Embedded Eigenvalues for Operators and Fredholm Properties of Pencils

Nifeen Hussain Altaweel, B.Sc., M.Sc.

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Abstract

The perturbation problem for operators is considered one of the differential equations with operator coefficients; a possible example of this problem is embedded eigenvalues, which serves as a prototype of this problem.

My research is concerned with two main tasks; first, highlighting the idea of the existence of embedded eigenvalues (trapped modes) of different operators. These include the stability of the embedded eigenvalues within the spectrum for the operator on a cylindrical domain. Common threads will be taken from these problems to subsequently develop a more generalised understanding of the existence of embedded eigenvalues.

The second task is to study the Fredholm properties of an operator pencil. In particular, we detect and approximate the spectra of the Fredholm operator pencils via a Green's kernel (contour integral) by considering exponential solutions of differential equations with operator coefficients. The arguments for this task act on a class of weighted function spaces which can be modelled on Sobolev spaces.

One of the main motivation behind this research is to gain a deeper understanding the development of aspects of the theory of ordinary differential equations with operator coefficients by concentrating on some specific examples of trapped modes.

The results of our first task showed that, in different cases, for sufficiently small potential functions our operator has an eigenvalue which is contained in the essential spectrum, and hence is an embedded eigenvalue.

According to the result of the second task, it was directly established that Fredholm operator pencil and the

index could be calculated without the need to consider the adjoint operator. Also, we leveraged certain concepts to go from the semi-Fredholm property to the Fredholm property using some of the results of the current thesis.

Declaration

I, Nifeen Hussain Altaweel, do hereby declare that, with the exception of any specific reference to other people's work which has been duly acknowledged, the work contained in this thesis "Embedded Eigenvalues for Operators and Fredholm Properties of Pencils" is the result of my own research carried out under the supervision of the Department of Mathematical and Statistics, at Lancaster University, Lancaster, the U.K. from October, 2015 to April, 2021. I further declare that this thesis' work, either in whole or in part, has not been presented for any other degree at this University or elsewhere.

Nifeen Hussain Altaweel

Dedication

This work is first and foremost dedicated to my Dad ”Hussain” through whom the Lord gave me life and which he protected to date. To my late Mum ”Jamila”, though you rest in perfect peace in the embrace of the Lord, this work is dedicated to you too.

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$B(H_2, H_0)$	Denotes the Hilbert space of all bounded linear operators (See Definition 4.4.1).
$\mathbb{C}$	The field of complex numbers (See Section 2.1).
$C^\infty(\mathbb{R})$	The class of all infinitely differentiable functions on $\mathbb{R}$ (See Remark 14).
$C_0^\infty(\mathbb{R})$	The class of all infinitely differentiable functions on $\mathbb{R}$ with compact support (See Remark 14).
$C_0^1(\mathbb{R})$	The class of all continuously differentiable functions on $\mathbb{R}$ (See Remark 14).
$C_0^0(\mathbb{R})$	The class of all continuous functions vanishing at infinity on $\mathbb{R}$ (See Remark 14).
$\mathrm{Dom}(A)$	The domain of a linear operator A (See Section 2.3).
$\mathrm{Dom}(A^*)$	The domain of the adjoint operator A (See Section 2.4).
$G(t)$	The Green's function (Green's kernel) (See Section 5.3).
$\{H_j\}_{j=0}^2$	The collection of Hilbert spaces (See Section 4.4).
$\{H_j^*\}_{j=0}^2$	The collection of dual Hilbert spaces (See Section 4.5).
$H^1(\mathbb{R})$	The space of complex-valued functions on $\mathbb{R}$ and is refinement of $L^2(\mathbb{R})$ (See Section 2.1).
$H^k(\mathbb{R})$	The space of complex-valued functions on $\mathbb{R}$ and $k = 0, 1, 2, ...$ (See Definition 2.1.2).
$H^k(\mathbb{R}^d)$	The space of complex-valued functions on $\mathbb{R}^d$ for $d \geq 1$ and $k = 0, 1, 2, ...$ (See Definition 2.1.3).
H_k	$H_k = \left\{\sum_{j=0}^\infty a_j u_j : ((1+\lambda_j^2)^{\frac{k}{4}} a_j)_{j\in\mathbb{N}_0} \in \ell^2(\mathbb{N}_0)\right\}$ is a linear subspace of the Hilbert space H for $k = 0, 1, ...$ (See Section 4.1).

Ker(A)	The kernel of a linear operator A (See Remark 5).
$L^2(\mathbb{R})$	It is a Hilbert space of complex-valued square integrable functions on $\mathbb{R}$ (See Section 2.1).
$L^\infty(\mathbb{R})$	It is a Banach space of complex-valued functions on $\mathbb{R}$ (See Remark 1).
ℓ^2	The space of square-summable sequences, which is a Hilbert space (See Section 4.1).
$\mathbb{N}_0 = \mathbb{N} \cup \{0\}$	The set of non-negative integers (See Section 4.1).
$P_{(\lambda-\epsilon,\lambda+\epsilon)}(A)$	The spectral projection of an operator A, for $\epsilon > 0$, and $\lambda \in \sigma_{dis}(A)$ (See Section 2.6).
$Q(A)$	The quadratic form domain of an operator A (See Definition 2.8.1).
Ran(A)	The range of a linear operator A (See Remark 5).
S	The symmetry operator (See Section 3.6).
$\widehat{u}(\tau)$	The Fourier transform of the function u for $\tau \in \mathbb{R}$ (See Remark 2).
V	The potential function (The relatively compact perturbation) (See Definition 2.6.8).
$W^{k,p}$	The Sobolev space is defined as the space of function $u \in L^p(\mathbb{R}^d)$ all of whose distributional derivative are also in $L^p(\mathbb{R}^d)$ for all multi-indices α which is normed by the expression $\|u\|_{W^{k,p}} = \sum_{0\leq|\alpha|\leq k} \|D^\alpha u\|_{L^p}$ and $1 \leq p \leq \infty$ (See Section 4.2).
$W^{k,\infty}$	The Sobolev space in the case $p = \infty$ with the norm using the essential supremum by $\|u\|_{W^{k,\infty}} = \max_{0\leq|\alpha|\leq k} \|D^\alpha u\|_{L^\infty}$ (See Section 4.2).
W^k	The Sobolev spaces be the space of distributions u on $\mathbb{R}$ with values in H_k such that $D_t^j \in L^2(\mathbb{R}, H_{k-j}), j = 0, 1, ..., k$ for $k = 0, 1, ...$ with the finite norm $\|u\|^2_{W^k(\mathbb{R})} = \int_{\mathbb{R}} \sum_{0\leq j\leq k} \left\| D_t^j u(t) \right\|^2_{H_{k-j}} dt$ (See Section 4.2).
$W^k_{\alpha,\beta}$	The weighted function spaces denotes the set of $u : \mathbb{R} \to H_k$ such that the finite norm $\|u\|^2_{W^k_{\alpha,\beta}} := \sum_{j=0}^k \int_{-\infty}^0 e^{2\alpha t} \left\| D_t^j u(t) \right\|^2_{H_{k-j}} dt + \sum_{j=0}^k \int_0^\infty e^{2\beta t} \left\| D_t^j u(t) \right\|^2_{H_{k-j}} dt$ for $k = 0, 1, 2, ...$ and $\alpha, \beta \in \mathbb{R}$ (See Section 4.2).

$W^0_{\alpha,\beta}$ — The weighted function spaces denotes the set of $u : \mathbb{R} \to H_0$ such that the finite norm $\|u\|^2_{W^0_{\alpha,\beta}} = \int_{-\infty}^0 e^{2\alpha t}\|u(t)\|^2_{H_0} dt + \int_0^\infty e^{2\beta t}\|u(t)\|^2_{H_0} dt$ for $\alpha, \beta \in \mathbb{R}$ (See Section 4.2).

$W^1_{\alpha,\beta}$ — The weighted function spaces denotes the set of $u : \mathbb{R} \to H_1$ such that the finite norm $\|u\|^2_{W^1_{\alpha,\beta}} = \int_{-\infty}^0 e^{2\alpha t}(\|D_t u(t)\|^2_{H_0} + \|u(t)\|^2_{H_1})dt + \int_0^\infty e^{2\beta t}(\|D_t u(t)\|^2_{H_0} + \|u(t)\|^2_{H_1})dt$ for $\alpha, \beta \in \mathbb{R}$ (See Section 4.2).

$W^2_{\alpha,\beta}$ — The weighted function spaces denotes the set of $u : \mathbb{R} \to H_2$ such that an equivalent finite norm $\|u\|^2_{W^2_{\alpha,\beta}} = \int_{-\infty}^0 e^{2\alpha t}\left[\left\|D_t^2 u(t)\right\|^2_{H_0} + \|u(t)\|^2_{H_2}\right] dt + \int_0^\infty e^{2\beta t}\left[\left\|D_t^2 u(t)\right\|^2_{H_0} + \|u(t)\|^2_{H_2}\right] dt$ for $\alpha, \beta \in \mathbb{R}$ (See Section 4.2).

$W^k_{\alpha,\alpha}$ — The weighted function spaces denotes the set of $u : \mathbb{R} \to H_k$ such that the finite norm $\|u\|^2_{W^k_{\alpha,\alpha}} := \sum_{j=0}^k \int_{-\infty}^\infty e^{2\alpha t}\left\|D_t^j u(t)\right\|^2_{H_{k-j}} dt$ for $k = 0, 1, 2, ...$ and $\alpha \in \mathbb{R}$ (See Section 4.2).

$W^0_{0,0}$ — The weighted function spaces is equal to $L^2(\mathbb{R}, H_0)$ (See Section 4.2).

$\Gamma(\mathcal{B}_A)$ — The projection of the spectrum of an operator pencil $\mathcal{B}_A$ is defined by $\{\Im\mu| \quad \mu \in \sigma(\mathcal{B}_A)\} \subseteq \mathbb{R}$ (See Definition 4.4.1).

$\rho(A)$ — The resolvent set of an operator A (See Section 2.6).

$\sigma(A)$ — The spectrum set of an operator A (See Section 2.6).

$\sigma_{dis}(A)$ — The discrete spectrum set of an operator A (See Section 2.6).

$\sigma_p(A)$ — The isolated point (discrete) spectrum set of an operator A (See Section 2.6).

$\sigma_c(A)$ — The continuous spectrum set of an operator A (See Section 2.6).

$\sigma_r(A)$ — The residual spectrum set of an operator A (See Section 2.6).

$\sigma_{ess}(A)$ — The essential spectrum set of an operator A (See Section 2.6).

$-\Delta$ — The Laplace operator is defined by $-\Delta = -\sum_{j=1}^d \frac{\partial^2}{\partial t_j^2}$ for $d \geq 1$ (See Chapter 2).

μ_0 — The eigenvalue of an operator pencil $\mathcal{B}_A$ (See Definition 4.4.1).

$\{\psi_{k,s}\}_{s=0}^{m_k-1}$	A canonical system of Jordan of $\mathcal{B}_A^*$ corresponding to $\overline{\mu_0}$ for $k = 1, ..., J$ (See Section 4.4).
$\{\varphi_{k,s}\}_{s=0}^{m_k-1}$	A canonical system of Jordan of $\mathcal{B}_A$ corresponding to μ_0 for $k = 1, ..., J$ (See Section 4.4).
$\kappa(A)$	The $\dim(\operatorname{Ker} A) \in \mathbb{N}_0 \cup \{\infty\}$ (See Section 5.1).
$\eta(A)$	The $\operatorname{Codim}(\operatorname{Ran} A) \in \mathbb{N}_0 \cup \{\infty\}$ (See Section 5.1).
$\Phi(H_2, H_0)$	The set of Fredholm operators (See Section 5.1).
$\Phi_-(H_2, H_0)$	The set of lower semi-Fredholm operators (See Section 5.1).
$\Phi_+(H_2, H_0)$	The set of upper semi-Fredholm operators (See Section 5.1).
$\operatorname{Index}(A)$	The index of Fredholm operator A (See Definition 5.1.3).
$\Upsilon(\mu)$	The Holomorphic function in a neighbourhood of μ_0 (See Definition 5.2.3).
$\mathcal{B}_A(\mu)$	The operator pencil (See Section 4.4).
$\mathcal{B}^*{}_A(\mu)$	The adjoint of an operator pencil (See Section 4.4).
$\mathcal{B}_A^{-1}(\mu)$	The inverse of an operator pencil (See Section 5.2.3).
$\Sigma_{\alpha,\beta}$	The linear span of the set of exponential solutions (See Section 5.4).
u_μ	The exponential solutions of $\mathcal{B}_A(D_t)u_\mu = 0$ (See Proposition 5.4.1).
v_μ	The exponential solutions of $\mathcal{B}^*{}_A(D_t)v_\mu = 0$ (See Proposition 5.4.1).
Ω	A bounded open set in $\mathbb{R}^d$ for $d \geq 1$ with a smooth boundary $\partial\Omega$ (See Section1.2.1).
∇	The distributional gradient (See Section 2.1).
∇^j	The distributional derivative for $0 \leq j \leq k$ and $k = 0, 1, ...$ (See Definition 2.1.2).
D^α	The distributional derivative is defined on $\mathbb{R}^d$ for $d \geq 1$ and a multi-index α (See Section 2.1.3).
$D_t = -i\dfrac{d}{dt}$	The operator which is defined on $\mathbb{R}$ (See Section 4.2).
χ_Λ	The indicator function defined on a Borel set Λ (See Section 2.5).

Chapter 1

Introduction

1.1 Overview

Most mathematical problems that require the theory of ordinary differential equations are generally challenging to solve [30]. This theory has a wide range of applications in physics and engineering sciences, such as heat conduction, meteorology, elasticity, plasticity theory, and thermodynamics. It impacts the development of different sciences and is considered one of the outstanding creation of the human imagination (see, for example, [32], [34], and [57]). In essence, this theory forms the basis of the solutions of many problems, for example, perturbation problems. A canonical example of these problems is the embedded eigenvalues for different operators. Our focus will be on mathematical problems involving the stability of trapped modes or eigenvalues embedded within continuous spectra for the Schrödinger operator or Laplace operator with a relatively compact perturbation. We will highlight the idea of the existence of embedded eigenvalues that occur in various applications arising in physics, in quantum mechanics, for instance, the eigenvalues of the energy operator correspond to the energy bonds states (See in [14]). It is known that these problems, that is, with embedded eigenvalues are generally challenging since the embedded eigenvalues (trapped modes) cannot be separated from the rest of the spectrum (see, for example, [30], [34], and [57]). The idea of this research is to develop aspects of the theory of ordinary differential equations with operator coefficients by (at least initially) concentrating on some particular examples of embedded eigenvalues. Common threads will be taken from these example problems to subsequently develop a more general mathematical theory. This thesis is concerned with two basic tasks which are in the embedded eigenvalues for operators and the Fredholm properties of pencils. Specifically, the first task described in this research is the development of the study of the existence of embedded eigenvalues (trapped modes) within spectra for the Laplace operator with a potential function $-\Delta - V$ on cylindrical domains $\mathbb{R} \times [-L, L]$. The second task of this research focuses on understanding the Fredholm properties of operator pencils $\mathcal{B}_{A},$, acting on weighted function spaces modelled on Sobolev spaces $W^k_{\alpha,\beta}$, for $k \in \mathbb{N}_0$ and $\alpha, \beta \in \mathbb{R}$.

The remainder of this introductory chapter is organised as follows: In Section 1.2, we give an outline of the stability of embedded eigenvalues for the Laplace operator with an added potential function satisfying symmetry conditions with respect to a cylindrical domain. In Section 1.3, the class of weighted function spaces are introduced with the study Fredholm properties of pencils. In Section 1.4, we outline the contributions of this thesis to literature. Finally, Section 1.5 gives the structure of this thesis.

1.2 The Stability of Embedded Eigenvalues for the Operator

It well-known that eigenvalues that belong to discrete spectrum are stable. This property is the basis of perturbation theory for eigenvalues. On the other hand, the behaviour of eigenvalues that are embedded in the continuous spectrum completely different (see [54]). An example of instability of embedded eigenvalues was given by Colin de Verdire [65]. In this work, we give an overview of the idea of stability of embedded eigenvalues, which means the study of the behaviour of the existence of eigenvalues in the continuous spectrum of the operator.

1.2.1 Introduction

A *waveguide*, which represents a unique distribution of transverse and longitudinal components of electric and magnetic fields (see [7]). From the mathematical point of view, a waveguide is defined as type of boundary condition on the wave equation such that the wave function must be equal to zero on the boundary and that the allowed region is finite in all diminsion but one (an unfinitely long cylinder is an example)(see [7]). We study a two-dimensional acoustic waveguide for the domain described by two parallel lines containing an abstraction of fairly general shape that is symmetric about the centreline of the waveguides. It is demonsrated that there exist at least one trapped mode of oscillation that corresponds to a local oscillation at particular frequency, in the absence of excitation, which decays with distance down the wavegides away from abstraction. Mathematically, this trapped mode is related to an eigenvalue of the Laplace operator in the waveguide. Our main aim is to show that trapped modes always exist. For waveguids, the eigenvalue associated with the trapped mode is said to be embedded in the continuous spectrum of the operator. As we shall see, the main difficulty with demonstrating this fact is that this eigenvalue is embedded in the spectrum, which prevent us from using the standard functional analysis technique. Normally, eigenvalues embedded in a continuous spectrum are a very rare occurrence; their study requires special methods and there must be particular reasons for their existence. In our case, we need to define the symmetry operator which allow us to reduce our consideration to the more simple problem for which the spectrum of the our operator. Furthermore, there are references to achievements made in the last 30 years with regards to the theorems on the existence of trapped modes, see [15]. In order to discuss this task, we highlight the idea of the existence of embedded eigenvalues for the Laplace operator Δ. Namely, $C^{\infty}(\mathbb{R})$ which is defined by the class of all infinitely differentiable functions on $\Omega \subset \mathbb{R}^d$ for $d \geq 1$. We seek to find pairs (λ, u) consisting of λ, which is an eigenvalue of the Laplace operator, and a non-zero function $u \in C^{\infty}(\Omega)$, which is the eigenfunction of the Laplace operator corresponding to the eigenvalue λ so that the following condition is satisfied:

$$\begin{cases} -\Delta u = \lambda u, & \text{in} \quad \Omega \\ \\ u \text{ satisfies Dirichlet boundary conditions} & \text{on} \quad \partial\Omega\,. \end{cases} \tag{1.1}$$

Such eigenvalue/eigenfunction pairs have creation properties which we will now explore. The eigenvalue problems involving the Laplace operator remind us of the basic result in the elementary theory of partial differential equations, which asserts that the problem possesses an unbounded sequence of eigenvalues. We have the following that:

Theorem 1.2.1. (General result for the Laplace operator on a bounded domain). The spectrum (which is defined in mathematics, particularly in functional analyis, is a generalisaition of the set of eigenvalues. Specifically a complex number λ which is said to be in the spectrum of a bounded operator A if $A-\lambda I$ is not invertable, where I is the identity operator) of the Laplace operator is discrete when Ω is a bounded open set in $\mathbb{R}^d$ for $d \geq 1$ with a smooth (or piecewise smooth) boundary $\partial\Omega$. By piecewise smooth, we mean that $\partial\Omega$ is a union of a finite number of smooth arcs or pieces of curves, for example, a rectangle (see [21]). Moreover, the eigenvalue of the problem (1.1) has an unbounded sequence of eigenvalues

$$0 \leq \lambda_0 \leq \lambda_1 \leq \ldots \leq \lambda_n \leq \ldots \cdot$$

($\lambda = 0$ occurs for Neumann boundary conditions). This celebrated result goes back to the Riesz-Fredholm theory of self-adjoint and compact operators in Hilbert spaces (see [35], pp. $378 - 380$). In what concerns λ_0 being the lowest eigenvalue of (1.1), can be characterised from a variational point of view as the minimum of the Rayleigh quotient, that is,

$$\lambda_0 = \inf_{u \in C^\infty(\Omega)} \frac{\displaystyle\int_\Omega |\nabla u|^2 \, dx}{\displaystyle\int_\Omega u^2 \, dx}, \tag{1.2}$$

where the infimum is taken over $C^\infty(\Omega)$ of the domain of the Laplace operator with Dirichlet and Neumann boundary conditions. Moreover, it is known that λ_0 is simple, that is, all the associated eigenfunctions are merely multiples of each other (see, for example, Gilbarg and Trudinger [11] and further details in Section 3.2 of this thesis).

1.2.2 The Spectrum and Essential Spectrum

Definition 1.2.1. Let A be a bounded self-adjoint operator and Λ a Borel set of $\mathbb{R}$ (which is defined as any set in space that can be formed from open sets through the operations of countable unions, countable intersections, and relative complements). $P_\Lambda \equiv \chi_\Lambda(A)$ is called a *spectral projection* of an operator A such that χ_Λ is an indicator function, i.e., a spectral projection is the image of Λ under an indicator function defined on its spectrum, which is hence an orthogonal projection on some closed subspace. See Section 2.6 of this thesis and [36] and the definition of χ_Λ is in theorem 2.7.2.

Definition 1.2.2. For a self-adjoint operator A, if $\lambda \in \sigma(A)$ and $P_{(\lambda-\epsilon,\lambda+\epsilon)}(A)$ is finite dimensional for some $\epsilon > 0$, $\lambda \in \sigma_{dis}(A)$ is a *discrete spectrum* of A, where $P_{(\lambda-\epsilon,\lambda+\epsilon)}(A)$ is a spectral projection of operator A.

The reader can see the associated definition in Section 2.6 for more details and [36].

Definition 1.2.3. The *essential spectrum* of the operator A is the complement in the spectrum of the discrete spectrum and is denoted by $\sigma_{ess}(A)$ i.e.,

$$\sigma_{ess}(A) = \sigma(A) \backslash \sigma_{dis}(A).$$

See the definition in Section 2.6 and [36].

The following theorem, we obtain the result for the relation between spectrum and essential spectrum, that is used in the first task.

Theorem 1.2.2. Let A be a self-adjoint operator and suppose $(a, b) \subset \sigma(A)$ for some open interval (a, b). Then,

$$(a, b) \subset \sigma_{ess}(A).$$

See the proof of this theorem in Section 2.6.

As per the definition of the essential spectrum, it is straightforward to observe the role of this spectrum in the following concepts:

Definition 1.2.4. A subset of Hilbert space is called a *relatively compact* if its closure is compact (see Section 2.6 of this thesis and [39]).

Theorem 1.2.3. Let A be a self-adjoint operator and let V be a relatively compact perturbation of A. Then,

- $A - V$ defined with $\text{Dom}(A - V) = \text{Dom}(A)$ is a closed operator.
- If V is symmetric, then $(A - V)$ is a self-adjoint operator.
- $\sigma_{ess}(A) = \sigma_{ess}(A - V)$.

See Section 2.6.3 of this thesis and its proof in [39] on page 113.

With a bit more work, the following regularity result shows that multiplication by V defines a relatively compact perturbation with respect to operator A.

Definition 1.2.5. (Cone property)

For each $u \in \Omega$ is the vertex of a cone contained in Ω and congruent to cone where Ω is union of congruent cones.

Theorem 1.2.4. Let Ω be domain in $\mathbb{R}^d$ for $d \geq 1$ and Ω has a cone property. Let $-\Delta$ be the Laplacian on Ω with any of the boundary conditions (Dirichlet, Neumann or a mixture (Dirichlet and Neumann)). Suppose V is a continuous function with bounded support then multiplication by V defines a relatively compact perturbation with respect to operator $-\Delta$.

The reader is referred to the proof of the previous theorem in Section 3.8 of the current thesis.

1.2.3 Embedded Eigenvalues for $-\Delta - V$.

Here, we need consider the Laplace operator, where the domain is $C_c^\infty(\Omega)$, which is smooth and has compactly supported functions on $\Omega \subseteq \mathbb{R}^d$ for $d \geq 1$, and which is dense in $L^2(\Omega)$. See [51]. For $u \in C^\infty(\Omega)$, we can define the Laplace operator by

$$-\Delta u = -\sum_{j=1}^{d} \frac{\partial^2 u}{\partial t_j^2}.$$

We note that $-\Delta$ is again a smooth compactly supported function, and is bounded and lies in $L^2(\Omega)$ for further details, see Section 2.8.2.

To set the scene, as a consequence of all the above concepts, this part has, as the underlying domain, the cylinder $\mathbb{R} \times [-L, L] = \{(t, s) | t \in \mathbb{R}, s \in [-L, L]\}$ and the Laplace operator on the cylinder which describes by

$$-\Delta = -(\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial s^2}).$$

This shows that the operator

$$-\Delta - V$$

on $\mathbb{R} \times [-L, L]$ has embedded eigenvalues for certain positive symmetric potential functions V. Moreover, the following theorem is considered a principal result in this task:

Theorem 1.2.5. (Has been published in March 2020)[41].

Consider on $\mathbb{R} \times [-L, L]$ the operator

$$-\Delta_D - V$$

with Dirichlet boundary conditions on $\mathbb{R} \times \{-L\}$ and $\mathbb{R} \times \{L\}$. Suppose V for a sufficiently small, non-negative continuous real valued function on $\mathbb{R} \times [-L, L]$ with bounded support, is symmetric, i.e.,

$$V(t, s) = V(t, -s)$$

for $t, s \in \mathbb{R} \times [-L, L]$. Then,

$$\sigma_{ess}(-\Delta_D - V) = [\lambda_1, \infty) \subseteq \sigma(-\Delta_D - V),$$

where $\lambda_1 = \dfrac{\pi^2}{4L^2}$, while there exists $\lambda > \lambda_1$ such that λ is an eigenvalue of $-\Delta_D - V$; more precisely there exists $u \neq 0$ and

$$u \in \text{Dom}(-\Delta_D - V) \subset L^2(\mathbb{R} \times [-L, L])$$

such that

$$(-\Delta_D - V)u = \lambda u.$$

Similarly, we can consider the operator

$$-\Delta_N - V$$

on $\mathbb{R} \times [-L, L]$ with Neumann boundary conditions on $\mathbb{R} \times \{-L\}$ and $\mathbb{R} \times \{L\}$. Suppose V for a sufficiently small, non-negative continuous real valued function on $\mathbb{R} \times [-L, L]$, with bounded support is symmetric, i.e,.

$$V(t, s) = V(t, -s)$$

for $t, s \in \mathbb{R} \times [-L, L]$. Then

$$\sigma_{ess}(-\Delta_N - V) = [\lambda_0, \infty) \subseteq \sigma(-\Delta_N - V),$$

where $\lambda_0 = 0$, while there exists $\lambda > \lambda_0$ such that λ is an eigenvalue of $-\Delta_N - V$; more precisely there exists $u \neq 0$ and

$$u \in \text{Dom}(-\Delta_N - V) \subset L^2(\mathbb{R} \times [-L, L])$$

such that

$$(-\Delta_N - V)u = \lambda u.$$

In both cases, for a sufficiently small V, the operator $-\Delta - V$ has an eigenvalue λ which is contained in the essential spectrum, and is hence an embedded eigenvalue. The arguments of this result combine the ideas discussed in Chapters 2 and 3; we can see the proof of this result at the end of Chapter 3.

1.3 Fredholm Properties of Pencils

Here, the idea underlying the second task is based on the theory of the ordinary differential equations with operator coefficients. We study Fredholm properties, which are related to the spectra of pencils. In particular, we detect and approximate the spectra of the Fredholm operator pencil via Green's kernel with power-exponential solutions for non-homogeneous equations. Then, we calculate the kernel and co-kernel explicitly to establish a Fredholm operator pencil and its index without consider its adjoint.

1.3.1 Operator pencil

First, in order to proceed with further results for Fredholm properties of pencils, one needs to consider the following space:

Definition 1.3.1. The the space H_k for $k = 0, 1, 2, ...$ is set by

$$H_k = \left\{ \sum_{j=0}^{\infty} a_j u_j : ((1+\lambda_j^2)^{\frac{k}{4}} a_j)_{j\in\mathbb{N}_0} \in \ell^2(\mathbb{N}_0) \right\}.$$

This is a linear subspace of the Hilbert space H with the norm given by

$$\|u\|_{H_k}^2 = \sum_{j=0}^{\infty} (1+\lambda_j^2)^{\frac{k}{2}} |a_j|^2,$$

for $u \in H_k$.

We have, for $j \leq k$ and $u \in H_k$. Then,

$$\|u\|_{H_j}^2 \leq \|u\|_{H_k}^2.$$

The reader can see the properties of these spaces in Section 4.1.

Now, we define the operator $D_t = -i\dfrac{d}{dt}$ on $\mathbb{R}$. We need to consider to get an idea of the solutions studied in this report the equation

$$(D_t^2 A_0 + D_t A_1 + A_2)U(t) = 0, \tag{1.3}$$

where A_j for $j = 0, 1, ..., k$ is a non-negative self-adjoint operator in a Hilbert space $L^2(\mathbb{R})$ with the domain $\mathrm{Dom}(A) = H_2(\mathbb{R})$. We are interested in the solutions of equation (1.3) which have the form

$$U(t) = e^{i\mu_0 t} \sum_{j=0}^{k} \frac{(it)^j}{j!} u_{k-j} \tag{1.4}$$

where μ_0 is a complex number, $u_k \in L^2(\mathbb{R})$ for $k = 0, 1, 2$ and $u_0 \neq 0$. By inserting $U(t)$ in (1.3) we arrive at the equations for u_0, u_1, u_2 :

$$(\mu^2 A_0 + \mu A_1 + A_2)u_0 = 0. \tag{1.5}$$

Non-trivial solutions of (1.4) are called eigenvectors of the quadratic operator pencil

$$\mathbb{C} \ni \mu \to \mathcal{B}_A = \mu^2 A_0 + \mu A_1 + A_2 : H_2(\mathbb{R}) \to L^2(\mathbb{R}), \tag{1.6}$$

which correspond to the eigenvalue μ_0 of the pencil. By $\mathbb{C}$ we denote the set of complex numbers and by operator pencils we call polynomial operator pencil in $\mu \in \mathbb{C}$ with operator coefficient.

A similar description of all solutions can be given the general equation

$$(\sum_{j=0}^{k} A_{k-j} D_t^{k-j})U(t) = 0, \tag{1.7}$$

with constant operator coefficients acting in a pair of Hilbert spaces $H_k(\mathbb{R}) \to L^2(\mathbb{R})$ for $k = 0, 1, 2, ...\cdot$

We introduce the operator pencil $\mathcal{B}_A$ by:

$$\mathcal{B}_A : \mathbb{C} \to B(H_k(\mathbb{R}), L^2(\mathbb{R})),$$

which is defined by

$$\mathcal{B}_A(\mu) = \sum_{j=0}^{k} A_{k-j} \mu^j.$$

where $A_j \in B(H_k(\mathbb{R}), L^2(\mathbb{R}))$. (By $B(H_k(\mathbb{R}), L^2(\mathbb{R}))$ we mean the space of linear bounded operators acting from $H_k(\mathbb{R})$ to $L^2(\mathbb{R})$) for $k = 0, 1, 2, ...\cdot$

Matrix polynomials are a good example of an operator pencil that act in finite-dimensional Hilbert spaces. The literature on this topic is extensive, but see, for example, [13] and [25]. There have been several studies on *the spectral theory* which is the study of spectra and related properties of operators in infinite dimensional Hilbert spaces which dealt with self-adjoint operators that appear in quantum mechanics and, indeed, in classical mechanics for conservative systems. The spectrum of a self-adjoint operator is real, and the related questions of interest are ones of the existence of the lower bounds to the spectrum and of the essential spectrum, the number of negative eigenvalues, the possible existence of spectral gaps, etc. However, even when dealing with conservative systems, it is sometimes more natural and convenient to consider a quadratic operator pencil (see, for example, [6]).

In particular, in this report, we have an operator pencil

$$\mathcal{B}_A : \mathbb{C} \to B(H_2, H_0),$$

which is defined as:

$$\mathcal{B}_A(\mu) = \mu^2 + A - \lambda, \tag{1.8}$$

where the collection of Hilbert spaces

$$\{H_j\}_{j=0}^2$$

with norm $\|.\|_j$ such that $H_2 \subset H_1 \subset H_0$, , and that substitute λ for A in the definition of $\mathcal{B}_A$. The above embeddings are dense since H_2 is dense to H_0. See example page 44 of the current thesis. We suppose the operator A is a bounded operator from H_j into H_0 for $j = 0, 1, 2$ and a scalar $\mu_0 \in \mathbb{C}$ is called an eigenvalue of $\mathcal{B}_A$, if $\mathcal{B}_A(\mu_0)$ is not injective. Hence, the eigenvalue problem is to find μ_0 and $u \neq 0$ and $u \in H_2$, such that:

$$\mathcal{B}_A(\mu_0)u = 0. \tag{1.9}$$

A specific example of non-trivial operator A satisfying this condition: e.g. Volterra integral operator may be defined a function $u \in L^2[0, 1]$ and a value $t \in [0, 1]$ is defined by

$$V(u)(t) = \int_0^t u(s)ds.$$

V is a bounded liner operator between Hilbert spaces with adjoint

$$V^*(u)(t) = \int_t^1 u(s)ds.$$

V is a Hilbert-Schmid operator, hence in particular is compact.

$$\sigma(V) = \{\lambda \in \mathbb{C} | V - \lambda I \quad \text{is not invertable}\}.$$

V has no eigenvalue and by the spectral theory of compact operator, its spectrum $\sigma(V) = 0$. See [12].

These problems are used to study the dispersion and damping properties of waves [49]. In a physical sense, we consider the wave equation, especially on a waveguide, to be a good example of an operator pencil. We have the domain defined on the three dimension $(t, z, x) \in \mathbb{R} \times \mathbb{R} \times \Omega$ where $\Omega \subset \mathbb{R}^2$ is a bounded domain with a boundary $\partial\Omega$. For the scalar field $\Psi(t, z, x)$ we have the wave equation,

$$\partial_t^2 \Psi - \partial_z^2 \Psi - \nabla_\Omega^2 \Psi = 0,$$

that allows us to obtain the solution,

$$\Psi(t, z, x) = e^{i\omega t} e^{i\mu z} \psi(x),$$

which gives

$$\nabla_\Omega^2 \psi - \mu^2 \psi + \omega^2 \psi = 0.$$

The operator pencil is considered by $\mathcal{B}_{(-\nabla_\Omega^2)}(\mu)$. Similarly, the eigenvalue problem is to find μ such that

$$\mathcal{B}_{(-\nabla_\Omega^2)}(\mu) = ((\nabla_\Omega^2 - \mu^2 + \omega^2)\psi)_{\nabla_\Omega^2}(\mu) = 0,$$

where $\mu = \omega^2$. See [40].

Now, the spectrum $\sigma(\mathcal{B}_A(\mu))$ of this operator function is the set of all $\mu \in \mathbb{C}$ such that $\sigma(\mathcal{B}_A)$ is not invertible in $B(H_2, H_0)$ and the resolvent set is defined as the complement $\rho(\mathcal{B}_A) = \mathbb{C} \setminus \sigma(\mathcal{B}_A)$ (see [9] and [58] for further details). The geometric and algebraic multiplicity of any $\mu_0 \in \sigma(\mathcal{B}_A)$ can be defined as $\dim\ker \mathcal{B}_A(\mu_0)$ and the sum of the length of a set of maximal Jordan chains corresponding to μ_0, respectively. (See Section 4.4 of this thesis and [58], for details).

In order to define the function space on which A given in (1.8) and values of parameters α and β, which are related to approximate eigenvalues of an operator pencil $\mathcal{B}_A$, we need to introduce the exponential weighted function spaces modelled on Sobolev spaces to examine the operator pencil.

Definition 1.3.2. We have a finite norm,

$$\|u\|^2_{W^k_{\alpha,\beta}} := \sum_{j=0}^{k} \int_{-\infty}^{0} e^{2\alpha t} \left\| D_t^j u \right\|^2_{H_{k-j}} dt + \sum_{j=0}^{k} \int_{0}^{\infty} e^{2\beta t} \left\| D_t^j u \right\|^2_{H_{k-j}} dt;$$

where $W^k_{\alpha,\beta}$ denotes the set of $u : \mathbb{R} \to H_k$, for $k \in \mathbb{N}_0$, and $\alpha, \beta \in \mathbb{R}$.

For further details, the reader is referred to Section 4.2 of this thesis and [31], [50], [51], [66]. As a consequence of the above concepts, it is easy to show that for $\alpha, \beta \in \mathbb{R}$, the operator

$$\mathcal{B}_A(D_t) = D_t^2 + A - \lambda,$$

defines a bounded map $W^2_{\alpha,\beta} \to W^0_{\alpha,\beta}$. With a bit more work, it is also possible to show that the inclusion $W^2_{\alpha,\beta} \hookrightarrow W^1_{\alpha',\beta'}$ defines a compact map whenever $\alpha' > \alpha$ and $\beta' < \beta$, the proof of which is given in Section 4.6. The projection of $\sigma(\mathcal{B}_A)$ onto the imaginary axis is of particular importance. It plays a significant role in this research and is denoted by $\Gamma(\mathcal{B}_A)$, that is,

$$\Gamma(\mathcal{B}_A) = \{\Im\mu | \mu \in \sigma(\mathcal{B}_A)\} \subset \mathbb{R}.$$

The above discussion implies that $\Gamma(\mathcal{B}_A)$ consists of isolated points and, given $\gamma \in \Gamma(\mathcal{B}_A)$, the total algebraic multiplicity of all those $\mu \in \sigma(\mathcal{B}_A)$ with $\Im\mu = \gamma$ is finite. See Section 4.4 and [9], and [58].

We determine all eigenvalues of $\mathcal{B}_A$ inside a given *closed contour* denotes by S_R and guarantee that, at each stage of approximation the equations are as well-conditioned as the original eigenvalue problem, which is in the form of equation (1.9). Now, we consider a simple definition of the *resolvent operator* (inverse operator) to investigate various results in the following definitions, which the reader can find in Chapter 5.

Definition 1.3.3. Let Ω be a domain in Complex plane $\mathbb{C}$. An operator function

$$\Upsilon(\mu) : \Omega \to B(H_2, H_0)$$

is referred to as holomorphic on Ω when it can be represented as a power series

$$\Upsilon(\mu) = \sum_{j=0}^{\infty} \Upsilon_j (\mu - \mu_0)^j, \quad \Upsilon_j \in B(H_2, H_0),$$

which is convergent in $B(H_2, H_0)$ in the neighbourhood of $\mu_0 \in \Omega$ (see Section 5.2 and [58]).

Definition 1.3.4. The resolvent operator can be represented as

$$\mathcal{B}_A^{-1}(\mu) = \sum_{k=1}^{J} \sum_{h=0}^{m_k-1} \frac{P_{k,h}}{(\mu - \mu_0)^{m_k - h}} + \Upsilon(\mu), \tag{1.10}$$

where,

$$P_{k,h} = \sum_{s=0}^{h} \langle ., \psi_{k,s} \rangle_{H_0} \varphi_{k,h-s},$$

where $\varphi_{k,s}$ is a canonical system of Jordan of $\mathcal{B}_A$ corresponding to μ_0, and $\psi_{k,s}$ is a canonical system of jordan of $\mathcal{B}_A^*$ (Adjoint pencil which is defined in Section 4.5) corresponding to $\overline{\mu_0}$ for $k = 1, 2, ..., J$ and $s = 0, ..., m_k - 1$, and Υ is a holomorphic function in the neighbourhood of μ_0.

See theorem 5.2.3, [58] and [59].

In addition, for certain functional $e^{it\mu}$, one is required to evaluate the integrals

$$\frac{1}{2\pi} \int_{\Im\mu} e^{it\mu} \mathcal{B}_A^{-1}(\mu) d\mu,$$

which is called *Green's Kernel.* Throughout this research, the Green function associated with $\mathcal{B}_A$ helps, among other things, to study the spectra of $\mathcal{B}_A$. In particular, we construct the Green's function and obtain asymptotic formula of this function at infinity based on the definition of $\mathcal{B}_A^{-1}(\mu)$ (Theorem 5.2.3 of the current thesis). Furthermore, we have the following regularity result relating to $\Gamma(\mathcal{B}_A)$, where $\alpha, \tau \in \mathbb{R}$, $\alpha \notin \Gamma(\mathcal{B}_A)$, the inverse operator which is defined in (1.10) and a bounded map of $\mathcal{B}_A : W^2_{\alpha,\alpha} \to W^0_{\alpha,\alpha}$ to give that

$$u(t) = \int_{\mathbb{R}} G(t - \tau) f(\tau) d\tau,$$

for $f \in W^0_{\alpha,\alpha} = L^2(\mathbb{R}, H_0)$. The reader can find the proof this result in Section 5.3. We can consider the ordinary differential equation with constant operator coefficients,

$$\mathcal{B}_A(D_t)u = f, \tag{1.11}$$

Therefore, the Green's function $G(t)$, which is considered the main object to investigeation, is the solution of (1.11). In other words, the numerical method we used for our purpose is based on integrals of the generalised resolvent $\mathcal{B}_A^{-1}$ by using the construction Green's kernel. The state-of-the-art results in the contour integration-based methods for solving non-linear matrix eigenvalue problems are presented in [38] and [63], and references therein. However, the results of the contour integration based on the solution of the methods for eigenvalue problems can be found in [64]. We use the, as obtained from the asymptotic formula for Green's function at infinity based on definition (1.10), to achieve new asymptotic representation of this function in the following formulae in (1.12) and (1.13) as $t \to \pm\infty$ of power-exponential solutions of (1.11) in the Sobolev space $W^0_{\alpha,\beta}$. A new Green's kernel is defined by

$$G^{(\beta)}(t) = \frac{1}{2\pi}\int_{\Im\mu=\beta} e^{it\mu}\mathcal{B}_A^{-1}(\mu)d\mu,$$

see the definition in Section 5.3.2 of the current thesis. To understand this relation between $G(t)$ and $G^{(\beta)}(t)$, consider the following theorem:

Theorem 1.3.1. Suppose there are no eigenvalues of the operator pencil $\mathcal{B}_A$ on the lines $\Im\mu = \beta$, and $\sum_{\alpha_\pm} = \{\mu \in \sigma(\mathcal{B}_A) : \Im\mu \lesseqgtr \alpha\}$. Then,

$$G(t) - G^{(\beta)}(t) = \sum_{\mu\in\sum_{\alpha_+}} e^{i\mu t}P_v(t), \tag{1.12}$$

$$G(t) - G^{(\beta)}(t) = -\sum_{\mu\in\sum_{\alpha_-}} e^{i\mu t}P_v(t), \tag{1.13}$$

where the operator $P_v(t)$ is defined by

$$P_v(t) = \frac{1}{2\pi}\int_{S_v} e^{it(\mu-\mu_v)}\mathcal{B}_A^{-1}(\mu)d\mu.$$

where S_v denotes the small circle centred μ_v. Further details are given in Section 5.3.2.

1.3.2 Fredholm operator pencil $\mathcal{B}_A$

Now, we base the following arguments on the Fredholm property which is related to the spectra of pencils. First, we can consider an operator $A \in B(H_2, H_0)$ to be a *Fredholm operator* if the dimensions of its null space Ker(A)

and of the orthogonal complement of its range Co-Ker$(A) = \text{Ran}(A)^{\perp}$ are finite (see [13]). Let $\Phi(H_2, H_0)$ denote the set of all Fredholm operators, where the number

$$\text{Index}(A) = \dim(\text{Ker}(A)) - \text{Codim}(\text{Ran}(A)),$$

is called the *index* of A. Subsequently, we assume that $\mathcal{B}_A(\mu) \in \Phi(H_2, H_0)$ for all $\mu \in \mathbb{C}$. If, in addition, the resolvent set of such $\mathcal{B}_A(\mu)$ is non-empty, the analytic Fredholm theorem, for example, [53] implies that the generalised resolvent $\mu \to \mathcal{B}_A^{-1}(\mu)$ is finitely meromorphic. This in turn implies that the spectrum $\sigma(\mathcal{B}_A)$ is countable and the geometric multiplicity of μ_0, that is $(\dim(\text{Ker}(\mathcal{B}_A))$, is finite. Moreover, the associated Jordan chains of generalised eigenvectors have finite length bounded by the algebraic multiplicity. We refer the readers to [53], [58] and Section 5.2 of this thesis for further details. The above discussion helps to obtain results for the Fredholm property of pencils $\mathcal{B}_A$ and the Fredholm index. The projection of $\sigma(\mathcal{B}_A)$ onto the imaginary axis has been related to the mapping properties of operator $\mathcal{B}_A$; we have the following theorem:

Theorem 1.3.2. (Published in [43], April, 2021. (Under review)).

Let $\Gamma = \Gamma(\mathcal{B}_A)$ and $\alpha, \beta \in \mathbb{R} \setminus \Gamma$. Set $\delta = \text{dist}(\alpha, \Gamma) > 0$. Then the map

$$\mathcal{B}_A(D_t) = D_t^2 + A - \lambda : W^2_{\alpha,\beta} \longrightarrow W^0_{\alpha,\beta} \tag{1.14}$$

is an isomorphism map.

Refer the reader can see the prove of this theorem in Section 4.6 of this thesis. This result is a special case of a general theory that has been developed for differential equations with operator coefficients (see [9] and [58]). The fact that Theorem 1.3.2 (or Theorem 4.6.2 of the current thesis) does not extend to $\alpha, \beta \in \mathbb{R} \setminus \Gamma$ has to do with the existence of exponential solutions of $\mathcal{B}_A(\mu_0)u = 0$, for $u \in W^2_{\alpha,\alpha}$ and these solutions give the link between the isomorphisms for different values for α and β. We consider $\Sigma_{\alpha,\beta}$ to denote the linear span of the set of all exponential solutions corresponding to $\mu_0 \in \sigma(\mathcal{B}_A)$. However, from the result (Theorem 5.3.7), which offers a new representation of $G(t)$ (see Section 5.3), we offer the following proposition which is found as the difference of two solutions of to (1.11)

Proposition 1.3.3. Let $f \in W^0_{\alpha,\alpha} \cap W^0_{\beta,\beta}$, $u_\alpha \in W^2_{\alpha,\alpha}$ and $u_\beta \in W^2_{\beta,\beta}$ be the solutions of

$$A^{(\alpha)}u_\alpha = f \quad \text{and} \quad A^{(\beta)}u_\beta = f,$$

respectively. Then

$$u_\alpha(t) - u_\beta(t) = \sum_{\mu\in\Sigma\alpha,\beta} \sum_{h=0}^{m_k-1} \int_{\mathbb{R}} e^{i\mu_0(t-s)} P_{k,h} f(s) ds.$$

The reader can find the proof this result in Section 5.4 of the current thesis. The following results are used to generalise theorem 1.3.2 to deal with $\mathcal{B}_A$ mapping between spaces:

Corollary 1.3.4. If $\alpha \leq \beta \in \mathbb{R}$. Then,

$$W^0_{\alpha,\beta} = W^0_{\alpha,\alpha} \cap W^0_{\beta,\beta}$$

while,

$$W^0_{\beta,\alpha} = W^0_{\alpha,\alpha} + W^0_{\beta,\beta}.$$

See the proof in Section 4.6 of this thesis. However, the result that establishes the semi-Fredholm property (See Theorem 5.5.1 of the current thesis) needs to provide the following result and certain concepts.

Theorem 1.3.5. For $\alpha, \beta \in \mathbb{R} \setminus \Gamma$, choose $\alpha < \alpha^{'}$ and $\beta^{'} < \beta$. Then, there exists c and for all $u \in W^2_{\alpha,\beta}$, such that

$$\|u\|_{W^2_{\alpha,\beta}} \leq c[\|\mathcal{B}_A(D_t)u\|_{W^0_{\alpha,\beta}} + \|u\|_{W^1_{\alpha^{'},\beta^{'}}}].$$

The reader can find the proof this result in Section 4.6 of this thesis. According to the Fredholm property of operator pencils through the set Γ:

Theorem 1.3.6. Let $\alpha, \beta \in \mathbb{R} \setminus \Gamma(\mathcal{B}_A)$, Suppose

$$A^{(\alpha)} = \mathcal{B}_A(D_t) : W^2_{\alpha,\alpha} \to W^0_{\alpha,\alpha}$$

and

$$A^{(\beta)} = \mathcal{B}_A(D_t) : W^2_{\beta,\beta} \to W^0_{\beta,\beta}$$

are isomorphism. Then $A^{(\alpha)}$ and $A^{(\beta)}$ are Fredholm with index 0. See Section 5.5 of the current thesis.

We obtain another results for the Fredholm property and semi-Fredholm property in Section 5.5, but the argument of Fredholm index and its dependence on the parameters α and β is considered the principal result in this section, which is in the following theorem:

Theorem 1.3.7. (Published in [42], April, 2021).

Suppose $\alpha < \beta \in \mathbb{R}\backslash\Gamma$. Then the maps

$$A^{(\alpha,\beta)} = \mathcal{B}_A(D_t) : W^2_{\alpha,\beta} \longrightarrow W^0_{\alpha,\beta}$$

and

$$A^{(\beta,\alpha)} = \mathcal{B}_A(D_t) : W^2_{\beta,\alpha} \longrightarrow W^0_{\beta,\alpha}$$

are Fredholm with

$$\operatorname{Index} A^{(\alpha,\beta)} : W^2_{\alpha,\beta} \longrightarrow W^0_{\alpha,\beta} = -|\Sigma_{\alpha,\beta}| = -\operatorname{Index} A^{(\beta,\alpha)} : W^2_{\beta,\alpha} \longrightarrow W^0_{\beta,\alpha}.$$

See the proof of this Theorem or (Theorem 5.5.5 in Section 5.5) of the current thesis.

1.4 Contribution of the Thesis

Despite the existence of several studies in the above areas of this research, there is still the need for further research aimed at tackling the challenges presently faced when adopting this approach. These areas are dependent on the development of the idea of the classical theory of ordinary differential equations with operator coefficients. Of particular interest are problems involving the stability of "trapped modes", or eigenvalues embedded within continuous spectra. In this setting, we obtain the results of the following types, which parallel those of the standard theory of the ordinary differential equations with operator coefficients:

1. Development of the construction of trapped modes for acoustic waveguides given by Evans et al. (1991), where this construction can be adapted to produce examples of embedded eigenvalues for Laplace operators with potential function.

2. The typical results which show the existence of embedded eigenvalues when we demonstrate the operator $-\Delta - V$ on a cylindrical domain with different boundary conditions (Dirichlet or Neumann) such that V is a symmetric, positive, and continuous function.

3. Using the Cauchy' Residue Theorem and inverse operator pencil to obtain asymptotic representation for Green's kernel at infinity.

4. The typical results which demonstrate the Fredholm property for operator pencil, where this property acts between certain weighted function spaces.

