

# AUSLANDER-REITEN THEORY OF FINITE DIMENSIONAL ALGEBRAS

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## INTRODUCTION

The goal of representation theory is to use linear algebra to study groups, rings and algebras. In these lecture notes, we consider representation theory of algebras. Quivers provide a useful concrete way to visualise algebras. Given a quiver, i.e. a directed graph, one can associate an algebra, called a *path algebra*, generated by the paths of the quiver. From the point of view of representation theory, the study of finite dimensional algebras reduces to the study of quotients of path algebras.

A central aim in representation theory of finite dimensional algebras is to classify all their modules and the morphisms between them. Due to the Krull-Schmidt theorem, the classification of modules can be reduced to the classification of indecomposable modules. That is, in some sense, indecomposable modules are the building blocks of all modules. It is then natural to ask when is a finite-dimensional algebra of *finite representation type*, i.e. when does it have finitely many indecomposable modules up to isomorphism? Gabriel's theorem [15] gives an elegant answer for path algebras of quivers without oriented cycles, also called *hereditary algebras*. This theorem is an example of an ADE classification, i.e. in terms of simply-laced Dynkin diagrams, which appear also in finite type classifications in many other areas of mathematics, including Lie algebras, root systems and cluster algebras.

Auslander-Reiten theory gives us a way to visualise the representation theory of a finite dimensional algebra using a quiver, called the Auslander-Reiten quiver. The vertices of this quiver correspond to the indecomposable modules and the arrows correspond to *irreducible morphisms*, which are the corresponding building blocks for the morphisms.

The aim of these lecture notes is to give a brief introduction to Auslander-Reiten theory and to provide methods for constructing Auslander-Reiten quivers. We present two methods to construct these quivers for some special classes of algebras. The first method is the *knitting algorithm*, which works for instance for hereditary algebras of finite representation type. The second method is a *geometric model* associated to (partial) triangulations of surfaces. This method, which has its origins in cluster-tilting theory [11], encodes the representation theory of an important class of algebras, called *gentle algebras*, which have been subject of intensive study since the 1980's due to the fact that they remain one of the relatively few classes of algebras for which the representation theory is computationally tractable.

The pre-requisites are a basic knowledge of linear algebra and rings and modules. Knowledge of the basic concepts of category theory is beneficial, but not essential. The list of references is not exhaustive, but it includes some of the main references for this subject. We refer the reader to [3, 4, 6, 24] for further study on quiver representations and Auslander-Reiten theory. The language of categories used in these theories is also nicely explained in [3, 24].

## 1. BOUND PATH ALGEBRAS

In this section we will associate algebras to quivers, i.e. directed graphs. From a representation-theoretic point of view, we will see that it is enough to study algebras associated to quivers.

**Definition 1.1.** A *quiver*  $Q = (Q_0, Q_1, s, t)$  consists of the following data:

- (1) a set  $Q_0$  of vertices,
- (2) a set  $Q_1$  of arrows between vertices,
- (3) two maps  $s, t : Q_1 \rightarrow Q_0$ , called *source* and *target*, respectively, such that, for each arrow  $\alpha : i \rightarrow j \in Q_1$ ,  $i = s(\alpha)$  and  $j = t(\alpha)$ .

A quiver is *finite* if  $Q_0$  and  $Q_1$  are finite sets. Throughout these notes, we will only consider finite quivers.

**Definition 1.2.** Let  $Q$  be a quiver.

- (1) A *path in  $Q$  of length  $\ell$*  is a sequence  $p = \alpha_1 \alpha_2 \cdots \alpha_\ell$ , with  $\alpha_i \in Q_1$  such that  $s(\alpha_i) = t(\alpha_{i-1})$ , for each  $i = 2, \dots, \ell$ . In particular,  $p$  has length 1 if and only if  $p \in Q_1$ .
- (2) We associate a path  $\varepsilon_i$  of length 0 to each vertex  $i$  of  $Q$ , which is called the *stationary path at  $i$* .
- (3) If  $s(\alpha_1) = t(\alpha_\ell)$ , then  $p$  is said to be an *oriented cycle*. An oriented cycle of length 1 is called a *loop*. An *acyclic* quiver is a quiver with no oriented cycles.

Sometimes we denote a path from  $i$  to  $j$  by  $i \rightsquigarrow j$ .

Throughout  $k$  denotes an algebraically closed field.

**Definition 1.3.** The *path algebra*  $kQ$  of  $Q$  is an algebra whose underlying vector space has all the paths of  $Q$  as basis and with multiplication defined on two basis elements given by concatenation of paths, i.e. given two paths  $p = \alpha_1 \cdots \alpha_\ell, p' = \alpha'_1 \cdots \alpha'_m$ ,

$$pp' = \begin{cases} \alpha_1 \cdots \alpha_\ell \alpha'_1 \cdots \alpha'_m & \text{if } t(\alpha_\ell) = s(\alpha'_1) \\ 0 & \text{otherwise.} \end{cases}$$

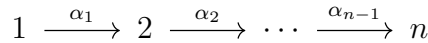
**Example 1.4.**

(1) Let  $Q$  be the quiver:



$kQ$  has basis given by  $\{\alpha^t \mid t \geq 0\}$ , where  $\alpha^0$  denotes the stationary path  $\varepsilon_1$ . The multiplication is given by  $\alpha^s \alpha^t = \alpha^{s+t}$ . The algebra  $kQ$  is isomorphic to the algebra  $k[x]$  of polynomials with one indeterminate.

(2) Let  $Q$  be the quiver



$kQ$  is generated by the paths  $\varepsilon_i (1 \leq i \leq n)$ ,  $\alpha_i (1 \leq i \leq n)$ ,  $\alpha_i \cdots \alpha_j (1 \leq i < j \leq n)$ , and it is isomorphic to the algebra of upper triangular  $n \times n$  matrices.

**Remark 1.5.** The path algebra  $kQ$  satisfies the following properties:

- (1)  $kQ$  has an identity  $1 = \sum_{i \in Q_0} \varepsilon_i$  if and only if  $Q_0$  is finite.
- (2)  $kQ$  is an associative algebra.
- (3)  $kQ$  is finite dimensional if and only if  $Q$  is finite and acyclic.

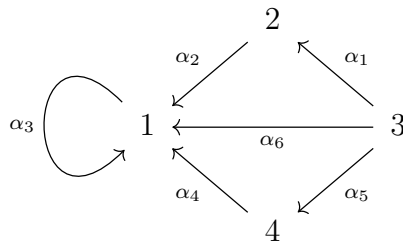
**Definition 1.6.** Let  $Q$  be a finite quiver.

- (1) The *arrow ideal*  $R_Q$  is the two-sided ideal of  $kQ$  generated by all arrows in  $Q$ .
- (2) An *admissible ideal*  $I$  is a two-sided ideal of  $kQ$  such that there is  $m \geq 2$  for which  $R_Q^m \subseteq I \subseteq R_Q^2$ .
- (3) Given an admissible ideal  $I$ , the quotient algebra  $kQ/I$  is said to be a *bound path algebra*.

The bound path algebra  $kQ/I$  is finite dimensional, since  $R_Q^m \subseteq I$  and it is connected (i.e. it is not the direct product of two algebras) because  $I \subseteq R_Q^2$ .

A *relation*  $\rho$  is a linear combination  $\rho = \sum_p \lambda_p p$  of paths, all with length at least two, and with same start and same endpoints. It is easy to check that any admissible ideal can be generated by a set of relations.

**Example 1.7.** Let  $Q$  be the quiver



The ideal  $I_1 = \langle \alpha_1 \alpha_2 - \alpha_5 \alpha_4, \alpha_6 \alpha_3, \alpha_2 \alpha_3, \alpha_3^4 \rangle$  is admissible since  $R_Q^5 \subseteq I_1 \subseteq R_Q^2$ .

The ideal  $I_2 = \langle \alpha_1 \alpha_2 - \alpha_5 \alpha_4, \alpha_6 \alpha_3, \alpha_2 \alpha_3 \rangle$  is not admissible because  $\alpha_3^m \notin I_2$ , for all  $m \geq 2$ .

The ideal  $I_3 = \langle \alpha_1 \alpha_2 - \alpha_6 \rangle$  is not admissible as  $\alpha_1 \alpha_2 - \alpha_6 \notin R_Q^2$ .

**Theorem 1.8.** Any finite dimensional algebra  $A$  is Morita equivalent to a bound path algebra  $kQ/I$ , i.e.  $\text{mod}(A) \simeq \text{mod}(kQ/I)$ .

For a proof, see [3, I.6.10, II.3.7].

