claim (ii) of Proposition 2.4.1 says us that for simple unitary VOSAs, a PCT operator determines the associated normalized invariant scalar product and vice versa. Nevertheless, we can still have different choices for the PCT operator or equivalently for the normalized invariant scalar product, which can give us different unitary structures. From this point of view, it is interesting to introduce the following result, which can be proved straightforwardly adapting [CKLW18, Proposition 5.19] with its immediate consequences [CKLW18, p. 38].

Proposition 2.4.2. *Let V be a simple VOSA. Let $(\theta, (\cdot|\cdot))$ and $(\theta', (\cdot|\cdot)')$ be two unitary structures on V. Then, there exists $h \in \mathrm{Aut}(V)$ such that:*

(i) $(a|b)' = (ha|hb)$ for all $a, b \in V$, that is, h realises a unitary isomorphism between $(V, \theta, (\cdot|\cdot))$ and $(V, \theta', (\cdot|\cdot)')$. As a consequence, if $\mathrm{Aut}(V) = \mathrm{Aut}_{(\cdot|\cdot)}(V)$, then the unitary structure on V is unique.

(ii) $\theta' = h^{-1}\theta h$; $\theta h \theta = h^{-1}$; $(a|ha) > 0$ for all non-zero $a \in V$.

To investigate the structure of the automorphism group of a unitary VOSA, it is useful to have the following uniqueness result for its conformal structure.

Proposition 2.4.3. *Let V be a vertex superalgebra with a conformal vector ν and a non-degenerate invariant bilinear form. Suppose that there exists another conformal vector ν' with a corresponding non-degenerate invariant bilinear form. If $L_0 = L'_0 =: \nu'_{(1)}$, then $\nu = \nu'$.*

Proof. The proof is obtained by applying [CKLW18, Proposition 4.8] to the vertex subalgebra $V_{\bar{0}}$. □

The following corollary is proved adapting the proof of [CKLW18, Corollary 4.11].

Corollary 2.4.4. *Let V be a VOSAs with a non-degenerate invariant bilinear form $(\cdot\,,\cdot)$ and such that $V_0 = \mathbb{C}\Omega$. Let g be either a linear or an antilinear map which preserves the vacuum vector and the (n)-product. Then, the following are equivalent:*

(i) g is grading-preserving, that is, $g(V_n) = V_n$ for all $n \in \frac{1}{2}\mathbb{Z}$.

(ii) g preserves $(\cdot\,,\cdot)$, that is: if g is linear, then $(g(a), g(b)) = (a, b)$ for all $a, b \in V$; if g is antilinear, then $(g(a), g(b)) = \overline{(a, b)}$ for all $a, b \in V$.

(iii) $g(\nu) = \nu$.

Therefore, we have all the ingredients to prove that:

Theorem 2.4.5. *Let V be a unitary VOSA. Then, $\mathrm{Aut}_{(\cdot|\cdot)}(V)$ is a compact subgroup of $\mathrm{Aut}(V)$. Moreover, if V is simple, then the followings are equivalent:*

(i) $(\cdot|\cdot)$ is the unique normalized invariant scalar product on V;

(ii) $\mathrm{Aut}_{(\cdot|\cdot)}(V) = \mathrm{Aut}(V)$;

(iii) θ commutes with every $g \in \mathrm{Aut}(V)$;

(iv) $\mathrm{Aut}(V)$ is compact;

(v) $\mathrm{Aut}_{(\cdot|\cdot)}(V)$ is totally disconnected.

Proof. The proof can be adapted from the one of [CKLW18, Lemma 5.20 and Theorem 3.21]. □

We conclude presenting a characterisation of the automorphism group of the real form of a VOSA, which we have defined in Remark 2.1.2:

Remark 2.4.6. Consider a simple unitary VOSA V. Following the proof of [CKLW18, Theorem 5.21], we have that $g \in \mathrm{Aut}(V)$ is unitary if and only if $(g\theta a|gb) = (\theta a|b)$ for all $a, b \in V$. On the other hand, $(\theta ga|gb) = (\theta a|b)$ for all $a, b \in V$ thanks to (ii)⇔(iii) of Corollary 2.4.4. It follows that g is unitary if and only if it commutes with θ. Hence, g is unitary if and only if it restricts to an automorphism of the real VOSA $V_{\mathbb{R}}$ defined in Remark 2.1.2. Conversely, every VOSA automorphism of $V_{\mathbb{R}}$ extends to an automorphism of its complexification V. Thus, we can identify $\mathrm{Aut}_{(\cdot|\cdot)}(V)$ and $\mathrm{Aut}(V_{\mathbb{R}})$.

2.5 The PCT theorem in the VOSA setting

The main topic of this section, inspired by [CKLW18, Section 5.2], is to characterise the unitarity of a VOSA equipped with a promising scalar product. The convenience of this approach is twofold. On the one hand, it is natural from the point of view of QFT. Indeed, we obtain also the VOSA version of the PCT theorem, see [SW64, Section 4.3]. On the other hand, to construct examples of unitary VOSAs, we often have to find a PCT operator for a given scalar product, which we start with. Therefore, it would be useful to have different methods to check the existence of a unitary structure, which avoid the direct search for a PCT operator.

As mentioned above, the Wightman axioms [SW64, Section 3.1] for QFT require for the unitarity at least that: (i) "the spacetime symmetries act unitarily"; (ii) "the adjoints of local fields are local". We translate these requirements introducing the following definitions respectively:

Definition 2.5.1. Let V be a VOSA with a normalized scalar product $(\cdot|\cdot)$.

(i) The pair $(V, (\cdot|\cdot))$ is said to have the **unitary Möbius symmetry** if

$$(L_n a|b) = (a|L_{-n} b) \qquad \forall a, b \in V \;\; \forall n \in \{-1, 0, 1\} . \tag{2.68}$$

(ii) An element $A \in \mathrm{End}(V)$ has an **adjoint** on V (with respect to $(\cdot|\cdot)$) if there exists A^+ in $\mathrm{End}(V)$ such that

$$(a|Ab) = (A^+ a|b) \qquad \forall a, b \in V . \tag{2.69}$$

It is clear that if an adjoint exists, then it will be unique and we will call it *the* adjoint of A on V. Of course, A can be treated as a densely defined operator on the Hilbert space completion $\mathcal{H}_{(V,(\cdot|\cdot))}$ of V and thus A^+ exists if and only if V is contained in the domain of the Hilbert space adjoint A^* of A, in which case $A^+ \subseteq A^*$. It is not difficult to verify that the set of elements having an adjoint on V is a unital subalgebra of $\mathrm{End}(V)$ closed with respect to the adjoint operation, that is, if $A, B \in \mathrm{End}(V)$, then

$$(\alpha A + \beta B)^+ = \overline{\alpha} A^+ + \overline{\beta} B^+ \;\; \forall \alpha, \beta \in \mathbb{C} , \quad (AB)^+ = B^+ A^+ , \quad A^{++} := (A^+)^+ = A . \tag{2.70}$$

We can easily adapt the proof of [CKLW18, Lemma 5.11] to obtain:

Lemma 2.5.2. *Let $(V, (\cdot|\cdot))$ have unitary Möbius symmetry. Then, for every homogeneous $a \in V$ and any $n \in \frac{1}{2}\mathbb{Z}$, a_n^+ exists on V. Moreover, for all $b \in V$, there exists a $N \in \frac{1}{2}\mathbb{Z}_{\geq 0}$ such that $a_{-n}^+ b = 0$ for all $n \geq N$.*

If $(V, (\cdot|\cdot))$ has the unitary Möbius symmetry, then Lemma 2.5.2 permits us to define for all $a \in V$, a parity-preserving field by

$$Y(a, z)^+ := \sum_{n\in\mathbb{Z}} a_{(n)}^+ z^{n+1} = \sum_{n\in\mathbb{Z}} a_{(-n-2)}^+ z^{-n-1} . \tag{2.71}$$

Consequently, We adapt (ii) of Definition 2.5.1 to the field setting, saying that for any given $a \in V$, the field $Y(a, z)$ has a **local adjoint** if for every $b \in V$, $Y(a, z)^+$ and $Y(b, z)$ are mutually local, that is, there exists $N \in \mathbb{Z}_{\geq 0}$ such that

$$(z - w)^N [Y(a, z)^+, Y(b, w)] = 0 .$$

Accordingly, we set $V^{(\cdot|\cdot)}$ as the subset of V consisting of all the elements $a \in V$ such that $Y(a, z)$ has a local adjoint. Then, retracing step by step the proofs of [CKLW18, Lemma 5.13 and Proposition 5.14], one proves that:

Proposition 2.5.3. *For every $a, b \in V$, $Y(a, z)^+$ and $Y(b, z)$ are mutually local if and only if $Y(a, z)$ and $Y(b, z)^+$ are mutually local. Moreover, $V^{(\cdot|\cdot)}$ is a vertex subalgebra of V.*

This allows us to prove the following technical result, just proceeding as in the proof of [CKLW18, Lemma 5.15].

Lemma 2.5.4. *Let $a \in V^{(\cdot|\cdot)}$ be a quasi-primary vector. Then, there exists a quasi-primary vector $\overline{a} \in V^{(\cdot|\cdot)}$ such that $d_{\overline{a}} = d_a$ and $z^{-2d_a} Y(a, z)^+ = Y(\overline{a}, z)$, that is, $a_n^+ = \overline{a}_{-n}$ for all $n \in \frac{1}{2}\mathbb{Z}$.*

Therefore, we prove the announced VOSA version of the PCT theorem:

Theorem 2.5.5. *Let V be a VOSA equipped with a normalized scalar product $(\cdot|\cdot)$ and such that $V_0 = \mathbb{C}\Omega$. Then, the following are equivalent:*

(i) $(V, (\cdot|\cdot))$ is a unitary VOSA.

(ii) $(V, (\cdot|\cdot))$ has unitary Möbius symmetry and every vertex operator has a local adjoint, that is, $V^{(\cdot|\cdot)} = V$.

Proof. We adapt the proof of [CKLW18, Theorem 5.16].

(i) $\Rightarrow$ (ii). Suppose that $(V, (\cdot|\cdot))$ is a unitary VOSA with PCT operator θ. By equation (2.44), we have that $(V, (\cdot|\cdot))$ has unitary Möbius symmetry. Moreover, by (2.41), for every homogeneous vector $a \in V$, we have that

$$a_{-n}^+ = (-1)^{2d_a^2 + d_a} \sum_{l\in\mathbb{Z}_{\geq 0}} \frac{(L_1^l \theta a)_n}{l!} \qquad \forall n \in \mathbb{Z} - d_a \tag{2.72}$$

and thus

$$Y(a,z)^+ = (-1)^{2d_a^2+d_a} \sum_{l\in\mathbb{Z}_{\geq 0}} \frac{Y(L_1^l\theta a, z) z^{2d_a-l}}{l!} \tag{2.73}$$

is mutually local with all fields $Y(b,z)$ for all $b\in V$. Then, $V^{(\cdot|\cdot)} = V$ by the arbitrariness of $a\in V$.

(ii) ⇒ (i). The first step is to prove the simplicity of V. Suppose that there exists a non-zero ideal $\mathscr{J}\subseteq V$. $L_0\mathscr{J}\subseteq\mathscr{J}$ implies that

$$\mathscr{J} = \bigoplus_{n\geq d} \mathscr{J}\cap V_n$$

for some $d\in\frac{1}{2}\mathbb{Z}$ such that $\mathscr{J}\cap V_d\neq\{0\}$. If $d=0$, then $\Omega\in\mathscr{J}$, which implies that $\mathscr{J}=V$, i.e., V is simple. Therefore, suppose $d\neq 0$ and let $a\in\mathscr{J}\cap V_d$ be non-zero, which says that a is quasi-primary. By Lemma 2.5.4, there exists $\overline{a}\in V_d$ such that $a_{-d}^+ = \overline{a}_d$. $\overline{a}_d a$ is non-zero because

$$(\Omega|\overline{a}_d a) = (\Omega|a_{-d}^+ a) = (a_{-d}\Omega|a) = (a|a)\neq 0\,.$$

Accordingly, $\overline{a}_d a = \overline{a}_d a_{-d}\Omega$ is a non-zero element of $\mathscr{J}\cap V_0$, which implies that $\Omega\in\mathscr{J}$, that is, the simplicity of V.

The remaining proof consists of constructing a PCT operator for $(\cdot|\cdot)$. Let $a\in V_1$. Since $V_0=\mathbb{C}\Omega$, then $L_{-1}L_1a=0$ and thus $(L_1a|L_1a)=0$ by unitary Möbius symmetry. This implies that $L_1a=0$ and $L_1V_1=\{0\}$ by the arbitrariness of $a\in V_1$. By (ii) and (iii) of Proposition 2.3.3, it follows that V has a unique normalized invariant bilinear form $(\cdot\,,\cdot)$, which is also non-degenerate by the simplicity of V. Now, $(V_n|V_m)=0$ and $(V_n,V_m)=0$ whenever $n\neq m$ thanks to unitary Möbius symmetry and (2.63) respectively. Then, there exists a unique antilinear $\mathbb{C}$-vector space map $\theta: V\mapsto V$, which is also grading-preserving and satisfies $(\cdot\,,\cdot) = (\theta(\cdot)|\cdot)$. θ is injective by the non-degeneracy of $(\cdot\,,\cdot)$. Moreover, by the fact that $\theta(V_n)\subseteq V_n$ for all $n\in\frac{1}{2}\mathbb{Z}$ and that every V_n is finite-dimensional, we have that θ is invertible, so that θ is a $\mathbb{C}$-vector space antilinear automorphism. To show that θ preserves the (n)-product, pick a quasi-primary vector $a\in V=V^{(\cdot|\cdot)}$. By Lemma 2.5.4, there exist a quasi-primary vector $\overline{a}\in V_{d_a}$ such that $a_n^+=\overline{a}_{-n}$ for all $n\in\frac{1}{2}\mathbb{Z}$. Furthermore,

$$\begin{aligned}(\theta(a_nb)|c) &= (a_nb,c) = (-1)^{2d_a^2+d_a}(b,a_{-n}c) = (-1)^{2d_a^2+d_a}(\theta(b)|a_{-n}c)\\ &= (-1)^{2d_a^2+d_a}(\overline{a}_n\theta(b)|c) \qquad \forall n\in\frac{1}{2}\mathbb{Z}\ \forall b,c\in V\,,\end{aligned}$$

which implies that $\theta a_n = (-1)^{2d_a^2+d_a}\overline{a}_n\theta$ for all $n\in\frac{1}{2}\mathbb{Z}$. Note that $\theta(\Omega)=\Omega$ because $V_0=\mathbb{C}\Omega$ and $(\Omega,\Omega)=(\Omega|\Omega)=1$. Then,

$$\theta(a) = \theta(a_{-d_a}\Omega) = (-1)^{2d_a^2+d_a}\overline{a}_{-d_a}\theta(\Omega) = (-1)^{2d_a^2+d_a}\overline{a}_{-d_a}\Omega = (-1)^{2d_a^2+d_a}\overline{a}$$

and thus, just collecting together the previous two calculations, $\theta a_{(n)} = \theta(a)_{(n)}\theta$ for all quasi-primary $a\in V$ and all $n\in\mathbb{Z}$. Note that, by unitary Möbius symmetry, θ commutes with L_n for all $n\in\{-1,0,1\}$ and vectors of the form $L_{-1}^k a$ with $k\in\mathbb{Z}_{\geq 0}$ and a quasi-primary span V. Hence, we can conclude that $\theta b_{(n)} = \theta(b)_{(n)}\theta$ for all $n\in\mathbb{Z}$ and all $b\in V$, just recalling that $[L_{-1},Y(b,z)] = Y(L_{-1}b,z)$ for all $b\in V$, see [Kac01, Corollary 4.4(c)]. To sum up, we have proved that θ is a $\mathbb{C}$-vector space antilinear automorphism, which preserves the grading and the (n)-product and realises the invariance property with respect to $(\cdot|\cdot)$ by construction. Then, we can conclude that θ is a PCT operator for $(\cdot|\cdot)$ thanks to Corollary 2.4.4 and Proposition 2.4.1. □

Thanks to the theorem above, we can prove, easily adapting the proof of [CKLW18, Proposition 5.17], the following:

Proposition 2.5.6. *Let V be a VOSA equipped with a normalized scalar product $(\cdot|\cdot)$ and such that $V_0=\mathbb{C}\Omega$. Then, the following are equivalent:*

(i) $(V,(\cdot|\cdot))$ is a unitary VOSA.

(ii) $Y(\nu,z)$ is a Hermitian field and V is generated by a family of Hermitian quasi-primary fields.

2.6 Unitary subalgebras

We introduce unitary subalgebras, cosets and graded tensor products of unitary VOSA, relying on some results introduced in Section 2.5.

Definition 2.6.1. A **unitary subalgebra** W of a unitary VOSA V is a vertex subalgebra of V such that:

(i) W is compatible with the grading, that is, $L_0 W \subseteq W$ or equivalently $W = \bigoplus_{n\in\frac{1}{2}\mathbb{Z}}(W\cap V_n)$;

(ii) $a^+_{(n)}b \in W$ for all $a, b \in W$ and all $n \in \mathbb{Z}$.

Note that $a^+_{(n)}$ exists for all $a \in W$ and all $n \in \mathbb{Z}$ by Lemma 2.5.2. Moreover, (ii) above is equivalent to say that $a^+_n b \in W$ for all $a, b \in W$ and all $n \in \frac{1}{2}\mathbb{Z}$, provided (i) above holds.

We characterise a unitary subalgebra with the following result.

Proposition 2.6.2. *Let W be a vertex subalgebra of a unitary VOSA V. Then, W is a unitary subalgebra of V if and only if $L_1 W \subseteq W$ and $\theta(W) \subseteq W$.*

Proof. The proof follows the one of [CKLW18, Proposition 5.23], just noting that we have to use (2.41) instead of [CKLW18, Eq. (84)] whenever required. □

Example 2.6.3. Let V be a unitary VOSA and $G \subseteq \mathrm{Aut}_{(\cdot|\cdot)}(V)$ be a closed subgroup. By Proposition 2.6.2, the vector superspace

$$V^G := \{a \in V \mid g(a) = a \ \ \forall g \in G\} \tag{2.74}$$

defines a unitary subalgebra, which is called a **fixed point subalgebra**. When G is finite, V^G is known as **orbifold subalgebra**. For example, the even subspace $V_{\bar{0}}$ is the orbifold subalgebra $V^{\{1_V,\Gamma_V\}}$.

Note that the projection operator e_W on a unitary subalgebra $W \subseteq V$ is a well-defined element of $\mathrm{End}(V)$ thanks to the grading compatibility of W and the finite dimensions of the eigenspaces V_n for all $n \in \frac{1}{2}\mathbb{Z}$. Then, proceeding as in the proofs of [CKLW18, Lemma 5.28 and Proposition 5.29], we can get the following structural result for unitary subalgebras.

Proposition 2.6.4. *Let V be a simple unitary VOSA with conformal vector ν, scalar product $(\cdot|\cdot)$ and PCT operator θ. Let W be a unitary subalgebra with the associated projector operator $e_W \in \mathrm{End}(V)$. Then, $[Y(a,z), e_W] = 0$ for all $a \in W$, $[L_n, e_W] = 0$ for all $n \in \{-1,0,1\}$, $[\theta, e_W] = 0$ and $e_W Y(a,z) e_W = Y(e_W a, z) e_W$ for all $a \in V$. If we define $\nu^W := e_W \nu \in W$, then:*

(i) $Y(\nu^W, z) = \sum_{n\in\mathbb{Z}} L^W_n z^{-n-2}$ is a Hermitian Virasoro field on V and such that $L^W_n = L_n|_W$ for all $n \in \{-1,0,1\}$;

(ii) ν^W is a conformal vector for the vertex superalgebra $(W, \Omega, T|_W, Y(\cdot,z)|_W)$, so that W has a VOSA structure;

(iii) $(W, \Omega, T|_W, Y(\cdot,z)|_W, \nu^W, (\cdot|\cdot), \theta|_W)$ is a simple unitary VOSA.

Some example of unitary VOSAs and unitary subalgebras are obtained by two general constructions which we introduce in the following.

For $j \in \{1,2\}$, let $(V^j, \Omega^j, Y_j, \nu^j)$ be two VOSAs. The **graded tensor product** $V^1 \hat{\otimes} V^2$, see [Kac01, Section 4.3], is the VOSA given by

$$(V^1 \otimes V^2, \Omega^1 \otimes \Omega^2, Y_1 \hat{\otimes} Y_2, \nu^1 \otimes \Omega^2 + \Omega^1 \otimes \nu^2)$$

where

$$(Y_1 \hat{\otimes} Y_2)(a^1 \otimes a^2, z) := Y_1(a^1, z)\Gamma_{V^1}^{p(a^2)} \otimes Y_2(a^2, z) \qquad \forall a^1 \otimes a^2 \in V^1 \otimes V^2 \,. \tag{2.75}$$

Clearly, the parity operator is given by $\Gamma_{V^1} \otimes \Gamma_{V^2}$. Consider the operators

$$Z_{V^1\hat{\otimes}V^2} = \frac{1_{V^1} \otimes 1_{V^2} - i\Gamma_{V^1} \otimes \Gamma_{V^2}}{1-i}\,, \quad Z_{V^1} = \frac{1_{V^1} - i\Gamma_{V^1}}{1-i}\,, \quad Z_{V^2} = \frac{1_{V^2} - i\Gamma_{V^2}}{1-i}\,.$$

It is not difficult to show that

$$\begin{aligned} Z_{V^1\hat{\otimes}V^2}(a^1\otimes a^2) &= (-1)^{p(a^1)p(a^2)} Z_{V^1}(a^1)\otimes Z_{V^2}(a^2) \\ Z^*_{V^1\hat{\otimes}V^2}(a^1\otimes a^2) &= (-1)^{p(a^1)p(a^2)} Z^*_{V^1}(a^1)\otimes Z^*_{V^2}(a^2) \end{aligned} \tag{2.76}$$

for all vectors $a^j \in V^j$ with given parities and with $j\in\{1,2\}$. Finally, by [Ten19, Proposition 2.20], if for $j\in\{1,2\}$, $(V^j,(\cdot|\cdot)_j,\theta_j)$ are unitary, then $(V^1\hat{\otimes}V^2,(\cdot|\cdot)_1(\cdot|\cdot)_2,\theta_1\otimes\theta_2)$ is unitary as well.

Another important tool is given by the **coset subalgebra** of a vertex subalgebra W of a vertex superalgebra V (it is called *centralizer* in [Kac01, Remark 4.6b]). It is the vertex subalgebra of V given by the vector subspace

$$W^c := \{a\in V \mid [Y(a,z),Y(b,w)]=0 \ \forall b\in W\} = \left\{a\in V \mid b_{(j)}a = 0 \ {}^{\forall b\in W}_{\forall j\in\mathbb{Z}_{\geq 0}}\right\} \tag{2.77}$$

where the last equality is obtained by the Borcherds commutator formula (2.21). Then, we have the following result:

Proposition 2.6.5. *Let V be a unitary VOSA with conformal vector ν and let W be a unitary subalgebra. Then, W^c is a unitary subalgebra of V. If V is simple, then we also have that:*

(i) $\nu=\nu^W+\nu^{W^c}$ and the operators $L_0^W=\nu^W_{(1)}$ and $L_0^{W^c}=\nu^{W^c}_{(1)}$ are simultaneously diagonalizable on V with non-negative eigenvalues.

(ii) Set $U:=\operatorname{span}\left\{a_{(-1)}b \mid a\in W\,,\ b\in W^c\right\}$. Then, U is a unitary subalgebra of V, unitarily isomorphic to $W\hat{\otimes}W^c$ through the map $a\otimes b\mapsto a_{(-1)}b$.

Proof. A straightforward adaptation of [CKLW18, Example 5.27] proves that W^c is a unitary subalgebra of V. Now, suppose that V is simple.

To prove (i), we adapt the proof of [CKLW18, Proposition 5.31]. Set $\nu'=\nu-\nu^W$ and let $a\in W^c$. Proposition 2.6.4 says us that $\nu'_{(j)}a=0$ for $j\in\{0,1\}$. Therefore, using the Borcherds commutator formula (2.21), $[\nu'_{(m)},a_{(n)}]=0$ for all $m\in\{0,1,2\}$ and all $n\in\frac{1}{2}\mathbb{Z}$. $\nu^W\in V_2$ as a consequences of Proposition 2.6.4 and thus $[L_0,L_0^W]=0$ by (2.26). This implies that the operator $z_1^{2L_0^W}:=e^{2\log(z_1)L_0^W}$ is well-defined on V. Using the commutation relations just found above, we have that

$$z_1^{2L_0}Y(a,z)z_1^{-2L_0} = z_1^{2L_0^W}Y(a,z)z_1^{-2L_0^W}\,, \qquad z_1^{2L_0^W}Y(\nu',z)z_1^{-2L_0^W}=Y(\nu',z)\,, \tag{2.78}$$

which we can use to prove that

$$\begin{aligned} z_1^{2L_0^W}[Y(\nu',z),Y(a,w)]z_1^{-2L_0^W} &= [Y(\nu',z),z_1^{2L_0}Y(a,w)z_1^{-2L_0}] \\ &= w^{2d_a}[Y(\nu',z),Y(a,z_1^2w)] \end{aligned}$$

which is zero by locality. By the arbitrariness of $a\in W$, we have $\nu'\in W^c$ by (2.77). With the same argument, we show that $\nu-\nu^{W^c}\in W^{cc}$. Therefore, for every $b\in W^c$ we have that

$$[Y(\nu',z),Y(b,w)] = [Y(\nu,z),Y(b,w)] = [Y(\nu^{W^c},z),Y(b,w)] \tag{2.79}$$

where we have used that $[Y(\nu-\nu^{W^c},z),Y(b,w)]=0$. This implies that $\nu'-\nu^{W^c}$ is in $W^c\cap W^{cc}$. $Y(\nu'-\nu^{W^c},z)$ commutes with $Y(\nu',w)$ and $Y(\nu^W,w)$ and thus with their difference $Y(\nu,w)$. Accordingly, $Y(\nu'-\nu^{W^c})$ commutes with every field $Y(a,w)$ with $a\in V$ because V is a representation of the Virasoro algebra. But V is simple and thus $Y(\nu'-\nu^{W^c},z)\in\mathbb{C}1_V$ by the Schur's Lemma, cf. [LL04, Proposition 4.5.5]. So the only possibilities is that $\nu'=\nu^{W^c}$, so proving the first claim of (i).

For the second claim of (i), note that L_0^W and $L_0^{W^c}$ commute and they are equal to their respective adjoints on V. Furthermore, they commute with L_0, which is diagonalizable on V with finite-dimensional eigenspaces. Then, L_0^W and $L_0^{W^c}$ are simultaneously diagonalizable on V with real eigenvalues, which we want to prove to be actually non-negative. Then, let $a\in V$ be a non-zero vector such that $L_0^Wa=sa$ and $L_0^{W^c}=ta$ for some $s,t\in\mathbb{R}$. Suppose by contradiction that s is negative. As a consequence of the unitarity of V and of the fact that $Y(\nu^W,z)$ is a Virasoro field, it is not difficult to prove that $(L_1^W)^na\neq 0$ for every $n\in\mathbb{Z}_{>0}$. Furthermore, $L_0(L_1^W)^na=(s+t-n)(L_1^W)^na$ for all $n\in\mathbb{Z}_{>0}$, which contradicts the fact that L_0 has non-negative eigenvalues. Accordingly, $s\geq 0$ and proceeding similarly, $t\geq 0$, which concludes the proof of (i).

Finally, (ii) is just [Ten19, Proposition 2.21]. □

2.7 Unitarity of $\widehat{\mathbb{Z}_2}$-graded simple current extensions

We discuss here the unitarity of simple current extensions of unitary VOSAs by the cyclic group of order two $\mathbb{Z}_2$. As a consequence, we state sufficient and necessary conditions to have a unitary structure on odd extensions of unitary VOSAs.

Consider an automorphism g of a VOSA V and suppose that g is an involution, that is, $g^2 = 1_V$. Then, we have a decomposition of the vector space V given by the group $\mathbb{Z}_2 \cong \{1_V, g\}$:

$$V = V^+ \oplus V^- , \quad V^+ := \{a \in V \mid g(a) = a\} = V^{\mathbb{Z}_2} , \quad V^- := \{a \in V \mid g(a) = -a\} . \tag{2.80}$$

V is known as a $\widehat{\mathbb{Z}_2}$**-graded simple current extension** of V^+, where $\widehat{\mathbb{Z}_2}$ is the dual group of $\mathbb{Z}_2$.

Lemma 2.7.1. *Let V be a simple VOSA with g an involution of* $\mathrm{Aut}(V)$*. Then, V^+ is a simple VOSA and V^- is an irreducible V^+-module.*

Proof. $V^+ = V^{\mathbb{Z}_2}$ is a vertex subalgebra of V thanks to Example 2.6.3. Moreover, $g(\nu) = \nu$ and thus V^+ is actually a VOSA with conformal vector ν. It is also clear that V^- is a V^+-module with the vertex operators $Y(a,z)|_{V^-}$ for $a \in V^+$.

Now, note that the proof of [Li94, Lemma 6.1.1] still works in the super case. Hence, we can use it to prove the simplicity of V^+ and the irreducibility of V^-. As a matter of fact, fix any non-zero vector $b \in V$, then by the simplicity of V, [Li94, Lemma 6.1.1] implies that V is linearly generated by vectors of type $a_n b$ where $a \in V$ and $n \in \frac{1}{2}\mathbb{Z}$. If we choose $b \in V^+$, then $a_n b \in V^p$ whenever $a \in V^p$ with $p \in \{+,-\}$. As a consequence, V^+ must be linearly generated by elements of type $a_n b$ where $a \in V^+$ and $n \in \frac{1}{2}\mathbb{Z}$. This is equivalent to say that V^+ is simple. Similarly, choosing $b \in V^-$, we can prove the irreducibility of V^-, concluding the proof. □

The following result is in part inspired by [DL14, Theorem 3.3], where the authors construct the entire structure of unitary VOA supposing some extra hypothesis, such as rationality, C_2-cofinite (see references therein) and the existence of a map whose characteristics make it suitable for constructing a PCT operator. Instead, our focus is on finding sufficient and necessary conditions to have a unitary structure starting with a pre-existent VOSA one.

Theorem 2.7.2. *Let V be a simple VOSA with g an involution of* $\mathrm{Aut}(V)$*. Then, V has a unitary structure with respect to which g is unitary if and only if V^+ is a unitary VOSA and V^- is a unitary V^+-module. In both cases, V^+ is simple and V^- is irreducible.*

Proof. Recall that a unitary VOSA W is simple if and only if $W_0 = \mathbb{C}\Omega$ by Proposition 2.3.6. Moreover, V^+ is a simple VOSA and V^- is an irreducible V^+-module thanks to Lemma 2.7.1.

If V and g are unitary, then V^+ is a simple unitary VOSA by Example 2.6.3. Moreover, V^- is an irreducible unitary V^+-module restricting the unitary structure of V to it.

Conversely, let $(\cdot|\cdot)_+$ and θ_+ realise the unitary structure on V^+. Denote by $(\cdot|\cdot)_-$ the scalar product on the unitary V^+-module V^-. Therefore, $(\cdot\,,\cdot)_+ := (\theta_+(\cdot)|\cdot)_+$ is a non-degenerate invariant bilinear form on V^+ and thus there exists a non-zero invariant bilinear form $(\cdot\,,\cdot)$ on V which extends $(\cdot\,,\cdot)_+$ thanks to (ii) of Proposition 2.3.3. Furthermore, $(\cdot\,,\cdot)$ is also non-degenerate by the simplicity of V, see (iii) of Proposition 2.3.3. It follows that the restriction $(\cdot\,,\cdot)_-$ of $(\cdot\,,\cdot)$ to V^- is a non-degenerate invariant bilinear form on it.

Consider the conjugate module $\overline{V^-}$ of the V^+-module V^- as defined in Section 2.2. Then, we have an isomorphism of V^+-modules induced by the following pairings:

$$\begin{array}{ccccc} V^- & \xrightarrow{\;f_{(\cdot\,,\cdot)_-}\;} & (V^-)' & \xrightarrow{\;f^{-1}_{(\cdot|\cdot)_-}\;} & \overline{V^-} \\ a & \longmapsto & [f_{(\cdot\,,\cdot)_-}(a)](\cdot) := (a,\cdot)_- & & \\ & & [f_{(\cdot|\cdot)_-}(b)](\cdot) := (b|\cdot)_- & \longleftarrow\!\!\shortmid & b \end{array} \tag{2.81}$$

Note that the V^+-module isomorphism (2.81) can be considered as an antilinear vector space automorphism of V^-, call it θ_-, with the property that

$$\theta_-(a_n b) = \theta_+(a)_n \theta_-(b) \qquad \forall a \in V^+ \;\; \forall b \in V^- \;\; \forall n \in \frac{1}{2}\mathbb{Z} . \tag{2.82}$$

Furthermore, by construction, we have that

$$(\theta_-(a)|b)_- = (a,b) \qquad \forall a,b \in V^- . \tag{2.83}$$

θ_-^2 is an automorphism of the irreducible V^+-module V^- and thus it must be a multiple of the identity by Schur's Lemma, cf. [LL04, Proposition 4.5.5]. Write $\theta_-^2 = r1_{V^-}$ with $r \in \mathbb{C}\backslash\{0\}$. Then, we have that

$$\overline{r}(a|a)_- = (\theta_-^2(a)|a)_- = (\theta_-(a),a) = (a,\theta_-(a)) = (\theta_-(a)|\theta_-(a))_- > 0 \quad \forall a \in V^-\backslash\{0\}$$

where we have used the symmetry of the bilinear form as in (i) of Proposition 2.3.3. It follows that r must be a positive number. Accordingly, we can renormalise the scalar product $(\cdot|\cdot)_-$ in such a way that $r = 1$. Consequently the "new" θ_- so obtained through that renormalisation is an involution.

The unitary structure on V is defined in the following way: the scalar product is $(\cdot|\cdot) := (\cdot|\cdot)_+ \oplus (\cdot|\cdot)_-$ and the PCT operator θ is given by $\theta_+ \oplus \theta_-$. By construction, $(g(a)|g(b)) = (a|b)$ for all $a,b \in V^+ \cup V^-$, that is g is unitary with respect to $(\cdot|\cdot)$. Moreover, $(\theta(\cdot)|\cdot) = (\cdot\,,\cdot)$, which assures us that $(\cdot|\cdot)$ is normalized and the invariant property holds, provided that θ is a well-defined VOSA automorphism. About this last point, we already have that θ is an antilinear vector space involution of V, which preserves the vacuum and the conformal vectors. So it remains to prove that θ respects the (n)-product. By construction, we already know that $\theta(a_nb) = \theta(a)_n\theta(b)$ whenever $a \in V^+$, $n \in \frac{1}{2}\mathbb{Z}$ and $b \in V$. Suppose that $a \in V^-$ and $n \in \frac{1}{2}\mathbb{Z}$. Then, we want to prove that $\theta(a_nb) = \theta(a)_n\theta(b)$ for all $b \in V$. First, we have that $\theta(L_mb) = \theta(\nu)_m\theta(b) = L_m\theta(b)$ for all $b \in V$ and all $m \in \mathbb{Z}$, that is, θ commutes with L_m for all $m \in \mathbb{Z}$. It follows that if $b \in V^+$, then

$$\begin{aligned}\theta(Y(a,z)b) &= \theta((-1)^{p(a)p(b)}e^{zL_{-1}}Y(b,-z)a) \\ &= (-1)^{p(a)p(b)}e^{zL_{-1}}Y(\theta(b),-z)\theta(a) = Y(\theta(a),z)\theta(b)\end{aligned} \tag{2.84}$$

where we have used the skew-symmetry (2.23) twice, proving that $\theta(a_nb) = \theta(a)_n\theta(b)$ whenever $b \in V^+$. Hence, using the invariance property of the bilinear form, we compute that

$$\begin{aligned}(Y(a,z)u|v) &= (\theta(Y(\theta(a),z)\theta(u))|v) \\ &= (Y(\theta(a),z)\theta(u),v) \\ &= (\theta(u),Y(e^{zL_1}(-1)^{2L_0^2+L_0}z^{-2L_0}\theta(a),z^{-1})v) \\ &= (u|Y(e^{zL_1}(-1)^{2L_0^2+L_0}z^{-2L_0}\theta(a),z^{-1})v) \qquad \forall u \in V^+ \;\; \forall v \in V\end{aligned} \tag{2.85}$$

which is the invariance property of the scalar product in a specific case. Furthermore, using (2.64) with (2.85), we get the inverse invariant formula

$$(Y(\theta(a),z)v|u) = (v|Y(e^{zL_1}(-1)^{2L_0^2+L_0}z^{-2L_0}a,z^{-1})u) \qquad \forall u \in V^+ \;\; \forall v \in V\,. \tag{2.86}$$

Consequently, if $b \in V^-$, then

$$\begin{aligned}(\theta(Y(a,z)b)|c) &= (Y(a,z)b,c) \\ &= (b,Y(e^{zL_1}(-1)^{2L_0^2+L_0}z^{-2L_0}a,z^{-1})c) \\ &= (\theta(b)|Y(e^{zL_1}(-1)^{2L_0^2+L_0}z^{-2L_0}a,z^{-1})c) \\ &= (Y(\theta(a),z)\theta(b)|c) \quad \forall c \in V^+\end{aligned}$$

where we have used the invariance property of the bilinear form for the second equality and the inverse invariant property (2.86) for the last one. By the non-degeneracy of $(\cdot|\cdot)$ on V^+, it follows that $\theta(a_nb) = \theta(a)_n\theta(b)$ whenever $b \in V^-$, concluding the proof that θ respects the (n)-product. Then, the invariance property of the scalar product $(\cdot|\cdot)$ follows just proceeding as in (2.85). □

As a corollary, we have a characterisation of the unitarity of a simple VOSA in terms of its even and odd parts.

Corollary 2.7.3. *Let V be a simple VOSA. Then, V is unitary if and only $V_{\overline{0}}$ is a unitary VOA and $V_{\overline{1}}$ is a unitary $V_{\overline{0}}$-module.*

Proof. It is an immediate consequence of Theorem 2.7.2, choosing g there as the parity operator Γ_V. □

Chapter 3

The construction of the irreducible graded-local conformal net

In Section 3.1 and Section 3.2, we construct a unique, up to isomorphism, irreducible graded-local conformal net from a simple unitary VOSA, provided suitable as well as natural analytic assumptions are given. We also outline the structures of the associated graded-local conformal nets on $S^{1(2)}$ and $\mathbb{R}$ in Section 3.3, which naturally arise in the graded case.

Comparing with the local case... The $\frac{1}{2}\mathbb{Z}$ gradation of VOSAs forces us to choose a test function space different from the local case, see [CKLW18, Chapter 6], to define smeared vertex operators associated to odd elements. This choice is done to have test functions with rapidly decreasing Fourier coefficients, which allow a good definition of smeared vertex operators, see (3.5). Other equivalent choices for the test function space are given in Section 3.3. Moreover, differently from the VOA case, the conformal vector of a simple energy-bounded unitary VOSA gives rise to a positive-energy strongly continuous projective unitary representation of $\mathrm{Diff}^+(S^1)^{(\infty)}$ which factors through the double cover $\mathrm{Diff}^+(S^1)^{(2)}$ instead of $\mathrm{Diff}^+(S^1)$. This fact affects the calculation of the adjoint action of such a representation on smeared vertex operators, which is a core step in the proof of the Möbius covariance of the net associated to the VOSA.

3.1 Energy bounds and smeared vertex operators

In this first part, we construct certain operator-valued distributions associated to the vertex operators of a unitary VOSA, called smeared vertex operators, assuming the analytic property of energy boundedness for the unitary VOSA itself.

Definition 3.1.1. Let $(V, (\cdot|\cdot))$ be a unitary VOSA. A vector $a \in V$ or equivalently its corresponding vertex operator $Y(a, z)$, is **energy-bounded** if there exist $s, k \in \mathbb{Z}_{>0}$ and a positive constant M such that

$$\|a_n b\| \leq M(|n| + 1)^s \left\|(L_0 + 1_V)^k b\right\| \qquad \forall n \in \frac{1}{2}\mathbb{Z} \ \forall b \in V \tag{3.1}$$

where for every $v \in V, \|v\| := (v|v)^{\frac{1}{2}}$ is the norm induced by the scalar product $(\cdot|\cdot)$. Accordingly, we say that V is **energy-bounded** if every $a \in V$ is energy-bounded.

Then, a nice property:

Proposition 3.1.2. *Let V be a unitary VOSA generated by a family of homogeneous elements satisfying the energy bounds. Then, V is energy-bounded too.*

Proof. The proof is the same as in [CKLW18, Proposition 6.1], just using the Borcherds identity for the superalgebras case (2.20) in [CKLW18, Eq. (103)]. This change does not affect the proof at all. □

An immediate corollary:

Corollary 3.1.3. *If V^1 and V^2 are energy-bounded VOSAs, then $V^1\hat{\otimes}V^2$ is energy-bounded too.*

The following proposition gives sufficient conditions of energy-boundedness and it will be useful in the production of examples in Chapter 6.

Proposition 3.1.4. *Let V be a simple unitary VOSA. Suppose that V is generated by $V_{\frac{1}{2}}\cup V_1\cup\mathfrak{F}$ where $\mathfrak{F}\subseteq V_2$ is a family of quasi-primary θ-invariant Virasoro vectors. Then, V is energy-bounded.*

Proof. For $V_1\cup\mathfrak{F}$, the result follows from [CKLW18, Proposition 6.3]. Accordingly, suppose that $V_{\frac{1}{2}}\neq\{0\}$. Considering only homogeneous elements $a,b\in V_{\frac{1}{2}}$, the Borcherds commutator formula (2.21) becomes

$$[a_m,b_k]=\sum_{j=0}^{\infty}\binom{m-\frac{1}{2}}{j}\left(a_{(j)}b\right)_{(m+k-1-j)}\qquad\forall m,k\in\mathbb{Z}-\frac{1}{2}\,. \tag{3.2}$$

Note that $a_{(j)}b\in V_{-j}$ for all $j\in\mathbb{Z}_{\geq0}$ and that V is of CFT type by Proposition 2.3.6. Hence, $a_{(j)}b=0$ for all $j\in\mathbb{Z}_{>0}$ and $a_{(0)}b=\alpha\Omega$ for some $\alpha\in\mathbb{C}$. Thanks to (2.43), we calculate that

$$-(\theta a|b)=-((\theta a)_{(-1)}\Omega|b)=-((\theta a)_{-\frac{1}{2}}\Omega|b)=(\Omega|a_{\frac{1}{2}}b)=(\Omega|a_{(0)}b)=\alpha(\Omega|\Omega)=\alpha\,.$$

Accordingly, (3.2) is equivalent to

$$[a_m,b_k]=-(\theta a|b)\delta_{m,-k}1_V\qquad\forall m,k\in\mathbb{Z}-\frac{1}{2}\,. \tag{3.3}$$

Then, we can proceed as in [BS90, Section 2] to get that every $a\in V_{\frac{1}{2}}$ satisfies the energy bounds (3.1) with $k=0$, that is, a_m is a bounded operator for all $m\in\mathbb{Z}-\frac{1}{2}$. Indeed, let b be a normalized vector in $\mathcal{H}^\infty$, then

$$\begin{aligned}\|a_mb\|^4&=(a_mb|a_mb)^2=(b|(-\theta a)_{-m}a_mb)^2\leq\left\|(-\theta a)_{-m}a_mb\right\|^2\\&=(a_mb|a_m(-\theta a)_{-m}a_mb)=(a_mb|(\theta a|\theta a)a_mb)+(a_mb|(\theta a)_{-m}a_m^2b)\\&=(\theta a|\theta a)\|a_mb\|^2\quad\forall m\in\mathbb{Z}-\frac{1}{2}\,.\end{aligned} \tag{3.4}$$

where we have used the Cauchy-Schwarz inequality in the first row; (3.3) for the last but one equality; the fact that $a_m^2=0$ for all $m\in\mathbb{Z}-\frac{1}{2}$ by (3.3) for the last one. Dividing by $\|a_mb\|^2$, we get the desired (energy) bounds, namely $s=k=0$ so that $Y(a,z)$ (and hence every element in $V_{\frac{1}{2}}$) is actually *bounded*, not only energy-bounded. Therefore, V is generated by the family of energy-bounded quasi-primary vectors $V_{\frac{1}{2}}\cup V_1\cup\mathfrak{F}$ and thus V is energy-bounded too by Proposition 3.1.2. ☐

The first step for the construction of the net is to define the Hilbert space of our theory:

Definition 3.1.5. Let $(V,(\cdot|\cdot))$ be a unitary VOSA. Then, $\mathcal{H}:=\mathcal{H}_{(V,(\cdot|\cdot))}$ is the Hilbert space completion of V with respect to the scalar product $(\cdot|\cdot)$. Hence, Γ and Z are the extensions to $\mathcal{H}$ of the operators Γ_V and Z_V respectively.

Let V be a unitary VOSA. For every $a\in V$, $a_{(n)}$ is an operator on $\mathcal{H}$ with dense domain V. Furthermore, $a_{(n)}$ is closable for all $n\in\mathbb{Z}$. Indeed, thanks to the invariance of the scalar product, $a_{(n)}$ has a densely defined adjoint for all $n\in\mathbb{Z}$. By definition, for all $n\in\frac{1}{2}\mathbb{Z}$, a_n is a closable operator on $\mathcal{H}$ with dense domain V as well.

Recall from Section 1.1 that we denote the Fréchet space of infinitely differentiable complex-valued functions on the circle by $C^\infty(S^1)$. Moreover, we set $C^\infty_\chi(S^1):=\chi C^\infty(S^1)$ with $\chi(z):=e^{i\frac{x}{2}}$ for all $z\in S^1$, where $z=e^{ix}$ for $x\in(-\pi,\pi]$. Now, suppose that V is also energy-bounded and consider $f\in C^\infty(S^1)$ and $g\in C^\infty_\chi(S^1)$ with their Fourier coefficients $\{\widehat{f}_n\mid n\in\mathbb{Z}\}$ and $\{\widehat{g}_n\mid n\in\mathbb{Z}-\frac{1}{2}\}$. Then, for every $a\in V_{\overline{0}}$ and every $b\in V_{\overline{1}}$, we define the operators $Y_0(a,f)$ and $Y_0(b,g)$ both with domain V by

$$Y_0(a,f)c:=\sum_{n\in\mathbb{Z}}\widehat{f}_na_nc\,,\quad Y_0(b,g)c:=\sum_{n\in\mathbb{Z}-\frac{1}{2}}\widehat{g}_nb_nc\qquad\forall c\in V\,. \tag{3.5}$$

Note that the latter operators are densely defined on $\mathcal{H}$ because the series in (3.5) converge in $\mathcal{H}$ thanks to the energy bounds and the rapidly decaying of the Fourier coefficients $\widehat{f}_n$ and $\widehat{g}_n$. Moreover, $Y_0(a,f)$ and $Y_0(b,g)$ have both densely defined adjoints thanks to the invariance of the scalar product. Then, we give the following definition:

Definition 3.1.6. Let $(V, (\cdot|\cdot))$ be an energy-bounded unitary VOSA. For all $a \in V_{\overline{0}}$, $b \in V_{\overline{1}}$ and all $f \in C^\infty(S^1)$, $g \in C^\infty_\chi(S^1)$, we define $Y(a,f)$ and $Y(b,g)$ as the closure of the operators (3.5) on the Hilbert space $\mathcal{H}$. We call them **smeared vertex operators**.

In order to define a net of von Neumann algebras associated to the smeared vertex operators, we need to find a common invariant core for them and their adjoints. This is what we do in the following, rewriting with more details the argument in [CKLW18, p. 47] and extending it to the super case (see also [Nel72, Section 1] and [Lok94, Section 1.5] for the construction of the core).

Working Hypothesis 3.1.7. Throughout the remaining of the current section, V is an energy-bounded unitary VOSA.

For every $k \in \mathbb{Z}_{\geq 0}$, set $\mathcal{H}^k$ the domain in $\mathcal{H}$ of the positive self-adjoint operator $(1_\mathcal{H} + L_0)^k$. Thanks to the closedness of $(1_\mathcal{H} + L_0)^k$, it is immediate to check that $\mathcal{H}^k$ is complete with respect to the scalar product $(\cdot|\cdot)_k := ((1_\mathcal{H}+L_0)^k \cdot |(1_\mathcal{H}+L_0)^k \cdot)$. Define V^k as the Hilbert space completion of V with respect to $(\cdot|\cdot)_k$ and consider the corresponding induced norm $\|\cdot\|_k$. Then, we have that $V^k = \mathcal{H}^k$. Indeed, if V were not $\|\cdot\|_k$-dense in $\mathcal{H}^k$, there would exist a non-zero vector $v \in \mathcal{H}^k \backslash V^k$ such that

$$((1_\mathcal{H} + L_0)^k v|(1_\mathcal{H} + L_0)^k a) = 0 \qquad \forall a \in V$$

or equivalently

$$(v|(1_\mathcal{H} + L_0)^{2k} a) = 0 \qquad \forall a \in V$$

which implies that $(v|b) = 0$ for all $b \in V$, which is impossible because V is $\|\cdot\|$-dense in $\mathcal{H}$ by construction. Define the Fréchet space

$$\mathcal{H}^\infty := \left(\bigcap_{k\in\mathbb{Z}_{\geq 0}} \mathcal{H}^k \,,\ \ \{\|\cdot\|_k \mid k \in \mathbb{Z}_{\geq 0}\} \right) . \tag{3.6}$$

As a first result, we have that:

Lemma 3.1.8. *$\mathcal{H}^\infty$ is a common core for all the smeared vertex operators $Y(a,f)$ and $Y(b,g)$ with $a \in V_{\overline{0}}$, $b \in V_{\overline{1}}$, $f \in C^\infty(S^1)$ and $g \in C^\infty_\chi(S^1)$. Moreover, $Y(a,f)$ and $Y(b,g)$ are continuous in $\mathcal{H}^\infty$ satisfying, for some positive integers M, k and s:*

$$\begin{aligned} \left\|Y(a,f)c\right\| &\leq M\|f\|_s \left\|(L_0 + 1_\mathcal{H})^k c\right\| \quad \forall c \in \mathcal{H}^\infty \\ \left\|Y(b,g)c\right\| &\leq M\|g\|_s \left\|(L_0 + 1_\mathcal{H})^k c\right\| \quad \forall c \in \mathcal{H}^\infty. \end{aligned} \tag{3.7}$$

Proof. V is contained in both the domains of $Y(a,f)$ and $Y(b,g)$ by construction and thanks to the energy bounds (3.1), they satisfy, for some positive integers M, k and s,

$$\begin{aligned} \left\|Y(a,f)c\right\| &\leq M\|f\|_s \left\|(L_0 + 1_\mathcal{H})^k c\right\| \quad \forall c \in V \\ \left\|Y(b,g)c\right\| &\leq M\|g\|_s \left\|(L_0 + 1_\mathcal{H})^k c\right\| \quad \forall c \in V. \end{aligned} \tag{3.8}$$

Recall that V is $\|\cdot\|_k$-dense in $\mathcal{H}^k$ for all $k \in \mathbb{Z}_{\geq 0}$ and thus in $\mathcal{H}^\infty$ as well. It follows that if $c \in \mathcal{H}^\infty$, then there exists a sequence $\{c^n \mid n \in \mathbb{Z}_{\geq 0}\}$ in V convergent to c in the Fréchet topology of $\mathcal{H}^\infty$. By the bounds (3.8) above, $\{Y(a,f)c^n \mid n \in \mathbb{Z}_{\geq 0}\}$ and $\{Y(b,g)c^n \mid n \in \mathbb{Z}_{\geq 0}\}$ are Cauchy sequences in $\mathcal{H}$ and thus they are convergent by the completeness of $\mathcal{H}$. By closeness of the smeared vertex operators, c is an element of their domains. As a consequence, we can extend the bounds (3.8) to the ones (3.7), from which the continuity statement follows. □

As a corollary of the above Lemma 3.1.8, one has that for all $a \in V_{\overline{0}}$, all $b \in V_{\overline{1}}$ and all $c \in \mathcal{H}^\infty$, the maps

$$\begin{aligned} C^\infty(S^1) \ni f &\longmapsto Y(a,f)c \in \mathcal{H} \\ C^\infty_\chi(S^1) \ni g &\longmapsto Y(b,g)c \in \mathcal{H} \end{aligned} \tag{3.9}$$

are continuous and linear, that is, they are operator-valued distributions.

The invariance of the common core $\mathcal{H}^\infty$ is a consequence of the following standard argument.

Lemma 3.1.9. *Let $a \in V_{\overline{0}}$ and $f \in C^\infty(S^1)$. For every $t \in \mathbb{R}$, define the $C^\infty(S^1)$-function $f_t(z) := f(e^{-it}z)$. Then, we have the following equalities:*

$$e^{itL_0}Y(a,f)c = Y(a,f_t)e^{itL_0}c \qquad \forall c \in \mathcal{H}^\infty \ \ \forall t \in \mathbb{R} \tag{3.10}$$

$$iL_0Y(a,f)c = -Y(a,f')c + iY(a,f)L_0c \qquad \forall c \in \mathcal{H}^\infty . \tag{3.11}$$

In particular, $Y(a,f)c \in \mathcal{H}^\infty$ for all $c \in \mathcal{H}^\infty$.

Proof. This is stated in [CKLW18, p. 47] (cf. the proof of Lemma 3.1.10 below). □

Lemma 3.1.10. *Let $b \in V_{\overline{1}}$ and $g = \chi h$ with $h \in C^\infty(S^1)$. For every $t \in \mathbb{R}$, define the $C^\infty(S^1)$-function $h_t(z) := h(e^{-it}z)$. Then, we have the following equalities:*

$$e^{itL_0}Y(b,g)c = Y(b,e^{-i\frac{t}{2}}\chi h_t)e^{itL_0}c \qquad \forall c \in \mathcal{H}^\infty \ \ \forall t \in \mathbb{R} \tag{3.12}$$

$$iL_0Y(b,g)c = -Y(b,g')c + iY(b,g)L_0c \qquad \forall c \in \mathcal{H}^\infty . \tag{3.13}$$

In particular, $Y(b,g)c \in \mathcal{H}^\infty$ for all $c \in \mathcal{H}^\infty$.

Proof. It is useful to note that e^{itL_0} is a bounded operator on $\mathcal{H}$ by the functional calculus of L_0 and in particular, it preserves V and $\mathcal{H}^\infty$, whereas $Y(b,g)c \in \mathcal{H}$ for all $c \in \mathcal{H}^\infty$. We prove (3.12) for homogeneous elements $b \in V_{\overline{1}}$ and $c \in V$ first. On the one hand, for all $t \in \mathbb{R}$, we can calculate that

$$e^{itL_0}Y(b,g)c = \sum_{n\in\mathbb{Z}-\frac{1}{2}} \widehat{g}_n e^{itL_0}b_nc = \sum_{n\in\mathbb{Z}-\frac{1}{2}} e^{it(d_c-n)}\widehat{g}_nb_nc \tag{3.14}$$

where we have used (3.5) and successively (2.26) to prove that $d_{b_nc} = d_c - n$ for all $n \in \mathbb{Z} - \frac{1}{2}$. On the other hand, just note that for all $t \in \mathbb{R}$,

$$\widehat{(\chi h_t)}_n = \widehat{(h_t)}_{\frac{2n-1}{2}} = e^{-i\frac{2n-1}{2}t}\widehat{h}_{\frac{2n-1}{2}} = e^{i\frac{t}{2}}e^{-int}\widehat{g}_n \quad \forall n \in \mathbb{Z} - \frac{1}{2} .$$

It follows that the two sides of (3.12) are equal when $c \in V$. Then, the general case with $c \in \mathcal{H}^\infty$ follows by Lemma 3.1.8.