5. Dependence of the Fredholm index on the parameters of the weighted function spaces.

The main objective of the first task is to develop aspects of the theory of differential equations, concentrating on particular examples of embedded eigenvalues; these include the stability of trapped modes within the continuous spectrum for operators. *Trapped modes* are localised oscilations which have finite energy and their existence in acoustic guides at wave numbers below the first antisymmetric cut-off has been well-documented (see [7]). For wave numbers above the cut-off the eigenvalue assioated with trapped mode is said to embdded in the continouous spectrum of the relevant operator. In a previous paper [15], it was demonstrated that existence of trapped modes is related to an eigenvalue of operator. In 1951, Ursell demonstrated the existence of trapped modes through a horizintal circular cylinder with a suffeciently small radius in water (see [22]). Jones used deep results on unbounded operator to extended Ursell's proof to a wide class of horizontal cylindrical obstacles in finite depths of water [14]. Jones' results, as applied to the water-wave problem, formed just a small part of his paper in which a number of results were obtained that showed the spectrum of the Laplace operator to satisfy the boundary conditions of semi-infinite domains [14]. Motivated by problems in water waves, a series of recent papers has been concerned with both demonstrating the existence of, and numerical algorithms for the computation of embedded eigenvalues (trapped modes) for different geometries. For example, Evans and Linton (1991) used some of the techniques described by the Ursell method (1951) to demonstrate the existence of trapped modes and provided a numerical technique for computing these modes in the vicinity of a vertical cylinder [16]. In 1993, Evan, Linton and Ursell considered the case of an abstract shape which can be described by two long parallel lines or walls of the channel, where it is not possible to seperate the problem into solutions (symmetric or antisymmetric) with respect to the centreplane, and showed that, in this case, a trapped mode could exist. Evan and Linton used a Green's function to construct a homogenenous equation for the trapped modes in the case of a cylinder and showed that the trapped modes frequencies agreed numerically with the previous results for the circular and recangular cross-section. Then, they identified these trapped mode frequencies as eigenvalues of the Laplace operator on an unbounded domain, and which established the existence of the smallest eigenvalue using the

Rayleigh quotient, (see Evans et al., 1993 [15]). Their results also prove the existence of trapped modes for thin obstacles aligned with the guide walls in higher dimensions. Further extensions to higher dimensions have been made by Linton and Mciver [7], who showed that trapped modes exist for axisymmetric bodies in cylindrical waveguides by exploiting the symmetry of the problem and looking for modes which have a specific azimuthal variation. However, in this work, we indicate in Chapter 3 how the method can be applied to the case of the the operator

$$-\Delta - V$$

with an additional positive symmetric potential function V on a cylindrical domain $\mathbb{R} \times [-L, L]$ with Dirichlet, or Neumann boundary conditions on $\mathbb{R} \times \{-L\}$ and $\mathbb{R} \times \{L\}$; that is, we successfully combined all ideas discussed in the thesis and proved for both cases that our operator has an embedded eigenvalue. This approach was applicable, for the sufficiently small V which satisfies the symmetry condition, when the operator $-\Delta - V$ has an eigenvalue λ which is contained in the essential spectrum in the cylindrical domain. Moreover, if no restrictions are placed on the symmetry of the solutions then the trapped modes occur at frequencies that correspond to eigenvalues that are embedded in the continuous spectrum of the Laplace operator with V. However, if the structure is symmetric about the centreline of the channel and the motion is split into symmetric and antisymmetric parts, then the operator may be decomposed, so that the essential spectrum of the antisymmetric part has a non-zero lower limit and the trapped mode corresponds to an eigenvalue which is below this value. In this case, standard variational methods may be used to prove the existence of trapped modes. The numerical method employed to determine the trapped mode frecunecies uses the ideas of Evans and Porter [17]. This result, which is developed in Section 3.9, predicts that the method can be applied to the case of a two-dimensional acoustic waveguide that can support trapped modes. Future work will investigate the structure in detail for the case where the deformed obstacles are of different geometries.

The second task of this thesis, as discussed in Chapters 4 and 5, is the consideration of the Fredholm operator and its properties with regards to pencils. The motivation behind this work came from applications to the mechanics and electrodynamics of continua. Fredholm operators, which were introduced by the Swedish mathematician Erik Ivar Fredholm (1866 − 1927), are useful for treatment perturbation problems that can be expressed as compact perturbations of invertible operators (see, for example, [40]).

However, there appears some areas of this task that are related to the results presented here. The earliest of these studies were focused on the problem of operator pencils on a domain with singularity on the boundary that appeared in [60]. The general approach to these problems was refined on the Sobloev spaces and was developed by differnt authours, for example, Maz'ya and Kozlov [58] and [60].

Furthermore, in Chapter 4, the fundamental results of the present research are related to the investigation of the quadratic operator pencil

$$\mathcal{B}_A(\mu) = \mu^2 + A - \lambda.$$

Theorems 4.6.1 and 4.6.2 are proved with regard to some of the properties of pencils. The latter (Theorem 4.6.2) is a special case of a general theory that has been developed for differential equations with operator coefficients (see [58]); for the operator pencil $\mathcal{B}_A$ it can also be obtained directly with elementary arguments if one moves to fourier space (see [10], [11] and [58]). We then determined some of the associated consequences, for examples, 4.6.4 and 4.6.6 of the current thesis could help to generalise Theorem 4.6.2. These consequences were introduced by Elton (see [10]), and were proved in the current thesis. The present work extends the existing work in its consideration of more general types of function spaces. Apart from filing numerous gaps in the existing collection of results, numerous new type of spaces are considered; perhaps the most important of these are the weighted function spaces $W^k_{\alpha,\beta}$ Sobolev spaces of arbitrary real order; see Section 4.2 for further information.

Finally, this task seeks to develop an approach considered in Chapter 5. Although the presumed existence of parallel results for the Fredholm properties of pencils has been remarked upon by several studies (see, for example, [9] and [58]); in this task, we deal with solutios of equation, where we systematically employ basic facts about the theory of Fredholpm operator pencils. One meet such power-exponential solutions in basic courses on ordinary differential equation with either scalar or matrix coefficient see [58]. In the infinite dimentional cases thes solutions also play an important role. In particular, they determine the asymptotics at infinity of arbitrary solutions and are used for constructing of the Green's kernel; see Maz'ya and Kozlov [58]. Furthermore, we have representations of Green's kernels of different types, where these new representations can be used to find the solutions of $\mathcal{B}_A(D_t)u = f$ in Sobolev space $W^0_{\alpha,\beta}$. The Fredholm property, which is related to the spectrum of the associated operator pencil $\mathcal{B}_A$ through the set of $\Gamma(\mathcal{B}_A)$ with the specialisation of the results of developing some arguments of Fredholm operators' indices with their applications, were developed by Kozlov, Maz'ya, and Rossmann in [59] and Elton in [9]. Here, we made the case of the Fredholm property of pencils dependent determination of the parameters α and β which move between components of $\mathbb{R} \setminus \Gamma$. Furthermore, there was a simplified setting which allowed for a simpler argument; for example, kernel and co-kernel can be calculated explicitly. This means that the Fredholm operator can be established directly and the index calculated without the need to consider an adjoint operator (see Theorem 5.5.5) in the current thesis. The techniques of the above work seems to be well suited to operators which appropriate bounded perturbation of differentiaion in one variable and even allows for the computations of the index and the charctrerisation of the kernel and co-kernels in certain cases. However these tchniques do not appear to generlise easily to cover differnt operators. In [9] there was an attempt to use techniques similar to those employed here, and Theorems 5.5.2, and 5.5.5 were established for the weighted function space $W^0_{\alpha,\beta}$. In this thesis, as in the majority of the literature cited above, the necessary Fredholm properties of pencils are proved locally with weighted function space results. Due to the more general function space setting of the present work, a more complete set of related weighted function space results has been obtained here. However, it should be pointed out that many of these results probably appear in the function space literature.

1.5 Structure of the Thesis

This thesis contains eight chapters with four main chapters $2, 3, 4$ and 5. Chapter 1 is the introductory chapter where we discuss the relevant historical and theoretical basis of our research.

Chapter 2 is primarily concerned with the study of the some fundamental notations that will be used during this thesis. After recalling the notions of bounded, unbounded and closed operators on a Hilbert space H, we present the basic concepts of adjoint for unbounded operator, and symmetric operators. Then, we focus on the spectrum of an operator with certain properties. We give the definition of an essential spectrum with an important result which is related to this kind of this spectrum. In Section 2.7, we present a compact operator with certain results that will be used during this thesis. Finally, we discuss Friedrichs extension theorem and its properties to build the Laplace operator.

The next chapter, Chapter 3 contains our first original work. We used the tools introduced in Chapter 2 to help investigate certain results. We introduce the concept of boundary conditions with a focus on the spectrum of the Laplacian for a bounded domain. Moreover, we give some examples on different domains (bounded and unbounded domains to compute the eigenvalues and eigenfunctions). We give the definition of the Symmetry operator. We detail the formulation of variational principle to calculate the following inequality:

$$\inf(\sigma(-\Delta - V)) < \lambda_0$$

in Section 3.7. In Section 3.8, there is an important theorem of the first task which is proved a relatively compact function V with respect to the Laplace operator in the Hilbert space $L^2(\Omega)$ where $\Omega \subset \mathbb{R}^d$ for $d \geq 1$. In Section 3.9, we demonstrate that the operator

$$-\Delta - V$$

on $\mathbb{R} \times [-L, L]$ has embedded eigenvalues for a sufficiently small real valued non-negative continuous function with bounded support and which is a symmetric function.

In Chapter 4, we give the definitions of Sobolev spaces and operator Pencil with some fundamental results. Section 4.1 defines the space H_k for $k = 0, 1, 2, ...$ with some of its properties. The majority of Section 4.2, is devoted to establishing the basic properties of the Sobolev spaces $W^k_{\alpha,\beta}$ for $k \in \mathbb{N}_0$ and $\alpha, \beta \in \mathbb{R}$ that are necessary in order to work with them. The operator pencil to which our results apply is introduced in Section 4.4 (see Definition 4.4.1). We then consider the adjoint of pencils with some of their properties. At the end of this chapter, we give the result, which is Theorem 1.3.1 (or Theorem 4.6.2). Additionally, it is shown that certain arguments can be used to help generalise previous theorem to deal with operator pencil mapping between Sobolev spaces in Corollary 1.3.4 (or Corollaries 4.6.4, and 4.6.6).

In Chapter 5, we give the definition of Fredholm operator and define its properties. Later, we give the definition of the resolvent operator of Fredholm operator pencil $\mathcal{B}_A^{-1}(\mu)$. We focus on the Green's function $G(t)$ and its properties, which are played a significant role in this thesis. Then, we obtain the asymptotic the formula of the Green's function at infinity based on Theorem 5.2.3 and we observe certain results that help to achieve a new asymptotic representation of this function as $t \to \pm\infty$ to the exponential solution of $\mathcal{B}_A(D_t)u = f$ in the Sobolev space $W^0_{\alpha,\beta}$. In the final section, we give some of the results of the semi-Fredholm property (see theorem 5.5.1) and index formula of the Fredholm operator pencils $\mathcal{B}_A$, which forms the basis of this chapter (see theorem 5.5.5).

In Chapter 6, there is an appendix which gives further examples.

In Chapter 7, we give a review of what we have achieved in this thesis, outline the limitations of the study and draw conclusions on our finding.

In Chapter 8, we have list of the publications including three internal thesis and publications and one external research.

Chapter 2

Fundmental Ideas and Preliminaries

In this chapter, we define the space H^k where $k = 0, 1, 2, ...$ in Section 2.1. In Section 2.2, we give some lemmas and theorems for the Fourier transform and inverse Fourier transform on $\mathbb{R}$ and $\mathbb{R}^d$ for $d \geq 1$. We give definitions of bounded and unbounded linear operators on the Hilbert space H in Section 2.3. In Section 2.4, we define the self-adjoint for unbounded operators, symmetric operators and essentially self-adjoint operators. We define the spectrum of the linear operator in Section 2.5. The definition of Essential spectrum is given in Section 2.6. In Section 2.7, we define a compact operator with some properties. In Section 2.8, we observe the Friedrichs extension theorem and the representation theorem that characterises the linear operator in the Hilbert space H.

2.1 Notations

To avoid confusion, we begin by making explicit some notations that will be frequently used during the current thesis. Let $L^2(\mathbb{R})$ be the Hilbert space of complex-valued square integrable functions on $\mathbb{R}$ with inner product

$$\langle u, v \rangle_{L^2(\mathbb{R})} = \int_{\mathbb{R}} \overline{u(t)} v(t) dt,$$

and where the norm is defined by

$$\|u\|^2_{L^2(\mathbb{R})} = \int_{\mathbb{R}} |u(t)|^2 dt.$$

We will also encounter the space in this research, denoted as $H^1(\mathbb{R})$, which is similar to the space of complex valued function and is refinement of $L^2(\mathbb{R})$ such that a function $u : \mathbb{R} \to \mathbb{C}$ is said to be in $H^1(\mathbb{R})$ if $u \in L^2(\mathbb{R})$ and its distributional gradient ∇u is a function that is in $L^2(\mathbb{R})$. Here, $\mathbb{C}$ is the field of complex numbers. Now, the inner product and norm of $H^1(\mathbb{R})$ are given by respectively,

$$\begin{aligned}\langle u, v\rangle_{H^1(\mathbb{R})} &= \langle u, v\rangle_{L^2(\mathbb{R})} + \langle \nabla u, \nabla v\rangle_{L^2(\mathbb{R})}\\ &= \int_{\mathbb{R}} \overline{u(t)} v(t) dt + \int_{\mathbb{R}} \overline{\nabla u(t)} \nabla v(t) dt,\end{aligned}$$

and

$$\begin{aligned}\|u\|^2_{H^1(\mathbb{R})} &= \|u\|^2_{L^2(\mathbb{R})} + \|\nabla u\|^2_{L^2(\mathbb{R})}\\ &= \int_{\mathbb{R}} |u(t)|^2 dt + \int_{\mathbb{R}} |\nabla u(t)|^2 dt.\end{aligned}$$

See [18], [44] and [52].

Remark 1. We have the following notes:

i) In this research, we encounter the space $L^\infty(\mathbb{R})$ which is defined a Banach space of complex-valued functions on $\mathbb{R}$ and its norm is defined by

$$\|u\|^2_{L^\infty(\mathbb{R})} = \sup_{t\in\mathbb{R}} |u(t)|,$$

where a function $u : \mathbb{R} \to \mathbb{C}$.

ii) The form $\langle ., . \rangle$ is antilinear in the first argument and linear in the second argument (see [18], [44] and [52]).

Now, we can define the space H^1 on $\mathbb{R}^d$ for $d \geq 1$.

Definition 2.1.1. The space $H^1(\mathbb{R}^d)$ for $d \geq 1$ can be defined as

$$H^1(\mathbb{R}^d) = \left\{ u \in L^2(\mathbb{R}^d) : \frac{\partial u}{\partial t_i} \in L^2(\mathbb{R}^d), i = 1, 2, ..., d \right\},$$

where $\frac{\partial u}{\partial t_i}$ is the distributional derivative. The space $H^1(\mathbb{R}^d)$ is equipped with the norm

$$\begin{aligned}\|u\|^2_{H^1(\mathbb{R}^d)} &= \|u\|^2_{L^2(\mathbb{R}^d)} + \sum_{i=1}^{d} \left\| \frac{\partial u}{\partial t_i} \right\|^2_{L^2(\mathbb{R}^d)}\\ &= \int_{\mathbb{R}^d} |u(t)|^2 dt + \sum_{i=1}^{d} \int_{\mathbb{R}^d} \left| \frac{\partial u}{\partial t_i} \right|^2 dt,\end{aligned}$$

(see [44], [50], [51] and [52]). Moreover, for $k = 0, 1, 2, ...$ the following definition is the generalisation of the previous notation where we define the space H^k on $\mathbb{R}$ and $\mathbb{R}^d$ for $d \geq 1$.

Definition 2.1.2. The space H^k where $k = 0, 1, 2, ...$ is defined by

$$H^k(\mathbb{R}) = \left\{u \in L^2(\mathbb{R}) : \nabla^j u \in L^2(\mathbb{R}), \quad 0 \leq j \leq k\right\},$$

and the norm of this space is defined by

$$\|u\|^2_{H^k(\mathbb{R})} = \sum_{0\leq j\leq k} \|\nabla^j u\|^2_{L^2(\mathbb{R})} = \sum_{0\leq j\leq k} \int_{\mathbb{R}} |\nabla^j u(t)|^2 dt,$$

where a distributional derivative as ∇^j for $0 \leq j \leq k$. In particular, $H^0(\mathbb{R}) = L^2(\mathbb{R})$, (see [18], [44] and [52]).

Definition 2.1.3. For $d \geq 1$, a distributional derivative can be represented as

$$D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial t_1^{\alpha_1} \partial t_2^{\alpha_2} .....\partial t_d^{\alpha_d}},$$

for a multi-index α which is a vector in $\mathbb{N}_0^d$. We can write

$$\alpha = (\alpha_1, \alpha_2, ..., \alpha_d),$$

and the *degree* of α as defined to be

$$|\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_d.$$

The space H^k where $k = 0, 1, 2, ...$ on $\mathbb{R}^d$ is defined by

$$H^k(\mathbb{R}^d) = \left\{u \in L^2(\mathbb{R}^d) : D^\alpha u \in L^2(\mathbb{R}^d) \quad \text{for all } |\alpha| \leq k\right\},$$

and the norm of this space is defined by

$$\|u\|^2_{H^k(\mathbb{R}^d)} = \sum_{0\leq|\alpha|\leq k} \int_{\mathbb{R}^d} |D^\alpha u|^2 dt = \sum_{0\leq|\alpha|\leq k} \|D^\alpha u\|^2_{L^2(\mathbb{R}^d)}.$$

See [44], [50], [51] and [52].

2.2 Fourier Transform in $\mathbb{R}$ and $\mathbb{R}^d$

In this section, we define the Fourier transform and its inverse in $\mathbb{R}$. Then, we give some lemmas and some theorems of the Fourier transform and inverse Fourier transform in $\mathbb{R}^d$ for $d \geq 1$.

Remark 2. The Fourier transform of the function u for $\tau \in \mathbb{R}$ is defined as

$$\widehat{u}(\tau) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-it\tau} u(t) dt,$$

and the inverse Fourier transform will be

$$u(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{it\tau} \widehat{u}(\tau) d\tau.$$

The Plancherel theorem for the functions in $L^2(\mathbb{R})$ gives

$$\|u\|^2_{L^2(\mathbb{R})} = \int_{\mathbb{R}} |u(t)|^2 dt = \int_{\mathbb{R}} |\widehat{u}(\tau)|^2 d\tau = \|\widehat{u}\|^2_{L^2(\mathbb{R})}.$$

See [50], [51] and [62]. The following result is considered useful to prove some arguments in the current thesis.

Lemma 2.2.1. For $\tau \in \mathbb{R}$, then

$$\widehat{\nabla^k u}(\tau) = (i\tau)^k \widehat{u}(\tau),$$

where ∇ is a distributional gradient.

Proof. We use the mathematical induction to prove that

$$\widehat{\nabla^k u}(\tau) = (i\tau)^k \widehat{u}(\tau).$$

We prove that the statement is true for $k = 0$, which is trivial. Assuming that the statement is true for $k - 1$, we have

$$\widehat{\nabla^{k-1} u}(\tau) = (i\tau)^{k-1} \widehat{u}(\tau).$$

Now, we will prove that the statement is true for k. Integrating by parts, we get

$$\begin{aligned} \widehat{\nabla^k u}(\tau) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \nabla^k u(t) e^{-it\tau} dt &= \frac{1}{\sqrt{2\pi}} (\nabla^{k-1} u(t) e^{-it\tau}) \mid_{-\infty}^{\infty} + \frac{i\tau}{\sqrt{2\pi}} \int_{\mathbb{R}} \nabla^{k-1} u(t) e^{-it\tau} dt \\ &= 0 + (i\tau) \widehat{\nabla^{k-1} u}(\tau) \\ &= (i\tau)^k \widehat{u}(\tau). \end{aligned}$$

Therefore,

$$\widehat{\nabla^k u}(\tau) = (i\tau)^k \widehat{u}(\tau),$$

for $\tau \in \mathbb{R}$. □

Now, the following Lemma proves that the equivalent norms of spaces.

Lemma 2.2.2. $u \in H^k(\mathbb{R})$ if and only if $(1+\tau^2)^{\frac{k}{2}}\widehat{u} \in L^2(\mathbb{R})$ and the norms $\|u\|_{H^k(\mathbb{R})}$ and

$$\left[\int_{\mathbb{R}} (1+\tau^2)^k |\widehat{u}(\tau)|^2\right]^{\frac{1}{2}} d\tau$$

are equivalent.

Proof. We need to find the constants c_1 and c_2 such that

$$c_1 \sum_{r=0}^{k} |\tau^r|^2 \leq (1+\tau^2)^k \leq c_2 \sum_{r=0}^{k} |\tau^r|^2. \tag{2.1}$$

We have

$$\sum_{r=0}^{k} |\tau^r|^2 = \sum_{r=0}^{k} |\tau^{2r}| = \sum_{r=0}^{k} \tau^{2r} \leq \sum_{r=0}^{k} \binom{k}{r} \tau^{2r} = (1+\tau^2)^k = \sum_{r=0}^{k} \binom{k}{r} \tau^{2r} \leq 2^k \sum_{r=0}^{k} \tau^{2r},$$

so we can take $c_1 = 1$ and $c_2 = 2^k$ where $1 \leq \binom{k}{r} \leq 2^k$ because $\binom{k}{r} \leq \sum_{s=0}^{k} \binom{k}{s} = (1+1)^k = 2^k$. Now, to complete this proof, we use the Plancherel theorem for functions in $L^2(\mathbb{R})$ for $k = 0, 1, 2, ...$;

$$\begin{aligned}
\|u\|^2_{H^k(\mathbb{R})} &= \int_{\mathbb{R}} |\nabla^k u(t)|^2 dt + \int_{\mathbb{R}} |\nabla^{k-1} u(t)|^2 dt + ... + \int_{\mathbb{R}} |u(t)|^2 dt \\
&= \|\nabla^k u\|^2_{L^2(\mathbb{R})} + \|\nabla^{k-1} u\|^2_{L^2(\mathbb{R})} + ... + \|u\|^2_{L^2(\mathbb{R})} \\
&= \|\widehat{\nabla^k u}\|^2_{L^2(\mathbb{R})} + \|\widehat{\nabla^{k-1} u}\|^2_{L^2(\mathbb{R})} + ... + \|\widehat{u}\|^2_{L^2(\mathbb{R})}.
\end{aligned}$$

By Lemma 2.2.1, to get that,

$$\begin{aligned}
&\|\widehat{\nabla^k u}\|^2_{L^2(\mathbb{R})} + \|\widehat{\nabla^{k-1} u}\|^2_{L^2(\mathbb{R})} + ... + \|\widehat{u}\|^2_{L^2(\mathbb{R})} \\
&\qquad = \int_{\mathbb{R}} |\widehat{\nabla^k u}(\tau)|^2 d\tau + \int_{\mathbb{R}} |\widehat{\nabla^{k-1} u}(\tau)|^2 d\tau + ... + \int_{\mathbb{R}} |\widehat{u}(\tau)|^2 d\tau \\
&\qquad = \int_{\mathbb{R}} (1 + \tau^{2k} + \tau^{2k-2} + ... + \tau^2) |\widehat{u}(\tau)|^2.
\end{aligned}$$

However, by using (2.1)

$$\begin{aligned}
\int_{\mathbb{R}} (1 + \tau^{2k} + \tau^{2k-2} + ... + \tau^2) |\widehat{u}(\tau)|^2 &\leq 1 \int_{\mathbb{R}} (1+\tau^2)^k |\widehat{u}(\tau)|^2 d\tau \\
&= \|(1+\tau^2)^{\frac{k}{2}} \widehat{u}\|^2_{L^2(\mathbb{R})}.
\end{aligned}$$

Similarly, on the other side

$$\int_{\mathbb{R}} (1+\tau^{2k}+\tau^{2k-2}+...+\tau^2)|\widehat{u}(\tau)|^2 \geq 2^{-k}\int_{\mathbb{R}}(1+\tau^2)^k|\widehat{u}(\tau)|^2 d\tau$$
$$= 2^{-k}\|(1+\tau^2)^{\frac{k}{2}}\widehat{u}\|^2_{L^2(\mathbb{R})}.$$

□

Now, the above definition of the Fourier transform can be generalised in $\mathbb{R}^d$ for $d \geq 1$. The Fourier transform $\widehat{u}(\tau)$ is defined by

$$\widehat{u}(\tau) = \frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d} e^{-i\tau\cdot t}u(t)dt,$$

and the inverse Fourier transform will be

$$u(t) = \frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d} e^{i\tau\cdot t}\widehat{u}(\tau)d\tau,$$

where τ, $t \in \mathbb{R}^d$ and $\tau \cdot t$ is the dot product of these vectors. The dot product is sometimes written as $\langle \tau, t\rangle$. The Plancherel theorem for the functions in $L^2(\mathbb{R}^d)$ gives

$$\|u\|^2_{L^2(\mathbb{R}^d)} = \int_{\mathbb{R}^d}|u(t)|^2dt = \int_{\mathbb{R}^d}|\widehat{u}(\tau)|^2d\tau = \|\widehat{u}\|^2_{L^2(\mathbb{R}^d)}.$$

Remark 3. If $\alpha = (\alpha_1, \alpha_2, ...\alpha_d) \in \mathbb{N}_0^d$ and $\tau \in \mathbb{R}^d$, set

$$\tau^\alpha = \prod_{j=1}^{d} \tau_j^{\alpha_j}.$$

Lemma 2.2.3. If $u \in H^k(\mathbb{R}^d)$ with multi-index α and $\tau \in \mathbb{R}^d$, then

$$\widehat{D^\alpha u}(\tau) = (i\tau)^\alpha \widehat{u}(\tau).$$

Proof. By definition 2.1.3, we have $\widehat{D^\alpha u}$, and use integration by parts

$$\widehat{D^\alpha u}(\tau) = \frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d} D^\alpha u(t)e^{-i\tau\cdot t}dt$$
$$= \frac{(-1)^{|\alpha|}}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d} u(t)D^\alpha e^{-i\tau\cdot t}dt,$$

Then, we have

$$\begin{aligned}
\widehat{D^\alpha u}(\tau) &= \frac{(-1)^{|\alpha|}}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} u(t) \frac{\partial^{|\alpha|} e^{-i\tau\cdot t}}{\partial t_1^{\alpha_1} \partial t_2^{\alpha_2} .....\partial t_d^{\alpha_d}} dt \\
&= \frac{(-1)^{|\alpha|}}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} u(t) \frac{\partial^{|\alpha|} (e^{-i\tau_1 t_1} e^{-i\tau_2 t_2} .....e^{-i\tau_d t_d})}{\partial t_1^{\alpha_1} \partial t_2^{\alpha_2} .....\partial t_d^{\alpha_d}} dt \\
&= \frac{(-1)^{|\alpha|}}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} u(t) \frac{\partial^{\alpha_1} e^{-i\tau_1 t_1}}{\partial t_1^{\alpha_1}} \frac{\partial^{\alpha_2} e^{-i\tau_2 t_2}}{\partial t_2^{\alpha_2}} ..... \frac{\partial^{\alpha_d} e^{-i\tau_d t_d}}{\partial t_d^{\alpha_d}} dt \\
&= \frac{(-1)^{|\alpha|}}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} u(t) (-i\tau)^{\alpha_1} e^{-i\tau_1 t_1} (-i\tau)^{\alpha_2} e^{-i\tau_2 t_2} .....(-i\tau)^{\alpha_d} e^{-i\tau_d t_d} dt.
\end{aligned}$$

To complete the proof, we use Remark 3 to get

$$\begin{aligned}
\widehat{D^\alpha u}(\tau) &= \frac{(-1)^{|\alpha|}}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} u(t)(-i)^{\alpha_1+\alpha_2+....+\alpha_d} \prod_{j=1}^{d} \tau_j^{\alpha_j} e^{-i(\tau_1 t_1 + \tau_2 t_2 + ...+\tau_d t_d)} dt \\
&= \frac{(-1)^{|\alpha|}}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} u(t)(-i)^{\alpha_1+\alpha_2+....+\alpha_d} \prod_{j=1}^{d} \tau_j^{\alpha_j} e^{-i\tau\cdot t} dt \\
&= (-1)^{|\alpha|}(-i)^{|\alpha|} \prod_{j=1}^{d} \tau_j^{\alpha_j} \widehat{u}(\tau) \\
&= (i)^{|\alpha|} \tau^\alpha \widehat{u}(\tau) \\
&= (i\tau)^\alpha \widehat{u}(\tau).
\end{aligned}$$

□

The following lemma is a generalisation of Lemma 2.2.2.

Lemma 2.2.4. There are constants c_1 and c_2, such that

$$c_1 \|u\|^2_{H^k(\mathbb{R}^d)} \le \|(1+\tau^2)^{\frac{k}{2}} \widehat{u}(\tau)\|^2_{L^2(\mathbb{R}^d)} \le c_2 \|u\|^2_{H^k(\mathbb{R}^d)},$$

holds for all $u \in H^k(\mathbb{R}^d)$.

Proof. We find the constants c_1 and c_2 such that

$$c_1 \sum_{|\alpha|\le k} |\tau^\alpha|^2 \le (1+|\tau|^2)^k \le c_2 \sum_{|\alpha|\le k} |\tau^\alpha|^2 \tag{2.2}$$

for $\tau \in \mathbb{R}^d$. Writing $\alpha = (j_1, j_2, ..., j_d)$ and setting $j_0 = k - |\alpha|$, we get $|\alpha| = j_1 + j_2 + ... + j_d$ and $k = j_0 + |\alpha|$.

It follows that $k = j_0 + j_1 + j_2 + ... + j_d$, so we have

$$\sum_{|\alpha|\le k} |\tau^\alpha|^2 = \sum_{|\alpha|\le k} \tau^{2\alpha} = \sum_{\substack{j_0,j_1,\dots,j_d\ge 0\\ j_0+j_1+\dots+j_d=k}} (\tau_1^2)^{j_1}(\tau_2^2)^{j_2}...(\tau_d^2)^{j_d}.$$

The observation $1 \le \binom{k}{j_0,j_1,j_2,...,j_d}$ and using the multinomial theorem, we get

$$\begin{aligned}
\sum_{\substack{j_0,j_1,\dots,j_d\ge 0\\ j_0+j_1+\dots+j_d=k}} (\tau_1^2)^{j_1}(\tau_2^2)^{j_2}...(\tau_d^2)^{j_d} &\le \sum_{\substack{j_0,j_1\dots,j_d\ge 0\\ j_0+j_1+\dots+j_d=k}} \binom{k}{j_0,j_1,j_2,...,j_d} 1^{j_0}(\tau_1^2)^{j_1}(\tau_2^2)^{j_2}...(\tau_d^2)^{j_d} \\
&= \sum_{\substack{j_0,j_1\dots,j_d\ge 0\\ j_0+j_1+\dots+j_d=k}} \frac{k!}{j_0!j_1!j_2!...j_d!} 1^{j_0}(\tau_1^2)^{j_1}(\tau_2^2)^{j_2}...(\tau_d^2)^{j_d} \\
&= (1+\tau_1^2+\tau_2^2+...+\tau_d^2)^k \\
&= (1+\tau^2)^k,
\end{aligned}$$

and we can take $c_1 = 1$.

On the other hand, by using the multinomial theorem again, we have

$$\begin{aligned}
(1+\tau^2)^k &= (1+\tau_1^2+\tau_2^2+...+\tau_d^2)^k \\
&= \sum_{\substack{j_0,j_1\dots,j_d\ge 0\\ j_0+j_1+\dots+j_d=k}} \binom{k}{j_0,j_1,j_2,...,j_d} 1^{j_0}(\tau_1^2)^{j_1}(\tau_2^2)^{j_2}...(\tau_d^2)^{j_d} \\
&\le (1+d)^k \sum_{|\alpha|\le k} |\tau^\alpha|^2,
\end{aligned}$$

where we have

$$\binom{k}{j_0,j_1,j_2,.....,j_d} \le \sum_{\substack{n_0,n_1\dots,n_d\ge 0\\ n_0+n_1+\dots+n_d=k}} \binom{k}{n_0,n_1,n_2,...,n_d} (1^{n_0}1^{n_1}1^{n_2}...1^{n_d}) = (1+d)^k,$$

and can take $c_2 = (1+d)^k$.

Now, we have,

$$\|u\|^2_{H^k(\mathbb{R}^d)} = \sum_{|\alpha|\le k} \|D^\alpha u\|^2_{L^2(\mathbb{R}^d)}.$$

By using the Plancherel theorem for functions in $L^2(\mathbb{R}^d)$ and Lemma 2.2.1, we get that

$$\begin{aligned}
\|u\|^2_{H^k(\mathbb{R}^d)} &= \sum_{|\alpha|\leq k} \|\widehat{D^\alpha u}\|^2_{L^2(\mathbb{R}^d)} \\
&= \sum_{|\alpha|\leq k} \int_{\mathbb{R}^d} |\widehat{D^\alpha u}(\tau)|^2 d\tau \\
&= \sum_{|\alpha|\leq k} \int_{\mathbb{R}^d} |(i\tau)^\alpha \widehat{u}(\tau)|^2 d\tau \\
&= \int_{\mathbb{R}^d} \left(\sum_{|\alpha|\leq k} |(i\tau)^\alpha|^2 \right) |\widehat{u}(\tau)|^2 d\tau \\
&= \int_{\mathbb{R}^d} \left(\sum_{|\alpha|\leq k} |\tau^\alpha|^2 \right) |\widehat{u}(\tau)|^2 d\tau.
\end{aligned}$$

From (2.2), it follows that

$$\begin{aligned}
\int_{\mathbb{R}^d} \left(\sum_{|\alpha|\leq k} |\tau^\alpha|^2 \right) |\widehat{u}(\tau)|^2 d\tau \leq 1 \int_{\mathbb{R}^d} (1+\tau^2)^k |\widehat{u}(\tau)|^2 d\tau \\
= \|(1+\tau^2)^{\frac{k}{2}} \widehat{u}\|^2_{L^2(\mathbb{R}^d)},
\end{aligned}$$

and similarly,

$$\begin{aligned}
\int_{\mathbb{R}^d} \left(\sum_{|\alpha|\leq k} |\tau^\alpha|^2 \right) |\widehat{u}(\tau)|^2 d\tau \geq \frac{1}{(1+d)^k} \int_{\mathbb{R}^d} (1+|\tau^2|)^k |\widehat{u}(\tau)|^2 d\tau \\
= (1+d)^{-k} \|(1+\tau^2)^{\frac{k}{2}} \widehat{u}\|^2_{L^2(\mathbb{R}^d)}.
\end{aligned}$$

Thus, the proof is complete. □

2.2.1 Example

In this section, we will define an isomorphism map on H^k for $k = 0, 1, ...$ by giving example, this definition will be used to investigate some results during the current thesis. The following map:

$$\nabla - 1 : H^1(\mathbb{R}) \to H^0(\mathbb{R}) = L^2(\mathbb{R}),$$

is an isomorphism map, by Lemma 2.2.1, we have

$$\widehat{(\nabla - 1)u} = \widehat{\nabla u} - \widehat{u} = (i\tau - 1)\widehat{u}(\tau).$$

Now, we can get

$$\|\widehat{(\nabla - 1)}u\|^2_{L^2(\mathbb{R})} = \int_{\mathbb{R}} |(i\tau - 1)\widehat{u}(\tau)|^2 d\tau.$$

It would imply,

$$\begin{aligned}
\|\widehat{(\nabla - 1)}u\|^2_{L^2(\mathbb{R})} &= \int_{\mathbb{R}} |(i\tau - 1)|^2 |\widehat{u}(\tau)|^2 d\tau \\
&= \int_{\mathbb{R}} (|\tau|^2 + 1)|\widehat{u}(\tau)|^2 d\tau \\
&= \|(\tau^2 + 1)^{\frac{1}{2}} \widehat{u}\|^2_{H^1(\mathbb{R})} \\
&= \|u\|^2_{H^1(\mathbb{R})}.
\end{aligned}$$

Therefore, it has be taken that $c_2 = c_1 = 1$.

Without invoking Fourier transform, we have, $L = \nabla - 1 : H^1(\mathbb{R}) \to H^0(\mathbb{R}) = L^2(\mathbb{R})$ where $\nabla = \dfrac{d}{dt}$. To prove the mapping is isomorphism we need to get two constants c_1 and c_2 such that for all $u \in H^1(\mathbb{R})$

$$c_1\|u\|_{H^1(\mathbb{R})} \leq \|Lu\|_{H^0(\mathbb{R})=L^2(\mathbb{R})} \leq c_2\|u\|_{H^1(\mathbb{R})},$$

where $\|u\|_{H^1(\mathbb{R})} = \int_{\mathbb{R}}(|\nabla u|^2 + |u|^2)dt$. We will prove this inequality we can consider

$$\begin{aligned}
\|Lu\|^2_{L^2(\mathbb{R})} = \|(\nabla - 1)u\|^2_{L^2(\mathbb{R})} &= \int_{\mathbb{R}} |(\nabla - 1)u(t)|^2 dt \\
&= \int_{\mathbb{R}} |(\nabla - 1)u(t)|^2 dt \\
&= \int_{\mathbb{R}} |\nabla u - u|^2 dt \\
&= \int_{\mathbb{R}} (|\nabla u|^2 - \nabla u\overline{u} - \overline{\nabla u}u + |u|^2) dt \\
&= \int_{\mathbb{R}} (|\nabla u|^2 + |u|^2) dt \\
&= \|u\|^2_{H^1(\mathbb{R})},
\end{aligned}$$

where we have noted that

$$\int_{\mathbb{R}} (\nabla u\overline{u} + \overline{\nabla u}u)dt = \int_{\mathbb{R}} (\nabla u)\overline{u} + (\nabla\overline{u})u dt = \int_{\mathbb{R}} \nabla(\overline{u}u)dt = \int_{\mathbb{R}} \nabla|u|^2 dt = [|u|^2]^{\infty}_{-\infty} = 0,$$

since $\overline{\nabla u} = \nabla \overline{u}$. Thus

$$\begin{aligned}\|Lu\|^2_{L^2(\mathbb{R})} &= \int_{\mathbb{R}} (|\nabla u|^2 + |u|^2) dt \\ &= \|u\|^2_{H^1(\mathbb{R})},\end{aligned}$$

and we can take $c_1 = c_2 = 1$. Therefore,

$$c_1 \|u\|_{H^1(\mathbb{R})} \leq \|\widehat{(\nabla - 1)} u\|_{L^2(\mathbb{R})} \leq c_2 \|u\|_{H^1(\mathbb{R})}.$$

2.3 Linear Operator

In this section, we introduce some of the concepts of bounded and unbounded operators on a Hilbert space H.

Definition 2.3.1. A *linear operator* A on a Hilbert space H is a pair consisting of a dense linear subspace $\mathrm{Dom}(A)$ of H together with a linear map $A : \mathrm{Dom}(A) \to H$, which maps linearly $\mathrm{Dom}(A)$ in H, that is

$$A(u + v) = A(u) + A(v)$$

$$A(cu) = cA(u)$$

for all $u, v \in \mathrm{Dom}(A)$, $c \in \mathbb{C}$, and $\mathrm{Dom}(A)$ denotes the domain of A. See [48] and [52].

Definition 2.3.2. A linear operator $A : \mathrm{Dom}(A) \to H$ is said to be a *bounded operator* if there exists a positive constant m such that

$$\|Au\|_H \leq m \|u\|_{\mathrm{Dom}(A)} \quad \text{for each } u \in \mathrm{Dom}(A). \tag{2.3}$$

The collection of all bounded linear operators from $\mathrm{Dom}(A)$ into H is denoted by $B(\mathrm{Dom}(A), H)$ or $B(H) = B(H, H)$. If A is a bounded linear operator, then its norm $\|A\|_{op}$ is the smallest k for which (2.3) holds, that is,

$$\|A\|_{op} = \sup_{u \neq 0} \frac{\|Au\|_H}{\|u\|_{\mathrm{Dom}(A)}}.$$

See [48] and [52].

Remark 4. For a bounded operator $A : \mathrm{Dom}(A) \to H$, and for all $u \in H$ and $u \neq 0$, it has

$$\frac{\|Au\|_H}{\|u\|_{\mathrm{Dom}(A)}} \leq \|A\|_{op}.$$

Thus, we get that

$$\|Au\|_H \leq \|A\|_{op} \|u\|_{\mathrm{Dom}(A)}. \tag{2.4}$$

This also holds when $u = 0$.

Lemma 2.3.1. Let H be a Hilbert space and $A, B : H \to H$ be bounded operators, then

$$\|AB\|_{op} \leq \|A\|_{op} \|B\|_{op}.$$

Definition 2.3.3. (Unbounded linear operator)

A linear operator A is *unbounded* if there exists a sequence $\{u_n\}_{n\in\mathbb{N}} \subset \mathrm{Dom}(A)$ that is convergent in H such that $\lim_{n\to\infty} \|Au_n\|_H \to \infty$. See the definition in [19].

Definition 2.3.4. (Closed operator)

The operator A on H with domain $\mathrm{Dom}(A)$ is called *closed* if for all sequences $\{u_n\}_{n\in\mathbb{N}}$ in $\mathrm{Dom}(A)$ with limit $u \in H$, and there exists $v \in H$ such that $\lim_{n\to\infty} Au_n = v$. It follows that $u \in \mathrm{Dom}(A)$ and that $Au = v$. See [19].

2.4 Adjoint for an Unbounded Linear Operator

Definition 2.4.1. Given $A : \mathrm{Dom}(A) \to H$ and $B : \mathrm{Dom}(B) \to H$ are densely defined linear operators (possibly unbounded operators), then we say B is the *adjoint* of A if for all $u \in \mathrm{Dom}(A)$ and $v \in \mathrm{Dom}(B)$, then $\langle Au, v\rangle = \langle u, Bv\rangle$. We write $B = A^*$. It is easy to show that the adjoint operator is always a closed operator (see [19]).

Definition 2.4.2. If $A : \mathrm{Dom}(A) \to H$ is a densely defined operator (possibly unbounded operator) on H, then A is a *symmetric operator* if

$$\langle Au, v\rangle = \langle u, Av\rangle \quad \text{for all} \quad u, v \in \mathrm{Dom}(A).$$

From the last two definitions, it is easy to conclude that if A is a symmetric operator, then A^* is a closed extension of it

$$A^* \mid_{\mathrm{Dom}(A)} = A.$$

It is a general fact that the domain of the adjoint operator $\mathrm{Dom}(A^*)$ contains the domain $\mathrm{Dom}(A)$.

Definition 2.4.3. An operator A is said to be *self-adjoint* if it is symmetric operator and $\mathrm{Dom}(A) = \mathrm{Dom}(A^*)$ (see [19]).

Definition 2.4.4. A symmetric operator A is called *essentially self-adjoint* if its closure $\overline{A}$ is a self-adjoint operator (see [19]).

Remark 5. We have the following notes:

- If $A : H \to H$ is a linear operator, $\mathrm{Ker}(A)$ and $\mathrm{Ran}(A)$ stand for the kernel and range of A respectively, which are defined by:

$$\mathrm{Ker}(A) = \{u \in \mathrm{Dom}(A) : Au = 0\}.$$

$$\mathrm{Ran}(A) = \{Au : u \in \mathrm{Dom}(A)\}.$$

- Every self-adjoint linear operator $A : H \to H$ is a symmetric operator. On the contrary, symmetric operators need not be self-adjoint operators and the reason is that A^* may be a proper extension of A, that is

$$\mathrm{Dom}(A^*) \neq \mathrm{Dom}(A).$$

  See [36].

Proposition 2.4.1. Let A be a symmetric operator on a Hilbert space. Then, the following properties are equivalent:

(1) The operator A is an essentially self-adjoint operator.

(2) We have $\mathrm{Ker}(A^* \pm i) = \{0\}$.

(3) The subspaces $\mathrm{Ran}(A \pm i)$ are dense in H.

See the book by Reed and Simon [36].

Remark 6. There are similar statements with $\pm i$ replaced by λ and $\overline{\lambda}$ for any fixed $\lambda \in \mathbb{C} \backslash \mathbb{R}$ that are also valid.

2.5 The Spectrum of a Linear Operator

In this section, we define the spectrum of a linear operator in the domain and some important concepts that are related to the spectrum. Suppose A is a closed densely defined linear operator on a Hilbert space H with domain $\mathrm{Dom}(A)$; the *resolvent set* of A denoted by $\rho(A)$ is the set of all complex numbers λ such that

$$(A - \lambda I) : \mathrm{Dom}(A) \to H \quad \text{is bijection,}$$

where I is the identity operator on the Hilbert space H. The *spectrum* of A denoted by $\sigma(A)$ is the complement of the resolvent set in $\mathbb{C}$, meaning that $\sigma(A) = \mathbb{C} \setminus \rho(A)$ (see [19]).

If $A - \lambda I$ is one-to-one and onto, then the open mapping theorem implies that $(A - \lambda I)^{-1}$ is bounded. Therefore, the operator

$$(A - \lambda I)^{-1} : H \to H$$

is called the *resolvent operator* and can be denoted as $R(\lambda, A)$ (see, for example, [19]).

Now, we focus on an important concept in the present research, which is the spectrum. It is known that the spectrum $\sigma(A)$ has three disjoint components:

$$\sigma_{dis}(A) \cup \sigma_c(A) \cup \sigma_r(A) = \sigma(A),$$

such that:

- the *discrete spectrum* $\sigma_{dis}(A)$ or $\sigma_p(A)$ *point spectrum* of A consists of all $\lambda \in \sigma(A)$ such that $(A - \lambda I)$ fails to be an injective equivalent $\mathrm{Ker}(A - \lambda I)$ is non-trivial. In this case, λ is called the eigenvalue of A, and the non-zero elements of $\mathrm{Ker}(A - \lambda I)$ are the corresponding eigenfunctions. See [8].

- $\sigma_c(A)$ is the *continuous spectrum* of A that consists of all $\lambda \in \sigma(A)$ such that $(A - \lambda I)$ is injective and does have a dense image in H, but it fails to be surjective. See [8].

- $\sigma_r(A)$ is the *residual spectrum*, that is, the collection of complex numbers λ such that $(A - \lambda I)$ is injective but does not have a dense image. See [8].

2.6 The Essential Spectrum

Here, we discuss one of the main types of the spectrum in a Hilbert space H, which is called the essential spectrum.

2.6.1 The Definition of the Essential Spectrum

We have already looked at the characterisation of the spectrum. Now, from a perturbation point of view, there is another characterisations of spectral decomposition that reduce for the self-adjoint operators in the Hilbert spaces H to the following:

- The discrete spectrum $\sigma_{dis}(A)$.
- The essential spectrum $\sigma_{ess}(A)$.

We have the class of operator on the Hilbert spaces which is called the projection.

Definition 2.6.1. Let $B(H)$ denotes the set of all bounded operators in a Hilbert space H and let $P \in B(H)$ and $P^2 = P$. Then, P is called *projection.* The range of projection is always a closed subspace on which P acts as the identity (see [36]).

The following definition of the spectral projection because it will be used to investigate the essential spectrum of the operator.

Definition 2.6.2. Let A be a bounded self-adjoint operator and Λ a Borel set of $\mathbb{R}$ (which is defined as any set in space that can be formed from open sets through the operations of countable unions, countable intersections, and relative complements). $P_\Lambda \equiv \chi_\Lambda(A)$ is called a *spectral projection* of an operator A such that χ_Λ is an indicator function. I.e., a spectral projection is the image of Λ under an indicator function defined on its spectrum, which is hence an orthogonal projection on some closed subspace. See [36].

Remark 7. χ_Λ is an indicator function of the single point λ, then the corresponding spectral projection χ_Λ for the operator A is indeed orthogonal projection on the kernel $A - \lambda I$, i.e. the eigenvector for λ. If λ is not eigenvalue, that the projection is 0.

Proposition 2.6.1. $\lambda \in \sigma(A)$ if and only if $P_{(\lambda-\epsilon,\lambda+\epsilon)}(A) \neq 0$ for any $\epsilon > 0$. See in [36].

Definition 2.6.3. If A is a self-adjoint operator and if $\lambda \in \sigma(A)$ and $P_{(\lambda-\epsilon,\lambda+\epsilon)}(A)$ is a finite dimensional for some $\epsilon > 0$, we say $\lambda \in \sigma_{dis}(A)$ is the *discrete spectrum* of A. See in [36].

Proposition 2.6.2. Let A be an self-adjoint operator. A real λ is in the discrete spectrum if and only if λ is an isolated point in $\sigma(A)$ and if the λ is eigenvalue of the finite multiplicity. See Section 3.3 and the standard reference in [36].

Definition 2.6.4. The *essential spectrum* of the operator A is the complement in the spectrum of the discrete spectrum and is denoted by $\sigma_{ess}(A,)$ that is

$$\sigma_{ess}(A) = \sigma(A) \backslash \sigma_{dis}(A).$$

Theorem 2.6.3. The essential spectrum of operator A is always closed.

Proof. See in [36]. □

Basic Examples

- Intuitively, the point of the essential spectrum of the operator A corresponds
  - either to a point in the continuous spectrum of an operator A,
  - to a limit point of a sequence of eigenvalues with finite multiplicity,
  - or to an eigenvalue of infinite multiplicity (see [36]).
- The Laplacian on $\mathbb{R}^d$ for $d \geq 1$ and $-\Delta$ is a self-adjoint operator on $L^2(\mathbb{R}^d)$. The spectrum is continuous and equal to $\overline{\mathbb{R}^+}$. The essential spectrum is also $\overline{\mathbb{R}^+}$, and the operator has no discrete spectrum. See Section 3.4 of the current thesis.

In the following result, we focus on the relationship between the spectrum and essential spectrum of the operator A. This theorem is used to prove the main result of the first task (see Section 3.9, Chapter 3 of the current thesis).

Theorem 2.6.4. Let A be a self-adjoint operator and suppose $(a, b) \subset \sigma(A)$ for some open interval (a, b). Then,

$$(a, b) \subset \sigma_{ess}(A).$$

Proof. Let $\lambda \in (a, b)$, $\epsilon > 0$ and $N \in \mathbb{N}$. Let $I_1, I_2, \cdots, I_N$ denote N non-empty open disjoint intervals contained in interval $(\lambda-\epsilon, \lambda+\epsilon) \cap (a, b)$. Now, $P_{I_j} \neq 0$, (because $P_{I_j} = 0$ would imply $I_j \cap \sigma(A) = \emptyset$). Hence, $\dim \operatorname{Ran} P_{I_j} \geq 1$. Because I_j is disjoint and contained in $(\lambda - \epsilon, \lambda + \epsilon)$, then

$$\dim \operatorname{Ran} P_{(\lambda-\epsilon,\lambda+\epsilon)} \geq \sum_{j=1}^{N} \dim \operatorname{Ran} P_{I_j} \geq N.$$

Because N was arbitrary, it follows that

$$\dim \operatorname{Ran} P_{(\lambda-\epsilon,\lambda+\epsilon)} = \infty.$$

From the definition of essential spectrum, we have that $\lambda \in \sigma_{ess}(A)$. Hence, $(a, b) \subset \sigma_{ess}(A)$. $\square$

2.6.2 Essential Spectrum of Self-adjoint Operators

In this section, we give the definition of a Wely sequence with an important theorem and nice example.

Definition 2.6.5. A sequence $\{u_n\}_{n\in\mathbb{N}}$ is called a *Wely sequence* for the operator A and λ if there exists $\{u_n\}_{n\in\mathbb{N}} \subset \operatorname{Dom}(A)$, such that $\|u_n\|_{\operatorname{Dom}(A)} = 1$ and $\lim_{n\to\infty} \|(A - \lambda I)u_n\|_{\operatorname{Dom}(A)} = 0$ (see the definition in [36] and [47]).

Theorem 2.6.5. (Weyl's Criterion)

Let A be a self-adjoint operator in a Hilbert space H. Then, $\lambda \in \sigma_{ess}(A)$ if and only if there exists a Weyl sequence $\{u_n\}_{n\in\mathbb{N}}$ for A and λ (see [36] and Theorem V11.12 in [47]).

Remark 8. If $\lambda \in \sigma_{pt}$ and we choose u_n to a single eigenfunction. I.e., Above statements are still true if the convergent is replaced by weak convergence.

In the following example, we use a self-adjoint operator $-\Delta$ to apply the Weyl Criterion and observe the relationship between this operator with an essential spectrum. See the properties of this operator in Section 2.8.2 in this Chapter.

Example 1. Let a self-adjoint operator $-\Delta = -\dfrac{\partial^2}{\partial t^2}$ on $\mathbb{R}$ and for $\lambda > 0$. Then, $\lambda \in \sigma_{ess}(-\Delta)$, by using Weyl's criterion.

