

# MOMENT-SEQUENCE TRANSFORMS

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*To Gadadhar Misra, master of operator theory*

ABSTRACT. We classify all functions which, when applied term by term, leave invariant the sequences of moments of positive measures on the real line. Rather unexpectedly, these functions are built of absolutely monotonic components, or reflections of them, with possible discontinuities at the endpoints. Even more surprising is the fact that functions preserving moments of three point masses must preserve moments of all measures. Our proofs exploit the semidefiniteness of the associated Hankel matrices and the complete monotonicity of the Laplace transforms of the underlying measures. As a byproduct, we characterize the entrywise transforms which preserve totally non-negative Hankel matrices, and those which preserve all totally non-negative matrices. The latter class is surprisingly rigid: such maps must be constant or linear. We also examine transforms in the multivariable setting, which reveals a new class of piecewise absolutely monotonic functions.

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## 1. INTRODUCTION

The ubiquitous encoding of functions or measures into discrete entities, such as sampling data, Fourier coefficients, Taylor coefficients, moments, and Schur parameters,

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leads naturally to operating directly on the latter ‘spectra’ rather than the original. The present article focuses on operations which leave invariant power moments of positive multivariable measures. To put our essay in historical perspective, we recall a few similar and inspiring instances.

The characterization of positivity preserving analytic operations on the spectrum of a self-adjoint matrix is due to Löwner in his groundbreaking article [33]. Motivated by the then-novel theory of the Gelfand transform and the Wiener–Levy theorem, in the 1950s Helson, Kahane, Katznelson, and Rudin identified all real functions which preserve Fourier transforms of integrable functions or measures on abelian groups [23, 28, 38]. Roughly speaking, these Fourier-transform preservers have to be analytic, or even absolutely monotonic. The absolute-monotonicity conclusion was not new, and resonated with earlier work of Bochner [9] and Schoenberg [42] on positive definite functions on homogeneous spaces. Later on, this line of thought was continued by Horn in his doctoral dissertation [25]. These works all address the question of characterizing real functions  $F$  which have the property that the matrix  $(F(a_{ij}))$  is positive semidefinite whenever  $(a_{ij})$  is, possibly with some structure imposed on these matrices. Schoenberg’s and Horn’s theorems deal with all matrices, infinite and finite, respectively, while Rudin et al. deal with Toeplitz-type matrices via results of Herglotz and Carathéodory.

In this article, we focus on functions which preserve moment sequences of positive measures on Euclidean space, or, equivalently, in the one-variable case, functions which leave invariant positive semidefinite Hankel kernels. As we show, these moment preservers are quite rigid, with analyticity and absolute monotonicity again being present in a variety of combinations, especially when dealing with multivariable moments. We state in detail in Section 2 our results for one-variable functions and domains and for moment sequences of measures on them, but first we present in Section 1.1 tabulated lists of our results in one and several variables.

The first significant contribution below is the relaxation to a minimal set of conditions, which are very accessible numerically, that characterize the positive definite Hankel kernel transformers in one variable. Specifically, Schoenberg proved that a continuous map  $F : (-1, 1) \rightarrow \mathbb{R}$  preserves positive semidefiniteness when applied to matrices of all dimensions, if and only if  $F$  is analytic and has positive Taylor coefficients [42]. Later on, Rudin was able to remove the continuity assumption [38]. In our first major result, we prove that a map  $F : (-1, 1) \rightarrow \mathbb{R}$  preserves positive semidefiniteness of all matrices if and only if it preserves this on Hankel matrices. Even more surprisingly, a refined analysis reveals that preserving positivity on Hankel matrices of rank at most 3 already implies the same conclusion.

Our result can equivalently be stated in terms of preservers of moment sequences of positive measures. Thus we also characterize such preservers under various constraints on the support of the measures. Furthermore, we examine the analogous problem in higher dimensions. In this situation, extra work is required to compensate for the failure of Hamburger’s theorem in higher-dimensional Euclidean spaces.

Our techniques extend naturally to totally non-negative matrices, in parallel to their natural connection to the Stieltjes moment problem. We prove that the entrywise transformations which preserve total non-negativity for all rectangular matrices, or all symmetric matrices, are either constant or linear. Furthermore, we show that the

entrywise preservers of totally non-negative Hankel matrices must be absolutely monotonic on the positive semi-axis. The class of totally non-negative matrices was isolated by M. Krein almost a century ago; he and his collaborators proved its significance for the study of oscillatory properties of small harmonic vibrations in linear elastic media [18, 19]. Meanwhile this chapter of matrix analysis has reached maturity and it continues to be explored and enriched on intrinsic, purely algebraic grounds [13, 14].

We conclude by classifying transformers of tuples of moment sequences, from which a new concept emerges, that of a piecewise absolutely monotonic function of several variables. In particular, our results extend original theorems by Schoenberg and Rudin. For more on the wider framework within which this article sits, we refer the reader to the survey [4].

Besides the classical works cited above delineating this area of research, we rely in the sequel on Bernstein's theory of absolutely monotone functions [7, 54], a related pioneering article by Lorch and Newman [32] and Carlson's interpolation theorem for entire functions [11].

The study of positive definite functionals defined on  $*$ -semigroups, with or without unit, led Stochel to a series of groundbreaking discoveries, complementing the celebrated Naimark and Sz. Nagy dilation theorems and, in particular, putting multivariate moment problems in a wider, more flexible framework [47, 48, 49]. A byproduct of his studies is a classification of positive definite functionals on the multiplicative semigroup  $(-1, 1)$  [48], culminating with a similar conclusion to our main one-dimensional result: these positive functionals are absolutely monotonic on  $(0, 1)$  with possibly discontinuous derivatives, of any order, at the origin.

As a final remark, we note that entrywise transforms of moment sequences were previously studied in a particular setting motivated by infinite divisibility in probability theory [26, 50]. The study of entrywise operations which leave invariant the cone of all positive matrices has also recently received renewed attention in the statistics literature, in connection to the analysis of big data. In that setting, functions are applied entrywise to correlation matrices to improve properties such as their conditioning, or to induce a Markov random-field structure. The interested reader is referred to [3, 21, 22] and the references therein for more details.

A companion to the present article [5] was recently completed, which extends the work here with definitive classifications of preservers of totally positive and totally non-negative kernels, and together with kernels having additional structure, such as those of Hankel [53] or Toeplitz [43] type, or generating series, such as Pólya frequency functions and sequences.

**1.1. Summary of main results.** Tables 1.1 and 1.2 below summarize the results proved in this article. The notation used below is explained in the main body of the article; see also the List of Symbols following this subsection.

In the one-variable setting, we have identified the positivity preservers acting on (i) all matrices, and (ii) all Hankel matrices, in the course of classifying such functions acting on (iii) moment sequences, i.e., all Hankel matrices arising from moment sequences of measures supported on  $[-1, 1]$ . Characterizations for all three classes of matrices are obtained with the additional constraint that the entries of the matrices lie in  $(0, \rho)$ ,  $(-\rho, \rho)$ , and  $[0, \rho)$ , where  $\rho \in (0, \infty]$ .

	$F[-]$ preserves positivity on:		
Domain $I$ , $\rho \in (0, \infty]$	$\cup_{N \geq 1} \mathcal{P}_N(I)$	$\mathcal{H}^+(I)$	$\mu \in \mathcal{M}([0, 1])$ or $\mathcal{M}([-1, 1])$ , $s_0(\mu) \in I \cap [0, \infty)$
$(0, \rho)$	Theorems 2.12, 4.4	Theorems 4.2, 4.4	Theorems 4.2, 4.4
$[0, \rho)$	Proposition 8.1	Proposition 8.1	Theorems 4.1
$(-\rho, \rho)$	Theorem 2.10	Theorems 6.1	Theorems 6.1

TABLE 1.1. The one-variable case.

We then extend each of the results in Table 1.1 to apply to functions acting on tuples of positive matrices or moment sequences: see Table 1.2.

	$F[-]$ preserves positivity on $m$ -tuples of elements in:		
Domain $I$ , $\rho \in (0, \infty]$	$\cup_{N \geq 1} \mathcal{P}_N(I)$	$\mathcal{H}^+(I)$	$\mu \in \mathcal{M}([0, 1])$ or $\mathcal{M}([-1, 1])$ , $s_0(\mu) \in I \cap [0, \infty)$
$(0, \rho)$	Theorem 9.6	Theorem 9.6	Theorem 9.6
$[0, \rho)$	Proposition 9.8	Proposition 9.8	Theorem 9.5
$(-\rho, \rho)$	Theorem 9.11 (see [16] for $\rho = \infty$ )	Theorem 9.11	Theorem 9.11

TABLE 1.2. The multivariable case.

In the one-variable setting, we do more than is recorded in Table 1.1, since our results cover various classes of totally non-negative matrices (Section 5), as well as the closed-interval settings of  $[0, \rho]$  and  $[-\rho, \rho]$  for  $\rho < \infty$  (Section 8). The multivariable case may contain products of open and closed intervals, but it would be rather cumbersome, and somewhat artificial, to consider them all. We do not pursue this direction in the present work.

In all of the above contexts, with the exception of functions on  $[0, \rho]^m$  (i.e., the (2, 3) entry in both tables), the characterizations are uniform: all such positivity preservers are necessarily analytic on the domain and absolutely monotonic on the closed positive orthant. The converse result holds trivially by the Schur product theorem. The one exceptional case reveals a richer family of ‘facewise absolutely monotonic maps’; see Section 9.2.

We have also improved on all of the above results, by significantly relaxing the hypotheses required to obtain absolute monotonicity.

Finally, and for completeness, we remark that Theorem 4.8 from our previous work [3], which is widely used herein, admits a generalization to all, possibly non-consecutive, integer powers, and again the bounds have closed form. This result is obtained through a careful analysis and novel results about Schur polynomials; we refer the reader to the recent paper by Khare and Tao [30] for more details.

**1.2. List of symbols.** For the convenience of the reader, we list some of the symbols used in this paper.

- Given a subset  $I \subset \mathbb{R}$ ,  $\mathcal{P}_N^k(I)$  is the set of positive semidefinite  $N \times N$  matrices with entries in  $I$  and of rank at most  $k$ . We let  $\mathcal{P}_N(I) := \mathcal{P}_N^N(I)$  and  $\mathcal{P}_N := \mathcal{P}_N(\mathbb{R})$ .
- $\mathcal{H}^+(I)$  denotes the set of positive semidefinite Hankel matrices of arbitrary dimension with entries in  $I$ .
- $\mathcal{H}_n^{++}$  denotes the set of  $n \times n$  totally non-negative Hankel matrices, and  $\mathcal{H}^{++}$  denotes the set of all totally non-negative Hankel matrices.
- $H^{(1)}$  denotes the truncation of a possibly semi-infinite matrix  $H$  obtained by excising the first column.
- $F[H]$  is the result of applying  $F$  to each entry of the matrix  $H$ .
- For  $K \subset \mathbb{R}$ , we denote by  $\text{Meas}^+(K)$  the set of admissible measures, i.e., non-negative measures  $\mu$  supported on  $K$  and admitting moments of all orders.
- The  $k$ th moment of a measure  $\mu$  is denoted by  $s_k(\mu)$ ; the corresponding moment sequence is  $\mathbf{s}(\mu) := (s_k(\mu))_{k \geq 0}$ . The associated Hankel moment matrix  $H_\mu$  has  $(i, j)$  entry  $s_{i+j}(\mu)$ . In particular, the moment sequence of  $\mu$  is the leading row and column of  $H_\mu$ .
- Given  $K \subset \mathbb{R}$ ,  $\mathcal{M}(K)$  denotes the set of moment sequences associated to elements of  $\text{Meas}^+(K)$ . For any  $k \geq 0$ ,  $\mathcal{M}_k(K)$  denotes the corresponding set of truncated moment sequences:  $\mathcal{M}_k(K) := \{(s_0(\mu), \dots, s_k(\mu)) : \mu \in \text{Meas}^+(K)\}$ .
- Given  $K \subset \mathbb{R}$  and a scalar  $\rho$  with  $0 < \rho \leq \infty$ ,  $\mathcal{M}^\rho(K)$  denotes the subset of  $\mathcal{M}(K)$  with moments  $s_j \in (-\rho, \rho)$  for all  $j \geq 0$ , and, for any  $k \geq 0$ , we let  $\mathcal{M}_k^\rho(K)$  denote the subset of  $\mathcal{M}_k(K)$  with  $s_j \in (-\rho, \rho)$  for  $j = 0, \dots, k$ .
- Given  $\rho$  with  $0 < \rho \leq \infty$ , an integer  $k \geq 0$ , and  $x \in [-1, 1)$ , we let  $\mathcal{M}^{\rho,+}(\{1, x\})$  and  $\mathcal{M}_k^{\rho,+}(\{1, x\})$  denote the subsets of  $\mathcal{M}(\{1, x\})$  and  $\mathcal{M}_k(\{1, x\})$ , respectively, with total mass  $s_0 < \rho$  and such that 1 and  $x$  both have positive mass.
- Given an integer  $m \geq 1$ , a function  $F : \mathbb{R}^m \rightarrow \mathbb{R}$  acts on tuples of moment sequences of admissible measures  $\mathcal{M}(K_1) \times \dots \times \mathcal{M}(K_m)$  as follows:

$$F[\mathbf{s}(\mu_1), \dots, \mathbf{s}(\mu_m)] := (F(s_k(\mu_1), \dots, s_k(\mu_m)))_{k \geq 0}. \quad (1.1)$$

- Given  $h > 0$  and an integer  $n \geq 0$ ,  $\Delta_h^n F$  denotes the  $n$ th forward difference of the function  $F$  with step size  $h$ .
- $\mathbf{1}_{m \times n}$  denotes the  $m \times n$  matrix with all entries equal to 1.
- $\mathbb{C}^+ := \{z \in \mathbb{C} : \Re z > 0\}$  denotes the right open half-plane.

