Mathematical
Inequalities
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MEANINGFUL AND CO–MEANINGFUL SEQUENCES

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Submitted to Math. Inequal. Appl.

Abstract. Two versions of the concept "meaningful" are distinguished, "strictly" and "power" meaningful. By establishing the converse of a theorem of Bennett, a full characterisation of strictly meaningful sequences is given. A dual concept "co-meaningful" is introduced (again in two versions), and an analogous characterisation is obtained.

1. Introduction

This study supplements the pioneering articles [2], [3] by Grahame Bennett. The starting point is the following result from [4].

THEOREM BJ. *For a function f on $[0,1]$, define, for $n \geq 1$,*

$$\alpha_n(f) = \frac{1}{n} \sum_{r=1}^{n} f\left(\frac{r}{n+1}\right),$$

$$\beta_n(f) = \frac{1}{n+1} \sum_{r=0}^{n} f\left(\frac{r}{n}\right).$$

If f is convex, then $\alpha_n(f)$ increases with n, and $\beta_n(f)$ decreases.

One way to generalise these expressions is as follows. Given a strictly positive sequence $a = (a_n)_{n\geq 1}$, write $A_n = \sum_{r=1}^{n} a_r$ and $C_n = A_n/n$, and define

$$\alpha_n(f,a) = \frac{1}{n} \sum_{r=1}^{n} f\left(\frac{a_r}{C_n}\right) \tag{1}$$

for $n \geq 1$. When $a_n = n$, $\alpha_n(f,a)$ reproduces $\alpha_n(f)$ (more exactly, it equates to $\alpha_n(f_1)$, where $f_1(x) = f(2x)$). For natural applications, a_n will be either increasing or decreasing.

For $\beta_n(f)$, we consider sequences starting with a term a_0, with $a_n > 0$ for all $n \geq 1$. We now write $A_n = \sum_{r=0}^{n} a_r$ and $D_n = A_n/(n+1)$, and define

$$\beta_n(f,a) = \frac{1}{n+1} \sum_{r=0}^{n} f\left(\frac{a_r}{D_n}\right) \tag{2}$$

Mathematics subject classification (2010): 26D15, 26A51.

Keywords and phrases: monotonic; average; convex; meaningful.

for $n \geq 1$. When $a_n = n$, $\beta_n(f,a)$ equates to $\beta_n(f_1)$, where $f_1(x) = f(2x)$. As this shows, we certainly do not exclude the possibility that $a_0 = 0$ (hence the exclusion of $\beta_0(f,a)$).

In fact, $\beta_n(f,a) = \alpha_{n+1}(f,b)$, where $b_n = a_{n-1}$ for $n \geq 1$. However, analogous statements and proofs for the two cases are better reflected by the notation of (2). We shall see that the term a_0 plays a special role.

In the papers cited, some profound results for $\alpha_n(f,a)$ are presented, but $\beta_n(f,a)$ is not considered.

Slightly adapting Bennett's terminology, we will say that the sequence $(a_n)_{n\geq 1}$ is *strictly meaningful* if $\alpha_n(f,a)$ increases with n for all convex functions f defined on the relevant interval. Actually, Bennett did not introduce a term for this property. He used the term *meaningful* for the case where $f(x)$ is restricted to the functions x^p for all real p (of course, with $\alpha_n(f,a)$ decreasing when $0 < p < 1$). To highlight the distinction, I will use the term *power meaningful* for this case: this is consistent with the concept of power majorisation versus majorisation, a topic that is also discussed in the papers mentioned. (I would really prefer to omit the word "strictly", but this would clash with Bennett's terminology.)

The "relevant interval" is, of course, the least interval containing all the points a_r/C_n for $r \leq n$. This will vary according to the sequence (a_n). In all examples of interest, f will in fact be convex on $(0,\infty)$.

In these terms, Theorem BJ says that the sequence $a_n = n$ is strictly meaningful. A sequence that is power meaningful but not strictly meaningful is $a_n = 2n-1$ ([2, Theorem 4] and [4]).