Firstly, we build a sequence $\{u_n\}_{n\in\mathbb{N}}$ of approximate eigenfunctions that satisfy the conditions of Weyl's criterion. Consider an explicit function $\varphi : \mathbb{R} \to \mathbb{R}$ such that $\varphi(t) = 0$ if $|t| > 1$ with $\|\varphi\|_{L^2(\mathbb{R})} = 1$, so $\|\varphi^{'}\|_{L^2(\mathbb{R})} < \infty$ and $\|\varphi^{''}\|_{L^2(\mathbb{R})} < \infty$. Set

$$\psi_n(t) = \frac{1}{\sqrt{n}}\varphi(\frac{t}{n}),$$

for $n \geq 1$. Then, to build the sequence, we get that $u_n(t)$ is defined by

$$u_n(t) = e^{ikt}\psi_n(t),$$

where $k^2 = \lambda$. Now, the norm of $\psi_n(t)$ is defined by

$$\|\psi_n\|^2_{L^2(\mathbb{R})} = \int_{\mathbb{R}} \left|\frac{1}{n^{\frac{1}{2}}}\varphi(\frac{t}{n})\right|^2 dt = \frac{1}{n}\int_{\mathbb{R}} \left|\varphi(\frac{t}{n})\right|^2 dt = \int_{\mathbb{R}} |\varphi(s)|^2 ds = \|\varphi\|^2_{L^2(\mathbb{R})},$$

and, the norm of the first derivative of $\psi_n(t)$ to get

$$\|\psi_n^{'}\|^2_{L^2(\mathbb{R})} = \int_{\mathbb{R}} \left|\frac{1}{n^{\frac{1}{2}}}\frac{1}{n}\varphi^{'}(\frac{t}{n})\right|^2 dt = \frac{1}{n}\frac{1}{n^2}\int_{\mathbb{R}} \left|\varphi^{'}(\frac{t}{n})\right|^2 dt = \frac{1}{n^2}\int_{\mathbb{R}} |\varphi^{'}(s)|^2 ds = \frac{1}{n^2}\|\varphi^{'}\|^2_{L^2(\mathbb{R})}.$$

Then, the second derivative of $\psi_n(t)$ has

$$\|\psi_n^{''}\|^2_{L^2(\mathbb{R})} = \int_{\mathbb{R}} \left|\frac{1}{n^{\frac{1}{2}}}\frac{1}{n^2}\varphi^{''}(\frac{t}{n})\right|^2 dt = \frac{1}{n}\frac{1}{n^4}\int_{\mathbb{R}} \left|\varphi^{''}(\frac{t}{n})\right|^2 dt = \frac{1}{n^4}\int_{\mathbb{R}} |\varphi^{''}(s)|^2 ds = \frac{1}{n^4}\|\varphi^{''}\|^2_{L^2(\mathbb{R})}.$$

Now, the norm of the sequence $\{u_n\}_{n\in\mathbb{N}}$ will be of the form

$$\|u_n\|^2_{L^2(\mathbb{R})} = \|e^{ikt}\psi_n\|^2_{L^2(\mathbb{R})} = \int_{\mathbb{R}} |e^{ikt}\psi_n(t)|^2 dt = \int_{\mathbb{R}} |e^{ikt}|^2|\psi_n(t)|^2 = \int_{\mathbb{R}} |\psi_n(t)|^2 dt = \|\psi_n\|^2_{L^2(\mathbb{R})};$$

where we use $|e^{ikt}| = 1$, and

$$\|u_n\|^2_{L^2(\mathbb{R})} = \|\psi_n\|^2_{L^2(\mathbb{R})} = \|\varphi\|^2_{L^2(\mathbb{R})} = 1.$$

Now, we will prove $\| - \Delta u_n - \lambda u_n\|_{L^2(\mathbb{R})} \to 0$ as $n \to \infty$. We can consider,

$$\begin{aligned}
(-\Delta - \lambda)u_n &= (-\Delta - \lambda)(e^{ikt}\psi_n(t)) \\
&= -\Delta(e^{ikt}\psi_n(t)) - \lambda(e^{ikt}\psi_n(t)) \\
&= k^2 e^{ikt}\psi_n(t) - 2ike^{ikt}\psi_n^{'}(t) - e^{ikt}\psi_n^{''}(t) - \lambda(e^{ikt}\psi_n(t)) \\
&= -2ike^{ikt}\psi_n^{'}(t) - e^{ikt}\psi_n^{''}(t).
\end{aligned}$$

Because $\lambda = k^2$ and for $k \in \mathbb{R}$. Thus, by using the triangle inequality

$$\begin{aligned} |-\Delta(e^{ikt}\psi_n(t)) - \lambda e^{ikt}\psi_n(t)|^2 &= |-2ike^{ikt}\psi_n^{'}(t) - e^{ikt}\psi_n^{''}(t)|^2 \\ &\leq |4k^2 e^{ikt}\psi_n^{'}(t)|^2 + 2|2ike^{ikt}\psi_n^{'}(t)||e^{ikt}\psi_n^{''}(t)| + |e^{ikt}\psi_n^{''}(t)|^2 \\ &\leq 2\left\{|4k^2 e^{ikt}\psi_n^{'}(t)|^2 + |e^{ikt}\psi_n^{''}(t)|^2\right\}, \end{aligned}$$

we can take the norm

$$\begin{aligned} \|-\Delta e^{ikt}\psi_n - \lambda e^{ikt}\psi_n\|_{L^2(\mathbb{R})}^2 &\leq 2(4k^2\|\psi_n^{'}\|_{L^2(\mathbb{R})}^2 + \|\psi_n^{''}\|_{L^2(\mathbb{R})}^2) \\ &\leq 2(4k^2\frac{1}{n^2}\|\varphi^{'}\|_{L^2(\mathbb{R})}^2 + \frac{1}{n^4}\|\varphi^{''}\|_{L^2(\mathbb{R})}^2). \end{aligned}$$

It follows that $\|-\Delta e^{ikt}\psi_n - \lambda e^{ikt}\psi_n\|_{L^2(\mathbb{R})} \to 0$ as $n \to \infty$.

Now, generalise this problem to obtain for a self-adjoint operator $-\Delta = -\sum_{i=1}^d \frac{\partial^2}{\partial t_i^2}$ on $\mathbb{R}^d$ for $d \geq 1$; we need to prove for $\lambda > 0$ that $\lambda \in \sigma_{ess}(-\Delta)$ by using Weyl's criterion. It will build a sequence $\{u_n\}_{n\in\mathbb{N}}$ of approximate eigenfunctions that satisfy the conditions of Weyl's criterion. Consider an explicit function $\varphi : \mathbb{R}^d \to \mathbb{R}$ such that $\varphi(t) = 0$ if $|t| > 1$, $\|\varphi\|_{L^2(\mathbb{R}^d)} = 1$, $\|\nabla\varphi\|_{L^2(\mathbb{R}^d)} < \infty$ and $\|\Delta\varphi\|_{L^2(\mathbb{R}^d)} < \infty$. Set

$$\psi_n(t) = \frac{1}{n^{\frac{d}{2}}}\varphi(\frac{t}{n}).$$

Then, we can build the sequence $\{u_n\}_{n\in\mathbb{N}}$ by

$$u_n(t) = e^{i\langle k,t\rangle}\psi_n(t),$$

where $|k|^2 = \lambda$ for $k = (k_1, k_2, ..., k_d)$ in $\mathbb{R}^d$. Recall that $t \in \mathbb{R}^d$.

Now, we can take the norm of $\psi_n(t)$ to get

$$\|\psi_n\|_{L^2(\mathbb{R}^d)}^2 = \int_{\mathbb{R}^d}\left|\frac{1}{n^{\frac{d}{2}}}\varphi(\frac{t}{n})\right|^2 dt = \frac{1}{n^d}\int_{\mathbb{R}^d}\left|\varphi(\frac{t}{n})\right|^2 dt = \int_{\mathbb{R}^d}|\varphi(s)|^2 ds = \|\varphi\|_{L^2(\mathbb{R}^d)}^2,$$

and, the norm of the first derivative of $\psi_n(t)$ to get

$$\|\nabla\psi_n\|_{L^2(\mathbb{R}^d)}^2 = \int_{\mathbb{R}^d}\left|\frac{1}{n^{\frac{d}{2}}}\frac{1}{n}\nabla\varphi(\frac{t}{n})\right|^2 dt = \frac{1}{n^d}\frac{1}{n^2}\int_{\mathbb{R}^d}\left|\nabla\varphi(\frac{t}{n})\right|^2 dt = \frac{1}{n^2}\int_{\mathbb{R}^d}|\nabla\varphi(s)|^2 ds = \frac{1}{n^2}\|\nabla\varphi\|_{L^2(\mathbb{R}^d)}^2.$$

Then, the second derivative of $\psi_n(t)$ has

$$\|\Delta\psi_n\|_{L^2(\mathbb{R}^d)}^2 = \int_{\mathbb{R}^d}\left|\frac{1}{n^{\frac{d}{2}}}\frac{1}{n^2}\Delta\varphi(\frac{t}{n})\right|^2 dt = \frac{1}{n^d}\frac{1}{n^4}\int_{\mathbb{R}^d}\left|\Delta\varphi(\frac{t}{n})\right|^2 dt = \frac{1}{n^4}\int_{\mathbb{R}^d}|\Delta\varphi(s)|^2 ds = \frac{1}{n^4}\|\Delta\varphi\|_{L^2(\mathbb{R}^d)}^2.$$

We have the norm of the sequence $\{u_n\}_{n\in\mathbb{N}}$

$$\|u_n\|^2_{L^2(\mathbb{R}^d)} = \|e^{i\langle k,t\rangle}\psi_n\|^2_{L^2(\mathbb{R}^d)} = \int_{\mathbb{R}^d} |e^{i\langle k,t\rangle}\psi_n(t)|^2 dt = \int_{\mathbb{R}^d} |e^{i\langle k,t\rangle}|^2 |\psi_n(t)|^2 = \int_{\mathbb{R}^d} |\psi_n(t)|^2 dt = \|\psi_n\|^2_{L^2(\mathbb{R}^d)},$$

where we use $|e^{i\langle k,t\rangle}| = 1$, and

$$\|u_n\|^2_{L^2(\mathbb{R}^d)} = \|\psi_n\|^2_{L^2(\mathbb{R}^d)} = \|\varphi\|^2_{L^2(\mathbb{R}^d)} = 1.$$

Now, we will prove $\|-\Delta u_n - \lambda u_n\|_{L^2(\mathbb{R}^d)} \to 0$ as $n \to \infty$. We consider that,

$$\begin{aligned}
(-\Delta - \lambda)u_n &= (-\Delta - \lambda)e^{i\langle k,t\rangle}\psi_n(t) \\
&= -\Delta(e^{i\langle k,t\rangle}\psi_n(t)) - \lambda(e^{i\langle k,t\rangle}\psi_n(t)) \\
&= |k|^2 e^{i\langle k,t\rangle}\psi_n(t) - 2ie^{i\langle k,t\rangle}k\cdot\nabla\psi_n(t) - e^{i\langle k,t\rangle}\Delta\psi_n(t) - \lambda(e^{i\langle k,t\rangle}\psi_n(t)),
\end{aligned}$$

where we have $|k|^2 = \lambda$ for $k \in \mathbb{R}^d$. Thus, by using triangle inequality

$$\begin{aligned}
|-\Delta e^{i\langle k,t\rangle}\psi_n(t) - \lambda e^{i\langle k,t\rangle}\psi_n(t)|^2 &= |-2ie^{i\langle k,t\rangle}k\cdot\nabla\psi_n(t) - e^{i\langle k,t\rangle}\Delta\psi_n(t)|^2 \\
&\leq |4|k|^2 e^{i\langle k,t\rangle}\nabla\psi_n(t)|^2 + 2|2ie^{i\langle k,t\rangle}k\cdot\nabla\psi_n(t)||e^{i\langle k,t\rangle}\Delta\psi_n(t)| \\
&\quad + |e^{i\langle k,t\rangle}\Delta\psi_n(t)|^2 \\
&\leq 2\left\{|4|k|^2 e^{i\langle k,t\rangle}\nabla\psi_n(t)|^2 + |e^{i\langle k,t\rangle}\Delta\psi_n(t)|^2\right\}.
\end{aligned}$$

We can take the norm,

$$\begin{aligned}
\|-\Delta e^{i\langle k,t\rangle}\psi_n - \lambda e^{i\langle k,t\rangle}\psi_n\|^2_{L^2(\mathbb{R}^d)} &\leq 2(4|k|^2\|\nabla\psi_n\|^2_{L^2(\mathbb{R}^d)} + \|\Delta\psi_n\|^2_{L^2(\mathbb{R}^d)}) \\
&\leq 2(4|k|^2\frac{1}{n^2}\|\nabla\varphi\|^2_{L^2(\mathbb{R}^d)} + \frac{1}{n^4}\|\Delta\varphi\|^2_{L^2(\mathbb{R}^d)}).
\end{aligned}$$

It follows that $\|-\Delta e^{i\langle k,t\rangle}\psi_n - \lambda e^{i\langle k,t\rangle}\psi_n\|^2_{L^2(\mathbb{R}^d)} \to 0$ as $n \to \infty$.

Remark 9. We have noted that:

- Let a self-adjoint operator $-\Delta = -\dfrac{\partial^2}{\partial t^2}$ on $\mathbb{R}$ and for $\lambda = 0$. Then, $\lambda \in \sigma_{dis}(-\Delta)$.
- For readers, we give a good example for an explicit function $\varphi : \mathbb{R} \to \mathbb{R}$ such that $\varphi(t) = 0$ if $|t| > 1$ with $\|\varphi\|_{L^2(\mathbb{R})} = 1$, so $\|\varphi'\|_{L^2(\mathbb{R})} < \infty$ and $\|\varphi''\|_{L^2(\mathbb{R})} < \infty$ in Appendix.1 of the current thesis.

2.6.3 Essential Spectrum and Relatively Compact Perturbation

The purpose of this part is to observe the relatively compact perturbation theory and its effect on the essential spectrum.

Definition 2.6.6. A subset of Hilbert space is called a *relatively compact* if its closure is compact (see [39]).

Definition 2.6.7. An operator is called a *compact map* if it is a linear operator from a Hilbert space to another such that the image under the linear operator of any bounded subset is relatively compact (see, for example, [39]).

Definition 2.6.8. An operator V with $\text{Dom}(A) \subset \text{Dom}(V)$ is called a *relatively compact perturbation* with respect to the self-adjoint operator A if and only if $V(A-i)^{-1}$ is compact (see [39]).

Remark 10. If V is a relatively compact, then $V(A-z)^{-1}$ is a compact for $z \in \rho(A)$, and if $V(A-z)^{-1}$ is a compact for some $z \in \rho(A)$, then V a is relatively compact. See [39].

Theorem 2.6.6. Let A be a self-adjoint operator and let V be a relatively compact perturbation of A. Then,

- $A+V$ defined with $\text{Dom}(A+V) = \text{Dom}(A)$ is a closed operator.
- If V is a symmetric operator, then $A+V$ is a self-adjoint operator.
- $\sigma_{ess}(A) = \sigma_{ess}(A+V)$.

Proof. See the proof in [39], pp. 113. □

Example 2. Let the operator $-\Delta$ is defined on $L^2(\mathbb{R}^3)$ by using the Foureir transform one can easily see that $\sigma_{ess}(-\Delta) = [0, \infty)$. Let $V \in L^2 + L^\infty$ then, $V(-\Delta+1)^{-1}$ is compact. For, we can find $V_n \in L^2$ with $V - V_n \in L^\infty$ and $\lim_{n\to\infty} \|V_n - V\|_\infty = 0$. Thus $V_n(-\Delta+1)^{-1}$ converges in norm to $V(-\Delta+1)^{-1}$ so we need only show that $V_n(-\Delta+1)^{-1}$ is compact for each n. But, $V_n(-\Delta+1)^{-1}$ is an integral operator with kernel $V_n(x)e^{-|x-y|}/4\pi|x-y|$, which is in $L^2(\mathbb{R}^6)$. Thus $V_n(-\Delta+1)^{-1}$ is Hilbert-Schmidt and so compact. Since $V(-\Delta+1)^{-1}$ is compact, V is relatively compact and so $\sigma_{ess}(-\Delta+V) = \sigma_{ess}(-\Delta) = [0, \infty)$. See more details in [39] and Section 3.3.

Remark 11. The specital case of Theorem 2.6.6 when V is a compact operator is Wely's original classical theorem. See [36]

Proposition 2.6.7. The essential spectrum $\sigma_{ess}(A)$ for a self-adjoint operator A satisfies:

- $\sigma_{ess}(A) \subset \sigma(A)$.
- $\sigma_{ess}(A)$ is closed,
- If V is a self-adjoint compact operator, then $\sigma_{ess}(A) = \sigma_{ess}(A + V)$

Proof. (see [36] and [45]). □

Lemma 2.6.8. The following statements are equivalent:

- $\lambda \in \sigma_{ess}(A)$.
- (Weyl Criterion) there exists a Weyl sequence $\{u_n\}_{n\in\mathbb{N}}$ for A and λ.
- λ is an eigenvalue of infinite multiplicity ($\dim(\mathrm{Ker}(A - \lambda I)) = \infty$), or there exists $\lambda_n \in \sigma(A)$ such that $\lambda_n \to \lambda$.
- For any self-adjoint compact operator V then, $\lambda \in \sigma(A + V)$. Refer the reader can see [36] and [45].

Remark 12. We have the following notes:

- With the definition of a discrete spectrum, we say that for a self-adjoint operator with a compact resolvent $(A - \lambda I)^{-1}$, the spectrum is reduced to the discrete spectrum. For a compact self-adjoint operator, the spectrum is discrete outside 0. This case that the discrete spectrum is not closed See in [3].
- The essential spectrum of an operator with a compact resolvent is empty. For example, Laplace operator $-\Delta$ on a bounded domain $\Omega = [-L, L]$. See Section 3.3 of this thesis.

2.7 Compact Operators

Here, we provide the definition for a compact operator with some properties. Later, we give some lemmas and theorems related to this operator and will be used in the current thesis.

Definition 2.7.1. Let H be the Hilbert space and $A : \mathrm{Dom}(A) \to H$ is called a *compact operator* if for every sequence $\{u_n\}$ in $\mathrm{Dom}(A)$ with $\|u_n\|_{\mathrm{Dom}(A)} \leq 1$ for all n there exists the subsequence $\{u_{n_i}\}$ for $i = 1, 2, ..., \infty$ such that $\{Au_{n_i}\}$ is convergent in H.

We define $B_c(\mathrm{Dom}(A), H) = \{A : \mathrm{Dom}(A) \to H : \text{ A is compact}\}$ and set $B_c(H) = B_c(H, H)$. By definition, a compact operator is a linear operator, and we have that all compact operators are bounded. Thus, it will turn out that $B_c(H) \subseteq B(H)$. In fact, we have $B_c(H)$ is a closed subspace of $B(H)$. The reader can see this definition in [1] and [57].

Remark 13. We have the following notes:

- The set of compact operators is a subspace of $B(H)$. In particular, each scalar multiple of a compact operator or the sum of two compact operators results in another compact operator (see [18] Satz II.3.2 (a) and [57]).

- If $A \in B_c(H)$ is a compact operator and $B \in B(H)$ is a linear bounded operator, then the superposition AB is a compact operator. To verify this, we employ the definition of a compact operator and omit the index i in the subsequence $\{u_{n_i}\}_{i\in\mathbb{N}}$. Let $\{u_n\}_{n\in\mathbb{N}}$ in H. Then, $\{Au_n\}$ is a bounded sequence as well. Because $\|Au_n\|_H \leq \|A\|_{op}\|u_n\|_H$ for each linear bounded operator A. Hence, there is $v \in H$ so that $\|BAu_n - v\|_H \to 0$ (see [57]).

- If $\dim(\mathrm{Dom}(A)) = \infty$ and $A : \mathrm{Dom}(A) \to H$ is invertible, then A is not compact (see [45]).

Definition 2.7.2. A sequence $\{u_n\}_{n\in\mathbb{N}}$ of continuous function of closed interval $I = [a, b]$ is an *uniformly bounded* if there exists m such that $|u_n(t)| \leq m$ for all n and $t \in I$.

Definition 2.7.3. A sequence $\{u_n\}_{n\in\mathbb{N}}$ from a closed interval $I = [a, b]$ to a Hilbert space H is said to be *equicontinuous* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $n \in \mathbb{N}$

$$|u_n(t) - u_n(s)| \leq \varepsilon,$$

whenever $|t - s| < \delta$ and $t, s \in I$.

See, for example, [1], pp. 54 and [57].

Lemma 2.7.1. Let $u \in L^2(\mathbb{R})$ with $\text{supp}(\hat{u}) \subseteq [-R, R]$. Then,

i) $\|u\|_{L^\infty(\mathbb{R})} \leq \sqrt{\frac{R}{\pi}}\|u\|_{L^2(\mathbb{R})}$. ($L^\infty$ is defined in Remark 1).

ii) $|u(h) - u(s)| \leq R|h - s|^{\frac{1}{2}}\|u\|_{L^2(\mathbb{R})}$, for all $h, s \in \mathbb{R}$.

Proof. i) Let $t \in \mathbb{R}$, and then, the inverse Fourier transform of $u(t)$ is

$$u(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \widehat{u}(\tau) e^{i\tau t} d\tau$$

by the Holder inequality and by the Parseval identity, we get

$$\begin{aligned} \|u\|^2_{L^\infty(\mathbb{R})} \leq \frac{1}{2\pi} \int_{-R}^{R} |\widehat{u}(\tau)| d\tau &\leq \frac{1}{2\pi} \int_{-R}^{R} |1|^2 d\tau \int_{-R}^{R} |\widehat{u}(\tau)|^2 d\tau \\ &\leq \frac{R}{\pi} \int_{-R}^{R} |\widehat{u}(\tau)|^2 d\tau \\ &= \frac{R}{\pi} \|\widehat{u}\|^2_{L^2(\mathbb{R})} \\ &= \frac{R}{\pi} \|u\|^2_{L^2(\mathbb{R})}. \end{aligned}$$

ii) For $h, s \in \mathbb{R}$, we have

$$u(h) - u(s) = \int_s^h \nabla u(t) dt,$$

so, we get that

$$|u(h) - u(s)|^2 \leq \left(\int_s^h |\nabla u(t)| dt \right)^2.$$

By the Holder inequality, we get

$$
\begin{aligned}
|u(h) - u(s)|^2 &\leq \left(\int_s^h |\nabla u(t)| dt \right)^2 \\
&\leq \int_s^h |1|^2 dt \int_s^h |\nabla u(t)|^2 dt \\
&\leq |h - s| \|\nabla u\|^2_{L^2(\mathbb{R})}.
\end{aligned}
$$

Because $\widehat{\nabla u}(\tau) = i\tau \widehat{u}(\tau)$ by Lemma 2.2.1, the Parseval identity and $\operatorname{supp}(\hat{u}) \subseteq [-R, R]$, we have

$$
\begin{aligned}
\|\nabla u\|^2_{L^2(\mathbb{R})} &= \|\widehat{\nabla u}\|^2_{L^2(\mathbb{R})} \\
&= \|\tau \widehat{u}(\tau)\|^2_{L^2(\mathbb{R})} \\
&= \int_{-R}^{R} |\tau|^2 |\widehat{u}(\tau)|^2 d\tau \\
&\leq R^2 \int_{\mathbb{R}} |\widehat{u}(\tau)|^2 d\tau \\
&\leq R^2 \|u\|^2_{L^2(\mathbb{R})}.
\end{aligned}
$$

Therefore,

$$
|u(h) - u(s)|^2 \leq R^2 |h - s| \|u\|^2_{L^2(\mathbb{R})}.
$$

□

Now, we consider some spaces and operators which are used to investigate some of concepts in the current research:

Remark 14. We have the following spaces:

- The space $C^\infty(\mathbb{R})$ denotes the class of all infinitely differentiable functions on $\mathbb{R}$.
- The space $C_0^\infty(\mathbb{R})$ denotes the space of all infinitely differentiable functions on $\mathbb{R}$ with compact support.
- The space $C_0^1(\mathbb{R})$ denotes the space of all continuously differentiable functions on $\mathbb{R}$ with not compact.
- The space $C_0^0(\mathbb{R})$ denotes the space of all continuous functions which vanish at infinity (i.e., with not compact). The reader will see all of the previous definitons of these spaces in [35], and [52].

Remark 15. For g in $C_0^\infty(\mathbb{R})$, an operator $g(D)$ is defined by

$$g(D)u(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} g(\tau)\widehat{u}(\tau)e^{it\tau} d\tau$$

for $t, \tau \in \mathbb{R}$.

Theorem 2.7.2. If f and g in $C_0^\infty(\mathbb{R})$. Then,

$$f(t)g(D) : L^2(\mathbb{R}) \to L^2(\mathbb{R})$$

is a compact map.

Proof. Choose R such that

$$\text{supp}(f) \subseteq [-R, R] \quad \text{and} \quad \text{supp}(g) \subseteq [-R, R].$$

Set $\chi_R = \chi_{[-R,R]}$ such that

$$\chi_{[-R,R]} = \begin{cases} 0 & \text{if} \quad t \notin [-R, R] \\ \\ 1 & \text{if} \quad t \in [-R, R], \end{cases}$$

(χ_R is an indicator function, it is in Definition 2.6.2), and we can write

$$f = f\chi_R \quad \text{and} \quad g = \chi_R g.$$

Then, we have

$$f(t)g(D) = f(t)\chi_{[-R,R]}(t)\chi_{[-R,R]}(D)g(D);$$

it is clear to observe $f(t) : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ and $g(D) : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ are bounded maps. (By the Lemma 2.7.1 $\|f\|_{L^\infty}$ and $\|g\|_{L^\infty}$ are bounded maps). To complete the proof, we need to prove the following lemma:

Lemma 2.7.3. We have,

$$\chi_{[-R,R]}(t)\chi_{[-R,R]}(D) : L^2(\mathbb{R}) \to L^2(\mathbb{R}), \tag{2.5}$$

is compact.

Proof. Choose a sequence $\{u_i\}_{i\in\mathbb{N}} \subseteq \{u \in L^2(\mathbb{R}) : \|u\|_{L^2(\mathbb{R})} \leq 1\}$, which is bounded. We can set

$$V_i = \chi_{[-R,R]}(t)\chi_{[-R,R]}(D)u_i(t) \in L^2(\mathbb{R}).$$

Because we have

$$\|\chi_{[-R,R]}\|_{L^\infty(\mathbb{R})} = \sup_{t\in[-R,R]} |\chi_{[-R,R]}(t)| = 1,$$

we have

$$\|V_i\|_{L^\infty(\mathbb{R})} \leq \|u_i\|_{L^2(\mathbb{R})} \leq 1.$$

To check this we observe

$$\begin{aligned}
\|V_i\|_{L^\infty(\mathbb{R})} &= \sup_{t\in\mathbb{R}} |V_i(t)| \\
&= \sup_{t\in\mathbb{R}} |\chi_{[-R,R]}(t)\chi_{[-R,R]}(D)u_i(t)| \\
&\leq \sup_{t\in\mathbb{R}} |\chi_{[-R,R]}(t)| \sup_{t\in\mathbb{R}} |\chi_{[-R,R]}(D)u_i(t)| \\
&= \sup_{t\in\mathbb{R}} |\chi_{[-R,R]}(D)u_i(t)|.
\end{aligned}$$

Now, let $w(t) = \chi_{[-R,R]}(D)u_i(t)$, and we have

$$\text{supp}(w) = \text{supp}\chi_{[-R,R]}(\tau) \cap \text{supp}(u_i)(\tau) \subseteq [-R, R],$$

and we have

$$|\widehat{w}(\tau)|^2 \leq \sup_{\tau\in[-R,R]} \chi_{[-R,R]}(\tau)|\widehat{u_i}(\tau)|^2.$$

Then,

$$\int_{\mathbb{R}} |\widehat{w}(\tau)|^2 d\tau \leq \int_{\mathbb{R}} |\widehat{u_i}(\tau)|^2 d\tau.$$

We can get the norm and by Parseval's identity have

$$\|\widehat{w}\|^2_{L^2(\mathbb{R})} \leq \|\widehat{u_i}\|^2_{L^2(\mathbb{R})} = \|u_i\|^2_{L^2(\mathbb{R})} \leq 1.$$

Therefore,

$$\|V_i\|_{L^\infty(\mathbb{R})} \leq \|u_i\|_{L^2(\mathbb{R})} \leq 1.$$

Now, by Lemma 2.7.1 we get

- $\|V_i\|_{L^\infty(\mathbb{R})} \leq \dfrac{R}{\pi}\|V_i\|_{L^2(\mathbb{R})}$,

- $|V_i(h) - V_i(s)| \leq R|h-s|^{\frac{1}{2}}\|V_i\|_{L^2(\mathbb{R})}$ for $h, s \in \mathbb{R}$.

These imply that $\{V_i\}_{i\in\mathbb{N}}$ is an equicontinuous family in $L^\infty(\mathbb{R})$. On the other hand, the boundedness implies that the functions are uniformly bounded in $L^\infty(\mathbb{R})$. Hence, by the Arzela-Ascoli theorem, there exists a subsequence $\{V_{i_m}\}_{m\in\mathbb{N}}$ that is uniformly convergent on the bounded interval $[-R, R]$ (see [52]). Because

$$\begin{aligned}\|\chi_R V_{i_n} - \chi_R V_{i_m}\|^2_{L^2(\mathbb{R})} &= \int_{-R}^{R} |\chi_R V_{i_n}(t) - \chi_R V_{i_m}(t)|^2 dt \\ &\leq 2R\|V_{i_n} - V_{i_m}\|^2_{L^\infty([-R,R])},\end{aligned}$$

we can get $\|V_{i_n} - V_{i_m}\|_{L^\infty([-R,R])} \to 0$ as $n, m \to \infty$. We have $\{\chi_R(t)V_{i_m}\}_{m\in\mathbb{N}} = \{\chi_R(t)\chi_R(D)u_{i_m}\}_{m\in\mathbb{N}}$ is a Cauchy sequence and is convergent in $L^2(\mathbb{R})$ for all $m \in \mathbb{N}$. Thus $\chi_{[-R,R]}(t)\chi_{[-R,R]}(D)$ is a compact map. □

Now, to complete our argument, we have that $f(t)$ and $g(D)$ are bounded operators and the operator $\chi_{[-R,R]}(t)\chi_{[-R,R]}(D)$ is a compact map. This implies, $\chi_{[-R,R]}(t)\chi_{[-R,R]}(D)$ is a bounded operator (see, for example, [57]). Therefore, the reader can see that as in [52] and above notes in this section, the composition (product) of two bounded operators is again a bounded operator. That is $f\chi_{[-R,R]}$ and $\chi_{[-R,R]}g$ are bounded operators, and it is easy to obtain this by Lemma 2.3.1 in Section 2.4 of this thesis,

$$\begin{aligned}\|f\chi_{[-R,R]}\chi_{[-R,R]}gu\|_{L^2(\mathbb{R})} &\leq \|f\chi_{[-R,R]}\|_{op}\|\chi_{[-R,R]}gu\|_{L^2(\mathbb{R})} \\ &\leq \|f\chi_{[-R,R]}\|_{op}\|\chi_{[-R,R]}g\|_{op}\|u\|_{L^2(\mathbb{R})}.\end{aligned}$$

It follows that by [57], the product of a compact operator with bounded operators is a compact operator. Thus, $f(t)g(D)$ is a compact map. See [4] Chapter 4. □

Lemma 2.7.4. Let f and $g \in C_0^0(\mathbb{R})$. Then,

$$f(t)g(D) : L^2(\mathbb{R}) \to L^2(\mathbb{R})$$

is a compact map.

Proof. Choose $\varphi, \psi \in C_0^\infty$ such that

- $\text{supp}(\varphi) \subseteq [-2, 2]$, $0 \leq \varphi \leq 1$ and $\varphi(t) = 1$ for $t \in [-1, 1]$.

- $\psi \geq 0$ and $\int \psi(t)dt = 1$.

For any $N \in \mathbb{N}$, we set the following sequences

$$\varphi_N(t) = \varphi(\frac{t}{N}) \quad \text{and} \quad \psi_N(t) = N\psi(tN).$$

Therefore, we put

$$f_N = \varphi_N(\psi_N * f) \quad \text{and} \quad g_N = \varphi_N(\psi_N * g).$$

Thus, $(f_N)_{N\in\mathbb{N}}$ and $(g_N)_{N\in\mathbb{N}} \subseteq C_0^\infty$ such that

$$\begin{aligned}
\|f_N - f\|_{L^\infty(\mathbb{R})} &= \sup_{t\in\mathbb{R}} |f_N(t) - f(t)| \\
&= \sup_{t\in\mathbb{R}} |\varphi_N(\psi_N * f)(t) - f(t)| \\
&\leq \sup_{t\in\mathbb{R}} |\varphi_N(t)(\psi_N * f)(t)| + \sup_{t\in\mathbb{R}} |f(t)| \\
&\leq \sup_{t\in\mathbb{R}} |\varphi(\frac{t}{N})(N\psi(tN) * f)(t)| + \sup_{t\in\mathbb{R}} |f(t)| \to 0
\end{aligned}$$

as $N \to \infty$, that is $\|f_N - f\|_{L^\infty(\mathbb{R})} \to 0$ as $N \to \infty$ (by definition of f).

Similarly, $\|g_N - g\|_{L^\infty(\mathbb{R})} \to 0 \quad \text{as} \quad N \to \infty$ (by definition of g).

Now, we can observe that by Lemma 2.7.1

$$\begin{aligned}
\|fg - f_N g_N\|_{op} &= \|fg - fg_N + fg_N - f_N g_N\|_{op} \\
&\leq \|f(g - g_N)\|_{op} + \|(f - f_N)g_N\|_{op} \\
&\leq \|f\|_{op}\|(g - g_N)\|_{op} + \|(f - f_N)\|_{op}\|g_N\|_{op} \\
&\leq \|f\|_{L^\infty(\mathbb{R})}\|g - g_N\|_{L^\infty(\mathbb{R})} + \|f - f_N\|_{L^\infty(\mathbb{R})}\|g_N\|_{L^\infty(\mathbb{R})} \to 0.
\end{aligned}$$

It follows that $f_N(t)g_N(D) \to f(t)g(D)$ in an operator norm as $N \to \infty$.

However, $f_N(t)g_N(D)$ is a compact map by Theoreom 2.7.2, and the set of compact operators is closed in an operator norm (see [29]). Therefore, $f(t)g(D)$ is compact. □

Lemma 2.7.5. Let $g \in C_0^0$ and $f, f^{'} \in C_0^1$. Then,

$$f(t)g(D) : H^1(\mathbb{R}) \to H^1(\mathbb{R})$$

is a compact map.

Proof. We can set $A_{f,g} = f(t)g(D)$.

First, we prove the map

$$A_{f,g} = f(t)g(D) : H^1(\mathbb{R}) \to L^2(\mathbb{R}) \tag{2.6}$$

is a compact map. We can consider the inclusion

$$i : H^1(\mathbb{R}) \hookrightarrow L^2(\mathbb{R})$$

is a bounded map. It is clear from definitions of the norms of $H^1(\mathbb{R})$ and $L^2(\mathbb{R})$ in Section 2.1. That is

$$\|u\|^2_{H^1(\mathbb{R})} = \|u\|^2_{L^2(\mathbb{R})} + \|\nabla u\|^2_{L^2(\mathbb{R})}.$$

Then, we can observe the map

$$A_{f,g} = f(t)g(D) : L^2(\mathbb{R}) \to L^2(\mathbb{R})$$

is a compact map from Lemma 2.7.4. Therefore,

$$A_{f,g} = f(t)g(D) : H^1(\mathbb{R}) \to L^2(\mathbb{R}) \tag{2.7}$$

is compact. Second, we have the derivative of $A_{f,g}$ being defined by

$$\begin{aligned}\nabla A_{f,g}u = \nabla f(t)g(D)u(t) &= f(t)g(D)\nabla u(t) + f^{'}(t)g(D)u(t) \\ &= A_{f,g}\nabla u + f^{'}g(D)u \\ &= A_{f,g}\nabla u + A_{f^{'},g}u.\end{aligned}$$

Now, we need to prove the map

$$\nabla A_{f,g}u = A_{f,g}\nabla u + A_{f^{'},g}u : H^1(\mathbb{R}) \to L^2(\mathbb{R})$$

is a compact map. First, we see the derivative of $A_{f,g}$

$$\nabla A_{f,g} : H^1(\mathbb{R}) \to L^2(\mathbb{R})$$

is a bounded map. And we have again

$$A_{f,g} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$$

is a compact map (from Lemma 2.7.4). We note also, the inclusion

$$i : H^1(\mathbb{R}) \hookrightarrow L^2(\mathbb{R})$$

is a bounded map, and finally, we have

$$A_{f',g} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$$

is a compact map again (from Lemma 2.7.4).

We observe the derivative of $A_{f,g}$ as composition

$$H^1(\mathbb{R}) \to L^2(\mathbb{R}) \to L^2(\mathbb{R}) + H^1(\mathbb{R}) \hookrightarrow L^2(\mathbb{R}) \to L^2(\mathbb{R}).$$

Because every step is bounded and compact, the map

$$\nabla A_{f,g} : H^1(\mathbb{R}) \to L^2(\mathbb{R}) \tag{2.8}$$

is a compact map.

Now, combining (2.7) and (2.8), we obtain

$$(1 + \nabla)A_{f,g} : H^1(\mathbb{R}) \to L^2(\mathbb{R}),$$

which is a compact map.

In fact, we conclude that

$$(1 + \nabla)^{-1} : L^2(\mathbb{R}) \to H^1(\mathbb{R})$$

is an isomorphism map. Hence,

$$A_{f,g} = f(t)g(D) : H^1(\mathbb{R}) \to H^1(\mathbb{R})$$

is a compact map. □

2.8 The Friedrichs Extension

The Friedrichs extension theorem says that a semi-bounded (or at least bounded below) symmetric operator is guaranteed to have a self-adjoint extension and, this gives us a particular distinguished example, namely the form closure. If we have A, which is a densely defined symmetric operator and we are looking for extensions that are self-adjoint, we need to enlarge the domain of this operator. The idea is to determine this domain, and then, we need to know some definitions related to find the extension of the self-adjoint operator.

2.8.1 Quadratic Form

One of the main results of the Riesz Lemma is that there is an injection between bounded quadratic forms and bounded operators. However, we are discussing this relationship between quadratic forms and unbounded operators (see [37]). Firstly, we have the definition of a quadratic form and its properties. The standard refrences are in ([57], Chapter VI) and ([37], Section VIII.6) or ([19], Section 4.4).

Definition 2.8.1. A *quadratic form* is a map $q : Q(A) \times Q(A) \to \mathbb{C}$ with domain $Q(A)$ in H such that

- $Q(A)$ is a dense linear subset of a Hilbert space H called the *form domain*.
- $q(\lambda_1 u_1 + \lambda_2 u_2, v) = \overline{\lambda_1} q(u_1, v) + \overline{\lambda_2} q(u_2, v)$ for $\lambda_1, \lambda_2 \in \mathbb{C}$ and $u_1, u_2, v \in Q(A)$.
- $q(u, \lambda_1 v_1 + \lambda v_2) = \lambda_1 q(u, v_1) + \lambda_2 q(u, v_2)$ for $\lambda_1, \lambda_2 \in \mathbb{C}$ and $u, v_1, v_2 \in Q(A)$.

If $q(u, v) = \overline{q(v, u)}$ for all $u, v \in Q(A)$, then q is said to be a *symmetric operator*. If $q(u, u) \geq 0$ for all $u \in Q(A)$, then q is *non-negative* and q is called *semi-bounded* by $m \in \mathbb{R}$ if $q(u, u) \geq m\|u\|^2_{Q(A)}$ for all $u \in Q(A)$.

It is easily shown that the positivity of q implies its semi-boundedness operator, and the semi-boundedness operator implies its symmetry. See [37], Section VIII.6. Now, we define the notion of the closedness of a quadratic form in analogy with that of the closedness of operators.

Definition 2.8.2. A quadratic form q in a Hilbert space H is said to be a *closed* if for any sequence $\{u_n\} \subset Q(A)$, and $u \in H$ with $\lim_{n\to\infty} u_n = u$ and $q(u_n - u_m, u_n - u_m) \to 0$ as $n, m \to \infty$, we have $u \in Q(A)$ and $q(u_n - u, u_n - u) \to 0$ as $n \to \infty$.

We say q is *closable* if for any sequence $\{u_n\} \subset Q(A)$ with $\lim_{n\to\infty} u_n = 0$ and $q(u_n - u_m, u_n - u_m) \to 0$ as $n, m \to \infty$, we have $q(u_n, u_n) \to 0$ as $n \to \infty$. See [46].

Lemma 2.8.1. Let $q : Q(A) \times Q(A) \to \mathbb{C}$ be a semi-bounded quadratic form in a Hilbert space H and choose $m \in \mathbb{R}$ such that $q(u,u) \geq m\|u\|^2_{Q(A)}$ for all $u \in Q(A)$. Let

$$\langle u, v\rangle_{Q(A)} = q(u,v) + (m+1)\langle u, v\rangle, \quad u, v \in Q(A).$$

Then, the following holds:

- q is closed if and only if $(Q(A), \langle \cdot, \cdot\rangle_{Q(A)})$ is a Hilbert space.
- If q is closable, then there is a closed extension $\widehat{q} : Q(\widehat{A}) \times Q(\widehat{A}) \to \mathbb{C}$ of q to a quadratic form $\widehat{q}$, and it is called the *clousre* denoted by $\widehat{q}$ of q, such that

  $$\widehat{q}(u,v) = q(u,v)$$

  for all $u, v \in Q(A)$. See the proof in [46].

We focus on quadratic forms and their closures, which are strongly connected with symmetric operators and their self-adjoint extensions. Moreover, we need to define the operator to be non-negative or semi-bounded operator.

Definition 2.8.3. An operator A is *semi-bounded* or *bounded below* iff there is some $m \in \mathbb{R}$ for which

$$\langle Au, u\rangle \geq m\|u\|^2_{\mathrm{Dom}(A)}$$

for all $u \in \mathrm{Dom}(A)$.

Definition 2.8.4. Let H be a Hilbert space. An operator A is called *non-negative* if $\langle Au, u\rangle \geq 0$ for all $u \in H$. We write $A \geq 0$ if A is non-negative and $A \leq B$ if $B - A \geq 0$.

Now, we see some theorems with propositions to constructs a specific self-adjoint extension from a quadratic form that is associated with a symmetric positive operator such as the Laplace operator.

Theorem 2.8.2. (Representation Theorem)

Suppose q is a closed semi-bounded quadratic form. Then, there is a unique semi-bounded self-adjoint operator

A such that $\mathrm{Dom}(A) \subset Q(A)$ and

$$q(u,v) = \langle Au, v\rangle$$

for all $v \in Q(A)$ and $u \in \mathrm{Dom}(A)$. Furthermore, if $v \in Q(A)$ and there exist $w \in H$ such that

$$q(u,v) = \langle u, w\rangle$$

for all $u \in Q(A)$, then $v \in \mathrm{Dom}(A)$ and $Av = w$. Refer the reader [36], pp. $278 - 279$.

Theorem 2.8.3. Let A be a non-negative self-adjoint operator in a Hilbert space H with quadratic form q. Then, there exists a map that is bounded for $v \in \mathrm{Dom}(A)$ if and only if $v \in Q(A)$ and also w such that

$$q(u,v) = \langle u, w\rangle$$

for all $u \in Q(A)$. In this case, we have $Av = w$.

See [19], pp. 81.

Proposition 2.8.4. If $\widehat{q}$ is a closed semi-bounded form, then there is a unique self-adjont operator $\widehat{A}$ so that $\mathrm{Dom}(\widehat{A}) \subset Q(\widehat{A})$ and

$$\widehat{q}(u,v) = \langle \widehat{A}u, v\rangle$$

if $v \in Q(\widehat{A})$ and $u \in \mathrm{Dom}(\widehat{A})$ (see, for example, [46]).

In the next theorem, we give a general theorem that constructs a specific self-adjoint extension called *the Friedrichs extension.* Friedrichs published the proof of the theorem in 1934. In fact, the first statement and proof of this theorem occured in the book Spektraltheorie der unendlichen Matrizen by Aurel Wintner (1929), and it is obtained from a quadratic form associated with a symmetric positive operator such as the Laplace operator (see [56]).

Theorem 2.8.5. (Friedrichs extension Theorem)

Let q be the quadratic form defined in the domain $\mathrm{Dom}(A)$ of a non-negative symmetric operator A by

$$q(u,v) = \langle Au, v\rangle \quad \text{for all} \quad u, v \in \mathrm{Dom}(A).$$

Then, q is a closable quadratic form, and its closure $\widehat{q}$ is the quadratic form of a unique self-adjoint operator $\widehat{A}$. $\widehat{A}$ is the only self-adjoint extension of A whose domain is contained in the form domain of $\widehat{q}$ (see [37], pp. $177 - 178$).

2.8.2 Application for Friedrichs Extension Theorem: The Laplace operator

We highlight one of the most important objects in the spectral theory of unbounded (differential) operators, namely the Laplace operator. Although there are different generalisations beyond this setting, we use it and restrict our attention to the Laplace operator in open subsets in space $\mathbb{R}^d$ for $d \geq 1$. It is not the case that the Laplace operator is an essentially self-adjoint operator in general therefore, we can not use the self -adjointness of Proposition 2.4.1. To do this, it is easy to apply the Laplace operator from Friedrichs extension Theorem, to show that there is a specific self-adjoint extension for this operator. We consider the Laplace operator, which is initially defined as follows:

- The domain is $C_c^\infty(\Omega)$, which is smooth and has compactly supported functions on $\Omega \subseteq \mathbb{R}^d$ for $d \geq 1$, which is dense in $L^2(\Omega)$. See [51].
- For $u \in C_c^\infty(\Omega)$, we use

$$-\Delta u = -\sum_{j=1}^{d} \frac{\partial^2 u}{\partial t_j^2}.$$

We note that $-\Delta$ is again a smooth compactly supported function, and it is bounded and lies in $L^2(\Omega)$. Now, we are progressing toward showing that $-\Delta$ is a self-adjoint operator or that a least there is a self-adjoint extension.

Proposition 2.8.6. Let $\Omega \subseteq \mathbb{R}^d$ be a non-empty open subset for $d \geq 1$ and $(C_c^\infty(\Omega), -\Delta)$ be the Laplace operator defined above. The following properties of the Laplace operator $-\Delta$:

(1) The Laplace operator is *symmetric* on $C_c^\infty(\Omega)$. That is we have $\langle -\Delta u, v\rangle = \langle u, -\Delta v\rangle$, for all $u, v \in C_c^\infty(\Omega)$.

(2) The Laplace operator is *non-negative* on $C_c^\infty(\Omega)$. That is we have $\langle -\Delta u, u\rangle \geq 0$, for all $u \in C_c^\infty(\Omega)$.

Proof. (1) Using integration by parts twice and the fact that the functions in the domain are compactly supported with respect to any fixed coordinate, we obtain

$$\langle -\Delta u, v\rangle = -\int_\Omega \overline{\Delta u(t)} v(t) dt = -\sum_{j=1}^{d} \int_\Omega \overline{\partial_{t_j}^2 u(t)} v(t) dt = -\sum_{j=1}^{d} \int_\Omega \overline{u(t)} \partial_{t_j}^2 v(t) dt = \langle u, -\Delta v\rangle,$$

for $u, v \in C_c^\infty(\Omega)$, so $-\Delta$ is a symmetric operator.

(2) We have

$$\langle -\Delta u, u\rangle = \int_\Omega \overline{-\Delta u(t)}u(t)dt = -\sum_{j=1}^{d}\int_\Omega \overline{\partial_{t_j}^2 u(t)}u(t)dt = \sum_{j=1}^{d}\int_\Omega |\partial_{t_j}u(t)|^2 dt = \sum_{j=1}^{d}\|\partial_{t_j}u\|_\Omega^2 \geq 0,$$

for all $u \in C_c^\infty(\Omega)$ so $-\Delta$ is a non-negative operator.

□

Remark 16. If there was not a negative sign in the definition, the Laplace operator would have been negative.

Theorem 2.8.7. Let $\Omega \subset \mathbb{R}^d$ a non-empty open subset for $d \geq 1$. Then, the Laplace operator $-\Delta$ admits self-adjoint extension.

Proof. This follows from the general previous theorem: Friedrichs extension Theorem. □

Chapter 3

Existence of Embedded Eigenvalues for Operator $-\Delta - V$.

This chapter has a main result of the first task of the current thesis. In Section 3.1, we introduce the concept of boundary conditions with some properties. We focus on the spectrum of the Laplacian in Section 3.2. Then, we give some examples in different domains to compute the eigenvalues and eigenfunctions from Section 3.3 to Section 3.5. In Section 3.6, we define a Symmetry operator S and its properties. Then, we consider the definition of the symmetry operator S with operator $(-\Delta - V)$ in Section 3.6.3. In Section 3.8, we discuss the Variational principle and calculate the $\inf(\sigma(-\Delta - V))$. In Section 3.9, we observe the result of the relatively compact perturbations with respect to the operator $-\Delta$. Finally, in Section 3.9, we give the main result: The Existence of embedded eigenvalues of operator $(-\Delta - V)$ on the Cylindrical domain $\mathbb{R} \times [-L, L]$.

3.1 Boundary Conditions

There are different types of boundary conditions that can be imposed on the boundary of the domain, for example, Dirichlet and Neumann boundary conditions (see, for example, [19], [53]). In the following part, we study Dirichlet and Neumann boundary conditions for Laplace operator $-\Delta$ which acting in $L^2(\Omega)$ where Ω is a open region on $\mathbb{R}^d$ for $d \geq 1$, we can observe the following:

- The Dirichlet Laplacian for Ω denoted by $-\Delta_D^\Omega$ as the unique self-adjoint operator in $L^2(\Omega)$ whose quadratic form is the closure of

$$q(u,v) = \int_\Omega \overline{\nabla u} \cdot \nabla v dt$$

  with domain $C_c^\infty(\Omega)$.

- The Neumann Laplacian for Ω denoted by $-\Delta_N^\Omega$ as the unique self-adjoint operator on $L^2(\Omega)$ whose quadratic form is

$$q(u,v) = \int_\Omega \overline{\nabla u} \cdot \nabla v dt$$

  with domain $H^1(\Omega) = \{u \in L^2(\Omega) \mid \nabla u \in L^2(\Omega)\}$, where ∇u is the distributional gradient.

Both these definitions for the Dirichlet and Neumann operators are equivalent when closing $C_c^\infty(\Omega)$ with the quadratic form q defined above, and using the self-adjoint operator given by Friedrichs extension theorem and the above definitions do not show their association with the boundary conditions (see Section 2.8). One way to understand this is to define A as the operator closure of ∇ over $C_c^\infty(\Omega)$. Closing via the operator norm means that both the functions and their gradients converge in L^2. The functions that converge in this norm must converge point-wise. Given any function in the domain of A, this requires that it both vanish on the boundary and have a distributional gradient. Then, A is defined on $H^1(\Omega)$. Here, A^* is the closure of $-\nabla$ defined on $C_c^\infty(\Omega)$. No boundary condition is imposed because the boundary term drops out in the definition of the adjoint because the domain of A requires all functions to vanish. The domain of the operator A^*A is a subset of $H^1(\Omega)$, and A^*A is a self-adjoint operator, so it must be the Dirichlet Laplacian by its uniqueness in Friedrichs extension Theorem. The domain of AA^* is a subset of $H^1(\Omega)$, so for the same reason, it must be the Neumann Laplacian. Refer the reader can see [19] and [53]. Now, we can see the most common boundary conditions are the following:

- Dirichlet boundary conditions: This is used for instance when your domain $\Omega \subset \mathbb{R}^2$ is a membrane and you fix its boundary as if Ω was a drum. Because you donot have any vibrations on the rim of a drum you must have $u|_{\partial\Omega} = 0$. See [19] and [53].

- Neumann boundary conditions: $\frac{\partial u}{\partial \nu}|_{\partial\Omega} = 0$, Here ν is the unit outward normal vector for the boundary $\partial\Omega$. This conditions can be used when a surface has a prescribed heat flux, such as perfect insulator(the heat doesnot go through the boundary). See [19] and [53].

3.2 Spectrum of the Laplacian

As in the Chapter 1, we consider the eigenvalue problem for the Laplacian on a domain. Namely, the space $C^\infty(\Omega)$ which is defined by the space of all classes of infinitely differentiable functions on $\Omega \subset \mathbb{R}^d$ for $d \geq 1$. We seek to find pairs (λ, u) consisting of λ, which is called an eigenvalue of the Laplace operator $-\Delta$, and a non-zero function $u \in C^\infty(\Omega)$ which is the eigenfunction of the Laplace operator $-\Delta$ corresponding to the eigenvalue λ so that the following condition is satisfied:

$$\begin{cases} -\Delta u = \lambda u, & \text{in} \quad \Omega \\ u \text{ satisfies Dirichlet conditions} & \text{on} \quad \partial\Omega\,. \end{cases} \tag{3.1}$$

Such eigenvalue/eigenfunction pairs have some very nice properties, some of which we will explore here. The study of eigenvalue problems involving the Laplace operator goes back to a basic result in the elementary theory of partial differential equations that asserts that the problem possesses an unbounded sequence of eigenvalues. See [21] and [25].