## 2. REPRESENTATIONS OF A BOUND PATH ALGEBRA

In the previous section we saw that quivers provide a nice way to visualise finite dimensional algebras. Now, we will explain how quivers can be used to visualise also modules and morphisms between modules.

Throughout this section  $Q$  denotes a finite, connected quiver and  $I$  an admissible ideal. Note that if  $I = 0$  is admissible, then  $Q$  must be acyclic.

**Definition 2.1.** A representation  $M = (M_i, \varphi_\alpha)_{i \in Q_0, \alpha \in Q_1}$  of  $Q$  is given by:

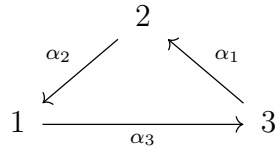
- $k$ -vector spaces  $M_i$ , for all  $i \in Q_0$ , and
- linear maps  $\varphi_\alpha : M_{s(\alpha)} \rightarrow M_{t(\alpha)}$ , for all  $\alpha \in Q_1$ .

Let  $p = \alpha_1 \cdots \alpha_\ell$  be a path in  $Q$  and  $M = (M_i, \varphi_\alpha)_{i \in Q_0, \alpha \in Q_1}$  be a representation of  $Q$ . We denote by  $\varphi_p$  the composition of linear maps  $\varphi_p = \varphi_{\alpha_\ell} \cdots \varphi_{\alpha_1}$ . Given a relation  $\rho = \sum_p \lambda_p p$  in  $I$ , we have  $\varphi_\rho = \sum_p \lambda_p \varphi_p$ .

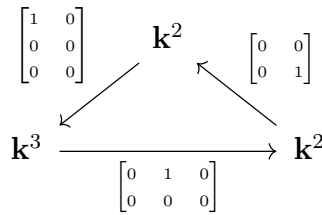
**Definition 2.2.** A representation  $M = (M_i, \varphi_\alpha)_{i \in Q_0, \alpha \in Q_1}$  of  $Q$  is said to be *bound by  $I$* , or to be a *representation of  $(Q, I)$* , if  $\varphi_\rho = 0$  for all  $\rho \in I$ .

A representation  $M$  is *finite dimensional* if  $M_i$  is finite dimensional, for all  $i \in Q_0$ . The *dimension vector* of  $M$  is the vector  $\underline{\dim} M = (\dim M_i)_{i \in Q_0}$ .

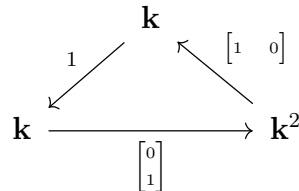
**Example 2.3.** Consider the quiver  $Q$ :



bound by  $I = \langle \alpha_1 \alpha_2, \alpha_2 \alpha_3, \alpha_3 \alpha_1 \rangle$ . The representation:



is bound by  $I$ . However, the representation given by



is not bound by  $I$ .

**Definition 2.4.** Let  $M = (M_i, \varphi_\alpha), N = (N_i, \psi_\alpha)$  be representations of  $(Q, I)$ .

- (1) A *morphism of representations*  $f : M \rightarrow N$  is a collection  $(f_i)_{i \in Q_0}$  of linear maps,  $f_i : M_i \rightarrow N_i$ , such that for each  $\alpha : i \rightarrow j \in Q_1$ , the following diagram

commutes:

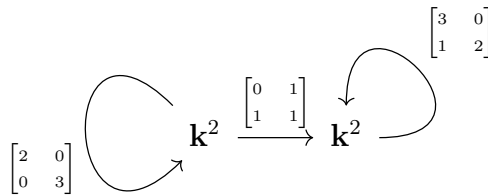
$$\begin{array}{ccc} M_i & \xrightarrow{\varphi_\alpha} & M_j \\ f_i \downarrow & & \downarrow f_j \\ N_i & \xrightarrow{\psi_\alpha} & N_j. \end{array}$$

(2) The morphism  $f = (f_i)_{i \in Q_0}$  is an *isomorphism* if each  $f_i$  is bijective.

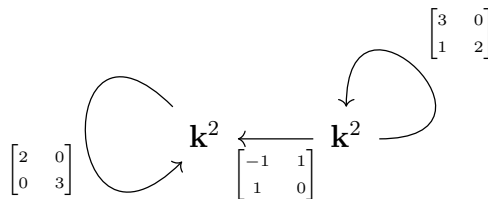
**Example 2.5.** Let  $Q$  be the quiver



The following represents a morphism of representations:



This morphism is bijective with inverse given by



We obtain the category  $\text{rep}(Q, I)$  of finite-dimensional bound quiver representations of  $(Q, I)$ , whose objects are finite-dimensional bound quiver representations and maps are given by morphisms of bound quiver representations.

Given a finite dimensional algebra  $A$ , we denote by  $\text{mod}(A)$  the category of finite dimensional right  $A$ -modules.

**Theorem 2.6.** *There is an equivalence of categories  $\text{mod}(\mathbf{k}Q/I) \simeq \text{rep}(Q, I)$ .*

*Proof.* Denote the algebra  $\mathbf{k}Q/I$  by  $A$  and write  $e_i = \varepsilon_i + I$ . We begin by constructing a functor  $F : \text{mod}(A) \rightarrow \text{rep}(Q, I)$ .

Given  $M \in \text{mod}(A)$ , we define  $F(M)$  to be the representation  $(M_i, \varphi_\alpha)$ , where  $M_i = Me_i$ , and  $\varphi_\alpha : M_{s(\alpha)} \rightarrow M_{t(\alpha)}$  is the map  $me_{s(\alpha)} \mapsto m\bar{\alpha} := m(\alpha + I)$ .

Note that each  $\varphi_\alpha$  is a  $\mathbf{k}$ -linear map since  $M$  is an  $A$ -module.

In order to check that  $F(M) \in \text{rep}(Q, I)$ , we need to show that  $F(M)$  is bound by  $I$ . Given a relation  $\rho = \sum_{p:i \rightsquigarrow j} \lambda_p p$  in  $I$ , we have

$$\begin{aligned} \varphi_\rho(me_i) &= \sum_{p:i \rightsquigarrow j} \lambda_p \varphi_p(me_i) \\ &= \sum_{p:i \rightsquigarrow j} \lambda_p m(p + I) \\ &= m \sum_{p:i \rightsquigarrow j} \lambda_p (p + I) \\ &= m(\rho + I) = m0 = 0. \end{aligned}$$

This defines  $F$  on the objects. Now, let  $f : M \rightarrow N$  be a morphism in  $\text{mod}(A)$ , and let  $F(M) = (M_i, \varphi_\alpha)$ ,  $F(N) = (N_i, \psi_\alpha)$ . We define  $F(f) = (f_i)_{i \in Q_0}$  by  $f_i(me_i) := f(m)e_i$ .

We need to check that  $f_j \varphi_\alpha = \psi_\alpha f_i$ , for each  $\alpha : i \rightarrow j \in Q_1$ . Indeed, given  $me_i \in M_i$ , we have:

$$\begin{aligned} f_j \varphi_\alpha(me_i) &= f_j(m\bar{\alpha}) = f(m\bar{\alpha})e_j \\ &= f(m)\bar{\alpha}e_j = \psi_\alpha(f(m)e_i) \\ &= \psi_\alpha(f_i(me_i)) = \psi_\alpha f_i(me_i). \end{aligned}$$

Therefore,  $F(f)$  is a morphism of representations.

It is easy to check that  $F(f)$  is indeed a (covariant) functor, i.e. that  $F(1_M) = 1_{F(M)}$ , for any  $A$ -module  $M$ , and  $F(gf) = F(g)F(f)$ , for  $f : L \rightarrow M$ ,  $g : M \rightarrow N \in \text{mod}(A)$ .

The next step is to construct a functor  $G : \text{rep}(Q, I) \rightarrow \text{mod}(A)$ . Given  $(M_i, \varphi_\alpha) \in \text{rep}(Q, I)$ , we define  $G(M_i, \varphi_\alpha) = M$  as follows. The underlying vector space of  $M$  is  $\bigoplus_{i \in Q_0} M_i$ . It is enough to define the right  $A$ -action on paths in  $Q$ . Let  $p$  be a path in  $Q$  and  $m = (m_i)_{i \in Q_0}$  be an element of  $M$ . If  $p = \varepsilon_i$  for some  $i$ , let  $mp := m_i$ , and if  $p$  has length  $\geq 1$ , we define  $mp$  to be the following element in  $M$ :

$$(mp)_k := \begin{cases} 0 & \text{if } k \neq t(p) \\ \varphi_p(m_{s(p)}) & \text{if } k = t(p). \end{cases}$$

In order to check that the  $A$ -action is well defined, we need to show that if  $\rho = \sum_{p:i \rightsquigarrow j} \lambda_p mp \in I$ , then  $m\rho = 0$ . Indeed, we have that  $m\rho$  is the element in  $M$  whose only possible non-zero coordinate is  $(m\rho)_j = \sum \lambda_p \varphi_p(m_i)$ . But  $\sum \lambda_p \varphi_p(m_i) = 0$  since  $(M_i, \varphi_\alpha)$  is bound by  $I$ .