**1.3. Organization.** The plan of the article is as follows. Section 2 recalls notation and reviews previous work, while Section 3 lists our main results for classical positive Hankel matrices transformers, which, in particular, go beyond previous classical results. Sections 4, 6, 7, and 8 are devoted to proofs, arranged by the domains of the entries of the relevant Hankel matrices. For these proofs, we work with measures with restricted total mass, which is reflected in the domains of the test sets of matrices, and helps unify previously known results. Thus, we end up showing stronger results than in Section 2; these results were tabulated in a concise form in Section 1.1 above. An additional strengthening involves severely reducing the supports of the test measures, which translates to rank constraints on the test sets of Hankel matrices and hence

stronger results. This technical point is not mentioned in the above tables, but is detailed in the aforementioned Sections 4, 6, 7, and 8 devoted to proofs.

Section 5 contains the classifications of preservers of total non-negativity for several different sets of matrices, in the dimension-free setting. Section 9 deals with multivariable transformers of Hankel kernels. Section 10 makes the natural link with Laplace transforms and interpolation of entire functions. The appendix is devoted to algebraic properties of adjugate matrices.

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## 2. PRELIMINARIES

We collect in this section the basic concepts and notation necessary for accessing the rest of the article. Bibliographical indications will rely on classical texts. We are fortunate to be able to refer to a few very recent outstanding monographs, including [40, 46].

**2.1. Matrices of moments.** Our raw material consists of structured matrices of moments and functions acting on them. In this subsection, we concentrate on the first. Henceforth  $N$  is a positive integer.

**Definition 2.1.** Given a subset  $I \subset \mathbb{R}$ , denote by  $\mathcal{P}_N(I)$  the set of positive semidefinite  $N \times N$  matrices with entries in  $I$ , and let  $\mathcal{P}_N := \mathcal{P}_N(\mathbb{R})$ .

The set  $\mathcal{P}_N$  is a convex cone, closed in the Euclidean topology of  $\mathbb{R}^{N \times N}$ . Schur's product theorem asserts  $A \circ B \in \mathcal{P}_N$  whenever  $A, B \in \mathcal{P}_N$ ; here  $A \circ B = (a_{ij}b_{ij})$  denotes the entrywise product of two equidimensional matrices  $A = (a_{ij})$  and  $B = (b_{ij})$ . For a proof it is sufficient to decompose  $B$  into a sum of rank-one positive matrices and follow the definition of matrix positivity.

Recall that a matrix is said to be *totally non-negative* if all its minors are non-negative. Totally non-negative matrices occur in a variety of areas; see [13] and the references therein. For instance, a well-known observation due to Schoenberg asserts that given vectors  $x_1, x_2, \dots, x_N$ , in an inner-product space, the corresponding matrix  $(\exp(-\|x_j - x_k\|^2))_{j,k=1}^N$  is totally non-negative.

**Definition 2.2.** For an integer  $n \geq 1$ , let  $\mathcal{H}_n^{++}$  denote the set of  $n \times n$  totally non-negative Hankel matrices, and let  $\mathcal{H}^{++} := \bigcup_{n \geq 1} \mathcal{H}_n^{++}$  denote the set of totally non-negative Hankel matrices of arbitrary size.

The moment problem, in the widely accepted meaning of the term, is arguably the quintessential inverse problem. It has a long history and continues to lead to unexpected impacts in pure and applied mathematics; see, for instance, [1, 31, 40, 45]. Moments of positive measures are in general observables, with a physical or probabilistic interpretation. These observed real numbers are not free, but are subject to an array of semi-algebraic constraints, which are generally hard to deal with directly. A convenient and numerically friendly approach is to organize the moments into matrices with redundant entries, the simplest case being associated to measures supported on subsets of the real line. We will start with this generic situation.

Let  $\mu$  be a non-negative measure on  $\mathbb{R}$ , rapidly decreasing at infinity, that admits moments of all orders; let its moment data and associated Hankel matrix be denoted as follows:

$$s_k(\mu) = s_k := \int_{\mathbb{R}} x^k d\mu, \quad \mathbf{s}(\mu) := (s_k(\mu))_{k \geq 0}, \quad H_\mu := \begin{pmatrix} s_0 & s_1 & s_2 & \cdots \\ s_1 & s_2 & s_3 & \cdots \\ s_2 & s_3 & s_4 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \quad (2.1)$$

All measures appearing in this paper are taken to be non-negative and are assumed to possess moments of all orders. We will henceforth call such measures *admissible*.

Throughout this paper, we allow matrices to be semi-infinite in both coordinates. We also identify without further comment the space of real sequences  $(s_0, s_1, \dots)$  and the corresponding Hankel matrices, as done in (2.1).

To verify the positivity of the matrix  $H_\mu$ , it is sufficient to observe that

$$0 \leq \int \left| \sum_{j=0}^N c_j x^j \right|^2 d\mu = \sum_{j,k=0}^N H_\mu(j,k) c_j c_k.$$

**Definition 2.3.** Given subsets  $I, K \subset \mathbb{R}$ , let  $\text{Meas}^+(K)$  denote the admissible measures supported on  $K$ , and let  $\mathcal{H}^+(I)$  denote the set of complex Hermitian positive semidefinite Hankel matrices with entries in  $I$ . We will henceforth use the adjective ‘positive’ to mean ‘complex Hermitian positive semidefinite’ when applied to matrices.

The following theorem combines classical results of Hamburger, Stieltjes, and Hausdorff.

**Theorem 2.4.** *A sequence  $\mathbf{s} = (s_k)_{k=0}^\infty$  is a moment sequence for an admissible measure on  $\mathbb{R}$  if and only if the Hankel matrix with first column  $\mathbf{s}$  is positive. In other words, the map  $\Psi : \mu \mapsto (s_k(\mu))_{k=0}^\infty$  is a surjection from  $\text{Meas}^+(\mathbb{R})$  onto  $\mathcal{H}^+(\mathbb{R})$ . Moreover,*

- (1) *restricted to  $\text{Meas}^+([0, \infty))$ , the map  $\Psi$  is a surjection onto the positive Hankel matrices with non-negative entries, such that removing the first column still yields a positive matrix;*
- (2) *restricted to  $\text{Meas}^+([-1, 1])$ , the map  $\Psi$  is a bijection onto the positive Hankel matrices with uniformly bounded entries;*
- (3) *restricted to  $\text{Meas}^+([0, 1])$ , the map  $\Psi$  is a bijection onto the positive Hankel matrices with uniformly bounded entries, such that removing the first column still yields a positive matrix.*

*Proof.* The first assertion is classical; for example, see Akhiezer's book [1, Theorems 2.1.1, 2.6.4, and 2.6.5]. For the last two statements, we simply remark that for an admissible measure  $\mu$ ,

$$s_{2n}(\mu) = \int_{[-1,1]} x^{2n} d\mu + \int_{\mathbb{R} \setminus [-1,1]} x^{2n} d\mu.$$

The first integral remains uniformly bounded as a function of  $n$ , while the second tends to infinity with  $n$  whenever the measure  $\mu$  has positive mass on  $\mathbb{R} \setminus [-1, 1]$ .  $\square$

**Definition 2.5.** In view of the above correspondence, we denote by  $\mathcal{M}(K)$  the set of moment sequences associated to measures in  $\text{Meas}^+(K)$ . Equivalently,  $\mathcal{M}(K)$  is the collection of first columns of Hankel matrices associated to admissible measures supported on  $K$ . We write  $H^{(1)}$  to denote the truncation of a matrix  $H$  in which the first column is excised.

For technical reasons which will become apparent from the proofs below, we introduce an additional parameter via the following definition.

**Definition 2.6.** Given  $0 < \rho \leq \infty$  and  $I \subset \mathbb{R}$ , let  $\mathcal{M}^\rho(I)$  denote the set of moment sequences  $(s_k(\mu))_{k=0}^\infty$  of admissible measures  $\mu$  supported on  $I$ , with all moments in  $(-\rho, \rho)$ . Also, for any  $n \geq 0$ , let  $\mathcal{M}_n^\rho(I)$  denote the corresponding set of *truncated moment sequences*  $(s_k(\mu))_{k=0}^n$ .

Note that  $\mathcal{M}^\rho(I) = \mathcal{M}(I)$  and  $\mathcal{M}_n^\rho(I) = \mathcal{M}_n(I)$  when  $\rho = \infty$ . Moreover, for a non-negative measure  $\mu$  supported on  $[-1, 1]$ , the mass  $s_0(\mu)$  dominates  $|s_k(\mu)|$  for all  $k \geq 0$ . Studying moment sequences of admissible measures having mass  $s_0 < \rho$  is therefore equivalent to working with Hankel matrices with entries in a bounded interval  $(-\rho, \rho)$ . This will be our approach in the remainder of the paper.

A simple characterization of rank-one Hankel matrices is stated below.

**Lemma 2.7.** *A rank-one  $N \times N$  matrix  $\mathbf{u}\mathbf{u}^T$ , with entries in any field, is Hankel if and only if either the successive entries of  $\mathbf{u}$  are in a geometric progression, or all entries but the last are 0. More precisely, the matrix  $\mathbf{u}\mathbf{u}^T$  is Hankel if and only if*

$$u_j = \begin{cases} u_1(u_2/u_1)^{j-1} & \text{if } u_1 \neq 0, \\ 0 & \text{if } u_1 = 0 \text{ and } 1 \leq j < N. \end{cases} \quad (2.2)$$

*Proof.* This is immediate for  $N \geq 2$ . For  $N > 2$ , each principal  $3 \times 3$  block submatrix of  $\mathbf{u}\mathbf{u}^T$  with successive rows and columns is of the form

$$\begin{pmatrix} u_{j-1}^2 & u_{j-1}u_j & u_{j-1}u_{j+1} \\ u_j u_{j-1} & u_j^2 & u_j u_{j+1} \\ u_{j+1}u_{j-1} & u_{j+1}u_j & u_{j+1}^2 \end{pmatrix},$$

whence  $u_{j-1}u_{j+1} = u_j^2$  for all  $j \geq 2$ . Identity (2.2) follows immediately.  $\square$

We invite the reader to find all positive measures on the real line which produce a rank-one Hankel matrix. In general, one can read off from a positive Hankel matrix whether the representing measure is unique, and estimate the shape of the support of the representing measure(s) (of utmost importance in polynomial optimization), and enter into the Lebesgue decomposition of the representing measure(s). We refer

to [1, 31, 40] for aspects of such refined analysis pertaining to the moment problem and its current applications.