Theorem 3.2.1. (General result for the Laplace operator on a bounded domain). The spectrum of the Laplace operator is discrete when Ω is a bounded open set in $\mathbb{R}^d$ for $d \geq 1$ with a smooth (or piecewise smooth) boundary $\partial\Omega$. By piecewise smooth, we mean that $\partial\Omega$ is the union of a finite number of smooth arcs or pieces of curves, for example, a rectangle (see [21]). Moreover, the eigenvalue problem (3.1) has an unbounded sequence of eigenvalues

$$0 \leq \lambda_0 \leq \lambda_1 \leq ... \leq \lambda_n \leq ...\cdot$$

This result goes back to the Riesz-Fredholm theory of self-adjoint and compact operators on Hilbert spaces (see [25] ,pp. $378 - 380$). In what concerns λ_0 being the lowest eigenvalue of problem (3.1), we remember that it can be characterised from a variational point of view as the minimum of the Rayleigh quotient, that is,

$$\lambda_0 = \inf_{u \in C^\infty(\Omega)} \frac{\displaystyle\int_\Omega |\nabla u(t)|^2 \, dt}{\displaystyle\int_\Omega |u(t)|^2 \, dt}, \tag{3.2}$$

where the infimum is taken over $C^\infty(\Omega)$ of the domain of the Laplace operator with Dirichlet and Neumann boundary conditions. Moreover, it is known that λ_0 is simple that is all the associated eigenfunctions are merely multiples of each other (see, e.g., Gilbarg and Trudinger [11]).

In the following sections, we aim to solve the eigenvalue equations with boundary conditions (Dirichlet, Neumann or mixture (Dirichlet and Neumann)) in different spaces.

3.3 Laplacian on a Bounded Domain

In this Section, we determine the eigenvalues and eigenfunctions of the problem $-\Delta u = \lambda u$ in one dimension, such as a closed interval $[0, b]$. Then, we generalise these examples from $[0, b]$ to an arbitrary interval $[a, b]$ for $a, b \in \mathbb{R}$ with the different boundary conditions.

Example 1

Consider the eigenvalue equation

$$-\Delta u = \lambda u \tag{3.3}$$

on an interval $[0, L]$ with Dirichlet boundary conditions $u(L) = u(0) = 0$. Then, we can consider three cases on λ :

- If $\lambda = 0$, the general solution is

  $$u(t) = At + B$$

  where A, B are constants then, $u(0) = B = 0$ and $u(L) = AL + B = 0$. It follows that 0 is not an eigenvalue for this problem.

- If $\lambda < 0$ so that $\lambda = -\mu^2 < 0$ for $\mu > 0$, then the eigenvalue equation

$$-\Delta u = \lambda u,$$

has the general solution of the form

$$u(t) = A\exp(-\mu t) + B\exp(\mu t).$$

Therefore, $u(t) = 0$ is a trivial function, and this problem has no negative eigenvalues.

- If $\lambda > 0$ and $\lambda = \mu^2$ such that $\mu > 0$, then the equation

$$-\Delta u = \lambda u$$

has the general solution

$$u(t) = A\cos(\mu t) + B\sin(\mu t).$$

To observe that, the eigenfunctions are

$$u_n(t) = \sin(\frac{n\pi}{L}t) \quad \text{for } n \geq 1;$$

with the eigenvalues

$$\lambda = (\frac{n\pi}{L})^2 \quad \text{for } n \geq 1.$$

Example 2

Consider the equation

$$-\Delta u = \lambda u \tag{3.4}$$

on $[0, L]$ with Neumann boundary conditions that are $u^{'}(L) = u^{'}(0) = 0$. Similarly, consider three cases on λ :

- If $\lambda = 0$, the general solution is

$$u(t) = At + B.$$

So $\lambda = 0$ is an eigenvalue with a corresponding eigenfunction $u = 1 \neq 0$ where we take $B = 1$ for convenience.

- If $\lambda < 0$ so that $\lambda = -\mu^2 < 0$ for $\mu > 0$, then the equation

$$-\Delta u = \lambda u$$

has a general solution

$$u(t) = A\cosh(-\mu t) + B\sinh(\mu t).$$

We have that,

$$u^{'}(t) = -\mu A\sinh(-\mu t) + \mu B\cosh(\mu t).$$

So, $\lambda < 0$ is not an eigenvalue for this problem.

- If $\lambda > 0$, let $\lambda = \mu^2$ then, the general solution is

$$u(t) = A\cos(\mu t) + B\sin(\mu t),$$

giving

$$u^{'}(t) = -\mu A\sin(\mu t) + \mu B\cos(\mu t).$$

Therefore,

$$\lambda = \lambda_n = \frac{n^2\pi^2}{L^2} \quad \text{for} \quad n \geq 0.$$

Therefore, the eigenfunctions are

$$u_n(t) = \cos(\frac{n\pi}{L}t) \quad \text{for} \quad n \geq 0.$$

Example 3

The eigenvalue problem

$$-\Delta u = \lambda u \tag{3.5}$$

on $[0, L]$ with mixed conditions (Dirichlet and Neumann boundary conditions), $u(L) = 0$ and $u^{'}(0) = 0$.

Consider three cases on λ :

- If $\lambda = 0$, the general solution is

$$u(t) = At + B,$$

where A and B are constants. Here, 0 is not an eigenvalue for this problem.

- If $\lambda < 0$, with $\lambda = -\mu^2 < 0$ for $\mu > 0$, then the general solution is

$$u(t) = A\exp(-\mu t) + B\exp(\mu t).$$

Then,

$$u^{'}(t) = -\mu A\exp(-\mu t) + \mu B\exp(\mu t).$$

This problem has no negative eigenvalues.

- If $\lambda > 0$ with $\lambda = \mu^2$, then the general solution is

$$u(t) = A\cos(\mu t) + B\sin(\mu t).$$

Then,

$$u^{'}(t) = -\mu A\sin(\mu t) + \mu B\cos(\mu t).$$

The eigenfunctions are

$$u_n(t) = \cos(\frac{n\pi}{L})t \quad \text{for } n = 1{,}3{,}5{,}...,$$

with eigenvalues

$$\lambda = \frac{n^2\pi^2}{4L^2} \qquad \text{for } n = 1{,}3{,}5{,}...\cdot$$

Remark 17. In the previous examples, we observe the eigenfunctions that have corresponding eigenvalues for the equation $-\Delta u = \lambda u$ on $[0, L]$ with Dirichlet, Neumann or mixed boundary conditions. We can generalise this equation to be on $[-L, L]$ to observe the eigenvalues in this domain. Instead, we simply specify that the solution must be the same for the two boundaries and the derivative. Also, this type of boundary condition will typically be on an interval of the form $[-L, L]$ instead of $[0, L]$. In summary, for the equation:

$$-\Delta u = \lambda u. \tag{3.6}$$

on $[-L, L]$ with Dirichlet boundary conditions, we can observe the eigenfunctions

$$\sin\frac{n\pi(t+L)}{2L} = \sin(\frac{n\pi t}{2L} + \frac{n\pi}{2}) = \begin{cases} (-1)^n \sin\frac{n\pi t}{2L} & n \text{ is even} \\ \\ (-1)^n \cos\frac{n\pi t}{2L} & n \text{ is odd} \end{cases}$$

with eigenvalues

$$\lambda = \frac{n^2\pi^2}{4L^2} \quad \text{for } n = 1{,}2{,}3{,}\ldots\cdot$$

In general, for any arbitrary closed interval $[a, b]$, the eigenfunctions will be

$$\sin\frac{n\pi}{b-a}(t-a),$$

with eigenvalues

$$\lambda_n = \frac{n^2\pi^2}{(b-a)^2}.$$

Similarly, the same steps can be followed for Neumann boundary conditions and mixed boundary conditions.

Note that: In an appendix $3, 4$ and 5, there are certain examples to calculate the eigenvalues and eigenfunctions.

3.4 Laplacian on an Unbounded Domain

The Partial differential equations when encountered on an unbounded domain require additional considerations (see [30]). In this section, we give an important problem occurring on unbounded domains when computing eigenvalues for the equation $-\Delta u = \lambda u$. In particular, we place an emphasis on the following steps of which the Fourier transform may be seen as coming from:

(a) Unbounded domain $\Omega \subset \mathbb{R}$, especially whole space domain.

(b) The appearance of a continuous spectrum for the partial differential operator.

(c) Fourier transform of arbitrary functions.

Consider the equation

$$-\Delta u = \lambda u,$$

on $\mathbb{R}$, we can rewrite the equation by

$$-(\frac{d^2u}{dt^2}) = \lambda u(t).$$

The main tool for solving this equation on an unbounded domain is the Fourier Transform

$$U : L^2(\mathbb{R}) \to L^2(\mathbb{R})$$

given by

$$Uu(t) = \int_{\mathbb{R}} u(t)e^{-2\pi it}dt.$$

Integrating by parts we have

$$U(\partial_{t_j}u)(t) = 2\pi tUu(t),$$

and would imply,

$$U(-\Delta u)(t) = 4\pi^2\|t\|^2 Uu(t),$$

for $u \in \text{Dom}(-\Delta)$.

Lemma 3.4.1. Now, we prove that $-\Delta$ is an essentially self-adjoint operator.

Proof. By showing $\text{Ran}(-\Delta + z)$ is a dense in $L^2(\mathbb{R})$, that is by using Proposition 2.4.1. Let $z = \pm i$ to show that $\text{Ran}(-\Delta \pm i)$ is a dense in $L^2(\mathbb{R})$. It is easily to prove its orthogonal complement, that is

$$\text{Ran}(-\Delta \pm i)^{\perp} = \{0\}.$$

Suppose $u \in L^2(\mathbb{R})$ be such that

$$\langle u, (-\Delta \pm i)v\rangle = 0,$$

for all $v \in \text{Dom}(-\Delta) = C_c^{\infty}(\mathbb{R})$. Because the Fourier transform is an unitary [33], we get

$$0 = \langle Uu, U(-\Delta \pm i)v\rangle.$$

For $v \in C_c^{\infty}(\mathbb{R})$, and $U(-\Delta u)(t) = 4\pi^2\|t\|^2 Uu(t)$ so this becomes

$$0 = \langle Uu, (4\pi^2\|t\|^2 \pm i)Uv\rangle.$$

We need to prove $u = 0$, we can get that $\langle (4\pi^2\|t\|^2 \mp i)Uu, Uv\rangle$, for all $v \in \mathrm{Dom}(-\Delta)$. Because the Fourier transform is unitary again, $\mathrm{Dom}(-\Delta)$ is dense in $L^2(\mathbb{R})$ and so is $U\,\mathrm{Dom}(-\Delta)$ is dense in $L^2(\mathbb{R})$, so this would imply $0 = (4\pi^2\|t\|^2 \mp i)Uu$, it follows, $u = 0$. Thus, $-\Delta$ is an essentially self-adjoint operator. $\square$

Also, the formula above shows the Laplace operator $(\mathrm{Dom}(-\Delta), -\Delta)$ is unitarily equivalent with the multiplication operator M on $U\,\mathrm{Dom}(-\Delta)$. This is an essentially self-adjoint operator. We can define the multiplication operator (D, M) acting on $L^2(\mathbb{R})$ by

$$Mu(t) = 4\pi^2\|t\|^2 u(t),$$

where

$$D = \{u \in L^2(\mathbb{R}) | \quad t \to \|t\|^2 u(t) \in L^2(\mathbb{R})\},$$

and $\|t\|^2$ is the norm of $\mathbb{R}$. Indeed (D, M) is a self-adjoint operator, and so is the closure of $(U\,\mathrm{Dom}(\Delta), M)$. Now, by using the inverse Fourier transform, it follows that the closure of $-\Delta$ is unitarily equivalent to (D, M). Thus, $Mu(t)$ is the Fourier transform of the function $-\dfrac{\partial^2 u}{\partial t^2}$, which is continuous of compact support in $L^2(\mathbb{R})$, meaning we can write

$$-\Delta = U^{-1}MU.$$

Finally, since the range (or the essential range) of the multiplication operator is $[0, \infty)$, it follows that

$$\sigma(-\Delta) = [0, \infty).$$

The spectrum is a continuous spectrum since it is clear that there is no eigenvalue of the multiplication operator (see, for example, [29]).

3.5 Laplacian on a Cylindrical Domain

Consider an unbounded domain of the form $\mathbb{R} \times \Omega$, where $\Omega \subseteq \mathbb{R}^{d-1}$ for $d \geq 1$ is a bounded domain and the eigenvalue equation

$$-\Delta u = \lambda u$$

is everywhere on $\mathbb{R}$ and the Laplacian coincides with the decoupled operator

$$-\Delta = (-\Delta_{\mathbb{R}}) \otimes I + I \otimes (-\Delta_{\Omega}) \tag{3.7}$$

on $L^2(\mathbb{R}) \otimes L^2(\Omega) = L^2(\mathbb{R} \times \Omega)$, where I denotes the identity operator for the appropriate spaces. The operator $-\Delta_{\Omega}$ is a self-adjoint Laplatian operator on a bounded region $\Omega \subseteq \mathbb{R}^{d-1}$ for $d \geq 1$. Now, we find the spectrum of $-\Delta$ we have: First, the spectrum of the operator $-\Delta_{\Omega}$ is

$$\sigma(-\Delta_{\Omega}) = \{\lambda_0, \lambda_1, ...\}.$$

The reader can refer back to Section 3.3. Second, the spectrum of the operator $-\Delta_{\mathbb{R}}$ is

$$\sigma(-\Delta_{\mathbb{R}}) = [0, \infty).$$

The reader can refer back to Section 3.4. In view of the equation (3.7) and [36, Corollary page 301], this is proved the straight strip has continuous spectrum starting from the first eigenvalue of the Laplacian

$$\begin{aligned} \sigma(-\Delta) &= \sigma(-\Delta_{\mathbb{R}}) + \sigma(-\Delta_{\Omega}) \\ &= [0, \infty) + \{\lambda_0, \lambda_1, ...\}. \end{aligned}$$

Therefore,

$$\begin{aligned} \sigma(-\Delta) &= \cup_{j=0}^{\infty} [0, \infty) + \lambda_j \\ &= \cup_{j=0}^{\infty} [\lambda_j, \infty) \\ &= [\lambda_0, \infty) \cup [\lambda_1, \infty) \cup ... \\ &= [\inf_{j=0} \lambda_j, \infty) \\ &= [\lambda_0, \infty). \end{aligned}$$

In view of 3.7 and [36, Thm VIII.33] this is shown the spectrum of $-\Delta$ on unbounded domain is the form

$$\sigma(-\Delta) = [\lambda_0, \infty),$$

where λ_0 is the smallest eigenvalue of $(-\Delta_{\Omega})$.

3.6 Symmetry Operator S

In this section, we investigate some properties of a symmetry operator S and understand the commutativity between a bounded operator and an unbounded operator.

3.6.1 Properties of the Symmetry operator S

Now, we consider a symmetry operator S when it acts on any function $u(t)$ in $L^2(\mathbb{R})$ is defined as

$$S : L^2(\mathbb{R}) \to L^2(\mathbb{R})$$

by

$$Su(t) = u(-t).$$

Lemma 3.6.1. There are some important properties of a Symmetry operator S:

- The symmetry operator is a bounded linear operator.
- $S^2 = I$ and S is a symmetric operator hence, self-adjoint operator and unitary.
- The eigenvalues of symmetry operator S are ± 1,

$$\sigma(S) \subseteq \{-1, 1\}.$$

- The eigenfunctions of the operator S are the symmetric (even function) or antisymmetric (odd function) with the respective eigenvalues $\lambda = +1$ and $\lambda = -1$. See, for example, [20], pp. $257 - 266$.

Proof. 1) We need to prove S is a bounded linear operator, that is, there exists a constant $m > 0$ such that

$$\|Su\|_{L^2(\mathbb{R})} \leq m\|u\|_{L^2(\mathbb{R})}.$$

We can consider the norm,

$$\|u\|^2_{L^2(\mathbb{R})} = \int_{-\infty}^{\infty} |u(t)|^2 dt.$$

Then,

$$\|Su\|^2_{L^2(\mathbb{R})} = \int_{-\infty}^{\infty} |Su(t)|^2 dt = \int_{-\infty}^{\infty} |u(-t)|^2 dt.$$

Now, let $-t = w$. It follows $dt = -dw$ such that

$$t = -\infty \Rightarrow w = \infty$$

and

$$t = \infty \Rightarrow w = -\infty.$$

Thus,

$$\|Su\|^2_{L^2(\mathbb{R})} = \int_{-\infty}^{\infty} |u(-t)|^2 dt = \int_{-\infty}^{\infty} |u(w)|^2 dw = \|u\|^2_{L^2(\mathbb{R})}.$$

Therefore, we can take $m = 1$, and S is a bounded operator.

2) Now, we want to prove S is a symmetric operator

$$\langle Su_1, u_2 \rangle = \langle u_1, Su_2 \rangle.$$

We can consider that

$$\begin{aligned} \langle Su_1, u_2 \rangle &= \textstyle\int_{-\infty}^{\infty} \overline{Su_1(t)} u_2(t) dt \\ &= \textstyle\int_{-\infty}^{\infty} \overline{u_1(-t)} u_2(t) dt. \end{aligned}$$

Now, let $-t = w$. It follows $dt = -dw$ such that

$$t = -\infty \Rightarrow w = \infty,$$

and

$$t = \infty \Rightarrow w = -\infty.$$

Therefore,

$$\begin{aligned} \langle Su_1, u_2 \rangle &= \int_{-\infty}^{\infty} \overline{u_1(w)} u_2(-w) dw \\ &= \int_{-\infty}^{\infty} Su_2(w) \overline{u_1(w)} dw \\ &= \int_{-\infty}^{\infty} \overline{u_1(w)} Su_2(w) dw \\ &= \langle u_1, Su_2 \rangle. \end{aligned}$$

So, S is symmetric hence S is a self-adjoint operator.

3) Suppose λ is an eigenvalue of S with eigenfunction u we have the eigenvalue equation

$$Su(t) = \lambda u(t).$$

That is,

$$S^2 u(t) = S \cdot Su(t) = S \cdot \lambda u(t) = \lambda^2 u(t).$$

But, $S^2 = I$ gives

$$u(t) = \lambda^2 u(t).$$

That is, $\lambda^2 = 1$. The only possible eigenvalues of S are

$$\lambda = +1, \quad Su(t) = u(t) \quad \text{even function,}$$

and

$$\lambda = -1, \quad Su(t) = -u(t) \quad \text{odd function.}$$

□

3.6.2 Commutativity

We say that a closed densely defined operator A is defined in a Hilbert space H has a symmetry operator S if

$$[A, S] = 0, \tag{3.8}$$

that is, A commutes with S. It is called the *commutativity* of an unbounded operator with a bounded operator (see, for example, [20]). The importance of commuting operators is that these operators have simultaneous eigenfunctions whenever there exist. Suppose $u \neq 0$ is an eigenfunction of A with eigenvalue λ, that is,

$$Au(t) = \lambda u(t).$$

Now, u may not be an eigenfunction of S. However, A commutes with S that is, $[A, S] = 0$. Then, we have

$$A(Su(t)) = S(Au(t)) = S(\lambda u(t)) = \lambda(Su(t)).$$

This equation states that the function $Su \neq 0$ is also an eigenfunction of the operator A with the eigenvalue λ, together with the eigenfunction u. Now, we set

$$u_{\pm} = \frac{1}{2}(I \pm S)u = \frac{1}{2}(u \pm Su).$$

Then, u_+ and u_- will be also eigenfunctions of A with an eigenvalue λ while

$$\begin{aligned} Su_{\pm} &= S(\frac{1}{2}(I \pm S)u) \\ &= \frac{1}{2}(Su \pm S^2 u) \\ &= \frac{1}{2}(Su \pm u) \\ &= \pm u. \end{aligned}$$

So, u_+ and u_- are eigenfunctions of S with the eigenvalues are $+1$ and -1. Also, note that

$$u(t) = \frac{1}{2}[u(t) + u(-t) - u(-t) + u(t)] = \frac{1}{2}[u(t) + u(-t)] + \frac{1}{2}[u(t) - u(-t)] = u_+ + u_-.$$

It follows that when we look for the eigenfunctions of A, it is sufficient to look for eigenfunctions that are simultaneous eigenfunctions of A and S. We have previously seen examples of an eigenvalue equation that is either symmetric or antisymmetric with respect to the origin with different boundary conditions. However, now, we consider the below example for an eigenvalue equation on the interval $[-L, L]$ with observation the symmetry operator when it affects on this equation. We observe the simultaneous eigenfunctions of operators A and S.

Example

Consider the operator $A = -\Delta$ and the eigenvalue problem is

$$-\Delta u = \lambda u$$

on the closed interval $[-L, L]$ with Dirichlet conditions $u(-L) = u(L) = 0$. As we saw in Section 3.3 for Example (1), the eigenfunctions are

$$u_n(t) = \sin\frac{n\pi(t+L)}{2L} = \sin(\frac{n\pi t}{2L} + \frac{n\pi}{2}) = \begin{cases} (-1)^n \sin\dfrac{n\pi t}{2L} & n \text{ is even} \\ \\ (-1)^n \cos\dfrac{n\pi t}{2L} & n \text{ is odd,} \end{cases} \tag{3.9}$$

with corresponding eigenvalues

$$\lambda_n = \frac{\pi^2 n^2}{4L^2} \quad \text{for} \quad n = 1,2,3,.... \tag{3.10}$$

Symmetry consideration

Now, we want to observe the simultaneous eigenfunctions of Laplace operator $-\Delta$ and S. We apply the symmetry operator S on

$$-\Delta u(t) = \lambda u(t).$$

It is easy to check the operators $-\Delta$ commutes with S, that is $[-\Delta, S] = 0$. Therefore, $u(-t) = u(t)$ or $u(-t) = -u(t)$ with $u(-L) = u(L) = 0$. This leads to separate the function into

1) $-\Delta u(t) = \lambda u(t)$, $u(-L) = u(L)$ and u is even.

2) $-\Delta u(t) = \lambda u(t)$, $u(-L) = -u(L)$ and u is odd.

If u is even, it means $u(-t) = u(t)$ and $u^{'}(0) = 0$. So, it is enough to consider the operator $A_e = -\Delta$ on $[0, L]$ with the mixed boundary conditions

$$u(L) = 0 \quad \text{and} \quad u^{'}(0) = 0.$$

Referring to the Section 3.3, Example (3). Then, the eigenfunctions will be

$$u_n^e(t) = \cos(\frac{n\pi}{L})t \quad \text{for} \quad n = 1,3,5,..., \tag{3.11}$$

with eigenvalues

$$\lambda_n^e = \frac{\pi^2 n^2}{4L^2} \quad \text{for } n = 1,3,5,...\cdot \tag{3.12}$$

Similarly, if u is odd, it means $u(-t) = -u(t)$ and $u(0) = 0$. So it is enough to consider $A_o = -\Delta$ on $[0, L]$ with Dirichlet boundary conditions

$$u(L) = 0 \quad \text{and} \quad u(0) = 0.$$

Refer back to the Section 3.3, Example (1). The eigenfunctions are

$$u_n^o(t) = \sin(\frac{n\pi}{L})t \quad \text{for } n \geq 1, \tag{3.13}$$

with the eigenvalues

$$\lambda_n^o = \frac{n^2\pi^2}{L^2} \quad \text{for } n \geq 1. \tag{3.14}$$

Symmetries can often be used to simplify the problem at hand. It means, at our work if we combine the spectrum of the operators A_e and A_o, we have

$$\begin{aligned}
&\sigma(A_e) \cup \sigma(A_o) \\
&\quad = \left\{ \frac{\pi^2 n^2}{4L^2}, \quad n = 1{,}3{,}5{,}... \right\} \cup \left\{ \frac{\pi^2 n^2}{L^2}, \quad n = 1{,}2{,}3{,}... \right\} \\
&\quad = \left\{ \frac{\pi^2 n^2}{4L^2}, \quad n = 1{,}3{,}5{,}... \right\} \cup \left\{ \frac{\pi^2 n^2}{4L^2}, \quad n = 2{,}4{,}6{,}... \right\} \\
&\quad = \left\{ \frac{\pi^2 n^2}{4L^2}, \quad n = 1{,}2{,}3{,}... \right\} \\
&\quad = \sigma(A).
\end{aligned}$$

Therefore, $\sigma(A) = \sigma(-\Delta) = \{\frac{\pi^2 n^2}{4L^2}, \quad n = 1{,}2{,}3{,}...\}$. This means that the eigenfunctions corresponding to the above eigenvalues are simultaneous eigenfunctions of $-\Delta$ and S. One of the goals of this example is to understand the role the symmetry property to observe the simultaneous eigenfunctions which preserves this property in the sense above. In other words, the symmetry operator S can always be split into even and odd functions, and if we add the spectrum of the even operator with the spectrum of odd operator together, we would get the main spectrum of Laplace operator on the interval $[-L, L]$. We again observe the spectrum of eigenvalues of Laplace operator (main operator) decompose into pairs of spectrum because of the presence of the symmetry operator.

3.6.3 For a Symmetric Potential V and $[-\Delta - V, S] = 0$

We have the operator $A = -\Delta - V$ with potential V, which is a symmetric (an even function) $V(t, s) = V(t, -s)$ for $t, s \in \mathbb{R}$, and we consider S is a symmetry operator in two dimensions defined by $S : L^2(\mathbb{R} \times [-L, L]) \to L^2(\mathbb{R} \times [-L, L])$ such that $Su(t, s) = u(t, -s)$. The operator of the form is

$$-\Delta - V = -\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial s^2} - V(t, s).$$

First we show this operator $-\Delta - V$ commutes with the operator S. In particular, we have $S(-\Delta u) = -\Delta Su$ and $SVu = VSu$. Since, V is symmetric and from the definition of S we have

$$\begin{aligned}(-\Delta - V)Su(t,s) &= (-\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial s^2} - V(t,s))Su(t,s) \\ &= (-\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial s^2} - V(t,s))u(t,-s) \\ &= -\frac{\partial^2 u}{\partial t^2}(t,-s) - \frac{\partial^2 u}{\partial s^2}(t,-s) - V(t,s)u(t,-s).\end{aligned}$$

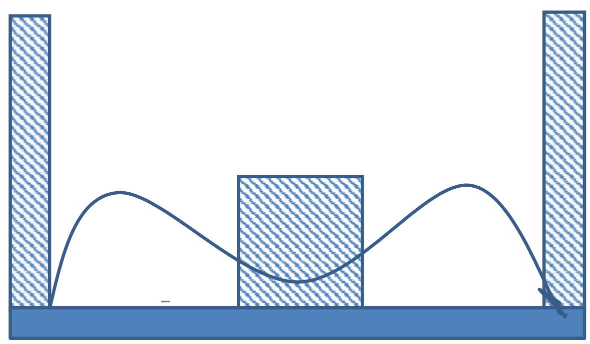
On the other hand,

$$\begin{aligned}S((-\Delta - V)u(t,s)) &= S(-\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial s^2} - V(t,s))u(t,s) \\ &= S(-\frac{\partial^2 u}{\partial t^2}(t,s) - \frac{\partial^2 u}{\partial s^2}(t,s) - V(t,s)u(t,s)) \\ &= -\frac{\partial^2 u}{\partial t^2}(t,-s) - \frac{\partial^2 u}{\partial s^2}(t,-s) - V(t,-s)u(t,-s),\end{aligned}$$

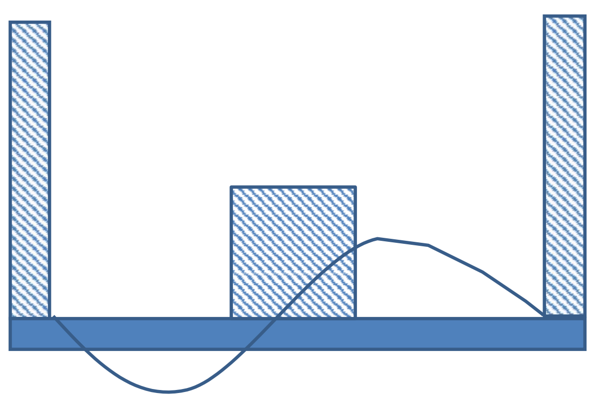
where $V(t,s) = V(t,-s)$. Therefore,

$$[-\Delta - V, S] = 0.$$

See Figure 3.1



(a) The Symmetrical state wavestate represented by two finite square wells.



(b) The Antisymmetrical wavestate.

Figure 3.1: The Symmetrical and Anti-symmetrical wavestate

3.6.4 Symmetry operator in Physics applications

The fact that the symmetry operator is Hermitian means that it is, technically, an observable. As above we observed the eigenstate of symmetry are particularly useful when symmetry commutes with unbounded operator.

Definition 3.6.1. (Ground state)

Ground state of a quantum-mechanical system its lowest energy state; the energy of the ground state is known as the zero-point energy of the system. See [34].

Symmetry as a Quantum Number

The fact that the symmetry operator is Hermitian means that it is technically, an observable. More pertinently, we can find eigenstate of the symmetry operator

$$s|\psi\rangle = \eta_\psi|\psi\rangle,$$

where η_ψ is called the symmetry of the state $|\psi\rangle$. Using the fact that $s^2 = 1$. The symmetry eigenstates are particularly useful when symmetry with the Hamiltonian H,

$$sHs^\dagger = H \Leftrightarrow [s, H] = 0.$$

In this case, the energy eigenstates can be assigned definite symmetry. This follows immediately when the energy level is non-degenerate. But, even when the energy level is degenerate, general theorems of linear algebra ensure that we can always pick a basis within the eigenspace which have definite symmetry. See [20].

Example: Harmonic Oscillator

As a simple example, we can consider the one-dimensional harmonic oscillator. The Hamiltonian is

$$H = \frac{1}{2m}p^2 + \frac{1}{2}mw^2x^2.$$

The simplest way to build the Hilbert space is introduce raising and lowering operator $a \sim (x + ip/mw)$ and $a^\dagger \sim (x - ip/mw)$ (up to a normalisation constant). The ground sate $|0\rangle$ obeys $a|0\rangle = 0$ while higher state are built by $|n\rangle \sim (a^\dagger)^n|0\rangle$. The Hamiltonian is invariant under symmetry $[s, H] = 0$ which means that all energy eigenstates must have a definite symmetry. Since the creation operator $a^\dagger$ is linear in x and p, we have

$$sa^\dagger s = -a^\dagger$$

This means that the symmetry of the state $|n+1\rangle$ is

$$s|n+1\rangle = sa^\dagger|n\rangle = -a^\dagger s|n\rangle \quad \text{then} \quad \eta_{n+1} = -\eta_n.$$

We learn that the excited states alternate in their symmetry. To see their absolute value we need only determine the symmetry of the ground state. This is

$$\psi_0(x) = \langle x|0\rangle \sim e^{(-\frac{mwx^2}{2h})}.$$

Since the ground state doesn't change under reflection we have $\eta_0 = +1$ and, in general $\eta_n = (-1)^n$. See [20].

3.7 Variational Principle and inf$(\sigma(-\Delta - V))$

The variational principle, is a very powerful tool when studying a self-adjoint linear operators A on a Hilbert space H. There are many things that can be proved by using the variational princilple methods, for example, the bounds for eigenvalues and monotonicity of eigenvalues (see, for example, [39]). For a given function w on a set $\Omega \subset \mathbb{R}^d$ for $d \geq 1$, we define the classical variational principle based on the Rayleigh functional of w on Ω as

$$\frac{\langle Aw, w\rangle}{\|w\|^2_{L^2(\Omega)}},$$

applies only to a semi-bounded operator (see, for example, [39]). In this section, we consider this principle to obtain a quantitv estimate of eigenvalues and for comparing the eigenvalues of different operators (see [19]). This principle allows us to calculate an upper bound for the ground state energy (The energy of the ground state is known as the zero-point energy of a quantum-mechanical system) by finding a trial wave function u for which the integral is minimised. More precisely, the goal of this section is to find $u \neq 0$ and $u \in L^2(\mathbb{R} \times \Omega)$ for $\Omega \subseteq \mathbb{R}^d$ and $d \geq 1$ such that the inequality

$$\langle(-\Delta - V)u, u\rangle < \lambda_0\|u\|^2_{L^2(\mathbb{R}\times\Omega)}$$

holds where λ_0 is an eigenvalue of the operator $-\Delta$. Note that $\langle(-\Delta - V)u, u\rangle$ is a quadratic form of the operator $-\Delta - V$ for this u. The proof of this is simple. We claim the ground state of the operator $-\Delta - V$ is less than the ground state energy λ_0 by using the Variational principle. We consider again the operator

$$-\Delta - V,$$

on $\mathbb{R} \times \Omega$ where a sufficiently regular, real-valued, continuous and bounded support function V on $C_0^\infty(\mathbb{R} \times \Omega)$ is called *potential* with $V \geq 0$. Let ψ_0 be a ground state eigenfunction with the ground state energy eigenvalue λ_0 of the operator $-\Delta_\Omega$. In particular,

$$-\Delta_\Omega \psi_0(s) = \lambda_0 \psi_0(s).$$

Lemma 3.7.1. We have,

$$\inf(\sigma(-\Delta - V)) < \lambda_0, \tag{3.15}$$

for $V \geq 0$ and $V \not\equiv 0$.

Proof. For $R > 0$, we can define a trial wave function u_R by

$$u_R(t, s) = \varphi_R(t)\psi_0(s),$$

where φ_R is defined by (see Figure 3.2)

$$\varphi_R(t) = \begin{cases} 1 & |t| \leq R \\ 2 - \dfrac{|t|}{R} & R < |t| < 2R \\ 0 & |t| \geq 2R \end{cases}$$

and the distributional gradient will be

$$\nabla u_R(t, s) = (\varphi_R^{'}(t)\psi_0(s), \varphi_R(t)\nabla_s \psi_0(s)).$$

The norm of u_R is then given by

$$\begin{aligned}\|u_R\|^2_{L^2(\mathbb{R})} &= \int_{-2R}^{2R} \varphi_R^2(t)dt \int_\Omega |\psi_0(s)|^2 ds \\ &= [2R + \frac{2R}{3}] \int_\Omega |\psi_0(s)|^2 ds \\ &= \frac{8R}{3} \int_\Omega |\psi_0(s)|^2 ds.\end{aligned}$$

In addition, the norm of distributional gradient ∇u_R is given by

$$\begin{aligned}\|\nabla u_R\|^2_{L^2(\mathbb{R})} = \langle -\Delta u_R, u_R\rangle &= \left\langle -(\frac{d^2}{dt^2} + \Delta_\Omega)u_R, u_R\right\rangle \\ &= \langle -\Delta_\Omega u_R, u_R\rangle + \left\|\frac{d}{dt}u_R\right\|^2_{L^2(\mathbb{R})} \\ &= \int_{-2R}^{2R} \varphi_R^2(t)dt\, \langle -\Delta_\Omega \psi_0, \psi_0, \rangle + \left\|\frac{d}{dt}u_R\right\|^2_{L^2(\mathbb{R})} + \\ &= \int_{-2R}^{2R} \varphi_R^2(t)dt\, \langle \lambda_0 \psi_0, \psi_0\rangle + \left\|\frac{d}{dt}u_R\right\|^2_{L^2(\mathbb{R})} \\ &= \int_{-2R}^{2R} \varphi_R^2(t)dt \int_\Omega \lambda_0 |\psi_0(s)|^2 ds + \int_{-2R}^{2R} |\varphi_R^{'}(t)|^2 dt \int_\Omega |\psi_0(s)|^2 ds \\ &= \frac{8R}{3}\lambda_0 \|\psi_0\|^2_{L^2(\mathbb{R})} + \frac{2}{R}\|\psi_0\|^2_{L^2(\mathbb{R})}.\end{aligned}$$

Given a non-trivial function $V \geq 0$, we can find $\epsilon > 0$, $I \subseteq \mathbb{R}$, and $\Pi \subseteq \Omega$ such that $V \geq \epsilon > 0$ for any $(t, s) \in I \times \Pi$, and

$$\int_\Pi |\psi_0(s)|^2 ds > 0.$$

Here, $\psi_0(s)$ is solution of the eigenvalue equation. However, by the Classical unique continuation principle for the eigenvalue equation, $(-\Delta - V)\psi_i = 0$ in a open set Ω for $i = 1, 2$ and if $\psi_1 = \psi_2$ on $\Omega^{'}$ where $\Omega^{'} \subseteq \Omega$ is open and non empty set, then $\psi_1 = \psi_2$ on Ω, meaning the difference of two solutions vanish at the space at some point and the solutions must be identical in all the space (see [39], pp. 240).

Now, V has a bounded support (by assumption), then there exists R_0 such that $\text{supp}(V) \subseteq [-R_0, R_0] \times \Omega$. If $R > R_0$, then

$$u_R(t, s) = 1 \cdot \psi_0(s)$$

for $(t, s) \in \text{supp}(V)$, so

$$V u_R = V\psi_0.$$

Therefore, we have

$$\langle Vu_R, u_R\rangle \geq \epsilon |I| \int_{\Pi} |\psi_0(s)|^2 ds = \delta.$$

Then,

$$\begin{aligned}
\inf(\sigma(-\Delta - V)) &< \frac{\int_{\Omega} |\nabla u_R|^2 dtds - \int_{\Omega} V(t)|u_R(t)|^2 dtds}{\int_{\Omega} |u_R(t)|^2 dtds} \\
&< \frac{(\frac{2}{R} + \frac{8R}{3}\lambda_0) \|\psi_0\|^2_{L^2(\mathbb{R})} - \delta}{\frac{8R}{3} \|\psi_0\|^2_{L^2(\mathbb{R})}} \\
&< \lambda_0 + \frac{3}{8R}(\frac{2}{R} - \frac{\delta}{\|\psi_0\|^2_{L^2(\mathbb{R})}}) \\
&< \lambda_0,
\end{aligned}$$

when $R > 2(\frac{\|\psi_0\|^2_{L^2(\mathbb{R})}}{\delta})$ since $\frac{3}{4R^2} - \frac{3\delta}{8R\|\psi_0\|^2_{L^2(\mathbb{R})}} < 0$.

Therefore,

$$\inf(\sigma(-\Delta - V)) < \lambda_0,$$

by the variational principle. □

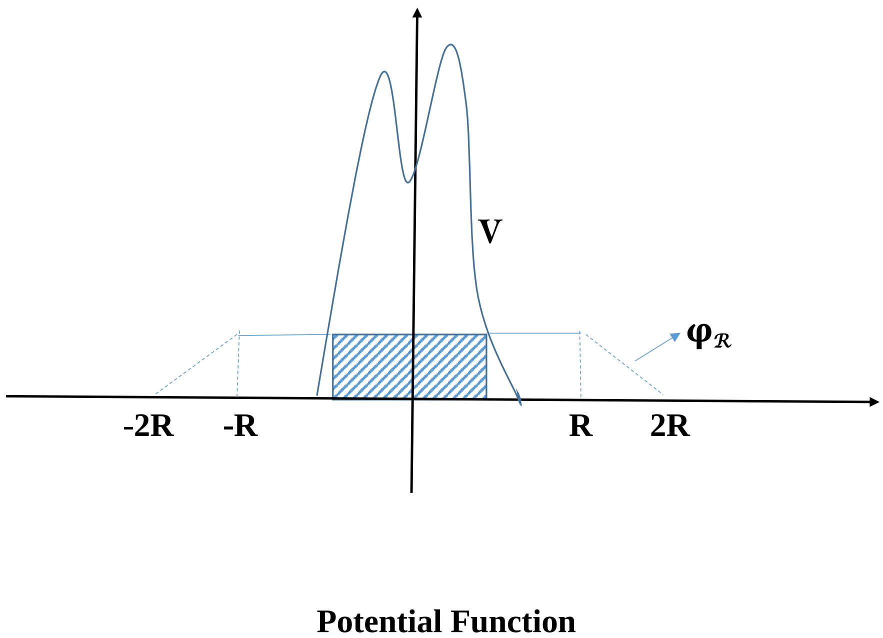


Figure 3.2: Potential function

3.8 Relatively Compact Perturbations

In this section, we observe the Relatively compact perturbations which, is mentioned in a simple way in Section 2.6.3; we will use this argument in the main result of the first task of the current thesis.

Theorem 3.8.1. Let A be a self-adjoint and semi-bounded operator on a Hilbert space H. An operator V with $\text{Dom}(A) \subseteq \text{Dom}(V)$ is a compact map from the $\langle \text{Dom}(A), \|.\|_{\text{Dom}(A)} \rangle$ into H if and only if V is a relatively compact perturbation with respect to the operator A, where the norm of the $\text{Dom}(A)$ is given by

$$\|u\|^2_{\text{Dom}(A)} = \|u\|^2_{L^2(\mathbb{R})} + \|Au\|^2_{L^2(\mathbb{R})}.$$

(See [39], pp. 113 and Section 2.6.3).

Definition 3.8.1. (Cone property)

For each $u \in \Omega$ is the vertex of a cone contained in Ω and congruent to cone where Ω is union of congruent cones.

Theorem 3.8.2. Let Ω be a domain in $\mathbb{R}^d$ for $d \geq 1$ and Ω has a cone property. Let $A = -\Delta$ be the Laplacian on Ω with any of the boundary conditions (Dirichlet, Neumann or a mixture). Suppose V is a continuous function with bounded support; then, multiplication by V defines a relatively compact perturbation with respect to the operator $-\Delta$.

Proof. First, we need to prove that multiplication by V defines a compact map

$$\text{Dom}(-\Delta) \to L^2(\Omega).$$

Since $-\Delta$ is a non-negative operator so that, $\langle -\Delta u, u \rangle \geq 0$ for $u \in \text{Dom}(-\Delta)$.

Consider the inclusion map,

$$\text{Dom}(-\Delta) \hookrightarrow Q(-\Delta);$$

we need to prove this map is continuous when $\text{Dom}(-\Delta)$ has the norm

$$\|u\|^2_{\text{Dom}(-\Delta)} = \|u\|^2_{L^2(\mathbb{R})} + \|\Delta u\|^2_{L^2(\mathbb{R})}$$

and $Q(-\Delta)$ has the norm

$$\|u\|^2_{Q(-\Delta)} = \langle -\Delta u, u \rangle + \|u\|^2_{L^2(\mathbb{R})}.$$

By the Cauchy-Schwartz inequality, it is easy to see

$$\begin{aligned}|\langle -\Delta u, u\rangle| &\leq \| -\Delta u\|_{L^2(\mathbb{R})}\|u\|_{L^2(\mathbb{R})} \leq \frac{1}{2}(\|u\|^2_{L^2(\mathbb{R})} + \|\Delta u\|^2_{L^2(\mathbb{R})}) \\ &= \frac{1}{2}\|u\|^2_{\text{Dom}(-\Delta)}.\end{aligned}$$

Thus, the inclusion map from $\text{Dom}(-\Delta)$ into $Q(-\Delta)$ is continuous.

We have $Q(-\Delta) \subseteq H^1(\Omega)$ (recall Section 2.8).

Suppose $\Omega^{'} \subseteq \Omega$ is a bounded open set. Then, the restriction gives us a compact embedding

$$H^1(\Omega) \to L^2(\Omega^{'}) = H^0(\Omega^{'})$$

by the Rellich-Kondarchov Theorem. See, for example, [50, Thm 6.2 Part 1 and 2, pp. 144]. We choose $\Omega^{'}$ such that

$$\text{supp}(V) \cap \Omega \subseteq \Omega^{'},$$

(recall that V has a bounded support). Here, $V \in L^\infty(\Omega^{'})$ multiplication by V gives us a bounded map

$$L^2(\Omega^{'}) \to L^2(\Omega^{'}).$$

Then, we can extend by 0 to get a bounded map

$$L^2(\Omega^{'}) \to L^2(\Omega).$$

Since $\text{supp}(V) \subseteq \Omega^{'}$. Thus, the composition

$$\text{Dom}(-\Delta) \hookrightarrow Q(-\Delta) \hookrightarrow H^1(\Omega) \to L^2(\Omega^{'}) \to L^2(\Omega),$$

is a simply multiplication by V as a map

$$\text{Dom}(-\Delta) \to L^2(\Omega).$$

Because every step is bounded and the restriction is compact. Therefore, the map

$$V : \text{Dom}(-\Delta) \to L^2(\Omega)$$

is a compact map. Second, by Theorem 3.8.1, it follows that V is a relatively compact perturbation with respect to the operator $-\Delta$. $\square$

3.9 Existence of Embedded Eigenvalues

In this section, we combine the ideas discussed in the previous sections and previous concepts to demonstrate that the operator

$$-\Delta - V$$

on $\mathbb{R} \times [-L, L]$ has embedded eigenvalues for some positive symmetric potential functions V.

Theorem 3.9.1. (Has been published in [41], March, 2020).
Consider on $\mathbb{R} \times [-L, L]$, the operator

$$-\Delta_D - V,$$

with Dirichlet boundary conditions on $\mathbb{R}\times\{-L\}$ and $\mathbb{R}\times\{L\}$. Suppose V denotes a sufficiently small non-negative continuous real valued function on $\mathbb{R} \times [-L, L]$ with bounded support which is symmetric

$$V(t, s) = V(t, -s)$$

for $t, s \in \mathbb{R} \times [-L, L]$. Then,

$$\sigma_{ess}(-\Delta_D - V) = [\lambda_1, \infty) \subseteq \sigma(-\Delta_D - V),$$

where $\lambda_1 = \dfrac{\pi^2}{4L^2}$ while there exists $\lambda > \lambda_1$ such that λ is an eigenvalue of $-\Delta_D - V$; more precisely, there exists $u \neq 0$ and

$$u \in \mathrm{Dom}(-\Delta_D - V) \subset L^2(\mathbb{R} \times [-L, L])$$

such that

$$(-\Delta_D - V)u = \lambda u.$$

Similarly, we can consider the operator

$$-\Delta_N - V,$$

on $\mathbb{R} \times [-L, L]$ with Neumann boundary conditions on $\mathbb{R} \times \{-L\}$ and $\mathbb{R} \times \{L\}$. Suppose V denotes a sufficiently small, non-negative continuous real valued function on $\mathbb{R} \times [-L, L]$ with bounded support which is symmetric

$$V(t, s) = V(t, -s)$$

for $t, s \in \mathbb{R} \times [-L, L]$. Then,

$$\sigma_{ess}(-\Delta_N - V) = [\lambda_0, \infty) \subseteq \sigma(-\Delta_N - V),$$

where $\lambda_0 = 0$ while there exists $\lambda > \lambda_0$ such that λ is an eigenvalue of $-\Delta_N - V$; more precisely, there exists $u \neq 0$ and

$$u \in \mathrm{Dom}(-\Delta_N - V) \subset L^2(\mathbb{R} \times [-L, L])$$

such that

$$(-\Delta_N - V)u = \lambda u.$$

Proof. If $V = 0$, then the operator $-\Delta_D$ on $\mathbb{R} \times [-L, L]$ has a continuous spectrum, in particular

$$\sigma(-\Delta_D) = [\lambda_1, \infty) = \left[\frac{\pi^2}{4L^2}, \infty\right),$$

with Dirichlet boundary conditions or

$$\sigma(-\Delta_N) = [\lambda_0, \infty) = [0, \infty)$$

with the second case Neumann boundary conditions (see Section 3.5). Now, suppose V is a symmetric non-negative continuous real valued function on $\mathbb{R} \times [-L, L]$ with bounded support. Making use of the symmetry (see Section 3.6), we can decompose the spectrum of the operator $-\Delta - V$ as

$$\sigma(-\Delta_D - V) = \sigma(A_o - V) \cup \sigma(A_e - V), \tag{3.16}$$

where $A_o = -\Delta_D$ on $\mathbb{R} \times [0, L]$ with Dirichlet boundary conditions on $\mathbb{R} \times \{0\}$ and $A_e = -\Delta_D$ on $\mathbb{R} \times [0, L]$ with Neumann boundary conditions on $\mathbb{R} \times \{0\}$; both operators have the original Dirichlet boundary conditions on $\mathbb{R} \times \{L\}$. To explain that, we have

1) $-\Delta_D u(t, s) = \lambda u(t, s)$, $u(-L, s) = u(L, s)$ and u is even.

2) $-\Delta_D u(t, s) = \lambda u(t, s)$, $u(-L, s) = -u(L, s)$ and u is odd.

If u is odd, it means $u(-t, s) = -u(t, s)$ and $u(0, s) = (0, s)$. So it is enough to consider $A_o = -\Delta_D$ on $\mathbb{R} \times [0, L]$ with Dirichlet boundary conditions

$$u(L, s) = (0, s) \quad \text{and} \quad u(0, s) = (0, s).$$

Refer back to the Section 3.3, Example (1). Similarly, if u is even, it means $u(-t,s) = u(t,s)$ and $u^{'}(0,s) = (0,s)$. So, it is enough to consider the operator $A_e = -\Delta_D$ on $\mathbb{R} \times [0, L]$ with the mixed boundary conditions

$$u(L,s) = (0,s) \quad \text{and} \quad u^{'}(0,s) = (0,s).$$

Referring to the Section 3.3, Example (3). To calculate the spectrum of A_o, we can use the result in Section 3.5 and Example (1) in Section 3.3 to get

$$\sigma(A_o) = [\mu_1, \infty) = \left[\frac{\pi^2}{L^2}, \infty\right).$$

Similarly, to calculate the spectrum of A_e, we can use the result in Section 3.5 and Example (3) in Section 3.3 to get that,

$$\sigma(A_e) = [\mu_0, \infty) = \left[\frac{\pi^2}{4L^2}, \infty\right).$$

Note that μ_1 is the smallest eigenvalue of the operator $-\dfrac{d^2}{ds^2}$ on $[0, L]$ with Dirichlet boundary conditions at 0 and L, but μ_0 is the smallest eigenvalue of the operator $-\dfrac{d^2}{ds^2}$ on $[0, L]$ with Neumann boundary conditions at 0 and Dirichlet boundary conditions at L. By Theorem 2.6.4, it follows that

$$\sigma_{ess}(A_o) = \sigma(A_o) = [\mu_1, \infty),$$

and

$$\sigma_{ess}(A_e) = \sigma(A_e) = [\mu_0, \infty).$$

Next, we have that V is a relatively compact perturbation of the operators A_o and A_e (by Theorems 3.8.2 and 2.6.6). Therefore,

$$\sigma_{ess}(A_o - V) = \sigma_{ess}(A_o) = [\mu_1, \infty).$$

Similarly,

$$\sigma_{ess}(A_e - V) = \sigma_{ess}(A_e) = [\mu_0, \infty).$$

By equality (3.16), it follows that

$$\begin{aligned}\sigma_{ess}(-\Delta - V) &= \sigma_{ess}(A_o - V) \cup \sigma_{ess}(A_e - V)\\ &= [\mu_1, \infty) \cup [\mu_0, \infty)\\ &= [\mu_0, \infty).\end{aligned}$$

Now, by the variational principle (see Section 3.7), we can observe that $\sigma(A_o - V)$ contains an eigenvalue below μ_1 i.e.,

$$\inf(\sigma(A_o - V)) < \mu_1.$$

However, if V is sufficiently small, this eigenvalue will be above μ_0 (note that $\mu_0 < \mu_1$). Combining our observations we can see that for a sufficiently small V, the operator $-\Delta_D - V$ has an eigenvalue in (μ_0, μ_1) hence an embedded eigenvalue.

Now, similarly consider the operator

$$-\Delta_N - V$$

on $\mathbb{R} \times [-L, L]$ with Neumann boundary conditions on $\mathbb{R} \times \{-L\}$ and $\mathbb{R} \times \{L\}$. We can follow the same steps of the Dirichlet boundary conditions case to decompose the spectrum of the operator $-\Delta_N - V$ as

$$\sigma(-\Delta_N - V) = \sigma(A_o^{'} - V) \cup \sigma(A_e^{'} - V), \tag{3.17}$$

where $A_o^{'} = -\Delta_N$ on $\mathbb{R} \times [0, L]$ with Dirichlet boundary conditions on $\mathbb{R} \times \{0\}$, and $A_e^{'} = -\Delta_N$ on $\mathbb{R} \times [0, L]$ with Neumann boundary conditions on $\mathbb{R} \times \{0\}$; both operators have the original Neumann boundary conditions on $\mathbb{R} \times \{0\}$.

1) $-\Delta_N u(t, s) = \lambda u(t, s)$, $u(-L, s) = u(L, s)$ and u is even.

2) $-\Delta_N u(t, s) = \lambda u(t, s)$, $u(-L, s) = -u(L, s)$ and u is odd.