The definition of  $G$  on morphisms is as follows: given  $f = (f_i) : (M_i, \varphi_\alpha) \rightarrow (N_i, \psi_\alpha)$ , we have  $G(f) : M \rightarrow N$  defined by  $G(f)(m) := (f_i(m_i))_{i \in Q_0}$ .

Clearly  $G(f)$  is linear as each  $f_i$  is linear. In order to show that  $G(f)$  is a module homomorphism, it is enough to check  $G(f)(ma) = G(f)(m)a$  for all  $m = m_i \in M_i$  and  $a = p + I \in A$ , where  $p$  is a path from  $i$  to  $j$ .

On the one hand, we have  $(ma)_k = 0$  for  $k \neq j$  and  $(ma)_j = \varphi_p(m_i)$ , and so

$$(G(f)(ma))_k = \begin{cases} 0 & \text{if } k \neq j \\ f_j \varphi_p(m_i) = \psi_p f_i(m_i) & \text{if } k = j. \end{cases}$$

On the other hand,  $(G(f)(m))_k = 0$  for  $k \neq i$ , and  $(G(f)(m))_i = f_i(m_i)$ , and so according to the definition of  $A$  action,

$$(G(f)(m)a)_k = \begin{cases} 0 & \text{if } k \neq j \\ \psi_p(f_i(m_i)) & \text{if } k = j. \end{cases}$$

It is easy to check that  $G$  is indeed a functor and that  $FG \simeq 1_{\text{rep}(Q,I)}$  and  $GF \simeq 1_{\text{mod}(A)}$ , thus giving the required equivalence of categories.  $\square$

### 3. REPRESENTATION FINITE HEREDITARY ALGEBRAS

The Krull-Schmidt theorem states that every module over a finite-dimensional algebra can be written as a direct sum of indecomposable modules in a unique way (up to isomorphism and changing the order). Therefore, in order to classify all the modules over an algebra, it is sufficient to classify the indecomposable ones.

In this section we discuss representation types of algebras, and discuss the simplest case one can hope for, which is when there are finitely many indecomposable modules.

#### Definition 3.1.

- (1) Given two representations  $M = (M_i, \varphi_\alpha)$ ,  $N = (N_i, \psi_\alpha)$  of  $Q$ , we can construct a new representation

$$M \oplus N := (M_i \oplus N_i, \begin{bmatrix} \varphi_\alpha & 0 \\ 0 & \psi_\alpha \end{bmatrix}),$$

called the *direct sum of  $M$  and  $N$* .

- (2) A representation  $M$  is *indecomposable* if  $M \neq 0$  and it cannot be written as a direct sum of two non-zero representations.

#### Example 3.2.

- (1) Let  $Q$  be the quiver  $1 \longrightarrow 2 \longrightarrow 3$ . The representation

$$M = \mathbf{k} \xrightarrow{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} \mathbf{k}^2 \xrightarrow{\begin{bmatrix} 0 & 1 \end{bmatrix}} \mathbf{k}$$

is not indecomposable since  $M \cong (\mathbf{k} \xrightarrow{1} \mathbf{k} \longrightarrow 0) \oplus (0 \longrightarrow \mathbf{k} \xrightarrow{1} \mathbf{k})$ .

- (2) Let  $Q$  be the quiver  $1 \rightrightarrows 2$ . We have

$$\mathbf{k} \xrightarrow{\begin{bmatrix} 1 \\ 2 \end{bmatrix}} \mathbf{k}^2 \cong (\mathbf{k} \xrightarrow[2]{1} \mathbf{k}) \oplus (0 \rightrightarrows \mathbf{k}).$$

#### Definition 3.3. An algebra $A$ is:

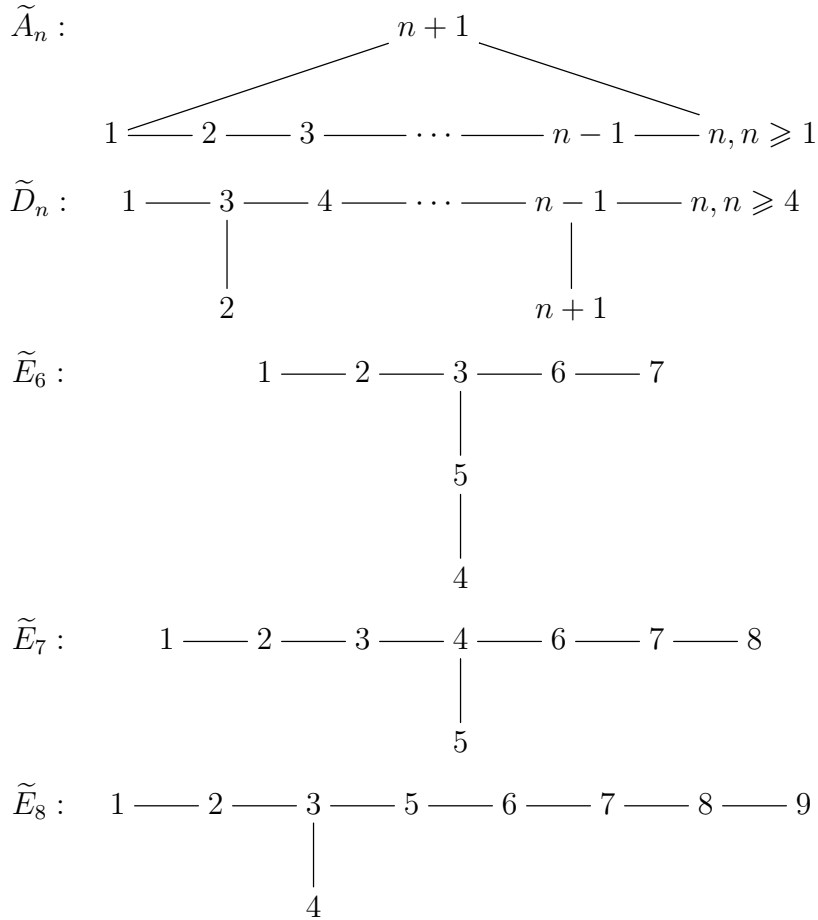
- (1) of *finite representation type* if, up to isomorphism, there are only finitely many indecomposable objects in  $\text{mod}(A)$ .  
(2) *hereditary* if  $A \simeq \mathbf{k}Q$ , for some finite, connected and acyclic quiver  $Q$ .

Representation finite hereditary algebras have been classified by Gabriel.

**Theorem 3.4** (Gabriel's theorem). *An hereditary algebra  $\mathbf{k}Q$  is of finite representation type if and only if  $Q$  is an orientation of an ADE diagram, i.e. the underlying graph of  $Q$  is of one of the following forms:*







**Example 3.5.** Let  $Q$  be the quiver  $1 \rightrightarrows 2$ . The indecomposable representations over  $kQ$  are of the following form:

$$\mathbf{k}^n \begin{array}{c} \xrightarrow{1} \\ \xrightarrow{J_{n,\lambda}} \end{array} \mathbf{k}^n, \quad \mathbf{k}^n \begin{array}{c} \xrightarrow{J_{n,0}} \\ \xrightarrow{1} \end{array} \mathbf{k}^n, \quad \mathbf{k}^{n+1} \begin{array}{c} \xrightarrow{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} \\ \xrightarrow{\quad \quad} \end{array} \mathbf{k}^n, \quad \mathbf{k}^n \begin{array}{c} \xrightarrow{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} \\ \xrightarrow{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} \end{array} \mathbf{k}^{n+1},$$

where  $n > 0$ , and  $J_{n,\lambda}$  denotes the nilpotent  $n \times n$  Jordan block corresponding to the eigenvalue  $\lambda \in k$ .

**Example 3.6.** The path algebra  $kQ$  associated to the quiver:

- $1 \longrightarrow 2$  is of finite type.
- $1 \rightrightarrows 2$  is of tame type.
- $1 \begin{array}{c} \rightrightarrows \\ \rightrightarrows \\ \rightrightarrows \end{array} 2$  is of wild type.

#### 4. AUSLANDER-REITEN THEORY

In this section we give a brief overview of Auslander-Reiten (AR) theory, giving the basic concepts and main results in order to define the AR-quiver and describe the knitting algorithm, which provides a method to construct the AR-quiver of the finite-representation hereditary algebras.

