Mathematical
Inequalities
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THE ℓ_p–NORM OF $C-I$, WHERE C IS THE CESÀRO OPERATOR

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Submitted to Math. Inequal. Appl.

Abstract. For the Cesàro operator C, it is known that $\|C-I\|_2 = 1$. Here we prove that $\|C-I\|_4 \le 3^{1/4}$ and $\|C^T - I\|_4 = 3$. Bounds for intermediate values of p are derived from the Riesz-Thorin interpolation theorem. An estimate for lower bounds is obtained.

1. Introduction and basic results

For a matrix operator A, we denote by $\|A\|_p$ the norm of A as an operator on the (real) sequence space ℓ_p. Let C be the Cesàro operator, so that for a sequence $x = (x_n)$, we have $Cx = y$, where

$$y_n = \frac{1}{n}(x_1 + x_2 + \ldots + x_n). \tag{1}$$

For the transpose C^T, we have $C^T x = y$, where

$$y_n = \sum_{k=n}^{\infty} \frac{x_k}{k}. \tag{2}$$

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Hardy's inequality [4, p. 239–241] states that $\|C\|_p = p^*$, where p^* is the conjugate index defined by $\frac{1}{p} + \frac{1}{p^*} = 1$. By duality, this implies that $\|C^T\|_p = p$ (this is known as Copson's inequality).

For $p = 2$, a stronger statement applies: $\|C-I\|_2 = 1$, where I is the identity matrix. This was proved in [3], using the fact that $(C-I)(C^T-I)$ is the diagonal matrix with entries $1 - \frac{1}{n}$, together with the Hilbert space property $\|AA^T\|_2 = \|A\|_2^2$. However, it can also be easily established by a slightly amended version of the direct method of [4]. This proof does not appear to be well known, and we will generalise it below, so we sketch it here.

Proof. We have $x_n = ny_n - (n-1)y_{n-1}$, hence $y_n - x_n = (n-1)(y_{n-1} - y_n)$. For any a, b, it is elementary that $b^2 - a^2 \ge 2a(b-a)$. (Here the proof for general p uses $b^p - a^p \ge pa^{p-1}(b-a)$, valid only for positive a, b.) So $2y_n(y_{n-1} - y_n) \le y_{n-1}^2 - y_n^2$, hence

$$2y_n(y_n - x_n) \le (n-1)(y_{n-1}^2 - y_n^2),$$

Mathematics subject classification (2010): 26D15, 47B37.
Keywords and phrases: Cesàro, Hardy, inequality, average.

equivalently

$$2x_ny_n - y_n^2 \geq ny_n^2 - (n-1)y_{n-1}^2.$$

Adding these inequalities for $1 \leq n \leq N$, we obtain

$$2\sum_{n=1}^{N} x_ny_n - \sum_{n=1}^{N} y_n^2 \geq Ny_N^2 \geq 0.$$

so that

$$\sum_{n=1}^{N} y_n^2 \leq 2\sum_{n=1}^{N} x_ny_n,$$

hence $\sum_{n=1}^{N}(y_n - x_n)^2 \leq \sum_{n=1}^{N} x_n^2$. (At this point, the proof in [4] applies Hölder's inequality.) $\square$

Our objective here is to consider $\|C-I\|_p$ and $\|C^T-I\|_p$ for other values of p. First, some simple facts. By Hardy's inequality and its dual, $p^*-1 \leq \|C-I\|_p \leq p^*+1$ and $p-1 \leq \|C^T-I\|_p \leq p+1$ for all $p \geq 1$. Also, if e_n is the nth unit vector, then for $p > 1$, both $\|Ce_n\|_p$ and $\|C^Te_n\|_p$ tend to 0 as $n \to \infty$, so $\|C-I\|_p$ and $\|C^T-I\|_p$ are not less than 1.

PROPOSITION 1. *We have* $\|C-I\|_\infty = \|C^T-I\|_1 = 2$.

Proof. Consider C^T-I first. The element $(C^T-I)e_n$ is given by column n:

$$(C^T-I)e_n = \left(\frac{1}{n}, \dots, \frac{1}{n}, \frac{1}{n} - 1, 0, 0, \dots\right),$$

in which $\frac{1}{n}$ occurs $n-1$ times. So $\|(C^T-I)e_n\|_1 = 2(1-\frac{1}{n})$, hence $\|C^T-I\|_1 = 2$.