We introduce corresponding terminology for $\beta_n(f,a)$. We will say that $(a_n)_{n\geq 0}$ is *strictly co-meaningful* if $\beta_n(f,a)$ decreases with n for all convex f, and *power co-meaningful* if $\beta_n(f,a)$ decreases with n for $f(x) = x^p$ with $p \geq 1$ and increases with n when $0 < p < 1$ (we exclude $p < 0$, to accommodate the possibility that $a_0 = 0$).

The main theorem in [2] establishes sufficient conditions for the property "power meaningful". In [3] it is shown that the same conditions actually imply "strictly meaningful" (though the theorem is not explicitly stated in these terms). A partial converse is given in [2] for "power meaningful", but not for "strictly meaningful". Here we will prove such a converse result, showing in fact that for monotonic sequences, the stated sufficient conditions are also necessary, so in fact give a full characterisation of monotonic strictly meaningful sequences.

We then explore the extent to which analagous results apply for $\beta_n(f,a)$.

There is a very satisfactory companion to the main theorem for increasing sequences, but there are also differences: for example, there are no non-trivial decreasing co-meaningful sequences (in either sense).

The article [1] presents a different generalisation of Theorem BJ, applying to sums of the form $(1/c_{n+1})\sum_{r=0}^{n} f(a_r/a_n)$. These results do not imply, or follow from, the ones considered here. Another generalisation, to weighted averages $\sum_{r=0}^{n} w_{n,r} f(r/n)$, is discussed in [5].

2. Strictly meaningful sequences: Bennett's theorem and the converse

Case (i) of [3, Theorem 1] establishes the following:

THEOREM B1. *If a_n and a_{n+1}/C_n both increase with n, or both decrease, then (a_n) is strictly meaningful.*

This is not how the theorem is actually stated in [3]. However, a scrutiny of the proof, together with formula (37), reveals that it is what is really proved. The statement is combined with a "case (ii)", which asserts, in different notation, that $(n-c)$ is power meaningful for $0 \le c \le \frac{1}{2}$.

The most obvious particular case is $a_n = n^\alpha$. For this sequence, we have

$$\frac{a_{n+1}}{C_n} = \frac{n(n+1)^\alpha}{S_n(\alpha)},$$

where $S_n(\alpha) = \sum_{r=1}^n r^\alpha$. By Theorem BJ, with $f(x) = x^\alpha$, we see that $S_n(\alpha)/[n(n+1)^\alpha]$ increases with n when $\alpha \ge 1$ or $\alpha < 0$, and decreases when $0 \le \alpha \le 1$. Hence (n^α) is strictly meaningful both for $0 \le \alpha \le 1$ and for $\alpha < 0$.

Other examples satisfying the condition are $a_n = n + c$, where $c > 0$, and $a_n = n(n+1)$.

Bennett also shows that if a_n and a_{n+1}/C_n both increase (or both decrease), then so does a_n/C_n [2, Lemma 3]. Further, the following converse is established [2, Theorem 6]: if (a_n) is power meaningful and increasing, then a_n/C_n is increasing (and similarly if decreasing). However, a_{n+1}/C_n may fail to be increasing; the sequence $a_n = 2n-1$ is a counter-example.

The focus in Bennett's papers is firmly on "power meaningful"; no converse result is stated for "strictly meaningful". Here we supply such a converse: we show that for a monotonic sequence to be strictly meaningful, the sufficient condition stated in Theorem B1 is also necessary.

THEOREM 1. *Suppose that (a_n) is strictly meaningful. If a_n is increasing (or decreasing), then so is a_{n+1}/C_n.*

Proof. Fix $n \ge 2$. Write $x_r = a_r/C_n$ and $y_r = a_r/C_{n+1}$. The statement $\alpha_n(f,a) \le \alpha_{n+1}(f,a)$ equates to

$$(n+1)\sum_{r=1}^{n} f(x_r) \le n \sum_{r=1}^{n+1} f(y_r). \tag{3}$$

Assume first that a_n is increasing; then C_n is increasing, so $x_n \ge y_n$. We show that

$$(n+1)x_n \le n y_{n+1} + y_n. \tag{4}$$

Let $f(t) = (t-y_n)^+$. By (3), we have $(n+1)(x_n - y_n) \le n(y_{n+1} - y_n)$, which equates to (4). Now suppose that a_n is decreasing. Then $x_n \le y_n$, and taking $f(t) = (y_n - t)^+$, we find $(n+1)(y_n - x_n) \le n(y_n - y_{n+1})$, so (4) is reversed.