To prove (3.13), we show that the right hand side of (3.12) is differentiable in $t = 0$. Let $c \in \mathcal{H}^\infty$. Since $\mathcal{H}^\infty$ is an invariant core for L_0 and e^{itL_0} for all $t \in \mathbb{R}$, we are allowed to write that

$$\left\| \frac{Y(b,e^{-i\frac{t}{2}}\chi h_t)e^{itL_0}c - Y(b,g)c}{t} + Y(b,g')c - iY(b,g)L_0c \right\|$$

is equal to

$$\left\| \frac{Y(b,e^{-i\frac{t}{2}}\chi h_t)e^{itL_0}c - Y(b,g)e^{itL_0}c + Y(b,g)e^{itL_0}c - Y(b,g)c}{t} + Y(b,g')c - iY(b,g)L_0c \right\| .$$

The latter is less than or equal to

$$\left\| Y(b, \frac{e^{-i\frac{t}{2}}\chi h_t - g}{t})e^{itL_0}c + Y(b,g')c \right\| + \left\| Y(b,g) \left[\frac{e^{itL_0}c - c}{t} - iL_0c \right] \right\| \tag{3.15}$$

which tends to zero as t tends to zero. Indeed, $(e^{itL_0}c - c)t^{-1}$ converges to iL_0c in $\mathcal{H}^\infty$ and thus the second term in (3.15) converges to zero by Lemma 3.1.8; whereas the first term in (3.15) is less than or equal to

$$\left\| Y(b, \frac{e^{-i\frac{t}{2}}\chi h_t - g}{t})[e^{itL_0}c - c] \right\| + \left\| Y(b, \frac{e^{-i\frac{t}{2}}\chi h_t - g}{t} + g')c \right\|$$

which tends to zero thanks to the bound (3.7) and the continuity (3.9). Thus, (3.12) is differentiable at $t=0$ with derivative

$$\begin{aligned} iL_0Y(b,g)c &= \frac{\mathrm{d}}{\mathrm{d}t}\left[e^{itL_0}Y(b,g)c\right]_{t=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\left[Y(b,e^{-i\frac{t}{2}}\chi h_t)e^{itL_0}c\right]_{t=0} \\ &= -Y(b,g')c + iY(b,g)L_0c \quad \forall c\in\mathcal{H}^\infty\,, \end{aligned}$$

which is the desired formula. □

As a clear consequence of Lemma 3.1.9 and Lemma 3.1.10, we have that $\mathcal{H}^\infty$ is a common invariant core for all the smeared vertex operators.

Remark 3.1.11. It is useful to note that formula (3.10) implies the following equality between operators: for all $a\in V_{\overline{0}}$ and all $f\in C^\infty(S^1)$, we have that

$$e^{itL_0}Y(a,f)e^{-itL_0} = Y(a,f_t) \qquad \forall t\in\mathbb{R}\,. \tag{3.16}$$

Similarly from formula (3.12), for all $b\in V_{\overline{1}}$ and all $g=\chi h$ with $h\in C^\infty(S^1)$, we have that

$$e^{itL_0}Y(b,g)e^{-itL_0} = Y(b,e^{-i\frac{t}{2}}\chi h_t) \qquad \forall t\in\mathbb{R}\,. \tag{3.17}$$

Finally, by (2.41), we can easily calculate that for all homogeneous $a\in V$, all $b,c\in V$ and all f either in $C^\infty(S^1)$ or in $C^\infty_\chi(S^1)$ depending on $p(a)$,

$$(Y(a,f)b|c) = (b|Y(a,f)^+c)\,, \tag{3.18}$$

where we have defined

$$Y(a,f)^+ := (-1)^{2d_a^2+d_a}\sum_{l\in\mathbb{Z}_{\geq 0}}\frac{Y(\theta L_1^l a,\overline{f})}{l!} = i^{2d_a}\sum_{l\in\mathbb{Z}_{\geq 0}}\frac{Y(Z\theta L_1^l a,\overline{f})}{l!}\,, \tag{3.19}$$

which is well-defined because $L_1^l a\in V_{d_a-l}$ for all $l\in\mathbb{Z}_{\geq 0}$ and thus the sum in (3.19) is finite. This implies that $Y(a,f)^+\subseteq Y(a,f)^*$ and thus $\mathcal{H}^\infty$ is an invariant core for the adjoints of all smeared vertex operators. Therefore, $\mathcal{H}^\infty$ is a common invariant core also for $Y(a,f)^*$ for all $a\in V$ and all f either in $C^\infty(S^1)$ or in $C^\infty_\chi(S^1)$ depending on $p(a)$.

3.2 Strong graded-locality and the irreducible graded-local conformal net on S^1

In this second part, we introduce the central concept of strong graded-locality for a (simple) unitary VOSA V and we construct the irreducible graded-local conformal net on S^1 associated to it.

Thanks to Subsection 3.1, we can give the following:

Definition 3.2.1. Let $(V,(\cdot|\cdot))$ be an energy-bounded unitary VOSA. Define a family of von Neumann algebras $\mathcal{A}_{(V,(\cdot|\cdot))}$ on the Hilbert space completion $\mathcal{H}:=\mathcal{H}_{(V,(\cdot|\cdot))}$ of V by

$$\mathcal{A}_{(V,(\cdot|\cdot))}(I) := W^*\left(\left\{Y(a,f)\,,\ Y(b,g)\ \middle|\ \begin{array}{l} a\in V_{\overline{0}}\,,\ f\in C^\infty(S^1)\,,\ \mathrm{supp}f\subset I\,, \\ b\in V_{\overline{1}}\,,\ g\in C^\infty_\chi(S^1)\,,\ \mathrm{supp}g\subset I \end{array}\right\}\right) \qquad \forall I\in\mathcal{J}\,,$$

where, for any index set $\mathcal{I}$ and any family of closed densely defined operators $\{A_j\mid j\in\mathcal{I}\}$ on a given Hilbert space $\mathcal{G}$,

$$W^*(\{A_j\mid j\in\mathcal{I}\}) := \bigvee_{j\in\mathcal{I}} W^*(A_j)\,, \tag{3.20}$$

$$W^*(A_j) := \{B\in B(\mathcal{G})\mid BA_j\subseteq A_jB\,,\ B^*A_j\subseteq A_jB^*\}' \qquad \forall j\in\mathcal{I} \tag{3.21}$$

are called the **von Neumann algebra generated by** $\{A_j\mid j\in\mathcal{I}\}$ and A_j respectively. Note that (3.21) is the smallest von Neumann algebra to which the operator A_j is affiliated, see [Ped89, Section 4.5 and Subsection 5.2.7] for details.

Then, we prove the cyclicity of the vaccum vector $\Omega \in V$ for the family $\mathcal{A}_{(V,(\cdot|\cdot))}$, which is part of the axiom **(F)** given in Section 1.3.

Proposition 3.2.2. *Let V be an energy-bounded unitary VOSA. The vacuum vector Ω of V is cyclic for the von Neumann algebra*

$$A_{(V,(\cdot|\cdot))}(S^1) := \bigvee_{I\in\mathcal{J}} \mathcal{A}_{(V,(\cdot|\cdot))}(I) \,. \tag{3.22}$$

Proof. Let E be the projection from $\mathcal{H}$ onto the closed subspace $\overline{\mathcal{A}_{(V,(\cdot|\cdot))}(S^1)\Omega} \subset \mathcal{H}$. We want to prove that EV is dense in $\mathcal{H}$, which implies the cyclicity of Ω. We split the proof of this fact into the following three steps.

First, we prove that $EV \subseteq V$. To do this, note that

$$L_0 a = Y(\nu, f)a = Y(\nu, f_1)a + Y(\nu, f_2)a \qquad \forall a \in V$$

where $f(z) = 1$ for all $z \in S^1$ and $f_j \in C^\infty(S^1, \mathbb{R})$ such that $f = f_1 + f_2$ and $\mathrm{supp} f_j \subset I_j$ for some $I_j \in \mathcal{J}$. Now, note that $E \in \mathcal{A}_{(V,(\cdot|\cdot))}(S^1)'$ and that L_0 is a self-adjoint operator. Therefore, using (3.19), we have that

$$\begin{aligned}(L_0 b|Ea) &= (Y(\nu, f_1)b|Ea) + (Y(\nu, f_2)b|Ea) \\ &= (Eb|Y(\nu, f_1)a) + (Eb|Y(\nu, f_2)a) \\ &= (Eb|L_0 a) = (b|EL_0 a) \qquad \forall a, b \in V\,.\end{aligned}$$

This implies that if $a \in V$ is homogeneous, then Ea is an eigenvector of L_0 in $\mathcal{H}$ with eigenvalue d_a, thus $EV \subseteq V$.

Second, we prove that $EV = E\mathcal{H} \cap V$. On the one hand, EV is contained in both $E\mathcal{H}$ and V and thus in their intersection. On the other hand, if $Ea \in E\mathcal{H} \cap V$ with $a \in \mathcal{H}$, then

$$Ea = E^2 a = E(Ea) \in EV$$

and therefore $E\mathcal{H} \cap V$ is contained in EV, that is, $EV = E\mathcal{H} \cap V$.

Third, we prove that $EV = V$. Let $a \in EV$ and $b \in V$. Suppose initially that b is even. If $f \in C^\infty(S^1)$ with $\mathrm{supp} f \subset I \in \mathcal{J}$, then $Y(b, f)$ is affiliated with $\mathcal{A}_{(V,(\cdot|\cdot))}(I)$. This implies that

$$EY(b, f)a = Y(b, f)Ea = Y(b, f)a \implies Y(b, f)a \in E\mathcal{H}$$

for all $f \in C^\infty(S^1)$ with $\mathrm{supp} f \subset I$ for some $I \in \mathcal{J}$. Therefore, we obtain that

$$V \ni b_n a = Y(b, f_n)a = Y(b, g_n)a + Y(b, h_n)a \in E\mathcal{H} \qquad \forall n \in \mathbb{Z}$$

with $f_n(z) = z^n$, $g_n, h_n \in C^\infty(S^1)$ such that $f_n = g_n + h_n$, $\mathrm{supp} g_n \subset I_g$ and $\mathrm{supp} h_n \subset I_h$ for some $I_g, I_h \in \mathcal{J}$. Thus,

$$b_n a \in E\mathcal{H} \cap V = EV \qquad \forall b \in V_{\bar{0}} \ \forall a \in EV\,. \tag{3.23}$$

Now, the identity operator is in $\mathcal{A}_{(V,(\cdot|\cdot))}$ and thus $E\Omega = \Omega$, which implies that $\Omega \in EV$. Therefore, we can choose $a = \Omega$ and $n = d_b$ in (3.23) to obtain that $b_{(-1)}\Omega = b \in EV$ for all $b \in V_{\bar{0}}$. In a very similar way, we can prove that $b \in EV$ for all $b \in V_{\bar{1}}$, which implies that $EV = V$. This concludes the proof of the cyclicity of Ω because V is dense in $\mathcal{H}$. □

Now, we are ready to discuss the covariance of the family $\mathcal{A}_{(V,(\cdot|\cdot))}$. The first step is to prove that $\mathcal{A}_{(V,(\cdot|\cdot))}$ is generated by the quasi-primary vectors of V.

Lemma 3.2.3. *Let V be an energy-bounded unitary VOSA. Let A be a bounded operator on $\mathcal{H}$ and let $I \in \mathcal{J}$. Then, we have two cases:*

- *($\bar{0}$) Let $a \in V_{\bar{0}}$. Then, $AY(a, f) \subset Y(a, f)A$ for all $f \in C^\infty(S^1)$ with $\mathrm{supp} f \subset I$ if and only if $(A^*c|Y(a, f)d) = (Y(a, f)^*c|Ad)$ for all $c, d \in V$ and all $f \in C^\infty(S^1, \mathbb{R})$ with $\mathrm{supp} f \subset I$.*
- *($\bar{1}$) Let $b \in V_{\bar{1}}$. Then, $AY(b, g) \subset Y(b, g)A$ for all $g \in C^\infty_\chi(S^1)$ with $\mathrm{supp} g \subset I$ if and only if $(A^*c|Y(b, g)d) = (Y(b, g)^*c|Ad)$ for all $c, d \in V$ and all $g \in C^\infty_\chi(S^1, \mathbb{R})$ with $\mathrm{supp} g \subset I$.*

Proof. ($\bar{0}$) This is exactly the proof of [CKLW18, Lemma 6.5].

($\bar{1}$) We can use the same proof of [CKLW18, Lemma 6.5] with minus changes. Indeed, it is sufficient to use formula (3.17) instead of (3.16), whenever the latter occurs.

□

Proposition 3.2.4. *Let V be a simple energy-bounded unitary VOSA. Let A be a bounded operator on $\mathcal{H}$ and let $I \in \mathcal{J}$. Then, we have that: $A \in \mathcal{A}_{(V,(\cdot|\cdot))}(I)'$ if and only if*

$$(A^*c|Y(a,f)d) = (Y(a,f)^*c|Ad) \quad \text{and} \quad (A^*c|Y(b,g)d) = (Y(b,g)^*c|Ad)$$

for all quasi-primary $a \in V_{\bar{0}}$ and all quasi-primary $b \in V_{\bar{1}}$, for all $c, d \in V$, for all $f \in C^\infty(S^1,\mathbb{R})$ with $\mathrm{supp} f \subset I$ and all $g \in C^\infty_\chi(S^1,\mathbb{R})$ with $\mathrm{supp} g \subset I$. In particular, for any $I \in \mathcal{J}$, $\mathcal{A}_{(V,(\cdot|\cdot))}(I)$ is equal to the von Neumann algebra generated by the sets

$$\left\{ Y(a,f) \mid a \in \bigcup_{k\in\mathbb{Z}} V_k\,,\ L_1 a = 0\,,\ f \in C^\infty(S^1,\mathbb{R})\,,\ \mathrm{supp} f \subset I \right\}$$

and

$$\left\{ Y(b,g) \mid b \in \bigcup_{k\in\mathbb{Z}-\frac{1}{2}} V_k\,,\ L_1 b = 0\,,\ g \in C^\infty_\chi(S^1,\mathbb{R})\,,\ \mathrm{supp} g \subset I \right\}.$$

Proof. The proof of [CKLW18, Proposition 6.6] still works in this case, just using some precautions due to the presence of odd elements. This means that we have to use the generalized formula for the adjoint smeared vertex operators, which we get by (3.19), namely,

$$Y(a,f)c = (-1)^{2d_a^2+d_a} Y(\theta a,f)^*c = i^{-2d_a} Y(\theta Z^* a,f)^*c \tag{3.24}$$

for all quasi-primary $a \in V$, for all $c \in V$ and all f either in $C^\infty(S^1,\mathbb{R})$ or $C^\infty_\chi(S^1,\mathbb{R})$ depending on $p(a)$. □

Working Hypothesis 3.2.5. Unless differently stated, V is a simple energy-bounded unitary VOSA till the end of the current section.

The next step is the definition of the representation of $\mathrm{Diff}^+(S^1)^{(\infty)}$ on $\mathcal{H}$. By Theorem 1.2.2, the positive-energy unitary representation of the Virasoro algebra on V, which we have from the conformal vector ν of the theory, gives rise to a positive-energy strongly continuous projective unitary representation of $\mathrm{Diff}^+(S^1)^{(\infty)}$, which factors through $\mathrm{Diff}^+(S^1)^{(2)}$ because $e^{i4\pi L_0} = 1_{\mathcal{H}}$. In particular, as well explained in [Tol99, Theorem 5.2.1 and Proposition 5.2.4], for all $t \in \mathbb{R}$, all $f \in C^\infty(S^1,\mathbb{R})$ and all $A \in B(\mathcal{H})$, we have that

$$U(\exp^{(2)}(tf)) = e^{itY(\nu,f)}\,, \qquad U(\exp^{(2)}(tf))AU(\exp^{(2)}(tf))^* = e^{itY(\nu,f)}Ae^{-itY(\nu,f)} \tag{3.25}$$

where $\exp^{(2)}(tf)$ is the lift to $\mathrm{Diff}^+(\mathrm{S}^1)^{(2)}$ of the one-parameter subgroup $\exp(tf)$ generated by the real smooth vector field $f\frac{\mathrm{d}}{\mathrm{d}x}$. Moreover, we have that $U(\gamma)\mathcal{H}^\infty = \mathcal{H}^\infty$ for all $\gamma \in \mathrm{Diff}^+(S^1)^{(2)}$ and that $U(r^{(2)}(t)) = e^{itL_0}$, where $r^{(2)}(t)$ is the lift to $\mathrm{Diff}^+(\mathrm{S}^1)^{(2)}$ of the rotation subgroup, where $t \in \mathbb{R}$ is the angle of rotation. In particular, $U(r^{(2)}(2\pi)) = e^{i2\pi L_0}$ acts on V as the parity operator Γ_V, then the former is exactly Γ.

Remark 3.2.6. For every real smooth vector field $f\frac{\mathrm{d}}{\mathrm{d}x}$, the unitary operator $U(\exp^{(2)}(tf)) = e^{itY(\nu,f)}$ defines a strongly continuous one-parameter unitary group on $\mathcal{H}$. Moreover, for any $c \in \mathcal{H}^\infty$, the function $\mathbb{R} \ni t \mapsto e^{itY(\nu,f)}c \in \mathcal{H}$ is differentiable with derivative $e^{itY(\nu,f)}iY(\nu,f)c$. Fix $c \in \mathcal{H}^\infty$ and $k \in \mathbb{Z}_{>0}$. By [Tol99, Theorem 5.2.1], for every $\gamma \in \mathrm{Diff}^+(S^1)^{(2)}$, $U(\gamma)$ preserves $\mathcal{H}^k$ and acts continuously on it. Choose a real smooth vector field $f\frac{\mathrm{d}}{\mathrm{d}x}$, if ${}_k\int$ on $\mathcal{H}^k$ and ${}_0\int$ on $\mathcal{H}$ are Riemann integrals, we have that for all $h > 0$,

$$\begin{aligned} {}_k\int_0^h e^{i(t+s)Y(\nu,f)}iY(\nu,f)c\,\mathrm{d}s &= {}_0\int_0^h e^{i(t+s)Y(\nu,f)}iY(\nu,f)c\,\mathrm{d}s \\ &= e^{i(t+h)Y(\nu,f)}c - e^{itY(\nu,f)}c \qquad \forall t \in \mathbb{R}\,. \end{aligned} \tag{3.26}$$

Set $A := Y(\nu, f)$, for h small enough, we have that

$$\begin{aligned}\left\|\frac{e^{i(t+h)A}c - e^{itA}c}{h} - e^{itA}iAc\right\|_k &= \frac{1}{h}\left\|_k \int_0^h (e^{isA} - 1_{\mathcal{H}^k})e^{itA}iAc\,\mathrm{d}s\right\|_k \\ &\leq \frac{1}{h}\int_0^h \left\|(e^{isA} - 1_{\mathcal{H}^k})e^{itA}iAc\right\|_k \mathrm{d}s \\ &< \frac{h}{h}\epsilon = \epsilon \qquad \forall t \in \mathbb{R}\,.\end{aligned} \tag{3.27}$$

That means that $\mathbb{R} \ni t \mapsto e^{itY(\nu,f)}c \in \mathcal{H}^k$ is differentiable with derivative $e^{itY(\nu,f)}iY(\nu,f)c$. By the arbitrariness of k, we have the same result for $\mathcal{H}^\infty$ in place of $\mathcal{H}^k$.

A further step is to define a representation of $\mathrm{Diff}^+(S^1)$ on $C^\infty(S^1)$ and one of $\mathrm{Diff}^+(S^1)^{(2)}$ on $C^\infty_\chi(S^1)$. The former is exactly [CKLW18, Eq. (119)], which we rewrite with more details in the following for our convenience.

Let $\gamma \in \mathrm{Diff}^+(S^1)$ and consider the function $X_\gamma : S^1 \to \mathbb{R}$ defined by the formula

$$X_\gamma(z) := -i\frac{\mathrm{d}}{\mathrm{d}x}\log(\gamma(e^{ix})) \qquad z = e^{ix},\ x \in (-\pi, \pi]\,. \tag{3.28}$$

Remark 3.2.7. Note that we can rewrite (3.28) as

$$X_\gamma(z) := -i\frac{\mathrm{d}}{\mathrm{d}x}\log(\gamma(e^{ix})) = \frac{\mathrm{d}\phi_\gamma}{\mathrm{d}x}(x) \qquad z = e^{ix},\ x \in (-\pi, \pi] \tag{3.29}$$

where $\phi_\gamma(\cdot) = -i\log(\gamma(\cdot))$, provided we have chosen any branch of the complex logarithm, is one of the diffeomorphisms in $\mathrm{Diff}^+(S^1)^{(\infty)}$, which are representatives of the $2\pi\mathbb{Z}$-class of γ, see Section 1.2.

Thanks to the orientation preserving of γ, it results that $X_\gamma(z) > 0$ for all $z \in S^1$. Furthermore, it is easy to prove that $X_\gamma \in C^\infty(S^1)$ and that

$$X_{\gamma_1\gamma_2}(z) = X_{\gamma_1}(\gamma_2(z))X_{\gamma_2}(z) \qquad \forall\gamma_1, \gamma_2 \in \mathrm{Diff}^+(S^1)\ \forall z \in S^1\,. \tag{3.30}$$

For every $d \in \mathbb{Z}$, we define a family of continuous operators $\{\beta_d(\gamma) \mid \gamma \in \mathrm{Diff}^+(S^1)\}$ on the Fréchet space $C^\infty(S^1)$ by

$$(\beta_d(\gamma)f)(z) := \Big[X_\gamma(\gamma^{-1}(z))\Big]^{d-1} f(\gamma^{-1}(z)) \qquad \forall f \in C^\infty(S^1)\ \forall z \in S^1\,. \tag{3.31}$$

Moreover, for every $d \in \mathbb{Z}_{>0}$, β_d defines a strongly continuous representation of $\mathrm{Diff}^+(S^1)$ on $C^\infty(S^1)$, preserving the subspace of real-valued functions $C^\infty(S^1,\mathbb{R})$. Finally, it is useful to write the proof of the following fact, which is analogous to [CKL08, Eq. (43)].

Lemma 3.2.8. *For every $f_1 \in C^\infty(S^1,\mathbb{R})$ and every $f_2 \in C^\infty(S^1)$, the map*

$$\mathbb{R} \ni t \longmapsto \beta_d(\exp(tf_1))f_2 \in C^\infty(S^1)$$

is differentiable and

$$\frac{\mathrm{d}}{\mathrm{d}t}\big[\beta_d(\exp(tf_1))f_2\big]_{t=0} = (d-1)f_1'f_2 - f_1f_2'\,. \tag{3.32}$$

Proof. Note that we can prove the theorem for $f_2 \in C^\infty(S^1,\mathbb{R})$ and then extend the result to $C^\infty(S^1)$ thanks to the linearity of $\beta_d(\gamma)$.

It is not difficult to prove the differentiability with respect to the Fréchet topology of $C^\infty(S^1)$ recalling that

$$\frac{\mathrm{d}}{\mathrm{d}t}\exp(tf) = f\frac{\mathrm{d}}{\mathrm{d}x}\exp(tf) \qquad \forall f \in C^\infty(S^1,\mathbb{R})\,. \tag{3.33}$$

We prove (3.32) by direct calculation. On the one hand, we have that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}X_{\exp(tf_1)}(\exp(-tf_1)(e^{ix})) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big[X_{\exp(-tf_1)}(e^{ix})\Big]^{-1} \\ &= \frac{-\frac{\mathrm{d}}{\mathrm{d}t}X_{\exp(-tf_1)}(e^{ix})}{\Big[X_{\exp(-tf_1)}(e^{ix})\Big]^2} \\ &= -i\frac{\frac{\mathrm{d}}{\mathrm{d}x}\frac{\big(f_1\frac{d}{dx}\exp(-tf_1)\big)(e^{ix})}{\exp(-tf_1)(e^{ix})}}{\Big[X_{\exp(-tf_1)}(e^{ix})\Big]^2} \qquad \forall e^{ix} \in S^1\end{aligned}$$

where we have used (3.30) and the positivity of X_γ for the first equality and (3.33) interchanging $\frac{\mathrm{d}}{\mathrm{d}x}$ and $\frac{\mathrm{d}}{\mathrm{d}t}$ for the third one. As a consequence, we have that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[X_{\exp(tf_1)}(\exp(-tf_1)(e^{ix}))\right]_{t=0} = f_1'(e^{ix}) \qquad \forall e^{ix} \in S^1 \,. \tag{3.34}$$

On the other hand,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} f_2(\exp(-tf_1)(e^{ix})) &= \frac{\mathrm{d}}{\mathrm{d}t}\widetilde{f_2}(\phi_{\exp(-tf_1)}(x)) \\ &= f_2'(e^{ix})\frac{\mathrm{d}}{\mathrm{d}t}\phi_{\exp(-tf_1)}(x) \\ &= f_2'(e^{ix})\frac{i\left(f_1\frac{d}{dx}\exp(-tf_1)\right)(e^{ix})}{\exp(-tf_1)(e^{ix})}\end{aligned}$$

where we have used Remark 3.2.7. Therefore, it is clear that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[f_2(\exp(-tf_1)(e^{ix}))\right]_{t=0} = -f_2'(e^{ix})f_1(e^{ix}) \qquad \forall e^{ix} \in S^1 \,. \tag{3.35}$$

To conclude the proof, note that equation (3.32) is now an easy consequence of the chain and product rules for the derivation. □

To define a representation of $\mathrm{Diff}^+(S^1)^{(2)}$ on $C^\infty_\chi(S^1)$, consider for every $\gamma \in \mathrm{Diff}^+(S^1)^{(2)}$, a representative of its $4\pi\mathbb{Z}$-class of diffeomorphisms $\phi_\gamma \in \mathrm{Diff}^+(S^1)^{(\infty)}$ so that $\gamma(e^{i\frac{x}{2}}) = e^{i\frac{\phi_\gamma(x)}{2}}$ for all $x \in (-2\pi, 2\pi]$, see Section 1.2. Thus, we define the function $Y_\gamma : S^1 \to \mathbb{C}$ as

$$Y_\gamma(z) := e^{i\frac{\phi_\gamma(x)-x}{2}}\,, \qquad z = e^{ix},\ \ x \in (-\pi, \pi]\,. \tag{3.36}$$

which is well-defined because it does not depend on the choice of the representative in the class of diffeomorphisms $\{\phi_\gamma + 4k\pi \mid k \in \mathbb{Z}\}$.

Lemma 3.2.9. *$Y_\gamma \in C^\infty(S^1)$ for all $\gamma \in \mathrm{Diff}^+(S^1)^{(2)}$ and*

$$Y_{\gamma_1\gamma_2}(z) = Y_{\gamma_1}(\dot{\gamma_2}(z))Y_{\gamma_2}(z) \qquad \forall \gamma_1, \gamma_2 \in \mathrm{Diff}^+(S^1)^{(2)}\ \ \forall z \in S^1\,. \tag{3.37}$$

Proof. It is useful to recall that every $\phi \in \mathrm{Diff}^+(S^1)^{(\infty)}$ satisfies

$$\phi(x + 2k\pi) = \phi(x) + 2k\pi \qquad \forall x \in \mathbb{R}\ \ \forall k \in \mathbb{Z}\,. \tag{3.38}$$

For all $\gamma \in \mathrm{Diff}^+(S^1)^{(2)}$, the extension of $\widetilde{Y_\gamma}$ to the whole real axis is clearly infinitely differentiable. Moreover, it is 2π-periodic, indeed

$$\widetilde{Y_\gamma}(x + 2\pi) = e^{i\frac{\phi_\gamma(x+2\pi)-(x+2\pi)}{2}} = e^{i\frac{\phi_\gamma(x)+2\pi-x-2\pi}{2}} = e^{i\frac{\phi_\gamma(x)-x}{2}} = \widetilde{Y_\gamma}(x) \qquad \forall x \in \mathbb{R} \tag{3.39}$$

where we have used (3.38) for the second equality. Thus $Y_\gamma \in C^\infty(S^1)$ for all $\gamma \in \mathrm{Diff}^+(S^1)^{(2)}$.

To prove (3.37), let $\gamma_1, \gamma_2 \in \mathrm{Diff}^+(S^1)^{(2)}$. Write $z = e^{ix}$ with $x \in (-\pi, \pi]$ and for every such x, consider $k_{\gamma_2}(x) \in 2\pi\mathbb{Z}$ such that $\phi_{\gamma_2}(x) + k_{\gamma_2}(x)$ is in $(-\pi, \pi]$. Then, we have that

$$\begin{aligned}Y_{\gamma_1\gamma_2}(z) &= e^{i\frac{\phi_{\gamma_1}(\phi_{\gamma_2}(x))-x}{2}} \\ &= e^{i\frac{\phi_{\gamma_1}(\phi_{\gamma_2}(x))-\phi_{\gamma_2}(x)}{2}}e^{i\frac{\phi_{\gamma_2}(x)-x}{2}} \\ &= e^{i\frac{\phi_{\gamma_1}(\phi_{\gamma_2}(x)+k_{\gamma_2}(x))-(\phi_{\gamma_2}(x)+k_{\gamma_2}(x))}{2}}Y_{\gamma_2}(z) \\ &= Y_{\gamma_1}(e^{i(\phi_{\gamma_2}(x)+k_{\gamma_2}(x))})Y_{\gamma_2}(z) \\ &= Y_{\gamma_1}(\dot{\gamma_2}(z))Y_{\gamma_2}(z) \qquad \forall z \in S^1\end{aligned}$$

where we have used the group properties of the representatives as explained in Section 1.2 for the first equality and (3.38) for the third one. □

The following definition is inspired by [Boc96, Appendix A] and [Lok94, Section 1.1] (cf. also [Pal10, Section 2, 3]). For all $d \in \mathbb{Z} - \frac{1}{2}$, we define a family of continuous operators $\{\alpha_d(\gamma) \mid \gamma \in \mathrm{Diff}^+(S^1)^{(2)}\}$ on the Fréchet space $C^\infty_\chi(S^1)$ in the following way: let $g = \chi h$ with $h \in C^\infty(S^1)$ and set

$$(\alpha_d(\gamma)g)(z) := \chi(z)Y_{\gamma^{-1}}(z)(\beta_d(\dot\gamma)h)(z) \qquad \forall z \in S^1 . \tag{3.40}$$

By Lemma 3.2.9, α_d defines a strongly continuous representation of $\mathrm{Diff}^+(S^1)^{(2)}$ on $C^\infty_\chi(S^1)$, which preserves $C^\infty_\chi(S^1, \mathbb{R})$ as it is explained in the following Remark 3.2.10.

Remark 3.2.10. Let γ be a generic element in $\mathrm{Diff}^+(S^1)^{(2)}$ and write every $z \in S^1$ as e^{ix} for a unique $x \in (-\pi, \pi]$. Then, we have that

$$Y_\gamma(z)\chi(z) = e^{i\frac{\phi_\gamma(x)}{2}} = \gamma(e^{i\frac{x}{2}}) = \epsilon_\gamma(z)\chi(\dot\gamma(z)) \qquad \forall z \in S^1 \tag{3.41}$$

where $\epsilon_\gamma(z) \in \{\pm 1\}$ is defined as follows: for all $z \in S^1$, there exist a unique $\phi_{\dot\gamma}$ in $\{\phi_\gamma + 2k\pi \mid k \in \mathbb{Z}\}$ such that $\phi_{\dot\gamma}(x)$ is in $(-\pi, \pi]$; then

$$\epsilon_\gamma(z) := e^{i\frac{\phi_{\dot\gamma}(x)-\phi_\gamma(x)}{2}} \qquad \forall z = e^{ix} \in S^1 . \tag{3.42}$$

Note that definition (3.42) does not depend on the representative ϕ_γ. Thus, we can rewrite (3.40) as

$$(\alpha_d(\gamma)g)(z) = \Big[X_{\dot\gamma}(\dot\gamma^{-1}(z))\Big]^{d-1} \epsilon_{\gamma^{-1}}(z)g(\dot\gamma^{-1}(z)) \qquad \forall g \in C^\infty_\chi(S^1),\ \forall z \in S^1 , \tag{3.43}$$

which clearly assures us that $\alpha_d(\gamma)$ preserves $C^\infty_\chi(S^1, \mathbb{R})$ for all $\gamma \in \mathrm{Diff}^+(S^1)^{(2)}$. Finally, note that (3.43) corresponds to the one in [Boc96, Eq. (78)] for $\mathrm{M\ddot{o}b}(S^1)$.

It remains a last property to prove for the family α_d, that is:

Lemma 3.2.11. *For every $f \in C^\infty(S^1, \mathbb{R})$ and every $g \in C^\infty_\chi(S^1)$, the map*

$$\mathbb{R} \ni t \longmapsto \alpha_d(\exp^{(2)}(tf))g \in C^\infty_\chi(S^1)$$

is differentiable and

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big[\alpha_d(\exp^{(2)}(tf))g\Big]_{t=0} = (d-1)f'g - fg' . \tag{3.44}$$

Proof. We can easily adapt the proof of Lemma 3.2.8 to the current case with some extra work. Indeed, it is quite clear the differentiability of the function

$$\mathbb{R} \ni t \longmapsto Y_{\exp^{(2)}(-tf)} \in C^\infty(S^1)$$

and consequently the one of $t \mapsto \alpha_d(\exp^{(2)}(tf))g$. Moreover, we have that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}Y_{\exp^{(2)}(-tf)}(e^{ix}) &= \frac{i}{2}Y_{\exp^{(2)}(-tf)}(e^{ix})\frac{\mathrm{d}}{\mathrm{d}t}\phi_{\exp^{(2)}(-tf)}(e^{ix}) \\ &= \frac{1}{2}Y_{\exp^{(2)}(-tf)}(e^{ix})\frac{\left(-f\frac{\mathrm{d}}{\mathrm{d}x}\exp(-tf)\right)(e^{ix})}{\exp(-tf)(e^{ix})} \qquad \forall e^{ix} \in S^1\end{aligned}$$

where, without loss of generality, we have chosen (cf. Remark 3.2.7)

$$\phi_{\exp^{(2)}(tf)}(e^{ix}) = -i\log(\exp(tf)(e^{ix})) \qquad \forall e^{ix} \in S^1 .$$

From the calculations above, it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big[Y_{\exp^{(2)}(-tf)}(e^{ix})\Big]_{t=0} = -\frac{i}{2}f(e^{ix}) \qquad \forall e^{ix} \in S^1 . \tag{3.45}$$

Therefore, using the product and chain rules for the derivative, we obtain that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big[\alpha_d(\exp^{(2)}(tf))g\Big]_{t=0} = -\frac{i}{2}\chi f h + \chi\left((d-1)f'h - fh'\right) = (d-1)f'g - fg' \tag{3.46}$$

remembering that if $g = \chi h$ with $h \in C^\infty(S^1)$ then $g' = \frac{i}{2}g + \chi h'$ by definition. □

The following proposition is the key result to prove the Möbius covariance of the family $\mathcal{A}_{(V,(\cdot|\cdot))}$.

Proposition 3.2.12. *Let V be a simple energy-bounded unitary VOSA. Then, we have that for all $I \in \mathcal{J}$:*

- *($\overline{0}$) if $a \in V_{\overline{0}}$ is a quasi-primary vector then $U(\gamma)Y(a,f)U(\gamma)^* = Y(a, \beta_{d_a}(\gamma)f)$ for all f in $C^\infty(S^1)$ with $\mathrm{supp} f \subset I$ and all $\gamma \in \mathrm{Möb}(S^1)$;*
- *($\overline{1}$) if $b \in V_{\overline{1}}$ is a quasi-primary vector then $U(\gamma)Y(b,g)U(\gamma)^* = Y(b, \alpha_{d_b}(\gamma)g)$ for all g in $C^\infty_\chi(S^1)$ with $\mathrm{supp} g \subset I$ and all $\gamma \in \mathrm{Möb}(S^1)^{(2)}$.*

Proof. The proof of ($\overline{1}$) is an adaptation of [CKL08, pp. 1100-1103], similar to what is done in the proof of [CKLW18, Proposition 6.4], which is exactly the case ($\overline{0}$).

First, from equations (2.26)-(2.28), we see that for every quasi-primary vector $b \in V$

$$[L_m, b_n] = ((d_b - 1)m - n)b_{m+n} \qquad \forall m \in \{-1, 0, 1\}\,. \tag{3.47}$$

Second, note that $\mathrm{Möb}(S^1)$ is generated by the one-parameter groups generated by the exponential map of the three real smooth vector fields $l_m \frac{\mathrm{d}}{\mathrm{d}x}$ given by

$$\begin{aligned} l_0(z) &= 1\,, & Y(\nu, l_0) &= L_0\,, \\ l_1(z) &= \frac{z+z^{-1}}{2}\,, & Y(\nu, l_1) &= \frac{L_1 + L_{-1}}{2}\,, \\ l_{-1}(z) &= \frac{z-z^{-1}}{2i}\,, & Y(\nu, l_{-1}) &= \frac{L_1 - L_{-1}}{2i}\,, \end{aligned} \tag{3.48}$$

that is, it is generated by $\exp(tl)$ with $l \in C^\infty(S^1, \mathbb{R})$ and such that $Y(\nu, l)$ is a linear combination of L_m with $m \in \{-1, 0, 1\}$. Consequently, $\mathrm{Möb}(S^1)^{(2)}$ is generated by the corresponding one-parameter subgroups $\exp^{(2)}(tl)$.

Fix a homogeneous quasi-primary odd vector $b \in V$, fix a function $g \in C^\infty_\chi(S^1)$ and consider any $l \in C^\infty(S^1, \mathbb{R})$ such that $\exp^{(2)}(tl) \in \mathrm{Möb}(S^1)^{(2)}$ as explained above. Thanks to Section 3.1, $\mathcal{H}^\infty$ is an invariant core for all the smeared vertex operators and therefore we can write, using (3.47) and (3.48),

$$i[Y(\nu, l), Y(b,g)]c = Y(b, (d_b - 1)l'g - lg')c \qquad \forall c \in \mathcal{H}^\infty\,. \tag{3.49}$$

By Lemma 3.2.11, Lemma 3.1.8 and the continuity in (3.9), we have that the map

$$\mathbb{R} \ni t \mapsto Y(b, \alpha_{d_b}(\exp^{(2)}(tl))g)c \in \mathcal{H}^\infty$$

is differentiable on $\mathcal{H}$ for all $c \in \mathcal{H}^\infty$. Moreover,

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big[Y(b, \alpha_{d_b}(\exp^{(2)}(tl))g)c\Big]_{t=0} = i[Y(\nu, l), Y(b,g)]c \tag{3.50}$$

by equation (3.49).

For every $c \in \mathcal{H}^\infty$ define

$$c(t) := Y(b, \alpha_{d_b}(\exp^{(2)}(tl))g)U(\exp^{(2)}(tl))c \tag{3.51}$$

which is a well-defined vector in $\mathcal{H}^\infty$ because this core is invariant for the smeared vertex operators and the representation U. Then, using again Lemma 3.1.8, the continuity (3.9) and Remark 3.2.6, it is possible to prove the differentiability of $t \mapsto c(t)$ on $\mathcal{H}$ and using (3.50), that

$$\frac{\mathrm{d}}{\mathrm{d}t} c(t)|_{t=0} = iY(\nu, l)c\,. \tag{3.52}$$

This means that $c(t)$ satisfies the Cauchy problem on $\mathcal{H}$

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} c(t) = iY(\nu, l)c(t) \\ c(0) = Y(b,g)c \end{cases} \tag{3.53}$$

whose unique solution is given by

$$c(t) = U(\exp^{(2)}(tl))Y(b,g)c\,. \tag{3.54}$$

It follows by the definition of $c(t)$ that

$$Y(b,\alpha_{d_b}(\exp^{(2)}(tl))g)U(\exp^{(2)}(tl))c = c(t) = U(\exp^{(2)}(tl))Y(b,g)c\,. \tag{3.55}$$

Thus, by the arbitrariness of c in the core $\mathcal{H}^\infty$, we can conclude that

$$U(\exp^{(2)}(tl))Y(b,g)U(\exp^{(2)}(-tl)) = Y(b,\alpha_{d_b}(\exp^{(2)}(tl))g) \tag{3.56}$$

which is the desired result. □

Remark 3.2.13. As in [CKLW18, Proposition 6.4], substituting quasi-primary vectors with just primary ones, we can prove the covariance properties $(\overline{0})$ and $(\overline{1})$ in Proposition 3.2.12 for all $\gamma \in \mathrm{Diff}^+(S^1)$ and all $\gamma \in \mathrm{Diff}^+(S^1)^{(2)}$ respectively.

Now, we can prove the Möbius covariance of the net $\mathcal{A}_{(V,(\cdot|\cdot))}$:

Corollary 3.2.14. *Let V be a simple energy-bounded unitary VOSA and $\mathcal{A}_{(V,(\cdot|\cdot))}$ be the associated family of von Neumann algebras on S^1. Then, $\mathcal{A}_{(V,(\cdot|\cdot))}$ is an irreducible graded-local Möbius covariant net on S^1 except for the graded-locality, which might not hold.*

Proof. We need to prove that $\mathcal{A}_{(V,(\cdot|\cdot))}$ satisfies properties **(A)-(D)** as in Section 1.3. The isotony **(A)** is clear from Definition 3.2.1 of the net. The Möbius covariance **(B)** with respect to the representation U as defined at page 47 is proved thanks to Proposition 3.2.12 together with the fact that $\mathcal{A}_{(V,(\cdot|\cdot))}$ is generated by quasi-primary vectors as showed by Proposition 3.2.4. As far U is a positive-energy representation, as required by **(C)**, follows by simplicity, cf. Proposition 2.3.6. The simplicity of V says us also that Ω is the unique (irreducibility) U-invariant vector, which is even cyclic thanks to Proposition 3.2.2, proving **(D)** and concluding the proof. □

As Corollary 3.2.14 above suggests, it is unknown whether the net $\mathcal{A}_{(V,(\cdot|\cdot))}$ satisfies the axiom of graded locality **(E)** as in Section 1.3. This is due to a more general fact, outlined by E. Nelson in [Nel59, Section 10], see also [RS80, Section VIII.5]. It says that for a given separable infinite-dimensional Hilbert space $\mathcal{H}$, there exist two self-adjoint operators A and B and a common invariant core $\mathcal{D}$ for them, such that $ABa = BAa$ for all $a \in \mathcal{D}$, but $W^*(A)$ is not a subset of $W^*(B)'$, see Definition 3.2.1 for notation. Instead, the converse statement is a well-known functional analytic fact, that is:

Proposition 3.2.15. *Let A and B be closed densely defined operators on a Hilbert space $\mathcal{H}$ and let $\mathcal{D}$ be a common invariant domain for them. Assume that $W^*(A) \subseteq W^*(B)'$. Then, $ABa = BAa$ for all $a \in \mathcal{D}$.*

In VOSA terms, the discussion above means that the locality of vertex operators is not enough to conclude the graded locality of the von Neumann algebras $\mathcal{A}_{(V,(\cdot|\cdot))}(I)$ with $I \in \mathcal{J}$. Therefore, we introduce the following important definition.

Definition 3.2.16. Let V be a unitary VOSA and let Z be the extension of the operator Z_V on V to the whole Hilbert space $\mathcal{H}$. Then, V is called **strongly graded-local** if it is energy-bounded and $\mathcal{A}_{(V,(\cdot|\cdot))}(I') \subseteq Z\mathcal{A}_{(V,(\cdot|\cdot))}(I)'Z^*$ for all $I \in \mathcal{J}$.

Imposing the above definition, we have the desired irreducible graded-local conformal net:

Theorem 3.2.17. *Let $(V,(\cdot|\cdot))$ be a simple strongly graded-local unitary VOSA. Then, the family $\mathcal{A}_{(V,(\cdot|\cdot))}$ is an irreducible graded-local conformal net on S^1. Furthermore, if $(\cdot|\cdot)'$ is another scalar product on V then $\mathcal{A}_{(V,(\cdot|\cdot))}$ and $\mathcal{A}_{(V,(\cdot|\cdot)')}$ are isomorphic graded-local conformal nets.*

Proof. We have already proved properties **(A)-(D)** and irreducibility as in Section 1.3 by Corollary 3.2.14. Then, it remains to prove properties **(E)** and **(F)**.

It is easy to prove that $\Gamma\mathcal{A}_{(V,(\cdot|\cdot))}(I)\Gamma = \mathcal{A}_{(V,(\cdot|\cdot))}(I)$ for all $I \in \mathcal{J}$ and thus the graded-locality **(E)** is assured by Definition 3.2.16.

To prove the diffeomorphism covariance **(F)** of $\mathcal{A}_{(V,(\cdot|\cdot))}$, we could proceed as explained in Remark 5.2.2. Note that that proof does not use the strong graded-locality, but it requires further results such as Theorem 5.2.1 and Remark 3.2.13. Therefore, for a matter of convenience, we present here below a simpler proof, which instead makes use of strong graded-locality. This alternative proof is obtained by adapting the argument in the proof of [Car04, Proposition 3.7 (b)], cf. also the proof of [CKL08, Theorem 33]. The adaptation is as follows. By Remark 1.2.1 and Definition 3.2.1, we have that, for any interval $I \in \mathcal{J}$, $U(\gamma) \in \mathcal{A}_{(V,(\cdot|\cdot))}(I)$ for all $\gamma \in \mathrm{Diff}(I)^{(2)}$. Now, fix an interval $I \in \mathcal{J}$ and let $\gamma \in \mathrm{Diff}^+(S^1)^{(2)}$ be such that $\gamma I = I$. For all $J \in \mathcal{J}$ containing $\overline{I}$, it is possible to find $\gamma^J \in \mathrm{Diff}(J)^{(2)}$ such that $\gamma^J|_I = \gamma|_I$ and $\gamma^{-1}\gamma^J \in \mathrm{Diff}(I')^{(2)}$. The latter implies that $U(\gamma^{-1}\gamma^J) \in \mathcal{A}_{(V,(\cdot|\cdot))}(I') \subseteq \mathcal{A}_{(V,(\cdot|\cdot))}(I)'$ by the argumentation above, the strong graded-locality of V and the fact that every operator $U(g)$ commutes with the operator Z. We then have that

$$\begin{aligned}\mathcal{A}_{(V,(\cdot|\cdot))}(J) &\supseteq U(\gamma^J)\mathcal{A}_{(V,(\cdot|\cdot))}(I)U(\gamma^J)^* \\ &= U(\gamma)U(\gamma^{-1}\gamma^J)\mathcal{A}_{(V,(\cdot|\cdot))}(I)U(\gamma^{-1}\gamma^J)^*U(\gamma)^* \\ &= U(\gamma)\mathcal{A}_{(V,(\cdot|\cdot))}(I)U(\gamma^{-1}\gamma^J)U(\gamma^{-1}\gamma^J)^*U(\gamma)^* \\ &= U(\gamma)\mathcal{A}_{(V,(\cdot|\cdot))}(I)U(\gamma)^* .\end{aligned} \tag{3.57}$$

By the external continuity (1.47), we have that

$$U(\gamma)\mathcal{A}_{(V,(\cdot|\cdot))}(I)U(\gamma)^* \subseteq \mathcal{A}_{(V,(\cdot|\cdot))}(I) \qquad \forall\gamma \in \mathrm{Diff}^+(S^1)^{(2)} \quad \text{s.t.} \quad \gamma I = I\,. \tag{3.58}$$

Now, let γ be an arbitrary diffeomorphism in $\mathrm{Diff}^+(S^1)^{(2)}$. We can always find $\widehat{\gamma} \in \mathrm{Möb}(S^1)^{(2)}$ such that $\widehat{\gamma}I = \gamma I$. Then, we have that

$$\begin{aligned}U(\gamma)\mathcal{A}_{(V,(\cdot|\cdot))}(I)U(\gamma)^* &= U(\widehat{\gamma})U(\widehat{\gamma}^{-1}\gamma)\mathcal{A}_{(V,(\cdot|\cdot))}(I)U(\widehat{\gamma}^{-1}\gamma)^*U(\widehat{\gamma})^* \\ &\subseteq U(\widehat{\gamma})\mathcal{A}_{(V,(\cdot|\cdot))}(I)U(\widehat{\gamma})^* \\ &= \mathcal{A}_{(V,(\cdot|\cdot))}(\widehat{\gamma}I) = \mathcal{A}_{(V,(\cdot|\cdot))}(\gamma I)\,,\end{aligned} \tag{3.59}$$

where we have used: (3.58) for the second step because $\widehat{\gamma}^{-1}\gamma I = I$; the Möbius covariance of $\mathcal{A}_{(V,(\cdot|\cdot))}$ for the second equality. By the arbitrariness of $I \in \mathcal{J}$, we can conclude that $\mathcal{A}_{(V,(\cdot|\cdot))}$ is diffeomorphism covariant.

Finally, the isomorphism between $\mathcal{A}_{(V,(\cdot|\cdot))}$ and $\mathcal{A}_{(V,(\cdot|\cdot)')}$ is given by Proposition 2.4.2. □

Definition 3.2.18. Let V be a simple strongly graded-local unitary VOSA. Then, $\mathcal{A}_V$ is the unique, up to isomorphism, irreducible graded-local conformal net arising from V as described by Theorem 3.2.17.

We also get that:

Theorem 3.2.19. *Let V be a simple strongly graded-local unitary VOSA. Then,* $\mathrm{Aut}(\mathcal{A}_V) = \mathrm{Aut}_{(\cdot|\cdot)}(V)$. *If* $\mathrm{Aut}(V)$ *is compact, then* $\mathrm{Aut}(\mathcal{A}_V) = \mathrm{Aut}(V)$.

Proof. The proof of [CKLW18, Theorem 6.9] can be used with the following prescriptions. We have to use Proposition A.1 instead of [CKLW18, Proposition A.1] when this occurs (contextually a straightforward modification of Proposition 3.2.15 is required). Moreover, we have to replace [CKLW18, Corollary 4.11 and Theorem 5.21] with our Corollary 2.4.4 and Theorem 2.4.5 respectively. □

We end the current section with the following conjecture:

Conjecture 3.2.20. *Every simple unitary VOSA V is strongly graded-local and thus it gives rise to a unique, up to isomorphism, irreducible graded-local conformal net $\mathcal{A}_V$.*

3.3 The induced graded-local conformal nets on $\mathbb{R}$ and $S^{1(2)}$

In this third part, we give an outline of the graded-local conformal nets on $\mathbb{R}$ and $S^{1(2)}$, which we obtain from $\mathcal{A}_V$ of Definition 3.2.1 as explained in Section 1.4. In particular, what we are interested in is the actions of $\mathrm{Diff}^+(S^1)$ and $\mathrm{Diff}^+(S^1)^{(2)}$ on the these nets, which give us a consistent description of the context.

First, we describe the net $\mathcal{A}_V^{\mathbb{R}}$ on $\mathbb{R}$. Recall from Section 1.1 that the intersection between $C_\chi^\infty(S^1)$ and $C^\infty(S^1)$ contains $C_c^\infty(S^1\backslash\{-1\})$. Thus, for all $a \in V$, we can define the corresponding smeared vertex operators with $\mathcal{H}^\infty$ as common invariant domain by:

$$Y(a,f)c := \sum_{n\in\frac{1}{2}\mathbb{Z}} \widehat{f}_n a_n c \qquad \forall f \in C_c^\infty(S^1\backslash\{-1\}) \ \forall c \in V\,. \tag{3.60}$$

Note that if we rewrite $f = \chi h$ with $h \in C_c^\infty(S^1\backslash\{-1\})$ in (3.60) above, then we obtain again the definition given in (3.5) for odd elements. Thus, we identify

$$\mathcal{A}_V^{\mathbb{R}}(I) \equiv W^*\left(\left\{Y(a,f) \mid a\in V\,,\ f\in C_c^\infty(S^1\backslash\{-1\})\,,\ \operatorname{supp} f\subset I\right\}\right) \qquad \forall I\in\mathcal{J}^{\mathbb{R}}\,. \tag{3.61}$$

The point is that if $\gamma \in \mathrm{Diff}^+(S^1)_I^{(\infty)}$ for any $I \in \mathcal{J}^{\mathbb{R}}$, then the diffeomorphism covariance given by $\alpha_d(\dot\gamma)$ reduces to the one of $\beta_d(\dot\gamma)$, allowing semi-integers values of d for $\beta_d(\dot\gamma)$. This is clear if we consider the identification (3.61) above and the expression of $\alpha_d(\dot\gamma)$ given in (3.43). Indeed, in this case $\epsilon_{\dot\gamma}(z) = 1$ for all $z\in S^1$.

As far as the net on $S^{1(2)}$ is concerned, we have to consider $C^\infty(S^{1(2)})$, as defined in Section 1.1, as space of test functions. Again, for all $a \in V$, we can define the following smeared vertex operators with $\mathcal{H}^\infty$ as common invariant domain by:

$$Y(a,f)c := \sum_{n\in\frac{1}{2}\mathbb{Z}} \widehat{f}_n a_n c \qquad \forall f \in C^\infty(S^{1(2)}) \ \forall c \in V\,, \tag{3.62}$$

where we are considering f as an $L^2(S^{1(2)})$-function (and provided we give a suitable definition of energy-bounds). Consequently, the net is given by

$$\mathcal{A}_V^{(2)}(I) \equiv W^*\left(\left\{Y(a,f) \mid a\in V\,,\ f\in C^\infty(S^{1(2)})\,,\ \operatorname{supp} f\subset I\right\}\right) \qquad \forall I\in\mathcal{J}^{(2)}\,. \tag{3.63}$$

Moreover, the action of $\gamma \in \mathrm{Diff}^+(S^1)^{(2)}$ is determined by

$$(\kappa_d(\gamma)f)(x) := \left[\frac{\mathrm{d}\phi_\gamma}{\mathrm{d}x}(\phi_{\gamma^{-1}}(x))\right]^{d-1} f(\phi_{\gamma^{-1}}(x)) \qquad \forall f\in C^\infty(S^{1(2)}) \ \forall x\in[-2\pi,2\pi]\,, \tag{3.64}$$

which is well-defined observing that $\frac{\mathrm{d}\phi_\gamma}{\mathrm{d}x}$ is a 2π-periodic function (see (3.38)). The above analysis becomes clearer if we recall the following facts. As explained in Section 1.1, we can identify $C^\infty(S^1)$ as the space of 2π-periodic functions $C_+^\infty(S^{1(2)})$, whereas $C_\chi^\infty(S^{1(2)})$ as the one of 2π-antiperiodic functions $C_-^\infty(S^{1(2)})$ to get the decomposition (1.21), which we rewrite here:

$$C^\infty(S^{1(2)}) = C_+^\infty(S^{1(2)}) \oplus C_-^\infty(S^{1(2)}) \cong C^\infty(S^1)\oplus C_\chi^\infty(S^1)\,.$$

In particular, the isomorphisms above are realised by

$$\begin{aligned} C^\infty(S^1)\ni f &\longmapsto f^{(2)}\in C_+^\infty(S^{1(2)})\,, \quad f^{(2)}(x) := f(e^{i\frac{x}{2}}) \\ C_\chi^\infty(S^1)\ni g=\chi h &\longmapsto g^{(2)} := \chi^{(2)}h^{(2)}\in C_-^\infty(S^{1(2)})\,, \quad \chi^{(2)}(x) := e^{i\frac{x}{2}} \end{aligned} \tag{3.65}$$

for all $x\in[-2\pi,2\pi]$. Under the above isomorphisms, definition (3.62) coincides with the one in (3.5). It is then clear how to rewrite $\mathcal{A}_V^{(2)}$ in terms of the above functions:

$$\mathcal{A}_V^{(2)}(I) \equiv W^*\left(\left\{Y(a,f)\,,\ Y(b,g) \;\middle|\; \begin{array}{l} a\in V_{\bar 0}\,,\ f\in C_+^\infty(S^{1(2)})\,,\ \operatorname{supp} f\subset I\,, \\ b\in V_{\bar 1}\,,\ g\in C_-^\infty(S^{1(2)})\,,\ \operatorname{supp} g\subset I \end{array}\right\}\right) \qquad \forall I\in\mathcal{J}^{(2)}\,. \tag{3.66}$$

Moreover, from equations 3.29 and 3.41, we deduce the following actions: for all $\gamma\in\mathrm{Diff}^+(S^1)^{(2)}$

$$\left(\beta_d^{(2)}(\gamma)f^{(2)}\right)(x) := \left[X_\gamma(\gamma^{-1}(e^{i\frac{x}{2}}))\right]^{d-1} f(\gamma^{-1}(e^{i\frac{x}{2}})) \qquad \forall f\in C^\infty(S^1) \tag{3.67}$$

$$\left(\alpha_d^{(2)}(\gamma)g^{(2)}\right)(x) := \gamma^{-1}(e^{i\frac{x}{2}})\left(\beta_d^{(2)}(\gamma)h^{(2)}\right)(x) \qquad \forall g\in C_\chi^\infty(S^1) \tag{3.68}$$

for all $x\in[-2\pi,2\pi]$, where X_γ is defined either as in (3.28) or equivalently as $X_{\dot\gamma}^{(2)}$. Note that even in this framework, the actions above define the same one and they summarise in (3.64) defined above.

