

ON THE NILRADICAL OF A LEIBNIZ ALGEBRA

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Abstract

The purpose of this short note is to correct an error which appears in the literature concerning Leibniz algebras L : namely, that $N(L/I) = N(L)/I$ where $N(L)$ is the nilradical of L and I is the Leibniz kernel. *Mathematics Subject Classification 2000*: 17B05, 17B20, 17B30, 17B50. *Key Words and Phrases*: Leibniz algebras, nilradical, radical, Frattini ideal, Leibniz kernel.

1 Introduction

An algebra L over a field F is called a *Leibniz algebra* if, for every $x, y, z \in L$, we have

$$[x, [y, z]] = [[x, y], z] - [[x, z], y]$$

In other words, the right multiplication operator $R_x : L \rightarrow L : y \mapsto [y, x]$ is a derivation of L . As a result, such algebras are sometimes called *right* Leibniz algebras, and there is a corresponding notion of *left* Leibniz algebra. Clearly, the opposite of a right Leibniz algebra is a left Leibniz algebra so,

for our purposes, it does not matter which is used. Every Lie algebra is a Leibniz algebra and every Leibniz algebra satisfying $[x, x] = 0$ for every element is a Lie algebra.

Put $I = \text{span}\{x^2 : x \in L\}$. Then I is an ideal of L and L/I is a Lie algebra called the *liesation* of L . We define the following series:

$$L^1 = L, L^{k+1} = [L^k, L] \text{ and } L^{(1)} = L, L^{(k+1)} = [L^{(k)}, L^{(k)}] \text{ for all } k = 2, 3, \dots$$

Then L is *nilpotent* (resp. *solvable*) if $L^n = 0$ (resp. $L^{(n)} = 0$) for some $n \in \mathbb{N}$. The *nilradical*, $N(L)$, (resp. *radical*, $R(L)$) is the largest nilpotent (resp. solvable) ideal of L .

In [10] it is claimed that $R(L/I) = R(L)/I$ and $N(L/I) = N(L)/I$. However, whereas the former is clearly true, the latter is false in general. The claim appears in the proof of [10, Proposition 4], which has two corollaries. Although this paper appears only to have been published on arxiv it has been quite widely cited. Moreover, the results following from this assertion have been quoted in [8, Proposition 4.4], [12, Propositions 3.3, 3.4 and Corollaries 3.5, 3.6] and referenced to [10], and the same incorrect assertion is used to prove two results in a recent book ([3, Proposition 2.1 and 2.2]). The results which Gorbatsevich uses this assertion to prove are true, as are [3, Proposition 2.1 and 2.2]. The purpose of this short note is to give a correct characterisation of $N(L/I)$ and to provide or reference proofs for the results in which the incorrect assertion is used.

Throughout, L will denote a finite-dimensional (right) Leibniz algebra over a field F . The *Frattini ideal* of L is the largest ideal of L contained in every maximal subalgebra of L . We will denote algebra direct sums by \oplus and vector space direct sums by $\dot{+}$.

2 The nilradical

The literature concerning Leibniz algebras is quite diverse and a number of results have been duplicated, so the survey articles [9], [10] and the new book [3] are useful. The nilradical of a Leibniz algebra is a well-defined object.

Theorem 2.1 *The sum of two nilpotent ideals of a Leibniz algebra is nilpotent.*

Proof. See [9, Theorem 5.14] or [5, Lemma 1.5]. \square

Corollary 2.2 *Any Leibniz algebra has a maximal nilpotent ideal containing all nilpotent ideals of L .*

Proof. A valid proof of this can also be found as [7, Corollary 4]. Note that the proof given in [10, Proposition 1] and [3, Proposition 2.1] is incorrect. \square

Next, we show that $N(L/I) \neq N(L)/I$ in general.

EXAMPLE 2.1 *Let $L = Fx + Fx^2$ where $[x^2, x] = x^2$ is the two-dimensional solvable cyclic Leibniz algebra, we have $I = Fx^2 = N(L)$ and L/I is the nilradical of L/I .*

The fact that $N(L) = I$ is not significant in the above example as the following class of algebras shows.

EXAMPLE 2.2 *Let $L = Fx_1 + \dots + Fx_r + Fx_{r+1} + \dots + Fx_n + Fy$ where $[x_i, y] = x_i$ for $r + 1 \leq i \leq n$ and all other products are zero. Then $I = Fx_{r+1} + \dots + Fx_n$, $N(L) = Fx_1 + \dots + Fx_n$ and $N(L/I) = L/I$.*

Nor is the fact that $N(L/I) = L/I$ significant in the above examples as can be seen by taking the direct sum of them with a simple Lie algebra.

Next we look for more information on $N(L/I)$. First note the following.

Lemma 2.3 *If $I \subseteq \phi(L)$ then $N(L/I) = N(L)/I$.*

Proof. Clearly $N(L)/I \subseteq N(L/I) = K/I$, say. But K is nilpotent, by [4, Theorem 5.5]. \square

Lemma 2.4 *If $I \not\subseteq \phi(L)$ then there is a subalgebra B of L such that $L = I + B$ and $I \cap B \subseteq \phi(B)$.*

Proof. This is [13, Lemma 7.1]. \square

Then, using the same notation as in Lemma 2.4, we have the following.

Theorem 2.5 *The nilradical of L/I is $N(L/I) = (I + N(B))/I$ and this is the same as $N(L)/I$ if and only if $R_n|_I$ is nilpotent for all $n \in N(B)$.*

Proof. We have

$$N\left(\frac{L}{I}\right) \cong N\left(\frac{B}{I \cap B}\right) = \frac{N(B)}{I \cap B} = \frac{N(B)}{I \cap N(B)} \cong \frac{I + N(B)}{I}.$$

But $(I + N(B))/I \subseteq N(L/I)$, so equality results.

Now, $I + N(B) = N(L)$ if and only if $N(B)$ acts nilpotently on the right on I . \square

3 Some results where $N(L/I) = N(L)/I$ were used

First we have the following analogue of a well-known result for Lie algebras. The only references to this in the literature of which we are aware are [10, Proposition 4] and [3, Proposition 2.2]. However, the proof in each case is incorrect, though the result is true.

Proposition 3.1 *Let L be a Leibniz algebra over a field of characteristic zero, R be its radical and N its nilradical. Then $[L, R] \subseteq N$.*

Proof. Since $N(L/\phi(L)) = N(L)/\phi(L)$ we can assume that L is ϕ -free. Then $L = \text{Asoc}(L) + V$, where $V = S \oplus Z(V)$ and S is a semisimple Lie algebra, by [6, Corollary 2.9]. Also $R = \text{Asoc}(L) + Z(V)$ and $N = \text{Asoc}(L)$, by [6, Theorem 2.4], from which the result is clear. \square

The above result has the following corollary which appears in several places in the literature. It occurs as [10, Corollaries 5 and 6] and [3, Corollaries 2.2 and 2.3] but the proofs are incorrect. It appears with correct proofs as [9, Corollary 6.8], [11, Corollary 3], [2, Theorem 4], [1, Theorem 2] and [4, Theorem 2.6].

Corollary 3.2 *([10, Corollary 5]) With same notation as in Proposition 3.1, $[R, R] \subseteq N$; in particular, $[R, R]$ is nilpotent; in fact, L is solvable if and only if $[L, L]$ is nilpotent.*

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