If u is even, it means $u(-t, s) = u(t, s)$ and $u^{'}(0, s) = (0, s)$. So, it is enough to consider the operator $A_e^{'} = -\Delta_N$ on $\mathbb{R} \times [0, L]$ with the Neumann boundary conditions

$$u^{'}(L, s) = (0, s) \quad \text{and} \quad u^{'}(0, s) = (0, s).$$

Referring to the Section 3.3, Example (2). Similarly, if u is odd, it means $u(-t,s) = -u(t,s)$ and $u^{'}(0,s) = (0,s)$. So it is enough to consider $A_o^{'} = -\Delta_N$ on $\mathbb{R} \times [0,L]$ with mixed boundary conditions

$$u(L,s) = (0,s) \quad \text{and} \quad u^{'}(0,s) = (0,s).$$

Refer back to the Section 3.3, Example (3).

However,

$$\sigma(A_o^{'}) = [\mu_1^{'}, \infty) = \left[\frac{\pi^2}{4L^2}, \infty\right),$$

and

$$\sigma(A_e^{'}) = [\mu_0^{'}, \infty) = [0, \infty).$$

Looking at the result in Section 3.5 and Examples (2), (3) in Section 3.3, $\mu_1^{'}$ is the smallest eigenvalue of the operator $-\dfrac{d^2}{ds^2}$ with Dirichlet boundary conditions at 0 and Neumann boundary conditions at L, but $\mu_0^{'}$ is the smallest eigenvalue of the operator $-\dfrac{d^2}{ds^2}$ with Neumann boundary conditions at 0 and L. By Theorem 2.6.4, it follows that

$$\sigma_{ess}(A_o^{'}) = \sigma(A_o^{'}) = [\mu_1^{'}, \infty),$$

and

$$\sigma_{ess}(A_e^{'}) = \sigma(A_e^{'}) = [\mu_0^{'}, \infty).$$

Next, V is a relatively compact perturbation of the operators $A_o^{'}$ and $A_e^{'}$ (by Theorems 3.8.2 and 2.6.6). Therefore,

$$\sigma_{ess}(A_o^{'} - V) = \sigma_{ess}(A_o^{'}) = [\mu_1^{'}, \infty),$$

and

$$\sigma_{ess}(A_e^{'} - V) = \sigma_{ess}(A_e^{'}) = [\mu_0^{'}, \infty).$$

By equality (3.17), it follows that

$$\begin{aligned}
\sigma_{ess}(-\Delta - V) &= \sigma_{ess}(A_o^{'} - V) \cup \sigma_{ess}(A_e^{'} - V) \\
&= [\mu_1^{'}, \infty) \cup [\mu_0^{'}\infty) \\
&= [\mu_0^{'}, \infty).
\end{aligned}$$

Now, by the variational principle method (see Section 3.7), we can observe that $\sigma(A_o^{'} - V)$ contains an eigenvalue below $\mu_1^{'}$.

$$\inf(\sigma(A_o^{'} - V)) < \mu_1^{'}.$$

However, if V is sufficiently small, this eigenvalue will be above $\mu_0^{'}$ (note that $\mu_0^{'} < \mu_1^{'}$). Then, for a sufficiently small V, the operator $-\Delta - V$ an eigenvalue in $(\mu_0^{'}, \mu_1^{'})$ hence an embedded eigenvalue. □

Remark 18. We have that:

- In both cases for sufficiently small V the operator $-\Delta - V$ has an eigenvalue λ which is contained in the essential spectrum hence an embedded eigenvalue.

- For sufficiently small V we can define by:

$$\|V\|_\infty < \mu_1 - \mu_0 = \mu_0,$$

where we have,

$$\inf(\sigma(-\frac{d^2}{ds^2} - V)) > \inf(\sigma(-\frac{d^2}{ds^2}) - \|V\|_{L^\infty}) = \mu_1 - \mu_1 + \mu_0 = \mu_0,$$

and μ_0 is the smallest eigenvalue of the operator $-\frac{d^2}{ds^2}$ with Dirichlet boundary conditions at 0 and L. Similarly for Neumann boundary condition case.

3.9.1 Application in Physics

- Infinity square well: In this case the barriers are infinitely high. We have, the problem consists of solving the time-independent Schrödinger equation, normally written for $\psi(t)$

$$(-\frac{\hbar^2}{2m}\nabla^2 + V(t) - E)\psi(t) = 0, \tag{3.18}$$

for $E \in \mathbb{R}$. That can be rearranged to give

$$\frac{d^2\psi}{dt^2} = \frac{2m}{\hbar^2}(V - E)\psi,$$

where h is the reduced planck constant, m is the mass, E the energy of the particle.

In regions (2) and (3). See 3.3, $\psi_2 = \psi_3 = 0$. (Otherwise, potential energy term goes to infinity).

In region (1), $V = 0 : \psi_1(t) = Ae^{ikt} + Be^{-ikt}$, we have $k = \sqrt{\frac{2mE}{h^2}}$.

Apply boundary conditions:

Match only the wave functions, not derivative. Since $\psi_2 = \psi_3 = 0$ and derivative are also zero, the wave function would have to be 0 as well.

The infinity large barrier step makes it so that we don't have to force derivative to much.

At $t = 0 : \psi_1(0) = \psi_2(0)$, and $A + B = 0$,

$\psi_1(t) = A[e^{ikt} - e^{-ikt}] = (i2A)\sin(kt) = A^{'}\sin(kt)$.

At $t = L : \psi_1(L) = \psi_3(L)$, and $A^{'} sin(kt) = 0$, $kL = n\pi$ where $n > 0$.

We can consider the energy:

$$E = n^2 \frac{\hbar^2 k^2}{2m}.$$

So only particular values of energy are allowed. For each allowed energy, there is a corresponding wave functions. An energy and its corresponding wave function define a state of the system. The lowest state is called ground state:

$$E_1 = \frac{\hbar^2 k^2}{2m}.$$

Now, the Symmetry of the potential energy function, we expect to see the symmetry properties of the potential show up in the physical characteristics of the particle moving in the potential:

$$\psi^*(\frac{L}{2} - t)\psi^*(\frac{L}{2} - t) = \psi(\frac{L}{2} + t)\psi(\frac{L}{2} + t),$$

$$\psi(\frac{L}{2} - t) = \pm\psi(\frac{L}{2} + t).$$

The wave function itself can be symmetric or anti-symmetric.

We use the symmetry as tool in helping to solve problems. In this case, we might have considered putting $t = 0$ at the center of the well. See Figure 3.3 and [34] and [49].

- Finite square well.

  Now, we consider the case of the finite well, where a potential region is confined by equal barriers an either of height V_0 The step potential is simply the of V_0 the height of the barrier. We have the following piecewise continuous finite potential energy:

$$V(t) = \begin{cases} V_0 & t < 0 \\ 0 & 0 \leq t \leq L \\ V_0 & L < t \end{cases}$$

  Now, we want to solve Schrödinger's equation for this potential to get the wavefunction and allowed energies for $E < V_0$.

  I will refer to the three regions 1, 2 and 3 with associated wavefunction ψ_1, ψ_2 , ψ_3. See Figure 3.3.

  In region (1) $\psi_1(t) = A^{'} e^{ikt} + B^{'} e^{-ikt}$,

  In region (2) $\psi_2(t) = C^{'} e^{\alpha_2 t} + D^{'} e^{-\alpha_2 t}$,

  In region (3) $\psi_3(t) = F^{'} e^{\alpha_3 t} + G^{'} e^{-\alpha_3 t}$.

  We have, $D^{'} = F^{'} = 0$, Exponential must remain finite for $t \to \pm\infty$, and $C^{'} = Ce^{\frac{\alpha L}{2}}$ and $G^{'} = Ge^{\frac{-\alpha L}{2}}$.

  Therefore,

  In region (1) $\psi_1(t) = A^{'} e^{ik_1 t} + B^{'} e^{-ik_1 t}$,

  In region (2) $\psi_2(t) = Ce^{\alpha(t+\frac{L}{2})}$,

  In region (3) $\psi_3(t) = Ge^{-\alpha(t-\frac{L}{2})}$.

It is convenient to define two variable (both positive) one for region (1) and (3) and one for region (2) they are wavenumbers:

$$\alpha_2^2 = \alpha_3^2 = \frac{2m}{h^2}(V_0 - E) = \alpha \quad \text{and} \quad k_1^2 = \frac{2m}{h^2}E = k,$$

and the Schrödeinger's equation becomes

$$\frac{d^2\psi_{0,2}}{dt^2} = k_0^2\psi_{0,2} \quad \text{for} \quad t < 0 \quad \text{or} \quad L < t.$$

And we have,

$$\frac{d^2\psi_1}{dt^2} = -k_1^2\psi_1 \quad \text{for} \quad 0 < t < L.$$

In this case the finite potential well is symmetrical, so symmetry can be exploited to reduce the necessary calculations.

Symmetric:

$\psi(-t) = \psi(t)$ $G = C$ and $A^{'} = B^{'}$ $\psi_1(t) = Acos(kt)$

Anti-symmetric:

$\psi(-t) = -\psi(t)$ $G = -C$ and $A^{'} = -B^{'}$ $\psi_1(t) = Asin(kt)$

Symmetric (even):

In region (1) $\psi_1(t) = Acos(kt)$,

In region (2) $\psi_2(t) = Ce^{\alpha(t+\frac{L}{2})}$,

In region (3) $\psi_3(t) = Ce^{-\alpha(t-\frac{L}{2})}$.

Anti-symmetric (odd):

In region (1) $\psi_1(t) = Asin(kt)$,

In region (2) $\psi_2(t) = Ce^{\alpha(t+\frac{L}{2})}$,

In region (3) $\psi_3(t) = Ce^{-\alpha(t-\frac{L}{2})}$.

The next step is to match boundary conditions. Then we can determine the allowed energies. Note that because of the symmetry, the information gained by matching boundary conditions at $t = \dfrac{L}{2}$ will be exactly the same as what is learned from matching at $t = -\frac{L}{2}$. Meaning that we can match at one side only. Symmetric(even) case:

$\psi_1(\frac{L}{2}) = \psi_3(\frac{L}{2})$ and $A\cos(\frac{kL}{2}) = C$.

$\frac{\partial\psi_1(t)}{\partial t}|_{t=\frac{L}{2}} = \frac{\partial\psi_3(t)}{\partial t}|_{t=\frac{L}{2}}$ and $Ak\sin(\frac{kL}{2}) = -\alpha C$.

To determine the allowed energies. We don't care that much about A and C. Take the coefficient out of the picture by dividing the second equation by the first. We get the characteristic equation for the symmetric case:

$k\tan(\frac{kL}{2}) = \alpha$. Then, we can solve this equation to get E. See [34].

Symmetric(odd) case: Matching boundary conditions at $x = \frac{L}{2}$. $\psi_1(\frac{L}{2}) = \psi_3(\frac{L}{2})$ and $A\sin(\frac{kL}{2}) = C$.

$\frac{\partial\psi_1(t)}{\partial t}|_{t=\frac{L}{2}} = \frac{\partial\psi_3(t)}{\partial t}|_{t=\frac{L}{2}}$ and $Ak\cos(\frac{kL}{2}) = -\alpha C$.

Dividing the first BC equation into the second gives the characteristic equation for the symmetric case:

$k\cot(\frac{kL}{2}) = \alpha$. The same mathematical steps to get E. See [34].

Without solving the entire problem we can make some conclusion about the wavefunction and the allowed energy level. Recall that for the region inside the well $V(t) = 0$ and equation 3.18 reduce to for an finite square well potential of width L the allowed are quantised and

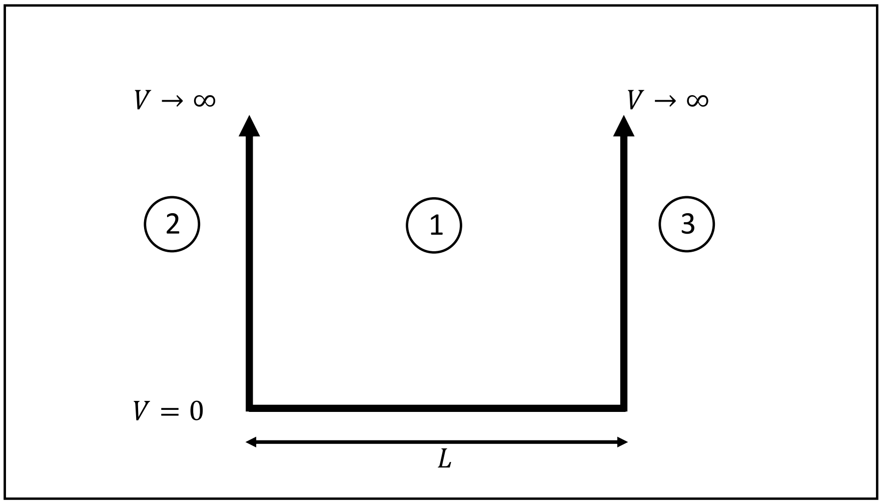
$$E\psi_2 = -\frac{h^2}{2m}\frac{d^2\psi^2}{dt^2},$$

the wavefunction is not zero outside the well.

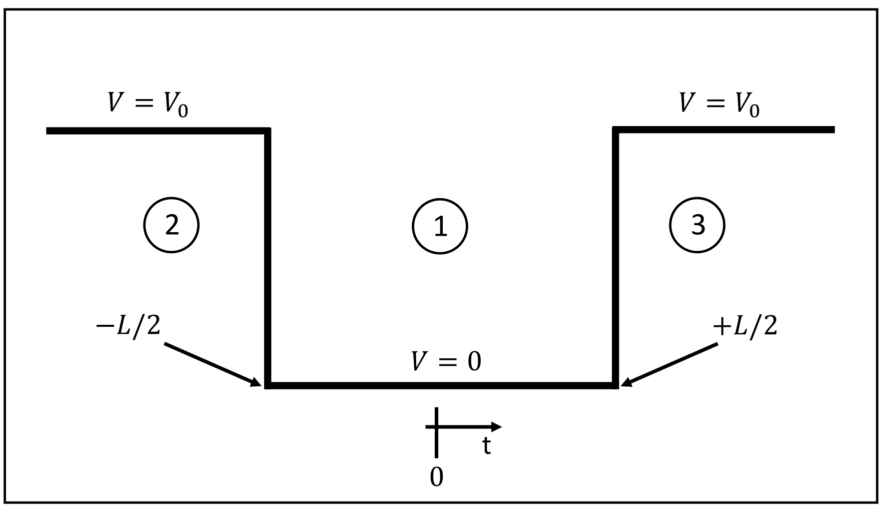
Note that: We consider a basic results: in one-dimensional potential there cannot be two or more bound state for any given energy:

- There is no degeneracy for bound states in one-dimensional potentials.
- The second result, the energy eigenstates $\psi(t)$ can be chosen to be real.
- If the potential is an even function of $t : V(-t) = V(t)$ the eigenstate can be chosen to be even or odd under $t \to -t$.

We note that: Energy is smaller than the potential for $t \to \pm\infty$; Energy is smaller than potential for $t \to -\infty$; Energy is larger than the potential for $t \to \infty$. The spectrum in this work has three kinds of spectrum in different regions discrete spectrum, continuous spectrum and non-degenerate, and continuous spectrum and doubly degenerate. See Figure 3.3, [34] and [49].



Infinitely deep square well



Finite height quantum well

Figure 3.3: Square quantum wells

Chapter 4

Operator pencil and its properties

In this chapter, we start to define the space H_k for $k = 0, 1, 2, ...$ with some properties for this space and some results. The Sobolev space $W^k_{\alpha,\beta}$ for $k \in \mathbb{N}_0$ and $\alpha, \beta \in \mathbb{R}$ are defined in Section 4.2, and we prove some arguments depend on the properties of these spaces. Then, we base on the operator pencil in Section 4.4. We define the class of the quadratic operator pencil $\mathcal{B}_A(\mu)$ to be studied in what follows. Next, we introduce some properties of the spectrum of the operator pencil $\mathcal{B}_A(\mu)$ and investigate the projection of the spectrum of this operator. In Section 4.5, we define the adjoint pencil, which is denoted by $\mathcal{B}^*{}_A(\mu)$ with its properties. Finally, in Section 4.6, the main results of the operator pencil which, will be used in Chapter 5 are presented.

4.1 The space H_k

Here, we define the space H_k for $k = 0, 1, 2, ...\cdot$

Definition 4.1.1. A set $\{u_j\}_{j\in\mathbb{N}_0}$ is an *orthonormal set* in a Hilbert space H if

$$\langle u_j, u_k\rangle = \delta_{j,k} = \begin{cases} 0 & \text{if} \quad j \neq k \\ 1 & \text{if} \quad j = k, \end{cases}$$

where δ_{jk} is the *Kronecker delta.*

Definition 4.1.2. A set $\{u_j\}_{j\in\mathbb{N}_0}$ is the *basis* for H if every $u \in H$ can be written uniquely in the form

$$u = \sum_{j=0}^{\infty} a_j u_j,$$

for constants a_j. If additionally $\{u_j\}_{j\in\mathbb{N}_0}$ is an orthonormal set, then, $\{u_j\}_{j\in\mathbb{N}_0}$ is an *orthonormal basis*. If $\{u_j\}_{j\in\mathbb{N}_0}$ is the basis, then it is a linearly independent set. Indeed, if $\sum_{j=0}^{\infty} a_j u_j = 0$, then $a_j = 0$.

Proposition 4.1.1. Let $\{u_j\}_{j\in\mathbb{N}_0}$ be an orthonormal set in a Hilbert space H. Then, the following statements are equivalent:

- $\{u_j\}_{j\in\mathbb{N}_0}$ is basis in H.
- For each $u \in H$. Then, $u = \sum_{j=0}^{\infty} \langle u, u_j \rangle u_j$.
- For each $u \in H$. Then, $\|u\|_H^2 = \sum_{j=0}^{\infty} |\langle u, u_j \rangle|^2$ (Parseval's identity).
- If $\langle u, u_j \rangle = 0$ for all $j \in \mathbb{N}_0$. Then, $u = 0$.
- The linear span of $\{u_j\}_{j\in\mathbb{N}_0}$ is dense in H. See [36] and [44].

Now, suppose A is a lower semi-bounded self-adjoint operator on a Hilbert space H and the operator A has a discrete spectrum; thus, $\sigma(A)$ consists of eigenvalues $\lambda_0 \leq \lambda_1 \leq \lambda_2, ...$ with corresponding orthonormal eigenfunctions $u_0, u_1, u_2, ... \in \mathrm{Dom}(A)$. In particular, $Au_j = \lambda_j u_j$ for all j, while $\{u_j\}_{j\in\mathbb{N}_0}$ is an orthonormal basis in H. Let q be the quadratic form defined on the form domain $Q(A)$ of the operator A

$$q(u, u) = \langle Au, u \rangle \quad \text{for all} \quad u \in \mathrm{Dom}(A) \subset Q(A).$$

Additionally, q is a semi-bounded quadratic form, so there exists $m \in \mathbb{R}$ with

$$q(u, u) \geq -m\|u\|_{Q(A)}^2.$$

The reader can refer back to Section 2.8.1 for more details.

- Now, the space H_k for $k = 0, 1, 2, ...$ is set by

$$H_k = \left\{ \sum_{j=0}^{\infty} a_j u_j : ((1 + \lambda_j^2)^{\frac{k}{4}} a_j)_{j\in\mathbb{N}_0} \in \ell^2(\mathbb{N}_0) \right\}.$$

This is a linear subspace of the Hilbert space H with the norm given by

$$\|u\|_{H_k}^2 = \sum_{j=0}^{\infty}(1+\lambda_j^2)^{\frac{k}{2}}|a_j|^2,$$

for $u \in H_k$.

- (The form $\langle .,. \rangle$ is anti-linear in the first argument and linear in the second argument). To observe that the mapping $u \to (\langle u, u_j \rangle)_{j\in\mathbb{N}_0}$ is an isometry from H into $\ell^2(\mathbb{N}_0)$; for $u = \sum_{j=0}^{\infty} a_j u_j$ (see, for example, [36]). Now, we can consider that

$$\begin{aligned}\langle u, u_j \rangle &= \langle \sum_{k=0}^{\infty} a_k u_k, u_j \rangle \\ &= \sum_{k=0}^{\infty} a_k \langle u_k, u_j \rangle \\ &= \sum_{\substack{k=0\\k\neq j}}^{\infty} a_k \cdot 0 + \sum_{\substack{k=0\\k=j}}^{\infty} a_k \cdot 1 = a_j.\end{aligned}$$

The space H_0 is given by

$$H_0 = \left\{ \sum_{j=0}^{\infty} a_j u_j : (a_j)_{j\in\mathbb{N}_0} \in \ell^2(\mathbb{N}_0) \right\}.$$

Note that $H_0 = H$, particularly $\{u_j\}_{j\in\mathbb{N}_0}$ is an orthonormal eigenbasis of H_0. Hence, the norm of $u \in H_0$, is the norm of H which by Parseval's identity we have

$$\|u\|_{H_0}^2 = \sum_{j=0}^{\infty} |\langle u, u_j \rangle|^2 = \sum_{j=0}^{\infty} |a_j|^2.$$

- The space H_1 is given by

$$H_1 = \left\{ \sum_{j=0}^{\infty} a_j u_j : ((1+\lambda_j^2)^{\frac{1}{4}} a_j)_{j\in\mathbb{N}_0} \in \ell^2(\mathbb{N}_0) \right\}.$$

Note that $H_1 = Q(A)$ is the form domain of A; we have

$$Au = \sum_{j=0}^{\infty} A(a_j u_j) = \sum_{j=0}^{\infty} a_j(Au_j) = \sum_{j=0}^{\infty} a_j \lambda_j u_j.$$

From the above discussion, the norm of the form domain $Q(A)$ is defined by (see Section 2.8.1).

Lemma 4.1.2. For $u \in Q(A)$ we have,

$$\|u\|_{Q(A)}^2 = \sum_{j=0}^{\infty}(\lambda_j + m + 1)|a_j|^2. \tag{4.1}$$

Proof. We can consider,

$$\begin{aligned}
\|u\|^2_{Q(A)} &= \langle Au, u\rangle + (m+1)\langle u, u\rangle \\
&= \langle \sum_{j=0}^{\infty} A a_j u_j, \sum_{j=0}^{\infty} a_j u_j\rangle + (m+1)\langle \sum_{j=0}^{\infty} a_j u_j, \sum_{j=0}^{\infty} a_j u_j\rangle \\
&= \langle \sum_{j=0}^{\infty} \lambda_j a_j u_j, \sum_{j=0}^{\infty} a_j u_j\rangle + (m+1)\langle \sum_{j=0}^{\infty} a_j u_j, \sum_{j=0}^{\infty} a_j u_j\rangle \\
&= \sum_{j,k=0}^{\infty} \overline{a_j} a_k \lambda_j \langle u_j, u_k\rangle + (m+1) \sum_{j,k=0}^{\infty} \overline{a_j} a_k \langle u_j, u_k\rangle \\
&= \sum_{j=0}^{\infty} \lambda_j |a_j|^2 + (m+1)|a_j|^2 \\
&= \sum_{j=0}^{\infty} (\lambda_j + m + 1)|a_j|^2.
\end{aligned}$$

□

Lemma 4.1.3. For constants c_1 and c_2 we have

$$c_1 \sum_{j=0}^{\infty} (1+\lambda_j^2)^{\frac{1}{2}} |a_j|^2 \le \sum_{j=0}^{\infty} (\lambda_j + m + 1)|a_j|^2 \le c_2 \sum_{j=0}^{\infty} (1+\lambda_j^2)^{\frac{1}{2}} |a_j|^2. \tag{4.2}$$

Proof. To prove the equivalent norms, it suffices to find c_1 and c_2 such that

$$c_1^2(1+\lambda_j^2) \le (\lambda_j + m + 1)^2 \le c_2(1+\lambda_j^2),$$

for all j. Now, we have

$$\begin{aligned}
(\lambda_j + m + 1)^2 &\le 3(\lambda_j^2 + m^2 + 1) \\
&= 3(\lambda_j^2 + 1) + 3m^2 \\
&\le 3(1+m^2)(\lambda_j^2 + 1).
\end{aligned}$$

So, we take $c_2 = \sqrt{3(1+m^2)}$. On the other hand, $\langle Au, u\rangle \ge -m\|u\|^2_{Q(A)}$ holds, for all $u \in Q(A)$. Taking $u = u_j$ gives $\langle Au_j, u_j\rangle \ge -m\|u_j\|^2_{Q(A)}$. It follows that $\langle \lambda_j u_j, u_j\rangle \ge -m\|u_j\|^2_{Q(A)}$. Then,

$$\lambda_j \langle u_j, u_j\rangle \ge -m\|u_j\|^2_{Q(A)} \quad \text{as} \quad \langle u_j, u_j\rangle = 1.$$

Therefore, $\lambda_j \geq -m$, hence $\lambda_j + m \geq 0$. Now, we have

$$\begin{aligned}\lambda_j^2 &= ((\lambda_j + m + 1) - (m+1))^2 \\ &\leq 2(\lambda_j + m + 1)^2 + 2(m+1)^2.\end{aligned}$$

To observe that $\lambda_j + m + 1 \geq 1$ and $2(m+1)^2 + 1 \geq 1 > 0$. Hence,

$$\begin{aligned}\lambda_j^2 + 1 &\leq 2(\lambda_j + m + 1)^2 + 2(m+1)^2 + 1 \\ &\leq 2(\lambda_j + m + 1)^2 + (2(m+1)^2 + 1)(\lambda_j + m + 1)^2 \\ &= (2(m+1)^2 + 3)(\lambda_j + m + 1)^2.\end{aligned}$$

Therefore, we can take $c_1 = \sqrt{2(m+1)^2 + 3}$. □

- The space H_2 is given by

$$H_2 = \left\{ \sum_{j=0}^{\infty} a_j u_j : ((1 + \lambda_j^2)^{\frac{1}{2}} a_j)_{j \in \mathbb{N}_0} \in \ell^2(\mathbb{N}_0) \right\}.$$

Note that $H_2 = \text{Dom}(A)$; and the norm on H_2 is the usual norm on the domain of A.

Lemma 4.1.4. For $u \in \text{Dom}(A)$ we have,

$$\|u\|^2_{\text{Dom}(A)} = \sum_{j=0}^{\infty} (1 + \lambda_j^2)|a_j|^2 = \|u\|^2_{H_2}. \tag{4.3}$$

Proof. We can consider,

$$\begin{aligned}\|u\|^2_{\text{Dom}(A)} &= \|Au\|^2_H + \|u\|^2_H \\ &= \langle \sum_{j=0}^{\infty} A a_j u_j, \sum_{j=0}^{\infty} A a_j u_j \rangle + \langle \sum_{j=0}^{\infty} a_j u_j, \sum_{j=0}^{\infty} a_j u_j \rangle \\ &= \langle \sum_{j=0}^{\infty} \lambda_j a_j u_j, \sum_{j=0}^{\infty} \lambda_j a_j u_j \rangle + \langle \sum_{j=0}^{\infty} a_j u_j, \sum_{j=0}^{\infty} a_j u_j \rangle \\ &= \sum_{j=0}^{\infty} |a_j \lambda_j|^2 + \sum_{j=0}^{\infty} |a_j|^2 \\ &= \sum_{j=0}^{\infty} (1 + \lambda_j^2)|a_j|^2 = \|u\|^2_{H_2}.\end{aligned}$$

□

Lemma 4.1.5. For $j \leq k$ and $u \in H_k$. Then,

$$\|u\|_{H_j}^2 \leq \|u\|_{H_k}^2.$$

Proof. By definition, we can observe

$$\|u\|_{H_j}^2 = \sum_{n=0}^{\infty} (1+\lambda_n^2)^{\frac{j}{2}} |a_n|^2,$$

and because $j \leq k$, we get

$$\begin{aligned}\sum_{n=0}^{\infty} (1+\lambda_n^2)^{\frac{j}{2}} |a_n|^2 &\leq \sum_{n=0}^{\infty} (1+\lambda_n^2)^{\frac{k}{2}} |a_n|^2 \\ &= \|u\|_{H_k}^2.\end{aligned}$$

□

Lemma 4.1.6. For $u \in H_2$. Then,

$$\|Au\|_{H_0} \leq \|u\|_{H_2}.$$

Proof. It is clear to observe that by equality (4.3). □

Example

Let $-\Delta$ be the Laplacian on $H_0 = L^2[-L, L]$ with any of the boundary conditions (Dirichlet, Neumann or mixture), where $-\Delta_{[-L,L]} = -\frac{\partial^2}{\partial t^2}$ is bounded operator from $H_2 = \mathrm{Dom}(-\Delta_{[-L,L]})$ into $H_0 = L^2[-L, L]$ such that $-\Delta\varphi_n = \lambda_n\varphi_n$, and $H_1 = Q(-\Delta_{[-L,L]})$ is a quadratic form domain for $-\Delta$. Therefore,

$$H_2 \subset H_1 \subset H_0,$$

$$\mathrm{Dom}(-\Delta_{[-L,L]}) \subset Q(-\Delta_{[-L,L]}) \subset L^2[-L, L].$$

In the case of Dirichlet boundary condition $\varphi(L) = \varphi(-L) = 0$, the eigenvalues are

$$\lambda_n = \frac{\pi^2 n^2}{4L^2} \quad \text{for } n = 1,2,3,...$$

with corresponding eigenfunctions

$$\varphi_n(t) = \sin \frac{n\pi}{2L}(t + L) \quad \text{for } n \geq 1.$$

4.2 Sobolev spaces

Sobolev space is a vector space of functions equipped with a norm that is a combination of L^p norms of the function itself as well as its derivatives up to a given order. The derivatives are understood in a suitable weak sense to make the space complete, thus a Banach space. We begin with the classical definition of Sobolev spaces:

Definition 4.2.1. Let $1 \le p \le \infty$, $k = 0, 1, ...$ Define the Sobolev space is defined as the space of function $u \in L^p(\mathbb{R}^d)$ all of whose distributional derivative are also in $L^p(\mathbb{R}^d)$ for all multi-indices α that satisfy $|\alpha| \le k$. This space is normed by the expression

$$\|u\|_{W^{k,p}} = \sum_{0\le|\alpha|\le k} \|D^\alpha u\|_{L^p}.$$

One can extend this to the case $p = \infty$ with the norm using the essential supremum by

$$\|u\|_{W^{k,\infty}} = \max_{0\le|\alpha|\le k} \|D^\alpha u\|_{L^\infty}.$$

For the simplicity and convenience of discuss, we will only deal in the case of one dimensional. In the one-dimensional case it is enough to assume that the $(k-1)$ derivative $u^{(k-1)}$ is differentiable almost everywhere and is equal almost everywhere to the Lebesgue integral of its derivative (this excludes irrelevant examples such as Cantor's function). Also, one of the most elegant and useful ways of measuring differentiability properties of functions is in terms of L^2 norms. One of the reason for this is L^2 is a Hilbert space and the other is the Fourier transform is unitary isomorphism on L^2. Now, from previous section and definition of space H_k for $k = 0, 1, ...$ we have that

Definition 4.2.2. By $L^2(\mathbb{R}, H_k)$ we denote the space vector valued function $u : \mathbb{R} \to H_k$ for $k = 0, 1, 2, ...$ with the finite norm

$$\|u\|^2_{L^2(\mathbb{R},H_k)} = \int_{\mathbb{R}} \|u(t)\|^2_{H_k} dt.$$

Definition 4.2.3. Let $W^k(\mathbb{R})$ be the space of distributions u on $\mathbb{R}$ with values in H_k such that $D_t^j \in L^2(\mathbb{R}, H_{k-j}), j = 0, 1, ..., k$. We equip $W^k(\mathbb{R}) = W^{k,2}(\mathbb{R})$ with the norm:

$$\|u\|^2_{W^k(\mathbb{R})} = \int_{\mathbb{R}} \sum_{0\le j\le k} \left\|D_t^j u(t)\right\|^2_{H_{k-j}} dt < \infty.$$

Note that $W^0 = L^2(\mathbb{R}, H_0)$. Each W^k is a Hilbert space with respect to the inner product

$$\langle u, v\rangle_{W^k} = \int_{\mathbb{R}} \sum_{0\le j\le k} \overline{D_t^j u(t)} D_t^j v(t) dt < \infty,$$

for $u, v \in W^k$. It is easy to prove that $u \in W^k(\mathbb{R})$ if and only if $u \to u(t)$ lies in $L^2(\mathbb{R}, H_k - j)$. Many problems of mathematical physics and variational calculus are not sufficient to deal the classical solutions of differential equations. It is necessary to introduce the weighted functions spaces. They were explicity defined in different references, for example, [31], [50], [51], and [66]. In particular, we introduce the exponential weight continuous function modelled on Sobolev spaces $W^k_{\alpha,\beta}$ on $\mathbb{R}$ which is played a big role in the current thesis.

4.3 Weighted Sobolev spaces $W^k_{\alpha,\alpha}$ and $W^k_{\alpha,\beta}$

We introduce the spaces we define the operator $D_t = -i\dfrac{d}{dt}$ on $\mathbb{R}$. For $\alpha, \beta \in \mathbb{R}$ and $k \in \mathbb{N}_0$. Let $W^k_{\alpha,\beta}$ denotes the set of $u : \mathbb{R} \to H_k$ such that

$$\|u\|^2_{W^k_{\alpha,\beta}} := \sum_{j=0}^{k} \int_{-\infty}^{0} e^{2\alpha t} \left\| D_t^j u(t)\right\|^2_{H_{k-j}} dt + \sum_{j=0}^{k} \int_{0}^{\infty} e^{2\beta t} \left\| D_t^j u(t)\right\|^2_{H_{k-j}} dt,$$

is finite.

For $k = 0$. Then, $W^0_{\alpha,\beta}$ is defined to be the set of $u : \mathbb{R} \to H_0$ such that

$$\|u\|^2_{W^0_{\alpha,\beta}} = \int_{-\infty}^{0} e^{2\alpha t} \|u(t)\|^2_{H_0} dt + \int_{0}^{\infty} e^{2\beta t} \|u(t)\|^2_{H_0} dt$$

is finite.

For $k = 1$. Then, $W^1_{\alpha,\beta}$ is defined to be the set of $u : \mathbb{R} \to H_1$ such that

$$\|u\|^2_{W^1_{\alpha,\beta}} = \int_{-\infty}^{0} e^{2\alpha t} (\|D_t u(t)\|^2_{H_0} + \|u(t)\|^2_{H_1}) dt + \int_{0}^{\infty} e^{2\beta t} (\|D_t u(t)\|^2_{H_0} + \|u(t)\|^2_{H_1}) dt$$

is finite.

For $k = 2$, an equivalent definition for $W^2_{\alpha,\beta}$ is the set of $u : \mathbb{R} \to H_2$ such that

$$\|u\|^2_{W^2_{\alpha,\beta}} = \int_{-\infty}^{0} e^{2\alpha t} \left[\left\|D_t^2 u(t)\right\|^2_{H_0} + \|u(t)\|^2_{H_2}\right] dt + \int_{0}^{\infty} e^{2\beta t} \left[\left\|D_t^2 u(t)\right\|^2_{H_0} + \|u(t)\|^2_{H_2}\right] dt$$

is finite.

Remark 19. For $\alpha \in \mathbb{R}$ and $k \in \mathbb{N}_0$, it is easy to define $W^k_{\alpha,\alpha}$, which denotes the set of $u : \mathbb{R} \to H_k$ by such that

$$\|u\|^2_{W^k_{\alpha,\alpha}} := \sum_{j=0}^{k} \int_{-\infty}^{\infty} e^{2\alpha t} \left\| D_t^j u(t) \right\|^2_{H_{k-j}} dt$$

is finite.

Now, the reader can see the proof of the next theorem to understand the equivalent of norms of space $W^2_{\alpha,\beta}$ when $\alpha = \beta$.

Theorem 4.3.1. For $\alpha \in \mathbb{R}$. Then, the norms

$$\|u\|^2_{W^2_{\alpha,\alpha}} = \int_{-\infty}^{\infty} e^{2\alpha t} (\left\| D_t^2 u(t) \right\|^2_{H_0} + \|u(t)\|^2_{H_2}) dt, \tag{4.4}$$

and

$$\int_{-\infty}^{\infty} e^{2\alpha t} (\left\| D_t^2 u(t) \right\|^2_{H_0} + \|D_t u(t)\|^2_{H_1} + \|u(t)\|^2_{H_2}) dt \tag{4.5}$$

are equivalent.

Proof. We find constants c_1 and c_2, which satisfy the following statement:

$$c_1 \|u\|^2_{W^2_{\alpha,\alpha}} \leq \int_{-\infty}^{\infty} e^{2\alpha t} (\left\| D_t^2 u(t) \right\|^2_{H_0} + \|D_t u(t)\|^2_{H_1} + \|u(t)\|^2_{H_2}) dt \leq c_2 \|u\|^2_{W^2_{\alpha,\alpha}}. \tag{4.6}$$

First, using the definitions of previous Section 4.1

$$\int_{-\infty}^{\infty} e^{2\alpha t} \|D_t u(t)\|^2_{H_1} dt = \sum_{j=0}^{\infty} \int_{-\infty}^{\infty} e^{2\alpha t} (1 + \lambda_j^2)^{\frac{1}{2}} |D_t a_j(t)|^2 dt,$$

we consider $u = \sum_{j=0}^{\infty} a_j u_j$ and $a_j : \mathbb{R} \to \mathbb{C}$. So $D_t u = \sum_{j=0}^{\infty} (D_t a_j) u_j$.

Now, integrating by parts, we get

$$\sum_{j=0}^{\infty} \int_{-\infty}^{\infty} e^{2\alpha t} (1 + \lambda_j^2)^{\frac{1}{2}} |D_t a_j(t)|^2 dt = \sum_{j=0}^{\infty} \int_{-\infty}^{\infty} e^{2\alpha t} (1 + \lambda_j^2)^{\frac{1}{2}} D_t a_j(t) \overline{D_t a_j(t)} dt.$$

Then we have,

$$\begin{aligned}\sum_{j=0}^{\infty}\int_{-\infty}^{\infty} e^{2\alpha t}(1+\lambda_j^2)^{\frac{1}{2}}|D_t a_j(t)|^2 dt &= \sum_{j=0}^{\infty}(1+\lambda_j^2)^{\frac{1}{2}} e^{2\alpha t}\overline{a_j(t)} D_t a_j(t)|_{-\infty}^{\infty} \\ &+ \sum_{j=0}^{\infty}\int_{-\infty}^{\infty}(1+\lambda_j^2)^{\frac{1}{2}} e^{2\alpha t}\overline{a_j(t)} D_t^2 a_j(t) dt \\ &- 2i\alpha \sum_{j=0}^{\infty}\int_{-\infty}^{\infty}(1+\lambda_j^2)^{\frac{1}{2}} e^{2\alpha t}\overline{a_j(t)} D_t a_j(t) dt \\ &= \sum_{j=0}^{\infty}\int_{-\infty}^{\infty}(1+\lambda_j^2)^{\frac{1}{2}} e^{2\alpha t}\overline{a_j(t)} D_t^2 a_j(t) dt \\ &- 2i\alpha \sum_{j=0}^{\infty}\int_{-\infty}^{\infty}(1+\lambda_j^2)^{\frac{1}{2}} e^{2\alpha t}\overline{a_j(t)} D_t a_j(t) dt.\end{aligned}$$

By Cauchy Schwartz inequality, we have

$$\begin{aligned}\left|\int_{-\infty}^{\infty}\sum_{j=0}^{\infty}(1+\lambda_j^2)^{\frac{1}{2}}[e^{2\alpha t}\overline{a_j(t)} D_t^2 a_j(t)] dt\right| &\le \left(\sum_{j=0}^{\infty}\int_{-\infty}^{\infty}(1+\lambda_j^2) e^{2\alpha t}|\overline{a_j(t)}|^2 dt\right)^{\frac{1}{2}} \left(\sum_{j=0}^{\infty}\int_{-\infty}^{\infty} e^{2\alpha t}|D_t^2 a_j(t)|^2 dt\right)^{\frac{1}{2}} \\ &= \left(\int_{-\infty}^{\infty} e^{2\alpha t}\sum_{j=0}^{\infty}(1+\lambda_j^2)|a_j(t)|^2 dt\right)^{\frac{1}{2}} \left(\int_{-\infty}^{\infty} e^{2\alpha t}\sum_{j=0}^{\infty}|D_t^2 a_j(t)|^2 dt\right)^{\frac{1}{2}} \\ &= \left(\int_{-\infty}^{\infty} e^{2\alpha t}\|u(t)\|_{H_2}^2 dt\right)^{\frac{1}{2}} \left(\int_{-\infty}^{\infty} e^{2\alpha t}\|D_t^2 u(t)\|_{H_0}^2 dt\right)^{\frac{1}{2}}.\end{aligned} \tag{4.7}$$

Similarly, using Cauchy Schwartz inequality for the other term gives

$$\begin{aligned}\left|-2i\alpha\sum_{j=0}^{\infty}\int_{-\infty}^{\infty}(1+\lambda_j^2)^{\frac{1}{2}} e^{2\alpha t}\overline{a_j(t)} D_t a_j(t) dt\right| &\le |2\alpha| \left(\int_{-\infty}^{\infty} e^{2\alpha t}\sum_{j=0}^{\infty}(1+\lambda_j^2)|\overline{a_j(t)}|^2 dt\right)^{\frac{1}{2}} \left(\int_{-\infty}^{\infty} e^{2\alpha t}|D_t a_j(t)|^2 dt\right)^{\frac{1}{2}} \\ &= |2\alpha| \left(\int_{-\infty}^{\infty} e^{2\alpha t}\|u(t)\|_{H_2}^2 dt\right)^{\frac{1}{2}} \left(\int_{-\infty}^{\infty} e^{2\alpha t}\|D_t u(t)\|_{H_0}^2 dt\right)^{\frac{1}{2}}.\end{aligned} \tag{4.8}$$

From (4.7) and (4.8), we have

$$\begin{aligned}\int_{-\infty}^{\infty} e^{2\alpha t}\|D_t u(t)\|_{H_1}^2 dt &\le \left(\int_{-\infty}^{\infty} e^{2\alpha t}\|u(t)\|_{H_2}^2 dt\right)^{\frac{1}{2}} \left(\int_{-\infty}^{\infty} e^{2\alpha t}\|D_t^2 u(t)\|_{H_0}^2 dt\right)^{\frac{1}{2}} \\ &+ |2\alpha| \left(\int_{-\infty}^{\infty} e^{2\alpha t}\|u(t)\|_{H_2}^2 dt\right)^{\frac{1}{2}} \left(\int_{-\infty}^{\infty} e^{2\alpha t}\|D_t u(t)\|_{H_0}^2 dt\right)^{\frac{1}{2}}.\end{aligned} \tag{4.9}$$

Therefore, we can write (4.9) by

$$\begin{aligned}\int_{-\infty}^{\infty} e^{2\alpha t}\|D_t u(t)\|_{H_1}^2 dt - |2\alpha| \left(\int_{-\infty}^{\infty} e^{2\alpha t}\|u(t)\|_{H_2}^2 dt\right)^{\frac{1}{2}} \left(\int_{-\infty}^{\infty} e^{2\alpha t}\|D_t u(t)\|_{H_0}^2 dt\right)^{\frac{1}{2}} \\ \le \left(\int_{-\infty}^{\infty} e^{2\alpha t}\|u(t)\|_{H_2}^2 dt\right)^{\frac{1}{2}} \left(\int_{-\infty}^{\infty} e^{2\alpha t}\|D_t^2 u(t)\|_{H_0}^2 dt\right)^{\frac{1}{2}}.\end{aligned} \tag{4.10}$$

Now, we can consider

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{2\alpha t}\|D_t u(t)\|_{H_1}^2 dt &= \int_{-\infty}^{\infty} e^{2\alpha t}\left[\left(\|D_t u(t)\|_{H_1}^2 dt\right)^{\frac{1}{2}} - |\alpha|\left(\|u(t)\|_{H_2}^2 dt\right)^{\frac{1}{2}} + |\alpha|\left(\|u(t)\|_{H_2}^2 dt\right)^{\frac{1}{2}}\right]^2 \\
&\le 2\int_{-\infty}^{\infty} e^{2\alpha t}\left(\left(\|D_t u(t)\|_{H_1}^2 dt\right)^{\frac{1}{2}} - |\alpha|\left(\|u(t)\|_{H_2}^2 dt\right)^{\frac{1}{2}}\right)^2 + 2\alpha^2\int_{-\infty}^{\infty} e^{2\alpha t}\|u(t)\|_{H_2}^2 dt \\
&= 2\int_{-\infty}^{\infty} e^{2\alpha t}\|D_t u(t)\|_{H_1}^2 dt - 4|\alpha|\left(\int_{-\infty}^{\infty} e^{2\alpha t}\|D_t u(t)\|_{H_1}^2 dt\right)^{\frac{1}{2}}\left(\int_{-\infty}^{\infty} e^{2\alpha t}\|u(t)\|_{H_2}^2 dt\right)^{\frac{1}{2}} \\
&+ 2\alpha^2\int_{-\infty}^{\infty} e^{2\alpha t}\|u(t)\|_{H_2}^2 dt + 2\alpha^2\int_{-\infty}^{\infty} e^{2\alpha t}\|u(t)\|_{H_2}^2 dt \\
&= 2\int_{-\infty}^{\infty} e^{2\alpha t}\|D_t u(t)\|_{H_1}^2 dt - 4|\alpha|\left(\int_{-\infty}^{\infty} e^{2\alpha t}\|D_t u(t)\|_{H_1}^2 dt\right)^{\frac{1}{2}}\left(\int_{-\infty}^{\infty} e^{2\alpha t}\|u(t)\|_{H_2}^2 dt\right)^{\frac{1}{2}} \\
&+ 4\alpha^2\int_{-\infty}^{\infty} e^{2\alpha t}\|u(t)\|_{H_2}^2 dt.
\end{aligned} \tag{4.11}$$

By (4.10), we get

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{2\alpha t}\|D_t u(t)\|_{H_1}^2 dt &\le 2\int_{-\infty}^{\infty} e^{2\alpha t}\|D_t u(t)\|_{H_1}^2 dt - 4|\alpha|\left(\int_{-\infty}^{\infty} e^{2\alpha t}\|D_t u(t)\|_{H_1}^2 dt\right)^{\frac{1}{2}}\left(\int_{-\infty}^{\infty} e^{2\alpha t}\|u(t)\|_{H_2}^2 dt\right)^{\frac{1}{2}} \\
&+ 4\alpha^2\int_{-\infty}^{\infty} e^{2\alpha t}\|u(t)\|_{H_2}^2 dt \\
&\le 2\left(\int_{-\infty}^{\infty} e^{2\alpha t}\|u(t)\|_{H_2}^2 dt\right)^{\frac{1}{2}}\left(\int_{-\infty}^{\infty} e^{2\alpha t}\|D_t^2 u(t)\|_{H_0}^2 dt\right)^{\frac{1}{2}} + 4\alpha^2\int_{-\infty}^{\infty} e^{2\alpha t}\|u(t)\|_{H_2}^2 dt \\
&\le \int_{-\infty}^{\infty} e^{2\alpha t}\|u(t)\|_{H_2}^2 dt + \int_{-\infty}^{\infty} e^{2\alpha t}\|D_t^2 u(t)\|_{H_0}^2 dt + 4\alpha^2\int_{-\infty}^{\infty} e^{2\alpha t}\|u(t)\|_{H_2}^2 dt \\
&\le (1+4\alpha^2)\left(\int_{-\infty}^{\infty} e^{2\alpha t}\|u(t)\|_{H_2}^2 dt + \int_{-\infty}^{\infty} e^{2\alpha t}\|D_t^2 u(t)\|_{H_0}^2 dt\right).
\end{aligned} \tag{4.12}$$

Substitute $\int_{-\infty}^{\infty} e^{2\alpha t}\|D_t u(t)\|_{H_1}^2 dt$ in (4.6) to consider

$$\int_{-\infty}^{\infty} e^{2\alpha t}\left[\left\|D_t^2 u(t)\right\|_{H_0}^2 + \|D_t u(t)\|_{H_1}^2 + \|u(t)\|_{H_2}^2 dt\right] \le (2+4\alpha^2)\left[\int_{-\infty}^{\infty} e^{2\alpha t}\|D_t^2 u(t)\|_{H_0}^2 dt + \int_{-\infty}^{\infty} e^{2\alpha t}\|u(t)\|_{H_2}^2 dt\right],$$

so, we can take $c_2 = 2 + 4\alpha^2$. On the other side, we get

$$\|u\|_{W_{\alpha,\alpha}}^2 = \int_{-\infty}^{\infty} e^{2\alpha t}\left[\left\|D_t^2 u(t)\right\|_{H_0}^2 + \|u(t)\|_{H_2}^2\right] dt \le \int_{-\infty}^{\infty} e^{2\alpha t}(\left\|D_t^2 u(t)\right\|_{H_0}^2 + \|D_t u(t)\|_{H_1}^2 + \|u(t)\|_{H_2}^2) dt.$$

So, we can take $c_1 = 1$. Thus, the proof is complete. □

Lemma 4.3.2. For $u \in W_{\alpha,\alpha}^0$. Then,

$$\|u\|_{W_{\alpha,\alpha}^0} = \|e^{\alpha t} u\|_{W_{0,0}^0}.$$

Proof. For $\alpha \in \mathbb{R}$, consider

$$\begin{aligned}\|u\|^2_{W^0_{\alpha,\alpha}} &= \int_{-\infty}^{\infty} e^{2\alpha t}\|u(t)\|^2_{H_0}dt \\ &= \int_{-\infty}^{\infty} \|e^{\alpha t}u(t)\|^2_{H_0}dt \\ &= \|e^{\alpha t}u\|^2_{W^0_{0,0}}.\end{aligned}$$

□

Lemma 4.3.3. There are the constants $c_1^{'}$ and $c_2^{'}$ such that

$$c_1^{'}\|u\|_{W^2_{\alpha,\alpha}} \leq \|e^{\alpha t}u\|_{W^2_{0,0}} \leq c_2^{'}\|u\|_{W^2_{\alpha,\alpha}}$$

for $u \in W^2_{\alpha,\alpha}$.

Proof. Note that $D_t(e^{\alpha t})u(t) = e^{\alpha t}(D_t - i\alpha)u(t)$. So,

$$\begin{aligned}\|e^{\alpha t}u\|^2_{W^2_{0,0}} &= \int_{-\infty}^{\infty}[\|D_t^2 e^{\alpha t}u(t)\|^2_{H_0} + \|e^{\alpha t}u(t)\|^2_{H_2}]dt \\ &= \int_{-\infty}^{\infty}[\|e^{\alpha t}(D_t - i\alpha)^2u(t)\|^2_{H_0} + \|e^{\alpha t}u(t)\|^2_{H_2}]dt \\ &= \int_{-\infty}^{\infty} e^{2\alpha t}[\|(D_t - i\alpha)^2u(t)\|^2_{H_0} + \|u(t)\|^2_{H_2}]dt \\ &= \int_{-\infty}^{\infty} e^{2\alpha t}[\|(D_t^2 - 2i\alpha D_t - \alpha^2)u(t)\|^2_{H_0} + \|u(t)\|^2_{H_2}]dt \\ &\leq 3\left(\int_{-\infty}^{\infty} e^{2\alpha t}[\|D_t^2u(t)\|^2_{H_0} + \|-2i\alpha D_tu(t)\|^2_{H_0} + \|-\alpha^2u(t)\|^2_{H_0} + \|u(t)\|^2_{H_2}]dt\right) \\ &= 3\left(\int_{-\infty}^{\infty} e^{2\alpha t}[\|D_t^2u(t)\|^2_{H_0} + 4\alpha^2\|D_tu(t)\|^2_{H_0} + \alpha^4\|u(t)\|^2_{H_0} + \|u(t)\|^2_{H_2}]dt\right) \\ &\leq 3(4\alpha^2 + \alpha^4 + 1)\left(\int_{-\infty}^{\infty} e^{2\alpha t}[\|D_t^2u(t)\|^2_{H_0} + \|D_tu(t)\|^2_{H_0} + \|u(t)\|^2_{H_0} + \|u(t)\|^2_{H_2}]dt\right).\end{aligned}$$

From Lemma 4.1.5

$$\begin{aligned}\|e^{\alpha t}u\|^2_{W^2_{0,0}} &\leq 3(4\alpha^2 + \alpha^4 + 1)\left(\int_{-\infty}^{\infty} e^{2\alpha t}[\|D_t^2u(t)\|^2_{H_0} + \|D_tu(t)\|^2_{H_1} + \|u(t)\|^2_{H_2} + \|u(t)\|^2_{H_2}]dt\right) \\ &\leq (2)(3)(4\alpha^2 + \alpha^4 + 1)\left(\int_{-\infty}^{\infty} e^{2\alpha t}[\|D_t^2u(t)\|^2_{H_0} + \|D_tu(t)\|^2_{H_1} + \|u(t)\|^2_{H_2}]dt\right) \\ &\leq 6(4\alpha^2 + \alpha^4 + 1)\|u\|^2_{W^2_{\alpha,\alpha}},\end{aligned}$$

we can take $c_2^{'} = \sqrt{6(\alpha^4 + 4\alpha^2 + 1)}$.