Chapter 4

The action of the dilation subgroup on smeared vertex operators

This whole chapter is dedicated to the proof of Theorem 4.0.4, which is crucial in the proof of Theorem 5.2.1 and in the development of the theory in Section 7.1. As a preparation, we need some facts about the relationship between the real and the complex picture. We use the notations and the conventions as explained in Section 1.1 and we put ourselves in the setting of Chapter 3.

Comparing with the local case... We point out that Theorem 4.0.4 can be considered as the VOSA counterpart of [CKLW18, Theorem B.6]. Nevertheless, the approach used to prove the former is completely different from the one used for the latter. Indeed, a crucial role in the proof of [CKLW18, Theorem B.6] is played by the theory of standard subspaces of a Hilbert space, see [Lon08] or [Lon08b]. Instead, we pursue a complex analytic approach without relying on the theory of standard subspaces of a Hilbert space. From this point of view, this can be also considered as a new proof of [CKLW18, Theorem B.6].

We calculate the **two-point function** $(Y(a,f)\Omega|Y(b,g)\Omega)$, where a and b are quasi-primary vectors of an energy-bounded unitary VOSA V and

$$Y(a,f)\Omega := \sum_{n\in\frac{1}{2}\mathbb{Z}} \widehat{f}_n a_n\Omega\,, \qquad Y(b,g)\Omega := \sum_{n\in\frac{1}{2}\mathbb{Z}} \widehat{g}_n b_n\Omega \tag{4.1}$$

for all $f,g\in C^\infty_c(S^1\backslash\{-1\},\mathbb{R})$. Specifically, we want to prove the following fact:

Proposition 4.0.1. *Let $(V,(\cdot|\cdot))$ be an energy-bounded unitary VOSA. Let $a,b\in V$ be quasi-primary vectors and $f,g\in C^\infty_c(S^1\backslash\{-1\},\mathbb{R})$. Then, we have the following formula:*

$$(Y(a,f)\Omega|Y(b,g)\Omega) = \frac{(a|b)\delta_{d_a,d_b}}{2\pi(2d_a-1)!}\int_0^{+\infty}\overline{\widehat{f^{\mathbb{R}}}(-p)}\widehat{g^{\mathbb{R}}}(-p)p^{2d_a-1}\mathrm{d}p\,, \tag{4.2}$$

where for all $h\in C^\infty_c(S^1\backslash\{-1\},\mathbb{R})$, $h^{\mathbb{R}}\in C^\infty_c(\mathbb{R},\mathbb{R})$ is defined by

$$h^{\mathbb{R}}(x) := \left(1+\frac{x^2}{4}\right)^{d_a-1} h\left(\frac{1+\frac{i}{2}x}{1-\frac{i}{2}x}\right)\,. \tag{4.3}$$

Remark 4.0.2. In the definition of $h^{\mathbb{R}}$ in equation (4.3), we have used the **Cayley transform** $C: S^1\backslash\{-1\}\longrightarrow\mathbb{R}$, which is the diffeomorphism defined by

$$C(z) := 2i\frac{1-z}{1+z}\,, \qquad C^{-1}(x) = \frac{1+\frac{i}{2}x}{1-\frac{i}{2}x}\,.$$

Furthermore, note that $\lim_{t\to\pm\pi} C(e^{it}) = \pm\infty$ and $\lim_{x\to\pm\infty} C^{-1}(x) = -1$. For further use, let us define the following isomorphisms between the space of functions $C_c^\infty(S^1\setminus\{-1\},\mathbb{R})$ and $C_c^\infty(\mathbb{R},\mathbb{R})$: for all $d\in\frac{1}{2}\mathbb{Z}$

$$C_c^\infty(S^1\setminus\{-1\},\mathbb{R}) \longleftrightarrow C_c^\infty(\mathbb{R},\mathbb{R})$$

$$h(z) \longrightarrow h^{\mathbb{R}}(x) := \left(1+\frac{x^2}{4}\right)^{d-1} h\left(\frac{1+\frac{i}{2}x}{1-\frac{i}{2}x}\right)$$

$$h^{\mathbb{C}}(z) := \left(\frac{(1+z)^2}{(4z)}\right)^{d-1} h\left(2i\frac{1-z}{1+z}\right) \longleftarrow h(x)\,.$$

Proof of Proposition 4.0.1. The proof is given by the following sequence of equalities **[A]**-**[D]**, which we prove separately below:

$$\begin{aligned}
(Y(a,f)\Omega|Y(b,g)\Omega) &\overset{[\mathbf{A}]}{=} (a|b)\delta_{d_a,d_b}\sum_{\substack{n\in\mathbb{Z}-d_a\\ n\leq -d_a}}\binom{d_a-n-1}{-n-d_a}\overline{\widehat{f}_n}\widehat{g}_n\\
&\overset{[\mathbf{B}]}{=} (a|b)\delta_{d_a,d_b}\lim_{\epsilon\to0^+}\oint_{S^1}\oint_{S^1}\frac{z^{d_a}w^{d_a}}{(z-(1-\epsilon)w)^{2d_a}}g(w)\overline{f(z)}\frac{\mathrm{d}w}{2\pi i w}\frac{\mathrm{d}z}{2\pi i z}\\
&\overset{[\mathbf{C}]}{=} \frac{(a|b)\delta_{d_a,d_b}}{(2\pi)^2 i^{2d_a}}\lim_{\epsilon\to0^+}\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{g^{\mathbb{R}}(y)\overline{f^{\mathbb{R}}(x)}}{[(1-\frac{\epsilon}{2}-i\frac{\epsilon}{4}y)x-y(1-\frac{\epsilon}{2})-i\epsilon]^{2d_a}}\,\mathrm{d}y\,\mathrm{d}x\\
&\overset{[\mathbf{D}]}{=} \frac{(a|b)\delta_{d_a,d_b}}{2\pi(2d_a-1)!}\int_0^{+\infty}\overline{\widehat{f^{\mathbb{R}}}(-p)}\widehat{g^{\mathbb{R}}}(-p)p^{2d_a-1}\,\mathrm{d}p\,.
\end{aligned}$$

Proof of **[A]**. From [Kac01, Eq. (4.1.2)], we have that $Y(a,z)\Omega = e^{zL_{-1}}a$ for all $a\in V$, which means

$$\sum_{n\in\mathbb{Z}-d_a} a_n\Omega z^{-n-d_a} = \sum_{l=0}^{+\infty}\frac{(L_{-1})^l a}{l!}z^l = \sum_{\substack{n\in\mathbb{Z}-d_a\\ n\leq -d_a}}\frac{(L_{-1})^{-n-d_a}a}{(-n-d_a)!}z^{-n-d_a}\,.$$

This implies that for any $a\in V$, we have that

$$a^{-n-d_a} := a_n\Omega = \begin{cases}\frac{(L_{-1})^{-n-d_a}a}{(-n-d_a)!} & n\leq -d_a\\ 0 & n > -d_a\,.\end{cases}$$

It follows that

$$\begin{aligned}
(Y(a,f)\Omega|Y(b,g)\Omega) &= (\sum_{n\in\mathbb{Z}-d_a}\widehat{f}_n a_n\Omega|\sum_{m\in\mathbb{Z}-d_b}\widehat{g}_m b_m\Omega)\\
&= \sum_{n\in\mathbb{Z}-d_a}\sum_{m\in\mathbb{Z}-d_b}\overline{\widehat{f}_n}\widehat{g}_m(a_n\Omega|b_m\Omega)\\
&= \sum_{\substack{n\in\mathbb{Z}-d_a\\ n\leq -d_a}}\sum_{\substack{m\in\mathbb{Z}-d_b\\ m\leq -d_b}}\overline{\widehat{f}_n}\widehat{g}_m(a^{-n-d_a}|b^{-m-d_b})\\
&= (a|b)\delta_{d_a,d_b}\sum_{\substack{n\in\mathbb{Z}-d_a\\ n\leq -d_a}}\binom{d_a-n-1}{-n-d_a}\overline{\widehat{f}_n}\widehat{g}_n
\end{aligned}$$

where we have used (2.44) and the formula $L_1 b^{-m-d_b} = (d_b-m-1)b^{-m-d_b-1}$, which can be easily deduced through an induction argument, for the last equality.

Proof of **[B]**. For all $\epsilon>0$ and $d_a\in\frac{1}{2}\mathbb{Z}$, consider the integral

$$\oint_{S^1}\oint_{S^1}\frac{z^{d_a}w^{d_a}}{(z-(1-\epsilon)w)^{2d_a}}g(w)\overline{f(z)}\frac{\mathrm{d}w}{2\pi i w}\frac{\mathrm{d}z}{2\pi i z} \tag{4.4}$$

where, for $d_a\in\mathbb{Z}-\frac{1}{2}$, the integrand has been extended to zero to $S^1\times S^1$ thanks to the smoothness and the compact support of f and g. By the change of variables $e^{ix} := z$ and

$e^{iy} := w$ with $x, y \in [-\pi, \pi]$, we rewrite (4.4), as

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{e^{i(x-y)d_a}}{(e^{i(x-y)}-(1-\epsilon))^{2d_a}} g(e^{iy})\overline{f(e^{ix})}\,\frac{\mathrm{d}y}{2\pi}\frac{\mathrm{d}x}{2\pi}\,. \tag{4.5}$$

For all $\epsilon > 0$ and $d_a \in \frac{1}{2}\mathbb{Z}$, define the following functions in $L^2(S^1) \cong L^2([-\pi,\pi])$ (see Section 1.1)

$$h_{\epsilon,d_a}(z) := \frac{z^{d_a}}{(z-(1-\epsilon))^{2d_a}}\,.$$

Recall that we denote by $\widetilde{f}$, $\widetilde{g}$ and $\widetilde{h_{\epsilon,d_a}}$ the images of f, g and h_{ϵ,d_a} respectively under the isometric isomorphism (1.10) between $L^2(S^1)$ and $L^2([-\pi,\pi])$; $*$ is the convolution between functions in $L^2([-\pi,\pi])$. Then, we can rewrite the integral (4.5) as

$$\int_{-\pi}^{\pi} (\widetilde{g} * \widetilde{h_{\epsilon,d_a}})(x)\overline{\widetilde{f}(x)}\,\frac{\mathrm{d}x}{(2\pi)^2} = \sum_{n\in\mathbb{Z}-d_a} \frac{(\widehat{\widetilde{g} * \widetilde{h_{\epsilon,d_a}}})_n}{2\pi}\overline{\widehat{f}_n} = \sum_{n\in\mathbb{Z}-d_a} \widehat{g}_n (\widehat{h_{\epsilon,d_a}})_n \overline{\widehat{f}_n} \tag{4.6}$$

where we have used (1.11) and Parseval's Theorem (1.12) first and the property (1.6) of the convolution $*$ after. It remains to calculate the Fourier coefficients of the function h_{ϵ,d_a}, which are

$$\begin{aligned}(\widehat{h_{\epsilon,d_a}})_n &= \oint_{S^1} \frac{z^{d_a-n-1}}{(z-(1-\epsilon))^{2d_a}}\frac{\mathrm{d}z}{2\pi i} \\ &= \begin{cases} \mathrm{Res}(h_{\epsilon,d_a}z^{-n-1}, 1-\epsilon) & n < d_a \\ \mathrm{Res}(h_{\epsilon,d_a}z^{-n-1}, 0) + \mathrm{Res}(h_{\epsilon,d_a}z^{-n-1}, 1-\epsilon) & n \geq d_a \end{cases}\end{aligned} \tag{4.7}$$

where $n \in \mathbb{Z} - d_a$. With an abuse of notation, we call $h_{\epsilon,d_a}z^{-n-1}$ the integrand in the second row of (4.7). Then, we calculate that

$$\begin{aligned}\mathrm{Res}(h_{\epsilon,d_a}z^{-n-1}, 1-\epsilon) &= \frac{1}{(2d_a-1)!}\lim_{z\to 1-\epsilon}\left[\frac{\mathrm{d}^{2d_a-1}}{\mathrm{d}z^{2d_a-1}}\left(z^{d_a-n-1}\right)\right] \\ &= \begin{cases} \binom{d_a-n-1}{-n-d_a}(1-\epsilon)^{-n-d_a} & n \leq -d_a \\ 0 & -d_a < n < d_a \\ \binom{d_a+n-1}{n-d_a}(-1)^{2d_a-1}(1-\epsilon)^{-n-d_a} & n \geq d_a \end{cases}\end{aligned} \tag{4.8}$$

whereas, for $n \geq d_a$

$$\begin{aligned}\mathrm{Res}(h_{\epsilon,d_a}z^{-n-1}, 0) &= \frac{1}{(n-d_a)!}\lim_{z\to 0}\left[\frac{\mathrm{d}^{n-d_a}}{\mathrm{d}z^{n-d_a}}\left(\frac{1}{(z-(1-\epsilon))^{2d_a}}\right)\right] \\ &= (-1)^{2d_a}(1-\epsilon)^{-n-d_a}\binom{n+d_a-1}{n-d_a}\,.\end{aligned} \tag{4.9}$$

From (4.8) and (4.9), we obtain that

$$(\widehat{h_{\epsilon,d_a}})_n = \begin{cases} \binom{d_a-n-1}{-n-d_a}(1-\epsilon)^{-n-d_a} & n \leq -d_a \\ 0 & n > -d_a \end{cases} \tag{4.10}$$

Putting (4.10) in (4.6), we have that the right hand side of equality **[B]** is

$$(a|b)\delta_{d_a,d_b}\lim_{\epsilon\to 0^+}\sum_{\substack{n\in\mathbb{Z}-d_a \\ n\leq -d_a}} (1-\epsilon)^{-n-d_a}\binom{d_a-n-1}{-n-d_a}\overline{\widehat{f}_n}\widehat{g}_n\,. \tag{4.11}$$

Therefore, noting that the binomial product in (4.11) is a polynomial in the variable n with degree $2d_a - 1$ and that $\widehat{f}_n$ and $\widehat{g}_n$ are rapidly decaying, we can swap the limit with the series in (4.11) to get the proof of equality **[B]**.

Proof of **[C]**. Just the change of variables given by the Cayley transform: $x := C(z)$ and $y := C(w)$.

Proof of **[D]**. Fix $d_a \in \frac{1}{2}\mathbb{Z}$. For all $\epsilon > 0$ and all $y \in \mathbb{R}$, set the $L^2(\mathbb{R})$-function

$$q_{\alpha_\epsilon(y)}(x) := \frac{1}{[(1-\frac{\epsilon}{2}-i\frac{\epsilon}{4}y)x - y(1-\frac{\epsilon}{2}) - i\epsilon]^{2d_a}} \quad \forall x \in \mathbb{R}\,, \quad \alpha_\epsilon(y) := \frac{y(1-\frac{\epsilon}{2})+i\epsilon}{1-\frac{\epsilon}{2}-i\frac{\epsilon}{4}y}\,.$$

Then, we can use the Plancherel Theorem, so that

$$\int_{\mathbb{R}} q_{\alpha_\epsilon(y)}(x)\overline{f^{\mathbb{R}}(x)}\,\mathrm{d}x = \int_{\mathbb{R}} \widehat{q_{\alpha_\epsilon(y)}}(p)\overline{\widehat{f^{\mathbb{R}}}(p)}\,\mathrm{d}p\,. \tag{4.12}$$

To calculate $\widehat{q_{\alpha_\epsilon(y)}}$, note that $\mathrm{Im}(\alpha_\epsilon(y)) > 0$ for all $0 < \epsilon < 1$ and all $y \in \mathbb{R}$. Thus, we can use the well-known procedure by Jordan's Lemma and Residue Theorem (e.g. see [AF03, Lemma 4.2.2]), to obtain

$$\widehat{q_{\alpha_\epsilon(y)}}(p) = \begin{cases} \sqrt{2\pi}i\mathrm{Res}(q_{\alpha_\epsilon(y)}e^{-ipz}, \alpha_\epsilon(y)) & p<0 \\ 0 & p>0 \end{cases} \tag{4.13}$$

where we have that for all $p < 0$,

$$\begin{aligned}\mathrm{Res}(q_{\alpha_\epsilon(y)}(z)e^{-ipz}, \alpha_\epsilon(y)) &= \frac{\beta_\epsilon(y)}{(2d_a-1)!}\lim_{z\to\alpha_\epsilon(y)}\left[\frac{\mathrm{d}^{2d_a-1}}{\mathrm{d}z^{2d_a-1}}(e^{-ipz})\right] \\ &= \beta_\epsilon(y)\frac{(-ip)^{2d_a-1}e^{-ip\alpha_\epsilon(y)}}{(2d_a-1)!}\,.\end{aligned} \tag{4.14}$$

where $\beta_\epsilon(y) := (1-\frac{\epsilon}{2}-i\frac{\epsilon}{4}y)^{-2d_a}$. Therefore, using (4.13) and (4.14) in (4.12), we have that

$$\int_{\mathbb{R}} q_{\alpha_\epsilon(y)}(x)\overline{f^{\mathbb{R}}(x)}\,\mathrm{d}x = \frac{(-i)^{2d_a+2}\sqrt{2\pi}}{(2d_a-1)!}\int_{-\infty}^0 \beta_\epsilon(y)e^{-ip\alpha_\epsilon(y)}\overline{\widehat{f^{\mathbb{R}}}(p)}p^{2d_a-1}\,\mathrm{d}p\,.$$

Thus, the left hand side of **[D]** is equal to

$$\frac{(a|b)\delta_{d_a,d_b}(-1)^{2d_a+1}}{(2d_a-1)!(2\pi)^{3/2}}\lim_{\epsilon\to0^+}\int_{\mathbb{R}}\int_{-\infty}^0 \beta_\epsilon(y)e^{-ip\alpha_\epsilon(y)}\overline{\widehat{f^{\mathbb{R}}}(p)}p^{2d_a-1}\,\mathrm{d}p\,g^{\mathbb{R}}(y)\,\mathrm{d}y\,. \tag{4.15}$$

We can swap the limit with the integrals in (4.15) because $f^{\mathbb{R}}, g^{\mathbb{R}}$ have compact support in $\mathbb{R}$. Moreover, for all $y \in \mathbb{R}$, $\alpha_\epsilon(y) \to y$ and $\beta_\epsilon(y) \to 1$ as $\epsilon \to 0^+$. Therefore (4.15) is equal to

$$\frac{(a|b)\delta_{d_a,d_b}(-1)^{2d_a+1}}{2\pi(2d_a-1)!}\int_{-\infty}^0 \overline{\widehat{f^{\mathbb{R}}}(p)}\widehat{g^{\mathbb{R}}}(p)p^{2d_a-1}\,\mathrm{d}p \tag{4.16}$$

which completes the proof after exchanging the variable p with $-p$. □

Fix $d \in \frac{1}{2}\mathbb{Z}$ and for all $\gamma \in \mathrm{M\ddot{o}b}(S^1)$, define the continuous linear operator acting on the Fréchet space $C^\infty(S^1)$:

$$(\beta_d(\gamma)f)(z) := \left[\frac{\mathrm{d}\gamma}{\mathrm{d}z}(\gamma^{-1}(z))\frac{\gamma^{-1}(z)}{z}\right]^{d-1} f(\gamma^{-1}(z)) \qquad \forall f \in C^\infty(S^1)\,. \tag{4.17}$$

Note that every $\beta_d(\gamma)$ preserves the subspace $C_c^\infty(S^1\backslash\{-1\})$. A dilation $\delta(\lambda) \in \mathrm{M\ddot{o}b}(S^1)$ of parameter $\lambda \in \mathbb{R}$ is given by the formulae

$$\delta(\lambda)(z) := \frac{z\cosh(\lambda/2)-\sinh(\lambda/2)}{-z\sinh(\lambda/2)+\cosh(\lambda/2)}\,, \qquad \delta(\lambda)^{-1}(z) = \frac{z\cosh(\lambda/2)+\sinh(\lambda/2)}{z\sinh(\lambda/2)+\cosh(\lambda/2)}\,. \tag{4.18}$$

Note that every dilation $\delta(\lambda)$ preserves the point $-1 \in S^1$ and thus it preserves $S^1\backslash\{-1\}$ too. A straightforward calculation gives us that

$$(\beta_d(\delta(\lambda))f)(z) = \left[\frac{(z\sinh(\lambda/2)+\cosh(\lambda/2))^2}{z}\delta(\lambda)^{-1}(z)\right]^{d-1} f(\delta(\lambda)^{-1}(z)) \tag{4.19}$$

for all $\lambda \in \mathbb{R}$ and all $f \in C^\infty(S^1)$. Then, we can state the following result:

Corollary 4.0.3. *Let $(V,(\cdot|\cdot))$ be an energy-bounded unitary VOSA. Let $a,b\in V$ be quasi-primary vectors and $f,g\in C_c^\infty(S^1\backslash\{-1\},\mathbb{R})$. Then, the following holds: for all $\lambda\in\mathbb{R}$*

$$(Y(a,\beta_{d_a}(\delta(\lambda))f)\Omega|Y(b,g)\Omega)=\frac{(a|b)\delta_{d_a,d_b}}{2\pi(2d_a-1)!}\int_0^{+\infty}\overline{e^{\lambda d_a}\widehat{f^{\mathbb{R}}}(-e^\lambda p)}\widehat{g^{\mathbb{R}}}(-p)p^{2d_a-1}\mathrm{d}p\,. \tag{4.20}$$

Proof. A straightforward calculation gives that for all $\lambda\in\mathbb{R}$,

$$\left(\beta_{d_a}(\delta(\lambda))f\right)^{\mathbb{R}}(x)=e^{\lambda(d_a-1)}f^{\mathbb{R}}(e^{-\lambda}x)\qquad\forall x\in\mathbb{R}\,.$$

Then, the result follows by Proposition 4.0.1 and observing that for all $\lambda\in\mathbb{R}$,

$$\left[\widehat{\left(\beta_{d_a}(\delta(\lambda))f\right)^{\mathbb{R}}}\right](p)=e^{\lambda(d_a-1)}\left[\widehat{f^{\mathbb{R}}(e^{-\lambda}\,\cdot)}\right](p)=e^{\lambda d_a}\widehat{f^{\mathbb{R}}}(e^\lambda p)\qquad\forall p\in\mathbb{R}\,.$$

□

Now, we are ready to formulate the main theorem of this chapter:

Theorem 4.0.4. *Let a be a quasi-primary vector of a simple energy-bounded unitary VOSA V. Define the operator $K:=i\pi\overline{(L_1-L_{-1})}$ and let $f\in C_c^\infty(S^1\backslash\{-1\},\mathbb{R})$ with* supp$f\subset S^1_+$*. Then, $Y(a,f)\Omega$ is in the domain of the operator $e^{\frac{K}{2}}$ and*

$$e^{\frac{K}{2}}Y(a,f)\Omega=i^{2d_a}Y(a,f\circ j)\Omega \tag{4.21}$$

where $j(z)=z^{-1}$ for all $z\in S^1$.

Let us fix some notation which we use throughout the proof:

- a,b are quasi-primary vectors in V;
- $\mathcal{H}:=\mathcal{H}_{(V,(\cdot|\cdot))}$ is the Hilbert space completion of V with respect to its invariant scalar product $(\cdot|\cdot)$ with induced norm $\|\cdot\|$;
- we use the notations as settled in Section 1.1;
- $\|\cdot\|_1$ is the usual norm in $L^1(\mathbb{R})$;
- we use the notations $\cdot^{\mathbb{R}}$ and $\cdot^{\mathbb{C}}$ as given in Remark 4.0.2.

Let U be the positive-energy strongly continuous unitary representation of $\text{Möb}(S^1)^{(\infty)}$ on $\mathcal{H}$ induced by the conformal vector ν as in p. 47. Then, we have that

$$e^{itK}=U(\delta^{(\infty)}(-2\pi t))\qquad\forall t\in\mathbb{R}\,. \tag{4.22}$$

As explained in Section 3.3, the actions of the representations α_d and β_d defined in (3.68) and (3.67) respectively, resume to the one of (4.17) when acting on $C_c^\infty(S^1\backslash\{-1\})$. Therefore, by Proposition 3.2.12 and Corollary 4.0.3, we obtain that for all $f,g\in C_c^\infty(S^1\backslash\{-1\},\mathbb{R})$

$$\begin{aligned}(Y(b,g)\Omega|e^{itK}Y(a,f)\Omega)&=(Y(b,g)\Omega|U(\delta^{(\infty)}(-2\pi t))Y(a,f)\Omega)\\&=C_{b,a}\int_0^{+\infty}\overline{\widehat{g^{\mathbb{R}}}(-p)}e^{-2\pi td_a}\widehat{f^{\mathbb{R}}}(-e^{-2\pi t}p)p^{2d_a-1}\mathrm{d}p\end{aligned} \tag{4.23}$$

for all $t\in\mathbb{R}$ and $C_{b,a}:=\frac{(b|a)\delta_{d_a,d_b}}{2\pi(2d_a-1)!}$. Our aim is to extend (4.23) by analyticity to some domain in $\mathbb{C}$ to get the desired expression for $e^{\frac{K}{2}}Y(a,f)\Omega$ by a limit procedure.

To further simplify the notation, for all $a\in V$ set the function

$$\phi_a(\cdot)\Omega:C_c^\infty(\mathbb{R},\mathbb{R})\ni h\longmapsto\phi_a(h)\Omega:=Y(a,h^{\mathbb{C}}\circ j)\Omega\in\mathcal{H}\,. \tag{4.24}$$

Note that for all $h\in C_c^\infty(S^1\backslash\{-1\},\mathbb{R})$, $(h\circ j)^{\mathbb{R}}(t)=h^{\mathbb{R}}\circ j_{\mathbb{R}}$, where $j_{\mathbb{R}}(t)=-t$ for all $t\in\mathbb{R}$. Therefore, excluding the case $\delta_{d_a,d_b}=0$, we obtain the formula: for all $f,g\in C_c^\infty(\mathbb{R},\mathbb{R})$ and $t\in\mathbb{R}$

$$\begin{aligned}F^{d_a}_{g,f}(t)&:=C_{b,a}^{-1}(\phi_b(g)\Omega|e^{itK}\phi_a(f)\Omega)\\&=C_{b,a}^{-1}(Y(b,g^{\mathbb{C}}\circ j)\Omega|e^{itK}Y(a,f^{\mathbb{C}}\circ j)\Omega)\\&=\int_0^{+\infty}\overline{\widehat{g}(p)}e^{-2\pi td_a}\widehat{f}(e^{-2\pi t}p)p^{2d_a-1}\mathrm{d}p\,.\end{aligned} \tag{4.25}$$

Hence, the goal is to find a suitable range for $z \in \mathbb{C}$, where the following expression makes sense as analytic function

$$F^{d_a}_{g,f}(z) := \int_0^{+\infty} \overline{\widehat{g}(p)} e^{-2\pi z d_a} \widehat{f}(e^{-2\pi z} p) p^{2d_a - 1} \mathrm{d}p \tag{4.26}$$

for all $f, g \in C_c^\infty(\mathbb{R}, \mathbb{R})$ with $\mathrm{supp} f \subset (-\infty, 0)$ and

$$\widehat{f}(\zeta) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-i\zeta x} \mathrm{d}x \tag{4.27}$$

for all $\zeta \in \mathbb{C}$, which is an entire function by [Rud87, Section 19.1(b)].

Note that for all $z \in \overline{D} := \overline{\left\{ z \in \mathbb{C} \mid -\frac{1}{2} < \mathrm{Im}(z) < 0 \right\}}$, $\widehat{f}_z(p) := \widehat{f}(e^{-2\pi z} p)$ is a rapidly decreasing function of p on $(0, +\infty)$, which assures us the convergence of the integral in (4.26) for all $z \in \overline{D}$. Indeed, if $z \in \overline{D}$ and $f \in C_c^\infty(\mathbb{R}, \mathbb{R})$ with $\mathrm{supp} f \subset (-\infty, 0)$, we have that for all $\alpha, \beta \in \mathbb{Z}_{\geq 0}$

$$\begin{aligned}
\left\| \widehat{f}_z \right\|_{\alpha,\beta} &:= \sup_{p \in (0,+\infty)} \left| p^\alpha \partial_p^\beta \widehat{f}_z(p) \right| \\
&= \sup_{p \in (0,+\infty)} \left| \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) p^\alpha \partial_p^\beta e^{-ie^{-2\pi z} p x} \mathrm{d}x \right| \\
&= \sup_{p \in (0,+\infty)} \left| \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} x^\beta f(x) p^\alpha (-ie^{-2\pi z})^\beta e^{-ie^{-2\pi z} p x} \mathrm{d}x \right| \\
&= \sup_{p \in (0,+\infty)} \left| \frac{(-1)^\alpha}{\sqrt{2\pi}} \int_{\mathbb{R}} x^\beta f^{(\alpha)}(x) (-ie^{-2\pi z})^{\beta-\alpha} e^{-ie^{-2\pi z} p x} \mathrm{d}x \right| \\
&= \sup_{p \in (0,+\infty)} \left| \frac{e^{2\pi(\alpha-\beta)z}}{\sqrt{2\pi}} \int_{\mathbb{R}} x^\beta f^{(\alpha)}(x) e^{-ie^{-2\pi z} p x} \mathrm{d}x \right| \\
&= \sup_{p \in (0,+\infty)} \left| \frac{e^{2\pi(\alpha-\beta)z}}{\sqrt{2\pi}} \int_{-\infty}^0 x^\beta f^{(\alpha)}(x) e^{-ie^{-2\pi \mathrm{Re}(z)} e^{-i2\pi \mathrm{Im}(z)} p x} \mathrm{d}x \right| \\
&\leq \sup_{p \in (0,+\infty)} \frac{\left| e^{2\pi(\alpha-\beta)z} \right|}{\sqrt{2\pi}} \int_{-\infty}^0 \left| x^\beta f^{(\alpha)}(x) \right| e^{-e^{-2\pi \mathrm{Re}(z)} \sin\left(2\pi \mathrm{Im}(z)\right) p x} \mathrm{d}x \\
&\leq \frac{\left| e^{2\pi(\alpha-\beta)z} \right|}{\sqrt{2\pi}} \int_{-\infty}^0 \left| x^\beta f^{(\alpha)}(x) \right| \mathrm{d}x \\
&= \frac{\left\| x^\beta f^{(\alpha)} \right\|_1}{\sqrt{2\pi}} \left| e^{2\pi(\alpha-\beta)z} \right| < +\infty
\end{aligned} \tag{4.28}$$

where we have used the integration by parts α times for the fourth equality.

The above analysis implies part of the following result:

Proposition 4.0.5. *Let $f, g \in C_c^\infty(\mathbb{R}, \mathbb{R})$ with $\mathrm{supp} f \subset (-\infty, 0)$. Define $F^{d_a}_{g,f}(z) : \overline{D} \to \mathbb{C}$ as in (4.26), (4.27) with $\overline{D} := \overline{\left\{ z \in \mathbb{C} \mid -\frac{1}{2} < \mathrm{Im}(z) < 0 \right\}}$. Then, we have the following facts:*

(i) For all f, g, $F^{d_a}_{g,f}$ is a well-defined function, analytic on D and continuous on $\overline{D}$.

(ii) For all $z \in \overline{D}$, $F^{d_a}_{g,f}(z)$ is linear in f and antilinear in g. Moreover, for all f, g, $F^{d_a}_{g,f}(z)$ is the unique analytic extension on D of $F^{d_a}_{f,g}(t)$ where $t \in \mathbb{R}$.

Proof. Throughout the proof, f and g are always functions in $C_c^\infty(\mathbb{R}, \mathbb{R})$ with $\mathrm{supp} f \subset (-\infty, 0)$. Moreover, we set $A(z) := e^{-2\pi z d_a}$.

Proof of (i). We have already proved that $F^{d_a}_{g,f}$ is a well-defined function. To prove analyticity, first note that

$$\begin{aligned}\frac{\partial}{\partial z}\left[\widehat{f}(e^{-2\pi z}p)\right] &= -2\pi e^{-2\pi z}p\left.\frac{\partial \widehat{f}}{\partial \zeta}\right|_{\zeta=e^{-2\pi z}p} \\ &= -2\pi e^{-2\pi z}p\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f(x)(-ix)e^{-ie^{-2\pi z}px}\mathrm{d}x \\ &= -2\pi p\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f(x)(-ie^{-2\pi z}x)e^{-ie^{-2\pi z}px}\mathrm{d}x \\ &= -2\pi p\partial_p\widehat{f}_z(p)\end{aligned} \tag{4.29}$$

for all $p \in (0,+\infty)$, which is a rapidly decreasing function of p on $(0,+\infty)$ by (4.28). Therefore, for all f, g, $\frac{\partial}{\partial z}F^{d_a}_{g,f}$ exists on D, i.e., for all f, g, $F^{d_a}_{g,f}$ is analytic on D. This also implies that for all f, g, $F^{d_a}_{g,f}$ is continuous on D.

Now, we prove that for all f, g, $F^{d_a}_{g,f}$ is continuous on $\overline{D}$. Let $\zeta \in \overline{D}$ and let D^r_ζ be a closed disk of radius $r > 0$ centred in ζ. Without lose of generality, consider as range for z the bounded region $D_\zeta := D^r_\zeta \cap \overline{D}$. Our aim is to prove that for all $\epsilon > 0$ there exist $\delta > 0$ such that

$$\left|F^{d_a}_{g,f}(z) - F^{d_a}_{g,f}(\zeta)\right| = \left|\int_0^{+\infty}\overline{\widehat{g}(p)}\left(A(z)\widehat{f}_z(p) - A(\zeta)\widehat{f}_\zeta(p)\right)p^{2d_a-1}\mathrm{d}p\right| < \epsilon \tag{4.30}$$

whenever $|z-\zeta| < \delta$. On the one hand, note that there exists a $p_0 > 0$ such that

$$\left|\int_{p_0}^{+\infty}\overline{\widehat{g}(p)}\left(A(z)\widehat{f}_z(p) - A(\zeta)\widehat{f}_\zeta(p)\right)p^{2d_a-1}\mathrm{d}p\right| < \frac{\epsilon}{2} \qquad \forall z \in D_\zeta\,. \tag{4.31}$$

Indeed, using that $g \in C^\infty_c(\mathbb{R},\mathbb{R})$, we can choose a $p_0 > 0$ such that

$$\int_{p_0}^{+\infty}\left|\widehat{g}(p)\right|p^{2d_a-1}\mathrm{d}p < \left[2\|f\|_1\max_{z\in D_\zeta}\left|A(z)\right|\right]^{-1}\frac{\epsilon}{2}\,. \tag{4.32}$$

Then, the left hand side of (4.31) is bounded by

$$\begin{aligned}&\int_{p_0}^{+\infty}\left|\widehat{g}(p)\right|\int_{-\infty}^{0}\left|f(x)\right|\left|A(z)e^{-ie^{-2\pi z}px} - A(\zeta)e^{-ie^{-2\pi\zeta}px}\right|\mathrm{d}xp^{2d_a-1}\mathrm{d}p \\ &\qquad\le \int_{p_0}^{+\infty}\left|\widehat{g}(p)\right|\int_{-\infty}^{0}\left|f(x)\right|\left(\left|A(z)\right|+\left|A(\zeta)\right|\right)\mathrm{d}xp^{2d_a-1}\mathrm{d}p \\ &\qquad\le 2\|f\|_1\max_{z\in D_\zeta}\left|A(z)\right|\int_{p_0}^{+\infty}\left|\widehat{g}(p)\right|p^{2d_a-1}\mathrm{d}p < \frac{\epsilon}{2} \qquad \forall z \in D_\zeta\,.\end{aligned} \tag{4.33}$$

On the other hand, let $x_0 < 0$ be such that $\mathrm{supp} f \subset (x_0, 0)$, then

$$\begin{aligned}&\left|\int_0^{p_0}\overline{\widehat{g}(p)}\left(A(z)\widehat{f}_z(p) - A(\zeta)\widehat{f}_\zeta(p)\right)p^{2d_a-1}\mathrm{d}p\right| \\ &\le \int_0^{p_0}\left|\widehat{g}(p)\right|\int_{x_0}^{0}\left|f(x)\right|\left|A(z)e^{-ie^{-2\pi z}px} - A(\zeta)e^{-ie^{-2\pi\zeta}px}\right|\mathrm{d}xp^{2d_a-1}\mathrm{d}p \\ &\le M_\zeta(z)\int_0^{p_0}\left|\widehat{g}(p)\right|p^{2d_a-1}\mathrm{d}p\int_{\mathbb{R}}\left|f(x)\right|\mathrm{d}x \qquad \forall z \in D_\zeta\end{aligned} \tag{4.34}$$

where $M_\zeta(z)$ is the function on D_ζ defined by

$$M_\zeta(z) := \max_{\substack{x_0\le x\le 0\\ 0\le p\le p_0}}\left|A(z)e^{-ie^{-2\pi z}px} - A(\zeta)e^{-ie^{-2\pi\zeta}px}\right|\,.$$

Note that $\left|A(z)e^{-ie^{-2\pi z}px} - A(\zeta)e^{-ie^{-2\pi\zeta}px}\right|$ is a continuous function in the three variables (z, p, x) on the compact domain $D_\zeta \times [0, p_0] \times [x_0, 0]$, which assures us that $M_\zeta(z)$ is non-negative, well-defined and continuous on D_ζ. Then, $M_\zeta(\zeta) = 0$ and by continuity, there exists $\delta > 0$ such that

$$M_\zeta(z) < \left[\|f\|_1\int_0^{p_0}\left|\widehat{g}(p)\right|p^{2d_a-1}\mathrm{d}p\right]^{-1}\frac{\epsilon}{2} \tag{4.35}$$

for all $z \in D_\zeta$ such that $|z-\zeta| < \delta$. Thus using equation (4.35) in (4.34), we obtain that

$$\left| \int_0^{p_0} \overline{\widehat{g}(p)} \left(A(z)\widehat{f}_z(p) - A(\zeta)\widehat{f}_\zeta(p) \right) p^{2d_a - 1} \mathrm{d}p \right| < \frac{\epsilon}{2} \tag{4.36}$$

for all $z \in D_\zeta$ such that $|z-\zeta| < \delta$. Therefore, the continuity of $F^{d_a}_{g,f}$ is proved estimating (4.30) by (4.31) and (4.36).

Proof of (ii). For all $z \in \overline{D}$, $F^{d_a}_{g,f}(z)$ is linear in f and antilinear in g by the properties of the scalar product. Then, it remains to show that $F^{d_a}_{g,f}(z)$ is the unique analytic extension on D of $F^{d_a}_{g,f}(t)$. Let $G(z)$ an analytic extension on D and set $H(z) := F^{d_a}_{g,f}(z) - G(z)$. $H(t) = 0$ for all $t \in \mathbb{R}$, then by the Schwarz reflection principle (cf. [SW64, p. 76]), $H(z)$ extends to $D \cup \mathbb{R} \cup D^{\mathrm{conj}}$, where D^{conj} are the set of complex conjugates of elements of D. Hence, $H(z) = 0$ in D by the identity theorem (cf. [AF03, p. 122]), i.e. for all f, g, $F^{d_a}_{g,f}(z)$ is the unique analytic extension on D of $F^{d_a}_{g,f}(t)$.

□

A further step in the proof of Theorem 4.0.4 is to use the function $F^{d_a}_{g,f}(z)$ to define an antilinear functional on $\mathcal{H}$. To this end, define the Hilbert spaces

$$\mathcal{H}_a := \overline{\left\{ \phi_a(g)\Omega \mid g \in C^\infty_c(\mathbb{R}, \mathbb{R}) \right\}}^{\|\cdot\|} \subset \mathcal{H}\,, \qquad \widehat{\mathcal{H}}_a := L^2((0,+\infty), p^{2d_a-1}\mathrm{d}p) \tag{4.37}$$

and denote by $\|\cdot\|_{2,a}$ the norm of $\widehat{\mathcal{H}}_a$. Note that $|C_{a,a}|^{-1} \|\phi_a(g)\Omega\| = \|\widehat{g}\|_{2,a}$ by (4.25). In particular, if $\{\phi_a(g_n)\Omega\}_{n\in\mathbb{Z}_{\geq 0}}$ is a convergent sequence in $\mathcal{H}$, then $\{\widehat{g_n}\}_{n\in\mathbb{Z}_{\geq 0}}$ will be a convergent sequence in $\widehat{\mathcal{H}}_a$. Let $\varphi \in \mathcal{H}_a$ and pick any sequence $\{\widehat{g_n}\}_{n\in\mathbb{Z}_{\geq 0}}$, convergent to some $g_\varphi \in \widehat{\mathcal{H}}_a$, such that $\varphi = \lim_n \phi_a(g_n)\Omega$. Define

$$F^{d_a}_{\varphi,f}(z) := \int_0^{+\infty} \overline{g_\varphi(p)} e^{-2\pi z d_a} \widehat{f}(e^{-2\pi z}p) p^{2d_a-1} \mathrm{d}p \tag{4.38}$$

where $f \in C^\infty_c(\mathbb{R}, \mathbb{R})$ with $\mathrm{supp} f \subset (-\infty, 0)$ and $z \in \overline{D}$. Then, the content of the following result is an extension of the properties proved in Proposition 4.0.5 to $F^{d_a}_{\varphi,f}(z)$.

Proposition 4.0.6. *Let $f \in C^\infty_c(\mathbb{R}, \mathbb{R})$ with $\mathrm{supp} f \subset (-\infty, 0)$, $z \in \overline{D}$ and $\varphi \in \mathcal{H}_a$ as defined in (4.37). Define $F^{d_a}_{\varphi,f}(z)$ as in (4.38). Then, the following properties hold:*

(i) For all f, φ, $F^{d_a}_{\varphi,f}$ is a well-defined function, analytic on D and continuous on $\overline{D}$.

(ii) For all $z \in \overline{D}$, $F^{d_a}_{\varphi,f}(z)$ is linear in f and antilinear in φ. Moreover, $F^{d_a}_{\varphi,f}(z) = F^{d_a}_{g,f}(z)$ for all $\varphi = \phi_a(g)\Omega$ with $g \in C^\infty_c(\mathbb{R}, \mathbb{R})$.

Proof. Throughout the proof, $f \in C^\infty_c(\mathbb{R}, \mathbb{R})$ with $\mathrm{supp} f \subset (-\infty, 0)$, $z \in \overline{D}$ and $\varphi \in \mathcal{H}_a$, unless it is differently specified. Moreover, we set $A(z) := e^{-2\pi z d_a}$.

Proof of (i). Let $\{\widehat{g_n}\}_{n\in\mathbb{Z}_{\geq 0}}$ be a sequence convergent to some $g_\varphi \in \widehat{\mathcal{H}}_a$ and such that $\varphi = \lim_n \phi_a(g_n)\Omega$. To prove that $F^{d_a}_{\varphi,f}(z)$ is a well-defined function, we have to show that the integral in (4.38) exists and that for all $\varphi \in \mathcal{H}_a$, g_φ is independent of the choice of the sequence $\{\widehat{g_n}\}_{n\in\mathbb{Z}_{\geq 0}}$. Equation (4.28) says us that for all f, φ, z, $F^{d_a}_{\varphi,f}(z)$ exists. To prove the independence from the sequence, let $\left\{\widehat{l_n}\right\}_{n\in\mathbb{Z}_{\geq 0}}$ be another sequence convergent to $l_\varphi \in \widehat{\mathcal{H}}_a$ and such that $\varphi = \lim_n \phi_a(l_n)\Omega$. Then, for all $\epsilon > 0$, there exists $N > 0$ such that

$$\begin{aligned}
\|g_\varphi - l_\varphi\|_{2,a} &\leq \|g_\varphi - \widehat{g_n}\|_{2,a} + \left\|\widehat{g_n - l_n}\right\|_{2,a} + \left\|\widehat{l_n} - l_\varphi\right\|_{2,a} \\
&= \|g_\varphi - \widehat{g_n}\|_{2,a} + |C_{a,a}|^{-1} \|\phi_a(g_n - l_n)\Omega\| + \left\|\widehat{l_n} - l_\varphi\right\|_{2,a} \\
&= \|g_\varphi - \widehat{g_n}\|_{2,a} + |C_{a,a}|^{-1} \|\phi_a(g_n)\Omega - \phi_a(l_n)\Omega\| + \left\|\widehat{l_n} - l_\varphi\right\|_{2,a} \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \qquad \forall n \geq N
\end{aligned} \tag{4.39}$$

where we have used (4.25) in the second raw. It follows that $F^{d_a}_{\varphi,f}(z)$ is well-defined.

Equation (4.29) implies that for all f, φ, $F^{d_a}_{\varphi,f}$ is analytic and therefore continuous on D.

To prove that for all f, φ, $F^{d_a}_{\varphi,f}$ is continuous on $\overline{D}$, let $\zeta \in \overline{D}$ and without loss of generality, consider $D_\zeta := D^r_\zeta \cap \overline{D}$ as range for z, where D^r_ζ is a closed disk of radius $r > 0$ centred in ζ. Then, our aim is to prove that for all $\epsilon > 0$ there exist $\delta > 0$ such that

$$\left| F^{d_a}_{\varphi,f}(z) - F^{d_a}_{\varphi,f}(\zeta) \right| < \epsilon \tag{4.40}$$

whenever $|z - \zeta| < \delta$. We have that for all f

$$\begin{aligned} \sup_{z \in D_\zeta} \left\| \widehat{f}_z \right\|_{2,a} &= \sup_{z \in D_\zeta} \left(\int_0^{+\infty} \left| \widehat{f}_z(p) \right|^2 p^{2d_a - 1} \mathrm{d}p \right)^{\frac{1}{2}} \\ &\leq \sup_{z \in D_\zeta} \frac{\left\| f^{(\alpha)} \right\|_1}{\sqrt{2\pi}} \left| e^{2\pi\alpha z} \right| \left(\int_0^{+\infty} p^{2d_a - 1 - 2\alpha} \mathrm{d}p \right)^{\frac{1}{2}} < +\infty \end{aligned} \tag{4.41}$$

where we have used (4.28) with $\beta = 0$ and a positive integer $\alpha > d_a$. Therefore, using Hölder's inequality, there exists $N > 0$ such that

$$\left| F^{d_a}_{g_n,f}(z) - F^{d_a}_{\varphi,f}(z) \right| \leq \left\| \widehat{g_n} - g_\varphi \right\|_{2,a} |A(z)| \left\| \widehat{f}_z \right\|_{2,a} < \frac{\epsilon}{3} \qquad \forall z \in D_\zeta \tag{4.42}$$

for all $n > N$. Then, choosing $n > N$, we can conclude that

$$\begin{aligned} \left| F^{d_a}_{\varphi,f}(z) - F^{d_a}_{\varphi,f}(\zeta) \right| &\leq \left| F^{d_a}_{\varphi,f}(z) - F^{d_a}_{g_n,f}(z) \right| + \left| F^{d_a}_{g_n,f}(z) - F^{d_a}_{g_n,f}(\zeta) \right| + \left| F^{d_a}_{g_n,f}(\zeta) - F^{d_a}_{\varphi,f}(\zeta) \right| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \end{aligned} \tag{4.43}$$

whenever $|z - \zeta| < \delta$ with $\delta > 0$ given by the continuity (4.30).

Proof of (ii). This is a straightforward consequence of the definition of $F^{d_a}_{\varphi,f}(z)$ and of part (i) above. □

Finally, we can prove Theorem 4.0.4.

Proof of Theorem 4.0.4. Our argument is inspired by [Ara76, Lemma 3.5]. Let $a \in V$ be a quasi-primary vector with conformal weight d_a. Let $z \in \overline{D}$, $\varphi \in \mathcal{H}_a$ as defined in (4.37) and $f \in C^\infty_c(\mathbb{R}, \mathbb{R})$ with $\operatorname{supp} f \subset (-\infty, 0)$. Then, consider $F^{d_a}_{\varphi,f}(z)$ as in (4.38), which is well-defined thanks to Proposition 4.0.6.

First, note that $\mathcal{H} = \mathcal{H}_a \oplus (\mathcal{H}_a)^\perp$ with respect to $(\cdot|\cdot)$ because $\mathcal{H}_a$ is a closed subspace of $\mathcal{H}$. Then, we set $F^{d_a}_{\varphi^\perp,f}(z) = 0$ for all $\varphi^\perp \in (\mathcal{H}_a)^\perp$ and extend it to the whole $\mathcal{H}$ by antilinearity.

Second, consider the group of unitary operators on $\mathcal{H}$ given by e^{itK} for all $t \in \mathbb{R}$ and looking at their action on $\phi_a(f)\Omega$, note that it preserve $\mathcal{H}_a$ and consequently its orthogonal complement $(\mathcal{H}_a)^\perp$ too. From [BR02, Corollary 2.5.23], it follows that there exist $D(e^{itK})^a_{\mathrm{ea}} \subset \mathcal{H}_a$ and $D(e^{itK})^{a,\perp}_{\mathrm{ea}} \subset (\mathcal{H}_a)^\perp$, two norm-dense subsets of entire analytic elements for the one-parameter group of isometries $t \mapsto e^{itK}$. Therefore, $D(K)_{\mathrm{ea}} := D(e^{itK})^a_{\mathrm{ea}} \oplus D(e^{itK})^{a,\perp}_{\mathrm{ea}}$ is a norm-dense subset of $\mathcal{H}$.

Third, let $\varphi = \psi + \psi^\perp \in D(K)_{\mathrm{ea}}$ and $\left\{\phi_a(g_n)\Omega\right\}_{n \in \mathbb{Z}_{\geq 0}}$ be a sequence in $D(e^{itK})^a_{\mathrm{ea}}$ such that $\psi = \lim_n \phi_a(g_n)\Omega$. Then, for all $t \in \mathbb{R}$, we have that

$$\begin{aligned} (e^{-itK}\varphi|\phi_a(f)\Omega) &= (e^{-itK}\psi|\phi_a(f)\Omega) + (e^{-itK}\psi^\perp|\phi_a(f)\Omega) \\ &= \lim_{n \to +\infty} (e^{-itK}\phi_a(g_n)\Omega|\phi_a(f)\Omega) \\ &= \lim_{n \to +\infty} C_{a,a} F^{d_a}_{\phi_a(g_n)\Omega,f}(t) \\ &= C_{a,a} \left(F^{d_a}_{\psi,f}(t) + F^{d_a}_{\psi^\perp,f}(t) \right) \\ &= C_{a,a} F^{d_a}_{\varphi,f}(t) \end{aligned} \tag{4.44}$$

where we have used (4.25) (K is self-adjoint), the orthogonality condition and the continuity given by (4.42). Thus, combine the Schwarz reflection principle and the identity theorem as in the proof of part (ii) of Proposition 4.0.5, we have for all f and all $\varphi \in D(K)_{\mathrm{ea}}$ that

$$(e^{-i\bar{z}K}\varphi|\phi_a(f)\Omega) = C_{a,a}F^{d_a}_{\varphi,f}(z) \qquad \forall z \in D \cup \mathbb{R}\,. \tag{4.45}$$

Now, using the analytic continuation to the boundary of (4.45), we obtain that

$$\begin{aligned}
(e^{\frac{K}{2}}\varphi|\phi_a(f)\Omega) &= C_{a,a}F_{\varphi,f}(-i/2) \\
&= \lim_{n\to+\infty} C_{a,a}F^{d_a}_{\phi_a(g_n)\Omega,f}(-i/2) \\
&= \lim_{n\to+\infty} C_{a,a}\int_0^{+\infty} \overline{\widehat{g_n}(p)}e^{i\pi d_a}\widehat{f}(-p)p^{2d_a-1}\mathrm{d}p \\
&= \lim_{n\to+\infty} (\phi_a(g_n)\Omega|e^{i\pi d_a}Y(a,f^{\mathbb{C}})\Omega) \\
&= (\varphi|e^{i\pi d_a}Y(a,f^{\mathbb{C}})\Omega) \qquad \forall \varphi \in D(K)_{\mathrm{ea}}
\end{aligned} \tag{4.46}$$

where we have used again the orthogonality condition and the continuity given by (4.42). Note that $e^{\frac{K}{2}}$ is self-adjoint because of the functional calculus of the self-adjoint operator K. Then, (4.46) implies, by definition of adjoint operator, that $\phi_a(f)\Omega$ is in the domain of $e^{\frac{K}{2}}$ and that

$$e^{\frac{K}{2}}Y(a,f^{\mathbb{C}}\circ j)\Omega = e^{\frac{K}{2}}\phi_a(f)\Omega = e^{i\pi d_a}Y(a,f^{\mathbb{C}})\Omega = i^{2d_a}Y(a,f^{\mathbb{C}})\Omega$$

for all $f \in C^\infty_c(\mathbb{R},\mathbb{R})$ with $\mathrm{supp} f \subset (-\infty,0)$. This is equivalent to

$$e^{\frac{K}{2}}Y(a,f\circ j)\Omega = i^{2d_a}Y(a,f)\Omega$$

for all $f \in C^\infty_c(S^1\backslash\{-1\},\mathbb{R})$ with $\mathrm{supp} f \subset S^1_-$, which implies the desired result. □

Chapter 5

Further results on the correspondence

In Section 5.1, we prove that there is a one-to-one correspondence between unitary subalgebras, see Section 2.6, of a simple strongly graded-local unitary VOSA V and covariant subnets, see Section 1.3, of the associated irreducible graded-local conformal net $\mathcal{A}_V$, see Definition 3.2.1. As a corollary, we have that also the coset and the fixed point constructions are preserved. In Section 5.2, we show that for a simple energy-bounded unitary VOSA, it is sufficient to test strong graded-locality on a generating set of quasi-primary vectors to conclude the strong graded-locality of the whole VOSA. Among other things, this result implies that the graded tensor products are carried over by the correspondence.

Comparing with the local case... The methods used to prove the results from Section 5.1 and Section 5.2 are close to the ones used in the local case [CKLW18, Chapter 7 and Chapter 8]. Nevertheless, extra difficulties are caused by the fact that the operator Z plays a crucial role, entering most of the calculations. Just as an example, in *Step 4* of the proof of Theorem 5.2.1, we prove that the PCT operator θ equals ZJ, where J is the modular conjugation associated to the couple $(\mathcal{A}_V(S^1_+), \Omega)$.

We set the following notation: for every homogeneous vector $a \in V$, define

$$\begin{aligned} C^\infty_a(S^1) &:= \begin{cases} C^\infty(S^1)\,, & d_a \in \mathbb{Z} \\ C^\infty_\chi(S^1)\,, & d_a \in \mathbb{Z} - \frac{1}{2} \end{cases} \qquad \iota_{d_a} := \begin{cases} \beta_{d_a}\,, & d_a \in \mathbb{Z} \\ \alpha_{d_a}\,, & d_a \in \mathbb{Z} - \frac{1}{2} \end{cases} \\ C^\infty_a(S^1, \mathbb{R}) &:= \begin{cases} C^\infty(S^1, \mathbb{R})\,, & d_a \in \mathbb{Z} \\ C^\infty_\chi(S^1, \mathbb{R})\,, & d_a \in \mathbb{Z} - \frac{1}{2} \end{cases} \end{aligned} \tag{5.1}$$

where β_{d_a} and α_{d_a} as defined in (3.31) and (3.40) respectively.

5.1 Unitary subalgebras and covariant subnets

The main result of this section is the following.

Theorem 5.1.1. *Let V be a simple strongly graded-local unitary VOSA. If W is a unitary subalgebra of V, then the simple unitary VOSA W is strongly graded-local and $\mathcal{A}_W$ embeds canonically as a covariant subnet of $\mathcal{A}_V$. Conversely, if $\mathcal{B}$ is a Möbius covariant subnet of $\mathcal{A}_V$, then $W := \mathcal{H}_\mathcal{B} \cap V$ is a unitary subalgebra of V such that $\mathcal{A}_W = \mathcal{B}$. In other terms, the map $W \mapsto \mathcal{A}_W$ gives a one-to-one correspondence between unitary subalgebras of V and covariant subnets of $\mathcal{A}_V$.*

Proof. The first and the second statement are obtained by adapting the proofs of [CKLW18, Theorem 7.1 and Theorem 7.4] respectively. We use the notation given in (5.1). As usual, $(\cdot|\cdot)$ is the normalized invariant scalar product on V and $\mathcal{H}$ is the Hilbert space of $\mathcal{A}_V$.

Suppose that W is a unitary subalgebra of V. By proposition 2.6.4, W with the restriction of $(\cdot|\cdot)$ is a simple unitary VOSA. Moreover, by definition of vertex subalgebra, for all $a \in W$, the vertex operator $Y_W(a,z)$ of W coincides with the restriction of the vertex operator $Y(a,z)$ of V and thus W is energy bounded.