#### 4.1. (Short) exact sequences and extensions.

**Definition 4.1.** Let  $A$  be a finite dimensional algebra.

A sequence of morphisms  $\cdots \longrightarrow M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \xrightarrow{f_3} \cdots$  in  $\text{mod}(A)$  is *exact* if  $\text{im} f_i = \ker f_{i+1}$ , for all  $i$ .

A *short exact sequence* (s.e.s. for short) is an exact sequence of the form

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0 .$$

In other words,  $f$  is injective,  $g$  is surjective and  $\text{im} f = \ker g$ . This is also called an *extension of  $N$  by  $L$* .

Note that, in an exact sequence, we have  $f_{i+1}f_i = 0$ , for all  $i$ .

**Example 4.2.** (1) Given a morphism  $f : M \rightarrow N$  of  $A$ -modules, the sequence

$$0 \longrightarrow \ker f \xrightarrow{i} M \xrightarrow{f} N \xrightarrow{p} \text{coker} f \longrightarrow 0 ,$$

where  $i$  is the inclusion and  $p$  is the projection, is exact, and

$$0 \longrightarrow \ker f \xrightarrow{i} M \xrightarrow{p} M/\ker f \longrightarrow 0$$

is short exact.

(2) Let  $Q$  be the quiver  $1 \longrightarrow 2$ , and consider the representations  $S(2) := 0 \longrightarrow \mathbf{k}$ ,  $M := \mathbf{k} \xrightarrow{1} \mathbf{k}$  and  $S(1) := \mathbf{k} \longrightarrow 0$ . Then

$$0 \longrightarrow S(2) \xrightarrow{(0,1)} M \xrightarrow{(1,0)} S(1) \longrightarrow 0 , \text{ and}$$

$$0 \longrightarrow S(2) \xrightarrow{(0,1)} S(1) \oplus S(2) \xrightarrow{(1,0)} S(1) \longrightarrow 0$$

are short exact sequences.

The following lemma, known as the splitting lemma, holds for any abelian category (see [3, Definition A.1.5] for the definition of abelian category).

**Lemma 4.3.** Given a s.e.s.  $0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$  in  $\text{mod}(A)$ , the following statements are equivalent:

(1)  $f$  is a split monomorphism (also called a section), i.e. there exists  $h : M \rightarrow L$  such that  $hf = 1_L$ ,

(2)  $g$  is a split epimorphism (also called a retraction), i.e. there exists  $h' : N \rightarrow M$  such that  $gh' = 1_N$ ,

(3) The sequence is equivalent to the s.e.s.  $0 \longrightarrow L \xrightarrow{i} L \oplus N \xrightarrow{p} N \longrightarrow 0$ , i.e. there is a commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & L & \xrightarrow{f} & M & \xrightarrow{g} & N \longrightarrow 0 \\ & & \parallel & & \downarrow \cong & & \parallel \\ 0 & \longrightarrow & L & \xrightarrow{i} & L \oplus N & \xrightarrow{p} & N \longrightarrow 0 . \end{array}$$

In this case, the s.e.s. is said to split.

The set of equivalence classes  $\text{Ext}^1(N, L)$  of extensions of  $N$  by  $L$ , with the equivalence relation defined in Lemma 4.3 (3), is an abelian group, whose zero element is the class of the split extension.

**Example 4.4.** Let  $Q$  be the quiver  $1 \rightrightarrows 2$ . The sequences

$$0 \longrightarrow S(2) \longrightarrow E \longrightarrow S(1) \longrightarrow 0, \text{ and}$$

$$0 \longrightarrow S(2) \longrightarrow E' \longrightarrow S(1) \longrightarrow 0,$$

where  $S(1) = \mathbf{k} \rightrightarrows 0$ ,  $S(2) = 0 \rightrightarrows \mathbf{k}$ ,  $E = \mathbf{k} \begin{smallmatrix} 1 \\ \longrightarrow \\ 0 \end{smallmatrix} \mathbf{k}$  and  $E' = \mathbf{k} \begin{smallmatrix} 0 \\ \longrightarrow \\ 1 \end{smallmatrix} \mathbf{k}$ , are non-equivalent short exact sequences.

**4.2. Simple, projective and injective representations.** Let  $A = \mathbf{k}Q/I$ , with  $Q$  a finite, connected quiver, and  $I$  an admissible ideal.

A *simple  $A$ -module* is a non-zero module that has no proper submodules.

**Proposition 4.5.** *The simple representations of  $(Q, I)$  are, up to isomorphism, of the form  $S(i) = (S(i)_j, \varphi_\alpha)$ , for each  $i \in Q_0$ , where  $\varphi_\alpha = 0$  for all  $\alpha \in Q_1$  and*

$$S(i)_j = \begin{cases} \mathbf{k} & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}.$$

An  $A$ -module  $P$  is *projective* if any s.e.s. ending at  $P$  splits, i.e.  $\text{Ext}^1(P, -) = 0^1$ . An  $A$ -module  $I$  is *injective* if any s.e.s. starting at  $I$  splits, i.e.  $\text{Ext}^1(-, I) = 0$ .

**Remark 4.6.**

- (1)  $P$  is projective if and only if for every epimorphism  $f : M \rightarrow N$  and every morphism  $g : P \rightarrow N$ , there is  $g' : P \rightarrow M$  such that  $g = fg'$ . In other words,  $\text{Hom}(P, -)^1$  maps surjective morphisms to surjective morphisms.
- (2)  $I$  is injective if and only if for every monomorphism  $u : L \rightarrow M$  and every morphism  $g : L \rightarrow I$ , there is  $g' : M \rightarrow I$  such that  $g = g'u$ . In other words,  $\text{Hom}(-, I)$  maps injective morphisms to surjective morphisms.

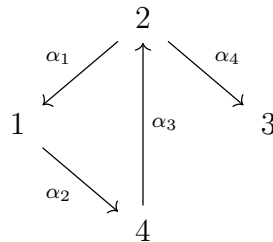
**Proposition 4.7.** *The projective representations of  $(Q, I)$  are, up to isomorphism, of the form  $P(i) = (P(i)_j, \varphi_\alpha)$ , for each  $i \in Q_0$ , where*

- $P(i)_j$  is the vector space generated by  $\{p + I \mid p \text{ path from } i \text{ to } j\}$ ,
- Given an arrow  $\alpha : j \rightarrow \ell$ ,  $\varphi_\alpha : P(i)_j \rightarrow P(i)_\ell$  is the linear map defined on the basis by composing the paths from  $i$  to  $j$  with the arrow  $\alpha$ .

Similarly, the injective representations of  $(Q, I)$  are, up to isomorphism, of the form  $I(i) = (I(i)_j, \varphi_\alpha)$ , for each  $i \in Q_0$ , where

- $I(i)_j$  is the vector space generated by  $\{p + I \mid p \text{ path from } j \text{ to } i\}$ ,
- Given an arrow  $\alpha : j \rightarrow \ell$ ,  $\varphi_\alpha : P(i)_j \rightarrow P(i)_\ell$  is the linear map defined on the basis by deleting the arrow  $\alpha$  from the paths from  $j$  to  $i$  which start with  $\alpha$  and sending to zero the remaining paths.