On the other side, we get

$$\begin{aligned}
\|u\|^2_{W^2_{\alpha,\alpha}} &= \int_{-\infty}^{\infty} e^{2\alpha t}[\|D_t^2 u(t)\|^2_{H_0} + \|u(t)\|^2_{H_2}]dt \\
&= \int_{-\infty}^{\infty} (\|e^{\alpha t} D_t^2 u(t)\|^2_{H_0} + \|e^{\alpha t} u(t)\|^2_{H_2})dt \\
&= \int_{-\infty}^{\infty} (\|(D_t + i\alpha)^2 e^{\alpha t} u(t)\|^2_{H_0} + \|e^{\alpha t} u(t)\|^2_{H_2})dt \\
&\leq 3\left(\int_{-\infty}^{\infty} [\|D_t^2 e^{\alpha t} u(t)\|^2_{H_0} + \|2i\alpha D_t e^{\alpha t} u(t)\|^2_{H_0} + \|\alpha^2 e^{\alpha t} u(t)\|^2_{H_0} + \|e^{\alpha t} u(t)\|^2_{H_2}]dt\right) \\
&= 3\left(\int_{-\infty}^{\infty} \|D_t^2 e^{\alpha t} u(t)\|^2_{H_0} + 4\alpha^2 \|D_t e^{\alpha t} u(t)\|^2_{H_0} + \alpha^4 \|e^{\alpha t} u(t)\|^2_{H_0} + \|e^{\alpha t} u(t)\|^2_{H_2}]dt\right) \\
&\leq 3(4\alpha^2 + \alpha^4 + 1)\left(\int_{-\infty}^{\infty} \|D_t^2 e^{\alpha t} u(t)\|^2_{H_0} + \|D_t e^{\alpha t} u(t)\|^2_{H_0} + \|e^{\alpha t} u(t)\|^2_{H_0} + \|e^{\alpha t} u(t)\|^2_{H_2} dt\right).
\end{aligned}$$

From Lemma 4.1.5

$$\begin{aligned}
\|u\|^2_{W^2_{\alpha,\alpha}} &\leq 3(4\alpha^2 + \alpha^4 + 1)\left(\int_{-\infty}^{\infty} \|D_t^2 e^{\alpha t} u(t)\|^2_{H_0} + \|D_t e^{\alpha t} u(t)\|^2_{H_1} + \|e^{\alpha t} u(t)\|^2_{H_2} + \|e^{\alpha t} u(t)\|^2_{H_2} dt\right) \\
&\leq (2)(3)(4\alpha^2 + \alpha^4 + 1)\int_{-\infty}^{\infty} [\|D_t^2 e^{\alpha t} u(t)\|^2_{H_0} + \|D_t e^{\alpha t} u(t)\|^2_{H_1} + \|e^{\alpha t} u(t)\|^2_{H_2}]dt \\
&\leq 6(4\alpha^2 + \alpha^4 + 1)\|e^{\alpha t} u\|^2_{W^2_{0,0}},
\end{aligned}$$

so we can take $c_1^{'} = \dfrac{1}{\sqrt{6(\alpha^4 + 4\alpha^2 + 1)}}$. □

Remark 20. We have the following notes:

- If $u \in L^2(\mathbb{R}, H_0) = W^0_{0,0}$, the Fourier transform of u is given by

$$\widehat{u}(\tau) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\tau t} u(t) dt.$$

  So, by the Plancherel Theorem, $\widehat{u} \in L^2(\mathbb{R}, H_0) = W^0_{0,0}$.

- For any $u : \mathbb{R} \to H_0$ can be written as $u(t) = \sum_{j=0}^{\infty} a_j(t) u_j$ for some $a_j : \mathbb{R} \to \mathbb{C}$ can be defined by $a_j(t) = \langle u(t), e_j \rangle$ (the form $\langle ., . \rangle$ is anti-linear in the first argument and linear in the second argument). So the Fourier transform of u is defined by

$$\widehat{u}(\tau) = \sum_{j=0}^{\infty} (\widehat{a})_j(\tau) e_j.$$

We can observe the following notation:

$$
\begin{aligned}
(\widehat{a})_j(\tau) &= \langle \widehat{u}(\tau), e_j \rangle \\
&= \langle \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\tau t} u(t)dt, e_j \rangle \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\tau t} \langle u(t), e_j \rangle dt \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\tau t} a_j(t)dt \\
&= \widehat{a}_j(\tau).
\end{aligned}
$$

- We can recall the norm of $\|\widehat{u}\|_{H_k}$ is given by

$$
\|\widehat{u}\|^2_{H_k} = \sum_{j=0}^{\infty} (1+\lambda_j^2)^{\frac{k}{2}} |\widehat{a}_j|^2.
$$

Lemma 4.3.4. $u \in W^2_{0,0}$ if and only if

$$
\|u\|^2_{W^2_{0,0}} = \sum_{j=0}^{\infty} \int_{-\infty}^{\infty} (\tau^4 + 1 + \lambda_j^2) |\widehat{a_j}(\tau)|^2 d\tau
$$

is finite.

Proof. We have

$$
\|u\|^2_{W^2_{0,0}} = \int_{-\infty}^{\infty} (\|D_t^2 u(t)\|^2_{H_0} + \|u(t)\|^2_{H_2}) dt < \infty. \tag{4.13}
$$

From the Plancherel theorem and Lemma 2.2.1, we get

$$
\begin{aligned}
\|u\|^2_{W^2_{0,0}} = \int_{-\infty}^{\infty} (\|D_t^2 u(t)\|^2_{H_0} + \|u(t)\|^2_{H_2}) dt &= \int_{-\infty}^{\infty} (\|\tau^2 \widehat{u}(\tau)\|^2_{H_0} + \|\widehat{u}(\tau)\|^2_{H_2}) d\tau \\
&= \int_{-\infty}^{\infty} (\sum_{j=0}^{\infty} \tau^4 |\widehat{a_j}(\tau)|^2 + \sum_{j=0}^{\infty} (1+\lambda_j^2) |\widehat{a_j}(\tau)|^2) d\tau \\
&= \sum_{j=0}^{\infty} \int_{-\infty}^{\infty} (\tau^4 + 1 + \lambda_j^2) |\widehat{a_j}(\tau)|^2 d\tau.
\end{aligned}
$$

Thus, the proof is complete. □

In the following arguments, we define the space $H^j(\mathbb{R}, H_{k-j})$ and we have a nice remark that will be used in the current thesis.

Definition 4.3.1. The space $H^j(\mathbb{R}, H_{k-j})$ is defined by:

$$H^j(\mathbb{R}, H_{k-j}) = \left\{u \in L^2(\mathbb{R}, H_{k-j}) : \nabla^l u \in L^2(\mathbb{R}, H_{k-j}), \quad 0 \le l \le j\right\}$$

and its norm is given by

$$\|u\|^2_{H^j(\mathbb{R}, H_{k-j})} = \int_{\mathbb{R}} \sum_{0 \le l \le j} \|\nabla^l u(t)\|^2_{H_{k-j}} dt,$$

where $u : \mathbb{R} \to H_{k-j}$.

Remark 21. For $\alpha, \tau \in \mathbb{R}$ and $k \in \mathbb{N}_0$, we have

$$u \in W^k_{\alpha,\alpha} \quad \text{if and only if} \quad e^{\alpha t} u \in W^k_{0,0} = \cap_{j=0}^k H^j(\mathbb{R}, H_{k-j}).$$

It follows that

$$\widehat{e^{\alpha t}u}(\tau) = \widehat{u}(\tau + i\alpha) \in L^2(\mathbb{R}, H_{k-j}),$$

where

$$\widehat{e^{\alpha t}u}(\tau) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{\alpha t} u(t) e^{-i\tau t} dt = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} u(t) e^{-it(\tau + i\alpha)} dt = \widehat{u}(\tau + i\alpha).$$

4.4 Operator pencil and its basic Properties

This section is considered the main section of this chapter. As in Chapter 1, we define the operator Pencil $\mathcal{B}_A$ and investigate some properties for this operator such as the spectrum of the operator pencil and the definition of the projection $\Gamma(\mathcal{B}_A)$.

Definition 4.4.1. Let A be as introduced in Section 4.1. An operator pencil

$$\mathcal{B}_A : \mathbb{C} \to B(H_2, H_0)$$

which is defined by

$$\mathcal{B}_A(\mu) = \mu^2 + A - \lambda \quad \text{for} \quad \mu \in \mathbb{C}, \tag{4.14}$$

where the collection of Hilbert spaces $H_2 \subset H_1 \subset H_0$. A is a bounded operator from H_2 into H_0 and a scalar $\mu_0 \in \mathbb{C}$ is called an *eigenvalue* of $\mathcal{B}_A$ if $\mathcal{B}_A(\mu_0)$ is not injective. Hence, the eigenvalue problem is to find μ_0 and

$u \neq 0$ and $u \in H_2$ such that

$$\mathcal{B}_A(\mu_0)u = 0, \tag{4.15}$$

and u is called an *eigenfunction* of $\mathcal{B}_A$.

Definition 4.4.2. The geometric and algebraic multiplicity of any $\mu_0 \in \sigma(\mathcal{B}_A)$ can be respectively defined as $\dim(\operatorname{Ker}\mathcal{B}_A)$ and the sum of the lengths of a set of maximal Jordan chains coresspoding to μ_0. See, for example, [9] and [58], pp. $406 - 407$.

Definition 4.4.3. Let $\mu_0 \in \mathbb{C}$. We say μ_0 is in the *spectrum* of $\mathcal{B}_A$ if $\mathcal{B}_A(\mu_0)$ is not invertible from H_2 to H_0.

$$\sigma(\mathcal{B}_A) = \{\mu_0 \in \mathbb{C} : \mathcal{B}_A(\mu_0) \text{ is not invertible}\}.$$

Proposition 4.4.1. Consider the operator pencil $\mathcal{B}_A$ given by (4.14). If $\lambda \notin \sigma(A)$, then the spectrum $\sigma(\mathcal{B}_A)$ is given as follows

- If $\lambda < \lambda_0$. Then,

$$\sigma(\mathcal{B}_A) = \{\pm i\sqrt{\lambda_j - \lambda}, \quad j \in \mathbb{N}_0\}.$$

- If $\lambda_{m-1} < \lambda < \lambda_m$ for some $m \in \mathbb{N}$, then

$$\sigma(\mathcal{B}_A) = \{\pm\sqrt{\lambda - \lambda_j}, \quad j = 0, 1, 2, ..., m-1\} \cup \{\pm i\sqrt{\lambda_j - \lambda}, \quad j = m, m+1, ...\}.$$

Proof. For $\mu_0 \in \mathbb{C}$, we have $\mu_0 \in \sigma(\mathcal{B}_A)$ if and only if

$$\mathcal{B}_A : H_2 \to H_0 \quad \text{ is not invertable,}$$

that is

$$A - (\lambda - \mu_0^2) : H_2 \to H_0 \quad \text{ is not invertable.}$$

This holds if and only if $\lambda - \mu_0^2 \in \sigma(A)$ or equivalently $\lambda - \mu_0^2 = \lambda_j$ for some $j \in \mathbb{N}_0$. The latter can be written as

$$\mu_0 = \begin{cases} \pm\sqrt{\lambda - \lambda_j} & \text{if} \quad \lambda > \lambda_j \\ \\ \pm i\sqrt{\lambda_j - \lambda} & \text{if} \quad \lambda < \lambda_j. \end{cases}$$

Therefore, we have two cases:

- If $\lambda < \lambda_0$, then

$$\sigma(\mathcal{B}_A) = \{\pm i\sqrt{\lambda_j - \lambda}, \quad j \in \mathbb{N}_0\}.$$

- If $\lambda_{m-1} < \lambda < \lambda_m$ for some $m \in \mathbb{N}$, then

$$\sigma(\mathcal{B}_A) = \{\pm\sqrt{\lambda - \lambda_j}, \quad j = 0, 1, 2, ..., m-1\} \cup \{\pm i\sqrt{\lambda_j - \lambda}, \quad j = m, m+1, ...\}.$$

□

Remark 22. The eigenvectors of $\mathcal{B}_A$ which are corresponding to eigenvalue

$$\mu_0 = \begin{cases} \pm\sqrt{\lambda - \lambda_j} & \text{if} \quad \lambda > \lambda_j \\ \\ \pm i\sqrt{\lambda_j - \lambda} & \text{if} \quad \lambda < \lambda_j. \end{cases}$$

are given by

$$\mathcal{B}_A(\mu_0)u = 0.$$

Then,

$$\mathcal{B}_A(\mu_0)u = (\mu_0^2 + A - \lambda)u = 0$$

for $u \neq 0$ and $u \in H_2$. It follows that

$$(-\lambda_j + A)u = 0.$$

Hence,

$$Au = \lambda_j u.$$

Therefore, u is the eigenfunction of A. See Section 4.1 of the current thesis.

Remark 23. We have the following notes:

- A collection of functions $\{\varphi_{k,s}\}$ for $k = 1, ..., J$ and $s = 0, ..., m_k - 1$ is called a *Jordan chains* corresponding to $\mu_0 \in \sigma(\mathcal{B}_A)$ if and only if $\varphi_{1,0}$ is an eigenfunction corresponding to $\mu_0 \in \sigma(\mathcal{B}_A)$ and the meromorphic function $\Phi(\mu)$ defined by

$$\Phi(\mu) = \sum_{s=0}^{m_k-1} \frac{\varphi_{k,s}}{(\mu - \mu_0)^{m_k - s}}$$

satisfies,

$$\mathcal{B}_A \Phi(\mu) = O(1),$$

as $\mu \to \mu_0$, for $k = 1, ..., J$. It can be seen that this is equivalent to the condition

$$\sum_{s=0}^{n} \frac{1}{s!} \partial_\mu^s \mathcal{B}_A(\mu_0) \varphi_{k,s} = 0$$

for $s = 0, 1, ..., m_k - 1$. See, for example, [10] and [58].

- By turning to the pencil $\mathcal{B}_A = \mu^2 + A - \lambda$, the operator A is a positive definite and the spectrum of A consists of the eigenvalues λ_j, satisfying $0 \leq \lambda_0 \leq \lambda_1 \leq .....$ and $\lambda_j \to \infty$. By J, we denote the multiplicity of λ_j and assume J is finite. Let $u_0, u_1, ...$ be the orthonormal eigenvectors of A corresponding to λ_j. See Section 4.1. The eigenvalues $\pm i\sqrt{\lambda_j - \lambda}$ and $\pm\sqrt{\lambda - \lambda_j}$ have geometric and algebraic multiplicities are equal to J. If $\lambda = \lambda_{m-1}$, then $0 \in \sigma(\mathcal{B}_A)$ has geometric multiplicity 1 and algebraic multiplicity 2. See, for example, [10] and [58], pp. $7 - 9$.

- In an Appendix.5, there is a good example of the operator pencil is defined from $\mathbb{C}$ to the set of all bounded operators $B(\mathbb{C}^2, \mathbb{C}^2) \cong M_{2\times 2}(\mathbb{C})$.

Definition 4.4.4. Let $\Gamma(\mathcal{B}_A)$ denote the projection of the spectrum of $\mathcal{B}_A$ onto the imaginary axis, that is,

$$\Gamma(\mathcal{B}_A) = \{\Im\mu| \quad \mu \in \sigma(\mathcal{B}_A)\} \subseteq \mathbb{R}.$$

Remark 24. We can observe from Proposition 4.4.1 for any $\lambda \notin \sigma(A)$ and the operator Pencil $\mathcal{B}_A(\mu)$ defined in (4.14). The projection of $\sigma(\mathcal{B}_A)$ as follows:

- If $\lambda < \lambda_0$, then

$$\Gamma(\mathcal{B}_A) = \{\pm\sqrt{\lambda_j - \lambda}, \quad j \in \mathbb{N}_0\}.$$

- If $\lambda_{m-1} < \lambda < \lambda_m$ for some $m \in \mathbb{N}$, then

$$\Gamma(\mathcal{B}_A) = \{0\} \cup \{\pm\sqrt{\lambda_j - \lambda}, \quad j = m, m+1, ...\}.$$

4.5 Adjoint Pencil

Definition 4.5.1. Let H be a Hilbert space and H^* be its *dual space*, which is defined as the space of all bounded linear functional on H and by the scalar product $\langle ., . \rangle$ is given on $H \times H^*$. This scalar product satisfies the following properties:

- The form $\langle ., . \rangle$ is a linear with respect to the first argument and an anti-linear with respect to the second argument.
- For all $u \in H$ and $v \in H^*$, we have
$$|\langle u, v \rangle| \leq \|u\|_H \|v\|_{H^*}.$$
- For any $\varphi \in H^*$, there exists $\psi \in H^*$ such that $\varphi(u) = \langle u, \psi \rangle$ for all $u \in H$ (see, for example, [58], pp. 404).

Definition 4.5.2. We introduce a collection of dual Hilbert spaces of $\{H_j^*\}_{j=0}^2$ with norm $\langle ., . \rangle$ such that
$$H_0^* \subset H_1^* \subset H_2^*.$$

An adjoint Pencil
$$\mathcal{B}_A^*(\mu) : \mathbb{C} \to B(H_0^*, H_2^*),$$
which is defined by
$$\mathcal{B}_A^* : \mu^2 + A^* - \lambda \quad \text{for} \quad \mu \in \mathbb{C},$$
where A^* is a bounded operator from H_0^* into H_2^* and an adjoint of A.

Remark 25. We have the following notes:

- If μ_0 is an eigenvalue of $\mathcal{B}_A$, then $\overline{\mu_0}$ is an eigenvalue of $\mathcal{B}_A^*$, and their geometric and algebraic multiplicity coincide.
- Therefore, the eigenvalues of $\mathcal{B}_A^*$ are defined by
$$\overline{\mu_0} = \begin{cases} \mp\sqrt{\lambda - \lambda_j} & \text{if} \quad \lambda > \lambda_j \\ \\ \mp i\sqrt{\lambda_j - \lambda} & \text{if} \quad \lambda < \lambda_j. \end{cases}$$

Therefore, we have two cases

– If $\lambda < \lambda_0$, then

$$\sigma(\mathcal{B}_A^*) = \{\mp i\sqrt{\lambda_j - \lambda}, \quad j \in \mathbb{N}_0\}.$$

– If $\lambda_{m-1} < \lambda < \lambda_m$ for some $m \in \mathbb{N}$, then

$$\sigma(\mathcal{B}_A^*) = \{\mp\sqrt{\lambda - \lambda_j} \quad j = 0, 1, 2, ..., m-1\} \cup \{\mp i\sqrt{\lambda_j - \lambda}, \quad j = m, m+1, ...\}.$$

Definition 4.5.3. Let $\Gamma(\mathcal{B}_A^*)$ denote the projection of spectrum of adjoint pencil $\mathcal{B}_A^*$ onto the imaginary axis

$$\Gamma(\mathcal{B}_A^*) = \{\Im\mu| \quad \mu \in \sigma(\mathcal{B}_A^*)\} \subseteq \mathbb{R}.$$

Remark 26. We have the following notes:

- We can observe from a previous Proposition 4.4.1 again for any $\lambda \notin \sigma(A^*)$ and the adjoint pencil

$$\mathcal{B}_A^*(\mu) = \mu^2 + A^* - \lambda.$$

  The projection of $\sigma(\mathcal{B}_A^*)$ is

  1) If $\lambda < \lambda_0$, then

$$\Gamma(\mathcal{B}_A^*) = \{\mp\sqrt{\lambda_j - \lambda}, \quad j \in \mathbb{N}_0\}.$$

  2) If $\lambda_{m-1} < \lambda < \lambda_m$ for some $m \in \mathbb{N}$, then

$$\Gamma(\mathcal{B}_A^*) = \{0\} \cup \{\mp\sqrt{\lambda_j - \lambda}, \quad j = m, m+1, ...\}.$$

- In fact, given the canonical system of Jordan chains corresponding to $\mu_0 \in \sigma(\mathcal{B}_A)$, we can find a unique canonical system of Jordan chains $\{\psi_{k,s}\}$, for $s = 0, ..., m_k - 1$ and $k = 1, ..., J$ corresponding to $\overline{\mu} \in \sigma(\mathcal{B}_A^*)$.

Example

We can consider the operator

$$\mathcal{B}_{\mathcal{A}} = A + \lambda^2 : \mathcal{H} \to H.$$

where A is self-adjoint operator in$H$ defined on the Hilbert space $\mathcal{H} \subset H$. Suppose A is positive definite and the spectrum of A consists the eigenvalues γ_k $k \geq 1$ satisfying $0 < \gamma_1 < \gamma_2 < ...$ and $\gamma_k \to \infty$. Denote by J_k the multiplicity of γ_k and J_k is finite. Let $\alpha_1^{(k)}, ...\alpha_{J_k}^{(k)}$ be the eigenvectors of A corresponding to γ_k. Clearly the spectrum of the pencil $\mathcal{B}_{\mathcal{A}}$ consists of the eigenvalues

$$\lambda_v = \begin{cases} i\sqrt{\gamma_v} & \text{for} \quad v = 1, 2, ... \\ \\ -i\sqrt{\gamma_v} & \text{for} \quad v = -1, -2, ..., \end{cases}$$

and the corresponding eigenvectors are given by

$$\varphi_v = \begin{cases} \alpha_s^{(v)} & \text{if} \quad v \geq 1 \\ \\ \alpha_s^{(-v)} & \text{if} \quad v \leq -1, \end{cases}$$

The equation for the generalised eigenvector $A\varphi_1 = -2\lambda\varphi_0$ where φ_0 is an eigenvector corresponding to λ. Furthermore, both the algebraic and geometric multiplicities are equal to $J_{|v|}$. The adjoint pencils $\mathcal{B}_{\mathcal{A}}^*$ has the same form $A + \lambda^2$ except that the new one A is a continuous extension of old one, with domain H and range $\mathcal{H}^*$. See [58].

4.6 Main Results of the operator pencil

In this section, we give many consequences of the operator pencil and its properties. Theorems 4.6.1 and 4.6.2 are provided the key results concerning in the next chapter. However, Theorem 4.6.2 is a result that has been developed for the theory of ordinary differential equations with operator coefficients (see, for example, [10] and [58]).

Theorem 4.6.1. For $\alpha, \beta \in \mathbb{R}$, then

$$\mathcal{B}_A(D_t) = D_t^2 + A - \lambda : W_{\alpha,\beta}^2 \to W_{\alpha,\beta}^0 \tag{4.16}$$

is a bounded map.

Proof. To prove the map $\mathcal{B}_A(D_t) : W^2_{\alpha,\beta} \to W^0_{\alpha,\beta}$ is bounded, we can observe from Section 4.2, $W^2_{\alpha,\beta}$ has an equivalent finite norm,

$$\|u\|^2_{W^2_{\alpha,\beta}} = \int_{-\infty}^{0} e^{2\alpha t}(\left\|D_t^2 u(t)\right\|^2_{H_0} + \|u(t)\|^2_{H_2})dt + \int_{0}^{\infty} e^{2\beta t}(\left\|D_t^2 u(t)\right\|^2_{H_0} + \|u(t)\|^2_{H_2})dt,$$

and $W^0_{\alpha,\beta}$ has a finite norm,

$$\|u\|^2_{W^0_{\alpha,\beta}} = \int_{-\infty}^{0} e^{2\alpha t} \|u(t)\|^2_{H_0} dt + \int_{0}^{\infty} e^{2\beta t} \|u(t)\|^2_{H_0} dt.$$

Now, we can consider the norm of $\mathcal{B}_A(D_t)$ in $W^0_{\alpha,\beta}$, then, we use Lemmas 4.1.5 and 4.1.6

$$\begin{aligned}
\|\mathcal{B}_A(D_t)u\|^2_{W^0_{\alpha,\beta}} &= \int_{-\infty}^{0} e^{2\alpha t} \left\|(D_t^2 + A - \lambda)u(t)\right\|^2_{H_0} dt + \int_{0}^{\infty} e^{2\beta t} \left\|(D_t^2 + A - \lambda)u(t)\right\|^2_{H_0} dt \\
&\le 3\left[\int_{-\infty}^{0} e^{2\alpha t}(\|D_t^2 u(t)\|^2_{H_0} + \|Au(t)\|^2_{H_0} + \|\lambda u(t)\|^2_{H_0})dt\right] \\
&+ 3\left[\int_{0}^{\infty} e^{2\beta}(\|D_t^2 u(t)\|^2_{H_0} + \|Au(t)\|^2_{H_0} + \|\lambda u(t)\|^2_{H_0})dt\right] \\
&\le 3\left[\int_{-\infty}^{0} e^{2\alpha t}(\|D_t^2 u(t)\|^2_{H_0} + \|u(t)\|^2_{H_2} + \lambda^2\|u(t)\|^2_{H_2})dt\right] \\
&+ 3\left[\int_{0}^{\infty} e^{2\beta}(\|D_t^2 u(t)\|^2_{H_0} + \|u(t)\|^2_{H_2} + \lambda^2\|u(t)\|^2_{H_2})dt\right] \\
&= 3\left[\int_{-\infty}^{0} e^{2\alpha t}(\|(D_t^2 u(t)\|^2_{H_0} + (1+\lambda^2)\|u(t)\|^2_{H_2})dt\right] \\
&+ 3\left[\int_{0}^{\infty} e^{2\beta}(\|(D_t^2 u(t)\|^2_{H_0} + (1+\lambda^2)\|u(t)\|^2_{H_2})dt\right] \\
&\le 3(1+\lambda^2)\left[\int_{-\infty}^{0} e^{2\alpha t}(\|(D_t^2 u(t)\|^2_{H_0} + \|u(t)\|^2_{H_2})dt\right] \\
&+ 3(1+\lambda^2)\left[\int_{0}^{\infty} e^{2\beta}(\|(D_t^2 u(t)\|^2_{H_0} + \|u(t)\|^2_{H_2})dt\right] \\
&\le 3(1+\lambda^2)\,\|u\|^2_{W^2_{\alpha,\beta}}.
\end{aligned}$$

Thus, the proof is complete. ☐

Theorem 4.6.2. (Published in [43], April, 2021. (Under review)).

Let $\Gamma = \Gamma(\mathcal{B}_A)$ and $\alpha \in \mathbb{R} \setminus \Gamma$. Set $\delta = \operatorname{dist}(\alpha, \Gamma) > 0$. Then, the map

$$\mathcal{B}_A(D_t) = D_t^2 + A - \lambda : W^2_{\alpha,\alpha} \longrightarrow W^0_{\alpha,\alpha}$$

is an isomorphism map.

Proof. To prove the mapping is an isomorphism map (see Section 2.2.1), we find the constants $c_1^{''}$ and $c_2^{''}$ such that

$$c_1^{''} \|v\|_{W^2_{\alpha,\alpha}} \leq \|\mathcal{B}_A(D_t)v\|_{W^0_{\alpha,\alpha}} \leq c_2^{''} \|v\|_{W^2_{\alpha,\alpha}} . \tag{4.17}$$

By Theorem 4.6.1, we get the first side

$$\|\mathcal{B}_A(D_t)v\|_{W^0_{\alpha,\alpha}} \leq c_2^{''}\|v\|_{W^2_{\alpha,\alpha}}.$$

However, we need to get $c_1^{''}$

$$c_1^{''} \|v\|_{W^2_{\alpha,\alpha}} \leq \|\mathcal{B}_A(D_t)v\|_{W^0_{\alpha,\alpha}} \leq c_2^{''} \|v\|_{W^2_{\alpha,\alpha}} ,$$

for all $v \in W^2_{\alpha,\alpha}$. Write $v = e^{-\alpha t}u$. We have

$$\begin{aligned}
\|\mathcal{B}_A(D_t)v\|^2_{W^0_{\alpha,\alpha}} &= \left\|(D_t^2 + A - \lambda)v\right\|^2_{W^0_{\alpha,\alpha}} \\
&= \|(D_t^2 + A - \lambda)(e^{-\alpha t}u)\|^2_{W^0_{\alpha,\alpha}} \\
&= \|e^{-\alpha t}((D_t + i\alpha)^2 + A - \lambda)u\|^2_{W^0_{\alpha,\alpha}}.
\end{aligned}$$

From Lemma 4.3.4, we get

$$\begin{aligned}
\|\mathcal{B}_A(D_t)v\|^2_{W^0_{\alpha,\alpha}} &= \|e^{-\alpha t}((D_t + i\alpha)^2 + A - \lambda)u\|^2_{W^0_{\alpha,\alpha}} \\
&= \|((D_t + i\alpha)^2 + A - \lambda)u\|^2_{W^0_{0,0}}.
\end{aligned}$$

Letting $u(t) = \sum_{j=0}^{\infty} a_j(t)u_j$, and back to Section 4.1, we get

$$\begin{aligned}
((D_t + i\alpha)^2 + A - \lambda)u(t) &= ((D_t + i\alpha)^2 + A - \lambda)\sum_{j=0}^{\infty} a_j(t)u_j \\
&= \sum_{j=0}^{\infty} \left[(D_t + i\alpha)^2 a_j(t)u_j + a_j(t)Au_j - \lambda a_j(t)u_j\right] \\
&= \sum_{j=0}^{\infty} \left[(D_t + i\alpha)^2 a_j(t)u_j + a_j(t)\lambda_j u_j - \lambda a_j(t)u_j\right] \\
&= \sum_{j=0}^{\infty} \left[(D_t + i\alpha)^2 a_j(t) + \lambda_j a_j(t) - \lambda a_j(t)\right] u_j,
\end{aligned}$$

so now, we have

$$\left\|((D_t + i\alpha)^2 + A - \lambda)u)\right\|^2_{W^0_{0,0}} = \sum_{j=0}^{\infty} \int_{\mathbb{R}} |((D_t + i\alpha)^2 + \lambda_j - \lambda)a_j(t)|^2 dt.$$

So, by Lemma 2.2.1 and Parseval's identity, we get

$$\begin{aligned}\sum_{j=0}^{\infty}\int_{\mathbb{R}}|((D_t+i\alpha)^2+\lambda_j-\lambda)a_j(t)|^2dt &= \sum_{j=0}^{\infty}\int_{\mathbb{R}}|((\tau+i\alpha)^2+\lambda_j-\lambda)\widehat{a}_j(\tau)|^2d\tau \\ &= \sum_{j=0}^{\infty}\int_{\mathbb{R}}|(\tau+i\alpha)^2+\lambda_j-\lambda|^2|\widehat{a}_j(\tau)|^2d\tau,\end{aligned}$$

that is

$$\|((D_t+i\alpha)^2+A-\lambda)u)\|^2_{W^0_{0,0}} = \sum_{j=0}^{\infty}\|((\tau+i\alpha)^2+\lambda_j-\lambda)\widehat{a}_j(\tau)\|^2_{L^2(\mathbb{R})}. \tag{4.18}$$

By Lemma 4.3.4 and finding c_1, then we get $c_1^{''}$ to satisfy (4.17)

$$c_1^2(\tau^4+\lambda_j^2+1) \le |(\tau+i\alpha)^2+\lambda_j-\lambda|^2. \tag{4.19}$$

First, we need to prove

$$\delta^4 \le |(\tau+i\alpha)^2+\lambda_j-\lambda|^2 = (\tau^2-\alpha^2+\lambda_j-\lambda)^2+4\alpha^2\tau^2, \tag{4.20}$$

for all $j \in \mathbb{N}_0$ and $\tau \in \mathbb{R}$. Fix j. So, we find the stationary points are given by

$$\begin{aligned}\frac{d}{d\tau}[(\tau^2-\alpha^2+\lambda_j-\lambda)^2+4\alpha^2\tau^2] &= 2\tau(\tau^2-\alpha^2+\lambda_j-\lambda+4\alpha^2) \\ &= 2\tau(\tau^2+3\alpha^2+\lambda_j-\lambda)=0.\end{aligned}$$

It follows that $\tau = 0$ or $\tau^2 = -3\alpha^2-\lambda_j+\lambda$. Then, we observe the Global (Absolute) minimum of $(\tau^2-\alpha^2+\lambda_j-\lambda)^2+4\alpha^2\tau^2$ in τ occurs when

$$\tau = 0 \quad \text{or} \quad \tau^2 = -3\alpha^2-\lambda_j+\lambda.$$

- If $\tau = 0$, then

$$(\tau^2-\alpha^2+\lambda_j-\lambda)^2+4\alpha^2\tau^2 = (-\alpha^2+\lambda_j-\lambda)^2.$$

- If $\tau^2 = -3\alpha^2-\lambda_j+\lambda$, then

$$\begin{aligned}(\tau^2-\alpha^2+\lambda_j-\lambda)^2+4\alpha^2\tau^2 &= 16\alpha^4-12\alpha^4+4\alpha^2(-\lambda_j+\lambda) \\ &= 4\alpha^2(\alpha^2-\lambda_j+\lambda).\end{aligned}$$

This case can only occurs if $4\alpha^2 \leq \alpha^2 - \lambda_j + \lambda$. The Global minimum points are

$$(0, (-\alpha^2 + \lambda_j - \lambda)^2) \quad \text{and} \quad (\tau^2, 4\alpha^2(\alpha^2 - \lambda_j + \lambda)).$$

See [23].

Now, we consider two cases on λ :

Case 1. If $\lambda_j > \lambda$, then $\lambda - \lambda_j < 0$, and it follows $-3\alpha^2 - \lambda_j + \lambda < 0$, but

$$\tau^2 = -3\alpha^2 - \lambda_j + \lambda.$$

This is contradiction. Because it follows that $\tau^2 < 0$, and $\alpha^2 - \lambda_j + \lambda < 4\alpha^2$. Hence,

$$4\alpha^2 \leq \alpha^2 - \lambda_j + \lambda$$

is not satisfied. Then, we have only one case $\tau = 0$ to get

$$(\tau^2 - \alpha^2 + \lambda_j - \lambda)^2 + 4\alpha^2\tau^2 \geq (-\alpha^2 + \lambda_j - \lambda)^2 \geq \delta^4.$$

Case 2. If $\lambda > \lambda_j$, then $\lambda - \lambda_j \geq 0$, so it follows that $0 \in \Gamma$ and hence $\alpha^2 \geq \delta^2 > 0$. Because

$$\delta = \text{dist}(\alpha, \Gamma) = \inf_{\gamma \in \Gamma} |\alpha - \gamma| \leq |\alpha - 0| = |\alpha|.$$

Therefore,

$$\alpha^2 - \lambda_j + \lambda \geq \alpha^2 \geq \delta^2.$$

So,

$$(-\alpha^2 + \lambda_j - \lambda)^2 \geq \delta^4 \quad \text{and} \quad 4\alpha^2(\alpha^2 - \lambda_j + \lambda) \geq 4\delta^4.$$

It follows that

$$(\tau^2 - \alpha^2 + \lambda_j - \lambda)^2 + 4\alpha^2\tau^2 \geq \delta^4. \tag{4.21}$$

Now, we can move back to proving (4.19), and we set the following constant:

$$c_{\alpha,\lambda} = \max\{2|\lambda + \alpha^2| + 1, 3\max\{0, -\lambda_j : j = 1, 2, ...\}\}.$$

Then, we have two cases:

Case (i) If $\tau^2 + \lambda_j \geq c_{\alpha,\lambda}$, we observe

$$\begin{aligned}\tau^2 - \alpha^2 + \lambda_j - \lambda &= \frac{1}{2}(\tau^2 + \lambda_j) + \frac{1}{2}(\tau^2 + \lambda_j - 2(\lambda + \alpha^2)) \\ &\geq \frac{1}{2}(\tau^2 + \lambda_j + 1) > 0.\end{aligned}$$

Now, we note that $\tau^2 + \lambda_j \geq -3\lambda_j$ (from the definition of constant $c_{\alpha,\lambda}$), so $\tau^2 + 1 \geq -4\lambda_j$ implies $\lambda_j \geq \frac{-1}{4}(\tau^2 + 1)$. It follows that

$$2\lambda_j(\tau^2 + 1) \geq \frac{-1}{2}(\tau^2 + 1)^2. \tag{4.22}$$

Substitute (4.22) in the following equality to observe

$$\begin{aligned}(\tau^2 + \lambda_j + 1)^2 &= (\tau^2 + 1)^2 + 2\lambda_j(\tau^2 + 1) + \lambda_j^2 \\ &\geq (\tau^2 + 1)^2 - \frac{1}{2}(\tau^2 + 1)^2 + \lambda_j^2 \\ &\geq \frac{1}{2}(\tau^2 + 1)^2 + \lambda_j^2 \\ &\geq \frac{1}{2}(\tau^4 + \lambda_j^2 + 1).\end{aligned}$$

Therefore,

$$\begin{aligned}(\tau^2 - \alpha^2 + \lambda_j - \lambda)^2 + 4\alpha^2\tau^2 &\geq \frac{1}{4}(\tau^2 + \lambda_j + 1)^2 \\ &\geq \frac{1}{4}(\frac{1}{2}(\tau^2 + 1)^2 + \lambda_j^2) \\ &\geq \frac{1}{8}(\tau^4 + \lambda_j^2 + 1)\end{aligned}$$

for all λ_j.

Case(ii). If $\tau^2 + \lambda_j < c_{\alpha,\lambda}$, then by (4.20), we can get

$$\begin{aligned}(\tau^2 - \alpha^2 + \lambda_j - \lambda)^2 + 4\alpha^2\tau^2 \geq \delta^4 &> \delta^4 \frac{(\tau^2 + \lambda_j)^2 + 1}{c_{\alpha,\lambda}^2 + 1} \\ &> \frac{\frac{\delta^4}{2}(\tau^4 + \lambda_j^2) + 1}{c_{\alpha,\lambda}^2 + 1} \\ &> \frac{\delta^4}{2(c_{\alpha,\lambda}^2 + 1)}(\tau^4 + \lambda_j^2 + 1),\end{aligned}$$

where we have noted that the same arguments of case (i),

$$\tau^2 + \lambda_j \geq -3\lambda_j,$$

so $\tau^2 \geq -4\lambda_j$. It implies $\lambda_j \geq \dfrac{-1}{4}\tau^2$, and all of these lead to $2\lambda_j\tau^2 \geq \dfrac{-1}{2}\tau^4$.

Then, we get

$$\begin{aligned}(\tau^2+\lambda_j)^2 &= (\tau^2)^2 + 2\lambda_j(\tau^2) + \lambda_j^2 \\ &\geq \tau^4 - \frac{1}{2}\tau^4 + \lambda_j^2 \\ &\geq \frac{1}{2}\tau^4 + \lambda_j^2 \\ &\geq \frac{1}{2}(\tau^4 + \lambda_j^2).\end{aligned}$$

Now, combining cases (i) and (ii), it follows that

$$(\tau^2 - \alpha^2 + \lambda_j - \lambda)^2 + 4\alpha^2\tau^2 \geq c_1^2(\tau^4 + \lambda_j^2 + 1),$$

for all $\tau, j \in \mathbb{R}$, with $c_1^2 = \min\left\{\dfrac{1}{8}, \dfrac{\delta^4}{2(c_{\alpha,\lambda}^2+1)}\right\}$.

Therefore, from (4.19) and Lemma 4.3.3, we get

$$\begin{aligned}\left\|((D_t+i\alpha)^2+\lambda_j-\lambda)a_j(t)\right\|^2_{W^0_{0,0}} &\geq c_1^2\sum_{j=0}^{\infty}\int_{\mathbb{R}}(\tau^4+\lambda_j^2+1)|\widehat{a_j}(\tau)|^2 d\tau \\ &= c_1^2\left\|u\right\|^2_{W^2_{0,0}} \\ &\geq c_1^2 c^{''}\left\|e^{-\alpha t}u\right\|^2_{W^2_{\alpha,\alpha}} \\ &= c_1^2 c^{''}\left\|v\right\|^2_{W^2_{\alpha,\alpha}},\end{aligned}$$

where we take $c_1^{''} = c_1^2 c^{''}$. Thus, the proof is complete. □

The following propositions and corollaries describe some properties of Sobolev spaces and which will be used in the last chapter of the current thesis.

Proposition 4.6.3. Let $\alpha, \beta, \alpha^{'}, \beta^{'} \in \mathbb{R}$ such that $\alpha \leq \alpha^{'}$ and $\beta^{'} \leq \beta$. Then,

$$W^0_{\alpha,\beta} \subset W^0_{\alpha^{'},\beta^{'}}. \tag{4.23}$$

Further, the inclusion map $i : W^0_{\alpha,\beta} \longrightarrow W^0_{\alpha^{'},\beta^{'}}$ is continuous.

Proof. It is clear that

$$\forall t \leq 0 \quad e^{2\alpha^{'} t} \leq e^{2\alpha t}$$

and

$$\forall t \geq 0 \quad e^{2\beta^{'} t} \leq e^{2\beta t},$$

whenever $\alpha \leq \alpha^{'}$ and $\beta^{'} \leq \beta$. With this, if $u(t) \in H_0$, we can write

$$\int_{-\infty}^{0} e^{2\alpha^{'} t} \|u(t)\|_{H_0}^2 dt + \int_{0}^{\infty} e^{2\beta^{'} t} \|u(t)\|_{H_0}^2 dt \leq \int_{-\infty}^{0} e^{2\alpha t} \|u(t)\|_{H_0}^2 dt + \int_{0}^{\infty} e^{2\beta t} \|u(t)\|_{H_0}^2 dt. \tag{4.24}$$

Now, assume that $u \in W^0_{\alpha,\beta}$. Then, the right-hand side of (4.24) is finite, and by (4.24) itself, its left-hand side (which is clearly positive) is also finite. We then deduce that $u \in W^0_{\alpha^{'},\beta^{'}}$. In other words, (4.24) is equivalent to

$$\|u\|_{W^0_{\alpha^{'},\beta^{'}}} \leq \|u\|_{W^0_{\alpha,\beta}},$$

which in turn, means that the inclusion map $i : W^0_{\alpha,\beta} \longrightarrow W^0_{\alpha^{'},\beta^{'}}$ is continuous. The proof is complete. □

The following corollary is a consequence of the previous result.

Corollary 4.6.4. Let $\alpha \leq \beta$. Then, we have

$$W^0_{\alpha,\beta} \subset W^0_{\alpha,\alpha} \cap W^0_{\beta,\beta}. \tag{4.25}$$

Proof. In fact, if $u \in W^0_{\alpha,\beta}$, then by (4.23), we immediately obtain $u \in W^0_{\alpha,\alpha}$ and $u \in W^0_{\beta,\beta}$. That is,

$$u \in W^0_{\alpha,\alpha} \cap W^0_{\beta,\beta}.$$

□

We now state the following result that ensures that the inverse inclusion of (4.25) holds without any condition between α and β.

Proposition 4.6.5. Let α and β be arbitrary real numbers. Then,

$$W^0_{\alpha,\alpha} \cap W^0_{\beta,\beta} \subset W^0_{\alpha,\beta}. \tag{4.26}$$

Proof. Let $u \in W^0_{\alpha,\alpha} \cap W^0_{\beta,\beta}$, then $u \in W^0_{\alpha,\alpha}$, and $u \in W^0_{\beta,\beta}$ and by definition,

$$\int_{-\infty}^{\infty} e^{2\alpha t}\|u(t)\|^2_{H_0}dt < \infty \text{ and } \int_{-\infty}^{\infty} e^{2\beta t}\|u(t)\|^2_{H_0}dt < \infty.$$

But, for any α and β, we can write

$$\int_{-\infty}^{0} e^{2\alpha t}\|u(t)\|^2_{H_0}dt + \int_{0}^{\infty} e^{2\beta t}\|u(t)\|^2_{H_0}dt \leq \int_{-\infty}^{\infty} e^{2\alpha t}\|u(t)\|^2_{H_0}dt + \int_{-\infty}^{\infty} e^{2\beta t}\|u(t)\|^2_{H_0}dt < \infty.$$

It follows that $u \in W^0_{\alpha,\beta}$. The proof is complete. □

Combining (4.25) and (4.26), we immediately obtain the following corollary:

Corollary 4.6.6. Suppose $\alpha \leq \beta$. Then,

$$W^0_{\alpha,\alpha} \cap W^0_{\beta,\beta} = W^0_{\alpha,\beta}$$

holds.

We now state another result of interest.

Proposition 4.6.7. Assume that $\alpha \leq \beta$. Then, we have

$$W^0_{\alpha,\alpha} + W^0_{\beta,\beta} \subset W^0_{\beta,\alpha}.$$

Proof. By (4.23) of Proposition 4.6.3, we immediately have

$$W^0_{\alpha,\alpha} \subset W^0_{\beta,\alpha} \text{ and } W^0_{\beta,\beta} \subset W^0_{\beta,\alpha}.$$

It follows that

$$W^0_{\alpha,\alpha} + W^0_{\beta,\beta} \subset W^0_{\beta,\alpha}, \tag{4.27}$$

because $W^0_{\beta,\alpha}$ is a linear vector space. □

Proposition 4.6.8. For $\alpha \leq \beta$. Then, we have

$$W^0_{\beta,\alpha} \subset W^0_{\alpha,\alpha} + W^0_{\beta,\beta}.$$

Proof. Suppose $w \in W^0_{\beta,\alpha}$. Define

$$u(t) = \begin{cases} w(t) & \text{for t} < 0 \\ \\ 0 & \text{for t} \geq 0, \end{cases}$$

and

$$v(t) = \begin{cases} 0 & \text{for} \quad t < 0 \\ \\ w(t) & \text{for} \quad t \geq 0. \end{cases}$$

Hence, $w(t) = u(t) + v(t)$. Now, by definitions of $u(t)$ and $v(t)$, we can observe,

$$\begin{aligned} \|u\|^2_{W^0_{\beta,\beta}} &= \int_{-\infty}^{0} e^{2\beta t} \|u(t)\|^2_{H_0} dt + \int_0^{\infty} e^{2\beta t} \|u(t)\|^2_{H_0} dt \\ &= \int_{-\infty}^{0} e^{2\beta t} \|w(t)\|^2_{H_0} dt \\ &\leq \int_{-\infty}^{0} e^{2\beta t} \|w(t)\|^2_{H_0} dt + \int_0^{\infty} e^{2\alpha t} \|w(t)\|^2_{H_0} dt \\ &= \|w\|^2_{W^0_{\beta,\alpha}} < \infty. \end{aligned}$$

Therefore, $u \in W^0_{\beta,\beta}$. Similarly,

$$\begin{aligned} \|v\|^2_{W^0_{\alpha,\alpha}} &= \int_{-\infty}^{0} e^{2\alpha t} \|v(t)\|^2_{H_0} dt + \int_0^{\infty} e^{2\alpha t} \|v(t)\|^2_{H_0} dt \\ &= \int_0^{\infty} e^{2\alpha t} \|w(t)\|^2_{H_0} dt \\ &\leq \int_{-\infty}^{0} e^{2\beta t} \|w(t)\|^2_{H_0} dt + \int_0^{\infty} e^{2\alpha t} \|w(t)\|^2_{H_0} dt \\ &= \|w\|^2_{W^0_{\beta,\alpha}} < \infty. \end{aligned}$$

Hence, $v \in W^0_{\alpha,\alpha}$. It follows that $u + v \in W^0_{\beta,\beta} + W^0_{\alpha,\alpha}$, that is $w \in W^0_{\beta,\beta} + W^0_{\alpha,\alpha}$. Hence,

$$W^0_{\beta,\alpha} \subset W^0_{\alpha,\alpha} + W^0_{\beta,\beta}. \tag{4.28}$$

The desired result is obtained. □

Combining (4.27) and (4.28), we immediately obtain the following corollary:

Corollary 4.6.9. Suppose $\alpha \leq \beta$. Then,

$$W^0_{\alpha,\alpha} + W^0_{\beta,\beta} = W^0_{\beta,\alpha}$$

holds.

Now, we provide an important corollary related to the properties of Sobolev spaces. Indeed, it will have a large role in proving some results in the next results of this thesis.

Remark 27. For t, α and $\beta \in \mathbb{R}$, we can define the weight continuous function $w_{\alpha,\beta}(t)$ by

$$w_{\alpha,\beta}(t) = \begin{cases} e^{\alpha t} & t \in (-\infty, 0) \\ \\ e^{\beta t} & t \in (0, \infty). \end{cases}$$

The multiplication by $w_{\alpha,\beta}$ gives an isomorphism map between $W^0_{\alpha,\beta}$ and $L^2 = W^0_{0,0}$ such that

$$\begin{aligned} \|w_{\alpha,\beta} u\|^2_{W^0_{0,0}} &= \int_{-\infty}^0 \|w_{\alpha,\beta} u(t)\|^2_{H_0} dt + \int_0^\infty \|w_{\alpha,\beta} u(t)\|^2_{H_0} dt \\ &= \int_{-\infty}^0 w^2_{\alpha,\beta} \|u(t)\|^2_{H_0} dt + \int_0^\infty w^2_{\alpha,\beta} \|u(t)\|^2_{H_0} dt. \end{aligned}$$

Corollary 4.6.10. If $\alpha^{'} > \alpha$ and $\beta^{'} < \beta$, then, the inclusion map

$$i : W^2_{\alpha,\beta} \hookrightarrow W^1_{\alpha^{'},\beta^{'}}$$

is a compact map.

Proof. We have $\alpha^{'} > \alpha$ and $\beta^{'} < \beta$. So,

$$\alpha^{'} - \alpha > 0 > \beta^{'} - \beta.$$

We have that the function $w_{\alpha^{'}-\alpha,\beta^{'}-\beta} \in C^\infty(\mathbb{R})$.

To prove

$$i : W^2_{\alpha,\beta} \hookrightarrow W^1_{\alpha^{'},\beta^{'}}$$

is compact. Firstly, the multiplication by $w_{\alpha,\beta}$ defines an isomorphism map

$$W^2_{\alpha,\beta} \to H_2.$$

Then, we consider the map

$$(1+D^2)^{\frac{1}{2}} : H_2 \to H_1$$

is an isomorphism map.

Again, the multiplication by $w_{\alpha'-\alpha,\beta'-\beta}$ defines a compact map

$$(1+D^2)^{\frac{-1}{2}} : H_1 \to H_1,$$

by Lemma 2.7.5. In fact, the multiplication $w_{-\alpha',-\beta'}$ gives an isomorphism map

$$H_1 \to W^1_{\alpha',\beta'}.$$

Hence, the inclusion as composition of

$$W^2_{\alpha,\beta} \to H_2 \to H_1 \to H_1 \to W^1_{\alpha',\beta'}.$$

Because the first step, second step and fourth step are isomorphism maps and the third step is a compact map. Therefore,

$$i : W^2_{\alpha,\beta} \hookrightarrow W^1_{\alpha',\beta'}$$

is a compact map. □

We finish this section by giving the following result for the operator Pencils $\mathcal{B}_A$ by using some arguments from previous results and some restriction on α, β.

Theorem 4.6.11. For $\alpha, \beta \in \mathbb{R} \setminus \Gamma$, choose $\alpha < \alpha^{'}$ and $\beta^{'} < \beta$. Then, there exists c, and for all $u \in W^2_{\alpha,\beta}$, such that

$$\|u\|_{W^2_{\alpha,\beta}} \leq c[\|\mathcal{B}_A(D_t)u\|_{W^0_{\alpha,\beta}} + \|u\|_{W^1_{\alpha',\beta'}}].$$

Proof. Choose $\chi^{\pm} \in C^{\infty}$. Where,

- $\operatorname{Ran} \chi^{\pm} \subseteq [0,1]$
- $\operatorname{supp}(\chi^{\pm}) \subseteq \pm[-1,\infty)$
- $\operatorname{supp}(\nabla\chi^{\pm}) \subseteq [-1,1]$

- $\chi^+ + \chi^- = 1$.

We set $u^{\pm} = \chi^{\pm} u$, and $v = \mathcal{B}_A(D_t)u$, so we consider $v^{\pm} = \chi^{\pm} v$, and $w^{\pm} = [\mathcal{B}_A(D_t), \chi^{\pm}]u$.