If e_W is the orthogonal projection onto the Hilbert space closure $\mathcal{H}_W$ of W, then $W = e_W V = \mathcal{H}_W \cap V$. Hence, for all $b \in V$ and all $f \in C_a^\infty(S^1)$, we have that

$$Y(a,f)e_W b \in \mathcal{H}_W\,, \qquad Y(a,f)^* e_W b \in \mathcal{H}_W\,.$$

Therefore,

$$\begin{aligned}(b|e_W Y(a,f)c) &= (Y(a,f)^* e_W b|c) = (Y(a,f)^* e_W b|e_W c)\\ &= (e_W b|Y(a,f)e_W c) = (b|Y(a,f)e_W c) \qquad \forall a \in W \ \ \forall b,c \in V\,.\end{aligned}$$

Recall that V is a core for every smeared vertex operator $Y(a,f)$ and thus the calculation above implies that $Y(a,f)$ with $a \in W$ and $f \in C_a^\infty(S^1)$ commutes with e_W. By Proposition 2.6.2, L_{-1}, L_0 and L_1 preserve W and thus we can prove that they commute with e_W, just proceeding as above. This implies that $e_W \in U(\text{Möb}(S^1)^{(\infty)})'$. Therefore, the subnet $\mathcal{B}_W$ of $\mathcal{A}_V$ defined by

$$\mathcal{B}_W(I) := \mathcal{A}_V(I) \cap \{e_W\}' \qquad \forall I \in \mathcal{J}$$

is Möbius covariant. Moreover, for all $I \in \mathcal{J}$, every smeared vertex operator $Y(a,f)$ with $a \in W$ and $f \in C_a^\infty(S^1)$ with $\text{supp}f \subset I$ is affiliated with $\mathcal{B}_W(I)$. Consequently, $\mathcal{H}_{\mathcal{B}_W} = \mathcal{H}_W$ and thus $\mathcal{B}_W$ becomes irreducible when restricted to $\mathcal{H}_W$ by Proposition 1.3.1. By twisted Haag duality (1.45), we have that

$$Z(\mathcal{B}_W(I)_{e_W})' Z^* = Z^* (\mathcal{B}_W(I)_{e_W})' Z = \mathcal{B}_W(I')_{e_W} \qquad \forall I \in \mathcal{J}\,.$$

Moreover,

$$\mathcal{D}(Y_W(a,f)) = e_W \mathcal{D}(Y(a,f)) = \mathcal{D}(Y(a,f)) \cap \mathcal{H}_W \qquad \forall a \in W \ \ \forall f \in C_a^\infty(S^1)$$

because every $Y(a,f)$ commutes with e_W and it coincides with $Y_W(a,f)$ on W. Accordingly, for every $a \in W$ and every $I \in \mathcal{J}$, $Y_W(a,f)$ must be affiliated with $Z\mathcal{B}_W(I')'_{e_W} Z^* = \mathcal{B}_W(I)_{e_W}$ whenever $f \in C_a^\infty(S^1)$ has $\text{supp}f \subset I$. Then, the following von Neumann algebras on $\mathcal{H}_W$

$$\mathcal{A}_W(I) := W^* \left(\left\{ Y_W(a,f) \mid a \in W\,, \ \ f \in C_a^\infty(S^1) \ \ \text{supp}f \subset I \right\} \right) \qquad \forall I \in \mathcal{J}$$

are contained in $\mathcal{B}_W(I)_{e_W}$ for all $I \in \mathcal{J}$ respectively. This means that W defines a simple strongly graded-local unitary VOSA because $\mathcal{B}_W(\cdot)_{e_W}$ is a graded-local Möbius covariant net. Then, by Theorem 3.2.17, $\mathcal{A}_W$ is an irreducible graded-local conformal net and by twisted Haag duality (1.45),

$$\mathcal{B}_W(I')_{e_W} = Z\mathcal{B}_W(I)'_{e_W} Z^* \subseteq Z\mathcal{A}_W(I)' Z^* = \mathcal{A}_W(I') \qquad \forall I \in \mathcal{J}$$

which implies that $\mathcal{A}_W(I) = \mathcal{B}_W(I)_{e_W}$ for all $I \in \mathcal{J}$, so concluding the proof of the first part.

Conversely, suppose that $\mathcal{B}$ is a Möbius covariant subnet of $\mathcal{A}_V$ and set $W := \mathcal{H}_{\mathcal{B}} \cap V$. Both $\Omega \in W$ and $L_n W \subseteq W$ for all $n \in \{-1,0,1\}$ because $\mathcal{H}_{\mathcal{B}}$ is globally invariant for the unitary representation U of $\text{Möb}(S^1)^{(\infty)}$ on $\mathcal{H}$. Now, if $a \in W$, then $a_{(-1)}\Omega = a \in W$ and

$$a_{(-n-1)}\Omega = \frac{1}{n}[L_{-1}, a_{(-n)}]\Omega = \frac{1}{n} L_{-1} a_{(-n)}\Omega \qquad \forall n \in \mathbb{Z}$$

thanks to the commutation relation (2.27). Using the inductive step, we can deduce that $a_{(n)}\Omega$ is in W for all $n \in \mathbb{Z}$. It follows that $Y(a,f)\Omega \in \mathcal{H}_{\mathcal{B}}$ for all $a \in W$ and all $f \in C_a^\infty(S^1)$. Let $e_{\mathcal{B}}$ be the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}_{\mathcal{B}}$ and for any $I \in \mathcal{J}$, let $\epsilon_{I'}$ be the unique vacuum-preserving normal conditional expectation of $\mathcal{A}_V(I')$ onto $\mathcal{B}(I')$, see e.g. [Lon03, Lemma 13]. If $a \in W$, $f \in C_a^\infty(S^1)$ with $\text{supp}f \subset I$ and $A \in \mathcal{A}_V(I')$, we have that

$$\begin{aligned}Y(a,f)e_{\mathcal{B}} ZAZ^*\Omega &= Y(a,f)Ze_{\mathcal{B}} AZ^*\Omega = Y(a,f)Z\epsilon_{I'}(A)Z^*\Omega\\ &= Z\epsilon_{I'}(A)Z^* Y(a,f)\Omega = Ze_{\mathcal{B}} AZ^* Y(a,f)\Omega\\ &= e_{\mathcal{B}} ZAZ^* Y(a,f)\Omega = e_{\mathcal{B}} Y(a,f) ZAZ^*\Omega\end{aligned} \tag{5.2}$$

where we have used that $\epsilon_{I'}(A)e_{\mathcal{B}} = e_{\mathcal{B}} A e_{\mathcal{B}}$ and that $\epsilon_{I'}(A) \in \mathcal{B}(I')$. By a slight modification of [CKLW18, Proposition 7.3], $Z\mathcal{A}_V(I')Z^*\Omega$ is a core for every $Y(a,f)$ with $a \in W$ and $f \in C_a^\infty(S^1)$

such that supp$f \subset I$, so that every such $Y(a,f)$ commutes with $e_{\mathcal{B}}$. Consequently, $Y(a,f)$ and $Y(a,f)^*$ with supp$f \subset I$ are affiliated with $\mathcal{B}(I) = \mathcal{A}_V(I) \cap \{e_{\mathcal{B}}\}'$. If $f \in C^\infty_a(S^1)$ (that is without any restriction on the support), we can find $g, h \in C^\infty_a(S^1)$ such that $f = g + h$, supp$g \subset I_g$ and supp$h \subset I_h$ for some $I_g, I_h \in \mathcal{J}$. Then, $Y(a,f)c = Y(a,g)c + Y(a,h)c$ for all $c \in \mathcal{H}^\infty$ thanks to the continuity (3.9). Hence, $Y(a,f)$ commutes with $e_{\mathcal{B}}$ for all $a \in W$ and all $f \in C^\infty_a(S^1)$ because $\mathcal{H}^\infty$ is a core for all smeared vertex operators. Now, proceeding similarly to the proof of Proposition 3.2.2, we can prove that $a_n b \in W$ and $a_n^+ b \in W$ for all $a, b \in W$. Recalling that $L_0 W \subseteq W$, we conclude that W is a unitary subalgebra of V. It follows also that $\mathcal{B} = \mathcal{A}_W$, which concludes the proof. □

A first corollary is:

Proposition 5.1.2. *Let V be a simple strongly graded-local unitary VOSA. If G is a closed subgroup of* $\mathrm{Aut}_{(\cdot|\cdot)}(V) = \mathrm{Aut}(\mathcal{A}_V)$*, then* $\mathcal{A}_{V^G} = \mathcal{A}_V^G$.

Proof. We can implement the same proof used for [CKLW18, Proposition 7.6], which we rewrite here in the following with the notation as in (5.1). For any $g \in G$, $gY(a,f)g^{-1} = Y(a,f)$ for all $a \in V^G$ and all $f \in C^\infty_a(S^1)$. Then, $g \in \mathcal{A}_{V^G}(I)$ for all $I \in \mathcal{J}$ and thus $\mathcal{A}_{V^G} \subseteq \mathcal{A}_V^G$. Conversely, there exists a unitary subalgebra W of V by Theorem 5.1.1, such that $\mathcal{A}_W = \mathcal{A}_V^G$. This implies that $W \subseteq V^G$ so that $\mathcal{A}_V^G = \mathcal{A}_W \subseteq A_{V^G}$, concluding the proof. □

Theorem 5.1.1 allows us to prove also that the coset constructions in the two settings are preserved by the correspondence, that is:

Proposition 5.1.3. *Let V be a simple strongly graded-local unitary VOSA and let W be a unitary subalgebra of V. Then,* $\mathcal{A}_{W^c} = \mathcal{A}_W^c$.

Proof. It is not difficult to check that the proof of [CKLW18, Proposition 7.8] works even in the super case with minor changes. We present a detailed version here below with the notation specified in (5.1).

First note that $\mathcal{A}_W$ and $\mathcal{A}_{W^c}$ are covariant subnets of $\mathcal{A}_V$ thanks to Proposition 2.6.5 and Theorem 5.1.1. By Proposition 2.6.4, we have two representations of the Virasoro algebra on V via the operators L_n^W and $L_n^{W^c}$ with $n \in \mathbb{Z}$. Thus, we can extend them to two representations U_W and U_{W^c} respectively on $\mathcal{H}$ of $\mathrm{Diff}^+(S^1)^{(\infty)}$ in the usual way, cf. p. 47 and references therein. In particular, we have that $U_X(\exp^{(\infty)}(tf)) = e^{itY(\nu^X,f)}$ for all $f \in C^\infty(S^1,\mathbb{R})$ with X equal to W or W^c. Now, we give some properties of these representations. For every $f \in C^\infty(S^1,\mathbb{R})$, both $Y(\nu^W,f)$ and $Y(\nu^{W^c},f)$ satisfy the linear energy bounds ($k=1$ in (3.7)). Moreover, by Proposition 2.6.5, the vertex operators $Y(\nu^W,z)$ and $Y(\nu^{W^c},z)$ commute. Thanks to these last two properties, we can apply the argument given in [BS90, Section 2] (based on [DF77, Theorem 3.1], see also [GJ87, Theorem 19.4.4]) to conclude that also $U_W(\gamma)$ and $U_{W^c}(\gamma)$ commute for all $\gamma \in \mathrm{Diff}^+(S^1)^{(\infty)}$. Moreover, we have that

$$U(\gamma) = U_W(\gamma)U_{W^c}(\gamma) \qquad \forall \gamma \in \mathrm{Diff}^+(S^1)^{(\infty)} . \tag{5.3}$$

Let $Y(a,g)$ be a smeared vertex operator with $a \in W$ and $g \in C^\infty_a(S^1)$, $f \in C^\infty(S^1,\mathbb{R})$ and $c \in \mathcal{H}^\infty$. For every $t \in \mathbb{R}$, define the vector in $\mathcal{H}^\infty$

$$c(t) := Y(a,g)U_{W^c}(\exp^{(\infty)}(tf))c = Y(a,g)e^{itY(\nu^{W^c},f)}c .$$

Due to Proposition 2.6.5, $[Y(\nu^{W^c},f), Y(a,g)] = 0$ and thus, proceeding as in the proof of Proposition 3.2.12, it is not difficult to prove that $t \mapsto c(t)$ is differentiable on $\mathcal{H}$ and satisfies the Cauchy problem on $\mathcal{H}$:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}c(t) = iY(\nu^{W^c},f)c(t) \\ c(0) = Y(a,g)c \end{cases}$$

which has as unique solution $e^{itY(\nu^{W^c},f)}Y(a,g)c$. Therefore, it must be

$$Y(a,g)U_{W^c}(\exp^{(\infty)}(tf))c = c(t) = e^{itY(\nu^{W^c},f)}Y(a,g)c = U_{W^c}(\exp^{(\infty)}(tf))Y(a,g)c .$$

By the arbitrariness of the choices made above, we have that $Y(a,g)$ and $U_{W^c}(\exp^{(\infty)}(tf))$ commute for all $a \in W$, all $f \in C^\infty(S^1,\mathbb{R})$, all $g \in C^\infty_a(S^1)$ and all $t \in \mathbb{R}$, that is, $U_{W^c}(\exp^{(\infty)}(tf)) \in \mathcal{A}_W(I)'$ for all $I \in \mathcal{J}$. By (5.3), what above implies that

$$U(\exp^{(2)}(tf))AU(\exp^{(2)}(tf))^* = U_W(\exp^{(\infty)}(tf))AU_W(\exp^{(\infty)}(tf))^*$$

for all $A \in \mathcal{A}_W(I)$ and all $I \in \mathcal{J}$. Similarly, we can prove that $U_W(\exp^{(\infty)}(tf)) \in \mathcal{A}_{W^c}(I)'$ for all $t \in \mathbb{R}$, all $f \in C^\infty(S^1, \mathbb{R})$ and all $I \in \mathcal{J}$. Pick $I \in \mathcal{J}$ and $A \in \mathcal{A}_{W^c}(I)$. For any $I_1 \in \mathcal{J}$, let $B \in \mathcal{A}_W(I_1)$ and choose a composition of exponentials of vector fields γ such that $\dot{\gamma}I_1 = I'$. Then, we have that

$$\begin{aligned} AZBZ^* &= U_W(\gamma)^* A U_W(\gamma) ZBZ^* = U_W(\gamma)^* AZU_W(\gamma)BZ^* \\ &= U_W(\gamma)^* AZB^\gamma U_W(\gamma) Z^* = U_W(\gamma)^* AZB^\gamma Z^* U_W(\gamma) \\ &= U_W(\gamma)^* ZB^\gamma Z^* A U_W(\gamma) = ZBZ^* U_W(\gamma)^* A U_W(\gamma) \\ &= ZBZ^* A \end{aligned} \tag{5.4}$$

where

$$B^\gamma := U_W(\gamma) B U_W(\gamma)^* = U(\dot{\gamma}) B U(\dot{\gamma})^* \in \mathcal{A}_W(I') \subseteq \mathcal{A}_V(I') .$$

The above means that for every $I \in \mathcal{J}$, every $A \in \mathcal{A}_{W^c}(I)$ commutes with $\mathcal{A}_W(I_1)$ for all $I_1 \in \mathcal{J}$, that is $\mathcal{A}_{W^c} \subseteq \mathcal{A}_W^c$.

Conversely, there exists a unitary subalgebra $\widetilde{W}$ of V such that $\mathcal{A}_W^c = \mathcal{A}_{\widetilde{W}}$ by Theorem 5.1.1. As a consequence, for every $a \in \widetilde{W}$ and every $f \in C_a^\infty(S^1)$, $Y(a,f)$ is affiliated with $Z\mathcal{A}_W(S^1)'Z^*$. This implies that $[Y(a,z), Y(b,w)] = 0$ for all $b \in W$ by Proposition 3.2.15, that is, $a \in W^c$. Therefore, we can conclude that $\widetilde{W} \subseteq W^c$ and thus $\mathcal{A}_W^c \subseteq \mathcal{A}_{W^c}$, which ends the proof. □

5.2 Strong graded-locality by quasi-primary vectors

Let V be a simple unitary energy-bounded VOSA. Let $\mathfrak{F}$ be a subset of V, then for every $I \in \mathcal{J}$ we define a von Neumann subalgebra of $\mathcal{A}_V(I)$ by

$$\mathcal{A}_{\mathfrak{F}}(I) := W^* \left(\left\{ Y(a,f)\,,\ Y(b,g) \;\middle|\; \begin{array}{l} a \in V_{\overline{0}} \cap \mathfrak{F}\,,\ f \in C^\infty(S^1)\,,\ \mathrm{supp} f \subset I\,, \\ b \in V_{\overline{1}} \cap \mathfrak{F}\,,\ g \in C_\chi^\infty(S^1)\,,\ \mathrm{supp} g \subset I \end{array} \right\} \right) . \tag{5.5}$$

Then, we can state the following useful criteria.

Theorem 5.2.1. *Let $\mathfrak{F}$ be a subset of a simple energy-bounded unitary VOSA V and assume that $\mathfrak{F}$ contains only quasi-primary vectors. Moreover, suppose that $\mathfrak{F}$ generates V and that there exists an $I \in \mathcal{J}$ such that $\mathcal{A}_{\mathfrak{F}}(I') \subseteq Z\mathcal{A}_{\mathfrak{F}}(I)'Z^*$. Then, V is strongly graded-local and $\mathcal{A}_{\mathfrak{F}}(I) = \mathcal{A}_V(I)$ for all $I \in \mathcal{J}$.*

Proof. The proof is an adaptation of the one of [CKLW18, Theorem 8.1], which is organised in five parts. We use the notation settled in (5.1).

Step 1. We prove two facts which will be useful throughout the proof. The net $\mathcal{A}_{\mathfrak{F}}$ is clearly isotonous and Möbius covariant thanks to Proposition 3.2.12. It follows that

$$\mathcal{A}_{\mathfrak{F}}(I') \subseteq Z\mathcal{A}_{\mathfrak{F}}(I)'Z^* \qquad \forall I \in \mathcal{J} . \tag{5.6}$$

Furthermore, we can prove that $\mathcal{A}_{\mathfrak{F}}(I) = \mathcal{A}_{\mathfrak{F}\cup\theta(\mathfrak{F})}(I)$ for all $I \in \mathcal{J}$. Indeed, let $a \in \mathfrak{F}$ and $f \in C_a^\infty(S^1)$ with $\mathrm{supp} f \subset I \in \mathcal{J}$, then $Y(a,f)$ is affiliated to $\mathcal{A}_{\mathfrak{F}}(I)$. Let $A \in B(\mathcal{H})$ be such that $Y(a,f)$ commutes with A and A^*. Then, we have

$$\begin{aligned} (A^*c|Y(\theta(a),f)d) &= \overline{(Y(\theta(a),f)d|A^*c)} = (-1)^{2d_a^2+d_a}\overline{(Y(a,f)^*d|A^*c)} \\ &= (-1)^{2d_a^2+d_a}\overline{(Ad|Y(a,f)c)} = (-1)^{2d_a^2+d_a}(Y(a,f)c|Ad) \\ &= (Y(\theta(a),f)^*c|Ad) \qquad \forall c,d \in V \end{aligned} \tag{5.7}$$

where we have used (3.19) for the second and the last equality, whereas Lemma 3.2.3 for the third one. The equality above is equivalent to say that $Y(\theta(a),f)$ commutes with A and A^* by Lemma 3.2.3. Thus, we can conclude that $Y(a,f)$ and $Y(\theta(a),f)$ are affiliated to the same von Neumann algebra, that is, $\mathcal{A}_{\mathfrak{F}}(I) = \mathcal{A}_{\mathfrak{F}\cup\theta(\mathfrak{F})}(I)$. From now onwards, we suppose, without loss of generality, that $\mathfrak{F} = \theta(\mathfrak{F})$.

Step 2. We prove that $\mathcal{A}_{\mathfrak{F}}$ defines an irreducible graded-local Möbius covariant net on S^1 acting on $\mathcal{H}$. For all $I \in \mathcal{J}$, define $\mathcal{P}_{\mathfrak{F}}(I)$ the algebra generated by all the smeared vertex operators $Y(a,f)$ where $a\in\mathfrak{F}$ and f in $C_a^\infty(S^1)$ with supp$f\subset I$. From the θ-invariance of $\mathfrak{F}$, we deduce that all $\mathcal{P}_\mathfrak{F}(I)$ are $*$-algebras and moreover they have $\mathcal{H}^\infty$ as invariant domain. Then, it makes sense to define for all $I\in\mathcal{J}$, $\mathcal{H}_\mathfrak{F}(I)$ as the closure of $\mathcal{P}_\mathfrak{F}(I)\Omega$. Remembering that $\mathcal{H}^\infty$ is an invariant domain for $U(\gamma)$ for all $\gamma\in\text{Möb}(S^1)^{(2)}$, we can apply

$$\begin{aligned}U(\gamma)Y(a^n,f^n)\cdots Y(a^1,f^1)\Omega &= U(\gamma)Y(a^n,f^n)U(\gamma)^*U(\gamma)\cdots Y(a^1,f^1)U(\gamma)^*\Omega\\ &= Y(a^n,\iota_{d_{a^n}}(\gamma)f^n)\cdots Y(a^1,\iota_{d_{a^1}}(\gamma)f^1)\Omega\end{aligned} \tag{5.8}$$

for all $\gamma\in\text{Möb}(S^1)^{(2)}$, where $\{a^m,f^m\}$ is any finite collection of homogeneous elements of $\mathfrak{F}$ and functions in $C_{a^m}^\infty(S^1)$. Passing to the closure, it follows that $U(\gamma)\mathcal{H}_\mathfrak{F}(I)=\mathcal{H}_\mathfrak{F}(\dot\gamma I)$ for all $\gamma\in\text{Möb}(S^1)^{(2)}$ and all $I\in\mathcal{J}$. Then

$$\overline{\bigcup_{\gamma\in\text{Möb}(S^1)^{(2)}} U(\gamma)\mathcal{H}_\mathfrak{F}(I)}=\mathcal{H}_\mathfrak{F}:=\overline{\mathcal{P}_\mathfrak{F}\Omega} \tag{5.9}$$

where $\mathcal{P}_\mathfrak{F}$ is the $*$-algebra with $\mathcal{H}^\infty$ as invariant core, generated by all smeared vertex operators $Y(a,f)$ with $a\in\mathfrak{F}$ and $f\in C_a^\infty(S^1)$ without any restriction on the support. Moreover, noting that for all $a\in\mathfrak{F}$ there exists an $f\in C_a^\infty(S^1)$ such that

$$a=a_{(-1)}\Omega=Y(a,f)\Omega\,,$$

we can conclude that $V\subset\mathcal{H}_\mathfrak{F}$ by the fact that $\mathfrak{F}$ is generating. Therefore, $\mathcal{H}_\mathfrak{F}$ is equal to $\mathcal{H}$. Now we adapt the proof of [Bor68, Theorem 1], cf. also [Lon08b, Theorem 3.2.1], to prove that $\mathcal{H}_\mathfrak{F}(I)=\mathcal{H}$ for all $I\in\mathcal{J}$, that is, a Reeh-Schlieder property for fields. Fix an $I\in\mathcal{J}$ and consider $v\in\mathcal{H}$ orthogonal to $\mathcal{H}_\mathfrak{F}(I)$. We want to prove that $v=0$. Let $I_0\in\mathcal{J}$ whose closure is contained in I, thus there exists a neighbourhood N of 0 small enough such that $r(t)I_0\subset I$ for all $t\in N$. We have

$$F(t):=(v|U(r^{(2)}(t))w)=0 \qquad \forall t\in N\ \ \forall w\in\mathcal{H}_\mathfrak{F}(I) \tag{5.10}$$

because $U(r^{(2)}(t))w\in\mathcal{H}_\mathfrak{F}(I)$ as we have proved above. The generator of the rotation subgroup is positive by construction and therefore by [HP57, Theorem 11.4.1], F can be extended to a continuous function, still called F, on the upper half-plane $\{z\in\mathbb{C}\mid \text{Im}(z)\geq 0\}$, which is analytic on $\{z\in\mathbb{C}\mid\text{Im}(z)>0\}$. By the identity theorem [AF03, p. 122], $F=0$, which implies that v is orthogonal to $U(r^{(2)}(t))\mathcal{H}_\mathfrak{F}(I)$ for all $t\in\mathbb{R}$. By [Lon08b, Proposition 1.4.1], the generator of the translation subgroup of $\text{Möb}(S^1)^{(2)}$ is positive. Thus we can apply the same argument above for the generator of rotations to the generator of translations and consequently conclude that v is orthogonal to $U(\gamma)\mathcal{H}_\mathfrak{F}(I)$ for all $\gamma\in\text{Möb}(S^1)^{(2)}$, just recalling that $\text{Möb}(S^1)^{(2)}$ is generated by rotations and translations. By (5.9), v is orthogonal to $\mathcal{H}_\mathfrak{F}=\mathcal{H}$, that is, $v=0$. This shows that $\mathcal{H}_\mathfrak{F}(I)=\mathcal{H}$ for all $I\in\mathcal{J}$. Now, let $a\in\mathfrak{F}$ and $Y(a,f)$ be affiliated with $\mathcal{A}_\mathfrak{F}(I)$ for some $I\in\mathcal{J}$. There exists a sequence of operators $\{A_n\}\subset\mathcal{A}_\mathfrak{F}(I)$ such that $\lim_{n\to+\infty}A_nc=Y(a,f)c$ for all $c\in\mathcal{H}^\infty$. Therefore, it is not difficult to see that $\overline{\mathcal{A}_\mathfrak{F}(I)\Omega\cap\mathcal{H}^\infty}$ is invariant for the left action of $\mathcal{P}_\mathfrak{F}(I)$. It follows that $\mathcal{P}_\mathfrak{F}(I)\Omega\subseteq\overline{\mathcal{A}_\mathfrak{F}(I)\Omega}$ and thus $\overline{\mathcal{A}_\mathfrak{F}(I)\Omega}=\mathcal{H}$ for all $I\in\mathcal{J}$. This means that the vacuum vector Ω is cyclic with respect to the net $\mathcal{A}_\mathfrak{F}$, which then defines an irreducible graded-local Möbius covariant net on S^1 acting on $\mathcal{H}$.

Step 3. We use the last three parts to prove that $\mathcal{A}_V(I)\subseteq\mathcal{A}_\mathfrak{F}(I)$ for all $I\in\mathcal{J}$, which will conclude the proof. Using the Möbius covariance of the net $\mathcal{A}_V$ given by Corollary 3.2.14, we can restrict to prove the inclusion above for the upper semicircle $S^1_+\in\mathcal{J}$ only. Accordingly, consider the Tomita-Takesaki modular theory associated to the von Neumann algebra $\mathcal{A}_\mathfrak{F}(S^1_+)$ and the vacuum Ω, that is, the modular operator Δ, the modular conjugation J and the Tomita operator $S=J\Delta^{\frac12}$, see [BR02, Definition 2.5.10]. Then, we are going to prove that ZJ defines an antilinear VOSA automorphism of V. Let j be the orientation-reversing isometry of S^1, that is, $j(z)=z^{-1}$ for all $z\in S^1$. By the Bisognano-Wichmann property, see Section 1.3, U extends to an antiunitary representation of $\text{Möb}(S^1)^{(\infty)}\ltimes\mathbb{Z}_2$, which we still denote by U, such that $U(j)=ZJ$ and

$$ZJ\mathcal{A}_\mathfrak{F}(I)ZJ=\mathcal{A}_\mathfrak{F}(j(I)) \qquad \forall I\in\mathcal{J} \tag{5.11}$$

$$ZJU(\gamma)ZJ=U(j\circ\dot\gamma\circ j) \qquad \forall\gamma\in\text{Möb}(S^1)^{(\infty)}\,. \tag{5.12}$$

Then, we obtain that L_n commutes with ZJ for every $n\in\{-1,0,1\}$ (consider the one-parameter subgroups $\exp(tl)$ as in the proof of Proposition 3.2.12 in (5.12) and derive them for $t=0$). Then, proceeding as in the first part of the proof of Proposition 3.2.2, we have that $ZJV \subseteq V$. This implies that for all homogeneous $a \in V$, the formal series

$$\Phi_a(z) := \sum_{n\in\mathbb{Z}-d_a} ZJa_nZJz^{-n-d_a} \tag{5.13}$$

is a well-defined field on V and moreover

$$[L_1, \Phi_a(z)] = \frac{\mathrm{d}}{\mathrm{d}z}\Phi_a(z)\,, \quad \Phi_a(z)\Omega|_{z=0} = ZJa\,, \quad \Phi_a(z)\Omega = e^{zT}ZJa\,. \tag{5.14}$$

Noting that (A.1) holds in the current setting, we can define, as it is done in Appendix A, the smeared fields

$$\Phi_a(f)c := \sum_{n\in\mathbb{Z}-d_a} \widehat{f}_nZJa_nZJc \qquad \forall f\in C_a^\infty(S^1)\ \forall c\in V\,.$$

Recall that every $\Phi_a(f)$ is closable (and we use the same symbol for its closure) thanks to the energy bounds and that they have $\mathcal{H}^\infty$ as common invariant domain. Then, note that if $c\in V$, we have

$$\Phi_a(f)c = ZJY(a,\overline{f}\circ j)ZJc \qquad \forall f\in C_a^\infty(S^1) \tag{5.15}$$

thanks to the fact that ZJ is a bounded operator which preserves V. Moreover, ZJ is an isometry which commutes with L_0 and so it preserves $\mathcal{H}^\infty$, giving us the following equality between operators

$$\Phi_a(f) = ZJY(a,\overline{f}\circ j)ZJ \qquad \forall f\in C_a^\infty(S^1)\,. \tag{5.16}$$

Now, $Y(a,\overline{f}\circ j)$ is affiliated to $\mathcal{A}_{\mathfrak{F}}(j(I))$ whenever supp$f\subset I$ for some $I\in\mathcal{J}$. Therefore, if supp$f\subset I$ for some $I\in\mathcal{J}$, then $ZJY(a,\overline{f}\circ j)ZJ = \Phi_a(f)$ is affiliated to $ZJ\mathcal{A}_{\mathfrak{F}}(j(I))ZJ$, which is exactly $\mathcal{A}_{\mathfrak{F}}(I)$ thanks to (5.11). Now, by the graded locality of $\mathcal{A}_{\mathfrak{F}}$ and by a slight modification of Proposition 3.2.15, we can prove that for all $a,b\in\mathfrak{F}$, $\Phi_a(z)$ and $Y(b,z)$ are mutually local in the Wightman sense as defined in Appendix A. By Proposition A.1, for all $a,b\in\mathfrak{F}$, $\Phi_a(z)$ and $Y(b,z)$ are mutually local in the vertex superalgebra sense. Then, combine the generating property of $\mathfrak{F}$ and Dong's Lemma [Kac01, Lemma 3.2], we can prove that $\Phi_a(z)$ and $Y(b,z)$ are mutually local (in the vertex superalgebras sense) for all $a\in\mathfrak{F}$ and all $b\in V$. Noting that $\Phi_a(z) = ZJY(a,z)ZJ$, we have that also $Y(a,z)$ and $\Phi_b(z)$ are mutually local for all $a\in\mathfrak{F}$ and all $b\in V$. Thus, using again that $\mathfrak{F}$ generates V and Dong's Lemma [Kac01, Lemma 3.2], we can conclude that $\Phi_a(z)$ and $Y(b,z)$ are mutually local for all $a,b\in V$. Using the uniqueness theorem for vertex superalgebras [Kac01, Theorem 4.4], it follows that $\Phi_a(z) = Y(ZJa,z)$ for all $a\in V$ and therefore ZJ defines an antilinear automorphisms of the VOSA V.

Step 4. Now, we want to prove the useful formula:

$$SY(a,f)\Omega = Y(a,f)^*\Omega \tag{5.17}$$

for all quasi-primary element $a\in V$ and all $f\in C_a^\infty(S^1)$ with supp$f\subset S^1_+$. To this aim, fix a homogeneous $a\in\mathfrak{F}$ and $f\in C_a^\infty(S^1)$ with supp$f\subset S^1_+$. On the one hand, $Y(a,f)$ is affiliated with $\mathcal{A}_{\mathfrak{F}}(S^1_+)$ and thus $J\Delta^{\frac12}Y(a,f)\Omega = Y(a,f)^*\Omega$, cf. [BR02, Proposition 2.5.9]. On the other hand, using the Bisognano-Wichmann property for $\mathcal{A}_{\mathfrak{F}}$ (1.42) first and Theorem 4.0.4 with the linearity of the maps in (3.9) after, we obtain

$$\theta Z\Delta^{\frac12}Y(a,f)\Omega = \theta Ze^{\frac{K}{2}}Y(a,f)\Omega = \theta(-1)^{2d_a^2+d_a}Y(a,f\circ j)\Omega = Y(a,f)^*\Omega\,.$$

Consequently, we have, using (2.40),

$$J\Delta^{\frac12}Y(a,f)\Omega = \theta Z\Delta^{\frac12}Y(a,f)\Omega \implies ZJ\Delta^{\frac12}Y(a,f)\Omega = \theta\Delta^{\frac12}Y(a,f)\Omega\,. \tag{5.18}$$

Using again that ZJ and θ both commute with L_n for all $n\in\{-1,0,1\}$ and the Bisognano-Wichmann property for $\mathcal{A}_{\mathfrak{F}}$, especially the implication that $\Delta^{\frac12} = e^{\frac{K}{2}}$ with K as in Theorem 4.0.4, we prove that $ZJ\Delta^{\frac12}ZJ$ and $\theta\Delta^{\frac12}\theta$ are both equal to $\Delta^{-\frac12}$. This implies from (5.18) that $ZJY(a,f)\Omega$ is equal to $\theta Y(a,f)\Omega$. Again ZJ and θ both commute with L_0 and thus $ZJY(a,\iota_{d_a}(r^{(2)}(t))f)\Omega$ is equal to $\theta Y(a,\iota_{d_a}(r^{(2)}(t))f)\Omega$ for all $t\in\mathbb{R}$. Therefore, using a partition of unity, we prove that $ZJY(a,f)\Omega$ is equal to $\theta Y(a,f)\Omega$ for all $f\in C_a^\infty(S^1)$ and consequently it must be $(ZJ)(a) = \theta(a)$. From the arbitrariness of $a\in\mathfrak{F}$, which generates V and the fact that ZJ and θ are antilinear automorphism, it follows that $ZJ = \theta$. By Theorem 4.0.4, we can conclude that for every quasi-primary element $a \in V$ and every $f \in C^\infty_a(S^1)$ with $\mathrm{supp} f \subset S^1_+$, $Y(a,f)\Omega$ is in the domain of the operator S and equation (5.17) holds.

Step 5. In this last step, we are going to prove that $Y(a,f)$ is affiliated to $\mathcal{A}_{\mathfrak{F}}(I)$ whenever $a \in V$ is any quasi-primary vector, $f \in C^\infty_a(S^1)$ with $\mathrm{supp} f \subset S^1_+$ and $I \in \mathcal{J}$ containing the closure of S^1_+. This will bring us to conclude that $\mathcal{A}_V(I) \subseteq \mathcal{A}_{\mathfrak{F}}(I)$ for all $I \in \mathcal{J}$, proving the theorem. Let $I \in \mathcal{J}$ containing the closure of S^1_+ and let $A \in \mathcal{A}_{\mathfrak{F}}(I') \subseteq \mathcal{A}_{\mathfrak{F}}(S^1_-)$. Then, using Proposition 3.2.12, that is, the Möbius covariance of the smeared vertex operators associated to quasi-primary elements, there exists $\delta > 0$ such that $e^{itL_0}Ae^{-itL_0}$ is still an element of $\mathcal{A}_{\mathfrak{F}}(S^1_-)$ for all $t \in (-\delta, \delta)$. Proceeding similarly to the proof of [CKLW18, Lemma 6.5] (cf. proof of Lemma 3.2.3), it is possible, for all $s \in (0,\delta)$, to construct an operator $A(\varphi_s)$ such that (cf. [DSW86, Lemma 5.3] for a similar argument)

$$A(\varphi_s) \in \mathcal{A}_{\mathfrak{F}}(S^1_-)\,, \quad A(\varphi_s)c \in \mathcal{H}^\infty \quad \forall c \in \mathcal{H}^\infty\,, \quad \lim_{s\to 0} A(\varphi_s)c = Ac \quad \forall c \in \mathcal{H}\,. \tag{5.19}$$

Now, let $X_1, X_2 \in \mathcal{P}_{\mathfrak{F}}(S^1_-)$ and $B \in \mathcal{A}_{\mathfrak{F}}(S^1_+) \subseteq Z\mathcal{A}_{\mathfrak{F}}(S^1_-)'Z^*$ by graded locality, we have

$$\begin{aligned}
(ZX_1^*A(\varphi_s)X_2\Omega|SB\Omega) &= (ZX_1^*A(\varphi_s)X_2\Omega|B^*\Omega) \\
&= (ZZ^*BZX_1^*A(\varphi_s)X_2\Omega|\Omega) \\
&= (ZX_1^*A(\varphi_s)X_2Z^*B\Omega|\Omega) \\
&= (B\Omega|ZX_2^*A(\varphi_s)^*X_1\Omega)\,.
\end{aligned}$$

where we have used (5.19), the properties of Z as in Section 2.1 and the commutation relation between smeared vertex operators and affiliated bounded operators. Because S is antilinear, the equation above implies that $ZX_1^*A(\varphi_s)X_2\Omega$ is in the domain of S^* and

$$S^*ZX_1^*A(\varphi_s)X_2\Omega = ZX_2^*A(\varphi_s)^*X_1\Omega\,. \tag{5.20}$$

Fix a quasi-primary vector $a \in V$ and a function $f \in C^\infty_a(S^1)$ with $\mathrm{supp} f \subset S^1_+$. For every $X_1, X_2 \in \mathcal{P}_{\mathfrak{F}}(S^1_-)$ and every $s \in (0,\delta)$, we have the following equalities

$$\begin{aligned}
(X_1\Omega|A(\varphi_s)Z^*Y(a,f)ZX_2\Omega) &= (X_1\Omega|A(\varphi_s)X_2Z^*Y(a,f)\Omega) \\
&= (ZX_2^*A(\varphi_s)^*X_1\Omega|Y(a,f)\Omega) \\
&= (S^*ZX_1^*A(\varphi_s)X_2\Omega|Y(a,f)\Omega) \\
&= (SY(a,f)\Omega|ZX_1^*A(\varphi_s)X_2\Omega) \\
&= (Y(a,f)^*\Omega|ZX_1^*A(\varphi_s)X_2\Omega) \\
&= (X_1Z^*Y(a,f)^*Z\Omega|A(\varphi_s)X_2\Omega) \\
&= (Z^*Y(a,f)^*ZX_1\Omega|A(\varphi_s)X_2\Omega) \\
&= (X_1\Omega|Z^*Y(a,f)ZA(\varphi_s)X_2\Omega)
\end{aligned}$$

where we have used: the twisted Haag duality (1.45) for $\mathcal{A}_{\mathfrak{F}}$ with an adaptation of Proposition 3.2.15 for the first and for the seventh equality; (5.20) for the third one; equation (5.17) for the fifth one; the usual properties of Z as in Section 2.1 and (5.19) in general. As we have proved in *Step 2* that $\mathcal{P}_{\mathfrak{F}}(S^1_-)\Omega$ is dense in $\mathcal{H}$ and therefore

$$A(\varphi_s)Z^*Y(a,f)ZX\Omega = Z^*Y(a,f)ZA(\varphi_s)X\Omega \qquad \forall X \in \mathcal{P}_{\mathfrak{F}}(S^1_-)\ \forall s \in (0,\delta)$$

which implies, thanks to the properties in (5.19), that $AX\Omega$ is in the domain of $Z^*Y(a,f)Z$ for all $X \in \mathcal{P}_{\mathfrak{F}}(S^1_-)$ and that

$$AZ^*Y(a,f)ZX\Omega = Z^*Y(a,f)ZAX\Omega \qquad \forall X \in \mathcal{P}_{\mathfrak{F}}(S^1_-)\,. \tag{5.21}$$

Moreover, $\mathcal{P}_{\mathfrak{F}}(S^1_-)\Omega$ is a core for all $(L_0 + 1_{\mathcal{H}})^k$ with $k \in \mathbb{Z}_{>0}$. Indeed, choose $I \in \mathcal{J}$ such that $\overline{I} \subset S^1_-$. Then, there exists a positive real number δ such that $e^{it}I \subset S^1_-$ for all $t \in (-\delta,\delta)$. By the Möbius covariance of the vertex operators, $U(r^{(2)}(t))\mathcal{P}_{\mathfrak{F}}(I)\Omega \subseteq \mathcal{P}_{\mathfrak{F}}(S^1_-)\Omega$ for all $t \in (-\delta,\delta)$, see *Step 2*. From [CKLW18, Lemma 7.2], it follows that $\mathcal{P}_{\mathfrak{F}}(S^1_-)\Omega$ is a core for every $(L_0 + 1_{\mathcal{H}})^k$ with $k \in \mathbb{Z}_{>0}$ and thus it is also a core for $Z^*Y(a,f)Z$ thanks to the energy bounds of Lemma 3.1.8. Therefore, (5.21) implies that $AZ^*Y(a,f)Z \subseteq Z^*Y(a,f)ZA$. By the arbitrariness of the choices done above, it follows that $Y(a,f)$ is affiliated with $Z\mathcal{A}_{\mathfrak{F}}(I')'Z^* = \mathcal{A}_{\mathfrak{F}}(I)$ (twisted Haag duality (1.45)) for all $I \in \mathcal{J}$ which contains the closure of S^1_+, all quasi-primary $a \in V$ and all $f \in C^\infty_a(S^1)$ with supp$f \subset S^1_+$. Applying Proposition 3.2.4, we get that $\mathcal{A}_V(S^1_+) \subseteq \mathcal{A}_{\mathfrak{F}}(I)$ for all $I \in \mathcal{J}$ which contains the closure of S^1_+. The external continuity (1.47) holds for the net $\mathcal{A}_{\mathfrak{F}}$ thanks to its Möbius covariance and thus

$$\bigcap_{\overline{S^1_+} \subset I} \mathcal{A}_{\mathfrak{F}}(I) = \mathcal{A}_{\mathfrak{F}}(S^1_+) \,.$$

Hence, $\mathcal{A}_V(S^1_+) \subseteq \mathcal{A}_{\mathfrak{F}}(S^1_+)$, which concludes the proof. □

Remark 5.2.2. Let V be a simple energy-bounded VOSA. As a representation of the Virasoro algebra, V is spanned by the conformal vector and by primary elements. Then, thanks to Theorem 5.2.1 and Remark 3.2.13, we can deduce the diffeomorphism covariance of the net $\mathcal{A}_{(V,(\cdot|\cdot))}$ without using the strong graded-locality assumption as it is done in the proof of Theorem 3.2.17.

Corollary 5.2.3. *Let V^1 and V^2 be simple strongly graded-local unitary VOSAs. Then, $V^1\hat{\otimes}V^2$ is strongly graded-local and $\mathcal{A}_{V^1\hat{\otimes}V^2} = \mathcal{A}_{V^1}\hat{\otimes}\mathcal{A}_{V^2}$.*

Proof. $V^1\hat{\otimes}V^2$ is energy-bounded by Corollary 3.1.3. Call $\mathfrak{F}^1$ and $\mathfrak{F}^2$ the sets of all quasi-primary vectors of V^1 and V^2 respectively. $V^1\hat{\otimes}V^2$ is generated by the set of quasi-primary vectors $\mathfrak{F} := (\mathfrak{F}^1 \otimes \Omega^2) \cup (\Omega^1 \otimes \mathfrak{F}^2)$. Moreover, $\mathcal{A}_{\mathfrak{F}}(I) = \mathcal{A}_{V^1}(I)\hat{\otimes}\mathcal{A}_{V^2}(I)$ for all $I \in \mathcal{J}$. In particular, if we call Z_1, Z_2 and Z the operators (1.40) for $\mathcal{A}_{V^1}, \mathcal{A}_{V^2}$ and $\mathcal{A}_{V^1}\hat{\otimes}\mathcal{A}_{V^2}$ respectively, then we have that

$$\begin{aligned}\mathcal{A}_{\mathfrak{F}}(I') = \mathcal{A}_{V^1}(I')\hat{\otimes}\mathcal{A}_{V^2}(I') &= \left(Z_1\mathcal{A}_{V^1}(I)'Z_1^*\right) \hat{\otimes} \left(Z_2\mathcal{A}_{V^2}(I)'Z_2^*\right) \\ &\subseteq Z\left(\mathcal{A}_{V^1}(I)\hat{\otimes}\mathcal{A}_{V^2}(I)\right)' Z^* = Z\mathcal{A}_{\mathfrak{F}}(I)'Z^* \qquad \forall I \in \mathcal{J}\end{aligned}$$

where we have used the twisted Haag duality (1.45) and (2.76). Then, we get the desired result just applying Theorem 5.2.1. □

The following result can be useful in the production of examples.

Theorem 5.2.4. *Let V be a simple unitary VOSA generated by $V_{\frac{1}{2}} \cup V_1 \cup \mathfrak{F}$ where $\mathfrak{F} \subseteq V_2$ is a family of quasi-primary θ-invariant Virasoro vectors. Then, V is strongly graded-local and $\mathcal{A}_V = \mathcal{A}_{V_{\frac{1}{2}}\cup V_1 \cup \mathfrak{F}}$.*

Proof. We use the notation settled in (5.1). First, by Proposition 3.1.4 and its proof, V is energy-bounded and in particular, every smeared vertex operator $Y(a,f)$ with $a \in V_{\frac{1}{2}} \cup V_1 \cup \mathfrak{F}$ and $f \in C^\infty_a(S^1)$ satisfies the energy bounds (3.7) with at most $k = 1$. Second, using Proposition A.1, we have that the vertex operators $Y(a,z)$ with $a \in V_{\frac{1}{2}} \cup V_1 \cup \mathfrak{F}$ are mutually local in the Wightman sense. Then, we use the argument in [BS90, Section 2] (based on [DF77, Theorem 3.1], see also [GJ87, Theorem 19.4.4]) to show that the family of von Neumann algebras $\mathcal{A}_{V_{\frac{1}{2}}\cup V_1 \cup \mathfrak{F}}(I)$ with $I \in \mathcal{J}$ satisfies the graded-locality condition required by Theorem 5.2.1. Therefore, we can conclude that V is strongly graded-local and that $\mathcal{A}_{V_{\frac{1}{2}}\cup V_1 \cup \mathfrak{F}} = \mathcal{A}_V$ by the same theorem. □

Chapter 6

Applications

The purpose of the current chapter is multiple. In Section 6.1 and Section 6.2, we show a systematic way to construct the most famous examples of graded-local conformal nets using all the power given by the theory developed throughout the present thesis. Contextually, we define and characterise a further correspondence between the two superconformal structures, which we have in the two settings. In the last Section 6.3, we introduce the rank-one lattice models and we present the classification results given by [CGH19].

Comparing with the local case... The local examples in Section 6.1 were already presented in [CKLW18, Chapter 8], but we give full details on how to construct the conformal nets from the VOAs. The same treatment is used for graded-local examples, which are obviously exclusive to this new setting. Similar discussion holds for superconformal structures and superconformal examples in Section 6.2, which do not have a local counterpart. We also point out that the well-known unitarity of some VOSAs is proved relying on the theory developed in Section 2.5, giving a different approach from the one in [AL17]. In Section 6.3, we prove the strong graded-locality of the odd rank-one lattice type VOSAs, giving also a new proof of the strong locality of the even ones. Nevertheless, the classification results for submodels are given for even rank-one lattice models only as presented in [CGH19].

All models constructed in the following sections refer to the representation theory of Lie superalgebras, see [Kac77], to produce the vertex superalgebras which we need. Considering that the way to build up vertex superalgebras from given Lie superalgebras is quite general, we propose it just here. The following setting is a little more general than what we really need and refers to [Kac01, Section 4.7 and Section 3.4] and the few notions about the calculus of formal distributions summarised in Section 2.1. Our hope is to give a broader range of constructions of examples of vertex superalgebras which rely on a general machinery. An alternative textbook to [Kac01] is [LL04, Section 6], where the examples of (just) VOAs which we need are constructed in a detailed way. There, the authors list also the main references where these non-super models are studied for the first time.

Recall that a **Lie superalgebra** is a vector superspace $\mathfrak{g} = \mathfrak{g}_{\overline{0}} \oplus \mathfrak{g}_{\overline{1}}$ (as usual, for $p \in \mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z}$, any vector $c \in \mathfrak{g}_p$ has parity $p(c) := p$ and it is called **even** or **odd** if $p(c)$ is $\overline{0}$ or $\overline{1}$ respectively) with a bilinear operation $[\cdot,\cdot]$, also called **bracket** or **supercommutator**, satisfying:

$$\begin{aligned} &\textbf{(Parity-preserving)} && [a,b] \in \mathfrak{g}_{p(a)+p(b)} \\ &\textbf{(Anticommutativity)} && [a,b] = -(-1)^{p(a)p(b)}[b,a] \\ &\textbf{(Jacobi identity)} && [a,[b,c]] = [[a,b],c] + (-1)^{p(a)p(b)}[b,[a,c]] \end{aligned} \tag{6.1}$$

for all vectors $a, b \in \mathfrak{g}$ with given parities $p(a), p(b) \in \mathbb{Z}_2$ respectively. Hereafter, with an abuse of notation, we use $(-1)^{p(a)p(b)}$ as in (6.1) to denote $(-1)^{p_a p_b}$, where $p_a, p_b \in \{0, 1\}$ are representatives of the remainder class of $p(a)$ and $p(b)$ in $\mathbb{Z}_2$ respectively. Note also that $g_{\overline{0}}$ is a ordinary Lie algebra. Moreover, the multiplication on the left by elements of $\mathfrak{g}_{\overline{0}}$ induces a structure of $\mathfrak{g}_{\overline{0}}$-module on $\mathfrak{g}_{\overline{1}}$.

A Lie superalgebra $\mathfrak{g}$ is called a **formal distribution Lie superalgebra** if it is spanned over $\mathbb{C}$ by coefficients of a family of $\mathfrak{g}$-valued mutually local formal distributions $\{a^\alpha(z) \mid \alpha \in A\}$, with A a set of indices. Then, $\mathfrak{g}$ is called **regular** if there exists an endomorphism T of $\mathfrak{g}$ over $\mathbb{C}$ such that $Ta^\alpha(z) = \partial_z a^\alpha(z)$ for all $\alpha \in A$. Equivalently, such T is an even derivation of the Lie superalgebra $\mathfrak{g}$ defined by $Ta^\alpha_{(n)} = -na^\alpha_{(n-1)}$ for all $n \in \mathbb{Z}$, where $a^\alpha(z) = \sum_{n\in\mathbb{Z}} a^\alpha_{(n)} z^{-n-1}$ for all $\alpha \in A$. Define a T-invariant subalgebra of $\mathfrak{g}$ by

$$\mathfrak{g}_{--} := \{a \in \mathfrak{g} \mid T^k a = 0 \ \ k \gg 0\} . \tag{6.2}$$

Note that from the definition of T, $\mathfrak{g}_{--}$ contains all the coefficients $a^\alpha_{(n)}$ with $n \in \mathbb{Z}_{\geq 0}$ and $\alpha \in A$. Consider a representation $\lambda : \mathfrak{g}_{--} \to \mathbb{C}$ of $\mathfrak{g}_{--}$ such that $\lambda(T\mathfrak{g}_{--}) = 0$. Define the induced $\mathfrak{g}$-module ($\langle\cdot\rangle$ means the ideal generated by)

$$\widetilde{V}^\lambda(\mathfrak{g}) := U(\mathfrak{g})/\langle a - \lambda(a) \mid a \in \mathfrak{g}_{--}\rangle \tag{6.3}$$

where $U(\mathfrak{g})$ is the universal enveloping algebra associated to $\mathfrak{g}$ (see [Jac79, Chapter V.1]), which is usually defined by

$$U(\mathfrak{g}) := \left(\bigoplus_{j=0}^{+\infty} \mathfrak{g}^{\otimes j}\right) \Big/ \langle ab - (-1)^{p(a)p(b)} ba - [a,b] \mid a, b \in \mathfrak{g}\rangle \ .$$

It is useful to recall that, if the set of indices A is ordered (which is the case we are going to deal with, e.g. [Kac77, Section 1.1.3]), then thanks to the Poincaré-Birkhoff-Witt theorem, see [Jac79, Section V.2], a linear basis for $U(\mathfrak{g})$ is given using the ordered monomials of degree $s \in \mathbb{Z}_{>0}$:

$$\{\Omega := [1], \ \ a^{\alpha_1}_{n_1} \cdots a^{\alpha_s}_{n_s}\Omega \mid \alpha_1 \leq \cdots \leq \alpha_s \, , \ \ n_1 \leq \cdots \leq n_s\} . \tag{6.4}$$

Noting that T can be extended to $U(\mathfrak{g})$ and pushed down to $\widetilde{V}^\lambda(\mathfrak{g})$, indicating it with T again, we have the following [Kac01, Theorem 4.7], which is obtain as a corollary of the existence theorem for vertex superalgebras [Kac01, Theorem 4.5]:

Theorem 6.0.1. *Let $(\mathfrak{g}, T)$ be a regular formal distribution Lie superalgebra with generating set $\{a^\alpha(z) \mid \alpha \in A\}$. Then, any induced module $\widetilde{V}^\lambda(\mathfrak{g})$ has a unique structure of vertex superalgebra determined by $\Omega := [1]$ as vacuum vector, the even derivation T as infinitesimal translation operator and generated by the state-field correspondence $Y(a^\alpha_{(-1)}\Omega, z) := a^\alpha(z)$ for all $\alpha \in A$.*

Vertex superalgebras so constructed are called **universal vertex superalgebras** associated to $\mathfrak{g}$.

Suppose also that $\mathfrak{g}$ is **graded**, that is, we have a diagonalizable derivation H of $\mathfrak{g}$, such that for some $\Delta_\alpha \in \mathbb{R}$ (which will be at most in $\frac{1}{2}\mathbb{Z}$ in our case)

$$Ha^\alpha(z) = (z\partial_z + \Delta_\alpha)a^\alpha(z) \, , \tag{6.5}$$

which we can rewrite coefficient-wise as

$$Ha^\alpha_n = -na^\alpha_n \, , \qquad a^\alpha(z) = \sum_{n\in\mathbb{Z}-\Delta_\alpha} a^\alpha_n z^{-n-\Delta_\alpha} \, . \tag{6.6}$$

Therefore, we can implement a simple quotient on $\widetilde{V}^\lambda(\mathfrak{g})$ by the maximal graded ideal, which produces a simple vertex superalgebra associated to $\mathfrak{g}$, which we denote by $V^\lambda(\mathfrak{g})$, cf. [Kac01, Section 3.4].

6.1 Starting examples

The easiest example of a vertex (super)algebra is $\mathbb{C}$ itself with vacuum vector $\Omega := 1$ and trivial structure, that is, $Y(c\Omega, z) := c1_\mathbb{C}$ for all $c \in \mathbb{C}$ and derivation $T := 0_\mathbb{C}$. It is called the **trivial vertex algebra**. It is clearly simple and of CFT type. Moreover, it can be made a unitary VOA just choosing the trivial conformal vector $\nu := 0$ and the usual scalar product on $\mathbb{C}$ with PCT operator given by the complex conjugation. This is also a first example of unitary subalgebra of any unitary VOSA. It is trivially strongly local and thus define an irreducible conformal net, the **trivial net** as in Section 1.3. Obviously, this is an example of covariant subnet of any graded-local conformal net too.

For more interesting examples, we introduce the following vertex superalgebras, which constitute the ground for constructing further ones. Specifically, we rewrite [CKLW18, Example 8.4 and Example 8.6], that is Virasoro models and the free boson respectively, with more details in Example 6.1.1 and Example 6.1.2 respectively. Then, we give the simplest super model possible, the free fermion, in Example 6.1.5. Therefore, we have Example 6.1.4 and Example 6.1.7 as straightforward applications of Corollary 5.2.3. Finally, we include [CKLW18, Example 8.7] in its super counterpart in Example 6.1.8. In the development of every example, we rely on the general theory presented at the beginning of Chapter 6.