The statement for $C-I$ follows by duality, or directly by taking x to be $e_1+\dots+e_{n-1}-e_n$: then $z_n = 2(1-\frac{1}{n})$. $\square$

Of course, it follows that $\lim_{p\to\infty}\|C-I\|_p = \lim_{p\to 1^+}\|C^T-I\|_p = 2$.

Bounds for intermediate values of p can now be derived from the *Riesz-Thorin interpolation theorem*. In the version we want (not the most general one), this states:

THEOREM RT. *Suppose that* $1 \leq q < r \leq \infty$ *and*

$$\frac{1}{p} = \frac{1-\theta}{q} + \frac{\theta}{r},$$

where $0 < \theta < 1$. *Suppose that* A *maps* ℓ_q *into* ℓ_q *and* ℓ_r *into* ℓ_r. *Then* A *maps* ℓ_p *into* ℓ_p, *and*

$$\|A\|_p \leq \|A\|_q^{1-\theta}\|A\|_r^{\theta}. \tag{3}$$

A proof can be seen in [2, chap. 1]. Note that the case $r = \infty$ simplifies to: if $p > q \geq 1$, then

$$\|A\|_p \leq \|A\|_q^{q/p}\|A\|_\infty^{1-q/p}. \tag{4}$$

An obvious consequence of the theorem is: *if* $\|A\|_p \geq \|A\|_{p_0}$ *for all* $p > p_0$, *then* $\|A\|_p$ *increases with* p *for* $p \geq p_0$.

For $C-I$ and C^T-I, we can deduce at once the following facts.

PROPOSITION 2. *For $p \geq 2$, $\|C-I\|_p$ increases with p and is not greater $2^{1-2/p}$. For $1 \leq p \leq 2$, $\|C^T - I\|_p$ decreases with p and is not greater than $2^{1-2/p^*} = 2^{2/p-1}$.*

We can derive bounds that are weaker, but easier to apply, as follows: by convexity of 2^x, we have $2^x < 1+x$ for $0 < x < 1$. Hence $\|C-I\|_p < \frac{2}{p^*}$ for $p > 2$ and $\|C^T - I\|_p < \frac{2}{p}$ for $1 < p < 2$.

However, the Riesz-Thorin theorem does not give the exact value when applied to C and C^T themselves, and we would not expect it to do so for $C-I$ and C^T-I.

The following conjecture seems plausible:

Conjecture (C): $\|C-I\|_p = p^* - 1 = 1/(p-1)$ for $1 < p \leq 2$, equivalently $\|C^T - I\|_p = p-1$ for $p > 2$.

This conjecture is discussed briefly in [1, p. 48]. After pointing out that the statement $\|C-I\|_p = 1$ for $p > 2$ is easily disproved by considering the p^*-norm of the rows, Bennett states that "similar examples" disprove conjecture (C). I cannot see that this is the case in any simple way, and it seems possible that this may have been an over-hasty remark. Regrettably, Bennett died in 2016, so is not available to elucidate.

2. The case $p = 4$

We now establish estimates for both operators for the case $p = 4$, by developing the method used for $\|C-I\|_2$.

THEOREM 1. *We have $\|C-I\|_4 \leq 3^{1/4}$.*

Proof. Choose $x \in \ell_4$ and let y_n be defined by (1). Then $y_n - x_n = (n-1)(y_{n-1} - y_n)$. By convexity of the function x^4, we have $b^4 - a^4 \geq 4a^3(b-a)$ for any a and b, positive or negative. So $y_{n-1}^4 - y_n^4 \geq 4y_n^3(y_{n-1} - y_n)$, hence

$$4y_n^3(y_n - x_n) \leq (n-1)(y_{n-1}^4 - y_n^4),$$

equivalently

$$4y_n^3 x_n - 3y_n^4 \geq n y_n^4 - (n-1)y_{n-1}^4.$$

Adding for $1 \leq n \leq N$, we obtain

$$4\sum_{n=1}^{N} y_n^3 x_n - 3\sum_{n=1}^{N} y_n^4 \geq N y_N^4 \geq 0.$$

Hence $\sum_{n=1}^N y_n^3(4x_n - 3y_n) \geq 0$. Write $y_n = x_n + z_n$. Then $\sum_{n=1}^N F(x_n, z_n) \geq 0$, where

$$F(x,z) = (x+z)^3(x-3z) = x^4 - 6x^2z^2 - 8xz^3 - 3z^4. \tag{5}$$

To deal with the term $8xz^3$, we use the inequality $-2xz \leq cx^2 + \frac{1}{c}z^2$, with c to be chosen. This gives $-8xz^3 \leq 4z^2(cx^2 + \frac{1}{c}z^2)$, so

$$F(x,z) \leq x^4 + (4c-6)x^2z^2 - \left(3 - \frac{4}{c}\right)z^4.$$

Choose $c = \frac{3}{2}$ to deduce that $F(x,z) \leq x^4 - \frac{1}{3}z^4$, hence $\sum_{n=1}^N z_n^4 \leq 3\sum_{n=1}^N x_n^4$. $\square$

Of course, the same estimate applies to $\|C^T - I\|_{4/3}$. Compare the bound $\sqrt{2}$ given by Proposition 2.