So

$$u = 1.u = (\chi^+ + \chi^-)u = \chi^+ u + \chi^- u = u^+ + u^-.$$

Similarly,

$$v = 1.v = (\chi^+ + \chi^-)v = \chi^+ v + \chi^- v = v^+ + v^-.$$

By definition of a commutator, we can observe

$$v^{\pm} = \chi^{\pm}\mathcal{B}_A(D_t)u = \mathcal{B}_A(D_t)\chi^{\pm}u - [\mathcal{B}_A(D_t), \chi^{\pm}]u = \mathcal{B}_A(D_t)u^{\pm} - w^{\pm}.$$

Now, we have $\text{supp}(u^+) \subseteq [-1, \infty)$ and $\text{supp}(u^-) \subseteq (-\infty, 1]$ and we can note by Corollary 4.6.6 the following: If $u \in W^2_{\alpha,\beta}$, then $u^- \in W^2_{\alpha,\alpha}$ and $u^+ \in W^2_{\beta,\beta}$.

Similarly, if $v \in W^0_{\alpha,\beta}$, then $v^- \in W^0_{\alpha,\alpha}$ and $v^+ \in W^0_{\beta,\beta}$.

In particular, we find a constant c_1 because $\chi^{\pm}$ is a bounded operator, by Lemma 2.3.1, and then by Proposition 4.6.3, we observe that

$$\|u^-\|_{W^2_{\alpha,\beta}} = \|\chi^- u\|_{W^2_{\alpha,\beta}} \leq \|\chi^-\|_{op}\|u\|_{W^2_{\alpha,\beta}} \leq c_1\|u^-\|_{W^2_{\alpha,\alpha}}.$$

Similarly,

$$\|u^+\|_{W^2_{\alpha,\beta}} = \|\chi^+ u\|_{W^2_{\alpha,\beta}} \leq \|\chi^+\|_{op}\|u\|_{W^2_{\alpha,\beta}} \leq c_1\|u^+\|_{W^2_{\beta,\beta}}.$$

To get constant c_2 and we have to use Proposition 4.6.3

$$\begin{aligned}
\|v^-\|_{W^0_{\alpha,\alpha}} + \|v^+\|_{W^0_{\beta,\beta}} &\leq c_2[\|v^-\|^2_{W^0_{\alpha,\beta}} + \|v^+\|^2_{W^0_{\alpha,\beta}}]^{\frac{1}{2}} \\
&\leq c_2[\|\chi^- v\|^2_{W^0_{\alpha,\beta}} + \|\chi^+ v\|^2_{W^0_{\alpha,\beta}}]^{\frac{1}{2}} \\
&\leq c_2[\|(\chi^- + \chi^+)v\|^2_{W^0_{\alpha,\beta}}]^{\frac{1}{2}} \\
&= c_2\|v\|_{W^0_{\alpha,\beta}}.
\end{aligned}$$

Also, by definition of a commutator, we have that for $t \in \mathbb{R}$,

$$
\begin{aligned}
[\mathcal{B}_A(D_t), \chi^{\pm}]u(t) &= [D_t^2 + A - \lambda, \chi^{\pm}]u(t) \\
&= (D_t^2 + A - \lambda)(\chi^{\pm}(t)u(t)) - \chi^{\pm}(t)(D_t^2 + A - \lambda)u(t) \\
&= D_t^2(\chi^{\pm}(t)u(t)) + A(\chi^{\pm}(t)u(t)) - \lambda(\chi^{\pm}(t)u(t)) - \chi^{\pm}(t)(D_t^2 u(t)) - \chi^{\pm}(t)Au(t) + \chi^{\pm}(t)\lambda u(t) \\
&= \chi^{\pm}(t)D_t^2 u(t) + D_t\chi^{\pm}D_t u(t) + D_t^2(\chi^{\pm}(u(t)) + D_t\chi^{\pm}D_t u(t) - \chi^{\pm}(t)(D_t^2 u(t)) \\
&= 2(D_t\chi^{\pm})(D_t u)(t) + (D_t^2(\chi^{\pm})(u(t)).
\end{aligned}
$$

Because we have $\operatorname{Ran}\chi^{\pm} \subseteq [0,1]$, $[\mathcal{B}_A(D_t), \chi^{\pm}]$ with coefficients that are bounded, and $\operatorname{supp}(\nabla\chi^{\pm}) \subseteq \pm[-1,1]$ (by assumption). Then, we can use the same argument of Theorem 4.6.1, Proposition 4.6.3 and Lemma 4.1.5 to prove the map

$$[\mathcal{B}_A(D_t), \chi^{\pm}] : W^1_{\alpha',\beta'} \to W^0_{\alpha,\beta}$$

is a bounded map whenever $\alpha < \alpha'$ and $\beta' < \beta$ To check this, we can look at the following:

$$
\begin{aligned}
\left\|(\mathcal{B}_A(D_t), \chi^{\pm})u\right\|^2_{W^0_{\alpha,\beta}} &= \int_{-\infty}^0 e^{2\alpha t}\left\|(\mathcal{B}_A(D_t), \chi^{\pm})u(t)\right\|^2_{H_0} dt + \int_0^\infty e^{2\beta t}\left\|(\mathcal{B}_A(D_t), \chi^{\pm})u(t)\right\|^2_{H_0} dt^2 \\
&= \int_{-\infty}^0 e^{2\alpha t}\left\|2(D_t u(t)) + u(t)\right\|^2_{H_0} dt + \int_0^\infty e^{2\beta t}\left\|2(D_t u)(t) + u(t)\right\|^2_{H_0} dt^2 \\
&\leq 4\left[\int_{-\infty}^0 e^{2\alpha t}(\|D_t u(t)\|^2_{H_0} + \|u(t)\|^2_{H_0})dt\right] + 4\left[\int_0^\infty e^{2\beta}(\|D_t u(t)\|^2_{H_0} + \|u(t)\|^2_{H_0}dt\right] \\
&= 4\left[\int_{-\infty}^0 e^{2\alpha t}(\|(D_t u(t)\|^2_{H_0} + \|u(t)\|^2_{H_1})dt\right] + 4\left[\int_0^\infty e^{2\beta}(\|(D_t u(t)\|^2_{H_0} + \|u(t)\|^2_{H_1})dt\right] \\
&\leq 4\left[\int_{-\infty}^0 e^{2\alpha' t}(\|(D_t u(t)\|^2_{H_0} + \|u(t)\|^2_{H_1})dt + \int_0^\infty e^{2\beta'}(\|(D_t u(t)\|^2_{H_0} + \|u(t)\|^2_{H_1})dt\right] \\
&= 4\left\|u\right\|^2_{W^1_{\alpha',\beta'}}.
\end{aligned}
$$

Now, we find constants $c_{3,0}, c_3$ such that by Proposition 4.6.3 and boundedness

$$\|w^-\|_{W^0_{\alpha,\alpha}} \leq c_{3,0}\|w^-\|_{W^0_{\alpha,\beta}} = c_{3,0}\|[\mathcal{B}_A(D_t), \chi^-]u\|_{W^0_{\alpha,\beta}} \leq 4c_3\|u\|_{W^1_{\alpha',\beta'}},$$

and similarly for

$$\|w^+\|_{W^0_{\beta,\beta}} \leq c_{3,0}\|w^+\|_{W^0_{\alpha,\beta}} = c_{3,0}\|[\mathcal{B}_A(D_t), \chi^+]u\|_{W^0_{\alpha,\beta}} \leq 4c_3\|u\|_{W^1_{\alpha',\beta'}}.$$

By Theorem 4.6.2, we set the maps $\mathcal{B}_A(D_t) : W^2_{\alpha,\alpha} \to W^0_{\alpha,\alpha}$ and $\mathcal{B}_A(D_t) : W^2_{\beta,\beta} \to W^0_{\beta,\beta}$ by $A^{(\alpha)}$ and $A^{(\beta)}$ are isomorphism maps, respectively. So we find c_4 such that (by assumption)

$$\begin{aligned}\|u^-\|_{W^2_{\alpha,\alpha}} &\leq c_4\|A^{(\alpha)}u^-\|_{W^0_{\alpha,\alpha}}\\ &\leq c_4(\|v^-\|_{W^0_{\alpha,\alpha}} + \|w^-\|_{W^0_{\alpha,\alpha}})\end{aligned}$$

and

$$\begin{aligned}\|u^+\|_{W^2_{\beta,\beta}} &\leq c_4\|A^{(\beta)}u^+\|_{W^0_{\beta,\beta}}\\ &\leq c_4[\|v^+\|_{W^0_{\beta,\beta}} + \|w^+\|_{W^0_{\beta,\beta}}].\end{aligned}$$

Put everything together, we get

$$\begin{aligned}\|u\|_{W^2_{\alpha,\beta}} \leq \|u^-\|_{W^2_{\alpha,\alpha}} + \|u^+\|_{W^2_{\beta,\beta}} &\leq c_1[\|u^-\|_{W^2_{\alpha,\alpha}} + \|u^+\|_{W^2_{\beta,\beta}}]\\ &\leq c_1c_4[\|v^-\|_{W^0_{\alpha,\alpha}} + \|v^+\|_{W^0_{\beta,\beta}} + \|w^-\|_{W^0_{\alpha,\alpha}} + \|w^+\|_{W^0_{\beta,\beta}}]\\ &\leq c_1c_4[c_2\|v\|_{W^0_{\alpha,\beta}} + 4c_3\|u\|_{W^1_{\alpha',\beta'}}].\end{aligned}$$

Hence,

$$\|u\|_{W^2_{\alpha,\beta}} \leq c_1c_4[c_2\|\mathcal{B}_A u\|_{W^0_{\alpha,\beta}} + 4c_3\|u\|_{W^1_{\alpha',\beta'}}].$$

The proof is complete. □

Conclusion

Weighted function spaces were introduced in the beginning of this chapter with some results. The operator Pencil was defined with some arguments that used the ideas from the theory differential equations, for example, Theorem 4.6.2. Then, we generalise this theorem to deal with $\mathcal{B}_A$ mapping between such spaces by proving the corollaries 4.6.6 and 4.6.9.

Chapter 5

The Fredholm Properties of Pencils and the Main Results

In the last Chapter, we start the prerequisites of the Fredholm operator and with the definition of the semi-Fredholm operator in Section 5.1. In Section 5.2, we introduce the definition of the Fredholm operator pencil and we observe the resolvent operator pencils (inverse operator) $\mathcal{B}_A^{-1}(\mu)$ with some properties. In Section 5.3, we investigate the Green's function $G(t)$ and we obtain asymptotic formula for this function at infinity which is based on Theorem 5.2.3. At the end of this chapter, we give certain results of the semi-Fredholm property, and main consequences which provide a key step for Fredholm properties of pencils, and the formula of the index (see Theorem 5.5.5).

5.1 Prerequisites of Fredholm Operators

Remark 28. We introduced a collection of Hilbert spaces $\{H_j\}_{j=0}^{2}$ with norm $\langle .,. \rangle_j$ as in Section 4.1 such that

$$H_2 \subset H_1 \subset H_0.$$

$B(H_2, H_0)$ is denoted the Hilbert space of all bounded linear operators. Let A be an operator in $B(H_2, H_0)$ and use the notations $\mathrm{Ker}(A)$ and $\mathrm{Ran}(A)$ for the set of kernel and the range of the operator A respectively. Let

$$\kappa(A) := \dim(\mathrm{Ker}\, A) \in \mathbb{N}_0 \cup \{\infty\}, \quad \text{and} \quad \eta(A) := \mathrm{Codim}(\mathrm{Ran}\, A) \in \mathbb{N}_0 \cup \{\infty\}.$$

Definition 5.1.1. An operator A which has a closed range and for which either $\kappa(A)$ or $\eta(A)$ is a finite-dimensional, it is called a *semi-Fredholm operator.*

Definition 5.1.2. (Fredholm operator)

A linear operator A. We say that:

- A is an *upper semi-Fredholm operator* if $\mathrm{Ran}(A)$ is closed in H_0 and $\kappa(A) < \infty$;
- A is a *lower semi-Fredholm operator* if $\mathrm{Ran}(A)$ is closed in H_0 and $\eta(A) < \infty$;
- A is a *Fredholm operator* if $\mathrm{Ran}(A)$ is closed in H_0, $\kappa(A) < \infty$, and $\eta(A) < \infty$.

  The sets of upper and lower semi-Fredholm operators set is denoted by $\Phi_+(H_2, H_0)$ and $\Phi_-(H_2, H_0)$ respectively, while the set of Fredholm operators set is denoted by $\Phi(H_2, H_0)$. See [45].

In particular, each Fredholm operator has a Fredholm index.

Definition 5.1.3. If A is a Fredholm operator, then the integer

$$\mathrm{Index}(A) = \kappa(A) - \eta(A)$$

is called the *Index* of A.

Lemma 5.1.1. Let A is a bounded linear operator, the following are equivalent:

- $\kappa(A)$ is finite dimensional and $\mathrm{Ran}(A)$ is closed.
- Every bounded sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq H_2$ with $\{Au_n\}_{n\in\mathbb{N}} \subseteq H_0$ convergent has a convergent subsequence (see, for example, [3] and [8]).

For properties of the Fredholm operator and their proofs. There exists a vast number of literatures on this topic, for example, [5], [24], and [55].

However, in the following part we shall apply some properties of the duality of Fredholm operator. We give quantities $\kappa(A)$ and $\eta(A)$ for an operator A with closed range are dual to each other and that a Fredholm operator and its dual operator have opposite Fredholm indices.

Proposition 5.1.2. (Adjoint Fredholm operator)

If $A \in B(H_2, H_0)$ is a Fredholm operator then $A^* \in B(H_0^*, H_2^*)$ is also Fredholm and

$$\text{Index}(A) = -\,\text{Index}(A^*).$$

Theorem 5.1.3. Let $A \in B(H_2, H_0)$ be operator with closed ranges then

$$\kappa(A^*) = \eta(A) \quad \text{and} \quad \eta(A^*) = \kappa(A).$$

See [45], [58] and the adjoint Fredholm operator is also used in [60].

Proposition 5.1.4. (Adjoint (semi-) Fredholm operator)

Let $A \in B(H_2, H_0)$ be an operator. Then:

- A is an upper semi-Fredholm operator if and only if A^* is a lower semi-Fredholm operator.
- A is a lower semi-Fredholm operator if and only if A^* is an upper semi-Fredholm operator.

Refer the reader can see the lecture notes of Banach spaces and thier operators, for example, [45].

5.2 Fredholm Operator Pencil

In this section, we define basic facts of the Fredholm operator Pencil and its adjoint; these are collected without proof. Then, we can structure of the formula of $\mathcal{B}_A^{-1}(\mu)$ near the pole.

Definition 5.2.1. We can consider the operator Pencil $\mathcal{B}_A$ which is defined in (in Section 4.5) such that

$$\mathcal{B}_A : \mathbb{C} \to B(H_2, H_0)$$

$$\mathcal{B}_A(\mu) = \mu^2 + A - \lambda \quad \text{for} \quad \mu \in \mathbb{C}.$$

is called Fredholm for all $\mu \in \mathbb{C}$, and it is invertible at least one value of μ (see, for example, [58] and [59]).

Theorem 5.2.1. Let Ω be in the domain $\mathbb{C}$. Suppose the operator Pencil $\mathcal{B}_A(\mu)$ satisfies the following conditions:

1) $\mathcal{B}_A(\mu) \in \Phi(H_2, H_0)$ for all $\mu \in \Omega$.

2) There exists a number $\mu \in \Omega$ such that the operator $\mathcal{B}_A(\mu)$ has a bounded inverse.

   Then, the spectrum of operator pencil $\mathcal{B}_A(\mu)$ consists of isoloted eigenvalues with finite algebraic multiplicity. See, for example, [58] and [59].

In what follows, we consider the operator pencil again and the definition of adjoint operator which is defined in Section 4.5.

Definition 5.2.2. The adjoint operator Pencil $\mathcal{B}_A^* : \mathbb{C} \to B(H_0^*, H_2^*)$ is a Fredholm operator for all $\overline{\mu} \in \mathbb{C}$ and invertible at least one value and therefore its spectrum is discrete. See [58].

Proposition 5.2.2. Let $\mathcal{B}_A$ be a Fredholm operator pencils. Then,

- $\mu_0 \in \mathbb{C}$ is an eigenvalue of $\mathcal{B}_A$ if and only if $\overline{\mu_0}$ is an eigenvalue of $\mathcal{B}_A^*$.

- The geometric and algebraic multiplicity of μ and $\overline{\mu}$ coincide.

Proof. The reader can see the proof of this proposition in [58] and [59]. $\square$

The main purpose in the following part is defined the inverse operator $\mathcal{B}_A^{-1}$ of operator pencil $\mathcal{B}_A$ near an eigenvalue μ_0, we need the notion of holomorphic function. Then, we consider some properties of this operator which will be used to investigate some arguments of this thesis.

Definition 5.2.3. Let Ω be a domain in Complex plane $\mathbb{C}$. An operator function

$$\Upsilon(\mu) : \Omega \to B(H_2, H_0)$$

is called holomorphic on Ω when it can be represented as a power series

$$\Upsilon(\mu) = \sum_{j=0}^{\infty} \Upsilon_j (\mu - \mu_0)^j, \quad \Upsilon_j \in B(H_2, H_0),$$

which is convergent in $B(H_2, H_0)$ in a neighbourhood of $\mu_0 \in \Omega$ (see [58]).

Theorem 5.2.3. Let μ_0 be an eigenvalue of $\mathcal{B}_A$ and let J and $m_1, ..., m_J$ be its geometric multiplicity and partial multiplicity respectively. Suppose that

$$\{\varphi_{k,s}\}, \quad s = 0, ..., m_k - 1, \quad k = 1, ..., J$$

is a canonical system of Jordan of $\mathcal{B}_A$ corresponding to μ_0. (Refer back to Section 4.4).

(i) There exits a unique

$$\{\psi_{k,s}\}, \quad s = 0, ..., m_k - 1, \quad k = 1, ..., J$$

is a canonical system of Jordan of $\mathcal{B}_A^*$ corresponding to $\overline{\mu_0}$. (Refer back to Section 4.5).

Such that in a neighbourhood of μ_0, the resolvent operator (inverse operator) can be represented as

$$\mathcal{B}_A^{-1}(\mu) = \sum_{k=1}^{J} \sum_{h=0}^{m_k-1} \frac{P_{k,h}}{(\mu - \mu_0)^{m_k - h}} + \Upsilon(\mu), \tag{5.1}$$

where,

$$P_{k,h} = \sum_{s=0}^{h} \langle ., \psi_{k,s} \rangle_{H_0} \varphi_{k,h-s}, \tag{5.2}$$

and Υ is a holomorphic function in the neighbourhood of μ_0.

(ii) The system $\{\psi_{k,s}\}$ is a canonical system of Jordan of $\mathcal{B}_A^*$ corresponding to $\overline{\mu_0}$ satisfies the bi-orthogonal condition that is,

$$\sum_{s=0}^{d} \sum_{n=s+1}^{m_k+s} \frac{1}{n!} (\mathcal{B}_A^{(n)}(\mu_0)\varphi_{k,m_k+s-n}, \psi_{j,d-s})_{H_0} = \delta_k \delta_d \tag{5.3}$$

for k, $j = 1, ..., J$, and $d = 0, ..., m_k - 1$.

(iii) Suppose $\psi_{j,0}, ..., \psi_{j,m_j-1}$ for $j = 1, ..., J$ is a collection of Jordan chain of $\mathcal{B}^*(A)$ corresponding to $\overline{\mu_0}$ which is subject to (5.3), then the collection $\psi_{j,0}, ..., \psi_{j,m_j-1}$ is a canonical system satisfying (i).

Proof. The reader can see the proof in [58] and [59]. □

Remark 29. We have the following notes:

- Let J and $m_1, ..., m_J$ be geometric multiplicity and partial multiplicity respectively of μ_0.

We have

$$\{\varphi_{k,s}\},\quad s=0,...,m_k-1,\quad k=1,...,J$$

is a canonical system of Jordan of $\mathcal{B}_A$ corresponding to μ_0. Let

$$\Phi_k(\mu)=\sum_{s=0}^{m_k-1}\varphi_{k,s}(\mu-\mu_0)^{s-m_k}$$

is the set generating system if and only if $\{\varphi_{k,s}\}$ is a conical set of Jordan chain.

We can consider the solution of the equation

$$\mathcal{B}_A(D_t)U=0,$$

of the form

$$U(t)=e^{i\mu_0 t}\sum_{n=0}^{m}\frac{(it)^n}{n!}u_{m-n} \tag{5.4}$$

By definition of a canonical system of Jordan chain of $\mathcal{B}_A$ corresponding to μ_0, one directly verifies (5.4) is a solution of $\mathcal{B}_A(D_t)U=0$ if and only if μ_0 is an eigenvalue of Pencil, u_0 is an eigenfunction corresponding to μ_0.

Then, the following collection

$$e^{i\mu_0 t}D_t^s\Phi_k(it),\quad s=0,...,m_k-1,\quad k=1,...,J$$

form the solutions, where

$$\Phi_k(z)=\sum_{h=0}^{m_k-1}\frac{z^h}{h!}\varphi_{k,s}.$$

- Similarly, we have

$$\{\psi_{k,s}\},\quad s=0,...,m_k-1,\quad k=1,...,J$$

is a canonical system of Jordan of $\mathcal{B}_A^*$ corresponding to $\overline{\mu_0}$.

However, we can consider the solution of the equation

$$\mathcal{B}^*{}_A(D_t)V=0,$$

of the form

$$V(t) = e^{i\overline{\mu}t} \sum_{n=0}^{m} \frac{(it)^n}{n!} v_{m-n} \tag{5.5}$$

Then the collection

$$e^{i\overline{\mu_0}t} D_t \Psi_k(it) \quad s = 0, ..., m_k - 1, \quad k = 1, ..., J$$

form the solutions where,

$$\Psi_k(z) = \sum_{h=0}^{m_k-1} \frac{z^h}{h!} \psi_{k,s}.$$

See, for example, [58], and [59].

5.3 Green's Kernel

What is a Green's function? Mathematically, it is the kernel of an integral operator that represent the inverse of a differential operator (see [25]). In this section, we construct bases to define the Green's function with some properties. Then, we obtain an asymptotic the formula of the Green's function at infinity based on Theorem 5.2.3 and we observe some results to achieve asymptotic new representation of this function as $t \to \pm\infty$ of exponential solution of $\mathcal{B}_A(D_t)u = f$ in the Sobolev space $W^0_{\alpha,\beta}$.

5.3.1 The Definition of a Green's Kernel

This section is devoted to estimate of Green's operator of the equation $\mathcal{B}_A(D_t) = f$. We observe the following assertion will use to define a Green's function of the resolvent operator and the bounded map $\mathcal{B}_A(D_t) : W^2_{\alpha,\alpha} \to W^0_{\alpha,\alpha}$.

Lemma 5.3.1. Suppose $\alpha \notin \Gamma(\mathcal{B}_A) = \Im(\sigma(\mathcal{B}_A))$, that is the line $\Im(\sigma(\mathcal{B}_A))$ does not contain eigenvalues of the operator Pencils $\mathcal{B}_A(\mu)$.

Then the Green's function is defined by

$$G(t) = \frac{1}{2\pi} \int_{\Im\mu=\alpha} e^{it\mu} \mathcal{B}_A^{-1}(\mu) d\mu. \tag{5.6}$$

Proof. We can set

$$A^{(\alpha)} = \mathcal{B}_A(D_t) = D_t^2 + A - \lambda : W^2_{\alpha,\alpha} \to W^0_{\alpha,\alpha},$$

and by Theorem 4.6.1, $A^{(\alpha)}$ is a bounded map.

If $u \in W^2_{\alpha,\alpha}$, and by Remark 21, in Section 4.2, we have that

$$\widehat{A^{(\alpha)}u}(\tau + i\alpha) = \mathcal{B}_A(\tau + i\alpha)\widehat{u}(\tau + i\alpha),$$

for $\alpha, \tau \in \mathbb{R}$.

Since $\alpha \notin \Gamma(\mathcal{B}_A) = \Im(\sigma(\mathcal{B}_A))$, and $\alpha, \tau \in \mathbb{R}$.

We have $\mathcal{B}_A(\tau + i\alpha)$ is an invertible for all $\tau \in \mathbb{R}$ by Theorem 5.2.1, (1), it follows that

$$\widehat{u}(\tau + i\alpha) = \mathcal{B}_A^{-1}(\tau + i\alpha)\widehat{f}(\tau + i\alpha), \tag{5.7}$$

where $f = A^{(\alpha)}u \in W^0_{\alpha,\alpha}$.

Now, by definition of the inverse of Fourier transform, by Remark 21 (again), and by (5.7), we can get that

$$e^{\alpha t}u(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\tau t}\widehat{u}(\tau + i\alpha)d\tau,$$

so,

$$e^{\alpha t}u(t) = \frac{1}{2\pi} \int\int_{\mathbb{R}^2} e^{i\tau t}\mathcal{B}_A^{-1}(\tau + i\alpha)e^{-is(\tau+i\alpha)}f(s)dsd\tau.$$

It follows that,

$$\begin{aligned} u(t) &= \frac{1}{2\pi} \int\int_{\mathbb{R}^2} e^{i(t-s)(\tau+i\alpha)}\mathcal{B}_A^{-1}(\tau + i\alpha)f(s)dsd\tau \\ &= \int_{\mathbb{R}} G(t-s)f(s)ds, \end{aligned}$$

where we can introduce the operator

$$G(t) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{it(\tau+i\alpha)}\mathcal{B}_A^{-1}(\tau + i\alpha)d\tau,$$

is called *Green's Kernel.*

For $\mu = \tau + i\alpha$, we can get that

$$G(t) = \frac{1}{2\pi} \int_{\Im\mu=\alpha} e^{it\mu}\mathcal{B}_A^{-1}(\mu)d\mu.$$

□

However, the following proposition, we observe the integral of the inverse operator is convergent in the norm of $B(H_0, H_2)$ to determine $G(t)$.

Proposition 5.3.2. For $\alpha \notin \Gamma(\mathcal{B}_A)$, i.e., the line $\Im(\sigma(\mathcal{B}_A))$ does not contain eigenvalues of the operator pencils $\mathcal{B}_A(\mu)$. Then for $t \neq 0$, the limit

$$\lim_{R\to\infty} \int_{-R+i\alpha}^{R+i\alpha} e^{it\mu} \mathcal{B}_A^{-1}(\mu) d\mu,$$

exists in $B(H_0, H_2)$.

Proof. We need to prove that

$$\lim_{R\to\infty} \int_{-R+i\alpha}^{R+i\alpha} e^{it\mu} \mathcal{B}_A^{-1}(\mu) d\mu \tag{5.8}$$

exists, by differentiating $\mathcal{B}_A^{-1}(\mu)$, Theorems 5.2.1, (1), and 4.6.2, we can get

$$\|D_\mu \mathcal{B}_A^{-1}(\mu)\|_{\mathbb{C}} \leq c|\mu|. \tag{5.9}$$

By using the integrating by parts and we know that $D_\mu e^{it\mu} = ite^{it\mu}$, we have

$$\begin{aligned}\lim_{R\to\infty} \int_{-R+i\alpha}^{R+i\alpha} e^{it\mu} \mathcal{B}_A^{-1}(\mu) d\mu = &-\frac{1}{t} \int_{-R+i\alpha}^{R+i\alpha} e^{it\mu} D_\mu \mathcal{B}_A^{-1}(\mu) d\mu \\ &- \frac{1}{it}(e^{it(R+i\alpha)} \mathcal{B}_A^{-1}(R) - e^{-it(R-i\alpha)} \mathcal{B}_A^{-1}(-R)).\end{aligned}$$

Therefore, by using (5.9) which implies that this operator sequence to get that,

$$-\frac{1}{t} \int_{\mathbb{R}} e^{it\mu} D_\mu \mathcal{B}_A^{-1}(\mu + i\alpha) d\mu,$$

in the space $B(H_0, H_2)$.

Thus for $t \neq 0$, we have that

$$\lim_{R\to\infty} \int_{-R+i\alpha}^{R+i\alpha} e^{it\mu} \mathcal{B}_A^{-1}(\mu) d\mu = -\frac{1}{t} \int_{\mathbb{R}} e^{it\mu} D_\mu \mathcal{B}_A^{-1}(\mu) d\mu, \tag{5.10}$$

where the last integral is absolute convergent in the norm of $B(H_0, H_2)$. See, for example, [27] and [58].

□

Remark 30. From above arguments, we have the operator

$$G(t) = \frac{1}{2\pi} \int_{\Im\mu=\alpha} e^{it\mu} \mathcal{B}_A^{-1}(\mu) d\mu.$$

explained in sense of the Cauchy integral. By (5.10) we get that,

$$G(t) = -\frac{1}{2\pi t} \int_{\Im\mu=\alpha} e^{it\mu} D_\mu \mathcal{B}_A^{-1}(\mu) d\mu$$

with absolute convergent in $B(H_0, H_2)$.

The following proposition, we have some properties of $G(t)$:

Proposition 5.3.3. For $\alpha \notin \Gamma(\mathcal{B}_A$ and we set $\Sigma_{\alpha_\pm} = \{\mu \in \sigma(\mathcal{B}_A) : \Im\mu \lesseqgtr \alpha\}$. Then, the operator

$$G(t) = \frac{1}{2\pi} \int_{\Im\mu=\alpha} e^{it\mu} \mathcal{B}_A^{-1}(\mu) d\mu$$

i) it does not depend on $\alpha \in \sum_{\alpha_\pm}$,

ii) For all t and $|t| \leq 1$,

$$\|D_t G(t)\|_{H_2} \leq c_\gamma e^{-\gamma t},$$

such that γ is an arbitrary number in $\sum_{\alpha_\pm}$.

Proof. See the proof in [58] and [61]. □

5.3.2 Representations for G(t)

Now, we observe the difference between $G(t)$ and $G^{(\beta)}(t)$.

According to Proposition 5.3.3(i), $G(t)$ does not depend on α, we can set $\Sigma_{\alpha_\pm} = \{\mu \in \sigma(\mathcal{B}_A) : \Im\mu \lesseqgtr \alpha\}$ and we consider $\Im\mu = \beta$.

A new Green's kernel is defined by

$$G^{(\beta)}(t) = \frac{1}{2\pi} \int_{\Im\mu=\beta} e^{it\mu} \mathcal{B}_A^{-1}(\mu) d\mu.$$

To understand this relation between $G(t)$ and $G^{(\beta)}(t)$, we have the following theorems.

Theorem 5.3.4. Let the operator is defined by

$$P_v(t) = \frac{1}{2\pi}\int_{S_v} e^{it(\mu-\mu_v)}\mathcal{B}_A^{-1}(\mu)d\mu,$$

where S_v is a small circle centred at the eigenvalue μ_v. Then, we have that

$$P_v(t) = i\sum_{k=1}^{J}\sum_{h=0}^{m_k-1}\frac{(it)^h}{h!}P_{k,h}, \tag{5.11}$$

where, $P_{k,h}$ is defined in (5.2), i.e.,

$$P_{k,h} = \sum_{s=0}^{h}\langle ., \psi_{k,s}\rangle_{H_0}\varphi_{k,h-s}, \tag{5.12}$$

and J be a geometric multiplicity of μ_0.

Proof. By using the definition of $\mathcal{B}_A^{-1}$ in (5.1), we have

$$\mathcal{B}_A^{-1}(\mu) = \sum_{k=1}^{J}\sum_{h=0}^{m_k-1}\frac{P_{k,h}}{(\mu-\mu_0)^{m_k-h}} + \Upsilon(\mu), \tag{5.13}$$

where,

$$P_{k,h} = \sum_{s=0}^{h}\langle ., \psi_{k,s}\rangle_{H_0}\varphi_{k,h-s}, \tag{5.14}$$

And, by using Cauchy's Residue theorem to solve the following integral, we get that

$$\begin{aligned}
\frac{1}{2\pi}\int_{S_v} e^{it(\mu-\mu_v)}\mathcal{B}_A^{-1}(\mu)d\mu &= 2\pi i\sum_{k=0}^{J} Res(e^{it(\mu-\mu_v)}\mathcal{B}_A^{-1}(\mu)) \\
&= i\sum_{k=0}^{J}\sum_{h=0}^{m_k-1}\frac{(it)^h}{h!}\lim_{\mu\to\mu_0}(\mu-\mu_0)^{h-m_k}e^{it(\mu-\mu_0)}\mathcal{B}_A^{-1}(\mu) \\
&= i\sum_{k=0}^{J}\sum_{h=0}^{m_k-1}\frac{(it)^h}{h!}\lim_{\mu\to\mu_0}(\mu-\mu_0)^{h-m_k}(e^{it(\mu-\mu_0)}\frac{P_{k,h}}{(\mu-\mu_0)^{m_k-h}} + \Upsilon(\mu_0)) \\
&= i\sum_{k=0}^{J}\sum_{h=0}^{m_k-1}\frac{(it)^h}{h!}P_{k,h}.
\end{aligned}$$

To get the operator,

$$P_v(t) = i\sum_{k=1}^{J}\sum_{h=0}^{m_k-1}\frac{(it)^h}{h!}P_{k,h}. \tag{5.15}$$

See for example, [58] and [61]. □

Remark 31. For the convenience of the readers, we give the example to compute the residue of the integral in the closed contour S_R in Appendix.6.

Theorem 5.3.5. Suppose there are no eigenvalues of the operator Pencil $\mathcal{B}_A$ on the lines $\Im\mu = \beta$, and $\Sigma_{\alpha_\pm} = \{\mu \in \sigma(\mathcal{B}_A) : \Im\mu \lesseqgtr \alpha\}$. Then

$$G(t) = \sum_{\mu \in \sum_{\alpha_+}} e^{i\mu t} P_v(t) + G^{(\beta)}(t), \tag{5.16}$$

$$G(t) = -\sum_{\mu \in \sum_{\alpha_-}} e^{i\mu t} P_v(t) + G^{(\beta)}(t). \tag{5.17}$$

Proof. Firstly, we need to prove (5.16), from Proposition 5.3.2 we have

$$\|D_\mu \mathcal{B}_A^{-1}(\mu)\|_{\mathbb{C}} \leq c|\mu|, \tag{5.18}$$

we can get the difference between $G(t)$ and $G^{(\beta)}(t)$, which equals to the sum of integrals in the operator

$$P_v(t) = \frac{1}{2\pi} \int_{S_v} e^{it(\mu-\mu_v)} \mathcal{B}_A^{-1}(\mu) d\mu,$$

which is multiplied by $e^{i\mu t}$, with the summation to get extended the eigenvalues $\mu \in \Sigma_{\alpha_\pm}$, we can get

$$G(t) - G^{(\beta)}(t) = \sum_{\mu \in \sum_{\alpha_+}} e^{i\mu t} P_v(t), \tag{5.19}$$

for $t > 0$.

Similarly, we can prove that the equation (5.17) for $t < 0$. ☐

Therefore, the formula (5.16) and (5.17) are the new representation of $G(t)$ as $t \to \pm\infty$. See for example, [58] and [61].

Theorem 5.3.6. For $k = 1, 2, ..., J$ and $s = 0, ..., m_k - 1$, and these conditions hold for all $\mu \in \Sigma_{\alpha_\pm}$ then we have

$$\sum_{h=0}^{m_k-1} \frac{(it - i\tau)^h}{h!} P_{k,m_k-1-s} = \sum_{h=0}^{m_k-1} \langle ., D_\tau \Psi(it)\psi_k \rangle_{H_0} D_t \Phi_k(it),$$

where Ψ_k and Φ_k are defined in Remark 29.

Proof. We have that in the second term of the right-hand side, to get

$$\sum_{h=0}^{m_k-1}\langle ., D_\tau\Psi(it)\psi_k\rangle_{H_0} D_t\Phi_k(it) = \sum_{h=0}^{m_k-1}\langle ., \sum_{j=0}^{m_k-1-s}\frac{(it)^j}{j!}\psi_{k,m_k-1-h-j}\rangle_{H_0}\sum_{s=0}^{h}\frac{(it)^s}{s!}\varphi_{k,h-s}$$
$$= \sum_{j=0}^{m_k-1}\sum_{s=0}^{m_k-1-j}\sum_{h=s}^{m_k-1-j}\frac{(-it)^j}{j!}\frac{(it)^s}{s!}\langle ., \psi_{k,m_k-1-h-j}\rangle_{H_0}\varphi_{k,h-s}$$
$$= \sum_{j=0}^{m_k-1}\sum_{s=0}^{m_k-1-j}\sum_{h=0}^{m_k-1-j-s}\frac{(-it)^j}{j!}\frac{(it)^s}{s!}\langle ., \psi_{k,m_k-1-h-j-s}\rangle_{H_0}\varphi_{k,h}.$$

We can take $n = s + j$ to obtain the last equality by,

$$\sum_{n=0}^{m_k-1}\sum_{s=0}^{n}\sum_{h=0}^{m_k-1-n-s}\frac{(-it)^{n-s}}{(n-s)!}\frac{(it)^s}{s!}\langle ., \psi_{k,m_k-1-n-h}\rangle_{H_0}\varphi_{k,h}$$
$$= \sum_{n=0}^{m_k-1}\frac{(it-i\tau)^n}{n!}\sum_{h=0}^{m_k-1-n}\langle ., \psi_{k,m_k-1-n-h}\rangle_{H_0}\varphi_{k,h}.$$

Since we have $P_{k,h}$ is defined in (5.2),

$$P_{k,h} = \sum_{s=0}^{h}\langle ., \psi_{k,s}\rangle_{H_0}\varphi_{k,h-s},$$

we have,

$$\sum_{h=0}^{m_k-1}\langle ., D_\tau\Psi(it)\psi_k\rangle_{H_0} D_t\Phi_k(it) = \sum_{n=0}^{m_k-1}\frac{(it-i\tau)^n}{n!}P_{k,m_k-1-s}.$$

Thus, the proof is complete. See for example, [58] and [61]. □

Theorem 5.3.7. For $k = 1, 2, ..., J$ and $s = 0, ..., m_k - 1$, and these conditions hold for all μ then the Green's kernel has new representation

$$G(t) - G^{(\alpha)}(t) = -i\sum_{\mu\in\Sigma_{\alpha_+}}\sum_{k=0}^{J}\sum_{h=0}^{m_k-1}e^{i\mu_0 t}\langle ., \psi_{k,s}\rangle_{H_0}\varphi_{k,h-s}.$$

Proof. By combining Theorems 5.3.5 and 5.3.6, we have directly

$$G(t) - G^{(\alpha)}(t) = -i\sum_{\mu\in\Sigma_{\alpha_+}}e^{i\mu_0 t}P_{k,h}$$
$$= -i\sum_{\mu\in\Sigma_{\alpha_+}}e^{i\mu_0 t}\sum_{k=0}^{J}\sum_{h=0}^{m_k-1}\frac{(it)^h}{h!}P_{k,h}$$
$$= -i\sum_{\mu\in\Sigma_{\alpha_+}}\sum_{k=0}^{J}\sum_{h=0}^{m_k-1}e^{i\mu_0 t}\langle ., \psi_{k,h-s}\rangle_{H_0}\varphi_{k,h-s},$$

where $\varphi_{k,s}$ is a canonical system of Jordan of $\mathcal{B}_A$ corresponding to μ_0, and $\psi_{k,s}$ is a canonical system of Jordan of $\mathcal{B}_A^*$ corresponding to $\overline{\mu_0}$ for $k = 1, 2, ..., J$ and $s = 0, ..., m_k - 1$, and these conditions hold for all $\mu \in \Sigma_{\alpha,\beta}$.

We can consider the function $G^{(\beta)}(t)$ by the following lemma:

Lemma 5.3.8. For $\alpha \notin \Gamma(\mathcal{B}_A)$, and we set $\Sigma_{\alpha_\pm} = \{\mu \in \sigma(\mathcal{B}_A) : \Im\mu \lesseqgtr \alpha\}$. Then,

$$G^{(\beta)}(t) = \begin{cases} i \sum_{\mu \in \Sigma_{\alpha_+}} Res(e^{it\mu}\mathcal{B}_A^{-1}(\mu); \mu) & \text{if} \quad t > 0 \\ \\ -i \sum_{\mu \in \Sigma_{\alpha_-}} Res(e^{it\mu}\mathcal{B}_A^{-1}(\mu); \mu) & \text{if} \quad t < 0. \end{cases}$$

The following Lemma, we can generalise the new representation of $G^{\beta}(t)$.

Lemma 5.3.9. Suppose $\alpha, \beta \in \mathbb{R} \setminus \Gamma(\mathcal{B}_A)$, and We have note that $\Sigma_{\beta_+} \subseteq \Sigma_{\alpha_+}$, $\Sigma_{\alpha_-} \subseteq \Sigma_{\beta_-}$, and $\Sigma_{\alpha,\beta} = \Sigma_{\alpha_+} \setminus \Sigma_{\beta_+} = \Sigma_{\beta_-} \setminus \Sigma_{\alpha_-}$. Then,

$$G^{(\beta)}(t) - G^{(\alpha)}(t) = \begin{cases} i \sum_{\mu \in \Sigma_{\beta_+}} \mathrm{Res}(e^{it\mu}\mathcal{B}_A^{-1}(\mu); \mu) - i \sum_{\mu \in \Sigma_{\alpha_+}} \mathrm{Res}(e^{it\mu}\mathcal{B}_A^{-1}(\mu); \mu) & t > 0 \\ \\ -i \sum_{\mu \in \Sigma_{\beta_-}} \mathrm{Res}(e^{it\mu}\mathcal{B}_A^{-1}(\mu); \mu) + i \sum_{\mu \in \Sigma_{\alpha_-}} \mathrm{Res}(e^{it\mu}\mathcal{B}_A^{-1}(\mu); \mu) & t < 0, \end{cases}$$

It follows,

$$G^{(\beta)}(t) - G^{(\alpha)}(t) = -i \sum_{\mu \in \Sigma_{\alpha,\beta}} \mathrm{Res}(e^{it\mu}\mathcal{B}_A^{-1}(\mu); \mu) \quad \text{for all} \quad t. \tag{5.20}$$

□

Lemma 5.3.10. If

$$\mathcal{B}_A^{-1}(\mu) = \langle ., \psi_{k,h-s} \rangle_{H_0} \varphi_{k,s} (\mu - \mu_0)^{h-m_k} + \Upsilon(\mu)$$

for μ is neighbourhood of μ_0. Then,

$$\mathcal{B}_A(\mu)\varphi_{k,s} = 0.$$

Proof. By Theorem 5.3.7 and by a Cauchy Integral Formula, we get that

$$P_{k,h} = \frac{1}{2\pi i} \oint_{S_v} \mathcal{B}_A^{-1}(\mu) d\mu,$$

where $\mathcal{B}_A^{-1}(\mu)$ is analytic function and S_v is a small circle.

Now, we get by the Contour Residue Theorem,

$$\begin{aligned}\mathcal{B}_A(\mu)P_{k,h} &= \mathcal{B}_A(\mu)\frac{1}{2\pi i}\oint_{S_v}\mathcal{B}_A^{-1}(\mu)d\mu \\ &= \frac{1}{2\pi i}\oint_{S_v}(I-(\mathcal{B}_A(\mu)-\mathcal{B}_A(\mu_0)))\mathcal{B}_A^{-1}(\mu)d\mu = 0,\end{aligned}$$

where $\mathcal{B}_A(\mu)-\mathcal{B}_A(\mu_0)$ is Holomorphic for μ near μ_0.

We can remove factor of $\mu-\mu_0$ from $\mathcal{B}_A(\mu)-\mathcal{B}_A(\mu_0)$. Then,

$$\mathcal{B}_A(\mu)\varphi_{k,h-s} = \frac{1}{\|\psi_{k,s}\|^2_{H_0}}\mathcal{B}_A(\mu)P_{k,h}\psi_{k,s} = 0,$$

where $P_{k,h} = \langle ., \psi_{k,s}\rangle\varphi_{k,h-s}$ since $\varphi_{k,h-s} \neq 0$ and $\psi_{k,s} \neq 0$ as otherwise $\mathcal{B}_A^{-1}(\mu)$ would have a removable singularity at μ_0. Hence, $\mu_0 \notin \sigma(\mathcal{B}_A)$. This is contradiction. □

Corollary 5.3.11. Similarly, the adjoint of $\mathcal{B}_A$ we have that

$$\mathcal{B}_A^*(\overline{\mu})\psi_{k,s} = 0,$$

such that $P_{k,h}^* = \langle ., \varphi_{k,h-s}\rangle_{H^*}\psi_{k,h-s}$ for $\psi_{k,s}$ is a canonical system of Jordan of $\mathcal{B}_A^*$ corresponding to $\overline{\mu_0}$ and $\varphi_{k,h-s}$ is a canonical system of Jordan of $\mathcal{B}_A$ corresponding to μ_0, to get that

$$\mathcal{B}_A^*(\overline{\mu})\psi_{k,s} = \frac{1}{\|\varphi_{k,s}\|^2_{H_0}}\mathcal{B}_A^*(\overline{\mu})P_{k,h}\varphi_{k,h-s} = 0$$

for all μ, $k = 1, ..., J$ and $s = 0, ..., m_k - 1$.

5.4 Exponential Solutions

We back to the previous arguments in Section 4.6 we have the fact that Theorem 4.6.2 does not extend to $\alpha \in \Gamma(\mathcal{B}_A)$ has to do with existence of exponential solutions of homogeneous equation

$$\mathcal{B}_A(D_t)u = 0, \tag{5.21}$$

for $u \in W^2_{\alpha,\alpha}$. See [10].

According Section 5.2, $\overline{\mu_0}$ is an eigenvalue of $\mathcal{B}_A^*$ and its geometric and algebraic multiplicities coincide with those of μ_0 is eigenvalue of $\mathcal{B}_A$. By Theorem 5.2.3 there exits $\{\psi_{k,s}\}_{s=0}^{m_k-1}$ is a canonical system of Jordan of $\mathcal{B}_A^*$ corresponding to $\overline{\mu_0}$ and $\{\varphi_{k,s}\}_{s=0}^{m_k-1}$ is a canonical system of Jordan of $\mathcal{B}_A$ corresponding to μ_0 and the canonical system $\{\psi_{k,s}\}_{s=0}^{m_k-1}$ satisfies the bi-orthogonality condition i.e.,

$$\sum_{s=0}^{d}\sum_{n=s+1}^{m_k+s}\frac{1}{n!}\mathcal{B}_A^{(n)}(\mu)\varphi_{k,m_k+s-n},\psi_{j,d-s})_{H_0}=\delta_k\delta_d,$$

for k and $d = 0, ..., m_k - 1$.

By Remark 29, we defined the solution of (5.21) if and only if μ_0 is an eigenvalue $\mathcal{B}_A$. Refer back to Section 5.2, [58] and [59].

However, in Section 5.3, Theorem 5.3.7 achieves to find the solution for the difference two solutions of non-homogeneous equation

$$\mathcal{B}_A(D_t)u = f. \tag{5.22}$$

We have $\alpha \leq \beta$ and $\Sigma_{\alpha,\beta}$ denote the linear span of the set of all exponential solutions corresponding to $\mu_0 \in \sigma(\mathcal{B}_A)$. Then, we have the following propositions:

Proposition 5.4.1. Let $\alpha \leq \beta \in \mathbb{R} \setminus \Gamma(\mathcal{B}_A)$ and suppose $f \in W^0_{\alpha,\alpha} \cap W^0_{\beta,\beta}$. Choose the unique $u_\alpha \in W^2_{\alpha,\alpha}$ and $u_\beta \in W^2_{\beta,\beta}$ such that

$$\mathcal{B}_A(D_t)u_\alpha = f \quad \text{and} \quad \mathcal{B}_A(D_t)u_\beta = f.$$

Then, the difference $u_\alpha - u_\beta$ lies in $\Sigma_{\alpha,\beta}$ (see, for example, [10] and [58]).

Proposition 5.4.2. For $\alpha, \beta \in \mathbb{R} \setminus \Gamma$, and we have the maps

$$A^{(\alpha)} = D_t^2 + A - \lambda : W^2_{\alpha,\alpha} \longrightarrow W^0_{\alpha,\alpha}, \tag{5.23}$$

$$A^{(\beta)} = D_t^2 + A - \lambda : W^2_{\beta,\beta} \longrightarrow W^0_{\beta,\beta}, \tag{5.24}$$

are isomorphisms.

Let $f \in W^0_{\alpha,\alpha} \cap W^0_{\beta,\beta}$, and $u_\alpha \in W^2_{\alpha,\alpha}$, $u_\beta \in W^2_{\beta,\beta}$ be the solutions of

$$A^{(\alpha)}u_\alpha = f \quad \text{and} \quad A^{(\beta)}u_\beta = f,$$

respectively. Then,

$$u_\alpha(t) - u_\beta(t) = \sum_{\mu \in \Sigma\alpha,\beta} \int_{\mathbb{R}} e^{i\mu_0(t-s)} P_{k,h} f(s) ds. \tag{5.25}$$

Proof. By Lemma 5.3.1, and Theorem 5.3.7 we can observe directly,

$$\begin{aligned} u_\alpha(t) - u_\beta(t) &= \int_{\mathbb{R}} G^{(\alpha)}(t-s) f(s) ds - \int_{\mathbb{R}} G^{(\beta)}(t-s) f(s) ds \\ &= \int_{\mathbb{R}} (G^{(\alpha)} - G^{(\beta)})(t-s) f(s) ds \\ &= \sum_{\mu \in \Sigma\alpha,\beta} \sum_{h=0}^{m_k-1} \int_{\mathbb{R}} e^{i\mu_0(t-s)} P_{k,h} f(s) ds. \end{aligned}$$

Where $P_{k,h} = \varphi_{k,h-s} \langle \psi_{k,s}, . \rangle_{H_0}$ for $\varphi_{k,h-s}$ is a canonical system of Jordan of $\mathcal{B}_A$ corresponding to μ_0, and $\psi_{k,h-s}$ is a canonical system of Jordan of $\mathcal{B}_A^*$ corresponding to $\overline{\mu_0}$ and $\Sigma_{\alpha,\beta}$ denote the linear span of the set of all exponential solutions corresponding to $\mu_0 \in \sigma(\mathcal{B}_A)$. □

Remark 32. For $\{\varphi_{k,s}\}_{s=0}^{m_k-1}$ is a canonical system of Jordan of $\mathcal{B}_A$ corresponding to μ_0, and $\{\psi_{k,h-s}\}_{s=0}^{m_k-1}$ is a canonical system of Jordan of $\mathcal{B}_A^*$ corresponding to $\overline{\mu_0}$ for $k = 1, ..., J$ and $s = 0, ..., m_k - 1$, and these conditions hold for all $\mu \in \Sigma_{\alpha,\beta}$, by Remark 29 and by Theorem 5.3.7, we can set

$$u_\mu(t) = -ie^{i\mu_0 t} \varphi_{k,s},$$

and

$$v_\mu(t) = e^{i\overline{\mu_0} t} \psi_{k,h-s},$$

for $k = 1, ..., J$ and $s = 0, ..., m_k - 1$.

Thus, u_μ and v_μ are called exponential solutions of $\mathcal{B}_A(D_t)u_\mu = 0$, and $\mathcal{B}_A^*(D_t)v_\mu = 0$, respectively, (see [58], pp. $10 - 11$).

Proposition 5.4.3. We have $u_\mu(t) = -ie^{i\mu_0 t}\varphi_{k,h-s}$, and $v_\mu(t) = e^{i\overline{\mu_0} t}\psi_{k,s}$, and by using Proposition 5.4.2, we can get

$$(A^{(\beta)})^{-1}f - (A^{(\alpha)})^{-1}f = \int_{\mathbb{R}} (G^{(\beta)} - G^{(\alpha)})(t-s)f(s)ds$$
$$= \sum_{\mu\in\Sigma\alpha,\beta} \sum_{h=0}^{m_k-1} \int_{\mathbb{R}} (-ie^{i\mu_0 t}\varphi_{k,h-s})\langle e^{i\overline{\mu_0}s}\psi_{k,s}, f(s)\rangle_{H_0} ds$$
$$= \sum_{\mu\in\Sigma\alpha,\beta} u_\mu \langle v_\mu, f\rangle_{H_0}.$$

5.5 Main Results

In this section, we aim to provide a key step for both Fredholm properties and further results for $\mathcal{B}_A$. We start by the result which is established the semi-Fredholm property in Theorem 5.5.1. Later, we observe consequences that are corresponding the change of the index formula of Fredholm operator.

The establishing the following property of (semi-Fredholm) is proved by using Corollary 4.6.10 and Theorem 4.6.11.

Theorem 5.5.1. Published in [42], April 30, 2021.