Example 6.1.1 (Virasoro models). The **Virasoro algebra** is the infinite dimensional Lie algebra

$$\mathfrak{Vir} := \bigoplus_{m\in\mathbb{Z}} \mathbb{C}L_m \oplus \mathbb{C}C \tag{6.7}$$

with commutation relations:

$$\begin{aligned} &[L_m, L_n] := (m-n)L_{m+n} + \frac{m^3-m}{12}\delta_{m,-n}C \qquad \forall m,n\in\mathbb{Z} \\ &[\mathfrak{Vir}, C] := 0\,. \end{aligned} \tag{6.8}$$

This Lie algebra can be realised as a central extension of the Witt algebra, see e.g. [Kac95, Section 7.3] and [KR87, Lecture 1] or [CKLW18, Section 3.1] for a quick survey. $(\mathfrak{Vir}, \mathrm{ad}L_{-1})$ is a regular formal distribution Lie superalgebra ($\mathrm{ad}L_{-1}$ is the adjoint action of L_{-1}) such that

$$\mathfrak{Vir}_{--} = \bigoplus_{n\geq -1} \mathbb{C}L_n \oplus \mathbb{C}C\,, \qquad \mathrm{ad}L_{-1}(\mathfrak{Vir}_{--}) = \bigoplus_{n\geq -1} \mathbb{C}L_n\,. \tag{6.9}$$

Consider the universal enveloping algebra $U(\mathfrak{Vir})$ associated to $\mathfrak{Vir}$ with $\Omega := [1]$ and define

$$\widetilde{V}^c(\mathfrak{Vir}) := U(\mathfrak{Vir})/\langle C-c\,,\ L_n\Omega \mid n\geq -1\rangle \qquad \forall c\in\mathbb{C}\,, \tag{6.10}$$

which are constructed from the $\mathfrak{Vir}_{--}$-representations λ_c on $\mathbb{C}$ such that $\lambda_c(\mathfrak{Vir}_{--}) = 0$ and $\lambda_c(C) = c$ respectively. $\widetilde{V}^c(\mathfrak{Vir})$ has a structure of vertex algebras for all $c\in\mathbb{C}$: it is linearly generated by the ordered elements

$$\{\Omega,\ L_{n_1}\cdots L_{n_s}\Omega \mid n_1\leq\cdots\leq n_s\leq -2\} \tag{6.11}$$

where Ω is also the vacuum vector, $T = L_{-1}$ (that is, the multiplication by L_{-1}) is the infinitesimal translation operator and the state-field correspondence generated by $\nu := L_{-2}\Omega$:

$$Y(\nu, z) = \sum_{n\in\mathbb{Z}} \nu_{(n)} z^{-n-1} = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}\,. \tag{6.12}$$

It is also clear that $\mathfrak{Vir}$ is graded by $\mathrm{ad}L_0$, making the simple graded quotient $V^c(\mathfrak{Vir})$ of $\widetilde{V}^c(\mathfrak{Vir})$ be a simple VOA with central charge $c\in\mathbb{C}$ of CFT type, where ν is the conformal vector, see [Kac01, Example 4.10]. Indeed, we have

$$L_0(L_{n_1}\cdots L_{n_s}\Omega) = (-\sum_{j=1}^{s} n_j)L_{n_1}\cdots L_{n_s}\Omega \qquad \forall s\in\mathbb{Z}_{\geq 0}\,. \tag{6.13}$$

$L(c) := V^c(\mathfrak{Vir})$ are called **Virasoro vertex operator algebras**. To put a unitary structure on $L(c)$, it is convenient to introduce the following Verma modules associated to $\mathfrak{Vir}$:

$$V(c,h) := U(\mathfrak{Vir})/\langle C-c\,,\ L_0\Omega - h\Omega\,,\ L_n\Omega \mid n\geq 1\rangle \qquad \forall c\in\mathbb{C}\ \forall h\in\mathbb{C}\,. \tag{6.14}$$

Every $V(c,h)$ contains a maximal ideal $J(c,h)$, see [KR87, Proposition 3.3(c)], so that we can define the corresponding irreducible $\mathfrak{Vir}$-module $L(c,h) := V(c,h)/J(c,h)$. Note that for all $c\in\mathbb{C}$, $L(c,0) = L(c)$ thanks to [Wan93, Lemma 4.2]. It is known, see [LL04, Theorem 6.1.9 and Theorem 6.1.12], that for all $c\in\mathbb{C}$, $V(c,h)$ and $L(c,h)$ are $L(c)$-modules for all $h\in\mathbb{C}$. Moreover, $L(c,h)$ for all $h\in\mathbb{C}$ exhaust all the irreducible $L(c)$-modules up to module isomorphism. By

[KR87, Proposition 3.4], we can construct a Hermitian form $(\cdot|\cdot)$ on $V(c,h)$, provided that c and h are real numbers, such that

$$(\Omega \mid \Omega) = 1, \quad (L_n a|b) = (a|L_{-n}b) \quad \forall a, b \in V(c,h), \quad \mathrm{Ker}(\cdot|\cdot) = J(c,h)\,. \tag{6.15}$$

Furthermore, it can be pushed down to a non-degenerate Hermitian form on $L(c,h)$ for all c and h in $\mathbb{R}$. By [KR87, Proposition 8.2 and Theorem 12.1], such non-degenerate Hermitian forms on $L(c,h)$ are positive definite if and only if

$$\begin{aligned} c \geq 1 \quad &\text{and} \quad h \geq 0 \\ c_m = 1 - \tfrac{6}{m(m+1)} \quad &\text{and} \quad h_{p,q}(m) = \tfrac{((m+1)p - mq)^2 - 1}{4m(m+1)} \end{aligned} \tag{6.16}$$

for all $m \geq 2$ and $1 \leq q \leq p \leq m-1$. In other words, $L(c,h)$ has a scalar product whenever (c,h) is as in (6.16). To sum up, for all values (6.16), $L(c)$ is a simple VOA of CFT type, equipped with a normalized scalar product and generated by the Hermitian quasi-primary field $Y(\nu, z)$, that is, $L(c)$ is a simple unitary VOA using Proposition 2.5.6. Moreover, it is possible to prove that every $L(c,h)$ is a unitary $L(c)$-module whenever we have (6.16), see [DL14, Theorem 4.4]. We point out that the unitarity of the Virasoro models was proved in [DL14, Theorem 4.2] by finding an explicit formula for the PCT operator. Indeed, this is the antilinear map which acts on generators as the identity:

$$\theta(L_{n_1} \cdots L_{n_s}\Omega) = L_{n_1} \cdots L_{n_s}\Omega \qquad \forall n_1 \leq \cdots \leq n_s \leq -2\,. \tag{6.17}$$

When c is as in (6.16), we can use Theorem 5.2.4 to deduce that $L(c)$ is strongly local and construct the irreducible conformal net $\mathrm{Vir}_c := \mathcal{A}_{L(c)}$, which is the well-known **Virasoro conformal net** of central charge c. Every Vir_c is the irreducible conformal net generated by the positive-energy strongly continuous projective unitary representation of $\mathrm{Diff}^+(S^1)^{(\infty)}$ and thus it is also an example of covariant subnet of any graded-local conformal net. In complete analogy, every $L(c)$ is a non-trivial example of a unitary subalgebra of any unitary VOSA as the one generated by its conformal vector.

Example 6.1.2 (The free boson). Let $\mathfrak{h}$ be a complex one-dimensional vector space, equipped with a non-degenerate symmetric bilinear form $B(\cdot,\cdot)$. We obtain a particular type of infinite dimensional Lie algebra, known as an affine Kac-Moody algebra (see [Kac95]), implementing a central extension of the affinization of $\mathfrak{h}$. Specifically, we have the complex vector space

$$\hat{\mathfrak{h}} := \mathfrak{h} \otimes_{\mathbb{C}} \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K \tag{6.18}$$

with the commutation relations: for every $x \in \mathfrak{h}$ and every $m \in \mathbb{Z}$, define $x_m := x \otimes t^m$, then

$$\begin{aligned} [a_m, b_n] &:= m\delta_{m,-n} B(a,b) K \qquad \forall a, b \in \mathfrak{h} \;\; \forall m, n \in \mathbb{Z} \\ [\hat{\mathfrak{h}}, K] &:= 0\,. \end{aligned} \tag{6.19}$$

$\hat{\mathfrak{h}}$ is known as an **oscillator** or **Heisenberg algebra**. We choose the generator $J \in \mathfrak{h}$ such that $B(J,J) = 1$, calling it the **current vector** of $\mathfrak{h}$ (see Example 6.1.8 for this choice of the name). $(\hat{\mathfrak{h}}, -\partial_t)$ is a regular formal distribution Lie superalgebra such that

$$\hat{\mathfrak{h}}_{--} = \mathfrak{h} \otimes_{\mathbb{C}} \mathbb{C}[t] \oplus \mathbb{C}K\,, \qquad \partial_t \hat{\mathfrak{h}}_{--} = \mathfrak{h} \otimes_{\mathbb{C}} \mathbb{C}[t]\,. \tag{6.20}$$

Consider the universal enveloping algebra $U(\hat{\mathfrak{h}})$ and define the $\hat{\mathfrak{h}}$-module

$$\widetilde{V}^1(\hat{\mathfrak{h}}) := U(\hat{\mathfrak{h}}) / \langle K - 1\,,\; J_m \mid m \geq 0 \rangle\,, \tag{6.21}$$

which is obtained from the $\hat{\mathfrak{h}}_{--}$-representation λ_1 on $\mathbb{C}$ such that $\lambda_1(\partial_t \hat{\mathfrak{h}}_{--}) = 0$ and $\lambda_1(K) = 1$. Then, $\widetilde{V}^1(\hat{\mathfrak{h}})$ has a structure of vertex algebra: it is linearly generated by the ordered elements

$$\{\Omega\,,\; J_{m_1} \cdots J_{m_s}\Omega \mid m_1 \leq \cdots \leq m_s \leq -1\} \tag{6.22}$$

where Ω is the vacuum vector, T is induced by $-\partial_t$ and the state-field correspondence generated by

$$Y(J_{-1}\Omega, z) = \sum_{n \in \mathbb{Z}} (J_{-1}\Omega)_{(n)} z^{-n-1} = \sum_{n \in \mathbb{Z}} J_n z^{-n-1}\,. \tag{6.23}$$

$\widetilde{V}^1(\hat{\mathfrak{h}})$ is also simple thanks to [Kac01, Theorem 3.5] or [LL04, Theorem 6.3.9]. By [Kac01, Proposition 4.10(a)] or [LL04, Theorem 6.3.2], $\widetilde{V}^1(\hat{\mathfrak{h}})$ is a simple VOA with central charge $c = 1$ of CFT type, where the conformal vector is given by $\nu := \frac{1}{2}J_{-1}J_{-1}\Omega$ and

$$L_0(J_{m_1}\cdots J_{m_s}\Omega) = (-\sum_{j=1}^{s} m_j)J_{m_1}\cdots J_{m_s}\Omega \qquad \forall s \in \mathbb{Z}_{\geq 0}\,. \tag{6.24}$$

$M(1) := \widetilde{V}^1(\hat{\mathfrak{h}})$ is called **oscillator** or **Heisenber vertex operator algebra**. By [KR87, Proposition 2.2 and Corollary 2.1], $M(1)$ can be equipped with a scalar product such that $(\Omega|\Omega) = 1$ and $(J_n a|b) = (a|J_{-n}b)$ for all $a, b \in M(1)$ and all $n \in \mathbb{Z}$. Therefore, using Proposition 2.5.6, $M(1)$ is a simple unitary VOA. As for the Virasoro algebra, Example 6.1.1, the unitarity of $M(1)$ was already proved in [DL14, Proposition 4.9], identifying the PCT operator θ as the antilinear operator such that

$$\theta(J_{m_1}\cdots J_{m_s}\Omega) = (-1)^s J_{m_1}\cdots J_{m_s}\Omega \qquad \forall m_1 \leq \cdots \leq m_s \leq -1\,. \tag{6.25}$$

By Theorem 5.2.4, $M(1)$ is a simple strongly local unitary VOA and thus there exists an irreducible conformal net $\mathcal{A}_{M(1)}$. Such net is usually called $\mathcal{A}_{U(1)}$ in the conformal net theory, known as the **free Bose chiral field net** or **$U(1)$-current net**, see [BMT88, Section 1B]. Those names derived from the fact that $\mathcal{A}_{U(1)}$ describes the chiral field theory of a **free boson**, also called **current** in a more general fashion, which is represented by the vertex operator $Y(J_{-1}\Omega, z)$.

Remark 6.1.3. We want to point out here a fact about the construction of the VOA of Example 6.1.2. Define the following Verma modules associated to $\hat{\mathfrak{h}}$:

$$M(k,h) := U(\hat{\mathfrak{h}})/\langle K-k\,,\ J_0-h\,,\ J_m \mid m \geq 1\rangle \qquad \forall k \in \mathbb{C}\ \forall h \in \mathbb{C}\,. \tag{6.26}$$

We have that for all $k \neq 0$, $M(k,0)$ is a simple VOA, which is also isomorphic to $M(1)$. $M(0,0)$ is a vertex algebra which simple quotient is the trivial vertex algebra, see [LL04, Theorems 6.3.2, 6.3.9 and Proposition 6.3.10] or [Kac01, Theorem 3.5(a) and Proposition 4.10(a)]. This means that if we consider $M(k,0)$ for any $k \in \mathbb{C}\backslash\{0\}$ in place of $M(1)$ in Example 6.1.2, we will obtain the same conformal net theory. Instead, for any fixed $k \neq 0$, $M(k,h)$ for all $h \in \mathbb{C}$ exhaust all the irreducible $M(1)$-modules up to module isomorphism thanks to [LL04, Theorem 6.3.9]. Moreover, such modules are all unitary whenever $h > 0$ thanks to [DL14, Propositions 4.9 and 4.10], cf. [KR87, Proposition 2.2 and Corollary 2.1].

What below is a straightforward application of Corollary 5.2.3.

Example 6.1.4 (*d* **free bosons**)**.** Applying Corollary 5.2.3 to Example 6.1.2, we have that

$$M(1)^d := \bigotimes_{j=1}^{d} M(1) \qquad \forall d \in \mathbb{Z}_{>0} \tag{6.27}$$

are simple unitary VOAs with central charge $c = d$, which are also strongly local. Moreover, the corresponding irreducible conformal nets satisfy

$$\mathcal{A}_{U(1)}^d := \bigotimes_{j=1}^{d} \mathcal{A}_{U(1)} = \mathcal{A}_{M(1)^d} \qquad \forall d \in \mathbb{Z}_{>0}\,. \tag{6.28}$$

$\mathcal{A}_{U(1)}^d$ represent the chiral field theory of d free bosons and thus we call them the d **free bosons net**. An equivalent way to construct $M(1)^d$ is implementing a similar construction of Example 6.1.2, starting differently with a d-dimensional complex vector space $\mathfrak{h}$, still equipped with a non-degenerate symmetric bilinear form. In that case, what we obtain is a simple unitary VOA isomorphic to $M(1)^d$ (see [DL14, Propostion 4.10]) and therefore the same corresponding conformal net theory.

Example 6.1.5 (**The free fermion**)**.** Let A be a complex vector superspace, such that $A_{\overline{0}} = \{0\}$ and $\dim_{\mathbb{C}} A_{\overline{1}} = 1$. Suppose that A is equipped with a non-degenerate skew-supersymmetric bilinear form $B(\cdot\,,\cdot)$, which turns out to be a non-degenerate symmetric one on $A_{\overline{1}}$ because

$$B(\varphi,\psi) = -(-1)^{p(\varphi)}B(\psi,\varphi) = B(\psi,\varphi) \qquad \forall \varphi,\psi \in A\,. \tag{6.29}$$

We consider the Clifford affinization of $(A, B(\cdot\,,\cdot))$, i.e., the Lie superalgebra

$$\mathfrak{f} := A \otimes_{\mathbb{C}} \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K \tag{6.30}$$

with commutation relations:

$$\begin{aligned} [\varphi_m, \psi_n] &:= B(\varphi, \psi)\delta_{m,-n}K \qquad \forall \varphi, \psi \in A \;\; \forall m, n \in \mathbb{Z} - \frac{1}{2} \\ [\mathfrak{f}, K] &:= 0 \end{aligned} \tag{6.31}$$

where $y_m := y \otimes t^{m-\frac{1}{2}}$, $p(y_m) = \overline{1}$ and $p(K) = \overline{0}$ for all $y \in A$ and all $m \in \mathbb{Z} - \frac{1}{2}$. $\mathfrak{f}$ is known as the **fermion algebra** and the basis element $\varphi \in A$ such that $B(\varphi, \varphi) = 1$ is called the **supercurrent vector** of A (see Example 6.1.8 for this choice of the name). $(\mathfrak{f}, -\partial_t)$ is a regular formal distribution Lie superalgebra such that

$$\mathfrak{f}_{--} = A \otimes_{\mathbb{C}} \mathbb{C}[t] \oplus \mathbb{C}K\,, \qquad \partial_t \mathfrak{f}_{--} = A \otimes_{\mathbb{C}} \mathbb{C}[t]\,. \tag{6.32}$$

Consider the universal enveloping algebra $U(\mathfrak{f})$ and the Verma module

$$\widetilde{V}^1(\mathfrak{f}) := U(\mathfrak{f})/\langle K - 1\,,\ \varphi_m \mid m \geq \frac{1}{2}\rangle\,, \tag{6.33}$$

which comes from the $\mathfrak{f}_{--}$-representation λ_1 on $\mathbb{C}$ such that $\lambda_1(\partial_t \mathfrak{f}_{--}) = 0$ and $\lambda_1(K) = 1$. Then, $\widetilde{V}^1(\mathfrak{f})$ has a structure of vertex superalgebra: it is linearly generated by the ordered elements with parities

$$\{\Omega, \varphi_{m_1} \cdots \varphi_{m_s}\Omega \mid m_1 \leq \cdots \leq m_s \leq -\frac{1}{2}\}, \quad p(\varphi_{m_1} \cdots \varphi_{m_s}\Omega) = \overline{s} \in \mathbb{Z}_2 \tag{6.34}$$

where Ω is the vacuum vector, T is induced by $-\partial_t$ and state-field correspondence obtainable from

$$Y(\varphi_{-\frac{1}{2}}\Omega, z) = \sum_{n\in\mathbb{Z}} (\varphi_{-\frac{1}{2}}\Omega)_{(n)} z^{-n-1} = \sum_{n\in\mathbb{Z}-\frac{1}{2}} \varphi_n z^{-n-\frac{1}{2}}\,. \tag{6.35}$$

Actually, $\widetilde{V}^1(\mathfrak{f})$ is simple, see [Kac01, Theorem 3.6] and carries a structure of VOSA of central charge $c = \frac{1}{2}$ of CFT type by [Kac01, Theprem 4.10(b)]. Specifically, the conformal vector is $\nu := \frac{1}{2}\varphi_{-\frac{3}{2}}\varphi_{\frac{1}{2}}\Omega$ with

$$L_0(\varphi_{m_1} \cdots \varphi_{m_s}\Omega) = (-\sum_{j=1}^{s} m_j)\varphi_{m_1} \cdots \varphi_{m_s}\Omega \qquad \forall s \in \mathbb{Z}_{\geq 0}\,. \tag{6.36}$$

$F := \widetilde{V}^1(\mathfrak{f})$ is known as the **neutral** or **real free fermion vertex operator superalgebra**. It is known, e.g. [KT85, Section 4] or [KR87, Lecture 4], that F has a scalar product $(\cdot|\cdot)$ such that

$$\begin{aligned} (\Omega|\Omega) &= 1 \\ (\varphi_m a|b) &= (a|\varphi_{-m}b) \quad \forall m \in \mathbb{Z} - \frac{1}{2} \;\; \forall a, b \in F \\ (L_n a|b) &= (a|L_{-n}b) \quad \forall n \in \mathbb{Z} \;\; \forall a, b \in F\,. \end{aligned} \tag{6.37}$$

Therefore, F has a unitary structure thanks to Proposition 2.5.6 because it is generated by its $\frac{1}{2}$-eigenspace $F_{\frac{1}{2}} = \mathbb{C}\varphi$. Also in this case, the unitarity of F was already well-known thanks to [AL17, Theorem 2.7], which gives us the PCT operator, that is, the antilinear map θ defined by the following action on ordered generators of F:

$$\theta(\varphi_{m_1} \cdots \varphi_{m_s}\Omega) = (-1)^s \varphi_{m_1} \cdots \varphi_{m_s}\Omega \qquad \forall s \in \mathbb{Z}_{\geq 0}\,. \tag{6.38}$$

By Theorem 5.2.4, F is a simple strongly graded-local unitary VOSA and thus we have a corresponding irreducible graded-local conformal net $\mathcal{F} := \mathcal{A}_F$, known as the **neutral** or **real free fermion net**. $\mathcal{F}$ is exactly the net of graded-local algebras of the chiral Ising model affiliated with the Majorana field, which gives the generators of the Araki's self-dual CAR-algebra, see [Boc96, Section 2.1] and [Ara70, Section 2]. In particular, the Majorana field $\psi(z)$ in [Boc96, Section 2.1] is our **supercurrent** $Y(\varphi_{-\frac{1}{2}}\Omega, z)$.

Also for the free fermion, we have the corresponding of Remark 6.1.3 and Example 6.1.4.

Remark 6.1.6. With the notation as in Example 6.1.5, define the $\mathfrak{f}$-modules

$$F(k) := U(\mathfrak{f})/\langle K - k\,,\ \varphi_m \mid m \geq \frac{1}{2}\rangle \qquad \forall k \in \mathbb{C}\,. \tag{6.39}$$

We have a structure of simple VOSA of CFT type with central charge $c = \frac{1}{2}$ on every $F(k)$ for all $k \in \mathbb{C}\backslash\{0\}$, whereas $F(0)$ is a vertex superalgebra which is not simple, see [Kac01, Theorems 3.6 and 4.10(b)]. For all $k \in \mathbb{C}\backslash\{0\}$, it is not difficult to verify that there exists a VOSA automorphism ϕ_k between F and $F(k)$ defined by

$$\phi_k(\varphi_{m_1}\cdots\varphi_{m_s}\Omega) := k^{-\frac{s}{2}}\varphi_{m_1}\cdots\varphi_{m_s}\Omega \qquad \forall s \in \mathbb{Z}_{\geq 0}\,. \tag{6.40}$$

It follows that F and $F(k)$ give rise to the same graded-local conformal net theory $\mathcal{F}$ whenever $k \in \mathbb{C}\backslash\{0\}$.

Example 6.1.7 (*d* **free fermions**)**.** Applying Corollary 5.2.3 to Example 6.1.5, we obtain that

$$F^d := \hat{\bigotimes}_{j=1}^{d} F \qquad \forall d \in \mathbb{Z}_{>0} \tag{6.41}$$

are simple strongly graded-local unitary VOSAs. In particular, their associated irreducible graded-local conformal nets are

$$\mathcal{F}^d := \hat{\bigotimes}_{j=1}^{d} \mathcal{F} = \mathcal{A}_{F^d} \qquad \forall d \in \mathbb{Z}_{>0}\,. \tag{6.42}$$

Obviously, $\mathcal{F}^d$ represents the graded-local conformal net theory of d free fermions and it is called the d **free fermions net**. Specifically, $\mathcal{F}^2$ is known as the **charged** or **complex free fermion net**. Note that we can obtain every VOSA F^d starting in Example 6.1.5 with a complex vector superspace A such that $A_{\overline{0}} = \{0\}$ and $\dim_{\mathbb{C}} A_{\overline{1}} = d$, equipped with a non-degenerate skew-supersymmetric bilinear form, which amounts again to a non-degenerate symmetric bilinear form on $A_{\overline{1}}$.

Of course, we can produce far more examples picking graded-local tensor products of free fermions, free bosons, Virasoro models or other well-known ones. The following is an example of that.

Example 6.1.8 (**Current and supercurrent algebra models**)**.** Let $\mathfrak{g} = \mathfrak{g}_{\overline{0}} \oplus \mathfrak{g}_{\overline{1}}$ be a finite dimensional simple Lie superalgebra such that $[\mathfrak{g}_{\overline{1}}, \mathfrak{g}_{\overline{1}}] = 0$ and with a non-degenerate symmetric bilinear form $B(\cdot\,,\cdot)$ such that

$$B(\mathfrak{g}_{\overline{0}}, \mathfrak{g}_{\overline{1}}) = 0\,,\ \ B([a,\varphi],\psi) = B(\varphi,[a,\psi]) \quad \forall a \in \mathfrak{g}_{\overline{0}}\ \forall \varphi,\psi \in \mathfrak{g}_{\overline{1}}\,. \tag{6.43}$$

According to [Li96, Section 4.3], the restrictions of $B(\cdot\,,\cdot)$ to $\mathfrak{g}_{\overline{0}}$ and $\mathfrak{g}_{\overline{1}}$ amount to a non-degenerate symmetric invariant bilinear form $B(\cdot\,,\cdot)_{\overline{0}}$ on $\mathfrak{g}_{\overline{0}}$ and to a non-degenerate symmetric bilinear form $B(\cdot\,,\cdot)_{\overline{1}}$ on the $\mathfrak{g}_{\overline{0}}$-module $\mathfrak{g}_{\overline{1}}$ such that

$$B(a\cdot\varphi,\psi)_{\overline{1}} = B(\varphi, a\cdot\psi)_{\overline{1}} \quad \forall a \in \mathfrak{g}_{\overline{0}}\ \forall \varphi,\psi \in \mathfrak{g}_{\overline{1}}\,. \tag{6.44}$$

Set $d_{\overline{0}} := \dim_{\mathbb{C}} \mathfrak{g}_{\overline{0}}$ and $d_{\overline{1}} := \dim_{\mathbb{C}} \mathfrak{g}_{\overline{1}}$. Consider the central extension $\widetilde{\mathfrak{g}}$ of the loop superalgebra associated to $\mathfrak{g}$, that is, the Lie superalgebra

$$\widetilde{\mathfrak{g}} := \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[t,t^{-1}] \oplus \mathbb{C}K \tag{6.45}$$

with the following commutation relations: for every $x \in \mathfrak{g}_{\overline{0}}$, $y \in \mathfrak{g}_{\overline{1}}$, $m \in \mathbb{Z}$ and $n \in \mathbb{Z} - \frac{1}{2}$, define $x_m := x \otimes t^m$ with $p(x_m) := \overline{0} =: p(K)$ and $y_n := y \otimes t^{n-\frac{1}{2}}$ with $p(y_n) := \overline{1}$, then

$$\begin{aligned} [a_m, b_n] &:= [a,b]_{m+n} + m\delta_{m,-n}B(a,b)K \qquad \forall a,b \in \mathfrak{g}_{\overline{0}}\ \forall m,n \in \mathbb{Z} \\ [a_m, \varphi_n] &:= [a,\varphi]_{m+n} \qquad \forall a \in \mathfrak{g}_{\overline{0}}\ \forall \varphi \in \mathfrak{g}_{\overline{1}}\ \forall m \in \mathbb{Z}\ \forall n \in \mathbb{Z} - \frac{1}{2} \\ [\varphi_m, \psi_n] &:= B(\varphi,\psi)K \qquad \forall \varphi,\psi \in \mathfrak{g}_{\overline{1}}\ \forall m,n \in \mathbb{Z} - \frac{1}{2} \\ [\widetilde{\mathfrak{g}}, K] &:= 0\,. \end{aligned} \tag{6.46}$$

$\widetilde{\mathfrak{g}}$ is called a **supercurrent algebra**. Accordingly, even and odd elements of a fixed orthonormal basis $\{J^1, \dots, J^{d_{\overline{0}}}, \varphi^1, \dots, \varphi^{d_{\overline{1}}}\}$ for $\mathfrak{g}$ are called **current** and **supercurrent vectors** respectively. Note that if $d_{\overline{0}} = 0$, we are in the case of Example 6.1.7 with $d = d_{\overline{1}}$. Similarly, if $d_{\overline{1}} = 0$, Example 6.1.4 with $d = d_{\overline{0}}$ is a special case of the present example, namely when $\mathfrak{g}_{\overline{0}}$ is a commutative Lie algebra. Anyway, $(\widetilde{\mathfrak{g}}, -\partial_t)$ is a formal distribution Lie superalgebra such that

$$\widetilde{\mathfrak{g}}_{--} = \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[t] \oplus \mathbb{C}K\,, \qquad \partial_t \widetilde{\mathfrak{g}}_{--} = \mathfrak{g}[t] := \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[t]\,. \tag{6.47}$$

Let us consider the universal enveloping algebra $U(\widetilde{\mathfrak{g}})$ and the following Verma modules at level $k \in \mathbb{C}$ associated to it:

$$\widetilde{V}^k(\mathfrak{g}) := U(\widetilde{\mathfrak{g}})/\langle K - k\,,\ \mathfrak{g}[t]\rangle\,, \tag{6.48}$$

which are clearly obtained trough the $\widetilde{g}_{--}$-representations λ_k on $\mathbb{C}$ such that $\lambda_k(\mathfrak{g}[t]) = 0$ and $\lambda_k(K) = k$ respectively. Every $\widetilde{V}^k(\mathfrak{g})$ has a vertex superalgebra structure: it is linearly generated by the ordered elements with parities:

$$\begin{aligned} &\{\Omega\,,\ J^{j_1}_{m_1} \cdots J^{j_s}_{m_s} \varphi^{h_1}_{n_1} \cdots \varphi^{h_r}_{n_r} \Omega \mid {}^{\,j_1 \leq \cdots \leq j_s \leq d_{\overline{0}}}_{\,m_1 \leq \cdots \leq m_s \leq -1}\,,\ {}^{\,h_1 \leq \cdots \leq h_r \leq d_{\overline{1}}}_{\,n_1 \leq \cdots \leq n_r \leq -\frac{1}{2}}\} \\ &p(J^{j_1}_{m_1} \cdots J^{j_s}_{m_s} \varphi^{h_1}_{n_1} \cdots \varphi^{h_r}_{n_r} \Omega) = \overline{r} \in \mathbb{Z}_2 \end{aligned} \tag{6.49}$$

where Ω is the vacuum vector, T is induced by $-\partial_t$ and the state-field correspondence obtained from

$$\begin{aligned} Y(J^j_{-1}\Omega, z) &= \sum_{n\in\mathbb{Z}} (J^j_{-1}\Omega)_{(n)} z^{-n-1} = \sum_{n\in\mathbb{Z}} J^j_n z^{-n-1} && \forall j \in \{1, \dots, d_{\overline{0}}\} \\ Y(\varphi^l_{-\frac{1}{2}}\Omega, z) &= \sum_{n\in\mathbb{Z}} (\varphi^l_{-\frac{1}{2}}\Omega)_{(n)} z^{-n-1} = \sum_{n\in\mathbb{Z}-\frac{1}{2}} \varphi^l_n z^{-n-\frac{1}{2}} && \forall l \in \{1, \dots, d_{\overline{1}}\}\,. \end{aligned} \tag{6.50}$$

Note that $H = -t\partial_t$ makes $\widetilde{\mathfrak{g}}$ be graded, so the simple quotients $V^k(\mathfrak{g})$ for all $k \in \mathbb{C}$ are simple vertex superalgebras. [Li96, Proposition 4.3.6] says that, for any fixed $k \in \mathbb{C}$, $V^k(\mathfrak{g})$ is isomorphic to the tensor product $V^k(\mathfrak{g}_{\overline{0}}) \otimes F^{d_{\overline{1}}}$. Let $h^\vee$ be the **dual Coxeter number** associated to the pair $(\mathfrak{g}_{\overline{0}}, B(\cdot\,,\cdot))$, see [Kac95, Chapter 6]. By [Kac01, Theorem 5.7], if $k \neq -h^\vee$, then every $V^k(\mathfrak{g}_{\overline{0}})$ is a simple VOA with central charge $c_k = \frac{kd_{\overline{0}}}{k+h^\vee}$ of CFT type. In particular, the conformal vectors are given by

$$\nu^k := \frac{1}{2(k+h^\vee)} \sum_{j=1}^{d_{\overline{0}}} J^j_{-1} J^j_{-1}\Omega \qquad \forall k \neq -h^\vee \tag{6.51}$$

and

$$L_0(J^{j_1}_{m_1} \cdots J^{j_s}_{m_s}\Omega) = (-\sum_{l=1}^s m_l) J^{j_1}_{m_1} \cdots J^{j_s}_{m_s}\Omega \qquad \forall s \in \mathbb{Z}_{\geq 0}\,. \tag{6.52}$$

$V^k(\mathfrak{g}_{\overline{0}})$ are called **current vertex operator algebras**. Consequently, $V^k(\mathfrak{g})$ are also simple VOSAs with central charge $c_k = \frac{d_{\overline{1}}}{2} + \frac{kd_{\overline{0}}}{k+h^\vee}$ of CFT type, called **supercurrent vertex operator superalgebras**. If $k \in \mathbb{Z}_{>0}$ and $k \neq 0$, then, see [Kac95, Theorem 11.7], $V^k(\mathfrak{g}_{\overline{0}})$ has a scalar product such that (eventually changing the linear basis of $\mathfrak{g}_{\overline{0}}$)

$$\begin{aligned} &(\Omega|\Omega) = 1 \\ &(J^j_n a|b) = (a|J^j_{-n} b) \qquad \forall j \in \{1, \dots, d_{\overline{0}}\}\ \forall n \in \mathbb{Z}\ \forall a, b \in V^k(\mathfrak{g}_{\overline{0}})\,. \end{aligned} \tag{6.53}$$

Therefore, thanks to the fact that $V^k(\mathfrak{g}_{\overline{0}})$ is generated by $V^k(\mathfrak{g}_{\overline{0}})_1 = \bigoplus_{j=1}^{d_{\overline{0}}} \mathbb{C}J^j_{-1}\Omega$, which give rise to Hermitian fields, we have that $V^k(\mathfrak{g}_{\overline{0}})$ is a simple unitary VOA by Proposition 2.5.6. Note that [DL14, Theorem 4.7] gives us also the PCT operator θ on ordered (eventually changed) generators:

$$\theta(J^{j_1}_{m_1} \cdots J^{j_s}_{m_s}\Omega) = (-1)^s J^{j_1}_{m_1} \cdots J^{j_s}_{m_s}\Omega \qquad \forall s \in \mathbb{Z}_{\geq 0}\,. \tag{6.54}$$

Further, by Theorem 5.2.4, $V^k(\mathfrak{g}_{\overline{0}})$ is a simple strongly local unitary VOA whenever $k \in \mathbb{Z}_{>0}$ and $k \neq -h^\vee$. By Corollary 5.2.3 and Example 6.1.7, it follows that $V^k(\mathfrak{g})$ is a simple strongly graded-local unitary VOSA whenever $k \in \mathbb{Z}_{>0}$ and $k \neq -h^\vee$. $\mathcal{A}_{V^k(\mathfrak{g})} = \mathcal{A}_{V^k(\mathfrak{g}_{\overline{0}})} \otimes \mathcal{F}^{d_{\overline{1}}}$ are known as **supercurrent algebra nets**. $\mathcal{A}_{V^k(\mathfrak{g}_{\overline{0}})}$ are usually called $\mathcal{A}_{G_k}$, that is, the **loop group conformal nets** associated to the level k positive-energy representation of the loop algebra of G, which is the compact connected simply connected real Lie group underlying $\mathfrak{g}_{\overline{0}}$, see [CKLW18, Example 8.7] and references therein.

6.2 Superconformal structures and further examples

We relate the superconformal structures which naturally arise in the graded-local conformal net and VOSA theories, see [CHL15, Definition 2.11] and [Kac01, Definition 5.9] respectively. To do this, we have to introduce the super-Virasoro models first:

Example 6.2.1 ($N = 1$ **super-Virasoro models**)**.** The **Neveu-Schwarz** or $N = 1$ **super-Virasoro algebra** NS is the Lie superalgebra

$$NS := \overbrace{\bigoplus_{m\in\mathbb{Z}} \mathbb{C}L_m \oplus \mathbb{C}C}^{\text{even}} \oplus \overbrace{\bigoplus_{n\in\mathbb{Z}-\frac{1}{2}} \mathbb{C}G_n}^{\text{odd}} \tag{6.55}$$

with commutation relations:

$$\begin{aligned} [L_m, L_n] &:= (m-n)L_{m+n} + \frac{m^3-m}{12}\delta_{m,-n}C \qquad \forall m,n\in\mathbb{Z} \\ [L_m, G_n] &:= \left(\frac{m}{2}-n\right)G_{m+n} \qquad \forall m\in\mathbb{Z}\ \forall n\in\mathbb{Z}-\frac{1}{2} \\ [G_m, G_n] &:= 2L_{m+n} + \frac{1}{3}\left(m^2-\frac{1}{4}\right)\delta_{m,-n}C \qquad \forall m,n\in\mathbb{Z}-\frac{1}{2} \\ [NS, C] &:= 0\,. \end{aligned} \tag{6.56}$$

$(NS, \mathrm{ad}L_{-1})$ is a regular formal distribution Lie superalgebra such that

$$\begin{aligned} NS_{--} &= \bigoplus_{m\geq -1} \mathbb{C}L_n \oplus \mathbb{C}C \oplus \bigoplus_{n\geq -\frac{1}{2}} \mathbb{C}G_n \\ \mathrm{ad}L_{-1}NS_{--} &= \bigoplus_{m\geq -1} \mathbb{C}L_n \oplus \bigoplus_{n\geq -\frac{1}{2}} \mathbb{C}G_n\,. \end{aligned} \tag{6.57}$$

Let $U(NS)$ be the universal enveloping algebra of NS and define the Verma modules associated to NS:

$$\widetilde{V}^c(NS) := U(NS)/\langle C-c\,, \bigoplus_{m\geq -1} \mathbb{C}L_m \oplus \bigoplus_{n\geq -\frac{1}{2}} \mathbb{C}G_n\rangle \qquad \forall c\in C\,, \tag{6.58}$$

which is obtained from the NS_{--}- representations λ_c on $\mathbb{C}$ defined by $\lambda_c(\mathrm{ad}L_{-1}NS_{--}) = 0$ and $\lambda_c(C) = c$ respectively. The vertex superalgebra structure on every $\widetilde{V}^c(NS)$ is given by: the following is a set of ordered linear generators with parities

$$\begin{aligned} &\{\Omega\,,\ L_{m_1}\cdots L_{m_s}G_{n_1}\cdots G_{n_r}\Omega \mid {}^{m_1\leq\cdots\leq m_s\leq -2}_{n_1\leq\cdots\leq n_r\leq -\frac{3}{2}}\} \\ &p(L_{m_1}\cdots L_{m_s}G_{n_1}\cdots G_{n_r}\Omega) = \overline{r}\in\mathbb{Z}_2\,. \end{aligned} \tag{6.59}$$

where Ω is the vacuum vector, $T := L_{-1}$ and state-field correspondence obtainable from

$$\begin{aligned} Y(\nu, z) &:= \sum_{n\in\mathbb{Z}} \nu_{(n)}z^{-n-1} = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}\,, \qquad \nu := L_{-2}\Omega \\ Y(\tau, z) &:= \sum_{n\in\mathbb{Z}} \tau_{(n)}z^{-n-1} = \sum_{n\in\mathbb{Z}-\frac{1}{2}} G_n z^{-n-\frac{3}{2}}\,, \qquad \tau := G_{-\frac{3}{2}}\Omega\,. \end{aligned} \tag{6.60}$$

NS is graded by $\mathrm{ad}L_0$ and thus the simple quotients $V^c(NS)$ produce simple VOSAs with central charge $c\in\mathbb{C}$ by [Kac01, Theorem 4.7 and Lemma 5.9], see also [Li96, Section 4.2] and [KW94, Section 3.1]. Specifically, the conformal vector is $\nu = L_{-2}\Omega$ and $\tau := G_{-\frac{3}{2}}\Omega$ is called a **superconformal vector**. Note from the commutation relations (6.56), that τ is a primary vector:

$$L_1\tau = -[G_{-\frac{3}{2}}, L_1]\Omega = 2G_{-\frac{1}{2}}\Omega = 0\,, \quad L_2\tau = -[G_{-\frac{3}{2}}, L_2]\Omega = \frac{5}{2}G_{\frac{1}{2}}\Omega = 0 \tag{6.61}$$

The $\frac{1}{2}\mathbb{Z}$-grading is given by the following action on the ordered generators

$$L_0(L_{m_1}\cdots L_{m_s}G_{n_1}\cdots G_{n_r}\Omega) = \left(-\sum_{j=1}^{s} m_j - \sum_{j=1}^{r} n_j\right) L_{m_1}\cdots L_{m_s}G_{n_1}\cdots G_{n_r}\Omega\,, \tag{6.62}$$

where s, r are non-negative integers. Moreover, by [FQS85], [GKO86, Section 4], [KW86, Theorem 5.1], if

$$\text{either} \quad c \geq \frac{3}{2} \quad \text{or} \quad c = \frac{3}{2}\left(1 - \frac{8}{m(m+2)}\right) \quad m \geq 2\,, \tag{6.63}$$

then there exists a scalar product on $V^c(NS)$ with respect to which $Y(\nu, z)$ and $Y(\tau, z)$ are Hermitian fields. Using Proposition 2.5.6, $V^c(NS)$, which is generated by ν and τ, is a simple unitary VOSA, provided that c is as in (6.63). [AL17, Section 2.2] gives an alternative way to prove the unitarity of $V^c(NS)$. In particular, the former shows that the PCT operator is the antilinear map which acts as the identity on ordered generators. It follows from [CKL08, Eqs. (26) and (27)] and Proposition 3.1.2 that $V^c(NS)$ is energy-bounded and $\mathcal{A}_{V^c(NS)}$ corresponds to the **Neveu-Schwarz algebra** or $N = 1$ **super-Virasoro net** SVir_c constructed in [CKL08, Section 6.3], which is an irreducible graded-local conformal net for all c as in (6.63) by [CKL08, Theorem 33] (in particular, $V^c(NS)$ is strongly graded-local, see [CKL08, p. 1100]).

The example of the $N = 1$ super-Virasoro models enables us to talk about the superconformal structure in both graded-local conformal net and VOSA settings. The question is whether a given VOSA or graded-local conformal net contains also a $N = 1$ super-Virasoro theory in addition to the Virasoro one. This is relevant when it comes to supersymmetry in CFT. The precise definitions are as follows, see [CHL15, Definition 2.11] and [Kac01, Definition 5.9] respectively:

Definition 6.2.2. A graded-local conformal net $\mathcal{A}$ with central charge c is said to be $N = 1$ **superconformal** if it contains SVir_c as Möbius covariant subnet and

$$U(\mathrm{Diff}^+(I)^{(\infty)}) \subset \mathrm{SVir}_c(I) \subseteq \mathcal{A}(I) \qquad \forall I \in \mathcal{J} \tag{6.64}$$

that is, SVir_c contains the Virasoro subnet Vir_c (see Example 6.1.1) given by the conformal symmetries.

Definition 6.2.3. A VOSA V with central charge c is said to be $N = 1$ **superconformal** if there exists a superconformal vector τ associated to the conformal vector ν, that is, the operators G_n with $n \in \mathbb{Z} - \frac{1}{2}$, given by $Y(\tau, z) = \sum_{n\in\mathbb{Z}-\frac{1}{2}} G_n z^{-n-\frac{3}{2}}$, satisfy the Neveu-Schwarz Lie superalgebra relation (6.56). If V has an invariant scalar product with respect to which τ is PCT-invariant, then we say that V is a **unitary** $N = 1$ **superconformal VOSA**.

Accordingly, if V is a simple $N = 1$ superconformal VOSA with central charge c, then the generated vertex subalgebra $W(\{\nu, \tau\})$ is isomorphic to $V^c(NS)$ of Example 6.2.1.

Remark 6.2.4. Trivially, $V^c(NS)$ with $c \in \mathbb{C}\backslash\{0\}$ is a simple $N = 1$ superconformal VOSA, which is unitary whenever c is as in (6.63). For c as in (6.63), every SVir_c is an $N = 1$ superconformal net.

To state a correspondence between the two superconformal structures, we need the following result:

Lemma 6.2.5. *Let $V^c(NS)$ be the Neveu-Schwarz VOSA with central charge $c \in \mathbb{C}\backslash\{0\}$. If g is either a linear or an antilinear automorphism of the vertex superalgebra $V^c(NS)$, which preserves the conformal vector, then $g(\tau) = \pm\tau$.*

Proof. By construction, $V^c(NS)_{\frac{3}{2}} = \mathbb{C}\tau$, where $\tau = G_{-\frac{3}{2}}\Omega$. This means that there exists $\alpha \in \mathbb{C} \setminus \{0\}$ such that $g(\tau) = \alpha\tau$. By commutation relations (6.56), we have that

$$G_{-\frac{1}{2}}\tau = 2\nu \;\Rightarrow\; \alpha^2 G_{-\frac{1}{2}}\tau = g(G_{-\frac{1}{2}}\tau) = 2g(\nu) = 2\nu = G_{-\frac{1}{2}}\tau$$

which implies $\alpha = \pm 1$, that is, the desired result. □

Corollary 6.2.6. *Let $V^c(NS)$ be the unitary Neveu-Schwarz VOSA with the central charge c as in the unitary series (6.63). The unitary structure of $V^c(NS)$ is unique with the unique PCT operator θ, given by the antilinear map which acts as the identity on ordered generators. Furthermore,*

$$\mathrm{Aut}(V^c(NS)) = \mathrm{Aut}_{(\cdot|\cdot)}(V^c(NS)) = \mathrm{Aut}(\mathrm{SVir}_c) \cong \mathbb{Z}_2\,.$$

Proof. From Theorem 2.4.5 and Proposition 2.4.2, we obtain the uniqueness of the unitary structure. The explicit form of θ is already given by [AL17, Theorem 2.6]. Anyway, it follows easily from (2.72) applied to τ and the fact that τ gives rise to a Hermitian field, see Example 6.2.1, which implies also its uniqueness. The last part is obtained by Theorem 3.2.19. □

Now we can prove the following correspondence:

Theorem 6.2.7. *Let V be a simple $N=1$ superconformal VOSA equipped with an invariant scalar product $(\cdot|\cdot)$ and PCT operator θ. If $W(\{\nu,\tau\})$ is a unitary subalgebra of V, then V is a simple unitary $N=1$ superconformal VOSA. Furthermore, V is a simple strongly graded-local unitary $N=1$ superconformal VOSA if and only if $\mathcal{A}_V$ is an irreducible $N=1$ superconformal net.*

Proof. By construction, $W:=W(\{\nu,\tau\})$ is isomorphic to a Neveu-Schwarz VOSA $V^c(NS)$ for some $c\in\mathbb{C}\setminus\{0\}$. Moreover, c must be a positive real number because of

$$0<(\nu|\nu)=(L_{-2}\Omega|\nu)=(\Omega|L_2\nu)=\frac{c}{2}$$

by [Kac01, Theorem 4.10 (a)]. The PCT operator θ of V restricts to an antilinear automorphism of W because W is unitary. By Lemma 6.2.5, $\theta(\tau)=\tau$, otherwise from (2.72) applied to τ and [Kac01, Proposition 5.9 (ii)], we have that

$$0<(\tau|\tau)=(G_{-\frac{3}{2}}\Omega|\tau)=(\Omega|(\theta\tau)_{\frac{3}{2}}\tau)=-\frac{2c}{3}<0$$

which is impossible. This means that $Y(\tau,z)$ is a Hermitian field of V and thus of W with respect to $(\cdot|\cdot)$. Hence, $(W,(\cdot|\cdot))$ realises a unitary representation of NS and therefore c must be in the unitary series (6.63), see Example 6.2.1 and references therein. This implies that V is a simple unitary $N=1$ superconformal VOSA.

Now, suppose that V is a simple strongly graded-local unitary VOSA. On the one hand, if V is also unitary $N=1$ superconformal, then by Theorem 5.1.1, $W:=W(\{\nu,\tau\})$ gives rise to a covariant subnet $\mathcal{A}_W$ of $\mathcal{A}_V$, containing the Virasoro subnet Vir_c arisen from the conformal vector $\nu\in W$. Then, $\mathcal{A}_V$ is a $N=1$ superconformal net. On the other hand, if $\mathcal{A}_V$ is a $N=1$ superconformal net, then it contains the covariant subnets $\mathrm{Vir}_c\subset\mathrm{SVir}_c$ for some c as in (6.63). By Theorem 5.1.1, the latter nets must derive from unitary subalgebras $L(c)\cong W(\{\nu\})$ (see Example 6.1.1) and $W:=W(\{\nu,\tau\})$ for a superconformal vector τ respectively. By the discussion above, τ is θ-invariant and thus V is unitary $N=1$ superconformal. □

Now, we are going to look at those automorphisms in the VOSA and the graded-local conformal net setting which preserve the respective superconformal structures.

Definition 6.2.8. Let V be a $N=1$ superconformal VOSA. We define $\mathrm{Aut}^{\mathrm{sc}}(V)$ as the closed subgroup of $\mathrm{Aut}(V)$ formed by those automorphisms which preserve the superconformal vector. If V is unitary, we set

$$\mathrm{Aut}^{\mathrm{sc}}_{(\cdot|\cdot)}(V):=\mathrm{Aut}^{\mathrm{sc}}(V)\cap\mathrm{Aut}_{(\cdot|\cdot)}(V)\,.$$

Thus, we have a result similar to Theorem 2.4.5:

Theorem 6.2.9. *Let V be a unitary $N=1$ superconformal VOSA. Then, $\mathrm{Aut}^{\mathrm{sc}}_{(\cdot|\cdot)}(V)$ is a compact subgroup of $\mathrm{Aut}(V)$. Moreover, if V is simple, then the following are equivalent:*

(i) *$(\cdot|\cdot)$ is the unique normalized invariant scalar product which makes V a simple unitary $N=1$ superconformal VOSA;*

(ii) *$(\cdot|\cdot)$ is the unique normalized invariant scalar product on V such that the corresponding PCT operator θ fixes the superconformal vector τ;*

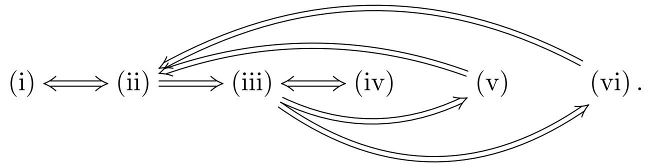
(iii) $\mathrm{Aut}^{\mathrm{sc}}_{(\cdot|\cdot)}(V)=\mathrm{Aut}^{\mathrm{sc}}(V)$;

(iv) *θ commutes with every $g\in\mathrm{Aut}^{\mathrm{sc}}(V)$;*

(v) $\mathrm{Aut}^{\mathrm{sc}}(V)$ *is compact;*

(vi) $\mathrm{Aut}^{\mathrm{sc}}_{(\cdot|\cdot)}(V)$ *is totally disconnected.*

Proof. The first claim follows from Theorem 2.4.5. For the "TFAE" part, we straightforwardly adapt the proof of [CKLW18, Theorem 5.21]. We prove the following implications:



(i)⇔(ii) is trivial by Definition 6.2.3.

(ii)⇒(iii) follows from the fact that if there exists $g \in \mathrm{Aut}^{\mathrm{sc}}(V) \setminus \mathrm{Aut}^{\mathrm{sc}}_{(\cdot|\cdot)}(V)$, then $(g(\cdot)|g(\cdot))$ defines a new invariant scalar product on V, which is different from $(\cdot|\cdot)$ and such that the corresponding PCT operator $g^{-1}\theta g$ fixes τ.

Regarding (iii)⇔(iv), we have that $g \in \mathrm{Aut}^{\mathrm{sc}}(V)$ is unitary if and only if $(g\theta a|gb) = (\theta a|b)$ for all $a, b \in V$. On the other hand, $(\theta ga|gb) = (\theta a|b)$ for all $a, b \in V$ thanks to (ii)⇔(iii) of Corollary 2.4.4. It follows that g is unitary if and only if it commutes with θ.

(iii)⇒(v) is immediate from the first part. Checking the construction of the automorphism $h \in \mathrm{Aut}(V)$ of Proposition 2.4.2 in the proof of [CKLW18, Proposition 5.19], we note that $h \in \mathrm{Aut}^{\mathrm{sc}}(V)$ if we are dealing with PCT operators which preserve τ. Then, the proofs of (v)⇒(ii) and (vi)⇒(ii) are as in the proof of [CKLW18, Proposition 5.21]. Also (iii)⇒(vi) is proved as in [CKLW18, Proposition 5.21]. □

Definition 6.2.10. If $\mathcal{A}$ is a $N = 1$ superconformal net, denote by $\mathrm{Aut}^{\mathrm{sc}}(\mathcal{A})$ the group of those automorphisms of the net which fix the elements of the covariant subnet SVir_c.

Note that Corollary 6.2.6 says us that, whenever c is chosen as in (6.63), then

$$\{1_{V^c(NS)}\} = \mathrm{Aut}^{\mathrm{sc}}_{(\cdot|\cdot)}(V^c(NS)) = \mathrm{Aut}^{\mathrm{sc}}(V^c(NS)) = \mathrm{Aut}^{\mathrm{sc}}(\mathrm{SVir}_c)\,. \tag{6.65}$$

More generally, proceeding similarly to the proof of Theorem 3.2.19, we have that:

Theorem 6.2.11. *Let V be a simple strongly graded-local unitary $N = 1$ superconformal VOSA. Then,* $\mathrm{Aut}^{\mathrm{sc}}_{(\cdot|\cdot)}(V) = \mathrm{Aut}^{\mathrm{sc}}(\mathcal{A}_V)$*. Moreover, if* $\mathrm{Aut}^{\mathrm{sc}}(V)$ *is compact, then* $\mathrm{Aut}^{\mathrm{sc}}(V) = \mathrm{Aut}^{\mathrm{sc}}(A_V)$*.*

Now, we can present further models, which show also a superconformal structure.

Example 6.2.12 (Kac-Todorov models). We present a special case of Example 6.1.8, known as Kac-Todorov models, see [KT85], [KW94, Section 2.1] and [Kac01, Theorem 5.9]. These objects occur when the Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ is chosen such that $\mathfrak{g}_{\bar{1}}$ is the adjoint module of $\mathfrak{g}_{\bar{0}}$, in which case $\mathfrak{g}_{\bar{1}} = \mathfrak{g}_{\bar{0}}$ as vector spaces and $d := d_{\bar{0}} = d_{\bar{1}}$, see [Li96, Remark 4.3.9]. The corresponding supercurrent algebra nets $\mathcal{A}_{V^k(\mathfrak{g})} = \mathcal{A}_{V^k(g_{\bar{0}})} \otimes \mathcal{F}^d$ are exactly the ones constructed in [CHL15, Section 6], see also [CH17, Example 5.14]. Moreover, those supercurrent algebra nets are irreducible $N = 1$ superconformal nets as proved in [CHL15, Proposition 6.2], thus the corresponding Kac-Todorov models $V^k(\mathfrak{g})$ are simple unitary $N = 1$ superconformal VOSAs thanks to Theorem 6.2.7.