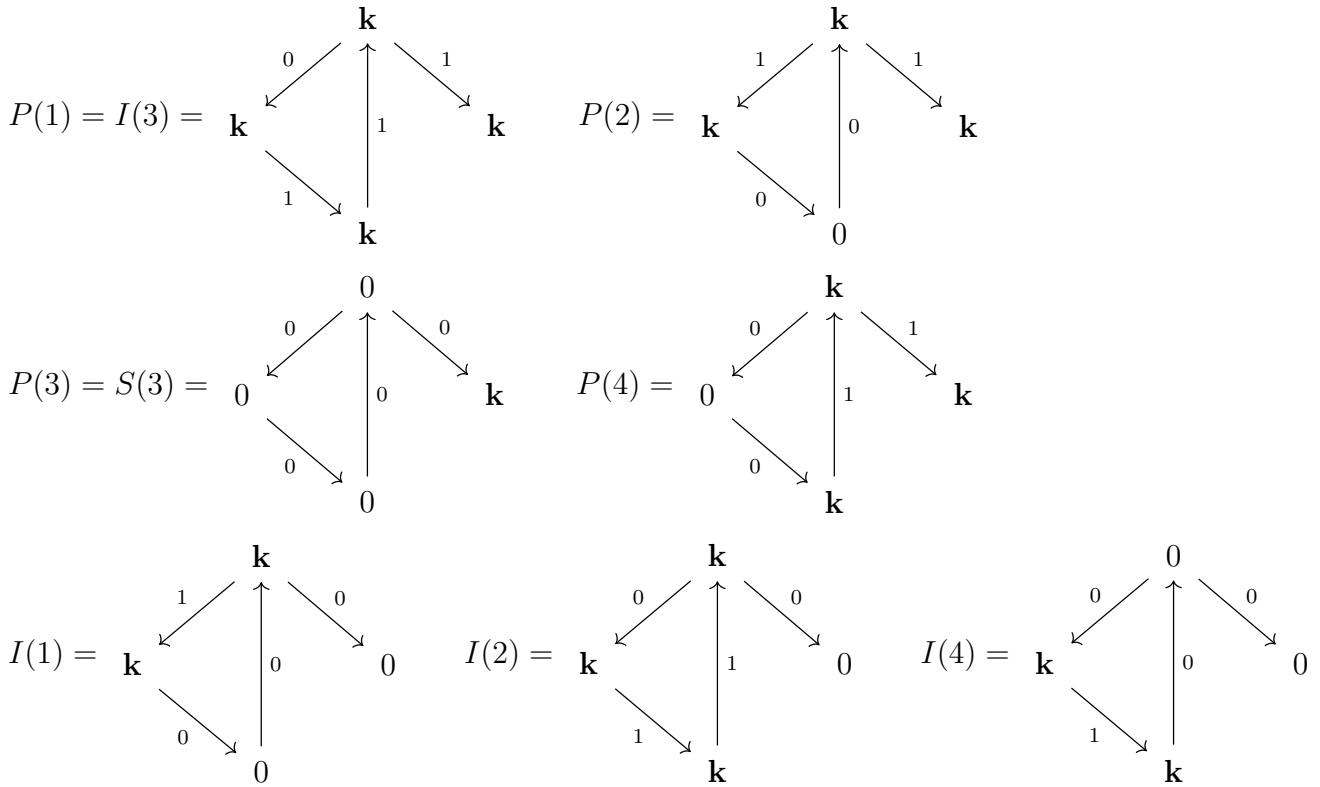
**Example 4.8.** Consider the algebra given by the quiver



<sup>1</sup>The reader can find the definition and basic results on Hom and Ext functors both in [3] and [24].

subject to the relations  $\alpha_3\alpha_1 = \alpha_1\alpha_2 = 0$ .

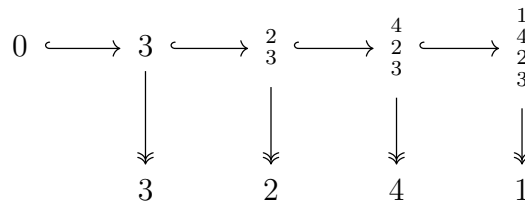
The projective and injective representations are as follows:



**Notation:** We will simplify the notation of an indecomposable representation, by encoding their composition series. For instance, the projective module  $P(1)$  in the example above can be denoted by:

$$P(1) = \begin{matrix} 1 \\ 4 \\ 2 \\ 3 \end{matrix}$$

meaning  $P(1)_i = \mathbf{k}$ , for all  $i \in Q_0$ , and there is an identity map from top to bottom, i.e.  $(P(1))_1 \xrightarrow{1} (P(1))_4 \xrightarrow{1} (P(1))_2 \xrightarrow{1} (P(1))_3$ . This module has a unique composition series given by:



**Theorem 4.9.** Given a representation  $M = (M_i, \varphi_\alpha)$  in  $\text{rep}(Q, I)$ , we have for all  $i \in Q_0$ :

$$\text{Hom}(P(i), M) \simeq M_i \simeq \text{Hom}(M, I(i)).$$

For a proof, see for instance [3, Lemma III.2.11].

**Corollary 4.10.** If  $0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$  is a s.e.s. in  $\text{rep}(Q, I)$ , then

$$\underline{\dim} M = \underline{\dim} L + \underline{\dim} N.$$

### 4.3. Irreducible morphisms and AR-sequences.

**Definition 4.11.** A morphism  $f : M \rightarrow N$  is *irreducible* if:

- $f$  is not a split monomorphism,
- $f$  is not a split epimorphism, and
- if  $f = gh$ , then  $h$  is a split monomorphism or  $g$  is a split epimorphism.

It is easy to check that an irreducible morphism is either injective or surjective, but not both.

**Lemma 4.12.** *A morphism is irreducible if and only if it admits no nontrivial factorisation.*

**Example 4.13.** Let  $Q$  be the quiver  $1 \longrightarrow 2 \longrightarrow 3$ . The map  $S(3) \xrightarrow{(0,0,1)} P(2)$  is irreducible. But the map  $S(3) \xrightarrow{(0,0,1)} P(1)$  is not irreducible as it factors nontrivially through  $P(2)$ .

**Definition 4.14.** A s.e.s.  $0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$  is an *AR-sequence* if the following conditions hold:

- (1)  $L, N$  are indecomposable;
- (2)  $f, g$  are irreducible morphisms.

**Remark 4.15.** An AR-sequence is also known as an *almost-split sequence*, in the sense that any map  $u : L \rightarrow U$  which is not a split monomorphism (resp. any map  $v : V \rightarrow N$  which is not split epimorphism) factors through  $f$  (resp.  $g$ ).

**Remark 4.16.**

- (1) An AR-sequence never splits. Therefore, no AR-sequence starts with an injective module or ends with a projective module.
- (2) An AR-sequence is uniquely determined, up to isomorphism, by each of its end terms.

**Theorem 4.17** (Auslander-Reiten theorem). *Let  $M$  be an indecomposable  $A$ -module.*

- (1) *If  $M$  is non-projective, there is an AR-sequence  $0 \longrightarrow \tau M \xrightarrow{f} E \xrightarrow{g} M \longrightarrow 0$  ending at  $M$ .*
- (2) *If  $M$  is non-injective, there is an AR-sequence  $0 \longrightarrow M \xrightarrow{f} E' \xrightarrow{g} \tau^{-1}M \longrightarrow 0$  starting at  $M$ .*

The module  $\tau M$  is called the *AR-translate* of  $M$ , and  $\tau^{-1}M$  is the *inverse AR-translate* of  $M$ .

We recommend [3, Section IV] for a proof of Theorem 4.17. Key tools in this proof are the *AR-formulas*, which describe the relationship between morphisms and extensions. Namely, for any pair of modules  $M, N \in \text{mod}(A)$ , we have:

$$\text{Ext}^1(M, N) \cong D \underline{\text{Hom}}(\tau^{-1}N, M) \cong D \overline{\text{Hom}}(N, \tau M).$$

Here,  $D$  is the standard  $\mathbf{k}$ -duality  $\text{Hom}_{\mathbf{k}}(-, \mathbf{k})$ ,  $\tau^{-1}I = 0$ , for all injective module  $I$ ,  $\tau P = 0$  for all projective module  $P$ , and the underlining (resp. overlining) means we are considering morphisms which do not factor through projective (resp. injective) modules.

When  $A$  is an hereditary algebra, the AR-formulas can be simplified to

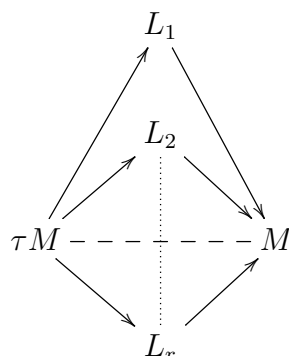
$$\text{Ext}^1(M, N) \cong D \text{Hom}(\tau^{-1}N, M) \cong D \text{Hom}(N, \tau M).$$

**4.4. The AR-quiver and the knitting algorithm.** Given a finite dimensional algebra  $A$ , we can record the information about  $\text{mod}(A)$  in a quiver, called the AR-quiver. In the case when  $A$  is of finite representation type, this quiver gives a complete picture of the representation theory of  $A$ .

**Definition 4.18.** The AR-quiver  $\Gamma(\text{mod}(A))$  of  $\text{mod}(A)$  is defined by:

- the vertices of  $\Gamma(\text{mod}(A))$  are the isomorphism classes of indecomposable  $A$ -modules,
- the arrows are the irreducible morphisms between the indecomposable modules.

Each AR-sequence  $0 \longrightarrow \tau M \longrightarrow L_1 \oplus \cdots \oplus L_r \longrightarrow M \longrightarrow 0$  is represented in the AR-quiver by a *mesh*:



The AR-quiver is a *translation quiver*, i.e. for each arrow  $M \rightarrow L$ , for which  $\tau^{-1}M \neq 0$  (resp.  $\tau L \neq 0$ ), there is an arrow  $L \rightarrow \tau^{-1}M$  (resp.  $\tau L \rightarrow M$ ).