Let $\alpha, \beta \in \mathbb{R} \setminus \Gamma(\mathcal{B}_A)$. Then, $\mathcal{B}_A(D_t) : W^2_{\alpha,\beta} \to W^0_{\alpha,\beta}$ is semi-Fredholm with a finite-dimensional kernel.

Proof. For the proof of this result, suppose we have a sequence satisfying the following terms:

- $\{u_i\}_{i\in\mathbb{N}} \subseteq W^2_{\alpha,\beta}$,
- $\|u_i\|_{W^2_{\alpha,\beta}} \leq 1$,
- $\mathcal{B}_A(D_t)u_i \to 0$ in $W^0_{\alpha,\beta}$.

First, with the constant c and by Theorem 4.6.11, we have

$$\|u_{i_n} - u_{i_m}\|_{W^2_{\alpha,\beta}} \leq c\left(\|\mathcal{B}_A(D_t)u_{i_n} - \mathcal{B}_A(D_t)u_{i_m}\|_{W^0_{\alpha,\beta}} + \|u_{i_n} - u_{i_m}\|_{W^1_{\alpha',\beta'}}\right), \tag{5.26}$$

for all $n, m \in \mathbb{N}$. By the first term of the right-hand side of (5.26), we can observe that

$$\|\mathcal{B}_A(D_t)u_{i_n} - \mathcal{B}_A(D_t)u_{i_m}\|_{W^0_{\alpha,\beta}} \to 0$$

as $n, m \to \infty$, as $\{\mathcal{B}_A(D_t)u_i\}_{i\in\mathbb{N}} \to 0$ in $W^0_{\alpha,\beta}$ (by assumption).

On the other hand, we choose $\alpha^{'} > \alpha$ and $\beta^{'} < \beta$, the inclusion

$$W^2_{\alpha,\beta} \hookrightarrow W^1_{\alpha^{'},\beta^{'}},$$

is a compact map by Corollary 4.6.10, hence we can find a subsequence $\{u_{i_n}\}_{n\in\mathbb{N}}$ which is convergent in $W^1_{\alpha^{'},\beta^{'}}$. Furthermore, it is true for the second term of the right-hand side of (5.26). That is,

$$\|u_{i_n} - u_{i_m}\|_{W^1_{\alpha^{'},\beta^{'}}} \to 0,$$

as $\{u_{i_n}\}_{n\in\mathbb{N}}$ is convergent in $W^1_{\alpha^{'},\beta^{'}}$.

Thus, $\{u_{i_n}\}$ for $n \in \mathbb{N}$ is a Cauchy sequence since $\{u_i\}_{i\in\mathbb{N}}$ is a bounded sequence in $W^2_{\alpha,\beta}$ (by assumption) and it has a convergent subsequence $\{u_{i_k}\}$ for $k \in \mathbb{N}$ (by Bolzano-Weierstrass) (see [28]) and we check that

$$\|u_{i_n} - u_i\|_{W^2_{\alpha,\beta}} \leq \|u_{i_n} - u_{i_k}\|_{W^2_{\alpha,\beta}} + \|u_i - u_{i_k}\|_{W^2_{\alpha,\beta}} \to 0.$$

i.e., $\{u_{i_n}\}_{n\in\mathbb{N}}$ is convergent in $W^2_{\alpha,\beta}$ as $n \to \infty$.

Summarising, we have shown any sequence that the sequence satisfying has a subsequence is convergent in $W^2_{\alpha,\beta}$. A standard argument (Lemma 5.1.1 of this thesis) can now use to show

$$\mathcal{B}_A(D_t) : W^2_{\alpha,\beta} \to W^0_{\alpha,\beta}$$

has a finite-dimensional kernel and a closed range.

□

Theorem 5.5.2. Let $\beta \in \mathbb{R}$. Then, the map $A^{(\beta)} = \mathcal{B}_A(D_t) : W^0_{\beta,\beta} \to W^0_{\beta,\beta}$ has a finite-dimensional kernel.

Proof. Choose $\alpha \in \mathbb{R} \setminus \Gamma(\mathcal{B}_A)$ with $\alpha \leq \beta$ and $A^{(\alpha)} = \mathcal{B}_A(D_t) : W^0_{\alpha,\alpha} \to W^0_{\alpha,\alpha}$ we have a continuous inclusion $i : W^0_{\alpha,\alpha} \hookrightarrow W^0_{\beta,\beta}$ from Proposition 4.6.3. So,

$$\operatorname{Ker} A^{(\beta)} \subseteq \operatorname{Ker} A^{(\alpha)}.$$

On the other hand, $\alpha \in \mathbb{R} \setminus \Gamma(\mathcal{B}_A)$ so, $\operatorname{Ker} A^{(\alpha)}$ must be finite-dimensional by Theorem 5.5.1. □

We complete this section with further consequences of Fredholm properties of Pencils $\mathcal{B}_A$ with some restriction on α, β.

Proposition 5.5.3. For $\alpha \leq \beta$, we can consider the maps

$$A^{(\alpha,\beta)} = \mathcal{B}_A(D_t) : W^2_{\alpha,\beta} \to W^0_{\alpha,\beta},$$

and

$$A^{(\beta,\alpha)} = \mathcal{B}_A(D_t) : W^2_{\beta,\alpha} \to W^0_{\beta,\alpha},$$

and we have that solutions $\{u_\mu : \mu \in \Sigma_{\alpha,\beta}\}$ and $\{v_\mu : \mu \in \Sigma_{\alpha,\beta}\}$ of the equations $\mathcal{B}_A(D_t)u_\mu = 0$ and $\mathcal{B}_A(D_t)v_\mu = 0$, respectively, and are linearly independent sets.

To observe the following claims:

- Claim (i):

$$\text{Ker } A^{(\alpha,\beta)} = \{u \in W^2_{\alpha,\beta} : A^{(\alpha,\beta)}u = 0\} = \{0\}.$$

Proof. Let $u \in \text{Ker } A^{(\alpha,\beta)} \subseteq W^2_{\alpha,\beta}$.

Since by Theorem 4.6.2

$$A^{(\alpha)} = \mathcal{B}_A(D_t) : W^2_{\alpha,\alpha} \to W^0_{\alpha,\alpha}$$

and

$$A^{(\beta)} = \mathcal{B}_A(D_t) : W^2_{\beta,\beta} \to W^0_{\beta,\beta},$$

are isomorphism.

That is,

$$\text{Ker } A^{(\alpha)} = \{0\} \quad \text{and} \quad \text{Ker } A^{(\beta)} = \{0\}.$$

By Corollary 4.6.6, we have

$$W^2_{\alpha,\beta} = W^2_{\alpha,\alpha} \cap W^2_{\beta,\beta}.$$

We have

$$u \in \text{Ker } A^{(\alpha)} \subset W^2_{\alpha,\alpha} \quad \text{and} \quad u \in \text{Ker } A^{(\beta)} \subset W^2_{\beta,\beta}.$$

Then, $u \in \text{Ker } A^{(\alpha,\beta)} \subseteq W^2_{\alpha,\beta}$.

That is,

$$\operatorname{Ker} A^{(\alpha,\beta)} = \{u \in W^2_{\alpha,\beta} : A^{(\alpha,\beta)}u = 0\} = \{0\}.$$

□

- Claim (ii):

$$\operatorname{Ran} A^{(\alpha,\beta)} = \{f \in W^0_{\alpha,\beta} : \langle v_\mu, f\rangle = 0 \quad \text{for all} \quad \mu \in \Sigma_{\alpha,\beta}\}.$$

Proof. Let $f \in W^0_{\alpha,\beta} = W^0_{\alpha,\alpha} \cap W^0_{\beta,\beta}$, by Corollary 4.6.6.

Then, we set

$$u_\alpha = (A^{(\alpha)})^{-1} f \in W^2_{\alpha,\alpha} \quad \text{and} \quad u_\beta = (A^{(\beta)})^{-1} f \in W^2_{\beta,\beta}.$$

If $\langle v_\mu, f\rangle = 0$ for all $\mu \in \Sigma_{\alpha,\beta}$,

then $u_\alpha = u_\beta$ by Proposition 5.4.3.

So, we have that

$$u_\alpha = u_\beta \in W^2_{\alpha,\alpha} \cap W^2_{\beta,\beta} = W^2_{\alpha,\beta},$$

and

$$f = A^{(\alpha,\beta)} u_\alpha \in \operatorname{Ran} A^{(\alpha,\beta)} \subset W^0_{\alpha,\beta}.$$

Furthermore, if $f = A^{(\alpha,\beta)} u_\mu \in W^0_{\alpha,\beta}$ for some $u_\mu \in W^2_{\alpha,\beta}$, then uniqueness of isomorphism inverse gives

$$u_\alpha = (A^{(\alpha)})^{-1} f = u_\mu = (A^{(\beta)})^{-1} f = u_\beta.$$

Hence, $\langle v_\mu, f\rangle = 0$ for all $\mu \in \Sigma_{\alpha,\beta}$ (u_μ is linearly independent set).

Then,

$$\operatorname{Ran} A^{(\alpha,\beta)} = \{f \in W^0_{\alpha,\beta} : \langle v_\mu, f\rangle = 0 \quad \text{for all} \quad \mu \in \Sigma_{\alpha,\beta}\}.$$

□

- Claim (iii):

$$\operatorname{Ker} A^{(\beta,\alpha)} = \operatorname{Span}\{u_\mu : \mu \in \Sigma_{\alpha,\beta}\}.$$

Proof. Suppose $A^{(\beta,\alpha)}u_\mu = 0$ for some $u_\mu \in W^2_{\beta,\alpha} = W^2_{\beta,\beta} + W^2_{\alpha,\alpha}$ by Corollary 4.6.9, we can write $u_\mu = u_\alpha + u_\beta$ for some $u_\alpha \in W^2_{\alpha,\alpha}$ and $u_\beta \in W^2_{\beta,\beta}$.

Then,

$$A^{(\alpha)}u_\alpha + A^{(\beta)}u_\beta = A^{(\beta,\alpha)}u_\mu = 0.$$

So,

$$f \in A^{(\beta)}u_\beta = -A^{(\alpha)}u_\alpha \in W^0_{\alpha,\alpha} \cap W^0_{\beta,\beta}.$$

Now, we have by Proposition 5.4.3,

$$u_\mu = u_\beta + u_\alpha = (A^{(\beta)})^{-1}f - (A^{(\alpha)})^{-1}f \in \mathrm{Span}\{u_\mu : \mu \in \Sigma_{\alpha,\beta}\}.$$

Therefore,

$$\mathrm{Ker}\, A^{(\beta,\alpha)} = \mathrm{Span}\{u_\mu : \mu \in \Sigma_{\alpha,\beta}\}.$$

□

- Claim (iv): We have

$$\mathrm{Ran}\, A^{(\beta,\alpha)} = W^0_{\beta,\alpha}.$$

Proof. Let $f \in W^0_{\beta,\alpha} = W^0_{\alpha,\alpha} + W^0_{\beta,\beta}$. So, $f = f_\alpha + f_\beta$ for some $f_\alpha \in W^0_{\alpha,\alpha}$ and $f_\beta \in W^0_{\beta,\beta}$, and $A^{(\alpha)}, A^{(\beta)}$ are isomorphism for $u_\mu \in W^2_{\beta,\alpha}$.

Then,

$$u_\mu = (A^{(\alpha)})^{-1}f_\alpha + (A^{(\beta)})^{-1}f_\beta \in W^2_{\beta,\alpha}.$$

and

$$A^{(\beta,\alpha)}u_\mu = A^{(\alpha)}(A^{(\alpha)})^{-1}f_\alpha + A^{(\beta)}(A^{(\beta)})^{-1}f_\beta = f_\alpha + f_\beta = f.$$

Therefore,

$$\mathrm{Ran}\, A^{(\beta,\alpha)} = W^0_{\beta,\alpha}.$$

□

Corollary 5.5.4. Let $\alpha, \beta \in \mathbb{R} \setminus \Gamma(\mathcal{B}_A)$, Suppose

$$A^{(\alpha)} : W^2_{\alpha,\alpha} \to W^0_{\alpha,\alpha}$$

and

$$A^{(\beta)} : W^2_{\beta,\beta} \to W^0_{\beta,\beta}$$

are isomorphism maps. Then, $A^{(\alpha)}$ and $A^{(\beta)}$ are Fredholm maps with index 0.

We finish this section, by the last result in the current thesis which shows how the index of the Fredholm maps $A^{(\alpha,\beta)}$ and $A^{(\beta,\alpha)}$ varies when we change α and β.

Theorem 5.5.5. Published in [42], April 30, 2021.

Suppose $\alpha < \beta \in \mathbb{R}\backslash\Gamma$. Then the maps

$$A^{(\alpha,\beta)} : W^2_{\alpha,\beta} \longrightarrow W^0_{\alpha,\beta}$$

and

$$A^{(\beta,\alpha)} : W^2_{\beta,\alpha} \longrightarrow W^0_{\beta,\alpha}$$

are Fredholm maps with

$$\operatorname{Index} A^{(\alpha,\beta)} : W^2_{\alpha,\beta} \longrightarrow W^0_{\alpha,\beta} = -|\Sigma_{\alpha,\beta}| = -\operatorname{Index} A^{(\beta,\alpha)} : W^2_{\beta,\alpha} \longrightarrow W^0_{\beta,\alpha}.$$

Proof. We have u_μ and v_μ consist of exponential functions with different exponents, then $\{u_\mu : \mu \in \Sigma_{\alpha,\beta}\}$ and $\{v_\mu : \mu \in \Sigma_{\alpha,\beta}\}$ are linearly independent set.

First we need to prove $A^{(\alpha,\beta)}$ is Fredholm:

By Claim (ii) $\operatorname{Ran} A^{(\alpha,\beta)} = \{v_\mu : \mu \in \Sigma_{\alpha,\beta}\}^\perp$, so

$\operatorname{Ran} A^{(\alpha,\beta)}$ is closed with the $\eta(A^{(\alpha,\beta)}) = |\Sigma_{\alpha,\beta}|$, and by claim (i) $\operatorname{Ker} A^{(\alpha,\beta)} = \{0\}$, it follows that $\kappa(A^{(\alpha,\beta)}) = 0$.

Then, we have that

$$A^{(\alpha,\beta)} : W^2_{\alpha,\beta} \longrightarrow W^0_{\alpha,\beta}$$

is a Fredholm map (by definition of the Fredholm in Section 5.1).

The index of this operator,

$$\operatorname{Index} A^{(\alpha,\beta)} = \kappa(A^{(\alpha,\beta)}) - \eta(A^{(\alpha,\beta)}) = 0 - |\Sigma_{\alpha,\beta}|.$$

Now, to prove that $A^{(\beta,\alpha)}$ is a Fredholm map, we observe that, by claim (iv) $\operatorname{Ran} A^{(\beta,\alpha)} = W^0_{\beta,\alpha}$ so, it is closed with $\eta(A^{(\beta,\alpha)}) = 0$.

Also, by Claim (iii), we observe $\kappa(A^{(\beta,\alpha)}) = |\Sigma_{\alpha,\beta}|$, that is, the dimension of kernel is finite.

Therefore,

$$A^{(\beta,\alpha)} : W^2_{\beta,\alpha} \longrightarrow W^0_{\beta,\alpha}$$

is a Fredholm map (by definition of the Fredholm in Section 5.1).

Therefore, the index of $A^{(\beta,\alpha)}$ is

$$\operatorname{Index} A^{(\beta,\alpha)} = \kappa(A^{(\beta,\alpha)}) - \eta(A^{(\beta,\alpha)}) = |\Sigma_{\alpha,\beta}| - 0.$$

□

Conclusion

In general, we observed the definition of inverse Fredholm operator Pencil. The properties of the Green's kernels were proved by Maz'ya and Kozlov [58]. Also, we had representations of Green's kernels of different types (5.16) and (5.17), these results used for the solutions of $\mathcal{B}_A(D_t)u = f$ in Sobolev spaces $W^0_{\alpha,\beta}$. At the end of this chapter, we considered the semi-Fredholm property, we observed the parameters α and β are varied so that move between components of $\mathbb{R}\backslash\Gamma$ then the index of the corresponding maps would change.

Chapter 6

Appendix

6.1 Appendix.1 (Explicit function)

Lemma 6.1.1. For any k we have

$$\lim_{u\to\infty} u^k \exp(-u) = 0.$$

Proof. For any fixed positive integer $N > 0$ and $u > 0$, we have

$$\exp(u) = \sum_{n=0}^{\infty} \frac{u^n}{n!} = 1 + u + \frac{u^2}{2} + \frac{u^3}{6} + ... \geq \frac{u^N}{N!}$$

Thus we can write

$$0 < u^k \exp(-u) < \frac{u^k}{\frac{u^N}{N!}} = N! u^{k-N}.$$

For $N > k$, we have $u^{k-N} \to 0$ as $u \to \infty$, therefore

$$\lim_{u\to\infty} u^k \exp(-u) = 0.$$

□

$Q1$: Find an explicit function $\varphi : \mathbb{R} \to \mathbb{R}$ such that $\varphi(x) = 0$ if $|x| > 1$, $\|\varphi\| = 1$, and $\|\varphi^{'}\| < \infty$, $\|\varphi^{''}\| < \infty$.

Solution: For $\varphi \in L^2(\mathbb{R})$ we can consider the function given by

$$\varphi(x) = \begin{cases} ce^{\left(\frac{-1}{1-x^2}\right)} & |x| < 1 \\ \\ 0 & \text{otherwise} \end{cases}$$

where c is a constant.

- We can see that from the definition $\varphi(x) = 0$ if $|x| > 1$.

- For the second condition, we can choose the constant c to the normalize $\varphi(x)$. Firstly, for $-1 < x < 1$ we have $-\infty < \frac{-2}{1-x^2} \leq 0$, therefore $0 < e^{\frac{-2}{1-x^2}} \leq 1$, therefore $0 < \int_{-1}^{1} e^{\frac{-2}{1-x^2}} dx \leq 2 < \infty$. We can now define

$$c = \sqrt{\frac{1}{\int_{-1}^{1} e^{\left(\frac{-2}{1-x^2}\right)} dx}}.$$

  Then

$$\begin{aligned} \|\varphi\|^2_{L^2(\mathbb{R})} &= \int_{-1}^{1} \left| ce^{\left(\frac{-1}{1-x^2}\right)} \right|^2 dx \\ &= |c|^2 \int_{-1}^{1} \left| e^{\left(\frac{-1}{1-x^2}\right)} \right|^2 dx \\ &= |c|^2 \int_{-1}^{1} e^{\left(\frac{-2}{1-x^2}\right)} dx = 1. \end{aligned}$$

- Now, we will prove the function is continuous and its derivatives are bounded; this allows us to show the norms of these derivatives are finite. We have

$$\varphi(x) = c \begin{cases} e^{\left(\frac{-1}{1-x^2}\right)} & -1 < x < 1 \\ \\ 0 & \text{otherwise} \end{cases}$$

  Since $x \to 1^-$ implies that $1 - x^2 \to 0^+$ so $u = \dfrac{1}{1-x^2} \to +\infty$ it follows that

$$\lim_{x \to 1^-} e^{\left(\frac{-1}{1-x^2}\right)} = \lim_{u \to \infty} e^{-u} = 0.$$

  Therefore the function φ is continuous at $x = -1$. Similarly, the function is continuous at $x = 1$, so φ is continuous at every point on the interval $[-1, 1]$.

Secondly, we will prove the first derivative of the function $\varphi(x)$ is continuous. We have

$$\varphi^{'}(x) = c \begin{cases} \dfrac{2x}{(1-x^2)^2} e^{\left(\frac{-1}{1-x^2}\right)} & -1 < x < 1 \\ \\ 0 & \text{otherwise} \end{cases}$$

Since $x \to 1^-$ implies $1 - x^2 \to 0^+$ so $u = \dfrac{1}{1-x^2} \to +\infty$ it follows from Lemma 6.1.1 that

$$\lim_{x \to 1^-} \frac{2x}{(1-x^2)^2} e^{\left(\frac{-1}{1-x^2}\right)} = \lim_{x \to 1^-} 2x \cdot \lim_{u \to \infty} u^2 e^{-u} = 2 \cdot 0 = 0.$$

Therefore the first derivative of the function φ is continuous at $x = -1$, and similarity at $x = 1$. Thus $\varphi^{'}$ is continuous at every point on the interval $[-1, 1]$, and hence the function is bounded. So, there exists $0 < M < \infty$ such that $|\varphi^{'}(x)| \leq M$. Hence

$$\|\varphi^{'}\| \leq \left(\int_{-1}^{1} M^2 \right)^{\frac{1}{2}} dx \leq \sqrt{2} M < \infty,$$

so, $\|\varphi^{'}\|^2_{L(\mathbb{R})}$ is finite.

Similarly, to show that the second derivative of the function $\varphi(x)$ is bounded it suffices to prove it is continuous. Now,

$$\varphi^{''}(x) = c \begin{cases} \dfrac{2(5x^4 - 4x^2 + 1)}{(1-x^2)^4} e^{\left(\frac{-1}{1-x^2}\right)} & -1 < x < 1 \\ \\ 0 & \text{otherwise} \end{cases}$$

Since $x \to 1^-$ implies $1 - x^2 \to 0^+$, so $u = \dfrac{1}{1-x^2} \to +\infty$, Lemma 1 gives

$$\lim_{x \to 1^-} \frac{2(5x^4 - 4x^2 + 1)}{(1-x^2)^4} e^{\left(\frac{-1}{1-x^2}\right)} = \lim_{x \to 1^-} (5x^4 - 4x^2 + 1) \cdot \lim_{u \to \infty} u^4 e^{-u} = 2 \cdot 0 = 0$$

It follows that the second derivative of the function φ is continuous at $x = -1$, and similarity at $x = 1$, so $\varphi^{''}$ is continuous at every point on the interval $[-1, 1]$ and hence the function is bounded. So, there exists $0 < M < \infty$ such that that $|\varphi^{''}(x)| \leq M$, which implies

$$\|\varphi^{''}\| \leq \left(\int_{-1}^{1} M^2 \right)^{\frac{1}{2}} dx \leq \sqrt{2} M < \infty,$$

thus, $\|\varphi^{''}\|_{L^2(\mathbb{R})}$ is finite.

6.2 Appendix.2 (Disc)

We compute the eigenvalues and eigenfunctions on a disc and use the polar coordinates r and θ. That is, $r = \sqrt{t^2 + s^2}$ and $\theta = \arctan \frac{s}{t}$.

We can consider the Disc

$$\Omega = \{0 \leq r < a, 0 \leq \theta \leq 2\pi\}.$$

Now, the partial derivative u with respect t and s, we have

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial r}\frac{t}{r} - \frac{\partial u}{\partial \theta}\frac{s}{r^2};$$

and

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial r}\frac{s}{r} + \frac{\partial u}{\partial \theta}\frac{t}{r^2}.$$

Therefore,

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial r^2}\frac{t^2}{r^2} + \frac{\partial u}{\partial r}\frac{s^2}{r^3} - 2\frac{\partial^2 u}{\partial r \partial \theta}\frac{ts}{r^3} + 2\frac{\partial u}{\partial \theta}\frac{ts}{r^4} + \frac{\partial^2 u}{\partial \theta^2}\frac{s^2}{r^4};$$

and,

$$\frac{\partial^2 u}{\partial s^2} = \frac{\partial^2 u}{\partial r^2}\frac{s^2}{r^2} + \frac{\partial u}{\partial r}\frac{t^2}{r^3} + 2\frac{\partial^2 u}{\partial r \partial \theta}\frac{ts}{r^3} - 2\frac{\partial u}{\partial \theta}\frac{ts}{r^4} + \frac{\partial^2 u}{\partial \theta^2}\frac{t^2}{r^4}.$$

It follows that the Laplacian applied to u has the form

$$-\Delta u = -\frac{\partial^2 u}{\partial s^2} - \frac{\partial^2 u}{\partial t^2} = -\frac{\partial^2 u}{\partial r^2} - \frac{1}{r}\frac{\partial u}{\partial r} - \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}.$$

Now, we find the solution of the eigenvalue equation

$$-\Delta u = \lambda u.$$

We can rewrite this equation as follows

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = -\lambda u, \tag{6.1}$$

with $u(r, \theta) = 0$ and $0 \leq \theta \leq 2\pi$.

By using a separation of variable, let

$$u(r,\theta) = R(r) \cdot \Phi(\theta).$$

Therefore, equation (6.1) becomes

$$\Phi(R_{rr} + \frac{1}{r}R_r) + \frac{1}{r^2}R\Phi_{\theta\theta} = -\lambda R\Phi.$$

Rearranging to separate the variables, we have

$$\frac{r^2}{R}(R_{rr} + \frac{1}{r}R_r + \lambda R) = -\frac{\Phi_{\theta\theta}}{\Phi}.$$

Because the right-hand side depends only on θ and the left-hand side depends only on r, both sides are equal to some constant. Therefore,

$$R_{rr} + \frac{1}{r}R_r + (\lambda - \frac{\mu}{r^2})R = 0. \tag{6.2}$$

Here, $R(a) = 0$ and $0 \leq r < a$.

And the another equation is

$$\Phi_{\theta\theta} + \mu\Phi = 0, \tag{6.3}$$

with $\Phi(\theta) = \Phi(2\pi)$ and $0 \leq \theta \leq 2\pi$.

The solution of equation (6.3) is given by

$$\Phi(\theta) = A\sin\sqrt{\mu}\theta + B\cos\sqrt{\mu}\theta.$$

Therefore, the boundary condition gives us $\sqrt{\mu} = n$ for $n \in \mathbb{N}_0$. Thus,

$$\Phi_n(\theta) = A_n \sin n\theta + B_n \cos n\theta$$

for arbitrary constants A_n and B_n.

Now, we can refer back to equation (6.2)

$$R_{rr} + \frac{1}{r}R_r + (\lambda - \frac{\mu}{r^2})R = 0,$$

with $R(a) = 0$ and $0 \leq r < a$. We set the variable $t = \sqrt{\lambda}r$ to get

$$\frac{dR}{dr} = \frac{dR}{dt} \cdot \frac{dt}{dr} = \sqrt{\lambda}\frac{dR}{dt}$$

and

$$\frac{d^2R}{dr^2} = \lambda \frac{d^2R}{dt^2}.$$

Then, we substitute in equation (6.2) to get that

$$t^2R_{tt} + tR_t + (t^2 - n^2)R = 0,$$

such that $R(\sqrt{\lambda}a) = 0$, and $0 \leq t < \sqrt{\lambda}a$. This is the Bessel differential equation, which has the solution $J_n(t)$, where $J_n(t)$ is the Bessel function of the first kind of order n, is defined by

$$J_n(t) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(n+k)!}(\frac{t}{2})^{n+2k}.$$

See [26]. Then, the solution of the equation where $t = \sqrt{\lambda}r$ is given by $R(r) = J_n(\sqrt{\lambda}r)$. Therefore,

$$u_n(r,\theta) = J_n(\sqrt{\lambda}r).[A_n \sin n\theta + B_n \cos n\theta].$$

In the case of Dirichlet boundary conditions

Consider $u(a,\theta) = 0$ implies that $J_n(\sqrt{\lambda}r) = 0$. We deduce that $\sqrt{\lambda}a$ is the zero of the Bessel function. However, $J_n(t)$ has an infinity sequence of positive zeros for $n = 0, 1, 2, ...$ and $m = 1, 2, 3, ...$, so we order them

$$0 < \alpha_{n,1} < \alpha_{n,2} < ... < \alpha_{n,m} < \alpha_{n,m+1} < ...\cdot$$

Thus, $\sqrt{\lambda_{n,m}}a = \alpha_{n,m}$ and the eigenvalues of the eigenvalue problem are given by

$$\lambda_{n,m} = (\frac{\alpha_{n,m}}{a})^2,$$

for $n = 0, 1, 2, 3, ...$ and $m = 1, 2, 3, ...$, (see [2]).

Remark 33. We have the following notes:

(1) If $n = 0$, then λ has multiplicity 1. This eigenvalue has a corresponding eigenfunction, which is simply the multiples of $J_0(\alpha_{0,m}r)$.

(2) If $n \neq 0$, then λ has multiplicity 2 and the eigenfunctions of the form:

$$u_{n,m}(r,\theta) = J_n((\frac{\alpha_{n,m}}{a})^2 r) \cdot (A_{n,m} \sin n\theta + B_{n,m} \cos n\theta), \quad n,m \in \mathbb{N},$$

with $A_{n,m}$ and $B_{n,m}$ as the arbitrary constants.

In the case of Neumann boundary conditions

$$u_{n,m}(r,\theta) = J_n(\sqrt{\lambda}r) \cdot (A_{n,m} \sin n\theta + B_{n,m} \cos n\theta), \quad n,m \in \mathbb{N}.$$

$$\frac{\partial u_{n,m}}{\partial r}(r,\theta) = 0 \quad \text{on the boundary condition.}$$

This implies

$$\sqrt{\lambda} J_n^{'}(\sqrt{\lambda}a) \cdot (A_{n,m} \sin n\theta + B_{n,m} \cos n\theta) = 0,$$

since the eigenvalue is not zero, so

$$J_n^{'}(\sqrt{\lambda}a) = 0.$$

The derivative of the Bessel function J_n has infinitely many positive zeros:

$$0 < \alpha_{n,1}^{'} < \alpha_{n,2}^{'} < ... < \alpha_{n,m}^{'} < \alpha_{n,m+1}^{'} < ...$$

for $n > 0$ and

$$0 = \alpha_{n,1}^{'} < \alpha_{n,2}^{'} < ... < \alpha_{n,m}^{'} < \alpha_{n,m+1}^{'} < ...$$

for $n = 0$. Thus, the eigenvalue of the Neumann eigenvalue problem is given by:

$$\lambda_{n,m} = (\frac{\alpha_{n,m}^{'}}{a})^2,$$

for $n = 0, 1, 2, 3, ...$ and $m = 1, 2, 3, ...$, (see [2]).

Remark 34. We have the following notes:

(1) If $n = 0$, then λ has multiplicity 1. This eigenvalue has corresponding eigenfunction, which is simply the multiples of $J_0(\alpha_{0,m}^{'} r)$.

(2) If $n \neq 0$, then λ has multiplicity 2 and the eigenfunctions of the form

$$u_{n,m}(r,\theta) = J_n((\frac{\alpha_{n,m}^{'}}{a})^2 r) \cdot (A_{n,m} \sin n\theta + B_{n,m} \cos n\theta), \quad n,m \in \mathbb{N},$$

with $A_{n,m}$ and $B_{n,m}$ as arbitrary constants.

6.3 Appendix.3 (Rectangular)

Now, we need to find the eigenvalues and eigenfunctions when we consider $\Omega = [0, L] \times [0, M]$ (product of intervals or a rectangle).

The two dimensional eigenvalue equation is

$$-\Delta u = \lambda u, \tag{6.4}$$

on Ω, i.e.,

$$u_{tt} + u_{ss} + \lambda u = 0, \tag{6.5}$$

such that

$$\begin{cases} u(0, s) = 0 = u(L, s), \quad 0 \leq s \leq M \\ \\ u(t, 0) = 0 = u(t, M), \quad 0 \leq t \leq L \end{cases}.$$

To find the eigenvalues, we solve the equation by separating the variables. Let $u(t, s) = T(t)S(s)$.

Then, the substitution in (6.5) gives

$$T^{''}S + TS^{''} + \lambda TS = 0.$$

Hence,

$$\frac{T^{''}}{T} + \frac{S^{''}}{S} + \lambda = 0.$$

Letting $\lambda = \mu^2 + \nu^2$ and using the boundary conditions.

The Dirichlet boundary conditions are $T(0) = T(L) = 0$ and $S(0) = S(M) = 0$.

Hence,

$$T^{''} + \mu^2 T = 0,$$

and

$$S^{''} + \nu^2 S = 0.$$

Therefore, the solutions of these are the same as for one dimensional eigenvalue equations: $\mu_n = \frac{n\pi}{L}$ and $\nu_m = \frac{m\pi}{M}$ for $m, n = 1, 2, 3, ...\cdot$ Hence,

$$T_n(t) = \sin(\frac{n\pi t}{L}),$$

and

$$S_m(s) = \sin(\frac{m\pi s}{M}).$$

To obtain the eigenfunctions

$$u(t, s)_{n,m} = \sin(\frac{n\pi t}{L}) \sin(\frac{m\pi s}{M}) \quad \text{for } n,m \geq 1,$$

with eigenvalues

$$\lambda_{n,m} = (\frac{n\pi}{L})^2 + (\frac{m\pi}{M})^2 \quad \text{for } n,m \geq 1.$$

Similarly, Neumann boundary conditions are $T^{'}(0) = T^{'}(L) = 0$ and $S^{'}(0) = S^{'}(M) = 0$. Therefore, the eigenfunctions are

$$u(t, s)_{n,m} = \cos(\frac{n\pi}{L}t) \cos(\frac{m\pi}{M}s) \quad \text{for } n,m \geq 0,$$

with eigenvalues

$$\lambda_{n,m} = (\frac{n\pi}{L})^2 + (\frac{m\pi}{M})^2 \quad \text{for } n,m \geq 0.$$

6.4 Appendix.4 (Circle)

The equation

$$-\Delta u = \lambda u,$$

on the unit circle

$$S^1 = \{(\cos\theta, \sin\theta), 0 \leq \theta \leq 2\pi\}.$$

In this case, we use a polar coordinate θ such that $1 = t^2 + s^2$ and $\theta = \arctan\frac{s}{t}$. The ordinary differential equation in polar coordinates is of the form;

$$-\Delta u(\theta) = \lambda u(\theta), \tag{6.6}$$

such that $-\Delta = -\dfrac{\partial^2}{\partial\theta^2}$ for S^1 with boundary conditions $u(0) = u(2\pi)$ and $u'(0) = u'(2\pi)$.

Consider three cases on λ :

If $\lambda = 0$, the general solution of the ordinary differential equation $-\Delta u = 0$ is

$$u(\theta) = A\theta + B,$$

where A, B are constants. Then, $u(0) = A(0) + B$ and $u(2\pi) = 2A\pi + B$. It follows that $2A\pi + B = B$, so $A = 0$.

This means $u(\theta) = B$ is constant and 0 is an eigenvalue for this problem with multiplicity 1.

Next, if $\lambda < 0$ then the general solution of the ordinary differential equation $-\Delta u(\theta) = \lambda u(\theta)$ is

$$u(\theta) = A\cosh\sqrt{-\lambda}\theta + B\sinh\sqrt{-\lambda}\theta.$$

Then,

$$u(0) = A\cosh\sqrt{-\lambda}(0) + B\sinh\sqrt{-\lambda}(0).$$

and

$$u(2\pi) = A\cosh\sqrt{-\lambda}(2\pi) + B\sinh\sqrt{-\lambda}(2\pi).$$

From the first boundary condition, we have

$$A = A\cosh\sqrt{-\lambda}2\pi + B\sinh\sqrt{-\lambda}2\pi. \tag{6.7}$$

Now, we have that

$$u^{'}(0) = \sqrt{-\lambda}A\sinh\sqrt{-\lambda}(0) + \sqrt{-\lambda}B\cosh\sqrt{-\lambda}(0),$$

$$u^{'}(2\pi) = \sqrt{-\lambda}A\sinh\sqrt{-\lambda}(2\pi) + \sqrt{-\lambda}B\cosh\sqrt{-\lambda}(2\pi).$$

From the second boundary condition, we have

$$\sqrt{-\lambda}B = \sqrt{-\lambda}A\sinh\sqrt{-\lambda}(2\pi) + \sqrt{-\lambda}B\cosh\sqrt{-\lambda}(2\pi). \tag{6.8}$$

From the equations (6.7) and (6.8) we have

$$\begin{cases} A = A\cosh\sqrt{-\lambda}(2\pi) + B\sinh\sqrt{-\lambda}(2\pi) \\ \sqrt{-\lambda}B = \sqrt{-\lambda}A\sinh\sqrt{-\lambda}(2\pi) + \sqrt{-\lambda}B\cosh\sqrt{-\lambda}(2\pi). \end{cases}$$

To solve the system of equations,

$$\begin{cases} A = A\cosh\sqrt{-\lambda}(2\pi) + B\sinh\sqrt{-\lambda}(2\pi) \\ B = A\sinh\sqrt{-\lambda}(2\pi) + \ B\cosh\sqrt{-\lambda}(2\pi), \end{cases}$$

or, equivalently,

$$\begin{cases} 0 = A(\cosh\sqrt{-\lambda}(2\pi) - 1) + B\sinh\sqrt{-\lambda}(2\pi) \\ 0 = A\sinh\sqrt{-\lambda}(2\pi) + B(\cosh\sqrt{-\lambda}(2\pi) - 1). \end{cases}$$

Multiply the first equation by $(-\sinh\sqrt{-\lambda}(2\pi))$ and the second equation by $(\cosh\sqrt{-\lambda}(2\pi) - 1)$. Then, this would imply

$$0 = -B\sinh^2\sqrt{-\lambda}(2\pi) + B(\cosh\sqrt{-\lambda}(2\pi) - 1)^2.$$

Therefore,

$$2B\cosh\sqrt{-\lambda}(2\pi) = 0, \tag{6.9}$$

(because $\cosh^2\theta - \sinh^2\theta = 1$. So, $B = 0$ as $\cosh\sqrt{-\lambda}(2\pi) \neq 0$). Then, we have

$$0 = A\sinh\sqrt{-\lambda}(2\pi) + B(\cosh\sqrt{-\lambda}(2\pi) - 1),$$

it would imply

$$A\sinh\sqrt{-\lambda}(2\pi) = 0,$$

(because $\sinh 0 = 0$ at $\lambda = 0$ for $A \neq 0$). It is impossible because $\lambda < 0$. This means the problem has no negative eigenvalues.

If $\lambda > 0$, then the general solution of the ordinary differential equation

$$-\Delta u(\theta) = \lambda u(\theta)$$

will be of the form

$$u(\theta) = A\cos\sqrt{\lambda}\theta + B\sin\sqrt{\lambda}\theta.$$

Then, we have

$$u(0) = A\cos\sqrt{\lambda}(0) + B\sin\sqrt{\lambda}(0),$$

and

$$u(2\pi) = A\cos\sqrt{\lambda}2\pi + B\sin\sqrt{\lambda}2\pi.$$

From the first boundary condition, we have

$$A = A\cos\sqrt{\lambda}2\pi + B\sin\sqrt{\lambda}2\pi. \tag{6.10}$$

Now, to get the derivative of u at $t = 0$

$$u^{'}(0) = -\sqrt{\lambda}A\sin\sqrt{\lambda}(0) + \sqrt{\lambda}B\cos\sqrt{\lambda}(0),$$

and at $t = 2\pi$

$$u^{'}(2\pi) = \sqrt{\lambda}A\sin\sqrt{\lambda}(2\pi) + \sqrt{\lambda}B\cos\sqrt{\lambda}(2\pi).$$

From the second boundary condition

$$\sqrt{\lambda}B = -\sqrt{\lambda}A\sin\sqrt{\lambda}(2\pi) + \sqrt{\lambda}B\cos\sqrt{\lambda}(2\pi). \tag{6.11}$$

From (6.10) and (6.11), we can obtain

$$\begin{cases} A = A\cos\sqrt{\lambda}(2\pi) + B\sin\sqrt{\lambda}(2\pi) \\ \\ \sqrt{\lambda}B = -\sqrt{\lambda}A\sin\sqrt{\lambda}(2\pi) + \sqrt{\lambda}B\cos\sqrt{\lambda}(2\pi), \end{cases}$$

or, equivalently,

$$\begin{cases} 0 = A(\cos\sqrt{\lambda}(2\pi) - 1) + B\sin\sqrt{\lambda}(2\pi) \\ \\ 0 = A\sin\sqrt{\lambda}(2\pi) + B(\cos\sqrt{\lambda}(2\pi) - 1). \end{cases}$$

Multiply the first equation by $(\sin\sqrt{\lambda}(2\pi))$ and the second equation by $(-\cos\sqrt{\lambda}(2\pi) - 1)$ of the above system to have

$$0 = B\sin^2\sqrt{\lambda}(2\pi) + B(\cos\sqrt{\lambda}(2\pi) - 1)^2.$$

Then, it would imply

$$B(\sin^2\sqrt{\lambda}(2\pi) + \cos^2\sqrt{\lambda}(2\pi)) - 2B\cos\sqrt{\lambda}(2\pi) + 2B = 0,$$

It follows that $2B\cos\sqrt{\lambda}(2\pi) = 0$ because $\cos^2\sqrt{\lambda}(2\pi) + \sin^2\sqrt{\lambda}(2\pi) = 1$.

So, $A\sin\sqrt{\lambda}(2\pi) = 0$ and $A \neq 0$, it would imply $\lambda = (\frac{n}{2})^2$ for $\lambda > 0$. Hence, there are positive eigenvalues for this problem, which are

$$\lambda = n^2 \quad \text{for } n = 1{,}2{,}3{,}...$$

The eigenfunctions are

$$u_n(\theta) = A^{'}\cos n\theta + B^{'}\sin n\theta \quad \text{for } n = 1{,}2{,}3{,}...,$$

where $A^{'}$ and $B^{'}$ are arbitrary constants.

6.5 Appendix.5 (Operator pencils)

Let us consider the following 2×2 matrix

$$\mathcal{A} = \begin{pmatrix} \mu^2 & \mu + 2 \\ \\ -\mu + 2 & \mu^2 + 2 \end{pmatrix}. \tag{6.12}$$

We can write it by $\mathcal{A}(\mu) = \mu^2 A_0 + \mu A_1 + A_2$, i.e.,

$$\mathcal{A} = \begin{pmatrix} \mu^2 & 0 \\ 0 & \mu^2 \end{pmatrix} + \begin{pmatrix} 0 & \mu \\ -\mu & 0 \end{pmatrix} + \begin{pmatrix} 0 & 2 \\ 2 & 2 \end{pmatrix}, \tag{6.13}$$

it follows that

$$\mathcal{A} = \mu^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \mu \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 2 \\ 2 & 2 \end{pmatrix}, \tag{6.14}$$

therefore,

$$A_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$A_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and

$$A_2 = \begin{pmatrix} 0 & 2 \\ 2 & 2 \end{pmatrix}$$

all A's are bounded operators from $\mathbb{C}^2 \to \mathbb{C}^2$, such $\mathcal{A}(\mu)$ is called a quadratic operator pencil and gives a mapping from $\mathbb{C}$ to the set of all bounded operators $B(\mathbb{C}^2, \mathbb{C}^2) \cong M_{2\times 2}(\mathbb{C})$, for $\mu \in \mathbb{C}$.

Firstly, We will find the spectrum of this operator For $\mu \in \mathbb{C}$. We have $\mu \in \sigma(\mathcal{A})$, iff $\mathcal{A} : \mathbb{C}^2 \to \mathbb{C}^2$ is not invertible i.e., $A_2 - (-\mu A_1 - \mu^2 A_0) : \mathbb{C}^2 \to \mathbb{C}^2$ is not invertible, it means the characteristic polynomial of matrix is

$$\det(\mathcal{A}(\mu)) = \mu^2(\mu^2+2) - (-\mu+2)(\mu+2) = 0,$$

it follows that

$$\mu^4 + 3\mu^2 - 4 = 0,$$

therefore,

$$\sigma(\mathcal{A}) = \{\pm 1, \pm 2i\},$$

and the projection of spectrum of operator pencil $\mathcal{A}$ onto the imaginary axis, that is

$$\Gamma(\mathcal{A}) = \{0\} \cup \{\pm 2\}.$$

We say that μ_0 is an eigenvalue of $\mathcal{A}$ if there exists $0 \neq u \in \mathbb{C}^2$ such that $\mathcal{A}(\mu_0)u = 0$ and u is called an eigenfunction of $\mathcal{A}$. To find these eigenvectors with respect to eigenvalues, we need to solve the homogeneous system and we will find them latter in this part.

6.6 Appendix.6 (Closed Contour)

Here, we have a closed contour S_R such that S_{R^+} gives a anti-clock wise contour and S_{R^-} gives a clock wise contour.

For μ be a sufficiently large, for a constant $c^{'}$, by the same arguments in the previous propositions, we can get that

$$\|\mathcal{B}_A^{-1}(\mu)\|_{\mathbb{C}} \leq c|\mu| \leq c^{'}. \tag{6.15}$$

For μ be a sufficiently large, for a constant $c^{'}$

$$\|\mathcal{B}_A^{-1}(\mu)\|_{\mathbb{C}} \leq c|\mu| \leq c^{'}. \tag{6.16}$$

For $\theta \in [0, \pi]$, let $\mu = i\alpha + Re^{i\theta}$ and $d\mu = iRe^{i\theta}d\theta$, then we can get,

$$|e^{it\mu}| = |e^{it(i\alpha + Re^{i\theta})}| = |e^{-\alpha t}e^{itRe^{i}\theta}|$$

$$= |e^{-\alpha t}e^{-tR\sin\theta}|.$$

For a sufficiently large R we obtain,

$$\|e^{it\mu}\mathcal{B}_A^{-1}(\mu)\|_{\mathbb{C}} \leq c^{'} e^{-\alpha t}e^{-tR\sin\theta}.$$

Now, for $t > 0$ it follows that,

$$\left|\int_{S_{R+}} e^{it\mu}\mathcal{B}_A^{-1}(\mu)d\mu\right| \leq c^{'} Re^{-\alpha t}\int_0^{\pi} e^{-tR\sin\theta}d\theta.$$

and $t < 0$ it follows that,

$$\left|\int_{S_{R-}} e^{it\mu}\mathcal{B}_A^{-1}(\mu)d\mu\right| \leq c^{'} Re^{-\alpha t}\int_0^{\pi} e^{+tR\sin\theta}d\theta.$$

Then the integral is absolutely convergent at 0 as $R \to \infty$, we have

$$\lim_{R\to\infty}\left|\int_{S_{R\pm}} e^{it\mu}\mathcal{B}_A^{-1}(\mu)d\mu\right| \leq \lim_{R\to\infty} c^{'} Re^{-\alpha t}\int_0^{\pi} e^{\pm tR\sin\theta}d\theta = 0.$$

From the previous Proposition 5.3.2 we have that

$$\lim_{R\to\infty}\int_{-R+i\alpha}^{R+i\alpha} e^{it\mu}\mathcal{B}_A^{-1}(\mu)d\mu, \tag{6.17}$$

exists to complete our argument.

If R is a sufficiently large, by the Cauchy's Residue Theorem gives

$$\int_{-R+i\alpha}^{R+i\alpha} e^{it\mu}\mathcal{B}_A^{-1}(\mu)d\mu + \int_{S_{R\pm}} e^{it\mu}\mathcal{B}_A^{-1}(\mu)d\mu = \pm 2\pi i\sum_{\mu\in\Sigma_\alpha^{\pm}} Res(e^{it\mu}\mathcal{B}_A^{-1}(\mu);\mu).$$

Chapter 7

Conclusion and Future Work

Here, we present chapter by chapter summary of the major problems tackled of this thesis, highlighting some salient points and limitations. We also suggest some follow up studies in order to surmount some identified challenges in this area.

7.1 Existence Eigenvalues for $-\Delta - V$ on Cylindrical Domain

The main work of the first task of this thesis was in Chapter 3. It was to gain a deeper understanding the development of aspects of the theory of partial differential equations with operator by concentrating on some particular examples of trapped modes. It was dependent on several studies on existence of trapped modes through horizontal circular cylinder sufficiently small radius in water, and which was proved by Ursell in (1951). Then, it was developed in (1991) by Evans and Linton when they used some techniques of Ursell method and they had been concerned with both existence of trapped modes and numerical algorithm. However, in this thesis we investigated the stability of embedded eigenvalues within spectrum for the operator

$$-\Delta - V$$

on cylindrical domain $\mathbb{R} \times [-L, L]$ for a sufficiently small, non-negative continuous real valued function V on $\mathbb{R} \times [-L, L]$ with bounded support, which is symmetric, i.e.,

$$V(t, s) = V(t, -s)$$

for $t, s \in \mathbb{R} \times [-L, L]$. The result of this part observed the above operator has an eigenvalue λ is contained in the essential spectrum, hence an embedded eigenvalue. Although the arguments which were appeared to have the embedded eigenvalues for the Laplacian operator with added a potential function on the cylindrical domain, there could be several other better arguments to do this and obtain even much better results by using previous studies. For example, some conditions on potentials V which adds to the different operator. It would be interesting to explore this further.

7.2 Operator pencil and Main Results

Chapter 4 was divided into two main parts: The first part defined the spaces H_k for $k = 0, 1, 2, ...$ Then, there were definitions of weighted functions spaces with some fundamental ideas. The majority of this part devoted to establishing the basic properties of the weighted functions spaces $W^k_{\alpha,\beta}$ for $k = 0, 1, 2, ...$ and $\alpha, \beta \in \mathbb{R}$ that were necessary in order to work with them. The second part of this chapter, there was some results of properties of pencils which were used to build some ideas of this research. We proved

$$\mathcal{B}_A(D_t) = D_t^2 + A - \lambda : W^2_{\alpha,\beta} \to W^0_{\alpha,\beta} \tag{7.1}$$

is an isomorphism. This result was special case of a general theory that has been developed for ordinary equations with operator coefficient. We also observed the fact of this theorem did not extend to $\alpha, \beta \in \mathbb{R} \setminus \Gamma$ has to do with the existence of exponential solutions of $\mathcal{B}_A(\mu_0)u = 0$, for $u \in W^2_{\alpha,\alpha}$ and these solutions gave a link between the the isomorphisms for different values for α, β. Then, there were some corollaries and lemmas at the end of this chapter which were proved some properties of Sobolev spaces.

7.3 Fredholm Properties of pencils

Chapter 5 of this thesis contained the Fredholm properties of operator pencils $\mathcal{B}_A$. In particular, we detected and approximated the spectra of operator pencils via Green's kernel with interesting of exponential solutions for equations $\mathcal{B}_A u = f$. We obtained some results for Fredholm property and semi-Fredholm property in Section 5.5. Then, we calculated the Kernels and Co-kernels explicitly to establish a Fredholm operator and its index without consider its adjoint. By using some arguments and techniques from previous studies, for example, [10] and [58]. The argument of Fredholm index and its dependence on the parameters α, β is considered the main result Theorem 5.5.5 in this section the maps

$$A^{(\alpha,\beta)} = \mathcal{B}_A(D_t) : W^2_{\alpha,\beta} \longrightarrow W^0_{\alpha,\beta}$$

and

$$A^{(\beta,\alpha)} = \mathcal{B}_A(D_t) : W^2_{\beta,\alpha} \longrightarrow W^0_{\beta,\alpha}$$

are Fredholm with

$$\operatorname{Index} A^{(\alpha,\beta)} : W^2_{\alpha,\beta} \longrightarrow W^0_{\alpha,\beta} = -|\Sigma_{\alpha,\beta}| = -\operatorname{Index} A^{(\beta,\alpha)} : W^2_{\beta,\alpha} \longrightarrow W^0_{\beta,\alpha}.$$

for $\alpha < \beta \in \mathbb{R}\backslash\Gamma$.

In conclusion, we focused on the classical theory of ordinary differential equations with operator coefficients in this research. In particular, we studied the perturbation problems for operators with existence embedded eigenvalues (trapped modes) which is related to an eigenvalue of different operators on cylindrical domain and then we studied a Fredholm propriety of operator pencils by using the Green's kernel to detect spectra of operator pencils. Again, these problems need more studies to be addressed first before a substantial progress could be made of the fact. There are studies will focus on this arguments in the future research for example, the stability these eigenvalues for different operators on different spaces and arguments uses to develop the formula of the index of a Fredholm map with many new applications to apply this theory of ordinary differential equations with operator coefficients.

Chapter 8

Publication

- Stability of embedded eigenvalues for operator N. Altaweel. (2019) . *The stability of embedded eigenvalues for Laplace operator*. Scholars' press. ID CMM-2019−103. https://onlinelibrary.wiley.com/journal. Printed by Schaltungsdienst lange o.H.G., Berlin.ISBN:978-613-8-92424-1 (Published in March,2020).
- Fredholm properties for Pencils N. Altaweel. (2021) *Fredholm properties for Pencils.* American Review of Mathematics and Statistics ID: MAS-1351 ISSN 2374 − 2348 (Print) 2374 − 2356 (Online) DOI: 10.15640/arms. Vol. 9 NO. 1. Publication date: April 30, 2021.
- Operator pencils and its properties: AIMS Press. The manuscript number Math20210668. April 08, 2021. (Under review).
- Numerical Range of Generalized Aluthge Transformation. AIMS Press. The manuscript number Math20210499. February 28, 2021. (Under review).

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