By the Riesz-Thorin theorem, we can deduce the following bounds on $[2,4]$ and $[4,\infty)$:

COROLLARY 1.1. *For $2 \le p \le 4$, we have $\|C - I\|_p \le 3^{1/2-1/p}$. For $p \ge 4$, we have $\|C - I\|_p \le 3^{1/p} 2^{1-4/p}$.*

Proof. For $2 < p < 4$, we have $\frac{1}{p} = \frac{1-\theta}{2} + \frac{\theta}{4}$ with $\theta = 2 - \frac{4}{p}$, so (3) gives the stated bound. For $p > 4$, the stated bound follows at once from (4). $\square$

The corresponding bounds for $\|C^T - I\|_p$ are $3^{1/p-1/2}$ for $\frac{4}{3} \le p \le 2$ and $3^{1-1/p} 2^{4/p-3}$ for $1 \le p \le \frac{4}{3}$.

We have no reason to suppose that $3^{1/4}$ is the exact value of $\|C - I\|_4$. We will present a lower bound for it later.

We now turn to C^T. As remarked earlier, it is clear that $\|C^T - I\|_4 \ge 3$. We now show that this is the exact value, in accordance with conjecture (C). The method has both similarities and differences to the case of $C - I$.

THEOREM 2. *We have $\|C^T - I\|_4 = 3$.*

Proof. Choose $x \in \ell_4$ and let y_n be defined by (2), so that $x_n = n(y_n - y_{n+1})$. Now $b^4 - a^4 \le 4b^3(b-a)$ for any a, b, so $y_n^4 - y_{n+1}^4 \le 4y_n^3(y_n - y_{n+1})$, hence

$$4y_n^3 x_n \ge n(y_n^4 - y_{n+1}^4),$$

equivalently

$$y_n^4 \le 4y_n^3 x_n + n y_{n+1}^4 - (n-1)y_n^4.$$

Adding, we obtain

$$\sum_{n=1}^{N} y_n^4 \le 4 \sum_{n=1}^{N} y_n^3 x_n + N y_{N+1}^4.$$

By Hölder's inequality applied to (2), $N y_{N+1}^4 \to 0$ as $N \to \infty$, so

$$\sum_{n=1}^{\infty} y_n^4 \le 4 \sum_{n=1}^{\infty} y_n^3 x_n.$$

Now write $y_n = x_n + z_n$. Then $\sum_{n=1}^{\infty} F(x_n, z_n) \ge 0$, where

$$F(x,z) = 4x(x+z)^3 - (x+z)^4 = 3x^4 + 8x^3 z + 6x^2 z^2 - z^4.$$

Again estimate the term $8x^3 z$ using $2xz \le cx^2 + \frac{1}{c} z^2$, with c to be chosen. This gives

$$F(x,z) \le (3+4c)x^4 + \left(6 + \frac{4}{c}\right) x^2 z^2 - z^4.$$

This time the choice of c will require a little more work. We have shown that

$$\sum_{n=1}^{\infty} z_n^4 \le (3+4c)\sum_{n=1}^{\infty} x_n^4 + \sum_{n=1}^{\infty}\left(6+\frac{4}{c}\right) x_n^2 z_n^2.$$

Write $\sum_{n=1}^{\infty} x_n^4 = X^2$ and $\sum_{n=1}^{\infty} z_n^4 = Z^2$ (so that $\|x\|_4 = X^{1/2}$). By the Cauchy-Schwarz inequality, $\sum_{n=1}^{\infty} x_n^2 z_n^2 \le XZ$, so

$$Z^2 \le (3+4c)X^2 + \left(6+\frac{4}{c}\right)XZ,$$

hence

$$\left[Z-\left(3+\frac{2}{c}\right)X\right]^2 \le g(c)X^2,$$

where

$$g(c) = \left(3+\frac{2}{c}\right)^2 + 3+4c = 12+4c+\frac{12}{c}+\frac{4}{c^2}.$$

We show that c can be chosen so that $g(c)^{1/2}+3+\frac{2}{c} = 9$: it then follows that $Z \le 9X$, so that $\|z\|_4 \le 3\|x\|_4$. The required equality is $g(c) = (6-\frac{2}{c})^2$, which simplifies to $c^2-6c+9=0$, satisfied by $c=3$. (We could have shortened the proof by simply taking $c=3$ in the first place, but it is arguably preferable to show how this choice is derived.) □

The Riesz-Thorin theorem delivers the following estimate for intermediate values.