In Section 9, we will treat multivariable moment problems. In that context, Hankel matrices are replaced by kernels with a Hankel-type property. The semigroup approach proves to be superior in the multivariable setting; see [6] for more details.

To conclude, we note that the study of Hankel matrices forms an important chapter of modern analysis, with ramifications for approximation theory, probability theory and control theory [35].

**2.2. Absolutely monotonic functions.** We turn now to operators on moments by identifying two relevant classes of functions.

Central to our study is the class of *absolutely monotonic entire functions*. These are entire functions with non-negative Taylor coefficients at every point of  $(0, \infty)$ . Equivalently, it is sufficient for such a function to have non-negative Taylor coefficients at zero. Their structure was unveiled in a fundamental memoir by Bernstein [7]; see also Widder's book [54] or the recent treatise [39].

One can restrict the absolute monotonicity definition to a finite interval, with the following outcome.

**Theorem 2.8** ([54, Chapter IV, Theorem 3a]). *If  $f$  is absolutely monotonic on  $[a, b]$ , then it can be extended analytically to the complex disc centered at  $a$  and of radius  $b - a$ .*

Recall that a function is said to be *completely monotonic* on an interval  $(a, b)$  if the map  $x \mapsto f(-x)$  is absolutely monotonic on  $(-b, -a)$ , i.e., if  $(-1)^k f^{(k)}(x) \geq 0$  for all  $x \in (a, b)$ . Similarly, a function is completely monotonic on an interval  $I \subset \mathbb{R}$  if it is continuous on  $I$  and is completely monotonic on the interior of  $I$ .

Complete monotonicity can also be defined using finite differences. Let  $\Delta_h^n f$  denote the  $n$ th forward difference of  $f$  with step size  $h$ :

$$\Delta_h^n f(x) := \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} f(x + kh).$$

Then  $f$  is completely monotonic on  $(a, b)$  if and only if  $(-1)^n \Delta_h^n f(x) \geq 0$  for all non-negative integers  $n$  and for all  $x, h$  such that  $a < x < x + h < \dots < x + nh < b$ . See [54, Chapter IV] for more details on completely monotonic functions. Such functions were also characterized in a celebrated result of Bernstein.

**Theorem 2.9** (Bernstein [54, Chapter IV, Theorem 12a]). *A function  $f : [0, \infty) \rightarrow \mathbb{R}$  is completely monotonic on  $0 \leq x < \infty$  if and only if*

$$f(x) = \int_0^\infty e^{-xt} d\mu(t)$$

*for some finite positive measure  $\mu$ .*

Atomic measures are not excluded in Bernstein's theorem, hence series of exponentials and Dirichlet series are an integral part of the theory of absolutely or completely monotonic functions. One of the major advantages of absolute monotonicity is the analytic extension of the respective function to a complex domain. We will exploit this quality further on in the present work.

**2.3. Matrix positivity transforms.** The main theme of our work is permanence properties of moment matrices  $A$  under *entrywise* operations. From the very beginning we warn the reader that our framework is in contrast to the classical functional calculus  $A \mapsto f(A)$  which is the subject of Löwner’s celebrated theorem: *a real function  $f$  preserves matrix ordering (i.e.,  $A \leq B$  implies  $f(A) \leq f(B)$ ) among self-adjoint matrices if and only if  $f$  extends analytically to the upper-half plane and it has positive imaginary part there.* For ample details and a dozen different proofs, see [12, 46].

Entrywise operations on matrices and kernels also have a long and interesting history, see [4]. We will provide the outlines of a few significant results.

Transformations which leave invariant Fourier transforms of various classes of measures on groups or homogeneous spaces were studied by many authors, including Schoenberg [42], Bochner [9], Helson, Kahane, Katznelson, and Rudin [23, 28]. From the latter works, Rudin extracted [38] an analysis of maps which preserve moment sequences for admissible measures on the torus; equivalently, these are functions which, when applied entrywise, leave invariant the cone of positive semidefinite Toeplitz matrices. Rudin’s result, originally proved by Schoenberg [42] under a continuity assumption, is as follows.

**Theorem 2.10** (Schoenberg, Rudin). *Given a function  $F : (-1, 1) \rightarrow \mathbb{R}$ , the following are equivalent.*

- (1) *Applied entrywise,  $F$  preserves positivity on the space of positive matrices with entries in  $(-1, 1)$  of all dimensions.*
- (2) *Applied entrywise,  $F$  preserves positivity on the space of positive Toeplitz matrices with entries in  $(-1, 1)$  of all dimensions.*
- (3) *The function  $F$  is real analytic on  $(-1, 1)$  and absolutely monotonic on  $(0, 1)$ .*

The facts that (3)  $\implies$  (1) and (3)  $\implies$  (2) follow from the Schur product theorem [44]. However, the converse results are highly nontrivial.

In the present paper, we consider moments of measures on the line rather than Fourier coefficients, so power moments rather than complex exponential moments. Hence we study functions  $F$  mapping moment sequences entrywise into themselves, i.e., such that for every admissible measure  $\mu$ , there exists an admissible measure  $\sigma = \sigma_\mu$  satisfying

$$F(s_k(\mu)) = s_k(\sigma) \quad \text{for all } k \geq 0.$$

Equivalently, by Theorem 2.4, we study entrywise endomorphisms of the cone of positive Hankel matrices with real entries. The following notion of entrywise calculus is central to this paper.

**Definition 2.11.** Given a domain  $D \subset \mathbb{R}$  and a function  $F : D \rightarrow \mathbb{R}$ , the function  $F[-]$  acts on the set of matrices with entries in  $D$ , by applying  $F$  entrywise:

$$F[A] := (F(a_{ij})) \quad \text{for the matrix } A = (a_{ij}).$$

The function  $F$  also acts entrywise on moment sequences with all moments in  $D$ , so that  $F[s(\mu)]_k := F(s_k(\mu))$  for all  $k \geq 0$ , and similarly for truncated moment sequences.

An observation on positivity preservers made by Löwner and developed by Horn [25] provides the following necessary condition for a function to preserve positivity on  $\mathcal{P}_N((0, \infty))$  when applied entrywise.

**Theorem 2.12** (Horn). *If a continuous function  $F : (0, \infty) \rightarrow \mathbb{R}$  is such that  $F[-] : \mathcal{P}_N((0, \infty)) \rightarrow \mathcal{P}_N(\mathbb{R})$ , then  $F \in C^{N-3}((0, \infty))$  and  $F^{(k)}(x) \geq 0$  for all  $x > 0$  and all  $0 \leq k \leq N - 3$ . Moreover, if it is known that  $F \in C^{N-1}((0, \infty))$ , then  $F^{(k)}(x) \geq 0$  for all  $x > 0$  and all  $0 \leq k \leq N - 1$ .*

The main idea in the proof is to develop into Taylor series a perturbation determinant

$$\det[F(a + tu_j u_k)]_{j,k=1}^N$$

and isolate the first non-zero coefficient as a universal constant times the product  $F(a)F'(a) \cdots F^{(N-3)}(a)$ . Our prior work in fixed dimension has amply exploited the symmetry and combinatorial flavor of similar determinants [3].

### 3. MAIN RESULTS IN 1D

We state in this brief section our main results, restricted to the one-variable case. The proofs will be given in subsequent sections with a gradual increase in technicality, which also applies the statements of these results. A leading thread is the isolation of minimal sets of matrices for the verification of preservers, without altering the conclusions. We remind the reader that all functions in this article act entry by entry on moment sequences and matrices.

The following theorem, the first in a series to be established below, gives an idea of the type of positive Hankel-matrix preservers we seek.

**Theorem 3.1.** *A function  $F : \mathbb{R} \rightarrow \mathbb{R}$  maps  $\mathcal{M}([-1, 1])$  into itself when applied entrywise, if and only if  $F$  is the restriction to  $\mathbb{R}$  of an absolutely monotonic entire function.*

In particular, Theorem 3.1 strengthens the Schoenberg–Rudin Theorem 2.10, by relaxing the assumptions in [38, 42] to require positivity preservation only for Hankel matrices. Theorem 3.1 is proved in Section 6 with three further strengthenings: we use test sets with at most three points (corresponding to Hankel test matrices of rank at most three), the measures are allowed to have a mass constraint, enabling us to classify functions  $F : (-\rho, \rho) \rightarrow \mathbb{R}$ , where  $0 < \rho \leq \infty$ , and we show that allowing functions to map entrywise into the co-domain  $\mathcal{M}(\mathbb{R})$  does not enlarge the class of preservers.

Our next result is a one-sided variant of the above characterizations, following Horn [25, Theorem 1.2]. Akin to Theorem 3.1, it arrives at the same conclusion under weaker assumptions than in [25].

**Theorem 3.2.** *A function  $F : [0, \infty) \rightarrow \mathbb{R}$  maps  $\mathcal{M}([0, 1])$  into itself when applied entrywise, if and only if  $F$  is absolutely monotonic on  $(0, \infty)$ , so non-decreasing, and  $0 \leq F(0) \leq \lim_{\epsilon \rightarrow 0^+} F(\epsilon)$ .*

In Section 4, we use results of Bernstein and Lorch–Newman to prove Theorem 3.2, and then provide a strengthening of it, Theorem 4.1, in the spirit described above after Theorem 3.1. Here, we can replace  $\mathcal{M}([0, 1])$  by test measures supported on at most two points.

Next, we provide a classification of the preservers of  $\mathcal{M}([0, \infty))$ , Theorem 3.3, which gives a Schoenberg-type characterization of functions preserving total non-negativity. It is akin to Theorem 3.2, and provides a connection between moment sequences, totally non-negative Hankel matrices, and their preservers; see Section 5 for the proof.

**Theorem 3.3.** *For a function  $F : [0, \infty) \rightarrow \mathbb{R}$ , the following are equivalent.*

- (1) *Applied entrywise, the function  $F$  preserves positive semidefiniteness on the set  $\mathcal{H}^{++}$  of all totally non-negative Hankel matrices.*
- (2) *Applied entrywise, the function  $F$  preserves the set  $\mathcal{H}^{++}$ .*
- (3) *Applied entrywise, the function  $F$  sends  $\mathcal{M}([0, \infty))$  to itself.*
- (4) *The function  $F$  agrees on  $(0, \infty)$  with an absolutely monotonic entire function and  $0 \leq F(0) \leq \lim_{\epsilon \rightarrow 0^+} F(\epsilon)$ .*

Our techniques lead to the following observation: the only non-constant maps which preserve the set of all totally non-negative matrices when applied entrywise are of the form  $F(x) = cx$ , where  $c > 0$ . See Theorem 5.7 for more details.

Returning to moment sequences, in the present paper we also study preservers of  $\mathcal{M}([-1, 0])$ , and show that these are classified as follows.

**Theorem 3.4.** *The following are equivalent for a function  $F : \mathbb{R} \rightarrow \mathbb{R}$ .*

- (1) *Applied entrywise,  $F$  maps  $\mathcal{M}([-1, 0])$  into  $\mathcal{M}((-\infty, 0])$ .*
- (2) *There exists an absolutely monotonic entire function  $\tilde{F}$  such that*

$$F(x) = \begin{cases} \tilde{F}(x) & \text{if } x \in (0, \infty), \\ 0 & \text{if } x = 0, \\ -\tilde{F}(-x) & \text{if } x \in (-\infty, 0). \end{cases}$$

It is striking to observe the possibility of a discontinuity at the origin, in both of the previous theorems. For the proof of this result, we refer the reader to Section 7.

We also derive a similar description of the functions that transform  $\mathcal{M}([-1, 0])$  into  $\mathcal{M}([0, \infty))$ ; see Theorem 7.3. In this variant, we observe that  $F$  may also be discontinuous at 0.

The arguments used to show Theorem 2.10 and its one-sided variant by Schoenberg, Rudin, and Horn do not carry over to our setting involving positive Hankel matrices. This is due to the fact that the hypotheses in Theorems 3.1 and 3.2 are significantly weaker.