The *knitting algorithm* is an algorithm that allows us to construct, in some special cases, the AR-quiver (or part thereof). One of these special cases is when  $A = \mathbf{k}Q$ , where the underlying graph of  $Q$  is ADE. It owes its name to the fact that it recursively constructs one mesh after the other, from left to right.

What follows is a description of this algorithm. We start by computing all the projective modules and their radicals.

The radical  $\text{rad}(M)$  of a module  $M$  is the intersection of all maximal submodules of  $M$ . The representation  $(P(i)'_j, \varphi'_\alpha)$  corresponding to the radical  $\text{rad}(P(i))$  of the projective  $P(i) = (P(i)_j, \varphi_\alpha)$  at  $i$  is such that  $P(i)'_j = P(i)_j$  if  $i \neq j$ ,  $P(i)_i$  is the vector space spanned by all nonconstant paths from  $i$  to  $i$ , and  $\varphi'_\alpha$  is the restriction of  $\varphi_\alpha$  to  $P(i)_{s(\alpha)}$ .

**Proposition 4.19.** *Every direct predecessor of  $P(i)$  in  $\Gamma(\text{mod}(A))$ , i.e. every indecomposable module  $X$  for which there is an irreducible morphism  $X \rightarrow P(i)$ , is a direct summand of  $\text{rad}(P(i))$ . In the case when  $A$  is hereditary, all predecessors of projective modules are projective modules.*

**Base step:**

- (1) Draw a vertex for each simple projective  $P(i)$ .
- (2) If  $P(i)$  is a summand of  $\text{rad}(P)$  for some projective  $P$ , then add a vertex corresponding to  $P$  and arrows from  $P(i)$  to  $P$  (the number of arrows equals the multiplicity of  $P(i)$  in  $\text{rad}(P)$ ).
- (3) Add vertices associated to remaining summands  $R$  of  $\text{rad}(P)$  and arrows  $R \rightarrow P$ .
- (4) Repeat previous steps for each  $R$ .

At this point we get a quiver  $\Delta_0$ .

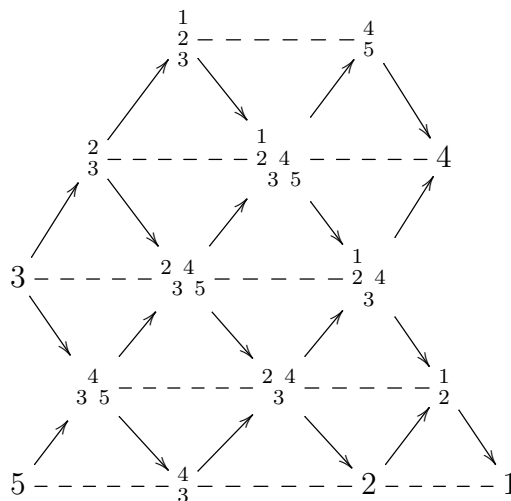
**Induction  $\Delta_n$  from  $\Delta_{n-1}$ :**

If  $X \in \Delta_{n-1}$  and all its direct predecessors are in  $\Delta_{n-1}$ , then:

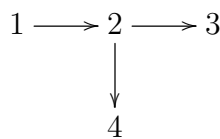
- (1) if  $X$  is a direct summand of  $\text{rad}(Q)$  for some projective  $Q$ , add a vertex associated to  $Q$  and arrows  $X \rightarrow Q$ .
- (2) if  $X$  is not injective, add a vertex corresponding to  $\tau^{-1}X$  and for each arrow  $X \rightarrow Y$ , add  $Y \rightarrow \tau^{-1}X$ .
- (3) Continue this procedure, and stop if you get negative integers in the dimension vector, or you get all the injectives.

If  $A$  is hereditary of finite representation type, it is known that each indecomposable  $A$ -module is uniquely determined by its dimension vector. Therefore, in order to calculate  $\tau^{-1}X$  in the knitting algorithm, one can simply use the formula  $\underline{\dim}\tau^{-1}X = \sum_{X \rightarrow Y} \underline{\dim}Y - \underline{\dim}X$ , by Corollary 4.10.

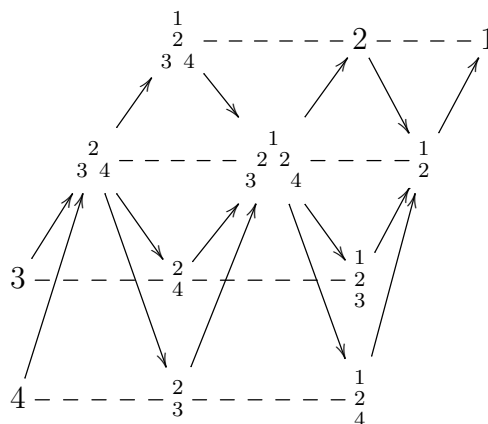
**Example 4.20.** Let  $Q$  be the quiver  $1 \longrightarrow 2 \longrightarrow 3 \longleftarrow 4 \longrightarrow 5$  of type  $A_5$ . The AR-quiver of  $kQ$  is given by:



**Example 4.21.** Let  $Q$  be the following quiver of type  $D_4$ :



The AR-quiver of  $kQ$  is given by:



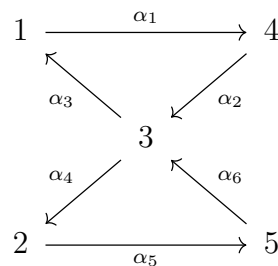
Here we have:

$$\begin{array}{c}
 1 \\
 2 \quad 2 \\
 3 \quad 4
 \end{array}
 =
 \begin{array}{c}
 \mathbf{k} \xrightarrow{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} \mathbf{k}^2 \xrightarrow{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} \mathbf{k} \\
 \downarrow \begin{bmatrix} 0 & 1 \end{bmatrix} \\
 \mathbf{k}
 \end{array}$$

## 5. GEOMETRIC MODELS

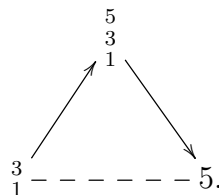
The knitting algorithm might not work when we start with a non-simple projective module.

For instance, consider the quiver  $Q$



together with the admissible ideal  $I = \langle \alpha_1\alpha_2, \alpha_2\alpha_3, \alpha_3\alpha_1, \alpha_4\alpha_5, \alpha_5\alpha_6, \alpha_6\alpha_4 \rangle$ , and let  $A = \mathbf{k}Q/I$ .

Suppose we start the knitting algorithm with  $P(5)$ , whose radical is  $\text{rad}(P(5)) = \begin{smallmatrix} 3 \\ 1 \end{smallmatrix}$ . This module is not the summand of the radical of any other projective module, and so according to the algorithm, we would knit the following mesh:



However, this mesh is not correct; the algorithm did not compute the irreducible morphism  $\begin{smallmatrix} 5 \\ 3 \\ 1 \end{smallmatrix} \rightarrow 3$ .

This section is devoted to a different way of computing the AR-quiver of certain classes of algebras, using the geometry of Riemann surfaces with boundary.

**5.1. Geometric model of type  $A_n$ .** We start by illustrating how to construct the AR-quiver of an hereditary algebra of type  $A_n$ , with the example

$$Q = 1 \longrightarrow 2 \longrightarrow 3 \longleftarrow 4 \longrightarrow 5 .$$

Consider a disc with  $8(= n + 3)$  marked points on its boundary, together with the triangulation  $T$ , i.e. maximal set of non-crossing diagonals, given in Figure 1.

Before associating an algebra to this data, we need to introduce some terminology and notation.

We call a curve  $\gamma$  in the marked disc (or any marked surface) an *arc* if it satisfies the following properties:

- The endpoints of  $\gamma$  are marked points on the boundary.
- $\gamma$  intersects the boundary of the surface only in its endpoints.
- $\gamma$  does not cut out a monogon or a digon.



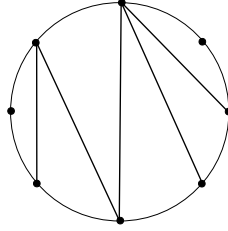


FIGURE 1. A triangulation of an octagon.

Given a marked point  $p$ , let  $m', m''$  be two points in the same boundary component of  $p$  such that  $m', m''$  are not marked points and  $p$  is the only marked point lying in the boundary segment  $\delta$  between  $m'$  and  $m''$ . Draw a curve  $c$  homotopic to  $\delta$  but lying in the interior of the disc except for its endpoints  $m'$  and  $m''$ . The *complete fan at  $p$*  is the sequence of diagonals in  $T$  which  $c$  crosses in the clockwise order.