Example 6.2.13 ($N = 2$ super-Virasoro models). The **Neveu-Schwarz $N = 2$ super-Virasoro algebra** $N2$ is the Lie superalgebra

$$N2 := \overbrace{\bigoplus_{m\in\mathbb{Z}} \mathbb{C}L_m \oplus \bigoplus_{m\in\mathbb{Z}} \mathbb{C}J_m \oplus \mathbb{C}C}^{\text{even}} \oplus \overbrace{\bigoplus_{n\in\mathbb{Z}-\frac{1}{2}} \mathbb{C}G_n^+ \oplus \bigoplus_{n\in\mathbb{Z}-\frac{1}{2}} \mathbb{C}G_n^-}^{\text{odd}}$$

with the commutation relations:

$$
\begin{aligned}
&[L_m, L_n] := (m-n)L_{m+n} + \frac{m^3-m}{12}\delta_{m,-n}C \qquad \forall m,n\in\mathbb{Z}\\
&[L_m, G^{\pm}_n] := \left(\frac{m}{2}-n\right)G^{\pm}_{m+n} \qquad \forall m\in\mathbb{Z}\ \forall n\in\mathbb{Z}-\frac{1}{2}\\
&[L_m, J_n] := -nJ_{m+n} \qquad \forall m,n\in\mathbb{Z}\\
&[G^+_m, G^-_n] := 2L_{m+n} + (m-n)J_{m+n} + \frac{1}{3}\left(m^2-\frac{1}{4}\right)\delta_{m,-n}C \qquad \forall m,n\in\mathbb{Z}-\frac{1}{2}\\
&[G^{\pm}_m, J_n] := \mp G^{\pm}_{m+n} \qquad \forall m\in\mathbb{Z}-\frac{1}{2}\ \forall n\in\mathbb{Z}\\
&[G^+_m, G^+_n] = 0 = [G^-_m, G^-_n] \qquad \forall m,n\in\mathbb{Z}-\frac{1}{2}\\
&[N2, C] := 0\,.
\end{aligned}
\tag{6.66}
$$

The construction of the unitary VOSA structure is similar to the one of Example 6.2.1, so we just sketch it here in the following. For every $c\in\mathbb{C}$, we define:

$$
\widetilde{V}^c(N2) := U(N2)/\langle \bigoplus_{m\in\mathbb{Z}_{\geq 0}-1}\mathbb{C}L_m \oplus \bigoplus_{m\in\mathbb{Z}_{\geq 0}}\mathbb{C}J_m \oplus \bigoplus_{n\in\mathbb{Z}_{\geq 0}-\frac{1}{2}}\mathbb{C}G^+_n \oplus \bigoplus_{n\in\mathbb{Z}_{\geq 0}-\frac{1}{2}}\mathbb{C}G^-_n\,,\ C-c\rangle
$$

where $U(N2)$ is the universal enveloping algebra of $N2$. Then, the maximal quotient module $V^c(N2)$ has a structure of simple VOSA with central charge $c\in\mathbb{C}$ of CFT type, where $\nu = L_{-1}\Omega$ is the conformal vector, see [Kac01, p. 182]. $V^c(NS)$ are call $N=2$ **super-Virasoro vertex operator superalgebras**. About unitarity, $V^c(NS)$ is a simple unitary VOSA whenever the central charge c is in the range:

$$
\text{either}\quad c\geq 3 \quad\text{or}\quad c=\frac{3n}{n+2}\quad \forall n\geq 0\,, \tag{6.67}
$$

see [CHKLX15, Theorem 3.2] and references therein. Note that

$$
Y(\nu,z)\,,\quad Y(J_{-1}\Omega,z)\,,\quad Y(G^-_{-\frac{3}{2}}\Omega,z)\quad\text{and}\quad Y(G^+_{-\frac{3}{2}}\Omega,z)
$$

are Hermitian fields. Now, energy-bounds are given by [CHKLX15, Eq. (3.1)] and Proposition 3.1.2, whereas the strong graded-locality is (in the proof of) [CHKLX15, Theorem 3.3]. For all c as in (6.67), we denote the irreducible graded-local conformal net $\mathcal{A}_{V^c(N2)}$ by $\mathrm{SVir2}_c$, known as $N=2$ **super-Virasoro net**. Note that $N2$ is a simple $N=1$ superconformal VOSA because it contains the Neveu-Schwarz $N=1$ super-Virasoro algebra NS of Example 6.2.1, setting $G_m := \frac{1}{\sqrt{2}}\left(G^+_m + G^-_m\right)$ for all $m\in\mathbb{Z}-\frac{1}{2}$. According to Theorem 6.2.7, if c is as in (6.67), then $V^c(N2)$ is a simple unitary $N=1$ superconformal VOSA and $\mathrm{SVir2}_c$ is a $N=1$ superconformal net.

Remark 6.2.14. In complete analogy with the $N=1$ superconformal structure introduced in Definition 6.2.2 and Definition 6.2.3, we can define, using the Neveu-Schwarz $N=2$ super-Virasoro algebra $N2$, the $N=2$ **superconformal nets** and the **unitary** $N=2$ **superconformal VOSAs** as done in [CHKLX15, Definition 3.5] and [Kac01, Section 5.9] respectively. Also in this case, we have a complete correspondence between the two $N=2$ superconformal structures in complete analogy with Theorem 6.2.7. In particular, $N2$ contains a copy of NS as unitary subalgebra as explained in Example 6.2.13. Therefore, thanks to Theorem 5.1.1, every $N=2$ superconformal net is also $N=1$ superconformal as well as every unitary $N=2$ superconformal VOSA is a unitary $N=1$ superconformal VOSA.

Remark 6.2.15. Let $V_{L_{2N}}$ be the even rank-one lattice type (simple unitary) VOA associated to the lattice L_{2N} (see e.g. [CGH19, Section II.B] and references therein) and let $V^k(\mathfrak{g})$ be the simple unitary VOA constructed from the Lie algebra $\mathfrak{g}:=\mathfrak{sl}(2,\mathbb{C})$ at level k as in [CKLW18, Example 8.7]. The interesting fact here is that, for all $c=\frac{3n}{n+2}$ with $n\in\mathbb{Z}_{\geq 0}$, the simple unitary VOSA $V^c(N2)$ can be constructed as the coset of the unitary subalgebra $V_{L_{2(n+2)}}$ of the tensor product $V^n(\mathfrak{g})\otimes F^2$, where F^2 is the charged free fermions of Example 6.1.7, see [CHKLX15, Theorem 3.2] and references therein. $V_{L_{2N}}$ as well as $V^n(\mathfrak{g})$ are strongly-local and therefore, by Proposition 5.1.3, Theorem 5.1.1 and Corollary 5.2.3, we have

$$
\mathrm{SVir2}_{\frac{3n}{n+2}} = \mathcal{A}^c_{V_{L_{2(n+2)}}} = \mathcal{A}^c_{U(1)_{2(n+2)}} \subset \mathcal{A}_{V^n(\mathfrak{g})}\otimes\mathcal{F}^2 \qquad \forall n\in\mathbb{Z}_{\geq 0}\,.
$$

We can produce far more examples picking the graded tensor product of the simple strongly graded-local unitary VOSAs considered above. A large exposition of this kind of examples of both graded-local conformal and $N=1$ superconformal nets are given in [CH17, Section 5.2].

6.3 Rank-one lattice models

We introduce and investigate the rank-one lattice models. Specifically, we construct the rank-one lattice type (unitary) VOSAs first, showing their strong graded-locality too. Second, we classify all the unitary subalgebras of the even models. As a consequence of Theorem 5.1.1, we obtain the complete classification of all covariant subnets of the corresponding even rank-one lattice conformal nets, also known as conformal net extensions of the U(1)-current net, see [BMT88, Section 4A].

By a **rank-one lattice** L we mean a positive definite integral lattice in $\mathbb{R}$. In other words, L is a free abelian group contained in $\mathbb{R}$ equipped with a non-degenerate symmetric bilinear form $B(\cdot\,,\cdot)$. Let $\mathfrak{h}_L := \mathbb{C}\otimes_{\mathbb{Z}} L$. Therefore, $\mathfrak{h}_L$ is a one-dimensional complex vector space equipped with a non-degenerate bilinear form, which is the bilinear extension of $B(\cdot\,,\cdot)$ to $\mathfrak{h}_L$. Denote the latter extension of $B(\cdot\,,\cdot)$ to $\mathfrak{h}_L$ by $(\cdot|\cdot)$. Example 6.1.2 says that every $\mathfrak{h}_L$ gives rise to the same, up to isomorphism, simple strongly local unitary VOA $M(1)$, known as the oscillator or Heisenberg VOA. Recall that we have a particular vector $J\in\mathfrak{h}_L$ such that $(J|J)=1$, called the current vector of $\mathfrak{h}_L$. Moreover, every rank-one lattice L can be identified with the rank-one lattice $L_M := \mathbb{Z}\sqrt{M}J$ for some positive integer M. Therefore, L_M is called **even** or **odd** if M is even or odd respectively.

Consider the **group algebra** $\mathbb{C}(L_M)$, which can be identified with the unital associative algebra generated by the formal set of elements $\{e^\alpha \mid \alpha\in L_M\}$ such that

$$e^\alpha e^\beta = e^{\alpha+\beta} \qquad \forall\alpha,\beta\in L_M \tag{6.68}$$

where $1 := e^0$ is the identity element.

By [Kac01, Theorem 5.5 and Proposition 5.5] and [AL17, Theorem 2.9], we have that for every positive integer M, the complex vector superspace

$$V_{L_M} := M(1)\otimes_{\mathbb{C}}\mathbb{C}(L_M) \tag{6.69}$$

has a structure of simple unitary VOSA with central charge $c=1$, known as **rank-one lattice type (unitary) VOSAs**. More specifically, they are called **even rank-one lattice type (unitary) VOAs** when L_M is even, otherwise they are called **odd rank-one lattice type (unitary) VOSAs** when L_M is odd. Note that, in the even case, these are "non super" models. In what follows, we are going to specify the unitary VOSA structure of rank-one lattice type VOSAs. A linear basis is

$$\begin{gathered}\{\Omega\otimes 1\,,\ J_{m_1}\cdots J_{m_s}\Omega\otimes e^\alpha \mid m_1\leq\cdots\leq m_s\leq -1\,,\ \alpha\in L_M\}\\ p(J_{m_1}\cdots J_{m_s}\Omega\otimes e^\alpha)=\overline{(\alpha|\alpha)}\in\mathbb{Z}_2\end{gathered} \tag{6.70}$$

where $\Omega\otimes 1$ is the vacuum vector. The infinitesimal translation operator T is defined by

$$\begin{aligned}T(J_m\Omega\otimes 1) &= mJ_{m-1}\Omega\otimes 1 \qquad \forall m\in\mathbb{Z}\\ T(\Omega\otimes e^\alpha) &= \alpha_{-1}\Omega\otimes e^\alpha \qquad \forall\alpha\in L_M\,.\end{aligned} \tag{6.71}$$

The state-field correspondence is

$$Y(J_{-1}\Omega\otimes 1,z) = \sum_{n\in\mathbb{Z}}(J_{-1}\Omega\otimes 1)_{(n)}z^{-n-1} = \sum_{n\in\mathbb{Z}} J_n z^{-n-1} \tag{6.72}$$

$$Y(\Omega\otimes e^\alpha,z) = e_\alpha z^{\alpha_0}E_+(\alpha,z)E_-(\alpha,z) \qquad \forall\alpha\in L_M \tag{6.73}$$

where

$$\begin{aligned}
J_0(v\otimes e^\beta) &= (J_0|\beta)v\otimes e^\beta \qquad \forall v\in M(1)\ \forall\beta\in L_M && (6.74)\\
J_n(v\otimes e^\beta) &= (J_nv)\otimes e^\beta \qquad \forall n\in\mathbb{Z}\backslash\{0\}\ \forall v\in M(1)\ \forall\beta\in L_M && (6.75)\\
e_\alpha(v\otimes e^\beta) &:= v\otimes e^{\alpha+\beta} \qquad \forall v\in M(1)\ \forall\alpha,\beta\in L_M && (6.76)\\
z^{\alpha_0}(v\otimes e^\beta) &:= z^{(\alpha|\beta)}v\otimes e^\beta \qquad \forall v\in M(1)\ \forall\alpha,\beta\in L_M && (6.77)\\
E_+(\alpha,z) &:= \exp\left(-\sum_{j<0}\frac{\alpha_j}{j}z^{-j}\right) \qquad \forall\alpha\in L_M && (6.78)\\
E_-(\alpha,z) &:= \exp\left(-\sum_{j>0}\frac{\alpha_j}{j}z^{-j}\right) \qquad \forall\alpha\in L_M\,. && (6.79)
\end{aligned}$$

The conformal vector is $\nu\otimes 1$, where $\nu:=\frac{1}{2}J_{-1}^2\Omega$ is the conformal vector of $M(1)$. If $Y(\nu\otimes 1,z)=\sum_{n\in\mathbb{Z}}L_nz^{-n-2}$, then

$$L_0(J_{n_1}\cdots J_{n_s}\Omega\otimes e^\alpha)=\left(\frac{(\alpha|\alpha)}{2}-\sum_{j=1}^s n_j\right)J_{n_1}\cdots J_{n_s}\Omega\otimes e^\alpha \qquad \forall s\in\mathbb{Z}_{\geq 0}\ \forall\alpha\in L_M \tag{6.80}$$

$$L_m=\frac{1}{2}\sum_{n\in\mathbb{Z}}J_nJ_{m-n}, \qquad \forall m\in\mathbb{Z}\setminus\{0\} \tag{6.81}$$

noting that $J_{-1}\Omega\otimes 1$ is primary of conformal weight 1, whereas $\Omega\otimes e^\alpha$ are primary of conformal weight $\frac{(\alpha|\alpha)}{2}$. It is also useful to note the commutation relations

$$[L_m,J_n]=-nJ_{m+n} \qquad \forall m,n\in\mathbb{Z}\,. \tag{6.82}$$

Before proceeding with the unitary structure, it is worthwhile to have some remarks about the VOSA structure just introduced.

Remark 6.3.1. It is important to point out that, in definitions (6.74) and (6.75), we have used the same symbols J_n with $n\in\mathbb{Z}$ to denote operators acting both on V_{L_M} and $M(1)$. Similarly, the Virasoro operators L_m with $m\in\mathbb{Z}$ of the conformal vector $\nu\otimes 1$ in equations (6.80) and (6.81) are different from the same symbol Virasoro operators of the conformal vector ν introduced in Example 6.1.2 for $M(1)$. At first glance, this choice can be difficult to understand, but it is well justified from the fact that those operators in (6.74), (6.75), (6.80) and (6.81) coincide with the corresponding ones of Example 6.1.2 when they act on the subspace $M(1)\otimes 1$ of V_{L_M}. Therefore, with the same abuse of notation used for all the operators J_n, we will write L_m with $m\in\mathbb{Z}$ to denote the Virasoro operators of both the conformal vectors $\nu\otimes 1$ and ν, making clear the context to correctly interpret them time by time.

Remark 6.3.2. In [Kac01, Theorem 5.5], the vertex superalgebra is constructed in a slightly more general way. Specifically, the **twisted group algebra** $\mathbb{C}_\epsilon(L_M)$ is used in place of $\mathbb{C}(L_M)$ in (6.69). The former is identified as the unital associative algebra linearly generated by $\{e^\alpha\mid\alpha\in L_M\}$ with identity $1:=e^0$ and multiplication:

$$e^\alpha e^\beta=\epsilon(\alpha,\beta)e^{\alpha+\beta} \qquad \forall\alpha,\beta\in L_M$$

where $\epsilon:L_M\times L_M\to\{\pm 1\}$ is a 2-cocycle, i.e.,

$$\begin{aligned}
\epsilon(\alpha,0)=\epsilon(0,\alpha)=1 \qquad \forall\alpha\in L_M\\
\epsilon(\beta,\gamma)\epsilon(\beta+\gamma,\alpha)=\epsilon(\beta,\alpha+\gamma)\epsilon(\gamma,\alpha) \qquad \forall\alpha,\beta,\gamma\in L_M
\end{aligned}$$

with the additional property

$$\epsilon(\alpha,\beta)\epsilon(\beta,\alpha)=(-1)^{(\alpha|\beta)+(\alpha|\alpha)(\beta|\beta)} \qquad \forall\alpha,\beta\in L_M\,.$$

By [Kac01, Theorem 5.5] again, the vertex superalgebra so constructed is isomorphic to the one obtained through the group algebra as in (6.69). Moreover, it is not difficult to check that the trivial 2-cocycle, that is $\epsilon=1$, is an admissible choice in our case. Therefore, it is not restrictive and it makes no difference to consider the construction implemented from (6.69).

Remark 6.3.3. For every positive integer M, denote by a generic I the identity operator $1_{V_{L_M}}$. From [FLM88, Eq.s (7.1.42), (3.2.17) and (2.1.11)], we have that, for any $\alpha \in L_M$,

$$E_+(\alpha, z) = I + \sum_{n=1}^{+\infty} \frac{\left(-\sum_{j<0} \frac{\alpha_j}{j} z^{-j}\right)^n}{n!} \tag{6.83}$$

$$\left(-\sum_{j<0} \frac{\alpha_j}{j} z^{-j}\right)^n = \sum_{m=n}^{\infty} \left(\sum_{\substack{j_1+\cdots+j_n=m \\ j_k>0}} \frac{\alpha_{-j_1}\cdots\alpha_{-j_n}}{j_1\cdots j_n} \right) z^m \tag{6.84}$$

$$E_-(\alpha, z) = I + \sum_{n=1}^{+\infty} \frac{\left(-\sum_{j>0} \frac{\alpha_j}{j} z^{-j}\right)^n}{n!} \tag{6.85}$$

$$\left(-\sum_{j>0} \frac{\alpha_j}{j} z^{-j}\right)^n = (-1)^n \sum_{m=n}^{\infty} \left(\sum_{\substack{j_1+\cdots+j_n=m \\ j_k>0}} \frac{\alpha_{j_1}\cdots\alpha_{j_n}}{j_1\cdots j_n} \right) z^{-m} . \tag{6.86}$$

Remark 6.3.4. Note that the operators e_α and z^{α_0} commute with $E_\pm(\alpha, z)$ for all $\alpha \in L_M$, see [FLM88, Eq. (7.1.37)].

As far as the unitary structure on every V_{L_M} is concerned, the invariant scalar product is given by the formula:

$$(v \otimes e^\alpha | w \otimes e^\beta) := (v|w)\delta_{\alpha,\beta} \qquad \forall v, w \in M(1) \ \forall \alpha, \beta \in L_M \tag{6.87}$$

where we are using the invariant scalar product on $M(1)$. Note that we are using the same symbol $(\cdot|\cdot)$ for the scalar products on L_M, $M(1)$ and V_{L_M}. Nevertheless, it will be clear from the context which is the correct scalar product to use time by time. The PCT operator θ is the antilinear map which acts on elements of the linear basis as

$$\theta(J_{n_1}\cdots J_{n_s}\Omega \otimes e^\alpha) = (-1)^s J_{n_1}\cdots J_{n_s}\Omega \otimes e^{-\alpha} \qquad \forall s \in \mathbb{Z}_{\geq 0} \ \forall \alpha \in L_M . \tag{6.88}$$

Now, we are ready to prove the strong graded locality of rank-one lattice type VOSAs.

Theorem 6.3.5. *For every positive integer M, V_{L_M} is a simple strongly graded-local unitary VOSA. In particular, $V_{L_1} \cong F^2$ and every V_{L_M} is a unitary subalgebra of F^{2M}. Consequently, every V_{L_M} gives rise to an irreducible graded-local conformal net $\mathcal{A}_{V_{L_M}}$ such that $\mathcal{A}_{V_{L_1}} \cong \mathcal{F}^2$ and every $\mathcal{A}_{V_{L_M}}$ is a covariant subnet of $\mathcal{F}^{2M}$.*

Proof. We have already showed that every V_{L_M} is a simple unitary VOSA. So, we have to prove the strong graded-locality. By [Kac01, Example 5.5a and Example 5.5b], V_{L_1} and the charged free fermion F^2 of Example 6.1.7 are isomorphic simple VOSAs. Therefore, we can conclude that V_{L_1} is strongly graded-local and $\mathcal{A}_{V_{L_1}} \cong \mathcal{F}^2$ is an irreducible graded-local conformal net. It is not difficult to show that, for every positive integer M, L_M can be identified with a sublattice of the lattice $L_1^M = \mathbb{Z}^M$. Moreover, from [Kac01, Remark 5.5b and Proposition 5.5], it should be clear that $V_{L_1^M}$ and F^{2M} are isomorphic simple VOSAs. Therefore, we can identify V_{L_M} with a unitary subalgebra of F^{2M}. Thanks to Theorem 5.1.1 and Example 6.1.7, it follows that, for every positive integer M, V_{L_M} is strongly graded-local and $\mathcal{A}_{V_{L_M}}$ is a covariant subnet of $\mathcal{F}^{2M}$. □

$\mathcal{A}_{V_{L_M}}$ are known as **rank-one lattice graded-local conformal nets**. In the even case, that is when $M = 2N$ with $N \in \mathbb{Z}_{>0}$, $\mathcal{A}_{V_{L_{2N}}}$ are irreducible (local) conformal nets, whose strong locality was already known from [CKLW18, Example 8.8]. In that case, the irreducible conformal nets are realised as extensions of the U(1)-current net $\mathcal{A}_{\mathrm{U}(1)}$ of Example 6.1.2 in [BMT88, Section 4A], where $\mathcal{A}_{V_{L_{2N}}}$ are called $\mathcal{A}_N$. For this reason, $\mathcal{A}_{V_{L_{2N}}}$ are usually denoted by $\mathcal{A}_{\mathrm{U}(1)_{2N}}$, see [Xu05, Section 4]. Accordingly, we will keep this notation for the remaining part of this thesis and we refer to every $\mathcal{A}_{\mathrm{U}(1)_{2N}}$ as a **U(1)-current extension net**.

Remark 6.3.6. It is also interesting to note that V_{L_3} and $V^1(N2)$ (see Example 6.2.13) are isomorphic simple VOSAs by [Kac01, Example 5.9c] and consequently $\mathrm{SVir2}_1 \cong \mathcal{A}_{V_{L_3}}$ can be seen as covariant subnet of $\mathcal{F}^6$.

The rest of the current section is dedicated to the classification theorems for unitary subalgebras of the simple unitary VOAs $V_{L_{2N}}$ for all $N \in \mathbb{Z}_{>0}$ and consequently for covariant subnets of the corresponding irreducible conformal nets $\mathcal{A}_{\mathrm{U}(1)_{2N}}$ for all $N \in \mathbb{Z}_{>0}$ respectively. We present some well-known unitary subalgebras of $V_{L_{2N}}$ first. Then, we move on to the classification results.

$M(1) \otimes_{\mathbb{C}} \mathbb{C}1$, which we call just $M(1)$, is a unitary subalgebra of every $V_{L_{2N}}$. The vacuum vector is $\Omega \otimes 1$, the conformal vector is $\nu \otimes 1$ and the PCT operator is the restriction of θ to $M(1)$, that is,

$$\theta(J_{n_1} \cdots J_{n_s}\Omega \otimes 1) = (-1)^s J_{n_1} \cdots J_{n_s}\Omega \otimes 1 \qquad \forall s \in \mathbb{Z}_{\geq 0}\,. \tag{6.89}$$

Let ϕ be the (linear) automorphism on $V_{L_{2N}}$ defined by the following action on the linear basis:

$$J_{n_1} \cdots J_{n_s}\Omega \otimes e^{\alpha} \longmapsto (-1)^s J_{n_1} \cdots J_{n_s}\Omega \otimes e^{-\alpha} \qquad \forall s \in \mathbb{Z}_{\geq 0}\ \forall \alpha \in L_{2N}\,, \tag{6.90}$$

which must not be confused with the antilinear PCT operator θ in (6.88). Note that ϕ is an involution and restricts to an involution of $M(1)$. Then, we define the orbifold subalgebras, see Example 2.6.3 (see also [DG98, Section 2]),

$$V_{L_{2N}}^{+} := V_{L_{2N}}^{\{I,\phi\}}\,, \qquad M(1)^{+} := M(1)^{\{I,\phi\}}\,.$$

We identify every $t \in \mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$ with its representative in $[0, 2\pi)$ and for any positive integer N, we define an action on $V_{L_{2N}}$ by automorphisms

$$g_{2N,t} := \exp\left(i\frac{t}{\sqrt{2N}}J_0\right) = \sum_{n\geq 0} \frac{\left(i\frac{t}{\sqrt{2N}}J_0\right)^n}{n!} \qquad \forall t \in \mathbb{T} \tag{6.91}$$

which act on generators of $V_{L_{2N}}$ in the following manner

$$v \otimes e^{\alpha} \longmapsto e^{i\frac{t}{\sqrt{2N}}(J|\alpha)} v \otimes e^{\alpha} \qquad \forall v \in M(1)\ \forall \alpha \in L_{2N}\,. \tag{6.92}$$

Note that J_0 leaves invariant the finite-dimensional eigenspaces of L_0 so that every $g_{2N,t}$ is a well-defined automorphism of $V_{L_{2N}}$, see [DN99, Section 2.3]. Moreover, for $\alpha = \sqrt{2N}J$, we have $e^{i\frac{t}{\sqrt{2N}}(J|\alpha)} = e^{it}$ so that $g_{2N,t}$ is not the identity when $t \neq 0$, $t \in [0, 2\pi)$. For the reader's benefit, we include the following well-known facts with proofs.

Proposition 6.3.7. *For any positive integer N, the automorphisms $g_{2N,t}$ and ϕ defined above give an embedding of the infinite dihedral group $D_\infty = \mathbb{T} \rtimes \mathbb{Z}_2$ into $\mathrm{Aut}_{(\cdot|\cdot)}(V_{L_{2N}})$. Moreover,*

$$V_{L_{2N}}^{\mathbb{T}} = M(1)\,, \tag{6.93}$$

$$V_{L_{2N}}^{D_\infty} = M(1)^{+}\,. \tag{6.94}$$

Proof. For every fixed N, let $G_{2N} \cong \mathbb{T}$ and $H \cong \mathbb{Z}_2$ be the groups generated by the automorphisms $g_{2N,t}$ for all $t \in \mathbb{T}$ and ϕ respectively. Using (6.90) and (6.92), it is easy to verify that $\phi g_{2N,t}\phi = g_{2N,t}^{-1}$ for all $g_{2N,t} \in G_{2N}$. It follows that the group generated by G_{2N} and H is isomorphic to D_∞.

Now we prove that the automorphisms $g_{2N,t}$ and ϕ are unitary. Recall from Theorem 2.4.5 that if V is a simple unitary VOA, then an automorphism g of V is unitary if and only if g commutes with the PCT operator. The above condition is easily verified on the linear basis (6.70) of $V_{L_{2N}}$. Thus, D_∞ embeds into $\mathrm{Aut}_{(\cdot|\cdot)}(V_{L_{2N}})$.

Note that (6.94) follows from (6.93) by definition. To prove (6.93), consider $a \in V_{L_{2N}}^{\mathbb{T}}$ and its linear decomposition into basis elements, that is,

$$a = \sum_{j=1}^{M} a_j v_j \otimes e^{\alpha_j}\,, \tag{6.95}$$

with $\mathbb{C} \ni a_j \neq 0$ for all j, v_j some of the vectors in (6.70). Applying a generic automorphism $g_{2N,t}$ to both sides of (6.95), we obtain

$$\sum_{j=1}^{M} a_j v_j \otimes e^{\alpha_j} = a = g_{2N,t}(a) = \sum_{j=1}^{M} e^{i\frac{t}{\sqrt{2N}}(J|\alpha_j)} a_j v_j \otimes e^{\alpha_j}\,. \tag{6.96}$$

By linear independence of $v_j \otimes e^{\alpha_j}$, (6.96) is satisfied if and only if $e^{i\frac{t}{\sqrt{2N}}(J|\alpha_j)} = 1$ for all j and $t \in \mathbb{T}$, i.e., $\alpha_j = 0$ for all j. To sum up, $a \in V_{L_{2N}}^{\mathbb{T}}$ if and only if $\alpha_j = 0$ for all j, which is equivalent to $a \in M(1)$. This completes the proof. □

In the following, for every fixed $N \in \mathbb{Z}_{>0}$, we identify $\mathbb{T}$ with the closed subgroup G_{2N} of $\mathrm{Aut}_{(\cdot|\cdot)}(V_{L_{2N}})$ as in the proof of Proposition 6.3.7. We also identify $\mathbb{Z}_k \subset \mathbb{T}$ with the cyclic subgroup of $\mathrm{Aut}_{(\cdot|\cdot)}(V_{L_{2N}})$ of order k generated by automorphisms $g_{N,\frac{2m\pi}{k}}$ for $m \in \{0, \dots, k-1\}$, and $D_k := \mathbb{Z}_k \rtimes \mathbb{Z}_2$ with the corresponding dihedral subgroup of $\mathrm{Aut}_{(\cdot|\cdot)}(V_{L_{2N}})$ of order k.

Note that L_{2Nk^2} is a sublattice of L_{2N} for all positive integers N and k. This inclusion induces an isomorphism of vector spaces

$$\iota : V_{L_{2Nk^2}} \longleftrightarrow V_{2N,k} := \bigoplus_{\alpha \in L_{2Nk^2}} M(1) \otimes_{\mathbb{C}} \mathbb{C}e^{\alpha} \subset V_{L_{2N}} . \tag{6.97}$$

Clearly, by definition (6.69), $V_{2N,k}$ is a unitary subalgebra of $V_{L_{2N}}$ and ι realises a unitary isomorphism. Therefore, from now onwards, we consider $V_{L_{2Nk^2}}$ as a unitary subalgebra of $V_{L_{2N}}$ without specifying the identification through ι.

Proposition 6.3.8. *For every positive integer N and k, we have that:*

$$V_{L_{2N}}^{\mathbb{Z}_k} = V_{L_{2Nk^2}}, \tag{6.98}$$

$$V_{L_{2N}}^{D_k} = V_{L_{2Nk^2}}^{+}. \tag{6.99}$$

Proof. First note that (6.99) follows from (6.98) by definition. Then, it is sufficient to prove that $V_{L_{2N}}^{\mathbb{Z}_k}$ is equal to $V_{2N,k}$ as in (6.97) to conclude. Using the same argument and notations as in the proof of Proposition 6.3.7, we have that $a \in V_{L_{2N}}^{\mathbb{Z}_k}$ if and only if $e^{i\frac{2\pi}{k\sqrt{2N}}(J|\alpha_j)} = 1$ for all j. $\alpha_j \in \mathbb{Z}\sqrt{2N}J$ for all j and thus the former condition is satisfied if and only if $\alpha_j \in \mathbb{Z}\sqrt{2N}kJ$ for all j. Then $V_{2N,k} = V_{L_{2N}}^{\mathbb{Z}_k}$ as desired. □

Remark 6.3.9. The closed subgroups of $D_\infty \subset \mathrm{Aut}_{(\cdot|\cdot)}(V_{L_{2N}})$ are the circle group $\mathbb{T}$, cyclic groups $\mathbb{Z}_k$, dihedral groups D_k and their conjugates $D_k^t := g_{2N,t} D_k g_{2N,-t}$ for all $t \in \mathbb{T}$. Note that $D_k^t = D_k^{t+\pi}$ for all $k \in \mathbb{Z}_{>0}$ and all $t \in \mathbb{T}$ ($g_{2N,\pi}$ is in the center of D_∞). Obviously $D_k = D_k^t$ for all $k \in \mathbb{Z}_{>0}$ and all $t \in \mathbb{Z}_k$. Thus, it is easy to verify that $V_{L_{2N}}^{D_k^t} = g_{2N,t}(V_{L_{2N}}^{D_k})$ for all $k \in \mathbb{Z}_{>0}$ and all $t \in \mathbb{T}$. In other words, $V_{L_{2N}}^{D_k}$ is unitarily isomorphic to $V_{L_{2N}}^{D_k^t}$ for all $k \in \mathbb{Z}_{>0}$ and all $t \in \mathbb{T}$.

Every $V_{L_{2N}}$ has a decomposition into Virasoro modules, introduced in Example 6.1.1, as described in [DG98, Section 2]. In particular, if N is not a **perfect square**, that is the square of a positive integer, then we have the following decompositions into irreducible Virasoro modules:

$$V_{L_{2N}} = \bigoplus_{p \geq 0} L(1, p^2) \oplus \bigoplus_{m>0} 2L(1, Nm^2) \tag{6.100}$$

$$M(1) = \bigoplus_{p \geq 0} L(1, p^2) \tag{6.101}$$

$$V_{L_{2N}}^{+} = \bigoplus_{p \geq 0} L(1, 4p^2) \oplus \bigoplus_{m>0} L(1, Nm^2) \tag{6.102}$$

$$M(1)^{+} = \bigoplus_{p \geq 0} L(1, 4p^2) . \tag{6.103}$$

Remark 6.3.10. As stated in [DG98, Section 2], the decompositions in (6.100) and (6.101) are due to the fact that the following isomorphisms of Virasoro modules hold

$$M(1) \otimes_{\mathbb{C}} \mathbb{C}e^{\pm\sqrt{2N}mJ} \cong L(1, Nm^2) \qquad \forall m \in \mathbb{Z}_{>0} . \tag{6.104}$$

Therefore, for all $m > 0$, $\Omega \otimes e^{\pm\sqrt{2N}mJ}$ form a basis for the subspace of $V_{L_{2N}}$ of primary vectors of conformal weight Nm^2 respectively. As a matter of fact, it is straightforward to check that the former elements are homogeneous of conformal weight Nm^2 using (6.80) and that $L_n(\Omega \otimes e^{\pm\sqrt{2N}mJ}) = 0$ for all $n > 0$ by (6.81). Then, they generate (as Virasoro modules) the two copies of $L(1, Nm^2)$. This also implies that every $L(1, Nm^2)$ in (6.102) is generated by the ϕ-invariant primary vector $\Omega \otimes (e^{\sqrt{2N}mJ} + e^{-\sqrt{2N}mJ})$ of conformal weight Nm^2.

We state the following Galois correspondence for unitary VOAs, which is crucial for the classification result. This is a variant of [CKLW18, Theorem 7.7], which can be entirely formulated and proved in the unitary VOA setting thanks to [DM99, Theorem 3].

Theorem 6.3.11. *Let V be a simple unitary VOA and G be a closed subgroup of $\mathrm{Aut}_{(\cdot|\cdot)}(V)$ which is topologically isomorphic to a Lie group. Then, the map $H \mapsto V^H$ gives a one-to-one correspondence between the closed subgroups H of G and the unitary subalgebras $W \subset V$ containing V^G.*

Proof. We first show that every unitary subalgebra $W \subseteq V$ which contains V^G is an *orthogonally complemented subVOA* in the sense of [DM99, Definition 2]. To this end it is rather straightforward to see that in our case the subspace in Eq. (1.3) in [DM99, Definition 2] coincides with the orthogonal complement

$$W^{\perp} := \{a \in V \mid (a|b) = 0 \text{ for all } b \in W\}$$

which is easily seen to be a W-submodule of V because of the invariance property of the scalar product and the fact that W is a unitary subalgebra. Now, for every closed subgroup H of G, V^H is a unitary subalgebra of V by Example 2.6.3. Moreover, since $\mathrm{Aut}_{(\cdot|\cdot)}(V)$ is compact by Theorem 2.4.5, G must be topologically isomorphic to a compact Lie group. Then, the result follows from [DM99, Theorem 3]. □

The unitary subalgebras of V_{L_2} have already been classified in [CKLW18, Theorem 8.13]. Here, we give a proof in the unitary VOA setting using the Galois correspondence stated in Theorem 6.3.11 instead of [CKLW18, Theorem 7.7].

First of all, recall that V_{L_2} is identified (cf. [CKLW18, Example 8.8]) with the simple unitary VOA $V^1(\mathfrak{sl}(2,\mathbb{C}))$ built from the affine Lie algebras $\mathfrak{sl}(2,\mathbb{C})_1$ associated to the Lie group SU(2), see Example 6.1.8 (see [GW84] and [PS86] for more information about loop groups). Consider the orthonormal generators J^a for $a \in \{x,y,z\}$ of the complex Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ and corresponding elements J_0^a of the associated loop algebra. Then, we have a group of unitary isomorphisms of $V^1(\mathfrak{sl}(2,\mathbb{C}))$, isomorphic to SO(3) and generated by operators of the form $e^{itJ_0^a}$ with $t \in \mathbb{T}$ and $a \in \{x,y,z\}$, see [Kac01, Remark 5.7a]. Then, $\mathrm{SO}(3) \subseteq \mathrm{Aut}_{(\cdot|\cdot)}(V_{L_2})$, see [CKLW18, p. 63]. Remember that the closed subgroups of SO(3) are, up to isomorphism, the circle group $\mathbb{T}$, all cyclic groups $\mathbb{Z}_k$, the infinite dihedral group D_∞, all finite dihedral groups D_k and the three platonic groups E_m. Finally, note that the action of D_∞ introduced in Proposition 6.3.7 can be embedded in SO(3) as described above, identifying the current vector J with $\sqrt{2}J^z$, cf. [BMT88, Section 5B]. Therefore, we have the following.

Theorem 6.3.12. *All unitary subalgebras of $V_{L_2} \cong V^1(\mathfrak{sl}(2,\mathbb{C}))$ are given by the family of fixed point subalgebras $V_{L_2}^H$ for every closed subgroup $H \subseteq \mathrm{SO}(3)$ together with the trivial subalgebra $\mathbb{C}\Omega \otimes 1$. In particular, $V_{L_2}^{\mathrm{SO}(3)} = L(1,0)$ and consequently $\mathrm{Aut}_{(\cdot|\cdot)}(V_{L_2}) = \mathrm{SO}(3)$.*

Proof. Let W be a nontrivial subalgebra. Then by [CKLW18, Lemma 8.10], $L(1,0) \subset W$. Moreover, by [CKLW18, Proposition 8.11] (cf. also [DG98, Corollary 2.4]) $L(1,0) = V_{L_2}^{SO(3)}$ and the claim follows from Theorem 6.3.11. □

For N different from a perfect square, the classification of unitary subalgebras of $V_{L_{2N}}$ will rely on Theorem 6.3.18 below, the key ingredient of our classification result, which states that any unitary subalgebra properly containing $L(1,0)$ contains also $M(1)^+$. To prove it, we begin with some preliminary results.

Lemma 6.3.13. *Let $N \in \mathbb{Z}_{>0}$.*

(i) The vector

$$u \otimes 1 := \left(\frac{1}{2}(J_{-1})^4\Omega - J_{-3}J_{-1}\Omega + \frac{3}{4}(J_{-2})^2\Omega\right) \otimes 1$$

is primary of conformal weight 4. In particular, if N is not a perfect square, then it is the highest weight vector of the unique irreducible Virasoro submodule $L(1,4)$ of $V_{L_{2N}}$ as in (6.100).

(ii) We also have

$$L_{-2}(\nu\otimes 1)=\left(\frac{1}{4}(J_{-1})^4\Omega+J_{-3}J_{-1}\Omega\right)\otimes 1$$

$$L_{-4}(\Omega\otimes 1)=\left(\frac{1}{2}(J_{-2})^2\Omega+J_{-3}J_{-1}\Omega\right)\otimes 1\,.$$

Proof. (i). From formula (6.80) it is clear that $u\otimes 1$ is homogeneous of conformal weight 4. According to (6.100), if $u\otimes 1$ is primary and N is not a perfect square, it will be the highest weight vector of the unique irreducible Virasoro submodules $L(1,4)$ of $V_{L_{2N}}$. We need to prove that $L_m u=0$ for all $m\geq 1$. Using equation (6.82), we see that

$$\begin{aligned}\frac{1}{2}L_m(J_{-1})^4\Omega=&\frac{1}{2}\Big(J_{m-1}(J_{-1})^3\Omega+J_{-1}J_{m-1}(J_{-1})^2\Omega\\&+(J_{-1})^2J_{m-1}J_{-1}\Omega+(J_{-1})^3J_{m-1}\Omega+(J_{-1})^4L_m\Omega\Big)\end{aligned}\tag{6.105}$$

$$-L_mJ_{-3}J_{-1}\Omega=-\left(3J_{m-3}J_{-1}\Omega+J_{-3}J_{m-1}\Omega+J_{-3}J_{-1}L_m\Omega\right)\tag{6.106}$$

$$\frac{3}{4}L_m(J_{-2})^2\Omega=\frac{3}{4}\left(2J_{m-2}J_{-2}\Omega+2J_{-2}J_{m-2}\Omega+(J_{-2})^2L_m\Omega\right).\tag{6.107}$$

Note that the last term on the right-hand side of each of equations (6.105) - (6.107) is 0, because $L_m\Omega=0$ for all $m\geq 1$. Moreover, by the commutation relations (6.19) we know J_j and J_k commute if $k\neq -j$. This means that for all $m\geq 5$ every term in equations (6.105) - (6.107) is zero since $J_j\Omega=0$ for all $j\geq 0$. Therefore, it remains to study the four cases $m\in\{1,2,3,4\}$ one by one. For $m=1$ we have

$$\begin{aligned}\frac{1}{2}L_1(J_{-1})^4\Omega=&2(J_{-1})^3J_0\Omega=0\\-L_1J_{-3}J_{-1}\Omega=&-3J_{-2}J_{-1}\Omega-J_{-3}J_0\Omega=-3J_{-2}J_{-1}\Omega\\\frac{3}{4}L_1(J_{-2})^2\Omega=&\frac{3}{4}\left(2J_{-1}J_{-2}\Omega+2J_{-2}J_{-1}\Omega\right)=3J_{-2}J_{-1}\Omega\end{aligned}$$

and the sum is clearly 0. For $m=2$,

$$\begin{aligned}\frac{1}{2}L_2(J_{-1})^4\Omega=&\frac{1}{2}\left(\,J_1(J_{-1})^3\Omega+J_{-1}J_1(J_{-1})^2\Omega+\right.\\&\left.+(J_{-1})^2J_1J_{-1}\Omega+(J_{-1})^3J_1\Omega\,\right)\\=&\frac{1}{2}\Big((3+2+1)(J_{-1})^2\Omega+4(J_{-1})^3J_1\Omega\Big)\\=&3(J_{-1})^2\Omega\\-L_2J_{-3}J_{-1}\Omega=&-3J_{-1}J_{-1}\Omega-J_{-3}J_1\Omega=-3(J_{-1})^2\Omega\\\frac{3}{4}L_2(J_{-2})^2\Omega=&\frac{3}{4}\left(2J_0J_{-2}\Omega+2J_{-2}J_0\Omega\right)=0\end{aligned}$$

and again the sum is 0. For $m=3$,

$$\begin{aligned}\frac{1}{2}L_3(J_{-1})^4\Omega=&2(J_{-1})^3J_2\Omega=0\\-L_3J_{-3}J_{-1}\Omega=&-3J_0J_{-1}\Omega-J_{-3}J_2\Omega=0\\\frac{3}{4}L_3(J_{-2})^2\Omega=&\frac{3}{4}\left(2J_1J_{-2}\Omega+2J_{-2}J_1\Omega\right)=0\,.\end{aligned}$$

Finally for $m=4$,

$$\begin{aligned}\frac{1}{2}L_4(J_{-1})^4\Omega=&2(J_{-1})^3J_3\Omega=0\\-L_4J_{-3}J_{-1}\Omega=&-3J_1J_{-1}\Omega-J_{-3}J_3\Omega=-3\Omega\\\frac{3}{4}L_4(J_{-2})^2\Omega=&\frac{3}{4}\left(2J_2J_{-2}\Omega+2J_{-2}J_2\Omega\right)=3\Omega\end{aligned}$$

which adds up to 0 as well.

(ii). By (6.82), we have that

$$L_{-2}\nu = \frac{1}{2}L_{-2}(J_{-1})^2\Omega = \frac{1}{2}\left(2J_{-3}J_{-1}\Omega + (J_{-1})^2 L_{-2}\Omega\right) = J_{-3}J_{-1}\Omega + \frac{1}{4}(J_{-1})^4\Omega$$

and using (6.81)

$$L_{-4}\Omega = \frac{1}{2}\sum_{j\in\mathbb{Z}} J_j J_{-4-j}\Omega = \frac{1}{2}\sum_{j=-3}^{-1} J_j J_{-4-j}\Omega = \frac{1}{2}\left(2J_{-3}J_{-1}\Omega + (J_{-2})^2\Omega\right) = J_{-3}J_{-1}\Omega + \frac{1}{2}(J_{-2})^2\Omega\,.$$

□

Proposition 6.3.14. *Let N, m be positive integers such that $g := \sqrt{2N}m \geq 2$. Let $b \in \mathbb{C}\setminus\{0\}$ and set*

$$e^g_b := \Omega \otimes \left(e^{gJ} + be^{-gJ}\right)\,.$$

Then

$$\left(\Omega\otimes e^{\pm gJ}\right)_{(g^2-2)}\left(\Omega\otimes e^{\mp gJ}\right) = (\pm gJ_{-1}\Omega)\otimes 1$$
$$(e^g_b)_{(g^2-5)}e^g_b = bv_g \otimes 1,$$

where

$$v_g := \frac{g^4}{12}(J_{-1})^4\Omega + \frac{2g^2}{3}J_{-3}J_{-1}\Omega + \frac{g^2}{4}(J_{-2})^2\Omega\,.$$

Proof. Note that the z^{-g^2+4}-coefficient of the formal z-series $Y(e^g_b, z)e^g_b$ corresponds to $(e^g_b)_{(g^2-5)}e^g_b$. We have that

$$Y(e^g_b, z)e^g_b = A + bB$$
$$A := Y(\Omega\otimes e^{gJ}, z)(\Omega\otimes e^{gJ}) + b^2 Y(\Omega\otimes e^{-gJ}, z)(\Omega\otimes e^{-gJ})$$
$$B := Y(\Omega\otimes e^{gJ}, z)(\Omega\otimes e^{-gJ}) + Y(\Omega\otimes e^{-gJ}, z)(\Omega\otimes e^{gJ}).$$

From formulae (6.85) and (6.86), we deduce that $E_-(cgJ, z)(\Omega\otimes e^{dgJ})$ is equal to

$$\left(\Omega + \sum_{n=1}^{+\infty}\frac{(-1)^n}{n!}\left(\sum_{m=n}^{\infty}\left(\sum_{\substack{j_1+\cdots+j_n=m\\ j_k>0}}\frac{(cg)^n J_{j_1}\cdots J_{j_n}\Omega}{j_1\cdots j_n}\right)z^{-m}\right)\right)\otimes e^{dgJ}, \tag{6.108}$$

for all $c, d \in \{-1, +1\}$. Formula (6.108) is equal to $\Omega\otimes e^{dgJ}$ because $J_j\Omega = 0$ for all $j \geq 0$ by construction. From Remark 6.3.4 and (6.73), we have that

$$Y(\Omega\otimes e^{\pm gJ}, z)(\Omega\otimes e^{\pm gJ}) = z^{g^2}\left(E_+(\pm gJ, z)\Omega\right)\otimes e^{\pm 2gJ} \tag{6.109}$$
$$Y(\Omega\otimes e^{\pm gJ}, z)(\Omega\otimes e^{\mp gJ}) = z^{-g^2}\left(E_+(\pm gJ, z)\Omega\right)\otimes 1 \tag{6.110}$$

From (6.84) we deduce that the lowest z-power in equation (6.109) is $g^2 > 0$. Thus we can restrict our attention just to B and therefore to equation (6.110) because $-g^2+4 \leq 0$. Then we have to consider the z^4-coefficient of $E_+(\pm gJ, z)\Omega$ in equation (6.110). This means that in (6.83) and (6.84), we can restrict the calculation to the cases $m = 4$ and $n \in \{1, 2, 3, 4\}$. There we find

$n = 1$: $\frac{\pm g}{4}J_{-4}$

$n = 2$: $\frac{1}{2}\left(\frac{(\pm g)^2}{4}(J_{-2})^2 + \frac{(\pm g)^2}{3}J_{-3}J_{-1} + \frac{(\pm g)^2}{3}J_{-1}J_{-3}\right)$

$n = 3$: $\frac{1}{3!}\left(\frac{(\pm g)^3}{2}J_{-2}(J_{-1})^2 + \frac{(\pm g)^3}{2}J_{-1}J_{-2}J_{-1} + \frac{(\pm g)^3}{2}(J_{-1})^2 J_{-2}\right)$

$n = 4$: $\frac{1}{4!}\left((\pm g)^4(J_{-1})^4\right)$.

Using the commutation relations (6.19), we obtain as z^4-coefficients of $E_+(\pm gJ, z)\Omega$ in equation (6.110), the elements

$$\pm\frac{g}{4}J_{-4}\Omega + \frac{g^2}{8}(J_{-2})^2\Omega + \frac{g^2}{3}J_{-3}J_{-1}\Omega \pm \frac{g^3}{4}J_{-2}(J_{-1})^2\Omega + \frac{g^4}{24}(J_{-1})^4\Omega\,. \tag{6.111}$$

In order to compute B, we have to add up the two versions of (6.111), so the summands with $\pm$ cancel while the other ones double, and we obtain for the z^{-g^2+4}-coefficient of $Y(e_b^g, z)e_b^g$:

$$bv_g \otimes 1 = b\left(\frac{g^4}{12}(J_{-1})^4\Omega + \frac{2g^2}{3}J_{-3}J_{-1}\Omega + \frac{g^2}{4}(J_{-2})^2\Omega\right)\otimes 1\,.$$

In the same manner, the z^{-g^2+1}-coefficient of the formal z-series $Y(\Omega\otimes e^{\pm gJ}, z)\left(\Omega\otimes e^{\mp gJ}\right)$ corresponds to $\left(\Omega\otimes e^{\pm gJ}\right)_{(g^2-2)}\left(\Omega\otimes e^{\mp gJ}\right)$ respectively. Thus we have to consider the z-coefficient of $E_+(\pm gJ, z)\Omega$ in equation (6.110). This corresponds to fixing $m = 1 = n$ in formulas (6.83) - (6.84) to obtain $\pm gJ_{-1}$. Thus $(\pm gJ_{-1}\Omega)\otimes 1$ are the desired z^{-g^2+1}-coefficients. □

Proposition 6.3.15. *For every nonzero complex number g, the primary vector $u\otimes 1$ in $V_{L_{2N}}$ of conformal weight 4 as in Lemma 6.3.13 is a linear combination of vectors $L_{-4}\Omega\otimes 1$, $L_{-2}\nu\otimes 1$ and $v_g\otimes 1$ as in Proposition 6.3.14.*

Proof. It is easy to see that the three vectors form a basis for the vector subspace of $V_{L_{2N}}$ generated by $(J_{-1})^4\Omega\otimes 1$, $J_{-3}J_{-1}\Omega\otimes 1$ and $(J_{-2})^2\Omega\otimes 1$. It follows that $u\otimes 1$ must be a linear combination of these three vectors. □

From previous results we obtain a generalization of [DG98, Theorem 2.9].

Corollary 6.3.16. *For any integer N which is not a perfect square, $V_{L_{2N}}^+$ is generated as a VOA by the conformal vector $\nu\otimes 1$ and the primary vector $e_1^{\sqrt{2N}} = \Omega\otimes\left(e^{\sqrt{2N}J} + e^{-\sqrt{2N}J}\right)$.*

Remark 6.3.17. Note that in [DG98], the authors denote by ω and β our conformal vector $\nu\otimes 1$ and the current vector J respectively (see [DG98, p. 264]). Moreover, their u^m stands for the highest weight vector of the irreducible Virasoro submodule $L(1, m^2)$ of $M(1)$ (see [DG98, p. 268]), thus their u^2 is our $u\otimes 1$ as in Lemma 6.3.13. Finally, e^n in [DG98, p. 269] coincides with our $e_1^{\sqrt{2N}}$, identifying n with N when they are different from a perfect square.

Proof of Corollary 6.3.16. Due to Remark 6.3.10 the vector $\Omega\otimes\left(e^{\sqrt{2N}J} + e^{-\sqrt{2N}J}\right)$ is in $V_{L_{2N}}^+$. Using Proposition 6.3.14, the vector $v_g\otimes 1\in V_{L_{2N}}^+$. Then, the result follows from Proposition 6.3.15 and [DG98, Theorem 2.9]. □

We are now ready to prove the key ingredient for the proof of our classification result.

Theorem 6.3.18. *Let N be an integer which is not a perfect square. Every unitary subalgebra W of $V_{L_{2N}}$ properly containing $L(1,0)$ contains also $M(1)^+$.*

Proof. Consider the decomposition of $V_{L_{2N}}$ as in (6.100). Keeping in mind the notational correspondence given in Remark 6.3.17, if W contains $L(1, 4p^2)$ for at least one $p > 0$, then the theorem follows from [DG98, Theorem 2.7(2)]. Similarly, if W contains $L(1, p^2)$ for at least one odd $p > 0$, then we have that $\bigoplus_{\substack{j=0\\ \text{even}}}^{2p} L(1, j^2) \subseteq W$ by [DG98, Lemma 2.6]. Then, the theorem follows from [DG98, Theorem 2.7(2)]. We therefore have to prove that W contains at least one $L(1, p^2)$ with $p > 0$.

Suppose for contradiction that W does not contain any $L(1, p^2)$ with $p > 0$, i.e., $W\cap L(1, p^2) = \{0\}$. This implies that W has the following decomposition into Virasoro modules

$$W = L(1,0)\oplus\bigoplus_{m>0} a_m L(1, Nm^2) \tag{6.112}$$

where $a_m\in\{0,1,2\}$.

We first want to prove that $a_m\neq 2$, for all m. If a_m were equal to 2 for some m then by Remark 6.3.10, $\Omega\otimes e^{\pm\sqrt{2N}mJ}\in W$. By Proposition 6.3.14, $J_{-1}\Omega\otimes 1$ would then lie in W, but

$J_{-1}\Omega \otimes 1$ is primary of conformal weight 1. In other words, $J_{-1}\Omega \otimes 1 \in L(1,1)$, which cannot be the case as $W \cap L(1,1) = \{0\}$ by assumption. Therefore, $a_m \in \{0,1\}$; moreover, at least one a_m equals 1 because by assumption $W \neq L(1,0)$. Fix such an m.

Second, from Remark 6.3.10, we know that, for this m, there exists a primary vector of conformal weight $g^2/2 := Nm^2$ in W which must be a linear combination of the form $\Omega \otimes \left(ae^{gJ} + be^{-gJ}\right)$, for some $a, b \in \mathbb{C}$. W is a unitary subalgebra, thus W must be invariant under the PCT operator θ. If a were 0 then $\theta\left(\Omega \otimes be^{-gJ}\right) = \Omega \otimes \bar{b}e^{gJ}$ would be in W, which means that $a_m = 2$; however, since $a_m = 1$, we find that $a \neq 0$. Similarly, $b \neq 0$; thus, up to rescaling, we can suppose $a = 1$. With $e_b^g = \Omega \otimes \left(e^{gJ} + be^{-gJ}\right) \in W$ as in Proposition 6.3.14, the vector $v_g \otimes 1$ must lie in W. Furthermore, $L_{-2}\nu \otimes 1$ and $L_{-4}\Omega \otimes 1$ are in W because they are vectors of $L(1,0)$. Thus by Proposition 6.3.15, $u \otimes 1$ lies in W. On the other hand, $u \otimes 1$ is the primary vector of conformal weight 4 generating $L(1,4)$, so we obtain $W \cap L(1,4) \neq \{0\}$, which leads to a contradiction. Therefore W must contain at least one $L(1,p^2)$ with $p > 0$, which concludes the proof of the theorem. □

Theorem 6.3.18 also allows us to explicitly calculate $\mathrm{Aut}_{(\cdot|\cdot)}(V_{L_{2N}})$ for N not a perfect square. We highlight that $\mathrm{Aut}(V_{L_{2N}})$ for all N has been calculated by [DN99, Theorem 2.1].

Corollary 6.3.19. *For any integer N which is not a perfect square, we have that*

$$\mathrm{Aut}_{(\cdot|\cdot)}(V_{L_{2N}}) = D_\infty \,. \tag{6.113}$$

Proof. We have $D_\infty \subseteq \mathrm{Aut}_{(\cdot|\cdot)}(V_{L_{2N}})$ by Proposition 6.3.7. Suppose for contradiction there exists $g \in \mathrm{Aut}_{(\cdot|\cdot)}(V_{L_{2N}}) \setminus D_\infty$ and let G be the (proper) closed abelian subgroup of $\mathrm{Aut}_{(\cdot|\cdot)}(V_{L_{2N}})$ generated by g. By the Galois correspondence given in Theorem 6.3.11 and the fact that $L(1,0)$ is the only proper unitary subalgebra of $M(1)^+ = V_{L_{2N}}^{D_\infty}$ as proved in [DG98, Corollary 2.8], we can deduce that $V_{L_{2N}}^{\mathrm{Aut}_{(\cdot|\cdot)}(V_{L_{2N}})} = L(1,0)$. Then, using again the Galois correspondence, $V_{L_{2N}}^G$ is a unitary subalgebra of $V_{L_{2N}}$ properly containing $L(1,0)$. By Theorem 6.3.18, $V_{L_{2N}}^G$ contains also $M(1)^+ = V_{L_{2N}}^{D_\infty}$, so G is a subgroup of D_∞ by the Galois correspondence, which is a contradiction, so $\mathrm{Aut}_{(\cdot|\cdot)}(V_{L_{2N}}) = D_\infty$. □

From formula (6.80), we deduce that for every $N \in \mathbb{Z}_{>0}$, the vector subspace of $V_{L_{2N}}$ of vectors of conformal weight 2 is spanned by

$$\begin{aligned} J_{-2}\Omega \otimes 1,\ \nu \otimes 1,\ J_{-1}\Omega \otimes e^{\sqrt{2}J},\ J_{-1}\Omega \otimes e^{-\sqrt{2}J} \quad &(N=1) \\ J_{-2}\Omega \otimes 1,\ \nu \otimes 1,\ \Omega \otimes e^{2J},\ \Omega \otimes e^{-2J} \quad &(N=2) \\ J_{-2}\Omega \otimes 1,\ \nu \otimes 1 \quad &(N>2)\,. \end{aligned} \tag{6.114}$$

Moreover, it is easy to verify by (6.81) that

$$\begin{aligned} L_1(J_{-1}\Omega \otimes e^{\pm\sqrt{2}J}) =& \pm\sqrt{2}\Omega \otimes e^{\pm\sqrt{2}J}, \\ L_1\left(J_{-2}\Omega \otimes 1\right) =& 2J_{-1}\Omega \otimes 1, \\ L_1(\nu \otimes 1) =& 0, \\ L_1(\Omega \otimes e^{\pm 2J}) =& 0. \end{aligned} \tag{6.115}$$

Now, let W be a unitary subalgebra of $V_{L_{2N}}$. Consider the conformal vector $\omega = e_W(\nu \otimes 1)$ of W as in Proposition 2.6.4 and let $Y(\omega, z) := \sum_{n\in\mathbb{Z}} L_n^\omega z^{-n-2}$ be the corresponding energy-momentum field. Using (i) of Proposition 2.6.4 together with [Kac01, Theorem 4.10(iv)] and the quasi-primarity of the conformal vector, we have that

$$\begin{aligned} L_0\omega =& L_0^\omega \omega = 2\omega \\ L_1\omega =& L_1^\omega \omega = 0\,. \end{aligned} \tag{6.116}$$

Thus, equations (6.116) imply that ω must be a quasi-primary vector in $V_{L_{2N}}$ of conformal weight 2. Hence, according to (6.114) and (6.115), recalling that all non-zero vectors on the right hand sides of (6.115) are linearly independent, ω must be equal to a multiple of $\nu \otimes 1$ for $N \neq 2$. Then, we have proved the following result (cf. also [Car99, Proposition 5.1]):

Proposition 6.3.20. *Let $N \neq 2$. Then, every nontrivial unitary subalgebra of $V_{L_{2N}}$ contains $L(1,0)$.*

Remark 6.3.21. For $N = k^2$ with k a positive integer, Proposition 6.3.20 follows also directly from Proposition 6.3.8 and Theorem 6.3.12.