In terms of A_n, (4) says

$$(n+1)\frac{na_n}{A_n} \le n\,\frac{(n+1)a_{n+1}}{A_{n+1}} + \frac{(n+1)a_n}{A_{n+1}},$$

hence

$$na_n A_{n+1} \leq (na_{n+1} + a_n) A_n.$$

Since $A_{n+1} = A_n + a_{n+1}$, this equates to

$$(n-1) a_n A_n \leq na_{n+1}(A_n - a_n) = na_{n+1} A_{n-1},$$

so

$$\frac{a_n}{C_{n-1}} = \frac{(n-1)a_n}{A_{n-1}} \leq \frac{na_{n+1}}{A_n} = \frac{a_{n+1}}{C_n}.$$

□

It is stated as an unsolved problem in [2, p. 575] whether (n^α) is power meaningful for all $\alpha > 1$. This is proved for the special case $\alpha = 3$ and stated without proof for $\alpha = 2$. Our Theorem disposes very easily of the corresponding question for "strictly meaningful".

COROLLARY 1. *If $\alpha > 1$, then (n^α) is not strictly meaningful.*

Proof. Note that $a_2/C_1 = 2^\alpha$ and

$$\frac{a_3}{C_2} = \frac{2.3^\alpha}{1+2^\alpha}.$$

By (strict) convexity of the function x^α, we have $2.3^\alpha < 2^\alpha + 4^\alpha$, hence $a_3/C_2 < a_2/C_1$. □

Another application of Theorem 1 is that $(n-c)$ is not strictly meaningful for $0 < c < 1$.

3. Co-meaningful sequences

We now investigate analagous results for co-meaningful sequences (in both senses). Recall that we are considering non-negative sequences $(a_n)_{n \geq 0}$ with $a_n > 0$ for $n \geq 1$, and that $A_n = \sum_{r=0}^n a_r$ and $D_n = A_n/(n+1)$. We start with some simple observations.

If $a_n = 1$ for all n, then $D_n = 1$ and $\beta_n(f, a) = f(1)$ for all n and any function f. So this sequence is strictly co-meaningful, in a trivial way. By Theorem BJ, the sequence $a_n = n$ is strictly co-meaningful.

If $f(x) = x$, then for any (a_n) as above,

$$\beta_n(f, a) = \frac{A_n}{(n+1)D_n} = 1$$

for all n. By Jensen's inequality, together with this identity, we have $\beta_n(f, a) \geq f(1)$ for all n and all convex functions f.

We start with necessary conditions, because these are established quite easily. There are some immediate contrasts with the meaningful case.

PROPOSITION 1. *If (a_n) is strictly co-meaningful, increasing and not constant, then $a_0 = 0$.*

Proof. Suppose that $a_0 > 0$. Since (a_n) is not constant, there exists $n \geq 2$ such that $D_n > D_{n-1}$, so that $a_0/D_n < a_0/D_{n-1}$. Let $f(x) = (a_0/D_{n-1} - x)^+$. Then $f(a_r/D_{n-1}) = 0$ for each r, so $\beta_{n-1}(f,a) = 0$, while $\beta_n(f,a) \geq \frac{1}{n+1} f(a_0/D_n) > 0$. □

For $p > 0$, write

$$\rho_{p,n}(a) = \left(\frac{1}{n+1} \sum_{r=0}^{n} a_r^p \right)^{1/p}.$$

It is elementary that for fixed n, $\rho_{p,n}(a) \to \max_{0 \leq r \leq n} a_r$ as $p \to \infty$. We deduce at once:

PROPOSITION 2. *(i) If (a_n) is increasing and power co-meaningful, then a_n/D_n decreases with n for $n \geq 1$.*

(ii) There are no non-constant, decreasing power co-meaningful sequences.