We show below how to further relax quite substantially the assumptions in Theorem 3.1 (Section 6), Theorem 3.2 (Section 4), and Theorem 3.4 (Section 7). In doing so, our goal is to understand the minimal amount of information that is equivalent to the requirement that a function preserves  $\mathcal{M}([0, 1])$  or  $\mathcal{M}([-1, 1])$  when applied entrywise. We will demonstrate that requiring a function to preserve moments for measures supported on at most three points, is equivalent to preserving moments for all measures. In particular, this shows that preserving positivity for positive Hankel matrices of rank at most three implies positivity preservation for all positive matrices.

This latter point prompts a comparison to the case of Toeplitz matrices considered in [38]. Rudin proved that Theorem 2.10(3) holds if  $F$  preserves positivity on a two-parameter family of Toeplitz matrices with rank at most 3, namely

$$\{(a + b \cos((i - j)\theta))_{i,j \geq 1} : a, b \geq 0, a + b < 1\}, \quad (3.1)$$

where  $\theta$  is a fixed real number such that  $\theta/\pi$  is irrational. Similarly, the present work shows that for power moments, it suffices to work with families of positive Hankel matrices of rank at most three. Theorem 6.1(1) contains the precise details.

4. MOMENT TRANSFORMERS ON  $[0, 1]$ 

Over the course of the next four sections, we will formulate and prove strengthened versions of the announced results.

Here, we provide two proofs of Theorem 3.2. The first is natural from the point of view of moments and Hankel matrices. The proof proceeds by first deriving from positivity considerations some inequalities satisfied by all moments transformers. We then obtain the desired characterization by appealing to classical results on completely monotonic functions. This is in the spirit of Lorch and Newman [32], who in turn are very much indebted to the original Hausdorff approach to the moment problem via summation rules and higher-order finite differences.

Using Theorem 2.9, we now provide our first proof of Theorem 3.2.

*Proof 1 of Theorem 3.2.* The ‘if’ part follows from two statements: (i) absolutely monotonic entire functions preserve positivity on all matrices of all orders, by the Schur product theorem; (ii) moment matrices from elements of  $\mathcal{M}([0, 1])$  have zero entries if and only if  $\mu = a\delta_0$  for some  $a \geq 0$ .

Conversely, suppose the function  $F$  preserves  $\mathcal{M}([0, 1])$  when applied entrywise, i.e., given any  $\mu \in \text{Meas}^+([0, 1])$ , there exists  $\sigma \in \text{Meas}^+([0, 1])$  such that

$$F(s_k(\mu)) = s_k(\sigma) \quad \text{for all } k \geq 0.$$

Let  $p(t) = a_0t^0 + \dots + a_d t^d$  be a real polynomial such that  $p(t) \geq 0$  on  $[0, 1]$ . Then,

$$0 \leq \int_0^1 p(t) d\sigma(t) = \sum_{k=0}^d a_k s_k(\sigma) = \sum_{k=0}^d a_k F(s_k(\mu)). \quad (4.1)$$

Here and below, we employ (4.1) with a careful choice of measure  $\mu$  and polynomial  $p$  to deduce additional information about the function  $F$ . In the present situation, fix finitely many scalars  $c_j$ ,  $t_j > 0$  and an integer  $n \geq 0$ , and set

$$p(t) = (1-t)^n \quad \text{and} \quad \mu = \sum_j e^{-t_j \alpha} c_j \delta_{e^{-t_j h}}, \quad (4.2)$$

where  $\alpha > 0$  and  $h > 0$ . Now let  $g(x) := \sum_j c_j e^{-t_j x}$ , and apply (4.1) to see that the forward finite differences of  $F \circ g$  alternate in sign. That is,

$$\sum_{k=0}^n (-1)^k \binom{n}{k} F \left( \sum_j c_j e^{-t_j \alpha - t_j k h} \right) \geq 0,$$

so  $(-1)^n \Delta_h^n (F \circ g)(\alpha) \geq 0$ . As this holds for all  $\alpha$ ,  $h > 0$  and all  $n \geq 0$ , it follows that  $F \circ g : (0, \infty) \rightarrow (0, \infty)$  is completely monotonic for all  $\mu$  as in (4.2). Using the weak density of such measures in  $\text{Meas}^+((0, \infty))$ , together with Bernstein’s theorem (Theorem 2.9), it follows that  $F \circ g$  is completely monotonic on  $(0, \infty)$  for all completely monotonic functions  $g : (0, \infty) \rightarrow (0, \infty)$ . Finally, a theorem of Lorch and Newman [32, Theorem 5] now gives that  $F : (0, \infty) \rightarrow (0, \infty)$  is absolutely monotonic.  $\square$

Our second proof of Theorem 3.2 involves a significant relaxation of its hypotheses. Our first observation is that, if  $F$  preserves positivity for  $2 \times 2$  matrices, and sends  $\mathcal{M}(\{1, u_0\})$  to  $\mathcal{M}(\mathbb{R})$  for a single  $u_0 \in (0, 1)$ , then  $F$  is absolutely monotonic on  $(0, \infty)$ . Further relaxation may be obtained by working with mass-constrained measures.

**Theorem 4.1.** *Fix scalars  $\rho$  and  $u_0$ , with  $0 < \rho \leq \infty$  and  $u_0 \in (0, 1)$ . Given a function  $F : [0, \rho) \rightarrow \mathbb{R}$ , the following are equivalent.*

- (1) *The map  $F[-]$  sends  $\mathcal{M}^\rho(\{1, u_0\}) \cup \mathcal{M}^\rho(\{0, 1\})$  into  $\mathcal{M}(\mathbb{R})$ , and  $F(a)F(b) \geq F(\sqrt{ab})^2$  for all  $a, b \in [0, \rho)$ .*
- (2) *The map  $F[-]$  sends  $\mathcal{M}^\rho([0, 1])$  into  $\mathcal{M}([0, 1])$ .*
- (3) *The function  $F$  agrees on  $(0, \rho)$  with an absolutely monotonic entire function and  $0 \leq F(0) \leq \lim_{\epsilon \rightarrow 0^+} F(\epsilon)$ .*

*If  $F$  is known to be continuous on  $(0, \rho)$ , then the second hypothesis in (1) may be omitted.*

Note that assertion (1) is *a priori* significantly weaker than the requirement that  $F$  preserves  $\mathcal{M}([0, 1])$ , at least when  $\rho = \infty$ , say. Moreover, hypothesis (3) here is the same as hypothesis (4) in Theorem 3.3, and Theorem 4.1 is used to prove that result in Section 5.

We now turn to proving Theorem 4.1. This requires results on functions preserving positivity for matrices of a fixed dimension, which we now develop.

As shown in [21, Theorem 4.1], the same result can be obtained by working only with a particular family of rank-two matrices, without the continuity assumption, and on any domain  $(0, \rho)$  as above. In the next theorem, Horn's hypotheses are relaxed even further by making appeal only to Hankel matrices.

**Theorem 4.2.** *Let  $F : I \rightarrow \mathbb{R}$ , where  $I := (0, \rho)$  and  $0 < \rho \leq \infty$ . Fix  $u_0 \in (0, 1)$  and an integer  $N \geq 3$ , and let  $\mathbf{u} := (1, u_0, \dots, u_0^{N-1})^T$ . Suppose  $F[-]$  preserves positivity on  $\mathcal{P}_2(I)$ , and  $F[A] \in \mathcal{P}_N(\mathbb{R})$  for the family of Hankel matrices*

$$\{A = a\mathbf{1}_{N \times N} + b\mathbf{u}\mathbf{u}^T : a \in [0, \rho), b \in [0, \rho - a), 0 < a + b < \rho\}. \quad (4.3)$$

*Then  $F \in C^{N-3}(I)$ , with*

$$F^{(k)}(x) \geq 0 \quad \text{for all } x \in I \quad (0 \leq k \leq N - 3),$$

*and  $F^{(N-3)}$  is a convex non-decreasing function on  $I$ . If, further,  $F \in C^{N-1}(I)$ , then  $F^{(k)}(x) \geq 0$  for all  $x \in I$  and  $0 \leq k \leq N - 1$ .*

*Finally, if  $F$  is assumed to be continuous on  $I$ , then the assumption that  $F$  preserves positivity on  $\mathcal{P}_2(I)$  is not necessary.*

**Remark 4.3.** In fact, our proof of Theorem 4.2 reveals that these hypotheses may be relaxed slightly, by replacing the test set  $\mathcal{P}_2((0, \rho))$  with the collection of rank-one matrices  $\mathcal{P}_2^1((0, \rho))$  and all matrices of the form

$$\begin{pmatrix} a & b \\ b & b \end{pmatrix} \quad \text{with } a > b > 0. \quad (4.4)$$

The proof of Theorem 4.2 relies on Lemma 2.7.

*Proof of Theorem 4.2.* If  $F \in C(I)$ , then the result follows by repeating the argument in [25, Theorem 1.2], but with the vector  $\alpha$  replaced by a vector  $\mathbf{u} \in \mathbb{R}^N$  as in Lemma 2.7.

Now suppose  $F$  is an arbitrary function, which is not identically zero on  $(0, \rho)$ ; we claim that  $F$  must be continuous. We first show that  $F(x) \neq 0$  for all  $x \in (0, \rho)$ . Indeed, suppose  $F(c) = 0$  for some  $c \in (0, \rho)$ . Given  $d \in (c, \rho)$ , define a sufficiently long

geometric progression  $u'_0 = c, \dots, u'_n = d$ , such that  $u'_{n+1} \in (d, \rho)$ . By considering the matrices

$$F[A_j], \quad \text{where } A_j := \begin{pmatrix} u'_j & u'_{j+1} \\ u'_{j+1} & u'_{j+2} \end{pmatrix}, \quad 0 \leq j \leq n-1,$$

we obtain that  $F(d) = 0$  for all  $d \in (c, \rho)$ . A similar argument applies to  $d \in (0, c)$ , showing that  $F \equiv 0$  on  $(0, \rho)$ .

Next, since  $F[-]$  preserves positivity on  $\mathcal{P}_2^1((0, \rho))$  and is positive on  $(0, \rho)$ , it follows that  $g : x \mapsto \log F(e^x)$  is midpoint convex on the interval  $(-\infty, \log \rho)$ . Moreover, applying  $F[-]$  to matrices of the form (4.4) shows that  $F$  is non-decreasing. Hence, by [37, Theorem 71.C], the function  $g$  is necessarily continuous on  $(-\infty, \log \rho)$ , and so  $F$  is continuous on  $(0, \rho)$ . This proves the result in the general case.  $\square$

Using the above result, we can now prove Theorem 4.2.

*Proof of Theorem 4.2.* If  $F \in C(I)$ , then the result follows by repeating the argument in [25, Theorem 1.2], but with the vector  $\alpha$  replaced by a vector  $\mathbf{u} \in \mathbb{R}^N$  as in Lemma 2.7.

Now suppose  $F$  is an arbitrary function, which is not identically zero on  $(0, \rho)$ ; we claim that  $F$  must be continuous. We first show that  $F(x) \neq 0$  for all  $x \in (0, \rho)$ . Indeed, suppose  $F(c) = 0$  for some  $c \in (0, \rho)$ . Given  $d \in (c, \rho)$ , define a sufficiently long geometric progression  $u'_0 = c, \dots, u'_n = d$ , such that  $u'_{n+1} \in (d, \rho)$ . By considering the matrices

$$F[A_j], \quad \text{where } A_j := \begin{pmatrix} u'_j & u'_{j+1} \\ u'_{j+1} & u'_{j+2} \end{pmatrix}, \quad 0 \leq j \leq n-1,$$

we obtain that  $F(d) = 0$  for all  $d \in (c, \rho)$ . A similar argument applies to  $d \in (0, c)$ , showing that  $F \equiv 0$  on  $(0, \rho)$ .

Next, since  $F[-]$  preserves positivity on  $\mathcal{P}_2^1((0, \rho))$  and is positive on  $(0, \rho)$ , it follows that  $g : x \mapsto \log F(e^x)$  is midpoint convex on the interval  $(-\infty, \log \rho)$ . Moreover, applying  $F[-]$  to matrices of the form (4.4) shows that  $F$  is non-decreasing. Hence, by [37, Theorem 71.C], the function  $g$  is necessarily continuous on  $(-\infty, \log \rho)$ , and so  $F$  is continuous on  $(0, \rho)$ . This proves the result in the general case.  $\square$

Finally, we turn to the proof of Theorem 4.1, which provides a second proof of Theorem 3.2 which is more informative. We first observe that Theorem 4.2 can be reformulated in terms of moment sequences, using the fact that the matrices occurring in the statement of the theorem can be realized as truncations of positive Hankel matrices; see Definition 2.5.