We can now associate a quiver  $Q_T$  to this triangulation, in the following way:

- vertices of  $Q_T$  are in one to one correspondence with diagonals of  $T$ . We will use the same notation for both.
- Given two vertices  $i$  and  $j$ , there is an arrow  $i \rightarrow j$  if and only if  $i$  and  $j$  share a marked point  $p$  and  $j$  is the immediate successor of  $i$  in the complete fan at  $p$ .

Note that we can associate a marked point to each arrow of  $Q_T$ . Namely, using the notation above, the marked point associated to the arrow  $i \rightarrow j$  is  $p$ .

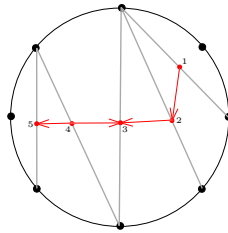


FIGURE 2. The quiver  $Q_T$  of the triangulation.

The quiver  $Q_T$  in Figure 2 is indeed  $Q$ , and in fact one can obtain any orientation of a Dynkin graph of type  $A_n$  from a triangulation of a disc with  $n + 3$  marked points on the boundary whose triangles are *outer-triangles*, i.e. triangles with at least one side on the boundary of the disc.

We will now describe how to obtain the AR-quiver of  $\mathbf{k}Q$  from this triangulation.

We will always consider arcs up to homotopy relative to their endpoints. Given an arc  $\gamma$  distinct from any diagonal of  $T$ , we define a representation  $M_\gamma = (M_i, \varphi_\alpha)$  of  $\mathbf{k}Q_T$ , as follows:

$$M_i = \begin{cases} \mathbf{k} & \text{if } \gamma \text{ crosses diagonal } i \\ 0 & \text{otherwise,} \end{cases} \quad \varphi_\alpha = \begin{cases} 1 & \text{if } M_{s(\alpha)} = M_{t(\alpha)} = \mathbf{k} \\ 0 & \text{otherwise.} \end{cases}$$

Irreducible morphisms correspond to pivoting one of the endpoints of an arc to its counterclockwise neighbour (*pivoting elementary move*). Given an arc  $\gamma$ , we define its translate  $\tau(\gamma)$  to be the arc obtained from  $\gamma$  by rotating both endpoints to their counterclockwise neighbour. In particular,  $M_\gamma = P(i)$  (resp.  $M_\gamma = I(j)$ ) if and only if  $\tau\gamma = i$  (resp.  $\tau^{-1}\gamma = j$ ).

A presentation of the AR-quiver of  $\text{mod}(\mathbf{k}Q_T)$  in terms of these combinatorics is presented in Figure 3

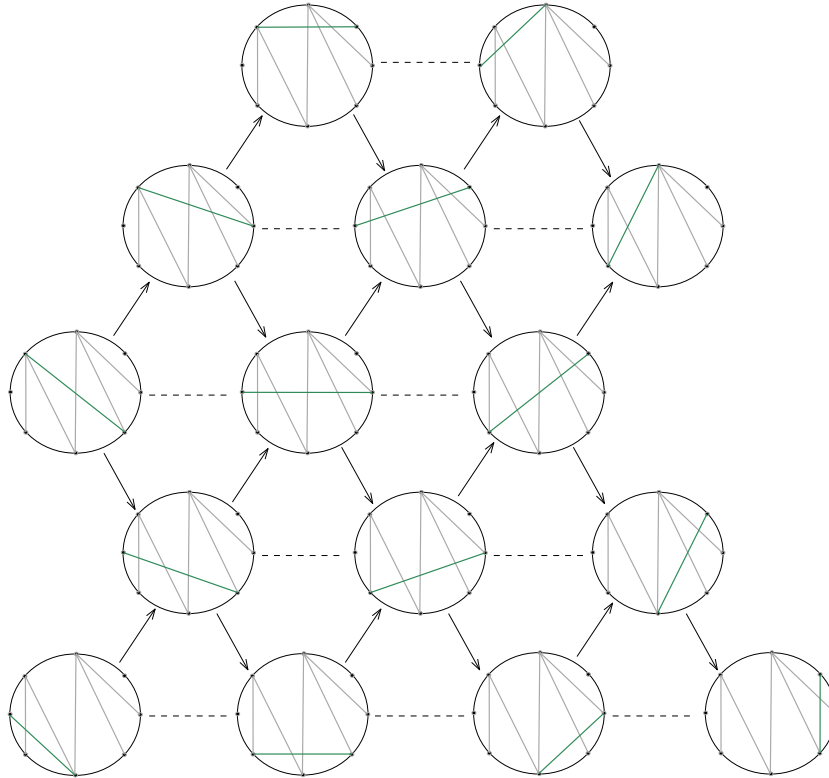


FIGURE 3. The geometric model of the AR-quiver of  $\text{mod}(kQ_T)$ .

Extensions have a nice description in terms of arcs. Indeed, there is an extension from  $N$  to  $M$  if and only if the corresponding arcs  $\gamma_N$  and  $\gamma_M$  cross each other as in Figure 4.

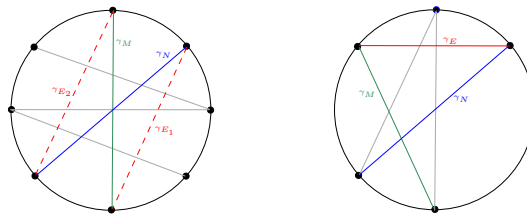


FIGURE 4. Extensions of  $N$  by  $M$  as crossings of  $\gamma_N$  and  $\gamma_M$ .

The summands of the middle term of the extension correspond to the arcs drawn in red in Figure 4.

**5.2. Geometric model for cluster-tilted algebras of type  $A_n$ .** Cluster-tilted algebras arise in the context of cluster-tilting theory. We refer the reader to [5] for a nice survey on this class of algebras.

Cluster-tilted algebras of type  $A_n$  are precisely the algebras associated to an arbitrary triangulation of the  $(n + 3)$ -gon.

An arbitrary triangulation  $T$  may include *inner triangles*, i.e. triangles whose three boundaries are all diagonals of  $T$ . The quiver  $Q_T$  is defined as above, but now we include relations  $\alpha\beta$ , if  $s(\alpha), t(\alpha) = s(\beta), t(\beta)$  are the boundaries of an inner triangle.

The algebra  $A$  at the start of this section is a cluster-tilted algebra of type  $A$ , which can be obtained from the triangulation in Figure 5.

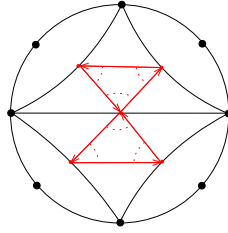


FIGURE 5. The triangulation associated to  $A$ .

Using the same rule for arcs, pivot elementary moves and translates, we are now able to compute the AR-quiver of  $\text{mod}(A)$  in terms of the geometric model (see Figure 6).

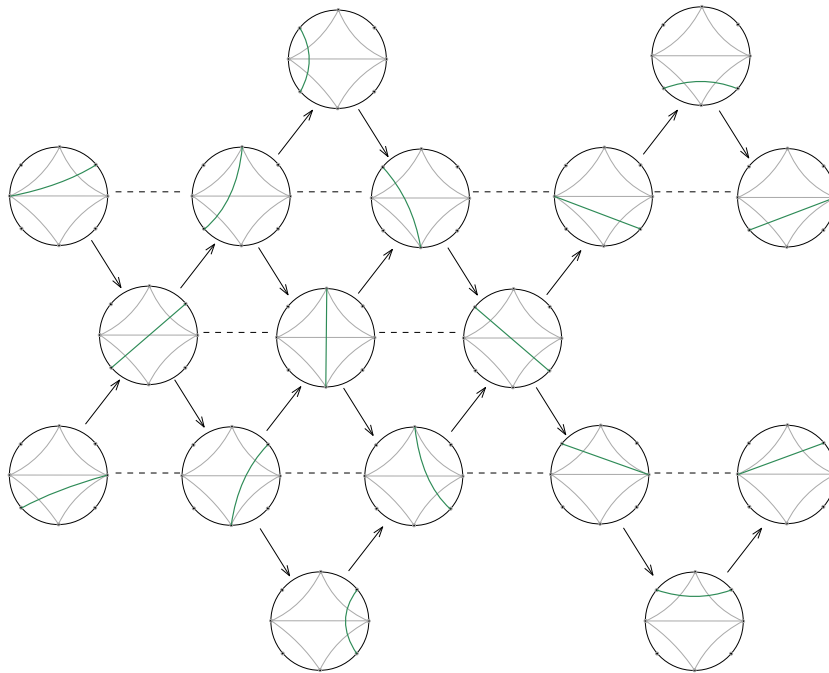


FIGURE 6. The geometric model of the AR-quiver of  $\text{mod}(A)$ .