Proof. (i) By the definition, for $p > 1$,

$$\frac{1}{(n+1)D_n^p} \sum_{r=0}^{n} a_r^p$$

decreases with n, so $\rho_{p,n}(a)/D_n$ decreases. Since (a_n) is increasing, $\rho_{p,n}(a) \to a_n$ as $p \to \infty$. Hence a_n/D_n decreases.

(ii) Now suppose that (a_n) is decreasing and $a_0 > 0$. Then $\rho_{p,n}(a) \to a_0$ as $p \to \infty$, so a_0/D_n decreases. Therefore D_n increases, so (D_n), hence also (a_n), is constant. □

Note. For "strictly co-meaningful, the method of Proposition 1 adapts easily to give a quick direct proof of both statements.

COROLLARY 2. *If (a_n) is increasing and power co-meaningful, with $a_0 = 0$, then $a_n \leq na_1$ for all $n \geq 1$.*

Proof. Then $a_1/D_1 = 2$, so $a_n/D_n \leq 2$ for $n \geq 1$. In other words, $(n+1)a_n \leq 2A_n$, hence $(n-1)a_n \leq 2A_{n-1}$. Now take $n \geq 2$ and assume that $a_r \leq ra_1$ for $r \leq n-1$. Then $A_{n-1} \leq \frac{1}{2}(n-1)na_1$. Hence $a_n \leq na_1$. □

With more care, one can show that $a_n - a_0 \leq n(a_1 - a_0)$ for $n \geq 1$.

These results enable us to dismiss a number of sequences very easily. Firstly, no non-constant increasing sequence with $a_0 > 0$ is strictly co-meaningful. Further:

EXAMPLE 1. By Corollary 2, (n^α) is not power co-meaningful for any $\alpha > 1$. (For this, we only need $a_2 \leq 2a_1$, which is even easier.)

EXAMPLE 2. Let $a_n = n + c$, where $c > 0$. Then $D_n = \frac{1}{2}n + c$, so

$$\frac{a_n}{D_n} = \frac{2(n+c)}{n+2c} = 2 - \frac{2c}{n+2c},$$

which increases with n. Hence (a_n) is not power co-meaningful.

EXAMPLE 3. Let $a_0 = 0$ and $a_n = n+1$ for $n \geq 1$. Then $A_3 = 9$, $A_4 = 14$, so $a_3/D_3 = \frac{16}{9}$, while $a_4/D_4 = \frac{25}{14} > \frac{16}{9}$, hence (a_n) is not power co-meaningful.

We now establish a companion result to Theorem B1, by essentially similar steps. The conclusion is that for non-constant, increasing sequences, the necessary conditions in Propositions 1 and 2 are also sufficient.

As in the meaningful case, the basis of the proof is the following Lemma on convex functions [2, Lemma 2].

LEMMA 1. *Let a, b, c, d be real numbers with $a < d$ and b, c in $[a,d]$. Also, let α, β, γ, δ be non-negative numbers such that*

$$\beta + \gamma = \alpha + \delta,$$

$$\beta b + \gamma c = \alpha a + \delta d.$$

Then for all convex functions f,

$$\beta f(b) + \gamma f(c) \leq \alpha f(a) + \delta f(d).$$

An instant proof (shorter than the one given in [2]) is as follows:

Proof. The hypotheses are equivalent to the statement that for any affine function $g(x) = mx + n$, we have $\beta g(b) + \gamma g(c) = \alpha g(a) + \delta g(d)$. Take g to be the affine function agreeing with f at a and d: then $f(b) \leq g(b)$ and $f(c) \leq g(c)$. The statement follows. □

We also require the following Lemma, corresponding to [2, Lemma 3] in the meaningful case. In most applications the conclusion is easily verified, so the reader may choose to defer the proof for now and move on to the main theorem.