**Theorem 4.4.** *Let  $F : I \rightarrow \mathbb{R}$ , where  $I = (0, \rho)$  and  $0 < \rho \leq \infty$ , and fix  $N \geq 3$ . Suppose  $F[-]$  maps the moment sequences in  $\mathcal{M}_{2N-2}^\rho(\{1, u_0\})$  with positive entries to*

$$\{(s_0(\mu), \dots, s_{2N-3}(\mu), s_{2N-2}(\mu) + t) : \mu \in \text{Meas}^+(\mathbb{R}), t \geq 0\}$$

*for some  $u_0 \in (0, 1)$ , and the moment sequences in  $\mathcal{M}_2^\rho(\{0, 1\}) \cup \mathcal{M}_2^\rho(\{u\})$  with positive entries to  $\mathcal{M}_2(\mathbb{R})$  for all  $u \in (0, 1)$ . Then  $F \in C^{N-3}(I)$ , with*

$$F^{(k)}(x) \geq 0 \quad \text{for all } x \in I \quad (0 \leq k \leq N-3),$$

*and  $F^{(N-3)}$  is a convex non-decreasing function on  $I$ . If, further, it is known that  $F \in C^{N-1}(I)$ , then  $F^{(k)}(x) \geq 0$  for all  $x > 0$  and  $0 \leq k \leq N-1$ .*

If  $F$  is continuous on  $I$ , then the assumption that  $F[-]$  maps elements of  $\mathcal{M}_2^\rho(\{u\})$  into  $\mathcal{M}_2(\mathbb{R})$  for all  $u \in (0, 1)$  may be omitted.

*Proof.* In view of Hamburger's Theorem for truncated moment sequences, a Hankel matrix with entries in the first and last columns given by

$$s_0, \dots, s_{N-1} \quad \text{and} \quad s_{N-1}, \dots, s_{2N-2}$$

is positive if and only if  $(s_0, \dots, s_{2N-3}) \in \mathcal{M}_{2N-3}(\mathbb{R})$ , and  $s_{2N-2} \geq \int x^{2N-2} d\mu$ , where  $\mu$  is any non-negative measure with the first  $2N-2$  moments equal to  $(s_0, \dots, s_{2N-3})$ . (For details, see Akhiezer's book [1, Theorem 2.6.3].)

Furthermore, in order to show continuity in Theorem 4.2 we only required  $2 \times 2$  submatrices, of the form (4.4) or of rank one. Moreover, every matrix in  $\mathcal{P}_2(\mathbb{R})$  is a truncated moment matrix.

These observations show that Theorem 4.4 is equivalent to Theorem 4.2.  $\square$

We now prove Theorem 4.1, with the help of Theorem 4.4.

*Proof of Theorem 4.1.* Clearly (2)  $\implies$  (1). Next, assume (3) holds, and suppose  $\mu \in \text{Meas}^+([0, 1])$  with  $s_0(\mu) < \rho$ . If  $\mu = a\delta_0$  for some  $a \geq 0$  then (2) is immediate; henceforth we will assume  $H_\mu$  has no zero entries, where  $H_\mu$  is as defined in (2.1). Now,  $F[H_\mu]$  is positive, by the Schur product theorem and the fact that the only moment matrices arising from elements of  $\mathcal{M}^\rho([0, 1])$  with zero entries come from  $\mathcal{M}^\rho(\{0\})$ . Clearly  $F[s(\mu)]$  is uniformly bounded, hence comes from a unique measure  $\sigma$  supported on  $[-1, 1]$ , by Theorem 2.4. Recalling Definition 2.5, we have that

$$F[H_\mu]^{(1)} = \sum_{n \geq 0} c_n [H_\mu^{(1)}]^{on},$$

where  $F(x) = \sum_{n \geq 0} c_n x^n$  by the hypotheses and Theorem 2.8. Note that  $F[H_\mu]^{(1)}$  is positive, by the above computation and Theorem 2.4, since  $\mu$  is supported on  $[0, 1]$ . By the same result,  $\sigma \in \text{Meas}^+([0, 1])$ , which gives (2).

It remains to show (1)  $\implies$  (3). It is immediate that mapping  $\mathcal{M}_2^\rho(\{0, 1\})$  into  $\mathcal{M}_2(\mathbb{R})$  is equivalent to mapping  $\mathcal{M}^\rho(\{0, 1\})$  into  $\mathcal{M}(\mathbb{R})$ . Thus, by Theorem 4.4, it holds that  $F^{(k)}(x) \geq 0$  for all  $x > 0$  and all  $k \geq 0$ . Theorem 2.8 now gives the result, apart from the assertion about  $F(0)$ , but this is immediate.  $\square$

We conclude this part by explaining why Theorem 4.1 provides a minimal set of rank-constrained positive semidefinite matrices for which positivity preservation is equivalent to absolute monotonicity.

**Definition 4.5.** For  $1 \leq k \leq N$ , let  $\mathcal{P}_N^k(I)$  denote the matrices in  $\mathcal{P}_N(I)$  of rank at most  $k$ .

**Remark 4.6.** A smaller set of rank-constrained matrices than that employed for Theorem 4.1 could not include a sequence of matrices in  $\bigcup_{N=1}^\infty \mathcal{P}_N^2([0, \rho])$  of unbounded dimension, hence would be contained in  $\mathcal{P}'_N := \bigcup_{n=1}^N \mathcal{P}_n^2([0, \rho]) \cup \bigcup_{n=1}^\infty \mathcal{P}_n^1([0, \rho])$  for some  $N \geq 1$ . However, as noted in the paragraphs preceding Proposition 4.11 below, the map  $x \mapsto x^\alpha$  preserves positivity on  $\mathcal{P}'_N$  for all  $\alpha \geq N-2$ , and such a function may be non-analytic.

**Remark 4.7.** The proof of Theorem 4.1 also strengthens a 1979 result of Vasudeva [52]. Vasudeva showed for  $I = (0, \infty)$  that if  $F : I \rightarrow \mathbb{R}$  preserves positivity entrywise on  $\mathcal{P}_N(I)$  for all  $N \geq 1$  then  $F$  is absolutely monotonic and so is represented by a convergent power series on  $I$ . The proof above shows that Vasudeva's result also holds if  $I$  is replaced by  $(0, \rho)$  for any  $\rho > 0$  and, for every  $N$ , the set  $\mathcal{P}_N(I)$  is replaced by the subset of Hankel matrices within it of rank at most 2.

**4.1. Hankel-matrix positivity preservers in fixed dimension.** We conclude this section by addressing briefly the fixed-dimension case for powers and analytic functions, as studied by FitzGerald and Horn, and also in previous work by the authors [3, 15, 20]. Our first result shows that considerations of Hankel matrices may be used to strengthen the main result in [3].

**Theorem 4.8.** *Fix  $\rho > 0$  and integers  $N \geq 1$  and  $M \geq 0$ , and let  $F(z) = \sum_{j=0}^{N-1} c_j z^j + c' z^M$  be a polynomial with real coefficients. The following are equivalent.*

- (1)  $F[-]$  preserves positivity on  $\mathcal{P}_N(\overline{D}(0, \rho))$ , where  $\overline{D}(0, \rho)$  is the closed disc in the complex plane with center 0 and radius  $\rho$ .
- (2) The coefficients  $c_j$  satisfy either  $c_0, \dots, c_{N-1}, c' \geq 0$ , or  $c_0, \dots, c_{N-1} > 0$  and  $c' \geq -\mathcal{C}(\mathbf{c}; z^M; N, \rho)^{-1}$ , where

$$\mathcal{C}(\mathbf{c}; z^M; N, \rho) := \sum_{j=0}^{N-1} \binom{M}{j}^2 \binom{M-j-1}{N-j-1}^2 \frac{\rho^{M-j}}{c_j}.$$

- (3)  $F[-]$  preserves positivity on Hankel matrices in  $\mathcal{P}_N^1((0, \rho))$ .

The strengthening here is the addition of the word ‘Hankel’ to hypothesis (3).

**Remark 4.9.** As the following proof of Theorem 4.8 shows, assumption (3) can be relaxed further, by assuming  $F$  preserves positivity on a distinguished family of Hankel matrices. More precisely, it can be replaced by

- (3')  $F[-]$  preserves positivity on two sequences of rank-one Hankel matrices,

$$\{b^n \rho \mathbf{u}(b) \mathbf{u}(b)^T, \rho \mathbf{u}(b^n) \mathbf{u}(b^n)^T : n \geq 1\}, \quad \text{for any fixed } b \in (0, 1),$$

where

$$\mathbf{u}(\epsilon) := (1 - \epsilon, (1 - \epsilon)^2, \dots, (1 - \epsilon)^N)^T, \quad \text{for any } \epsilon \in (0, 1). \quad (4.5)$$

Note that  $\mathbf{u}(\epsilon) \mathbf{u}(\epsilon)^T \in \mathcal{P}_N^1(\mathbb{R})$  is Hankel, by Lemma 2.7.

Thus, Remark 4.9 gives a notable reduction of the  $N$ -dimensional parameter space,  $\mathcal{P}_N^1((0, \rho))$ , to the countable subset of Hankel matrices required in (3'). If  $N > 1$ , this is indeed minimal information required to derive Theorem 4.8(2), since the extreme critical value  $\mathcal{C}(\mathbf{c}; z^M; N, \rho)$  cannot be attained on any finite set of matrices in  $\mathcal{P}_N^1((0, \rho))$ .

As a first step towards the proof of Theorem 4.8, we recall from [3, Lemma 2.4] that, under suitable differentiability assumptions, the conclusions of Theorem 4.2 still hold if one considers only rank-one matrices. We now formulate a slightly stronger version of this result.

**Proposition 4.10.** *Let  $F \in C^\infty((-\rho, \rho))$ , where  $0 < \rho \leq \infty$ . Fix a vector  $\mathbf{u} \in (0, \sqrt{\rho})^N$  with distinct coordinates, and suppose  $F[b_n \mathbf{u} \mathbf{u}^T] \in \mathcal{P}_N(\mathbb{R})$  for a positive real sequence  $b_n \rightarrow 0^+$ . Then the first  $N$  non-zero derivatives of  $F$  at 0 are strictly positive.*

The assumptions and conclusions of this result are similar to those of Theorem 4.2 above; a common generalization of both results can be found in [29].

*Proof.* For ease of exposition, we will assume  $F$  has at least  $N$  non-zero derivatives at 0, say of orders  $m_1 < \dots < m_N$ , where  $m_1 \geq 0$ . By results on generalized Vandermonde determinants [17, Chapter XIII, §8, Example 1], the vectors  $\{\mathbf{u}^{om_j} : 1 \leq j \leq N\}$  are linearly independent. Now, by Taylor's theorem,

$$F[b_n \mathbf{u} \mathbf{u}^T] = \sum_{j=1}^N \frac{F^{(m_j)}(0)}{m_j!} b_n^{m_j} \mathbf{u}^{om_j} (\mathbf{u}^{om_j})^T + o(b_n^{m_N}). \quad (4.6)$$

For each  $1 \leq k \leq N$ , choose  $\mathbf{v}_k \in \mathbb{R}^N$  such that  $\mathbf{v}_k^T \mathbf{u}^{om_j} = \delta_{j,k} m_j!$ . Then,

$$b_n^{-m_k} \mathbf{v}_k^T F[b_n \mathbf{u} \mathbf{u}^T] \mathbf{v}_k = m_k! F^{(m_k)}(0) + o(b_n^{m_N - m_k}) \geq 0,$$

and letting  $n \rightarrow \infty$  concludes the proof.  $\square$

We now use Proposition 4.10 to prove the theorem.

*Proof of Theorem 4.8.* In view of Remark 4.9 and [3, Theorem 1.1], it suffices to show that (3')  $\implies$  (2).