In the following we investigate the case $N = 2$ to be able to complete the classification.

First of all, we know from [DGH98, Lemma 3.1] (put $2J = \alpha$ there), cf. also the proof of [DMZ94, Theorem 6.3] putting $2J = \beta$ there, that V_{L_4} contains at least two distinct copies of $L\left(\frac{1}{2}, 0\right)$ generated by the Virasoro vectors

$$\omega_0 := \frac{\nu \otimes 1}{2} + \frac{\Omega \otimes \left(e^{2J} + e^{-2J}\right)}{4} \tag{6.117}$$

$$\omega_\pi := \frac{\nu \otimes 1}{2} - \frac{\Omega \otimes \left(e^{2J} + e^{-2J}\right)}{4} . \tag{6.118}$$

Let W_0 and W_π be the vertex subalgebras of V_{L_4} generated by ω_0 and ω_π respectively. Then, thanks to [CKLW18, Example 5.26], they are unitary subalgebras of V_{L_4}, unitarily isomorphic to the unitary Virasoro VOA $L\left(\frac{1}{2}, 0\right)$, see Example 6.1.1. Therefore, our goal is to prove the following result:

Theorem 6.3.22. *For every $t \in \mathbb{T}$, the vector*

$$\omega_t := \frac{\nu \otimes 1}{2} + \frac{\Omega \otimes \left(e^{it}e^{2J} + e^{-it}e^{-2J}\right)}{4} \in V_{L_4} \tag{6.119}$$

is a Virasoro vector with central charge $\frac{1}{2}$, generating a unitary subalgebra $W_t \subset V_{L_4}$ unitarily isomorphic to $L\left(\frac{1}{2}, 0\right)$. Moreover, if W is a nontrivial unitary subalgebra of V_{L_4} which does not contain $L(1, 0)$, then $W = W_t = g_{4,t}(W_0)$ for some $t \in \mathbb{T}$.

To prove Theorem 6.3.22, we need the following result.

Lemma 6.3.23. *Let $W \subset V_{L_4}$ be a nontrivial unitary subalgebra which does not contain $L(1, 0)$, then*

$$\omega = e_W(\nu \otimes 1) \in \left\{\omega_t \mid t \in \mathbb{T}\right\}$$

Proof. First of all, $L(1, 0) \not\subset W$ implies that the conformal vector ω of W must be different from $\nu \otimes 1$. Moreover, it must be different from a multiple of $\nu \otimes 1$ because W is nontrivial and because it has to satisfy $L_0^\omega \omega = 2\omega$ as in (6.116). Now, consider the energy-momentum field $Y(\omega, z) := \sum_{n\in\mathbb{Z}} L_n^\omega z^{-n-2}$ corresponding to ω. Then, (6.116) implies that ω must be a linear combination of the three quasi-primary vectors of V_{L_4} in (6.114), that is

$$\omega = a\nu \otimes 1 + b\Omega \otimes e^{2J} + d\Omega \otimes e^{-2J}$$

for some $a, b, d \in \mathbb{C}$. Note that either $b \neq 0$ or $d \neq 0$ because ω is not a multiple of $\nu \otimes 1$. Using that $\theta(\omega) = \omega$ (see Proposition 2.6.4) we obtain

$$\omega = a\nu \otimes 1 + \Omega \otimes \left(be^{2J} + \overline{b}e^{-2J}\right) = a\nu \otimes 1 + be^2$$

for some $a \in \mathbb{R}$, $b \in \mathbb{C} \setminus \{0\}$ and $e^2 := e^2_{\frac{\overline{b}}{b}} = \Omega \otimes \left(e^{2J} + \frac{\overline{b}}{b}e^{-2J}\right)$.

We want to calculate $L_0^\omega \omega$. Consider

$$\sum_{n\in\mathbb{Z}} L_n^\omega \omega z^{-n-2} = Y(\omega, z)\omega = aY(\nu \otimes 1, z)\omega + bY(e^2, z)\omega . \tag{6.120}$$

Then, we have that

$$L_0^\omega \omega = aL_0\omega + abC_{-2,\nu} + b^2 C_{-2,e^2} = 2a\omega + abC_{-2,\nu} + b^2 C_{-2,e^2} \tag{6.121}$$

where $C_{-2,\nu}$ and C_{-2,e^2} are z^{-2}-coefficients of $Y(e^2, z)(\nu \otimes 1)$ and $Y(e^2, z)e^2$ respectively.

Proceeding as in the proof of Proposition 6.3.14, we find

$$C_{-2,e^2} = \frac{\overline{b}}{b} 4(J_{-1})^2 \Omega \otimes 1 = \frac{\overline{b}}{b} 8\nu \otimes 1 .$$

To calculate $C_{-2,\nu}$, consider

$$\begin{aligned} Y(\Omega\otimes e^{2J},z)(\nu\otimes 1) =&E_+(2J,z)E_-(2J,z)\nu\otimes e^{2J}\\ \frac{\overline{b}}{b}Y(\Omega\otimes e^{-2J},z)(\nu\otimes 1) =&\frac{\overline{b}}{b}E_+(-2J,z)E_-(-2J,z)\nu\otimes e^{-2J} \end{aligned}$$

where we have used firstly Remark 6.3.4 and secondly formula (6.77). Using commutation relations (6.19) and the fact that $J_j\Omega = 0$ for all $j \geq 0$ by construction, we get

$$\begin{aligned} J_j(\nu\otimes e^{\pm 2J}) =&\frac{1}{2}J_j\,(J_{-1})^2\,\Omega\otimes e^{\pm 2J} = 0 \quad \forall j\geq 2\\ (J_1)^j\,(\nu\otimes e^{\pm 2J}) =&\frac{1}{2}\,(J_1)^j\,(J_{-1})^2\,\Omega\otimes e^{\pm 2J} = 0 \quad \forall j\geq 3\\ (J_1)^2\,(\nu\otimes e^{\pm 2J}) =&\frac{1}{2}\,(J_1)^2\,(J_{-1})^2\,\Omega\otimes e^{\pm 2J} = \Omega\otimes e^{\pm 2J}\\ J_1(\nu\otimes e^{\pm 2J}) =&\frac{1}{2}J_1\,(J_{-1})^2\,\Omega\otimes e^{\pm 2J} = J_{-1}\Omega\otimes e^{\pm 2J}\,. \end{aligned}$$

Together with (6.85) and (6.86), this implies

$$\begin{aligned} &E_-(\pm 2J,z)(\nu\otimes e^{\pm 2J}) =\\ &=\nu\otimes e^{\pm 2J} + \frac{-1}{1!}\cdot\frac{\pm 2J_1(\nu\otimes e^{\pm 2J})}{1}z^{-1} + \frac{(-1)^2}{2!}\cdot\frac{(\pm 2J_1)^2\,(\nu\otimes e^{\pm 2J})}{1\cdot 1}z^{-2}\\ &=\nu\otimes e^{\pm 2J} \mp \left(2J_{-1}\Omega\otimes e^{\pm 2J}\right)z^{-1} + \left(2\Omega\otimes e^{\pm 2J}\right)z^{-2}\,. \end{aligned}$$

Using (6.83) and (6.84), we find

$$E_+(\pm 2J,z) = I \pm 2J_{-1}z + O(z^2),$$

thus

$$\begin{aligned} Y(\Omega\otimes e^{2J},z)(\nu\otimes 1) =&E_+(2J,z)E_-(2J,z)\nu\otimes e^{2J}\\ =&\left(2\Omega\otimes e^{2J}\right)z^{-2} + O(z^{-1}) \end{aligned}$$

and

$$\begin{aligned} \frac{\overline{b}}{b}Y(\Omega\otimes e^{-2J},z)(\nu\otimes 1) =&\frac{\overline{b}}{b}E_+(-2J,z)E_-(-2J,z)\nu\otimes e^{-2J}\\ =&\frac{\overline{b}}{b}\left(2\Omega\otimes e^{-2J}\right)z^{-2} + O(z^{-1}). \end{aligned}$$

We deduce that

$$C_{-2,\nu} = 2e^2\,. \tag{6.122}$$

We can therefore rewrite identity (6.121):

$$L_0^\omega\omega = 2a\omega + 2abe^2 + 8|b|^2\nu\otimes 1 = 2(a^2+4|b|^2)\nu\otimes 1 + 4abe^2 \tag{6.123}$$

Imposing equation (6.116), namely $L_0^\omega\omega = 2\omega$, and using (6.123), we have the following identity

$$2a\nu\otimes 1 + 2be^2 = 2(a^2+4|b|^2)\nu\otimes 1 + 4abe^2$$

which has precisely the solutions

$$a=\frac{1}{2},\quad b=\frac{e^{it}}{4},\quad t\in\mathbb{T}.$$

Thus $\omega\in\{\omega_t:\ t\in\mathbb{T}\}$. □

Proof of Theorem 6.3.22. First, note that $\omega_t = g_{4,t}(\omega_0)$ for all $t\in\mathbb{T}$, where $g_{4,t}$ as in (6.91) is a unitary automorphism of V_{L_4}. Thus, $\omega_t\in V_{L_4}$ are still Virasoro vectors of central charge $\frac{1}{2}$ and by [CKLW18, Example 5.26], they generate unitary subalgebras $W_t = g_{4,t}(W_0)$ of V_{L_4} unitarily isomorphic to $L\left(\frac{1}{2},0\right)$.

Second, we have to show that every unitary subalgebra $W \subset V_{L_4}$ is of the above type. Lemma 6.3.23 proves that $\omega = e_W(\nu \otimes 1)$ must be of the form ω_t, for some $t \in \mathbb{T}$. Accordingly, W is a unitary VOA with conformal vector ω_t and hence its central charge is $c = \frac{1}{2}$. On the other hand $L\left(\frac{1}{2},0\right)$ is the unique, up to isomorphism, unitary VOA with central charge $c = \frac{1}{2}$, see e.g. [DL15]. It follows that $W = W_t$. □

Remark 6.3.24. Note that every W_t is contained in $V_{L_4}^{D_1^t}$ for all $t \in \mathbb{T}$. By Remark 6.3.9, $D_1^t = D_1^{t+\pi}$ for all $t \in \mathbb{T}$. Moreover, under the identification $V_{L_4}^+ = V_{L_4}^{D_1} = W_0 \otimes W_\pi$ given by [DGH98, Lemma 3.1 (ii)], we have that $V_{L_4}^{D_1^t} = W_t \otimes W_{t+\pi}$ for all $t \in \mathbb{T}$.

Then, we can prove the classification theorem.

Theorem 6.3.25. *The non-trivial unitary subalgebras W of the even rank-one lattice type VOAs $V_{L_{2N}}$ are classified as follows. Apart from the Virasoro subalgebra $L(1,0)$, we have:*

(i) If $N = k^2$ for some positive integer k, then after the identification of $V_{L_{2N}}$ with $V_{L_2}^{\mathbb{Z}_k}$, $W = V_{L_2}^H$ for some closed subgroup $H \subseteq SO(3)$ containing $\mathbb{Z}_k$.

(ii) If $N > 2$ is not a perfect square then $W = V_{L_{2N}}^H$ for some closed subgroup $H \subseteq D_\infty$.

(iii) If $N = 2$, then either $W = V_{L_4}^H$ for some closed subgroup $H \subseteq D_\infty$ or $W = W_t$ for some $t \in \mathbb{T}$.

Proof. First of all, note that $L(1,0) \subset V_{L_{2N}}$ for all $N \in \mathbb{Z}_{>0}$.

(i) The case $k = 1$ is Theorem 6.3.12. By Proposition 6.3.8, $V_{L_{2k^2}} = V_{L_2}^{\mathbb{Z}_k}$ for all $k \in \mathbb{Z}_{>0}$ is a unitary subalgebra of V_{L_2}. Then, the result follows from the Galois correspondence in Theorem 6.3.11 and from Theorem 6.3.12.

(ii) By Proposition 6.3.20, every unitary subalgebra of $V_{L_{2N}}$ contains $L(1,0)$ and thus contains also $M(1)^+$ by Theorem 6.3.18. On the other hand, $M(1)^+ = V_{L_{2N}}^{D_\infty}$ by Proposition 6.3.7 and therefore the result follows from the Galois correspondence in Theorem 6.3.11.

(iii) If $L(1,0) \subset W$, then we can proceed as in (ii) above. The other case is covered by Theorem 6.3.22.

□

Now, we can transpose Theorem 6.3.25 to the family of even rank-one lattice conformal nets $\mathcal{A}_{\mathrm{U}(1)_{2N}}$ with $N \in \mathbb{Z}_{>0}$ thanks to Theorem 5.1.1 and Theorem 3.2.19.

First, we point out that such an extension is maximal if and only if N is not a multiple of a perfect square. As a matter of fact, if $N = N'k^2$ for positive integers N' and k, then $\mathcal{A}_{\mathrm{U}(1)_{2N'k^2}} \subset \mathcal{A}_{\mathrm{U}(1)_{2N'}}$, cf. [BMT88, p. 37].

Second, by Theorem 3.2.19, there is a well-defined action of D_∞ on $\mathcal{A}_{\mathrm{U}(1)_{2N}}$ for every N in $\mathbb{Z}_{>0}$. Let $\mathcal{A}^+_{\mathrm{U}(1)_{2N}} := \mathcal{A}^{D_1}_{\mathrm{U}(1)_{2N}}$ and $\mathcal{A}^+_{\mathrm{U}(1)} := \mathcal{A}^{D_1}_{\mathrm{U}(1)}$, then we have the conformal net results corresponding to Proposition 6.3.7, Proposition 6.3.8 and Remark 6.3.9, namely

$$\begin{aligned}
\mathcal{A}^{\mathbb{T}}_{\mathrm{U}(1)_{2N}} &= \mathcal{A}_{\mathrm{U}(1)}\,, && (6.124)\\
\mathcal{A}^{D_\infty}_{\mathrm{U}(1)_{2N}} &= \mathcal{A}^+_{\mathrm{U}(1)}\,, && (6.125)\\
\mathcal{A}^{\mathbb{Z}_k}_{\mathrm{U}(1)_{2N}} &= \mathcal{A}_{\mathrm{U}(1)_{2Nk^2}}\,, && (6.126)\\
\mathcal{A}^{D_k}_{\mathrm{U}(1)_{2N}} &= \mathcal{A}^+_{\mathrm{U}(1)_{2Nk^2}}\,, && (6.127)\\
\mathcal{A}^{D_k^t}_{\mathrm{U}(1)_{2N}} &= g_{2N,t}\mathcal{A}^+_{\mathrm{U}(1)_{2Nk^2}}(g_{2N,t})^{-1}\,. && (6.128)
\end{aligned}$$

for all positive integers N and k. Furthermore, we have a well-defined action of SO(3) on $\mathcal{A}_{\mathrm{U}(1)_2}$, which extends the one of D_∞. Set $\mathcal{A}_t := \mathcal{A}_{W_t} \cong \mathrm{Vir}_{\frac{1}{2}}$ for all $t \in \mathbb{T}$ with W_t as in Theorem 6.3.22. Then, we can state the following classification result:

Theorem 6.3.26. *The non-trivial conformal subnets $\mathcal{A}$ of the even rank-one lattice conformal nets $\mathcal{A}_{U(1)_{2N}}$ are classified as follows. Apart from the Virasoro conformal net* Vir_1*, we have:*

(i) If $N = k^2$ for some positive integer k, then after the identification $\mathcal{A}_{\mathrm{U}(1)_{2k^2}} = \mathcal{A}^{\mathbb{Z}_k}_{\mathrm{U}(1)_2}$, $\mathcal{A} = \mathcal{A}^H_{\mathrm{U}(1)_2}$ for some closed subgroup $H \subseteq SO(3)$ containing $\mathbb{Z}_k$.

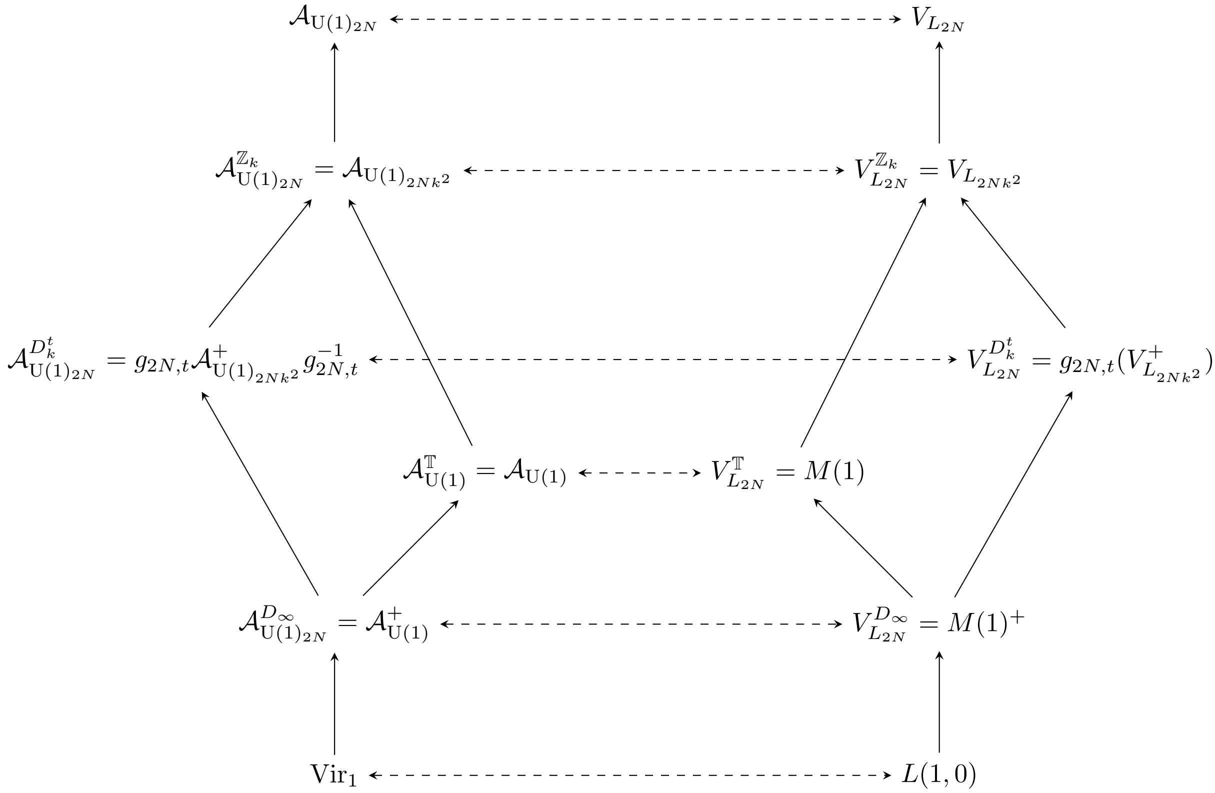
(ii) If $N > 2$ is not a perfect square then $\mathcal{A} = \mathcal{A}^H_{U(1)_{2N}}$ for some closed subgroup $H \subseteq D_\infty$.

(iii) If $N = 2$, then either $\mathcal{A} = \mathcal{A}^H_{U(1)_4}$ for some closed subgroup $H \subseteq D_\infty$ or $\mathcal{A} = \mathcal{A}_t$ for some $t \in \mathbb{T}$.

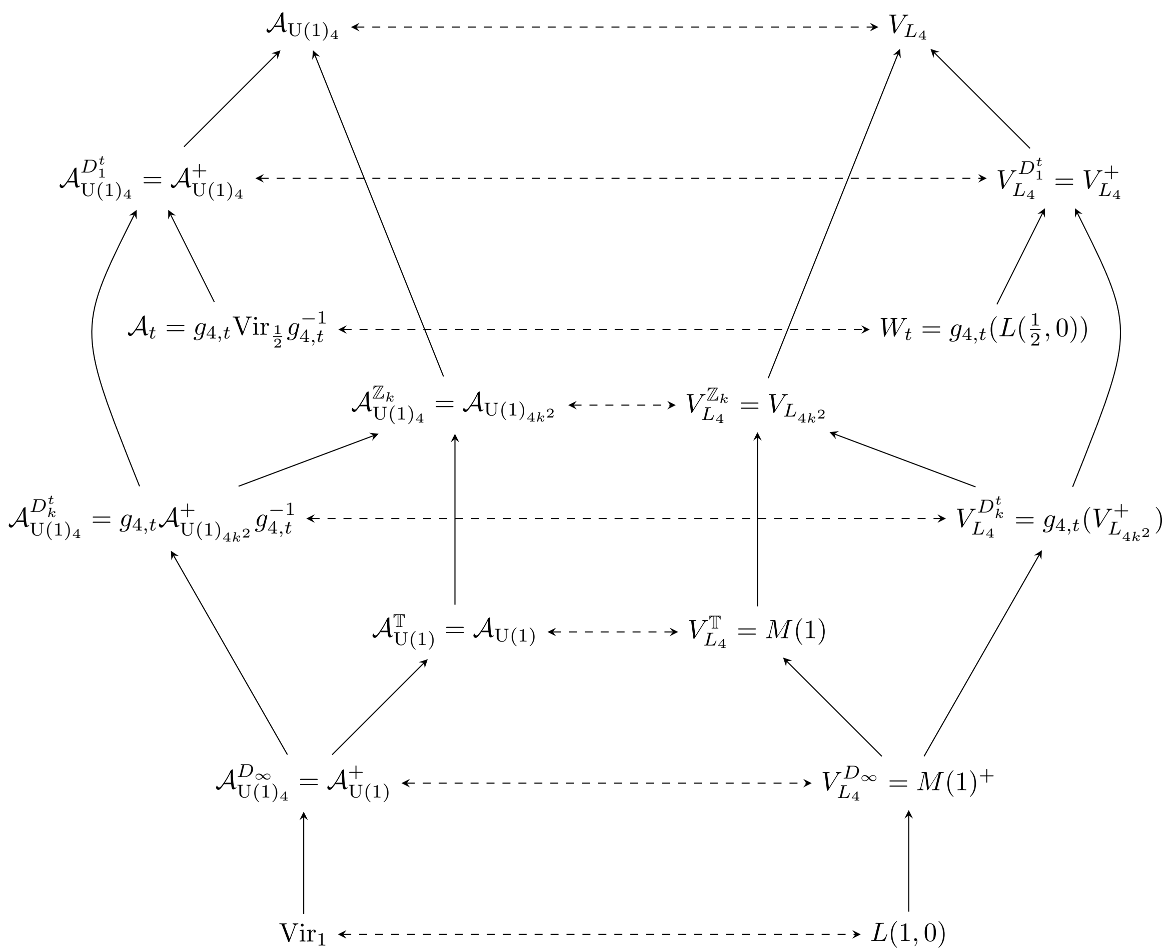
Regarding the case $N = 1$, recall that the classification has been known for a while, cf. [Car99, Theorem 3.2] and [Reh94, Proposition 5]. Recall also that $\mathcal{A}_{\mathrm{U}(1)_2}$ is isomorphic to the level 1 loop group net $\mathcal{A}_{\mathrm{SU}(2)_1}$, see [BMT88, Section 5B].

We support the understanding of these classification results by presenting two graphical realisations of the cases (ii) and (iii) of Theorem 6.3.25 and Theorem 6.3.26, that is, whenever $N \geq 2$, but different from a perfect square.

Case $k^2 \neq N > 2$



Case $N = 2$


$\mathcal{A}_{\mathrm{U}(1)_4}$
V_{L_4}
$\mathcal{A}_{\mathrm{U}(1)_4}^{D_1^t} = \mathcal{A}_{\mathrm{U}(1)_4}^{+}$
$V_{L_4}^{D_1^t} = V_{L_4}^{+}$
$\mathcal{A}_t = g_{4,t}\mathrm{Vir}_{\frac{1}{2}}g_{4,t}^{-1}$
$W_t = g_{4,t}(L(\frac{1}{2},0))$
$\mathcal{A}_{\mathrm{U}(1)_4}^{\mathbb{Z}_k} = \mathcal{A}_{\mathrm{U}(1)_{4k^2}}$
$V_{L_4}^{\mathbb{Z}_k} = V_{L_{4k^2}}$
$\mathcal{A}_{\mathrm{U}(1)_4}^{D_k^t} = g_{4,t}\mathcal{A}_{\mathrm{U}(1)_{4k^2}}^{+}g_{4,t}^{-1}$
$V_{L_4}^{D_k^t} = g_{4,t}(V_{L_{4k^2}}^{+})$
$\mathcal{A}_{\mathrm{U}(1)}^{\mathbb{T}} = \mathcal{A}_{\mathrm{U}(1)}$
$V_{L_4}^{\mathbb{T}} = M(1)$
$\mathcal{A}_{\mathrm{U}(1)_4}^{D_\infty} = \mathcal{A}_{\mathrm{U}(1)}^{+}$
$V_{L_4}^{D_\infty} = M(1)^{+}$
Vir_1
$L(1,0)$

Chapter 7

Reconstruction of the VOSA and functoriality

In Section 7.1, we show under which conditions we can recover a VOSA theory from an irreducible graded-local conformal net. Then, we look at the correspondence between VOSAs and graded-local conformal nets from a functorial point of view in Section 7.2.

Comparing with the local case... The content of Section 7.1 is a generalisation of the results proposed in [CKLW18, Chapter 9]. Apart from the usual difficulties due to the gradation of the objects involved, those results generalise from the local case as one would expect. A partial novelty is represented by Theorem 7.1.1, which proves that two irreducible graded-local conformal nets arisen from two distinct VOSAs cannot be isomorphic. Of course, this is a corollary of Theorem 7.1.9, but it is proved without introducing the theory of FJ smeared vertex operators. To the author's knowledge, the (expected) categorical result of Section 7.2 has never been put down in those terms, even for the local case.

7.1 Reconstruction of the VOSA theory

First, we give an "injectivity" result for the correspondence established by Theorem 3.2.17, that is, we prove that if an irreducible graded-local conformal net arises from two simple strongly graded-local unitary VOSAs, then they must have the same VOSA structure. Second, we prove again that result, but this time we explicitly reconstruct the VOSA structure of a simple strongly graded-local unitary VOSA V from its irreducible graded-local conformal net $\mathcal{A}_V$. In this way, we introduce a more general theory, which allows us to give sufficient conditions in order that a given irreducible graded-local conformal net arises from a simple strongly graded-local unitary VOSA.

Theorem 7.1.1. *Let V and W be two simple strongly graded-local unitary VOSAs, giving rise to the same graded-local conformal net theory. Then, V and W have the same VOSA structure.*

Proof. By assumption, V and W give rise to the same irreducible graded-local conformal net up to isomorphism. Then, without loss of generality, we can assume that $\mathcal{H}_V = \mathcal{H}_W$, $\Omega^V = \Omega^W$ and $U_{\mathcal{A}_V} = U_{\mathcal{A}_W}$. This implies that the generators of the rotation subgroup coincide, that is $L_0^V = L_0^W$. Let $\mathcal{H}_V^{\mathrm{fin}}$ denote the subspace of finite energy vectors, i.e., sums of eigenvectors of L_0^V. Then, we have that $V = \mathcal{H}_V^{\mathrm{fin}}$. Since $L_0^V = L_0^W$, we get $V = \mathcal{H}_V^{\mathrm{fin}} = \mathcal{H}_W^{\mathrm{fin}} = W$ as vector superspaces.

Now, we claim that $Y_V(a,z) = Y_W(a,z)$ for all $a \in V$. For any vector $a \in V$, clearly we have $Y_V(a,z)\Omega^W = e^{zT_W}a$. Moreover, for any $b \in W$, the vertex operators $Y_V(a,z)$ and $Y_W(b,z)$ are mutually local in the Wightman sense and thus they are mutually local in the vertex superalgebra sense thanks to Proposition A.1. It follows by the uniqueness theorem for vertex superalgebras [Kac01, Theorem 4.4], that $Y_V(a,z) = Y_W(a,z)$. By the arbitrariness of $a \in V$, the claim follows. Since $L_n^V = L_n^W$ for all $n \in \mathbb{Z}$, we finally get that $\nu^V = \nu^W$ so that V and W coincide as VOSAs. □

The fundamental idea for the theory we are going to present in the following is to construct certain quantum fields starting from an irreducible graded-local Möbius covariant net. This construction relies on the idea developed in [FJ96] by Fredenhagen and Jörß, who construct certain pointlike localized fields associated to an irreducible Möbius covariant net by a scaling limit procedure. Instead, we use and naturally generalise the alternative construction based on the Tomita-Takesaki modular theory for von Neumann algebras (see e.g. [BR02, Chapter 2.5]) presented in [CKLW18, Chapter 9].

For the reader's benefit, we report the following standard argument.

Remark 7.1.2. Let $\mathcal{N}$ be a von Neumann algebra on an Hilbert space $\mathcal{H}$ with a cyclic and separating vector $\Omega \in \mathcal{H}$ for $\mathcal{N}$. As usual, we denote by S, Δ and J the Tomita operator, the modular operator and the modular conjugation associated to the pair $(\mathcal{N}, \Omega)$ respectively. Recall that S is an antilinear operator such that $SB\Omega = B^*\Omega$ for all $B \in \mathcal{N}$, $S^*A\Omega = A^*\Omega$ for all $A \in \mathcal{N}'$ and $S = J\Delta^{\frac{1}{2}}$. For an arbitrary vector $a \in \mathcal{H}$, we define the following linear operator with dense domain:

$$\mathscr{L}_a^0 : \mathcal{N}'\Omega \longrightarrow \mathcal{H}\,, \quad A\Omega \longmapsto Aa\,. \tag{7.1}$$

If $a \in \mathcal{D}(S)$, the domain of S, then

$$\begin{aligned}(\mathscr{L}_{Sa}^0 A\Omega|B\Omega) &= (ASa|B\Omega) = (a|S^*A^*B\Omega)\\ &= (B^*A\Omega|a) = (A\Omega|Ba)\\ &= (A\Omega|\mathscr{L}_a^0 B\Omega) \qquad \forall A, B \in \mathcal{N}'\,,\end{aligned} \tag{7.2}$$

that is, $\mathscr{L}_{Sa}^0 \subseteq (\mathscr{L}_a^0)^*$. Hence, $\mathscr{L}_a^0$ and $\mathscr{L}_{Sa}^0$ are both closable and their respective closures $\mathscr{L}_a$ and $\mathscr{L}_{Sa}$ satisfy $\mathscr{L}_{Sa} \subseteq \mathscr{L}_a^*$ for all $a \in \mathcal{D}(S)$. We also have that all the operators $\mathscr{L}_a$ and $\mathscr{L}_{Sa}$ above are affiliated with $\mathcal{N}$.

The operator $\mathscr{L}_a$ with $a \in \mathcal{D}(S)$ of Remark 7.1.2 can be interpreted in certain situations, see [Car05], as abstract analogue of smeared vertex operators, see also [BBS01]. We explain this point of view in the following, constructing a family of localized fields associated to a given irreducible graded-local Möbius covariant net $\mathcal{A}$ on S^1. Let $\mathcal{H}$ be the vacuum Hilbert space of $\mathcal{A}$ and U be the positive-energy strongly continuous unitary representation of $\text{Möb}(S^1)^{(\infty)}$. Recall that the conformal Hamiltonian L_0, that is, the infinitesimal generator of the one-parameter subgroup of $U(\text{Möb}(S^1)^{(\infty)})$ of rotations, is a positive self-adjoint operator on $\mathcal{H}$. By the Vacuum Spin-Statistic theorem (1.46), U factors through a representation of $\text{Möb}(S^1)^{(2)}$, which we denote by the same symbol. Accordingly, the spectrum of L_0 is contained in $\frac{1}{2}\mathbb{Z}_{\geq 0}$ and we use the usual definition

$$\mathcal{H}^{\text{fin}} := \bigoplus_{n \in \frac{1}{2}\mathbb{Z}_{\geq 0}} \text{Ker}(L_0 - n1_{\mathcal{H}}) \subseteq \mathcal{H}\,. \tag{7.3}$$

It is known that U differentiates uniquely to a positive-energy unitary representation of the Virasoro algebra $\mathfrak{Vir}$ on $\mathcal{H}^{\text{fin}}$ and thus it restricts to a unitary representation of $\mathfrak{sl}(2,\mathbb{C})$ on $\mathcal{H}^{\text{fin}}$, the complexification of $\mathfrak{sl}(2,\mathbb{R})$, see Section 1.2 and references therein. The latter representation is spanned by the operators L_{-1}, L_0 and L_1 on $\mathcal{H}$, satisfying $L_1 \subseteq L_{-1}^*$ and the usual commutation relations

$$\begin{aligned}[L_1, L_{-1}] &= 2L_0\\ [L_1, L_0] &= L_1\\ [L_{-1}, L_0] &= -L_{-1}\,.\end{aligned} \tag{7.4}$$

A vector $a \in \mathcal{H}$ is called **quasi-primary** if $L_1 a = 0$ and if it is **homogeneous** of **conformal weight** $d_a \in \frac{1}{2}\mathbb{Z}$, that is, $L_0 a = d_a a$, so that in particular, $a \in \mathcal{H}^{\text{fin}}$. Fix a quasi-primary vector $a \in \mathcal{H}$. With a standard induction argument, we can prove that the commutation relations (7.4) imply that for all $n \in \mathbb{Z}_{\geq 0}$, the vectors

$$a^n := \frac{L_{-1}^n a}{n!} \in \mathcal{H}^{\text{fin}} \tag{7.5}$$

satisfy

$$L_0 a^n = (n + d_a)a^n \quad \forall n \in \mathbb{Z}_{\geq 0}\,, \qquad L_1 a^n = (2d_a + n - 1)a^{n-1} \quad \forall n \in \mathbb{Z}_{>0}\,. \tag{7.6}$$

Moreover, it is also easy to prove that

$$\|a^n\|^2 = (a^n|a^n) = \binom{2d_a+n-1}{n}\|a\|^2 \qquad \forall n \in \mathbb{Z}_{\geq 0}\,. \tag{7.7}$$

Define

$$\mathcal{H}^a := \overline{\{a^n \mid n \in \mathbb{Z}_{\geq 0}\}}^{\|\cdot\|} \subseteq \mathcal{H}\,. \tag{7.8}$$

By (7.7), we can prove that for all $f \in C_a^\infty(S^1)$ (see (5.1) for the notation), the series

$$\sum_{n\in\mathbb{Z}_{\geq 0}} \widehat{f}_{-n-d_a} a^n$$

converges in $\mathcal{H}^a$ to an element, which we call $a(f)$. Furthermore, $f \mapsto a(f)$ defines a linear continuous map from $C_a^\infty(S^1)$ to $\mathcal{H}^a$ and thus proceeding as done to prove Proposition 3.2.12, we have that:

Proposition 7.1.3. *Let $a \in \mathcal{H}$ be a quasi-primary vector. Then, for all $I \in \mathcal{J}$, we have that $U(\gamma)a(f) = a(\iota_{d_a}(\dot\gamma)f)$ for all $\gamma \in \mathrm{Möb}(S^1)^{(\infty)}$ and all $f \in C_a^\infty(S^1)$ with $\mathrm{supp} f \subset I$.*

Now, consider for every interval $I \in \mathcal{J}$, the Tomita operator $S_I = J_I \Delta_I^{\frac{1}{2}}$ associated to the von Neumann algebra $\mathcal{A}(I)$. By the Möbius covariance of the net, we get that

$$U(\gamma)S_I U(\gamma)^* = S_{\dot\gamma I}\,, \qquad U(\gamma)J_I U(\gamma)^* = J_{\dot\gamma I}\,, \qquad U(\gamma)\Delta_I U(\gamma)^* = \Delta_{\dot\gamma I} \tag{7.9}$$

for all $I \in \mathcal{J}$ and all $\gamma \in \mathrm{Möb}(S^1)^{(\infty)}$. Furthermore, by the Bisognano-Wichmann property (1.42), we have that

$$\Delta_{S^1_+}^{it} = e^{iKt} \quad \forall t \in \mathbb{R}\,, \qquad \Delta_{S^1_+}^{\frac{1}{2}} = e^{\frac{K}{2}}\,, \qquad K := i\pi\overline{(L_1 - L_{-1})} \tag{7.10}$$

where K is the infinitesimal generator of the one-parameter subgroup of dilations. We set $\theta := ZJ_{S^1_+}$ and we note that θ commutes with L_{-1}, L_0 and L_1. (Clearly, θ will be the PCT operator of our unitary VOSA theory, cf. the proof of Theorem 5.2.1.)

It should be not difficult to recognize that an equivalent of Theorem 4.0.4 can be derived for $a(f)$ in place of $Y(a,f)\Omega$ there. Indeed, despite it is stated in the energy bounded unitary VOSA setting, $a(f)$ maintains all the necessary properties of $Y(a,f)\Omega$ there. Then, we state:

Theorem 7.1.4. *Let $a \in \mathcal{H}$ be a quasi-primary vector and let $f \in C_c^\infty(S^1\backslash\{-1\},\mathbb{R})$ with $\mathrm{supp} f \subset S^1_+$. Then, $a(f)$ is in the domain of the operator $e^{\frac{K}{2}}$ and*

$$e^{\frac{K}{2}} a(f) = i^{2d_a} a(f\circ j) \tag{7.11}$$

where $j(z) = z^{-1}$ for all $z \in S^1$.

As an application, we get

Corollary 7.1.5. *Let $a \in \mathcal{H}$ be a quasi-primary vector and let $I \in \mathcal{J}$. Then, $a(f)$ is in the domain of the operator S_I for all $f \in C_a^\infty(S^1)$ with $\mathrm{supp} f \subset I$ and*

$$S_I a(f) = (-1)^{2d_a^2+d_a}(\theta(a))(\overline{f})\,.$$

Proof. By Theorem 7.1.4, (7.10) and the linearity of the map $f \mapsto a(f)$, we have that

$$\Delta_{S^1_+}^{\frac{1}{2}} a(f) = e^{\frac{K}{2}} a(f) = i^{2d_a} a(f\circ j) \qquad \forall f \in C_c^\infty(S^1\backslash\{-1\}) \ \text{ with } \ \mathrm{supp} f \subset S^1_+\,. \tag{7.12}$$

Then, $a(f)$ is in the domain of $S_{S^1_+}$ and

$$S_{S^1_+} a(f) = J_{S^1_+}\Delta_{S^1_+}^{\frac{1}{2}} a(f) = (-1)^{2d_a^2+d_a}(\theta(a))(\overline{f}) \tag{7.13}$$

for all $f \in C_c^\infty(S^1 \setminus \{-1\})$ with supp$f \subset S^1_+$. Let $I \in \mathcal{J}$ and choose $\gamma \in \text{Möb}(S^1)^{(\infty)}$ such that $\dot{\gamma} I = S^1_+$. By the covariance of the modular theory (7.9) and Proposition 7.1.3, we get that $a(f)$ is in the domain of S_I for all $f \in C_a^\infty(S^1)$ with supp$f \subset I$ because

$$\begin{aligned} S_I a(f) &= U(\gamma)^* S_{S^1_+} U(\gamma) a(f) \\ &= U(\gamma)^* S_{S^1_+} a(\iota_{d_a}(\dot{\gamma}) f) \\ &= (-1)^{2d_a^2+d_a} U(\gamma)^* (\theta(a))(\overline{\iota_{d_a}(\dot{\gamma}) f}) \\ &= (-1)^{2d_a^2+d_a} (\theta(a))(\overline{f}) \, , \end{aligned} \tag{7.14}$$

which concludes the proof. □

By Remark 7.1.2, we can construct the closed operator $\mathscr{L}^I_{a(f)}$ affiliated to $\mathcal{A}(I)$ for all $I \in \mathcal{J}$, where $a \in \mathcal{H}$ is a quasi-primary vector and $f \in C_a^\infty(S^1)$ with supp$f \subset I$.

Definition 7.1.6. For every quasi-primary vector $a \in \mathcal{H}$, every $I \in \mathcal{J}$ and every $f \in C_a^\infty(S^1)$ with supp$f \subset I$, $\mathscr{L}^I_{a(f)}$ is called a **Fredenhagen-Jörß** (shortly **FJ**) **smeared vertex operator** and it is denoted by $Y_I(a,f)$.

The notation introduced by Definition 7.1.6 is justified by the following properties of FJ smeared vertex operators, which follow directly from the properties of the operators $\mathscr{L}^I_{a(f)}$ constructed above.

Proposition 7.1.7. *Let $a \in \mathcal{H}$ be a quasi-primary vector. Then, the FJ smeared vertex operators $Y_I(a,f)$ have the following properties:*

(i) for every fixed $I \in \mathcal{J}$ and $b \in \mathcal{A}(I)'\Omega$, the map

$$\{f \in C_a^\infty(S^1) \mid \text{supp} f \subset I\} \ni f \longmapsto Y_I(a,f) b \in \mathcal{H} \tag{7.15}$$

is an operator valued distribution, that is, it is linear and continuous;

(ii) if $I_1 \subseteq I_2$ in $\mathcal{J}$, then $Y_{I_1}(a,f) \subseteq Y_{I_2}(a,f)$ for all $f \in C_a^\infty(S^1)$ with supp$f \subset I_1$, so that the operator valued distribution $f \mapsto Y_{I_1}(a,f)$ as defined in (7.15) extends to the one $f \mapsto Y_{I_2}(a,f)$;

(iii) the following Möbius covariance property holds:

$$U(\gamma) Y_I(a,f) U(\gamma)^* = Y_{\dot{\gamma} I}(a, \iota_{d_a}(\dot{\gamma}) f) \tag{7.16}$$

for all $\gamma \in \text{Möb}(S^1)^{(\infty)}$, all $I \in \mathcal{J}$ and all $f \in C_a^\infty(S^1)$ with supp$f \subset I$;

(iv) the following hermiticity formula holds:

$$(-1)^{2d_a^2+d_a} Y_I(\theta(a), \overline{f}) \subseteq Y_I(a,f)^* \tag{7.17}$$

for all $I \in \mathcal{J}$ and all $f \in C_a^\infty(S^1)$ with supp$f \subset I$.

Thanks to the distribution property stated above, we can write the usual formal formula for FJ smeared vertex operators:

$$Y_I(a,f) = \oint_{S^1} Y_I(a,z) f(z) z^{d_a} \frac{dz}{2\pi i z} \tag{7.18}$$

for every quasi-primary vector $a \in \mathcal{H}$ of conformal weight d_a, every $I \in \mathcal{J}$ and every $f \in C_a^\infty(S^1)$ with supp$f \subset I$. Accordingly, we have families of type $\{Y_I(a,z) \mid I \in \mathcal{J}\}$, which we call **FJ vertex operators** or **FJ fields**. As for the local case, it is unknown whether the FJ smeared vertex operators admit a common invariant domain and this prevents us to extend an FJ vertex operator $\{Y_I(a,z) \mid I \in \mathcal{J}\}$ to a unique distribution $\widetilde{Y}(a,z)$. This also implies that the FJ fields cannot be consider as quantum fields in the sense of Wightman as in [SW64, Chapter 3].

Proposition 7.1.8. *Let $\mathcal{A}$ be an irreducible graded-local Möbius covariant net. Then, $\mathcal{A}$ is generated by its FJ smeared vertex operators, which means that*

$$\mathcal{A}(I) = W^* \left(\left\{ Y_{I_1}(a,f) \mid a \in \mathcal{H}^{\text{fin}} \, , \; L_1 a = 0 \, , \; f \in C_a^\infty(S^1) \; \text{supp} f \subset I_1 \, , \; I_1 \subseteq I \right\} \right) \tag{7.19}$$

Proof. We trivially adapt the argument for [CKLW18, Proposition 9.1]. For every $I \in \mathcal{J}$, call $\mathcal{B}(I)$ the right hand side of (7.19), so defining a graded-local Möbius covariant subnet $\mathcal{B}$ of $\mathcal{A}$. By definition, every $a(f)$ with $f \in C^\infty_a(S^1)$ belongs to the vacuum Hilbert space $\mathcal{H}_\mathcal{B} := \overline{\mathcal{B}(S^1)\Omega}$ of $\mathcal{B}$. Since the representation U of $\text{Möb}(S^1)^{(\infty)}$ on $\mathcal{H}$ is reducible, all the $a(f)$ span a dense subset of $\mathcal{H}$. Consequently, $\mathcal{H}$ must coincide with $\mathcal{H}_\mathcal{B}$ and thus $\mathcal{A} = \mathcal{B}$. □

Now, we can prove the second of the three most important results of this chapter. As previously anticipated, we show that the FJ smeared vertex operators of an irreducible graded-local conformal net arisen from a simple strongly graded-local unitary VOSA are actually the ordinary smeared vertex operators. Note that as a corollary, we obtain Theorem 7.1.1.

Theorem 7.1.9. *Let V be a simple strongly graded-local unitary VOSA and consider the corresponding irreducible graded-local conformal net $\mathcal{A}_V$. Then, for every quasi-primary vector $a \in V$, we have that $Y_I(a,f) = Y(a,f)$ for all $I \in \mathcal{J}$ and all $f \in C^\infty_a(S^1)$ with* $\text{supp}f \subset I$. *In particular, one can recover the VOSA structure on $V = \mathcal{H}^{\text{fin}}$ from $\mathcal{A}_V$.*

Proof. We easily adapt the proof of [CKLW18, Theorem 9.2]. Note that $Y(a,f)$ is affiliated with $\mathcal{A}(I)$ and thus its domain contains $\mathcal{A}(I)'\Omega$, which contains $Z\mathcal{A}(I')\Omega \cap \mathcal{H}^\infty$ by Haag duality. By a slight modification of [CKLW18, Proposition 7.3], the latter is a core for $Y(a,f)$, implying that also the former is a core for it. By definition, $\mathcal{A}(I)'\Omega$ is a core for every FJ smeared vertex operator $Y_I(a,f)$. [Kac01, Eq. (4.1.2)] says that $Y(b,z)\Omega = e^{zL_{-1}}b$ for all $b \in V$. Thus, $Y(a,f)\Omega = a(f)$ and

$$Y(a,f)A\Omega = AY(a,f)\Omega = Aa(f) = Y_I(a,f)A\Omega \qquad \forall A \in \mathcal{A}(I)' .$$

It follows that $Y(a,f)$ and $Y_I(a,f)$ coincide in a common core and thus they must be the same. □

Now, we can move to the last part of this chapter, which consists of the reconstruction of a simple strongly graded-local unitary VOSA V from a given irreducible graded-local conformal net $\mathcal{A}$ such that $\mathcal{A} = \mathcal{A}_V$. To reach this goal, we note that a necessary condition is that for every quasi-primary vector $a \in \mathcal{H}$, the corresponding FJ vertex operator $\{Y_I(a,z) \mid I \in \mathcal{J}\}$ is **energy-bounded**, that is, there exists a positive real number M and positive integers k and s such that

$$\left\|Y_I(a,f)b\right\| \leq M\|f\|_s \left\|(L_0 + 1_\mathcal{H})^k b\right\| \tag{7.20}$$

for all $I \in \mathcal{J}$, all $f \in C^\infty_a(S^1)$ with $\text{supp}f \subset I$ and all $b \in \mathcal{A}(I)'\Omega \cap \mathcal{H}^\infty$. We are going to prove that the energy bound condition for the FJ smeared vertex operators is actually a sufficient condition.

We say that a family of quasi-primary vectors $\mathfrak{F} \subset \mathcal{H}$ generates $\mathcal{A}$ if the corresponding FJ smeared vertex operators generate the graded-local von Neumann algebras, that is, if

$$\mathcal{A}(I) = W^*\Big(\{Y_{I_1}(a,f) \mid a \in \mathfrak{F}\,,\ f \in C^\infty_a(S^1)\ \text{supp}f \subset I_1\,,\ I_1 \in \mathcal{J}\,,\ I_1 \subseteq I\}\Big) \quad \forall I \in \mathcal{J}\,.$$

Then, we present the third main result of this chapter:

Theorem 7.1.10. *Let $\mathcal{A}$ be an irreducible graded-local conformal net. Suppose that $\mathcal{A}$ is generated by a family of quasi-primary vectors $\mathfrak{F}$. Suppose also that $\mathfrak{F}$ is θ-invariant and the corresponding FJ vertex operators are energy-bounded. Furthermore, assume that* $\text{Ker}(L_0 - n1_\mathcal{H})$ *is finite-dimensional for all $n \in \frac{1}{2}\mathbb{Z}_{\geq 0}$. Then, the complex vector space $V := \mathcal{H}^{\text{fin}}$ has a structure of simple strongly graded-local unitary VOSA such that $\mathcal{A} = \mathcal{A}_V$.*

Proof. The following proof retraces the one of [CKLW18, Theorem 9.3] with just few adjustments where necessary.

As for the smeared vertex operators in Section 3.1, we can prove that the energy bound condition implies that $\mathcal{H}^\infty$ is a common invariant core for the operators $Y_I(a,f)$ with $I \in \mathcal{J}$, $a \in \mathfrak{F}$ and $f \in C^\infty_a(S^1)$ with $\text{supp}f \subset I$. Let $\{I_1, I_2\} \subset \mathcal{J}$ be a cover of S^1 and let $\{\varphi_1, \varphi_2\} \subset C^\infty(S^1, \mathbb{R})$ be a partition of unity on S^1 subordinate to $\{I_1, I_2\}$, that is, $\text{supp}\varphi_j \subset I_j$ for all $j \in \{1,2\}$ and $\sum_{j=1}^2 \varphi_j(z) = 1$ for all $z \in S^1$. Then, for every $a \in \mathfrak{F}$ and every $f \in C^\infty_a(S^1)$, we define the operators $\widetilde{Y}(a,f)$ on $\mathcal{H}$ with domain $\mathcal{H}^\infty$ by

$$\widetilde{Y}(a,f)b = \sum_{j=1}^2 Y_{I_j}(a,\varphi_j f)b \qquad b \in \mathcal{H}^\infty\,. \tag{7.21}$$

Let $\{\widetilde{I}_1, \widetilde{I}_2\} \subset \mathcal{J}$ be a second cover of S^1 and let $\{\widetilde{\varphi_1}, \widetilde{\varphi_2}\} \subset C^\infty(S^1, \mathbb{R})$ be a partition of unity on S^1 subordinate to $\{\widetilde{I}_1, \widetilde{I}_2\}$. Then, we have that

$$\begin{aligned}\widetilde{Y}(a,f)b &= \sum_{j=1}^{2} Y_{I_j}(a, \varphi_j f)b = \sum_{j,k=1}^{2} Y_{I_j}(a, \widetilde{\varphi_k}\varphi_j f)b \\ &= \sum_{k,j=1}^{2} Y_{\widetilde{I}_k}(a, \varphi_j \widetilde{\varphi_k} f)b = \sum_{k=1}^{2} Y_{\widetilde{I}_k}(a, \widetilde{\varphi_k} f)b \qquad \forall b \in \mathcal{H}^\infty\end{aligned}$$

where we have used (ii) of Proposition 7.1.7 for the third equality. This means that the definition (7.21) of $\widetilde{Y}(a,f)$ is independent of the choice of the cover of S^1 and of the partition of unity subordinate to it. As a consequence, we have that $\widetilde{Y}(a,f) = Y_I(a,f)$ for all $a \in \mathfrak{F}$, all $I \in \mathcal{J}$ and all $f \in C_a^\infty(S^1)$ with supp$f \subset I$. This implies the state-field correspondence $\widetilde{Y}(a,f)\Omega = a(f)$ for all $a \in \mathfrak{F}$ and all $f \in C_a^\infty(S^1)$. Moreover, we get the Möbius covariance property and the adjoint relation for the operators $\widetilde{Y}(a,f)$ with $f \in C_a^\infty(S^1)$ as stated in (iii) and (iv) of Proposition 7.1.7 respectively.

By hypothesis, for every $a \in \mathfrak{F}$, the FJ vertex operator $\{Y_I(a,z) \mid I \in \mathcal{J}\}$ satisfies the energy bounds with a positive real number M and positive integers s and k. By the convolution theorem for the Fourier series, for every $\varphi \in C^\infty(S^1)$ and every $f \in C_a^\infty(S^1)$, it is not difficult to prove that

$$\left|\widehat{(\varphi f)}_n\right| \leq \sum_{j \in \mathbb{Z} - d_a} \left|\widehat{f}_j\right| \left|\widehat{\varphi}_{n-j}\right| \qquad \forall n \in \mathbb{Z} - d_a \,.$$

Hence,

$$\begin{aligned}\|\varphi f\|_s &= \sum_{n \in \mathbb{Z} - d_a} (|n|+1)^s \left|\widehat{(\varphi f)}_n\right| \\ &\leq \sum_{n \in \mathbb{Z} - d_a} \sum_{j \in \mathbb{Z} - d_a} (|n|+1)^s \left|\widehat{f}_j\right| \left|\widehat{\varphi}_{n-j}\right| \\ &= \sum_{m \in \mathbb{Z}} \sum_{j \in \mathbb{Z} - d_a} (|m+j|+1)^s \left|\widehat{f}_j\right| \left|\widehat{\varphi}_m\right| \\ &\leq \|\varphi\|_s \|f\|_s \,,\end{aligned}$$

implying that

$$\left\|\widetilde{Y}(a,f)b\right\| = \left\|\sum_{j=1}^{2} Y_{I_j}(a, \varphi_j f)b\right\| \leq M \left(\sum_{j=1}^{2} \left\|\varphi_j\right\|_s\right) \|f\|_s \left\|(L_0 + 1_{\mathcal{H}})^k b\right\| \tag{7.22}$$

for all $f \in C_a^\infty(S^1)$ and all $b \in \mathcal{H}^\infty$. This means that the operators $\widetilde{Y}(a,f)$ with $f \in C_a^\infty(S^1)$ satisfy the energy bounds with the positive integers s and k and the positive real number $\widetilde{M} := M\left(\sum_{j=1}^{2} \left\|\varphi_j\right\|_s\right)$.