LEMMA 2. *If a_n is increasing and a_n/D_n is decreasing for $n \geq 1$, then a_{n+1}/D_n is decreasing.*

Proof. Note that

$$1 - \frac{A_{n-1}}{A_n} = \frac{a_n}{A_n} \qquad \text{and} \qquad \frac{A_{n+1}}{A_n} - 1 = \frac{a_{n+1}}{A_n},$$

so that

$$a_{n+1}\left(1 - \frac{A_{n-1}}{A_n}\right) = a_n\left(\frac{A_{n+1}}{A_n} - 1\right). \tag{5}$$

Let

$$U_n = \frac{A_{n-1}}{n+1}\left(\frac{a_n}{D_{n-1}} - \frac{a_{n+1}}{D_n}\right) = \frac{n}{n+1}a_n - \frac{A_{n-1}}{A_n}a_{n+1},$$

$$V_n = D_{n+1}\left(\frac{a_n}{D_n} - \frac{a_{n+1}}{D_{n+1}}\right) = \frac{n+1}{n+2}\,\frac{A_{n+1}}{A_n}a_n - a_{n+1}.$$

By hypothesis, $V_n \geq 0$. By (5),

$$
\begin{aligned}
U_n - V_n &= \left(\frac{n}{n+1} - \frac{n+1}{n+2} \frac{A_{n+1}}{A_n} + \frac{A_{n+1}}{A_n} - 1 \right) a_n \\
&= \left(\frac{1}{n+2} \frac{A_{n+1}}{A_n} - \frac{1}{n+1} \right) a_n \\
&= \frac{a_n}{A_n} (D_{n+1} - D_n) \\
&\geq 0,
\end{aligned}
$$

since $D_{n+1} \geq D_n$. Hence $U_n \geq 0$. □

THEOREM 2. *Let (a_n) be non-negative, increasing, not constant, with $a_1 > 0$. Then (a_n) is strictly co-meaningful if and only if $a_0 = 0$ and a_n/D_n decreases with n for $n \geq 1$.*

Proof. We have already shown that the conditions are necessary. Assume that they are satisfied, and let f be convex. We have to show that $\beta_n(f,a) \leq \beta_{n-1}(f,a)$ for $n \geq 2$. This equates to

$$
n \sum_{r=0}^{n} f\left(\frac{a_r}{D_n} \right) \leq (n+1) \sum_{r=0}^{n-1} f\left(\frac{a_r}{D_{n-1}} \right). \tag{6}
$$

The master stroke in [3] is the introduction of extra terms that cancel. Copying this idea, we will prove an inequality of the form

$$
nf\left(\frac{a_r}{D_n} \right) + J_{r+1} \leq (n+1) f\left(\frac{a_r}{D_{n-1}} \right) + J_r \tag{7}
$$

for $0 \leq r \leq n-1$, for certain terms J_r to be chosen (with $J_0 = 0$). Adding these inequalities, we then obtain

$$
n \sum_{r=0}^{n-1} f\left(\frac{a_r}{D_n} \right) + J_n \leq (n+1) \sum_{r=0}^{n-1} f\left(\frac{a_r}{D_{n-1}} \right).
$$

To recapture (6), we require $J_n = nf(a_n/D_n)$. This is ensured by taking $J_r = rf(D_{r-1}E_n)$ for $1 \leq r \leq n$, where $E_n = a_n/(D_{n-1}D_n)$ (also $J_0 = 0$). So we will prove (7), with J_r defined in this way.

When $r = 0$, since $a_0 = D_0 = 0$, both sides of (7) equate to $(n+1)f(0)$. (Remark: if we had $a_0 > 0$, then convexity of f would force the opposite inequality to (7) at this point!)