Assume (3') holds, and consider first the sequence  $b^n \rho \mathbf{u}(b) \mathbf{u}(b)^T$ . If  $0 \leq M < N$ , then the result follows from Proposition 4.10, since the critical value is precisely  $\mathcal{C}(\mathbf{c}; z^M; N, \rho) = c_M^{-1}$ . Now suppose  $M \geq N$ . Again using Proposition 4.10, either  $c_0, \dots, c_{N-1}$  and  $c'$  are all non-negative, or else we have that  $c_0, \dots, c_{N-1} > 0 > c_M$ .

In the latter case, to prove that  $c_M \geq -\mathcal{C}(\mathbf{c}; z^M; N, \rho)^{-1}$ , we use the sequence  $\rho \mathbf{u}(b^n) \mathbf{u}(b^n)^T$ , where  $\mathbf{u}(b^n)$  is defined as in (4.5). Let  $\mathbf{u}_n := \sqrt{\rho} \mathbf{u}(b^n)$  for  $n \geq 1$ . Then [3, Equation (3.11)] implies that  $0 \leq \det |c_M|^{-1} F[\mathbf{u}_n \mathbf{u}_n^T]$ , and so

$$|c_M|^{-1} \geq \sum_{j=0}^{N-1} \frac{s_{\mu(M,N,j)}(\mathbf{u}_n)^2}{c_j},$$

where  $\mu(M, N, j)$  is the hook partition  $(M - N + 1, 1, \dots, 1, 0, \dots, 0)$ , with  $N - j - 1$  ones after the first entry and then  $j$  zeros, and  $s_{\mu(M,N,j)}$  is the corresponding Schur polynomial. As  $n \rightarrow \infty$ , so  $\mathbf{u}_n \rightarrow \sqrt{\rho}(1, \dots, 1)^T$ . The Weyl Character Formula in type  $A$  gives that  $s_{\mu(M,N,j)}(1, \dots, 1) = \binom{M}{j} \binom{M-j-1}{N-j-1}$ , and it follows that

$$|c_M|^{-1} \geq \sum_{j=0}^{N-1} \binom{M}{j}^2 \binom{M-j-1}{N-j-1}^2 \frac{\rho^{M-j}}{c_j} = \mathcal{C}(\mathbf{c}; z^M; N, \rho).$$

Thus (2) holds, and this concludes the proof.  $\square$

Finally, we consider the question of which real powers preserve positivity on  $N \times N$  Hankel matrices. Recall that the Schur product theorem guarantees that integer powers  $x \mapsto x^k$  preserve positivity on  $\mathcal{P}_N((0, \infty))$ . It is natural to ask if any other real powers do so. In [15], FitzGerald and Horn solved this problem, and uncovered an intriguing transition. In their main result, they show that the power function  $x \mapsto x^\alpha$  preserves positivity entrywise on  $\mathcal{P}_N((0, \infty))$  if and only if  $\alpha$  is a non-negative integer or  $\alpha \geq N - 2$ . The value  $N - 2$  is known in the literature as the *critical exponent* for preserving positivity.

As shown in [20], the critical exponent remains unchanged upon restricting the problem to preserving positivity on  $\mathcal{P}_N^k((0, \infty))$  for any  $k \geq 2$ . More precisely, for each non-integral  $\alpha \in (0, N - 2)$ , there exists a rank-two matrix  $A \in \mathcal{P}_N^2((0, \infty))$  such that  $A^{\circ\alpha} \notin \mathcal{P}_N$ ; see [20] for more details.

As we now show, the result does not change when restricted to the set of positive semidefinite Hankel matrices.

**Proposition 4.11.** *Let  $2 \leq k \leq N$  and let  $\alpha \in \mathbb{R}$ . The following are equivalent.*

- (1) *The power function  $x \mapsto x^\alpha$  preserves positivity when applied entrywise to Hankel matrices in  $\mathcal{P}_N^k((0, \infty))$ .*
- (2) *The power  $\alpha$  is a non-negative integer or  $\alpha \geq N - 2$ .*

*Moreover, there exists a Hankel matrix  $A \in \mathcal{P}_N^2((0, \infty))$  such that  $A^{\circ\alpha} \notin \mathcal{P}_N$  for all non-integral  $\alpha \in (0, N - 2)$ .*

*Proof.* By the main result in [27], for pairwise distinct real numbers  $x_1, \dots, x_N > 0$ , the matrix  $((1 + x_i x_j)^\alpha)_{i,j=1}^N$  is positive semidefinite if and only if  $\alpha$  is a non-negative integer or  $\alpha \geq N - 2$ . The result now follows immediately, by Lemma 2.7.  $\square$

Note that replacing  $(0, \infty)$  with  $(0, \rho)$  for some  $\rho$  with  $0 < \rho < \infty$  leads to the same classification of entrywise powers preserving positivity on the reduced test set.

## 5. TOTALLY NON-NEGATIVE MATRICES

With a better understanding of the endomorphisms of moment sequences of positive measures, we turn next to the structure of preservers of total non-negativity, in both the fixed-dimension and dimension-free settings. Recall that a rectangular matrix is *totally non-negative* if every minor is a non-negative real number.

We begin with the well-known fact that moment sequences of positive measures on  $[0, \infty)$  are in natural correspondence with totally non-negative Hankel matrices.

**Lemma 5.1.** *A real sequence  $(s_k)_{k=0}^\infty$  is the moment sequence of a positive measure on  $[0, \infty)$  if and only if the corresponding semi-infinite Hankel matrix  $H := (s_{i+j})_{i,j=0}^\infty$  is totally non-negative. The measure is supported on  $[0, 1]$  if and only if the entries of  $H$  are uniformly bounded.*

*Proof.* The first claim is a consequence of well-known results in the theory of moments [18, 45], as outlined in the introduction to [14]. For measures on  $[0, 1]$ , the result now follows via Theorem 2.4(3).  $\square$

Lemma 5.1 also has a finite-dimensional version, which will be required in the proof of Theorem 3.3.

**Lemma 5.2** ([14, Corollary 3.5]). *Let  $A$  be an  $N \times N$  Hankel matrix. Then  $A$  is totally non-negative if and only if  $A$  and its truncation  $A^{(1)}$  have non-negative principal minors.*

With Lemmas 5.1 and 5.2 in hand, we can now establish our characterization of positivity preservers on  $\mathcal{H}^{++}$ .

*Proof of Theorem 3.3.* Suppose (1) holds, and let  $A \in \mathcal{H}^{++}$ . Then both  $F[A]$  and  $F[A]^{(1)} = F[A^{(1)}]$  have non-negative principal minors, so  $F[A] \in \mathcal{H}^{++}$ , by Lemma 5.2. Thus (1)  $\implies$  (2).

That (2)  $\implies$  (3) follows directly from Lemma 5.1. Next, suppose (3) holds and let  $a > 0$  and  $b \geq 0$ . Applying  $F[-]$  to the first few moments of the measure  $a\delta_{\sqrt{b/a}}$  shows that  $F(a)F(b) \geq F(\sqrt{ab})^2$ . By Theorem 4.1, we conclude that (4) holds.

Finally, suppose (4) holds and let  $H \in \mathcal{H}_N^{++}$  for some  $N \geq 1$ . If every entry of  $H$  is non-zero, then  $F[H]$  is positive semidefinite, by the Schur product theorem. Otherwise, suppose  $H$  has a zero entry. Denote the entries in the first row and last column of  $H$  by  $s_0, \dots, s_{N-1}$  and  $s_{N-1}, \dots, s_{2N-2}$ , respectively. By considering  $2 \times 2$  minors, it is easy to show that

$$s_0 = 0 \implies s_1 = 0 \iff s_2 = 0 \iff \dots \iff s_{2N-3} = 0 \iff s_{2N-2} = 0.$$

Consequently, if (4) holds and an entry of  $H$  is zero, then  $F[H] \in \mathcal{P}_N$ .  $\square$

**Remark 5.3.** While Theorem 3.3 is more natural to state for functions with domain  $[0, \infty)$ , the proof goes through verbatim for  $F : [0, \rho) \rightarrow \mathbb{R}$ , where  $0 < \rho < \infty$ . In this case, the test set  $\mathcal{H}^{++}$  in the first two assertions of Theorem 3.3 (but not the target set) must be replaced by its subset of matrices with entries in  $[0, \rho)$ .

Next we examine the class of polynomial maps that, when applied entrywise, preserve total non-negativity for Hankel matrices of a fixed dimension. First, note that the analogue of the Schur product theorem holds for totally non-negative Hankel matrices [14, Theorem 4.5]; this also follows from Lemma 5.2. Second, note that the Hankel matrix  $H_\epsilon := \mathbf{u}(\epsilon)\mathbf{u}(\epsilon)^T$  is totally non-negative for all  $\epsilon \in (0, 1)$ , where  $\mathbf{u}(\epsilon)$  was defined in (4.5):

$$\mathbf{u}(\epsilon) := (1 - \epsilon, \dots, (1 - \epsilon)^N)^T.$$

This holds because the elements of  $H_\epsilon$  are all positive, and the  $k \times k$  minors of  $H_\epsilon$  vanish if  $k \geq 2$ . As a consequence, Proposition 4.10 implies that if  $F$  is a polynomial which preserves positive semidefiniteness on  $\mathcal{H}_N^{++}$ , then the first  $N$  non-zero coefficients of  $F$  must be positive.

The following result shows that the next coefficient can be negative, with the same threshold as in Theorem 4.8.

**Theorem 5.4.** *Let  $\rho, N, M$  and  $F(z) = \sum_{j=0}^{N-1} c_j z^j + c' z^M$  be as in Theorem 4.8. The following are equivalent.*

- (1)  $F[-]$  preserves total non-negativity for elements of  $\mathcal{H}_N^{++}$  with entries in  $[0, \rho)$ .
- (2) The coefficients  $c_j$  satisfy either  $c_0, \dots, c_{N-1}, c' \geq 0$ , or  $c_0, \dots, c_{N-1} > 0$  and  $c' \geq -\mathcal{C}(\mathbf{c}; z^M; N, \rho)^{-1}$ , where

$$\mathcal{C}(\mathbf{c}; z^M; N, \rho) := \sum_{j=0}^{N-1} \binom{M}{j}^2 \binom{M-j-1}{N-j-1}^2 \frac{\rho^{M-j}}{c_j}.$$

- (3)  $F[-]$  preserves positivity for rank-one elements of  $\mathcal{H}_N^{++}$  with entries in  $(0, \rho)$ .

*Proof.* Clearly (1)  $\implies$  (3), and (3)  $\implies$  (3'), where the assertion (3') is as in Remark 4.9. That (3')  $\implies$  (2) follows from the proof of Theorem 4.8.

To prove (2)  $\implies$  (1), first observe from Theorem 4.8 that  $F[-]$  preserves positivity on  $\mathcal{P}_N([0, \rho])$ . Given any matrix  $A \in \mathcal{H}_N^{++}$  with entries in  $[0, \rho)$ , let  $B$  denote the  $N \times N$  matrix obtained by deleting the first column and last row of  $A$ , and then adding a last row and column of zeros. Both  $A$  and  $B$  are positive semidefinite, and

therefore so are  $F[A]$  and  $F[B]$ . Hence  $F[A]$  and  $F[A]^{(1)} = F[A^{(1)}]$  have non-negative principal minors, since the principal minors of the latter are included in those of  $F[B]$ . Lemma 5.2 now gives the result.  $\square$

It is trivial that the Hadamard (entrywise) power  $H^{\circ\alpha}$  is totally non-negative for all  $H \in \mathcal{H}_1^{++} \cup \mathcal{H}_2^{++}$  if and only if  $\alpha \geq 0$ . For higher dimensions, the situation is as follows.

**Theorem 5.5** ([14, Theorem 5.11 and Example 5.5]). *Let  $\alpha \in \mathbb{R}$  and  $N \geq 2$ . The power function  $x^\alpha$  preserves  $\mathcal{H}_N^{++}$  if and only if  $\alpha$  is a non-negative integer or  $\alpha \geq N-2$ .*

Thus the set of powers preserving total non-negativity for Hankel matrices coincides with the set of powers preserving positivity on  $\mathcal{P}_N([0, \infty))$ , as identified by FitzGerald and Horn [15].