COROLLARY 2.1. *For $2 \le p \le 4$, we have $\|C^T - I\|_p \le 3^{2-4/p}$. For $\frac{4}{3} \le p \le 2$, we have $\|C-I\|_p \le 3^{4/p-2}$.*

To derive a simpler, but weaker bound, note that the convex function 3^{2-x} lies below its linear interpolation $5-2x$ for $1 \le x \le 2$. Hence $3^{2-4/p} \le 5-\frac{8}{p}$ for $2 \le p \le 4$. Meanwhile, it is not hard to show that $3^{2-4/p}$ is strictly greater than the conjectured value $p-1$ for $2 < p < 4$.

One would hope to be able to extend Theorems 1 and 2 to other values. However, our methods do not adapt readily even to the case $p=6$.

3. Lower bounds

We return to the question of lower bounds for $\|C-I\|_p$ for $p>2$.

PROPOSITION 3. *For $p \ge 2$,*

$$\|C-I\|_p \ge \left(\frac{2^{p-1}-1}{p-1}\right)^{1/p}. \tag{6}$$

Proof. Fix n and let $x = e_1 + \cdots + e_n - e_{n+1} - \cdots - e_{2n}$. Let $y = Cx$ and $z = y - x$. For $1 \le r \le n$, we have $y_{n+r} = (n-r)/(n+r)$, hence $z_{n+r} = 2n/(n+r)$. Hence

$$\sum_{k=1}^{2n} z_k^p = (2n)^p \sum_{r=1}^{n} \frac{1}{(n+r)^p}.$$

By integral estimation,

$$\sum_{r=1}^{n} \frac{1}{(n+r)^p} > \int_{n+1}^{2n} \frac{1}{t^p}\,dt = \frac{1}{p-1}\left(\frac{1}{(n+1)^{p-1}} - \frac{1}{(2n)^{p-1}}\right),$$

so

$$\begin{aligned}\frac{\sum_{k=1}^{2n} z_k^p}{\sum_{k=1}^{2n} |x_k|^p} &> \frac{(2n)^{p-1}}{p-1}\left(\frac{1}{(n+1)^{p-1}} - \frac{1}{(2n)^{p-1}}\right) \\ &= \frac{1}{p-1}\left(\frac{(2n)^{p-1}}{(n+1)^{p-1}} - 1\right),\end{aligned}$$

which tends to $(2^{p-1}-1)/(p-1)$ as $n \to \infty$. □

In particular, $\|C - I\|_4 \ge (\frac{7}{3})^{1/4}$.

Note that the estimate in (6) reproduces the correct value 1 for $p = 2$. One can derive the somewhat simpler lower bound $2(1-\frac{1}{p})/(p-1)^{1/p}$, which can be compared with the upper bound $2(1-\frac{1}{p})$ noted after Proposition 2.

In the light of these results, there would appear to be no obvious candidate to conjecture for the exact value of $\|C - I\|_p$ for $p > 2$.

4. The continuous case

In the continuous case, C is the operator defined by $(Cf)(x) = \frac{1}{x}\int_0^x f(t)\,dt$, with dual $(C^T f)(x) = \int_x^\infty \frac{f(t)}{t}\,dt$. Hardy's inequality still applies. So do all our estimations, with routine adjustments to the proofs. For example, in Theorem 1, (5) becomes $3\int_0^X (Cf)^4 \le 4\int_0^X (Cf)^3 f$, and the proof concludes as before.

For $p = 2$ in the continuous case, it was shown in [5] that $C - I$ is actually isometric: $\|(C-I)f\|_2 = \|f\|_2$ for all f, and similarly for $C^T - I$. Of course, this is not true in the discrete case. Indeed, $(C^T - I)e_1 = 0$. For C, the problem is more interesting. In finite dimensions, one simply has $(C-I)e = 0$, where $e = (1,1,\ldots,1)$. However, in infinite dimensions, the author has been able to show that $\|(C-I)x\|_2 \ge (1/\sqrt{2})\|x\|_2$ for all x in ℓ_2; this constant is attained by $x = (1,-1,0,0,\ldots)$.

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