For $1 \leq r \leq n-1$, we apply Lemma 1, with

$$
\alpha = r, \quad \beta = n, \quad \gamma = r+1, \quad \delta = n+1,
$$

$$
a = D_{r-1}E_n, \quad b = \frac{a_r}{D_n}, \quad c = D_r E_n, \quad d = \frac{a_r}{D_{n-1}}.
$$

We verify that the conditions are satisfied: then (7) follows. Clearly, $\beta+\gamma=\alpha+\delta$. Since $(r+1)D_r=A_r$, we have

$$\gamma c-\alpha a=(A_r-A_{r-1})E_n=a_rE_n,$$

while

$$\delta d-\beta b=a_r\left(\frac{n+1}{D_{n-1}}-\frac{n}{D_n}\right)=a_r\frac{A_n-A_{n-1}}{D_{n-1}D_n}=\frac{a_ra_n}{D_{n-1}D_n}=a_rE_n.$$

Since D_n increases with n, we have $a\le c$ and $b\le d$. The inequality $c\le d$ equates to $(a_nD_r)/D_n\le a_r$, hence to $(a_n/D_n)\le(a_r/D_r)$, which is true, by hypothesis. The inequality $a\le b$ equates to $(a_nD_{r-1})/D_{n-1}\le a_r$. For $r=1$, this just says $a_1\ge 0$. For $r>1$, it equates to $(a_n/D_{n-1})\le(a_r/D_{r-1})$, which is true by Lemma 2. □

COROLLARY 3. *The sequence* $(n^\alpha)_{n\ge 0}$ *is strictly co-meaningful if* $0\le\alpha\le 1$. *Hence if* $S_n(\alpha)=\sum_{r=0}^n r^\alpha$, *then the expression*

$$(n+1)^{p-1}\frac{S_n(\alpha p)}{S_n(\alpha)^p}$$

decreases with n *if* $p\ge 1$, *and increases if* $0\le p\le 1$.

Proof. We have

$$\frac{a_n}{D_n}=\frac{n^\alpha(n+1)}{S_n(\alpha)}.$$

By Theorem BJ, applied to $f(x)=x^\alpha$, this decreases with n when $0\le\alpha\le 1$. The second statement simply records what it means to say that (n^α) is power co-meaningful. (We remark that the statement that (n^α) is power meaningful says that the opposite monotonicities hold when $(n+1)^{p-1}$ is replaced by n^{p-1}.) □

EXAMPLE 4. Let $a=(0,1,1,\ldots)$. Then $A_n=n$, so $D_n=n/(n+1)$ and $a_n/D_n=1+\frac{1}{n}$. This is decreasing, so a is strictly co-meaningful. For convex f, this says that

$$\beta_n(f,a)=\frac{1}{n+1}f(0)+\frac{n}{n+1}f\left(\frac{n+1}{n}\right)$$

decreases with n. Not surprisingly, this follows directly from the definition of convexity applied to the points 0, $(n+1)/n$ and $n/(n-1)$.

The following generalisation of Corollary 3 corresponds to [2, Theorem 7]:

THEOREM 3. *Suppose that* (a_n) *is increasing, with* $a_0=0$ *and* $a_n>0$ *for* $n\ge 1$, *and that* a_n/D_n *decreases with* n *for* $n\ge 1$. *Then* (a_n^α) *is strictly co-meaningful for* $0<\alpha\le 1$.

Proof. Write $S_n = \sum_{r=0}^{n} a_r^\alpha$. By Theorem 2, (a_n) is strictly co-meaningful. For the concave function $f(x) = x^\alpha$, this says that

$$\frac{1}{n+1}\left(\frac{n+1}{A_n}\right)^\alpha S_n$$

increases with n, so $A_n^\alpha (n+1)^{1-\alpha}/S_n$ decreases. By hypothesis, $(n+1)^\alpha a_n^\alpha / A_n^\alpha$ decreases. Hence $(n+1)a_n^\alpha/S_n$ decreases: this is the condition for (a_n^α) to be strictly co-meaningful. □

By Theorem 2 and Proposition 2, all increasing, power co-meaningful sequences with $a_0 = 0$ are strictly co-meaningful. We now give an example of a power co-meaningful sequence with $a_0 > 0$: by Proposition 1, it is not strictly co-meaningful.

EXAMPLE 5. The sequence $(1,2,2,\ldots)$ is power co-meaningful.