**Remark 5.6.** We note that Theorem 5.5 follows from a result of Jain [27, Theorem 1.1], since for  $x \in (0, 1)$ , the semi-infinite Hankel matrix  $(1 + x^{i+j})_{i,j=0}^\infty$  arises as the moment matrix of the measure  $\delta_1 + \delta_x$ , and is therefore totally non-negative, by Lemma 5.1.

We conclude this section by examining entrywise preservers of total non-negativity in the general setting, where the matrices are not assumed to have a Hankel structure, or to be symmetric or even square. By Theorem 3.3, every such preserver must be absolutely monotonic on  $(0, \infty)$ . However, it is not immediately clear how to proceed further with non-symmetric matrices; the analogue of the Schur product theorem no longer holds in this situation, as noted in [14, Example 4.3].

Our next result shows that, when working with rectangular or symmetric matrices, the set of functions preserving total non-negativity is very rigid.

**Theorem 5.7.** *Suppose  $F : [0, \infty) \rightarrow \mathbb{R}$ . The following are equivalent.*

- (1) *Applied entrywise, the function  $F$  preserves total non-negativity on the set of all rectangular matrices of arbitrary size.*
- (2) *Applied entrywise, the function  $F$  preserves total non-negativity on the set of all real symmetric matrices of arbitrary size.*
- (3) *The function  $F$  is constant or linear. In other words, there exists  $c \geq 0$  such that either  $F(x) \equiv c$ , or else  $F(x) = cx$  for all  $x \geq 0$ .*

Contrast this result, especially hypothesis (2), with Theorem 3.3.

We defer the proof of Theorem 5.7 until we have more closely examined the case of entire maps. This will give what is needed to overcome the main technical difficulty in proving Theorem 5.7.

Recall from [14, Section 5] that if  $A$  is a totally non-negative matrix which is  $3 \times 3$ , or symmetric and  $4 \times 4$ , then the Hadamard power  $A^{\circ\alpha}$  is totally non-negative for all  $\alpha \geq N-2$ , where  $N$  is the number of rows of  $A$ .

For larger matrices, very few entire functions preserve total non-negativity.

**Theorem 5.8.** *Let  $F(x) = \sum_{n=0}^\infty c_n x^n$  be entire with real coefficients. The entrywise map  $F[-]$  preserves total non-negativity for  $4 \times 4$  matrices if and only if  $F(x) \equiv c_0$  with  $c_0 \geq 0$ , or  $F(x) = c_1 x$  for all  $x \geq 0$  with  $c_1 \geq 0$ . The same conclusion holds if  $F[-]$  preserves total non-negativity for symmetric  $5 \times 5$  matrices.*

*Proof.* First we consider the  $4 \times 4$  case. Note that one implication is immediate, so suppose  $F[-]$  preserves total non-negativity and is not constant. Let  $A_y := y \text{Id}_3 \oplus \mathbf{0}_{1 \times 1}$ ,

where  $y \geq 0$  and  $\text{Id}_k$  denotes the  $k \times k$  identity matrix for  $k \geq 1$ . Observing that  $F[A_y]$  is totally non-negative, it follows that  $F(y) \geq F(0) \geq 0$  for all  $y \geq 0$ . If, moreover,  $y > 0$  is such that  $F(y) > F(0)$ , then from the same observation we conclude that  $F(0) = c_0 = 0$ .

Next, suppose for contradiction that  $F(x) = c_m x^m + O(x^{m+1})$ , where  $m > 1$  and  $c_m \neq 0$ . We make use of the matrix studied in [14, Example 5.9],

$$M := \begin{pmatrix} 3 & 6 & 14 & 36 \\ 6 & 14 & 36 & 98 \\ 14 & 36 & 98 & 276 \\ 36 & 98 & 284 & 842 \end{pmatrix}, \quad (5.1)$$

and let  $A(x) := \mathbf{1}_{4 \times 4} + xM$ . By the analysis in [14, Example 5.9], the matrix  $A(x)$  is totally non-negative for all  $x \geq 0$ , while for every real  $\alpha > 1$  there exists  $\delta_\alpha > 0$  such that

$$\det A(x)^{\circ\alpha} < 0 \quad \text{for all } x \in (0, \delta_\alpha).$$

Fix  $z \in (0, \delta_m)$ , let  $t > 0$ , and note that

$$F[tA(z)] = c_m t^m A(z)^{\circ m} + t^{m+1} C(t, z)$$

for some  $4 \times 4$  matrix  $C(t, z)$ . Since the matrix on the left-hand side is totally non-negative, it follows that

$$0 \leq t^{-4m} \det F[tA(z)] = c_m^4 \det A(z)^{\circ m} + O(t).$$

Letting  $t \rightarrow 0^+$  gives a contradiction. Hence  $c_1 \neq 0$ .

Finally, note that

$$F[tA(x)] = \sum_{n=1}^{\infty} c_n t^n (\mathbf{1}_{4 \times 4} + xM)^{\circ n} = \sum_{n=1}^{\infty} c_n t^n \sum_{j=0}^n \binom{n}{j} x^j M^{\circ j} = \sum_{j=0}^{\infty} \beta_j(t) x^j M^{\circ j},$$

where  $t \geq 0$  and  $\beta_j(t) := \sum_{n=j}^{\infty} c_n \binom{n}{j} t^n$ . Using a Laplace expansion, it is not hard to see that

$$\det F[tA(x)] = \det M_4(t) + O(x^5), \quad \text{where } M_4(t) := \sum_{j=0}^4 \beta_j(t) x^j M^{\circ j}.$$

If  $R$  is a commutative unital ring containing  $x$  and  $\alpha_1, \dots, \alpha_4$  then Appendix A gives that

$$\det M_4 = -57168 \alpha_0 \alpha_1^2 \alpha_2 x^4 + O(x^5), \quad \text{where } M_4 := \sum_{j=0}^4 \alpha_j x^j M^{\circ j}. \quad (5.2)$$

Taking  $R = \mathbb{R}[t, x]$  and  $\alpha_j = \beta_j(t)$ , we have that  $M_4$  equals  $M_4(t)$ . Since  $F[tA(x)]$  is totally non-negative for all  $x \geq 0$ , dividing through by  $x^4$  and letting  $x \rightarrow 0^+$ , it follows that  $\beta_0(t)\beta_1(t)^2\beta_2(t)$  vanishes on an interval. Since  $\beta_j(t) = F^{(j)}(t)/j!$ , each  $\beta_j$  is also entire; thus at least one  $\beta_j \equiv 0$ , whence  $\beta_2(t) \equiv 0$ . It follows that  $c_n = 0$  for all  $n \geq 2$ , as claimed. That  $c_1 \geq 0$  is now follows by considering  $F[\text{Id}_4]$ .

This concludes the proof for  $4 \times 4$  totally non-negative matrices. The proof for symmetric  $5 \times 5$  matrices now follows, as [14, Example 5.10] gives a  $5 \times 5$  symmetric totally non-negative matrix containing the matrix  $A(x)$  as a  $4 \times 4$  minor.  $\square$

With this result in hand, we can now complete the outstanding proof in this section.

*Proof of Theorem 5.7.* Clearly (3)  $\implies$  (1)  $\implies$  (2). Suppose (2) holds. Then, by Theorem 3.3, the function  $F$  is absolutely monotonic on  $(0, \infty)$ , and  $F(0) \geq 0$ . If  $F$  is not constant, then  $F(y) > F(0)$  for some  $y > 0$ . As  $F[y \text{Id}_3]$  is totally non-negative, looking at  $2 \times 2$  minors now shows that  $F(0) = 0$ .

To see that  $F$  is continuous at 0, note first that

$$A := \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

is totally non-negative. If  $L := \lim_{x \rightarrow 0^+} F(x)$ , then  $\det F[tA] \rightarrow -L^3 \geq 0$  as  $t \rightarrow 0^+$ , whence  $L = 0$ , as desired.

Thus  $F$  has the form required to apply Theorem 5.8, so  $F(x) = c_1 x$  for all  $x \in [0, \infty)$ , as required.  $\square$

## 6. MOMENT TRANSFORMERS ON $[-1, 1]$

Equipped with the one-sided result from Theorem 4.1, we now classify the functions which map the set  $\mathcal{M}([-1, 1])$  into  $\mathcal{M}(\mathbb{R})$  when applied entrywise. The goal of this section is to prove the following strengthening of Theorem 3.1, in the spirit of Theorem 4.1.

**Theorem 6.1.** *Let  $F : (-\rho, \rho) \rightarrow \mathbb{R}$ , where  $0 < \rho \leq \infty$ . The following are equivalent.*

- (1)  $F[-]$  maps the sequences  $\bigcup_{u \in (0, 1)} \mathcal{M}^\rho(\{-1, u, 1\})$  into  $\mathcal{M}(\mathbb{R})$ .
- (2)  $F[-]$  maps the sequences  $\bigcup_{u \in (0, 1)} \mathcal{M}^\rho(\{-1, u, 1\})$  into  $\mathcal{M}([-1, 1])$ .
- (3)  $F[-]$  maps  $\mathcal{M}^\rho([-1, 1])$  into  $\mathcal{M}(\mathbb{R})$ .
- (4)  $F$  is the restriction to  $(-\rho, \rho)$  of an absolutely monotonic entire function.

Recall that Schoenberg and Rudin's result, Theorem 2.10, characterizes positivity preservers for matrices with entries in  $(-1, 1)$ . As a consequence of Theorem 6.1, we obtain the following generalization of Theorem 2.10 with a much reduced test set.

**Corollary 6.2.** *The hypotheses of Theorem 2.10 are equivalent to  $F[-]$  preserving positivity on Hankel matrices arising from moment sequences, with entries in  $(-1, 1)$  and rank at most 3. Furthermore, this theorem holds with  $(-1, 1)$  and  $(0, 1)$  replaced by  $(-\rho, \rho)$  and  $(0, \rho)$ , respectively, whenever  $\rho \in (0, \infty]$ .*

The proof of Theorem 6.1 requires new ideas, as previous techniques to prove analogous results are not amenable to the more general Hankel setting; see Remark 6.7.

As a first step, we obtain the following lemma; together with Theorem 2.4, it explains why assertion (1) in Theorem 6.1 can be relaxed to assertion (2).

Recall the notion of truncated moment sequence from Definition 2.6.

**Lemma 6.3.** *If  $F : (-\rho, \rho) \rightarrow \mathbb{R}$  maps entrywise the sequences  $\mathcal{M}_2^\rho(\{-1, 1\})$  into  $\mathcal{M}_2(\mathbb{R})$ , then  $F$  is locally bounded. If  $F$  is known to be locally bounded on  $(0, \rho)$ , then the set  $\mathcal{M}_2^\rho(\{-1, 1\})$  may be replaced by  $\mathcal{M}_2^\rho(\{-1\})$ .*

*Proof.* Akin to the proof of Theorem 4.2, the assumption implies that  $F$  is non-decreasing, whence locally bounded, on  $(0, \rho)$ . Now let  $\mu = a\delta_{-1}$  for any  $a \in (0, \rho)$ . By considering the leading principal  $2 \times 2$  submatrix of  $F[H_\mu]$ , where  $H_\mu$  is the Hankel matrix (2.1) associated to the measure  $\mu$ , it follows that  $|F(-a)| \leq F(a)$ .  $\square$

The next step is to use assertion (2) in Theorem 6.1 to establish the continuity of  $F$  on  $(-\rho, \rho)$ .

**Proposition 6.4.** *Fix  $v_0 \in (0, 1)$  and suppose the function  $F : (-\rho, \rho) \rightarrow \mathbb{R}$ , where  $0 < \rho \leq \infty$ , maps entrywise*

$$\mathcal{M}_2^\rho(\{-1, 1\}) \cup \mathcal{M}_3^\rho(\{-1, v_0\}) \cup \bigcup_{u \in (0, 1)} \mathcal{M}_4^\rho(\{1, u\})$$

into  $\mathcal{M}_2([-1, 1]) \cup \mathcal{M}_3([-1, 1]) \cup \mathcal{M}_4([-1, 1])$ . Then  $F$  is continuous on  $(-\rho, \rho)$ .