For every $n \in \mathbb{Z}$ and every $m \in \mathbb{Z} - \frac{1}{2}$, define $e_n \in C^\infty(S^1)$ and $e_m \in C_\chi^\infty(S^1)$ by $e_n(z) := z^n$ and $e_m(z) := \chi(z) z^{m-\frac{1}{2}}$ with $z \in S^1$ respectively. For any $a \in \mathfrak{F}$ and any $n \in \mathbb{Z} - d_a$, set $a_n := \widetilde{Y}(a, e_n)$. Thus, we have that

$$\|a_n b\| \leq \widetilde{M}(|n|+1)^s \left\|(L_0 + 1_{\mathcal{H}})^k b\right\| \qquad \forall n \in \mathbb{Z} - d_a \ \forall b \in \mathcal{H}^\infty \,.$$

By definition, we get that $a_{-d_a}\Omega = a(e_{-d_a}) = a$ for all $a \in \mathfrak{F}$. The Möbius covariance property implies that $e^{itL_0} a_n e^{-itL_0} = e^{-int} a_n$ for all $a \in \mathfrak{F}$, all $t \in \mathbb{R}$ and all $n \in \mathbb{Z} - d_a$. This implies that $[L_0, a_n]b = -n a_n b$ for all $a \in \mathfrak{F}$, all $n \in \mathbb{Z} - d_a$ and all $b \in \mathcal{H}^\infty$ and thus a_n preserves $\mathcal{H}^{\mathrm{fin}}$ for all $a \in \mathfrak{F}$ and all $n \in \mathbb{Z} - d_a$. By the covariance property, we also have the remaining commutation relations $[L_{-1}, a_n]b = (-n - d_a + 1)a_{n-1}b$ and $[L_1, a_n]b = (-n + d_a - 1)a_{n+1}b$ for all $a \in \mathfrak{F}$, all $n \in \mathbb{Z} - d_a$ and all $b \in \mathcal{H}^\infty$. Define the linear span

$$V := \langle a^1_{n_1} \cdots a^l_{n_l} \Omega \mid l \geq 0 \,, \ a^j \in \mathfrak{F} \,, \ n_j \in \mathbb{Z} - d_{a^j} \ \forall j \in \{1, \cdots, l\} \rangle \tag{7.23}$$

We show that $V = \mathcal{H}^{\text{fin}}$. Let $\mathcal{H}_V$ be the closure of V in $\mathcal{H}$ and e_V be the corresponding orthogonal projection onto $\mathcal{H}_V$. For any $f \in C_a^\infty(S^1)$, $\sum_{n\in\mathbb{Z}-d_a} \widehat{f}_n e_n$ converges to f in $C_a^\infty(S^1)$ and hence

$$\widetilde{Y}(a,f)b = \sum_{j\in\mathbb{Z}-d_a} \widehat{f}_n a_n b \qquad \forall a \in \mathfrak{F}\ \forall f \in C_a^\infty(S^1)\ \forall b \in \mathcal{H}^\infty .$$

It follows that $\widetilde{Y}(a,f)b$ and $\widetilde{Y}(a,f)^*b$ are in $\mathcal{H}_V$ for all $a \in \mathfrak{F}$, all $f \in C_a^\infty(S^1)$ and all $b \in \mathcal{H}^\infty$. Considering that $\mathrm{Ker}(L_0 - n1_{\mathcal{H}})$ is finite-dimensional for all $n \in \frac{1}{2}\mathbb{Z}_{\geq 0}$, we get that $e_V\mathcal{H}^{\text{fin}} = V$. Consequently, we have that $[e_V, \widetilde{Y}(a,f)]b = 0$ for all $a \in \mathfrak{F}$, all $f \in C_a^\infty(S^1)$ and all $b \in V$. But $\mathcal{H}^{\text{fin}}$ is a core for every FJ smeared vertex operator, which means that $e_V Y_I(a,f) \subseteq Y_I(a,f)e_V$ for all $a \in \mathfrak{F}$, all $I \in \mathcal{J}$ and all $f \in C_a^\infty(S^1)$ with supp$f \subset I$. Then, using that $\mathfrak{F}$ generates $\mathcal{A}$, it must be $e_V = 1_{\mathcal{H}}$ by the irreducibility of $\mathcal{A}$. This obviously means that $V = \mathcal{H}^{\text{fin}}$.

What we have just proved says us that the formal series

$$\Phi_a(z) := \sum_{n\in\mathbb{Z}-d_a} a_n z^{-n-d_a} \qquad \forall a \in \mathfrak{F} \tag{7.24}$$

are translation covariant (with respect to the even endomorphism $T := L_{-1}$) parity-preserving fields on V, which are also mutually local (in the vertex superalgebra sense) thanks to the graded-locality of $\mathcal{A}$ and Proposition A.1. Then, we have a unique vertex superalgebra structure on V thanks to the existence theorem for vertex superalgebra [Kac01, Theorem 4.5] with vertex operators $Y(a,z) := \Phi_a(z)$ for all $a \in \mathfrak{F}$. Moreover, we have a unitary representation of the Virasoro algebra on V by operators L_n with $n \in \mathbb{Z}$ by differentiating the representation U of $\mathrm{Diff}^+(S^1)^{(\infty)}$ associated to $\mathcal{A}$, see Theorem 1.2.2. Consequently, $L(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$ is a local field on V, which is also mutually local with respect to every $Y(a,z)$ with $a \in V$ thanks to the graded-locality of $\mathcal{A}$. Furthermore, $L(z)\Omega = e^{zL_{-1}}L_{-2}\Omega$ and thus $L(z) = Y(\nu,z)$ with $\nu := L_{-2}\Omega$ by the uniqueness theorem for vertex superalgebras [Kac01, Theorem 4.4.]. Clearly, ν is a conformal vector which makes V a VOSA.

The scalar product on $\mathcal{H}$ restricts to a normalized scalar product on V having the unitary Möbius symmetry as defined in (i) of Definition 2.5.1. Furthermore, we have that

$$Y(a,z)^+ = (-1)^{2d_a^2+d_a} Y(\theta(a),z) \qquad \forall a \in \mathfrak{F}$$

where $Y(a,z)^+$ is the adjoint vertex operator as defined in (2.71). This means that every $Y(a,z)^+$ with $a \in \mathfrak{F}$ is local and mutually local with respect to every vertex operator of V. Set

$$\mathfrak{F}_+ := \left\{ \frac{a + (-1)^{2d_a^2+d_a}\theta(a)}{2} \mid a \in \mathfrak{F} \right\}, \quad \mathfrak{F}_- := \left\{ -i\frac{a - (-1)^{2d_a^2+d_a}\theta(a)}{2} \mid a \in \mathfrak{F} \right\}.$$

Note that V is generated by the family of Hermitian quasi-primary fields $\{Y(a,z) \mid a \in \mathfrak{F}_+ \cup \mathfrak{F}_-\}$ and thus V is unitary by Proposition 2.5.6. Furthermore, V is simple by (iv) of Proposition 2.3.3 because $V_0 = \mathbb{C}\Omega$. V is also energy-bounded thanks to Proposition 3.1.2 because it is generated by the family $\mathfrak{F}$ of energy-bounded elements. $\mathcal{A}_{\mathfrak{F}}$ as defined in (5.5) coincides with $\mathcal{A}$ by hypothesis and thus we can conclude that V is strongly graded-local and that $\mathcal{A}_V = \mathcal{A}$ by Theorem 5.2.1. □

We conjecture that:

Conjecture 7.1.11. *For every irreducible graded-local conformal net $\mathcal{A}$, there exists a simple strongly graded-local unitary VOSA V such that $\mathcal{A} = \mathcal{A}_V$.*

7.2 Functoriality

It seems natural to express the correspondence given by Theorem 3.2.17 and Theorem 7.1.10 in a categorical language. This is what we do in this chapter. We refer the reader to [McL98] as general reference for category theory.

On the one hand, we define the category **V**:

- objects are simple strongly graded-local unitary VOSAs, denoted by capital letters V, W, Q and so on;

- an arrow between objects W and V is a $\mathbb{C}$-vector superspace linear map $\phi \in \mathrm{Hom}(W, V)$ such that $\phi(a_{(n)}b) = \phi(a)_{(n)}\phi(b)$ for all $a, b \in W$ and all $n \in \mathbb{Z}$, $\phi(\Omega^W) = \Omega^V$, $\phi L_1^W = L_1^V \phi$ and $\phi\theta_W = \theta_V \phi$.

Lemma 7.2.1. *If ϕ is an arrow between objects W and V of $\mathbf{V}$, then $\phi L_{-1}^W = L_{-1}^V \phi$ and $\phi L_0^W = L_0^V \phi$. Moreover, $(\phi(a)|\phi(b))_V = (a|b)_W$ for all $a, b \in W$. In particular, $\phi(W)$ is a unitary subalgebra of V unitarily isomorphic to W.*

Proof. Recall that L_{-1}^W and L_{-1}^V are the infinitesimal translation operators of the corresponding VOSAs and thus

$$\phi(L_{-1}^W a) = \phi(a_{(-2)}\Omega^W) = \phi(a)_{(-2)}\phi(\Omega^W) = \phi(a)_{(-2)}\Omega^V = L_{-1}^V \phi(a) \qquad \forall a \in W .$$

Then, $\phi L_0^W = L_0^V \phi$ follows from (2.24). Now, W and V have a unique normalized non-degenerate invariant bilinear form by (iii) of Proposition 2.3.3, which we call $(\cdot\,,\cdot)_W$ and $(\cdot\,,\cdot)_V$ respectively. Define a bilinear form on W by $\widetilde{(\cdot\,,\cdot)}_W := (\phi(\cdot), \phi(\cdot))_V$. Using (2.62) and that $\phi L_1^W = L_1^V \phi$, it is easy to see that $\widetilde{(\cdot\,,\cdot)}_W$ is invariant. Moreover, $\widetilde{(\cdot\,,\cdot)}_W$ is non-zero and normalized because

$$\widetilde{(\Omega^W, \Omega^W)}_W = (\phi(\Omega^W), \phi(\Omega^W))_V = (\Omega^V, \Omega^V)_V = 1$$

and thus $\widetilde{(\cdot\,,\cdot)}_W = (\cdot\,,\cdot)_W$ by (iii) of Proposition 2.3.3. Then, $(\phi(a)|\phi(b))_V = (a|b)_W$ for all $a, b \in W$ if and only if $(\phi(\theta_W(a))|\phi(b))_V = (\theta_W(a)|b)_W$ for all $a, b \in W$, where

$$(\theta_W(a)|b)_W = (a, b)_W = \widetilde{(a, b)}_W = (\phi(a), \phi(b))_V = (\theta_V(\phi(a))|\phi(b))_V \qquad \forall a, b \in W . \tag{7.25}$$

Hence, if $\phi\theta_W = \theta_V\phi$, then $(\phi(a)|\phi(b))_V = (a|b)_W$ for all $a, b \in W$. The properties above imply that $\phi(W)$ is a unitary subalgebra of V, unitarily isomorphic to W. ☐

On the other hand, we consider the category $\mathbf{A}$:

- objects are irreducible graded-local conformal nets, denoted by symbols $\mathcal{A}, \mathcal{B}$ and so on;
- an arrow between objects $\mathcal{B}$ and $\mathcal{A}$ is a Hilbert space isometry $g \in \mathrm{Isom}(\mathcal{H}_\mathcal{B}, \mathcal{H}_\mathcal{A})$ such that $g(\Omega_\mathcal{B}) = \Omega_\mathcal{A}$, $g\mathcal{B}(I)g^* \subseteq \mathcal{A}(I)$ for all $I \in \mathcal{J}$ and $gU_\mathcal{B}(\gamma) = U_\mathcal{A}(\gamma)g$ for all $\gamma \in \mathrm{M\ddot{o}b}(S^1)^{(\infty)}$;
- set $\mathbf{A}_{\mathrm{eb}}$ as the full subcategory of $\mathbf{A}$, whose objects are all irreducible graded-local conformal nets satisfying the hypothesis of Theorem 7.1.10.

Note that, by the uniqueness of the diffeomorphism representation, see [CKL08, Corollary 11], $U_\mathcal{B}(\gamma) = g^* U_\mathcal{A}(\gamma) g$ for all $\gamma \in \mathrm{Diff}^+(S^1)$. Moreover, $g\mathcal{B}g^*$ is a covariant subnet of $\mathcal{A}$ isomorphic to $\mathcal{B}$.

Finally, we propose the following categorical result.

Theorem 7.2.2. *There exists a fully faithful functor $F : \mathbf{V} \to \mathbf{A}$, which is also injective on objects. Furthermore, F restricts to an isomorphism of categories between $\mathbf{V}$ and $\mathbf{A}_{\mathrm{eb}}$.*

Proof. We use the notation as in (5.1).

Let $\phi \in \mathrm{Hom}(W, V)$ be an arrow of $\mathbf{V}$. From Lemma 7.2.1, we have that

$$\|a\|_W = \left\|\phi(a)\right\|_V \qquad \forall a \in W , \tag{7.26}$$

which implies that ϕ is injective. Then, using a standard functional analytic argument, we can uniquely extend ϕ to an isometry g_ϕ from $\mathcal{H}_{(W,(\cdot|\cdot)_W)}$ to $\mathcal{H}_{(V,(\cdot|\cdot)_V)}$. By assumption, we have that $g_\phi(\Omega^W) = \Omega^V$. If $e_{g_\phi} := g_\phi g_\phi^*$ is the projection onto $g_\phi(\mathcal{H}_{(W,(\cdot|\cdot)_W)})$, then by the defining properties of the arrow ϕ and Lemma 7.2.1, we have that

$$g_\phi Y_W(a, f) g_\phi^* = Y_V(g_\phi(a), f) e_{g_\phi} \qquad \forall a \in W \ \ \forall f \in C_a^\infty(S^1)$$

and

$$g_\phi L_n^W g_\phi^* = L_n^V e_{g_\phi} \qquad \forall n \in \{-1, 0, 1\} .$$

$\phi(W)$ is a unitary subalgebra of V by Lemma 7.2.1 and thus $Y_V(g_\phi(a), f)$ for all $a \in W$ and L_n for all $n \in \{-1, 0, 1\}$ commute with e_{g_ϕ} by Proposition 2.6.4. As a consequence, $g_\phi \mathcal{A}_{(W,(\cdot|\cdot)_W)}(I) g_\phi^* \subseteq \mathcal{A}_{(V,(\cdot|\cdot)_V)}(I)$ for all $I \in \mathcal{J}$ and $g_\phi U_{\mathcal{A}_{(W,(\cdot|\cdot)_W)}}(\gamma) = U_{\mathcal{A}_{(V,(\cdot|\cdot)_V)}}(\gamma) g_\phi$ for all $\gamma \in \mathrm{M\ddot{o}b}(S^1)^{(\infty)}$.

Now, it should be clear how to define the functor $F : \mathbf{V} \to \mathbf{A}$: this is the map which associates to every simple strongly graded-local unitary VOSA $(V, (\cdot|\cdot))$, the corresponding irreducible graded-local conformal net $\mathcal{A}_{(V,(\cdot|\cdot))}$ defined by Theorem 3.2.17 and to every arrow ϕ, the unique isometry g_ϕ constructed above. By Theorem 3.2.19, if $\phi = 1_{(V,(\cdot|\cdot)_V)}$, then $g_\phi = 1_{\mathcal{H}_{(V,(\cdot|\cdot)_V)}}$ and thus $F(1_{(V,(\cdot|\cdot)_V)}) = 1_{\mathcal{H}_{(V,(\cdot|\cdot)_V)}}$ by construction. Moreover,

$$g_{\phi\circ\psi}(a) = (\phi\circ\psi)(a) = \phi(\psi(a)) = (g_\phi \circ g_\psi)(a) \qquad \forall\psi \in \mathrm{Hom}(Q, W)\ \ \forall\phi \in \mathrm{Hom}(W, V)\ \ \forall a \in Q$$

and thus it must be $F(\phi\circ\psi) = F(\phi)\circ F(\psi)$ for all arrows $\psi \in \mathrm{Hom}(Q, W)$ and $\phi \in \mathrm{Hom}(W, V)$ thanks to the uniqueness of the extension of $\phi\circ\psi$ from Q to $\mathcal{H}_{(Q,(\cdot|\cdot)_Q)}$. Therefore, F is a well-defined functor.

Let $\phi, \psi \in \mathrm{Hom}(W, V)$ be arrows of $\mathbf{V}$ such that $F(\phi) = F(\psi)$. In particular, this means that

$$\phi(a) = g_\phi(a) = g_\psi(a) = \psi(a) \qquad \forall a \in W\,,$$

which is the faithfulness of F.

Conversely, let $g \in \mathrm{Isom}(\mathcal{H}_{(W,(\cdot|\cdot)_W)}, \mathcal{H}_{(V,(\cdot|\cdot)_V)})$ be an arrow between $\mathcal{A}_{(W,(\cdot|\cdot)_W)}$ and $\mathcal{A}_{(V,(\cdot|\cdot)_V)}$ in the category $\mathbf{A}$. Let $e_g := gg^*$ be the projection onto $g(\mathcal{H}_{(W,(\cdot|\cdot)_W)})$. By the defining properties of g, $gL_n^W g^* = L_n^V e_g$ for all $n \in \{-1, 0, 1\}$. Moreover, for every $n \in \{-1, 0, 1\}$, L_n^V and e_g commute because $g\mathcal{A}_{(W,(\cdot|\cdot)_W)}g^*$ is a covariant subnet of $\mathcal{A}_{(V,(\cdot|\cdot)_V)}$. $g(W) = g(\mathcal{H}_{(W,(\cdot|\cdot)_W)}) \cap V$ because g is an isometry and thus $g(W)$ is a unitary subalgebra of V thanks to Theorem 5.1.1. Now, $gY_W(a, z)g^*$ is a parity-preserving field on $g(W)$, whenever $a \in W$. Therefore, by Proposition 3.2.15, Proposition A.1 and the fact that $g\mathcal{A}_{(W,(\cdot|\cdot)_W)}g^*$ is graded-local, $gY_W(a, z)g^*$ is mutually local in the vertex superalgebra sense with all $Y_W(b, z)$, where $b \in W$. By [Kac01, Remark 1.3]:

$$gY_W(a, z)g^*\Omega^V = gY_W(a, z)\Omega^W = ge^{zL_{-1}^W}a = e^{zL_{-1}^V}g(a) \qquad \forall a \in W\,.$$

By the uniqueness theorem for vertex superalgebra [Kac01, Theorem 4.4], we deduce that $gY_W(a, z)g^*$ must be equal to $Y_V(g(a), z)$ for all $a \in W$. Then, g restricts to a $\mathbb{C}$-vector superspace linear map from W to V, which preserves the (n)-product and such that $g(\Omega^W) = \Omega^V$ and $gL_n^W = L_n^V g$ for all $n \in \{-1, 0, 1\}$. As an isometry, g preserves the scalar products and $(\cdot|\cdot)_V$ is non-degenerate on $g(W)$ because this is a unitary subalgebra of V. Then, with an argument as in (7.25) and using that $\theta_V g(W) \subseteq g(W)$, we can conclude that $g\theta_W = \theta_V g$, so that g restricts to an arrow between W and V. This prove that F is a fully faithful functor.

We can conclude that F is injective on objects thanks to Theorem 7.1.1 or Theorem 7.1.9. Moreover, Theorem 7.1.10 says that F is also surjective on objects when we restrict the codomain of F to $\mathbf{A}_{\mathrm{eb}}$. Therefore, we have proved that $F : \mathbf{V} \to \mathbf{A}_{\mathrm{eb}}$ is a fully faithful functor both injective and surjective on objects, i.e., an isomorphism between the categories $\mathbf{V}$ and $\mathbf{A}_{\mathrm{eb}}$. □

Appendix A

Vertex superalgebra locality and Wightman locality

We investigate the relationship between the locality axiom in the vertex superalgebra framework and in the Wightman one. Indeed, as it is well explained in [Kac01, Chapter 1], the vertex superalgebra axioms are motivated by the Wightman axioms for a QFT. It is therefore likely to have an equivalence between the locality axioms in the two frameworks at least under certain assumptions. Those assumptions are represented by a suitable energy bound condition for fields in the spirit of Section 3.1.

Comparing with the local case... The following framework generalises the one of [CKLW18, Appendix A]. Some complications are given by odd elements, which must be treated in a different way from the even ones due to the $\mathbb{Z}-\frac{1}{2}$ gradation. This issue is somewhat analogous to the one about the definition of smeared vertex operators in Section 3.1.

Let L_0 be a self-adjoint operator on a Hilbert space $\mathcal{H}$ with spectrum contained in $\frac{1}{2}\mathbb{Z}_{\geq 0}$. Define the algebraic direct sum

$$V := \bigoplus_{n\in\frac{1}{2}\mathbb{Z}_{\geq 0}} \mathrm{Ker}(L_0 - n1_{\mathcal{H}}) ,$$

which is a dense subspace of $\mathcal{H}$. Then, denote by $\mathcal{H}^\infty$ the dense subspace of $\mathcal{H}$ given by the smooth vectors of L_0 (see [GW85], [Tol99] or Section 3.1), namely

$$\mathcal{H}^\infty := \bigcap_{k\in\mathbb{Z}_{\geq 0}} \mathcal{H}^k$$

where $\mathcal{H}^k$ are the domain of the operators $(L_0+1_{\mathcal{H}})^k$ respectively. Let $a:=\{a_n \mid n\in\mathbb{Z}\}$ and $b:=\{b_n \mid n\in\mathbb{Z}-\frac{1}{2}\}$ be two families of operators on $\mathcal{H}$ with V as a common domain and such that

$$\begin{aligned} e^{itL_0}a_ne^{-itL_0} &= e^{-int}a_n \qquad \forall t\in\mathbb{R}\ \forall n\in\mathbb{Z} \\ e^{itL_0}b_ne^{-itL_0} &= e^{-int}b_n \qquad \forall t\in\mathbb{R}\ \forall n\in\mathbb{Z}-\frac{1}{2}\,. \end{aligned} \tag{A.1}$$

It follows that

$$\begin{aligned} a_n\mathrm{Ker}(L_0-j1_{\mathcal{H}}) &\subseteq \mathrm{Ker}(L_0-(j-n)1_{\mathcal{H}}) \qquad \forall n\in\mathbb{Z}\ \forall j\in\frac{1}{2}\mathbb{Z}_{\geq 0} \\ b_n\mathrm{Ker}(L_0-j1_{\mathcal{H}}) &\subseteq \mathrm{Ker}(L_0-(j-n)1_{\mathcal{H}}) \qquad \forall n\in\mathbb{Z}-\frac{1}{2}\ \forall j\in\frac{1}{2}\mathbb{Z}_{\geq 0}\,. \end{aligned}$$

Consequently, all the operators a_n and b_n restrict to endomorphisms of V with the property that for all $c\in V$ there exists $N>0$ such that $a_n=b_n=0$ for all $n>N$. In other words, the formal series

$$\Phi_a(z) := \sum_{n\in\mathbb{Z}} a_nz^{-n}\,, \qquad \Phi_b(z) := \sum_{n\in\mathbb{Z}-\frac{1}{2}} b_nz^{-n-\frac{1}{2}}$$

are fields on V as defined in [Kac01, Section 3.1], see also Chapter 2. Moreover, assume that $\Phi_a(z)$ and $\Phi_b(z)$ satisfy the following **energy bounds**: there exist positive real M and $s, k \in \mathbb{Z}_{\geq 0}$ such that

$$\|a_n c\| \leq M(1+|n|)^s \left\|(L_0 + 1_{\mathcal{H}})^k c\right\| \qquad \forall n \in \mathbb{Z} \ \forall c \in V$$
$$\|b_n c\| \leq M(1+|n|)^s \left\|(L_0 + 1_{\mathcal{H}})^k c\right\| \qquad \forall n \in \mathbb{Z} - \frac{1}{2} \ \forall c \in V\,.$$

Thus, we define the following operators:

$$\Phi_a(f)c := \sum_{n\in\mathbb{Z}} \widehat{f}_n a_n c\,, \qquad \Phi_b(f)c := \sum_{n\in\mathbb{Z}-\frac{1}{2}} \widehat{f}_{\frac{2n-1}{2}} b_n c \qquad \forall f \in C^\infty(S^1) \ \forall c \in V\,.$$

The closures of the above operators, which we still denote by the same symbols, have $\mathcal{H}^\infty$ as common invariant domain, see Section 3.1. We call $\Phi_a(f)$ and $\Phi_b(f)$ **smeared fields**. We say that $a, \Phi_a(z)$ and $\Phi_a(f)$ are **even**, whereas $b, \Phi_b(z)$ and $\Phi_b(f)$ are **odd**. We say that a has **parity** $p(a) = \overline{0}$, whereas that b has parity $p(b) = \overline{1}$.

Now, let a^1 and a^2 be two families of operators with given parities as above. Let $[\cdot,\cdot]$ be the **graded commutator**, that is,

$$[\Phi_{a^1}(\cdot), \Phi_{a^2}(\cdot)] := \Phi_{a^1}(\cdot)\Phi_{a^2}(\cdot) - (-1)^{p(a^1)p(a^2)}\Phi_{a^2}(\cdot)\Phi_{a^1}(\cdot)\,, \tag{A.2}$$

where, with an abuse of notation, we are using $(-1)^{p(a^1)p(a^2)}$ as in (A.2) to denote $(-1)^{p_{a^1}p_{a^2}}$, where $p_{a^1}, p_{a^2} \in \{0,1\}$ are representatives of the remainder class of $p(a^1)$ and $p(a^2)$ in $\mathbb{Z}_2$ respectively. We say that the fields $\Phi_{a^1}(z)$ and $\Phi_{a^2}(z)$ are, accordingly with Section 2.1, **mutually local in the vertex superalgebra sense** if there exists an integer $N \geq 0$ such that

$$(z-w)^n[\Phi_{a^1}(z), \Phi_{a^2}(w)]c = 0 \qquad \forall n \geq N \ \forall c \in V\,, \tag{A.3}$$

whereas, we say that the fields $\Phi_{a^1}(z)$ and $\Phi_{a^2}(z)$ are **mutually local in the Wightman sense** (cf. [SW64, Section 3.1]) if

$$[\Phi_{a^1}(f_1), \Phi_{a^2}(f_2)]c = 0 \qquad \forall c \in V \tag{A.4}$$

whenever $f_1, f_2 \in C^\infty(S^1)$ are such that $\mathrm{supp} f_1 \subset I$ and $\mathrm{supp} f_2 \subset I'$ for some $I \in \mathcal{J}$.

We can now state the correspondence between the two locality definitions given above:

Proposition A.1. *Let a^1 and a^2 be two families of operators with given parities on $\mathcal{H}$ with V as common domain and satisfying the energy bounds. Then, the fields $\Phi_{a^1}(z)$ and $\Phi_{a^2}(z)$ are mutually local in the vertex superalgebras sense if and only if they are mutually local in the Wightman sense.*

Proof. We are going to present here an adaptation of the proof of [CKLW18, Proposition A.1].

For every $c, d \in V$, define the following formal series in the two formal variables z and w:

$$\varphi_{c,d}(z,w) := (d|[\Phi_{a^1}(z), \Phi_{a^2}(w)]c) = \sum_{\substack{n\in\mathbb{Z}+p_1\\ m\in\mathbb{Z}+p_2}} (d|[a^1_n, a^2_m]c) z^{-n-p_1} w^{-m-p_2}$$

where p_k are either 0 if $p(a^k) = \overline{0}$ or $\frac{1}{2}$ if $p(a^k) = \overline{1}$ for $k \in \{1,2\}$. $\varphi_{c,d}(z,w)$ can be considered as a formal distribution on $S^1 \times S^1$ as in [Kac01, Section 2.1], that is, $\varphi_{c,d}$ is a linear functional on the complex vector space of the trigonometric polynomials in two variables. Thanks to the energy bounds, $\varphi_{c,d}$ extends to a unique ordinary distribution on $S^1 \times S^1$ by continuity, that is, a continuous linear functional on the Fréchet space $C^\infty(S^1 \times S^1)$, which we call $\varphi_{c,d}(f)$ for $f \in C^\infty(S^1 \times S^1)$. (Recall that $C^\infty(S^1 \times S^1)$ is isomorphic to the completion of the algebraic tensor product $C^\infty(S^1) \otimes C^\infty(S^1)$.) The energy bounds assure us that there exists an $N > 0$ such that for every $c, d \in V$, there exists $M_{c,d} > 0$ such that

$$\left|\varphi_{c,d}(f)\right| \leq M_{c,d}\|f\|_N\,, \qquad \max_{\substack{|s|\leq N\\ x,y\in\mathbb{R}}} \left|\partial^s f(e^{ix}, e^{iy})\right| \qquad \forall f \in C^\infty(S^1 \times S^1) \tag{A.5}$$

where, as usual, $s = (s^1, s^2)$, $s^j \in \mathbb{Z}_{\geq 0}$ is a multi-index with $|s| = s^1 + s^2$ and $\partial^s := (\partial_x)^{s^1}(\partial_y)^{s^2}$.

Suppose that the fields $\Phi_{a^1}(z)$ and $\Phi_{a^2}(z)$ are mutually local in the vertex superalgebra sense. By [Kac01, Theorem 2.3(i)], the ordinary distribution $\varphi_{c,d}$ has support in the diagonal

$$D := \{(z,w) \in S^1 \times S^1 \mid z - w = 0\}.$$

It follows that $(d|[\Phi_{a^1}(f^1), \Phi_{a^2}(f^2)]c) = 0$ for all $c, d \in V$, whenever $f^1, f^2 \in C^\infty(S^1)$ with $\mathrm{supp} f^1 \subset I \in \mathcal{J}$ and $\mathrm{supp} f^2 \subset I'$. Due to the energy bounds, we can extend the identity above to all $c \in \mathcal{H}^\infty$ (cf. Lemma 3.1.8) and thus the fields $\Phi_{a^1}(z)$ and $\Phi_{a^2}(z)$ are mutually local in the Wightman sense.

Conversely, suppose that $\Phi_{a^1}(z)$ and $\Phi_{a^2}(z)$ are mutually local in the Wightman sense, then the formal distribution $\varphi_{c,d}$ has support in the diagonal D as above. Moreover, (A.5) says us that $\varphi_{c,d}$ is a distribution of *order* N in the sense of [Rud91, p. 156]. We would like to apply [Rud91, Theorem 6.25]. To this aim, consider a finite open cover $\{U_j\}$ of the diagonal D of the torus $S^1 \times S^1$ in such a way that every U_j is diffeomorphic to the open square $(-\pi, \pi) \times (-\pi, \pi)$ of $\mathbb{R}^2$ and such that $D \cap U_j$ is diffeomorphic to a diagonal of such a square. Complete $\{U_j\}$ to a finite open cover of $S^1 \times S^1$, which we denote by the same symbols. Let h_j be a partition of unity of $S^1 \times S^1$ with respect to the finite open cover $\{U_j\}$ and set $\varphi^j_{c,d} := h_j \varphi_{c,d}$, so that $\varphi_{c,d} = \sum_j \varphi^j_{c,d}$. Every $\varphi^j_{c,d}$ can be considered as a distribution in the angular variables $(x, y) \in (-\pi, \pi) \times (-\pi, \pi)$ and can be easily extended to a distribution on $\mathbb{R}^2$. Applying the change of variable $l := x - y$ and $m := x + y$, we obtain distributions $\phi^j_{c,d}$ on $\mathbb{R}^2$ with support in $\{(l, m) \in \mathbb{R}^2 \mid l = 0\}$. Smearing a distribution $\phi^j_{c,d}$ with a test function g in the variable $m \in \mathbb{R}$, we obtain a new distribution in the variable $l \in \mathbb{R}$ with support in $\{0\}$ only. Then, we are in the case of [Rud91, Theorem 6.25], which says us that $\phi^j_{c,d}(l, g) = \sum_{s=1}^N \widetilde{k}^j_s(g) \partial^s \delta(l)$. One can check that for every s and j, $\widetilde{k}^j_s$ is a distribution in the variable $m \in \mathbb{R}$. Thus, for every j, we can rewrite $\varphi_{c,d}(z, w) = \sum_{s=1}^N k^j_s(w) \partial^s_w \delta(z - w)$ for some distributions k^j_s in the variable w. By [Kac01, Theorem 2.3], we can deduce that $(z - w)^N \varphi_{c,d}(z, w) = 0$ for all $c, d \in V$, that is, the fields $\Phi_{a^1}(z)$ and $\Phi_{a^2}(z)$ are mutually local in the vertex superalgebra sense. $\square$

Bibliography

[AF03] M. J. Ablowitz and A. S. Fokas. *Complex Variables. Introduction and Applications. Second Edition.* (Cambridge University Press, New York, 2003).

[AL17] C. Ai, X. Lin. On the unitary structures of vertex operator superalgebras. *J. Alg.* **487**, 217-243 (2017).

[Ara70] H. Araki. On quasi free states of CAR and Bogoliubov automorphisms. *Publ. RIMS Kyoto Univ.* **6**, 385-442 (1970/71).

[Ara76] H. Araki. Positive cone, Radon-Nikodým theorems, relative Hamiltonian and the Gibbs condition in statistical mechanics. An application of the Tomita-Takesaki theory. In: C*-algebras and their applications to statistical mechanics and quantum field theory. 64-100. (North-Holland, Amsterdam, 1976).

[Ara10] H. Araki. *Mathematical Theory of Quantum Fields.* (Oxford University Press Inc., New York, 2010).

[Bar54] V. Bargmann. On unitary ray representations of continuous groups. *Ann. of Math.* **59**(1), 1-46 (1954).

[BKLR15] M. Bischoff, Y. Kawahigashi, R. Longo and K.-H. Rehren. *Tensor Categories and Endomorphisms of von Neumann Algebras. with Applications to Quantum Field Theory* (Springer International Publishing, 2015).

[Boc96] J. Böckenhauer. Localized endomorphisms of the chiral Ising model. *Commun. Math. Phys.* **117**, 265-304 (1996).

[Bor86] R. E. Borcherds. Vertex algebras, Kac-Moody algebras, and the Monster. *Proc. Natl. Acad. Sci. USA* **83**, 3068-3071 (1986).

[Bor68] H.-J. Borchers. On the converse of the Reeh-Schlieder theorem. *Commun. Math. Phys.* **10**, 269-273 (1968).

[BBS01] H.-J. Borchers, D. Buchholz and B. Schroer. Polarization-free generators and the S-matrix. *Commun. Math. Phys.* **219**(1), 125-140 (2001).

[BY90] H.-J. Borchers and J. Yngvason. Positivity of Wightman functionals and the existence of local nets. *Commun. Math. Phys.* **127**(3), 607-615 (1990).

[BY92] H.-J. Borchers and J. Yngvason. From quantum fields to local von Neumann algebras. *Rev. Math. Phys.* **Special Issue**, 15-47 (1992).

[BZ63] H.-J. Borchers and W. Zimmermann. On the self-adjointness of field operators. *Nuovo Cimento* **XXXI**, 1047-1059 (1963).

[Bos05] H. Bostelmann. Phase space properties and the short distance structure in quantum field theory. *J. Math. Phys.* **46**(5), 052301, pp. 17 (2005).

[BR02] O. Bratteli and D. W. Robinson. *Operator Algebras and Quantum Statistical Mechanics 1. C^*- and W^*- Algebras. Symmetry Groups. Decomposition of States. Second Edition.* (Springer-Verlag Berlin Heidelberg New York, 2002).

[BR02b] O. Bratteli and D. W. Robinson. *Operator Algebras and Quantum Statistical Mechanics 2. Equilibrium States, Models in Quantum Statistical Mechanics. Second Edition.* (Springer-Verlag Berlin Heidelberg New York, 2002).

[BGL93] R. Brunetti, D. Guido and R. Longo. Modular structure and duality in conformal quantum field theory. *Commun. Math. Phys.* **156**, 201-219 (1993).

[BMT88] D. Buchholz, G. Mack and I. T. Todorov. The current algebra on the circle as a germ of local field theories. *Nucl. Phys. B(Proc. Suppl.)* **5B**, 20-56 (1988).

[BS90] D. Buchholz and H. Schulz-Mirbach. Haag duality in conformal quantum field theory. *Rev. Math. Phys.* **2**(1), 105-125 (1990).

[Car98] S. Carpi. Absence of subsystems for the Haag-Kastler net generated by the energy-momentum tensor in two-dimensional conformal field theory. *Lett. Math. Phys.* **45**(3), 259-267 (1998).

[Car99] S. Carpi. Classification of subsystems for the Haag-Kastler nets generated by $c = 1$ chiral current algebras. *Lett. Math. Phys.* **47**(4), 353-364 (1999).

[Car04] S. Carpi. On the representation theory of Virasoro nets. *Commun. Math. Phys.* **244**(2), 261-284 (2004).

[Car05] S. Carpi. Intersecting Jones projections. *Int. J. Math.* **16**(6), 687-691 (2005).

[CCH13] S. Carpi, R. Conti and R. Hillier. Conformal nets and KK-theory. *Ann. Funct. Anal.* **4**(1), 11-17 (2013).

[CCHW13] S. Carpi, R. Conti, R. Hillier and M. Weiner. Representations of conformal nets, universal C*-algebras and K-theory. *Commun. Math. Phys.* **320**(1), 275-300 (2013).

[CGH19] S. Carpi, T. Gaudio and R. Hillier. Classification of unitary vertex subalgebras and conformal subnets for rank-one lattice chiral CFT models. *J. Math. Phys.* **60**(9), 093505, pp. 20 (2019).

[CGH21] S. Carpi, T. Gaudio and R. Hillier. From vertex operator superalgebras to graded-local conformal nets and back. *In preparation.*

[CH17] S. Carpi and R. Hillier. Loop groups and noncommutative geometry. *Rev. Math. Phys.* **29**(9), 1750029, pp. 42 (2017).

[CHKLX15] S. Carpi, R. Hillier, Y. Kawahigashi, R. Longo and F. Xu. $N = 2$ superconformal nets. *Commun. Math Phys.* **336**(3), 1285-1328 (2015).

[CHL15] S. Carpi, R. Hillier and R. Longo. Superconformal nets and noncommutative geometry. *J. Noncommut. Geom.* **9**(2), 391-445 (2015).

[CKL08] S. Carpi, Y. Kawahigashi and R. Longo. Structure and classification of superconformal nets. *Ann. Henri Poincaré* **9**(6), 1069–1121 (2008).

[CKLW18] S. Carpi, Y. Kawahigashi, R. Longo and M. Weiner. From vertex operator algebras to conformal nets and back. *Mem. Amer. Math. Soc.* **254**(1213), pp. vi+85 (2018).

[CW05] S. Carpi and M. Weiner. On the uniqueness of diffeomorphism symmetry in Conformal Field Theory. *Commun. Math. Phys.* **258**(1), 203-221 (2005).

[CS99] J. H. Conway and N. J. A. Sloane. *Sphere Packings, Lattices and Groups. Third Edition.* (Springer-Verlag New York, Inc., 1999).

[DMS97] P. Di Francesco, P. Mathieu and D. Sénéchal. *Conformal Field Theory.* (Springer-Verlag New York, Inc., 1997).

[DVVV89] R. Dijkgraaf, C. Vafa, E. Verlinde, and H. Verlinde. The operator algebra of orbifold models. *Commun. Math. Phys.* **123**(3), 485–526 (1989).

[Don93] C. Dong. Vertex algebras associated with even lattices. *J. Algebra* **161**(1), 245-265 (1993).

[DG98] C. Dong and R. L. Griess. Rank one lattice type vertex operator algebras and their automorphism groups. *J. Algebra* **208**(1), 262-275 (1998).

[DGH98] C. Dong, R. L. Griess and G. Höhn. Framed vertex operator algebras, codes and the Moonshine module. *Commun. Math. Phys.* **193**(2), 407-448 (1998).

[DJX14] C. Dong, X. Jiao and F. Xu. Mirror extensions of vertex operator algebras. *Commun. Math. Phys.* **329**(1), 263-294 (2014).

[DL14] C. Dong and X. Lin. Unitary vertex operator algebras. *J. Algebra* **397**, 252-277 (2014).

[DL15] C. Dong and X. Lin. The extensions of $L_{sl_2}(k,0)$ and preunitary vertex operator algebras with central charge $c < 1$. *Commun. Math. Phys.* **340**(2), 613-637 (2015).

[DM99] C. Dong and G. Mason. Quantum Galois theory for compact Lie groups. *J. Algebra* **214**(1), 92-102 (1999).

[DMZ94] C. Dong, G. Mason and Y. Zhu. Discrete series of the Virasoro algebra and the Moonshine module. In: Algebraic groups and their generalizations: quantum and infinite-dimensional methods. *Proc. Sympos. Pure Math.* **56**(2), 295-316. (AMS, Providence, 1994).

[DN99] C. Dong and K. Nagatomo. Automorphism groups and twisted modules for lattice vertex operator algebras. In: Recent developments in quantum affine algebras and related topics. *Contemp. Math.* **248**, 117–133. (AMS, Providence, 1999).

[DX06] C. Dong and F. Xu. Conformal nets associated with lattices and their orbifolds. *Adv. Math.* **206**(1), 279-306 (2006).

[DHR69I] S. Doplicher, R. Haag and J. E. Roberts. Fields, observables and gauge transformations. I. *Commun. Math. Phys.* **13**(1), 1-23 (1969).

[DHR69II] S. Doplicher, R. Haag and J. E. Roberts. Fields, observables and gauge transformations. II. *Commun. Math. Phys.* **15**(3), 173-200 (1969).

[DHR71] S. Doplicher, R. Haag and J. E. Roberts. Local observables and particle statistics. I. *Commun. Math. Phys.* **23**(3), 199-230 (1971).

[DHR74] S. Doplicher, R. Haag and J. E. Roberts. Local observables and particle statistics. II. *Commun. Math. Phys.* **35**(1), 49-85 (1974).

[DR89] S. Doplicher and J. E. Roberts. A new duality theory for compact groups. *Invent. Math.* **98**(1), 157-218 (1989).

[DF77] W. Driessler and J. Fröhlich. The reconstruction of local observable algebras from the Euclidean Green's functions of relativistic quantum field theory. *Ann. Inst. H. Poincaré, Sect. A (N. S.)* **27**(3), 221-236 (1977).

[DSW86] W. Driessler, S. J. Summers and E. H. Wichmann. On the connection between quantum fields and von Neumann algebras of local operators. *Commun. Math. Phys.* **105**(1), 49-84 (1986).

[Dun07] J. F. Duncan. Super-moonshine for Conway's largest sporadic group. *Duke Math. J.* **139**(2), 255-315 (2007).

[Dun08] J. F. Duncan. Moonshine for Rudvalis's sporadic group I. `https://arxiv.org/abs/math/0609449`.

[EMS20] J. van Ekeren, S. Möller and N. R. Scheithauer. Construction and classification of holomorphic vertex operator algebras. *J. reine angew. Math.* **759**, 61-99 (2020).

[Eps70] D. B. A. Epstein. The simplicity of certain groups of homeomorphisms. *Compositio Math.* **22**(2), 165-173 (1970).

[FR20] C. J. Fewster and K. Rejzner. Algebraic Quantum Field Theory. (2020) In: Progress and Visions in Quantum Theory in View of Gravity. Bridging Foundations of Physics and Mathematics. (Birkhäuser Basel, 2020).

[Fol08] G. B. Folland. *Quantum Field Theory. A Tourist Guide for Mathematicians.* (AMS, Providence, 2008).

[FH81] K. Fredenhagen and J. Hertel. Local algebras of observables and pointlike localized fields. *Commun. Math. Phys.* **80**(4), 555-561 (1981).

[FJ96] K. Fredenhagen and M. Jorß. Conformal Haag-Kastler nets, pointlike localized fields and the existence of operator product expansions. *Commun. Math. Phys.* **176**(3), 541-554 (1996).

[FR16] K. Fredenhagen and K. Rejzner. Quantum field theory on curved spacetimes: Axiomatic framework and examples. *J. Math. Phys.* **57**(3), 031101, pp.38 (2016).

[FHL93] I. B. Frenkel, Y.-Z. Huang and J. Lepowsky. On axiomatic approaches to vertex operator algebras and modules. *Mem. Amer. Math. Soc.* **104**(494), pp. viii+64 (1993).

[FLM84] I. B. Frenkel, J. Lepowsky and A. Meurman. A natural representation of the Fisher-Griess Monster with the modular function J as character. *Proc. Natl. Acad. Sci. USA* **81**(10), 3256-3260 (1984).

[FLM88] I. B. Frenkel, J. Lepowsky and A. Meurman. *Vertex Operator Algebras and the Monster.* (Academic Press, Inc., London, 1988).

[FQS85] D. Friedan, Z. Qiu and S. Shenker. Superconformal invariance in two dimensions and the tricritical Ising model. *Phys. Lett. B* **151**(1), 37-43 (1985).

[FV70] S. Fubini and G. Veneziano. Duality in operator formalism. *Nuovo Cimento A* **67**, 29-47 (1970).

[FST89] P. Furlan, G. M. Sotkov and I. T. Todorov. Two-dimensional Conformal Quantum Field Theory. *Riv. del Nuovo Cimento.* **12**(6), 1-202 (1989).

[GF93] F. Gabbiani and J. Fröhlich. Operator algebras and conformal field theory. *Commun. Math. Phys.* **155**(3), 569-640 (1993).

[Gan06] T. Gannon. *Moonshine Beyond the Monster. The Bridge Connecting Algebra, Modular Forms and Physics.* (Cambridge University Press, New York, 2006).

[Gin88] P. Ginsparg. Curiosities at $c = 1$. *Nucl. Phys. B* **295**(2), 153–170 (1988).

[GJ87] J. Glimm and A. Jaffe. *Quantum Physics. A Functional Integral Point of View. Second Edition.* (Springer-Verlag New York Inc., 1987).

[GKO86] P. Goddard, A. Kent and D. Olive. Unitary representations of the Virasoro and super-Virasoro algebras. *Commun. Math. Phys.* **103**(1), 105-119 (1986).

[GW84] R. Goodman and N. R. Wallach. Structure and unitary cocycle representations of loop groups and the group of diffeomorphisms of the circle. *J. Reine Angew. Math.* **347**, 69-133 (1984).

[GW85] R. Goodman and N. R. Wallach. Projective unitary positive-energy representations of $\mathrm{Diff}(S^1)$. *J. Funct. Anal.* **63**(3), 299-321 (1985).

[Gui19I] B. Gui. Unitarity of the modular tensor categories associated to unitary vertex operator algebras, I. *Commun. Math. Phys.* **366**(1), 333-396 (2019).

[Gui19II] B. Gui. Unitarity of the modular tensor categories associated to unitary vertex operator algebras, II. *Commun. Math. Phys.* **372**(3), 893-950 (2019).

[Gui20] B. Gui. Categorical extensions of conformal nets. *Commun. Math. Phys.* **383**(2), 763-839 (2020).

[Gui21] B. Gui. Q-systems and extensions of completely unitary vertex operator algebras. *Int. Math. Res. Not.* (2021). https://doi.org/10.1093/imrn/rnaa300.

[Haag96] R. Haag. *Local Quantum Physics. Fields, Particles, Algebras. Second Revised and Enlarged Edition.* (Springer-Verlag Berlin Heidelberg New York, 1996)

[Ham82] R. S. Hamilton. The inverse function theorem of Nash and Moser. *Bull. Amer. Math. Soc. (N. S.)* **7**(1), 65-222 (1982).

[Hen18] A. Henriques. Chiral conformal field theory. (Course notes, 2018). http://andreghenriques.com/Teaching/CFT-2018.pdf.

[Her71] M.-R. Herman. Simplicité du groupe des difféomorphismes de classe C^∞, isotopes à l'identité, du tore de dimension n. *C.R. Acad. Sci. Paris Sér. A-B* **273**, A232-A234 (1971).

[HP57] E. Hille and R. S. Phillips. *Functional Analysis and Semi-Groups.* Col. Pub. **31**. (AMS, Providence, 1957).

[Hua08] Y.-Z. Huang. Rigidity and modularity of vertex tensor categories. *Commun. Contemp. Math.* **10**(1), 871-911 (2008).

[Jac79] N. Jacobson. *Lie Algebras.* (Dover Publications, Inc., New York, 1979).

[Jon83] V. F. R. Jones. Index for subfactors. *Invent. Math.* **72**, 1-25 (1983).

[Jor96] M. Jorß. The construction of pointlike localized charged fields from conformal Haag-Kastler nets. *Lett. Math. Phys.* **38**(3), 257-274 (1996).

[Kac77] V. G. Kac. Lie superalgebras. *Adv. Math.* **26**(1), 8-96 (1977).

[Kac95] V. G. Kac. *Infinite dimensional Lie algebras. Third Edition.* (Cambridge University Press, Cambridge, 1995).

[Kac01] V. G. Kac. *Vertex algebras for beginners.* ULS **10**. (AMS, Providence, 2001).

[KR87] V. G. Kac and A. K. Raina. *Bombay Lectures on Highest Weight Representations of Infinite Dimensional Lie Algebras.* (World Scientific Publishing, Singapore, 1987).

[KT85] V. G. Kac and I. T. Todorov. Superconformal current algebras and their unitary representations. *Commun. Math. Phys.* **102**(2), 337-347 (1985).

[KW86] V. G. Kac and M. Wakimoto. Unitarizable highest weight representations of Virasoro, Neveu-Schwarz and Ramond algebras. In: Conformal Groups and Related Symmetries Physical Results and Mathematical Background. *Lecture Notes in Phys.* **261**, 345–371. (Springer Berlin Heidelberg, 1986).

[KW94] V. G. Kac and W. Wang. Vertex operator superalgebras and their representations. In: Mathematical Aspects of Conformal and Topological Field Theories and Quantum Groups. *Contemp. Math.* **175**, 161–191. (AMS, Providence, 1994).

[Kaw10] Y. Kawahigashi. From operator algebras to superconformal field theory. *J. Math. Phys.* **51**(1), 015209, pp. 20 (2010).

[KL04] Y. Kawahigashi and R. Longo. Classification of local conformal nets. Case $c < 1$. *Ann. of Math.* **160**(2), 493-522 (2004).

[KL06] Y. Kawahigashi and R. Longo. Local conformal nets arising from framed vertex operator algebras. *Adv. Math.* **206**(2), 729-751 (2006).

[Kir89] E. B. Kiritsis. Proof of the completeness of the classification of rational conformal theories with $c = 1$. *Phys. Lett. B* **217**(4), 427–430 (1989).

[Kös02v2] S. Köster. Conformal transformations as observables. https://arxiv.org/abs/math-ph/0201016v2.

[Kös02] S. Köster. Conformal transformations as observables. *Lett. Math. Phys.* **61**(5), 187-198 (2002).

[LL04] J. Lepowsky and H.-S. Li. *Introduction to Vertex Operator Algebras and Their Representation.* (Birkhäuser Boston, Inc., Boston, 2004).

[Li94] H.-S. Li. Symmetric invariant bilinear forms on vertex operator algebras. *J. Pure Appl. Algebra.* **96**(3), 279-297 (1994).

[Li96] H.-S. Li. Local systems of vertex operators, vertex superalgebras and modules. *J. Pure Appl. Algebra.* **109**(2), 143-195 (1996).

[Lok94] T. M. L. Loke. *Operator Algebras and Conformal Field Theory of the Discrete Series Representation of* $\mathrm{Diff}(S^1)$. (PhD Thesis, Cambridge, 1994).

[Lon94] R. Longo. A duality for Hopf algebras and for subfactors. I. *Commun. Math. Phys.* **159**(1), 133-150 (1994).

[Lon03] R. Longo. Conformal subnets and intermediate subfactors. *Commun. Math. Phys.* **237**(1-2), 7-30 (2003).

[Lon08] R. Longo. Real Hilbert subspaces, modular theory, $\mathrm{SL}(2,\mathbb{R})$ and CFT. In: Von Neumann algebras in Sibiu, 33-91. (Theta Ser. Adv. Math., 10, Theta, Bucharest, 2008).

[Lon08b] R. Longo. Lectures on Conformal Nets. *In preparation.* `https://www.mat.uniroma2.it/longo/lecture-notes.html`.

[McL98] S. Mc Lane. *Categories for the Working Mathematician. Second Edition.* (Springer-Verlag New York, Inc., New York, 1998).

[Mil84] J. Milnor. Remarks on infinite-dimensional Lie groups. In: Relativity, groups and topology II. *Les Houches* **Session XL**, 1007-1057 (1983). (Elsevier, New York, 1984).

[NS15] K.-H. Neeb and H. Salmasian. Classification of positive energy representations of the Virasoro group. *Int. Math. Res. Not. IMRS* **2015**(18), 8620-8656 (2015).

[Nel59] E. Nelson. Analytic vectors. *Ann. of Math.* **70**(3), 572-615 (1959).

[Nel72] E. Nelson. Time-ordered operator products and sharp-time quadratic forms. *J. Funct. Anal.* **11**, 211-219 (1972).

[Pal10] S. Palcoux. Neveu-Schwarz and operators algebras III. Subfactors and Connes fusion. `https://arxiv.org/abs/1010.0076`.

[Ped89] G. K. Pedersen. *Analysis Now.* (Springer-Verlag, New York, 1989).

[PS95] M. E. Peskin and D. V. Schroeder. *An Introduction to Quantum Field Theory.* (Perseus Books, Reading, 1995).

[PS86] A. Pressley and G. Segal. *Loop groups.* (Oxford University Press, New York, 1986).

[RS75] M. Reed and B. Simon. *Methods of Modern Mathematical Physics I: Fourier Analysis, Self-Adjointness.* (Academic Press, London, UK, 1975).

[RS80] M. Reed and B. Simon. *Methods of Modern Mathematical Physics II: Functional Analysis. Revised and Enlarged Edition.* (Academic Press, London, UK, 1980).

[Reh94] K.-H. Rehren. A new view of the Virasoro algebra. *Lett. Math. Phys.* **30**(2), 125-130 (1994).

[Rej16] K. Rejzner. *Perturbative Algebraic Quantum Field Theory. An Introduction for Mathematicians.* (Springer International Publishing, 2016).

[Roi04] M. Roitman. Invariant bilinear forms on a vertex algebra. *J. Pure Appl. Algebra* **194**(3), 329-345 (2004).

[Rud87] W. Rudin. *Real and Complex Analysis. Third Edition.* (McGraw-Hill Book Co., New York, 1987).

[Rud91] W. Rudin. *Functional Analysis. Second Edition.* (McGraw-Hill, Inc., New York, 1991).

[Sch98] N. R. Scheithauer. Vertex algebras, Lie algebras, and superstrings. *J. Algebra* **200**(2), 363-403 (1998).

[Ste94] G. Sterman. *An Introduction to Quantum Field Theory.* (Cambridge University Press, Cambridge, 1994).

[SW64] R. F. Streater and A. S. Wightman. *PCT, Spin and Statistics, and All That.* (W. A. Benjamin, Inc., New York, 1964).

[Ten17] J. E. Tener. Construction of the unitary free fermion Segal CFT. *Commun. Math. Phys.* **355**(2), 463-518 (2017).

[Ten19] J. E. Tener. Geometric realization of algebraic conformal field theories. *Adv. Math.* **349**, 488-563 (2019).

[Ten19b] J. E. Tener. Representation theory in chiral conformal field theory: from fields to observables. *Sel. Math. (N. S.)* **25**(5)(76), pp. 82 (2019).

[Thu74] W. Thurston. Foliations and groups of diffeomorphisms. *Bull. Amer. Math. Soc.* **80**(2), 304-307 (1974).

[Tol99] V. Toledano Laredo. Integrating unitary representations of infinite-dimensional Lie groups. *J. Funct. Anal.* **161**(2), 478-508 (1999).

[Wan93] W. Wang. Rationality of Virasoro vertex operator algebras. *Int. Math. Res. Not. IMRN* (7), 197-211 (1993).

[Wei05I] S. Weinberg. *The Quantum Theory of Fields. Vol I. Foundations.* (Cambridge University Press, Cambridge, 2005).

[Wei05II] S. Weinberg. *The Quantum Theory of Fields. Volume II. Modern Applications.* (Cambridge University Press, Cambridge, 2005).

[Wei05III] S. Weinberg. *The Quantum Theory of Fields. Volume III. Supersymmetry.* (Cambridge University Press, Cambridge, 2005).

[Wei05] M. Weiner. *Conformal Covariance and Related Properties of Chiral QFT.* (PhD Thesis, Università degli studi di Roma "Tor Vergata", 2005). `https://arxiv.org/abs/math/0703336`.

[Wig59] E. P. Wigner. *Group Theory. And Its Application to the Quantum Mechanics of Atomic Spectra. Translated from the German by J. J. Griffin. Expanded and Improved Edition.* (Academic Press Inc. (London) Ltd., London, 1959).

[Xu05] F. Xu. Strong additivity and conformal nets. *Pacific J. Math.* **221**(1), 167-199 (2005).

[Xu07] F. Xu. Mirror extensions of local nets. *Commun. Math. Phys.* **270**(3), 835-847 (2007).

[Xu98] X. Xu. *Introduction to Vertex Operator Superalgebras and Their Modules.* (Kluwer Academic Publishers, Dordrecht, 1998).

[Yam14] H. Yamauchi. Extended Griess algebras and Matsuo-Norton trace formulae. In: Conformal Field Theory, Automorphic Forms and Related Topics. *Contrib. Math. Comput. Sci.* **8**, 75-107. (Springer, Heidelberg, 2014).