(Note: the repetition of 2 has the effect that $\beta_n(f,a)$ is a combination of only two values of f. However, this does not mean that the result is trivial, as the following shows.) We will actually work with $\beta_{n-1}(f,a)$: we have to show that it decreases (or increases) for $n \geq 2$. Now $D_{n-1} = (2n-1)/n$, so

$$\beta_{n-1}(f,a) = \frac{1}{n} f\left(\frac{n}{2n-1}\right) + \frac{n-1}{n} f\left(\frac{2n}{2n-1}\right).$$

For $f(t) = t^p$, we have $\beta_{n-1}(f,a) = h(\frac{1}{n})$, where

$$h(x) = \frac{x + 2^p(1-x)}{(2-x)^p} = \frac{c - dx}{(2-x)^p},$$

where $c = 2^p$, $d = 2^p - 1$. We will show that $h(x)$ increases on $[0, \frac{1}{2}]$ if $p \geq 1$ and decreases if $0 < p \leq 1$. Now

$$(2-x)^{p+1} h'(x) = -d(2-x) + p(c - dx) = (pc - 2d) - (p-1)dx.$$

So our result will follow if we can show that $pc - 2d \geq \frac{1}{2}(p-1)d$, equivalently $2pc \geq (p+3)d$, for $p \geq 1$, and the reverse for $0 < p \leq 1$. Now

$$2pc - (p+3)d = 2p.2^p - (p+3)(2^p - 1) = (3+p) - (3-p)2^p.$$

This is clearly non-negative if $p \geq 3$. To finish, we show that

$$2^p \leq \frac{3+p}{3-p} \tag{8}$$

for $1 \leq p < 3$, together with the reverse for $0 \leq p \leq 1$. Note that equality holds when $p = 0$ and $p = 1$. Let $\phi(p) = p \log 2$ and $\psi(p) = \log(3+p) - \log(3-p)$. Then ϕ is linear, while

$$\psi'(p) = \frac{1}{3+p} + \frac{1}{3-p} = \frac{6}{9-p^2},$$

which increases with p on $[0,3)$, so ψ is convex on this interval. Since ϕ coincides with ψ at 0 and 1, it follows that $\psi(p) \leq \phi(p)$ on $[0,1]$ and $\psi(p) \geq \phi(p)$ on $[1,3)$, hence (8).

(In the same way, one can show that $\beta_n(f,a)$ decreases with n when $-1 \leq p < 0$, but we did not include this case in our definition of power co-meaningfulness.)

Similar reasoning establishes that $(1,r,r,\ldots)$ is power co-meaningful for any $r > 1$. In the inequality corresponding to (8), 2 is replaced by r and 3 by $(r+1)/(r-1)$.

Finally, we give an example to show that when $a_0 > 0$, the condition that a_n/D_n is decreasing does not ensure that (a_n) is power co-meaningful.

EXAMPLE 6. Consider the sequence $(2,3,\frac{10}{3},\frac{10}{3},\ldots)$. The number $\frac{10}{3}$ is chosen to ensure that $a_2/D_2 = a_1/D_1 = \frac{6}{5}$: it is the largest choice of a_2 compatible with the condition $a_2/D_2 \leq a_1/D_1$. Since D_n is increasing, it follows that a_n/D_n decreases for all $n \geq 1$. Since $D_1 = \frac{5}{2}$ and $D_2 = \frac{25}{9}$, we have

$$\beta_1(f,a) = \tfrac{1}{2}f(\tfrac{4}{5}) + \tfrac{1}{2}f(\tfrac{6}{5}), \qquad \beta_2(f,a) = \tfrac{1}{3}f(\tfrac{18}{25}) + \tfrac{1}{3}f(\tfrac{27}{25}) + \tfrac{1}{3}f(\tfrac{6}{5}).$$

With $f(x) = x^2$, we find

$$25^2[\beta_2(f,a) - \beta_1(f,a)] = \tfrac{1}{3} \times 18^2 + \tfrac{1}{3} \times 27^2 - (8+6) \times 25 = 108 + 243 - 350 = 1.$$

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