Note that when we have inner triangles, we can get a new type of crossing, see Figure 7.

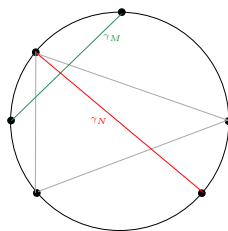


FIGURE 7. Crossing associated to an inner triangle.

However, this type of crossing does not give rise to an extension, and so all extensions are described in the same way as we have seen above. For more details see [13].

5.3. **Geometric model for gentle algebras.** We will now consider two possible generalisations of this combinatorial construction: on the one hand we can consider partial triangulations instead (i.e. any set of non-crossing diagonals), and on the other hand we can consider other surfaces.

Let  $C$  be the bound path algebra given by  $1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$  bound by  $\alpha\beta$ . This algebra can be obtained from the partial triangulation of a disc in Figure 8.

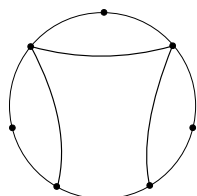


FIGURE 8. The partial triangulation of the algebra  $C$ .

The quiver is obtained in the same way as before. The relations are given by composition of two arrows in the same region. Note that this rule applied to an arbitrary triangulation of the disc gives rise to the same rule described in the previous subsection.

For partial triangulations, not every arc gives rise to an indecomposable module and two different arcs may give rise to the same indecomposable module. Therefore, we need to define *permissible arcs* and *equivalence of arcs* (this is not the same as homotopy).

An arc is permissible if each consecutive crossing corresponds to an arrow in the quiver. See Figure 9 for a counter-example.

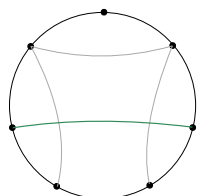


FIGURE 9. An arc which is not permissible

Two arcs are isomorphic if they intersect the same diagonals of the partial triangulation (see Figure 10).

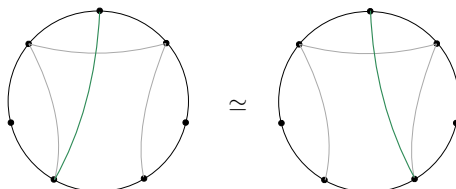


FIGURE 10. Isomorphic permissible arcs.

Indecomposable modules are therefore in bijection with equivalence classes of permissible arcs.

If we perform a pivot elementary move as described in the previous subsections, we may get an isomorphic arc. Hence, an irreducible morphism corresponds to a sequence of pivot elementary moves until one gets a non-equivalent arc.

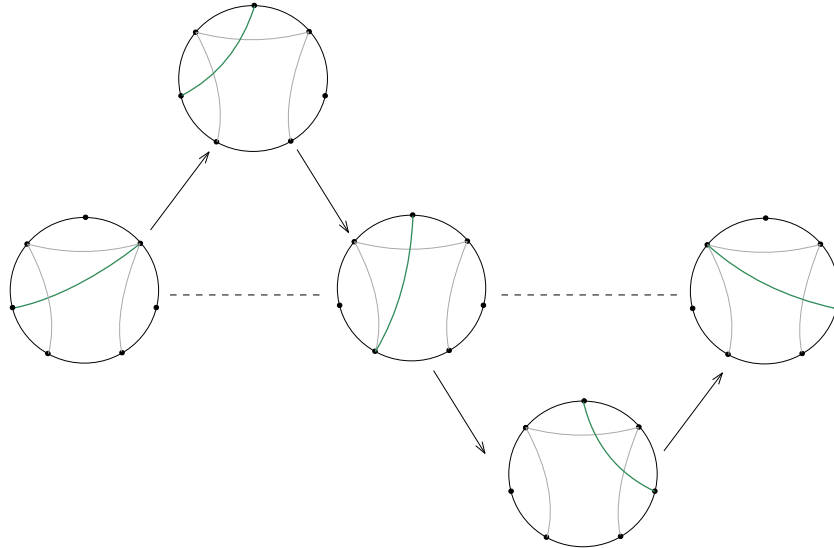


FIGURE 11. The geometric model of the AR-quiver of  $\text{mod}(C)$ .

The AR-quiver of  $\text{mod}(C)$  is given in Figure 11.

Now, let us consider an example coming from an annulus (see Figure 12).

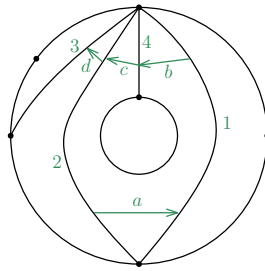
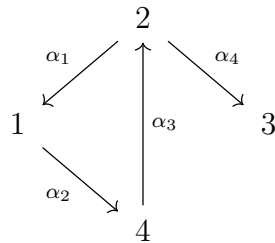


FIGURE 12. A partial triangulation in an annulus and corresponding quiver.

The quiver of the algebra  $D$  associated to this partial triangulation is defined as previously. But we refine the definition of relations as follows: the composition of two arrows with different marked points is zero and if  $\alpha$  is a loop, i.e. its start and endpoints correspond to a loop arc of the partial triangulation, then  $\alpha^2 = 0$ .

The algebra  $D$  is then



bound by the relations  $\alpha_3\alpha_1 = \alpha_1\alpha_2 = 0$ . By refining the notions of permissible arcs, equivalence of arcs, and pivot elementary moves, we get the AR-quiver of  $D$  as in Figure 13.

An algebra associated to an unpunctured surface with a finite set of marked points on the boundary is called a *tiling algebra*. It turns out that these algebras are precisely gentle algebras.

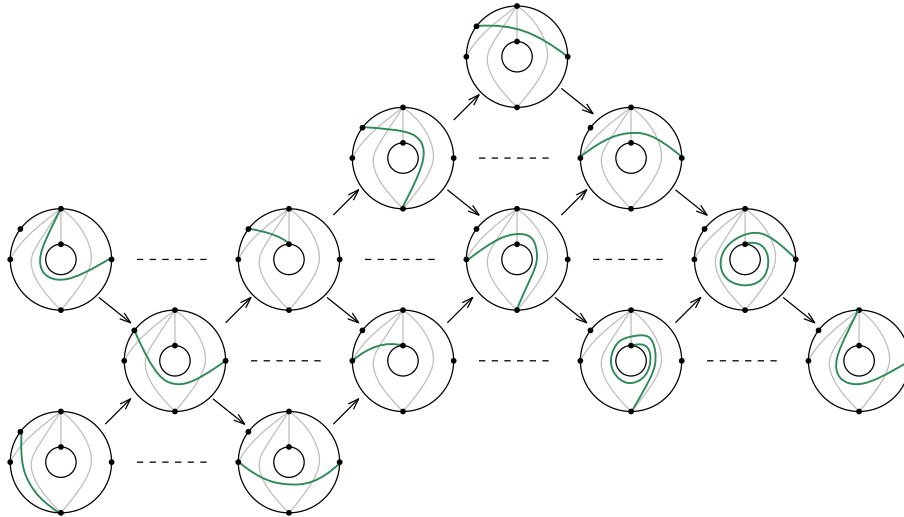


FIGURE 13.  $\Gamma(\text{mod}(D))$ .

**Definition 5.1.** A finite dimensional algebra  $A$  is *gentle* if it admits a presentation  $A = \mathbb{k}Q/I$  satisfying the following conditions:

- (1) Each vertex of  $Q$  is the source of at most two arrows and the target of at most two arrows.
- (2) For each arrow  $\alpha$  in  $Q$ , there is at most one arrow  $\beta$  in  $Q$  such that  $\alpha\beta \notin I$ , and there is at most one arrow  $\gamma$  such that  $\gamma\alpha \notin I$ .
- (3) For each arrow  $\alpha$  in  $Q$ , there is at most one arrow  $\delta$  in  $Q$  such that  $\alpha\delta \in I$ , and there is at most one arrow  $\mu$  such that  $\mu\alpha \in I$ .
- (4)  $I$  is generated by paths of length 2.

Gentle algebras first appeared in the context of tilting theory [4] (see also [3, Section IX]), where iterated tilted algebras of types  $A$  and  $\tilde{A}$  were observed to satisfy the properties above. Gentle algebras, which are tame, remain one of the relatively few classes of algebras for which the representation theory is computationally tractable. Partly due to this reason, there has been widespread interest in this class of algebras in many different contexts, such as Fukaya categories [17], dimer models [8], enveloping algebras of Lie algebras [18] and cluster theory [2, 16, 19]. We refer the reader to [7, 9, 12, 13, 21] for examples of recent developments in this area.

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