*Proof.* As  $F$  maps  $\mathcal{M}_2^\rho(\{-1, 1\})$  into  $\mathcal{M}_2([-1, 1])$ , considering

$$\mu = \frac{a+b}{2}\delta_1 + \frac{a-b}{2}\delta_{-1} \quad \text{and} \quad \nu = b\delta_1$$

shows that  $F(a) \geq F(b) \geq 0$  whenever  $0 \leq a \leq b < \rho$ . It follows immediately that  $F$  maps  $\mathcal{M}_2^\rho(\{0, 1\})$  into  $\mathcal{M}_2(\mathbb{R})$ . Then, by Theorem 4.4 for  $N = 3$  and our assumptions,  $F$  is continuous, non-negative, and non-decreasing on  $(0, \rho)$ . In particular,  $F$  has a right-hand limit at 0, and

$$0 \leq F(0) \leq \lim_{\epsilon \rightarrow 0^+} F(\epsilon). \quad (6.1)$$

We now fix  $v_0 \in (0, 1)$  and use the truncated moment sequences in  $\mathcal{M}_3^\rho(\{-1, v_0\})$  to prove two-sided continuity of  $F$  at all points in  $(-\rho, 0]$ . Fix  $\beta \in [0, \rho)$ , and for  $b$  such that  $0 < b < (\rho - \beta)/(1 + v_0)$ , let

$$a := \beta + bv_0 \quad \text{and} \quad \mu = a\delta_{-1} + b\delta_{v_0}.$$

By assumption, we have that  $F[-] : \mathcal{M}_3^\rho(\{-1, v_0\}) \rightarrow \mathcal{M}_3([-1, 1])$ , so there exists  $\sigma \in \text{Meas}^+[-1, 1]$  such that

$$(F(s_0(\mu)), \dots, F(s_3(\mu))) = (s_0(\sigma), \dots, s_3(\sigma)).$$

If the polynomials  $p_\pm(t) := (1 \pm t)(1 - t^2)$  then,

$$\int_{-1}^1 p_\pm(t) d\sigma \geq 0,$$

since  $p_\pm(t)$  are non-negative on  $[-1, 1]$ . Hence (4.1) gives that

$$F(a+b) - F(a+bv_0^2) \geq \pm (F(-a+bv_0) - F(-a+bv_0^3)),$$

or, equivalently,

$$F(\beta + b + bv_0) - F(\beta + bv_0 + bv_0^2) \geq |F(-\beta) - F(-\beta - b(v_0 - v_0^3))|.$$

Letting  $b \rightarrow 0^+$  and using the continuity of  $F$  on  $(0, \rho)$ , we conclude that  $F$  is left continuous at  $-\beta$ . We proceed similarly to show right continuity of  $F$  at  $-\beta$ ; let

$$a := \beta + bv_0^3 \quad \text{and} \quad \mu = a\delta_{-1} + b\delta_{v_0},$$

where  $b$  is such that  $0 < b < (\rho - \beta)/(1 + v_0^3)$ , and take  $b \rightarrow 0^+$  as before.  $\square$

**Remark 6.5.** The integration trick (4.1) used in the proof of Proposition 6.4 shows that certain linear combinations of moments are non-negative. The integral it employs may also be expressed using the quadratic form given by the Hankel moment matrix for the ambient measure. To see this, suppose  $\sigma$  is a non-negative measure on  $[-1, 1]$  with moments of all orders, and let  $H_\sigma := (s_{j+k}(\sigma))_{j,k \geq 0}$  be the associated Hankel moment matrix. If  $f : [-1, 1] \rightarrow \mathbb{R}_+$  is continuous then so its radical  $\sqrt{f} : [-1, 1] \rightarrow \mathbb{R}_+$ ,

and the latter can be uniformly approximated on  $[-1, 1]$  by a sequence of polynomials  $p_n(t) = \sum_{j=0}^{d_n} c_{n,j} t^j$ .

Thus

$$\int_{-1}^1 f \, d\sigma = \lim_{n \rightarrow \infty} \int_{-1}^1 p_n(t)^2 \, d\sigma = \lim_{n \rightarrow \infty} \sum_{j,k \geq 0} c_{n,j} c_{n,k} \int_{-1}^1 t^{j+k} \, d\sigma = \lim_{n \rightarrow \infty} \mathbf{v}_n^T H_\sigma \mathbf{v}_n,$$

where

$$\mathbf{v}_n := (c_{n,0}, c_{n,1}, \dots, c_{n,d_n}, 0, 0, \dots)^T \quad (n \geq 1).$$

Now, since the matrix  $H_\sigma$  is positive, the limit on the right-hand side is non-negative and so  $\int_{-1}^1 f \, d\sigma \geq 0$ .

With continuity in hand, we can now complete the proof of Theorem 6.1.

*Proof of Theorem 6.1.* Clearly (4)  $\implies$  (3)  $\implies$  (1) and (2)  $\implies$  (1); that (1)  $\implies$  (2) follows from the remarks preceding Lemma 6.3. Now suppose (1) holds. By Proposition 6.4, the function  $F$  is continuous on  $(-\rho, \rho)$ , so Theorem 4.1 gives that  $F$  agrees on  $(0, \rho)$  with a power series  $\tilde{F}$  having non-negative Maclaurin coefficients, which is convergent on the disc  $D(0, \rho)$ .

Let  $\mu := a\delta_{-1} + e^x \delta_{e^{-h}}$ , where  $a \in (0, \rho)$ ,  $x < \log(\rho - a)$ , and  $h > 0$ , and let the polynomial  $p_{\pm, n}(t) := (1 \pm t)(1 - t^2)^n$ . Then  $p_{\pm, n}(t)$  is non-negative for all  $t \in [-1, 1]$  and all  $n \geq 0$ . Applying (4.1) gives that

$$\left| \sum_{k=0}^n \binom{n}{k} (-1)^k F(a + e^{x-2kh}) \right| \geq \left| \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} F(-a + e^{x-(2k+1)h}) \right|. \quad (6.2)$$

Let  $H_{\pm, a}(x) := F(\pm a + e^x)$  and suppose  $F$  is smooth; dividing (6.2) by  $h^n$  and taking  $h \rightarrow 0^+$ , we see that

$$|H_{+, a}^{(n)}(x)| \geq |H_{-, a}^{(n)}(x)|.$$

Since  $H_{+, a}$  is real analytic, we conclude that the Taylor series for  $H_{-, a}$  has a positive radius of convergence everywhere, so  $H_{-, a}$  is real analytic on  $(-\infty, \log(\rho - a))$ . The change of variable  $x = \log(y + a)$  has a convergent power-series expansion for  $|y| < a$ . It follows that  $y \mapsto F(y)$  is real analytic on  $(-\rho, \rho)$ , hence is the restriction of  $\tilde{F}$ .

When  $F$  is not necessarily smooth, we may use a mollifier argument. Fix  $0 < \rho' < \rho$  and let  $G := F|_{(-\rho', \rho')}$ . For any  $\delta \in (0, \rho - \rho')$ , choose  $g_\delta \in C^\infty(\mathbb{R})$  such that  $g_\delta$  is non-negative, supported on  $(0, \delta)$ , and integrates to 1, and let

$$F_\delta(x) := \int_0^\delta G(x+t) g_\delta(t) \, dt \quad \text{for all } x \in (-\rho', \rho').$$

As the function  $x \mapsto G(x+t)$  satisfies hypothesis (1) of the theorem with  $\rho$  replaced by  $\rho'$ , so does the smooth function  $F_\delta$ ; let  $\tilde{F}_\delta$  be an analytic function on the disc  $D(0, \rho')$  which is absolutely monotonic on  $(0, \rho')$  and agrees on  $(-\rho', \rho')$  with  $F_\delta$ . Since

$$|F(x) - F_\delta(x)| = \left| \int_0^\delta (G(x) - G(x+t)) g_\delta(t) \, dt \right| \leq \sup_{0 \leq t \leq \delta} |G(x) - G(x+t)|$$

for all  $x \in (-\rho', \rho')$ , it follows that  $F_\delta$  converges to  $F$  locally uniformly on  $(-\rho', \rho')$  as  $\delta \rightarrow 0^+$ . The function  $\tilde{F}_\delta$  is absolutely monotonic, so  $|\tilde{F}_\delta(z)| \leq \tilde{F}_\delta(a)$  whenever  $|z| \leq a < \rho'$ , and  $\tilde{F}_\delta(a) \rightarrow F(a)$  as  $\delta \rightarrow 0^+$ . Hence  $\{\tilde{F}_\delta(z) : \delta > 0\}$  is uniformly

bounded on  $\overline{D}(0, a)$ , and therefore forms a normal family. Thus for some sequence  $\delta_n \rightarrow 0^+$ , the functions  $\widetilde{F}_{\delta_n}$  converge locally uniformly to a function  $\widetilde{F}$  that is analytic on  $D(0, \rho')$ , and  $F$  agrees with  $\widetilde{F}$  on  $(-\rho', \rho')$ . As this argument holds for all  $\rho' \in (0, \rho)$ , the proof is complete.  $\square$

**Remark 6.6.** The proof of Theorem 6.1 requires measures whose support contains the point 1, in order to be able to employ the mollifier argument to move from continuous to smooth functions.

**Remark 6.7.** Recall that Rudin [38] showed that  $F$  must be analytic on  $(-1, 1)$  and absolutely monotonic on  $(0, 1)$  if  $F[-]$  preserves positivity for the two-parameter family of Toeplitz matrices defined in (3.1). A natural strategy to prove Theorem 6.1 would be to show that there exists  $\theta \in \mathbb{R}$  with  $\theta/\pi$  irrational, such that the matrices  $(\cos((i-j)\theta))_{i,j=1}^n$  can be embedded into positive Hankel matrices, for all sufficiently large  $n$ . However, this is not possible: given  $0 < m_1 < m_2$  such that  $\cos(m_1\theta) < 0$  and  $\cos(m_2\theta) < 0$ , if there were a measure  $\mu \in \text{Meas}^+([-1, 1])$  such that  $\cos(m_j\theta) = s_{k_j}(\mu)$  for  $j = 1$  and  $j = 2$ , then, by the Toeplitz property,  $k_1$ ,  $k_2$ , and  $k_1 + k_2$  must all be odd, which is impossible.

## 7. MOMENT TRANSFORMERS ON $[-1, 0]$

We now study the structure of endomorphisms of  $\mathcal{M}([-1, 0])$ . The following result strengthens Theorem 3.4 and reveals that such functions may be discontinuous at the origin, in contrast to Theorem 6.1.

**Theorem 7.1.** *Given  $u_0 \in (0, 1)$  and  $F : (-\rho, \rho) \rightarrow \mathbb{R}$ , where  $0 < \rho \leq \infty$ , the following are equivalent.*

- (1)  $F[-]$  maps  $\mathcal{M}^\rho(\{-1, -u_0\})$  into  $\mathcal{M}((-\infty, 0])$  and

$$\mathcal{M}_4^\rho(\{-1, 0\}) \cup \bigcup_{u \in (0, 1)} \mathcal{M}_4^\rho(\{-u\})$$

into  $\mathcal{M}_4((-\infty, 0])$ .

- (2)  $F[-]$  maps  $\mathcal{M}^\rho([-1, 0])$  into  $\mathcal{M}([-1, 0])$ .  
(3) There exists an absolutely monotonic entire function  $\widetilde{F}$  such that

$$F(x) = \begin{cases} \widetilde{F}(x) & \text{if } x \in (0, \rho), \\ 0 & \text{if } x = 0, \\ -\widetilde{F}(-x) & \text{if } x \in (-\rho, 0). \end{cases}$$

In particular, the function  $F$  is odd, but may be discontinuous at 0.

*Proof.* To show that (3)  $\implies$  (2), note first that if  $\mu \in \text{Meas}^+([-1, 0])$ , so that  $\mu = a\delta_0$  for some  $a$ , then  $F[H_\mu] = H_{F(a)\delta_0}$ , so we may assume  $\mu$  is not of this form, whence the Hankel matrix  $H_\mu$  has no zero entries, and the moment sequence alternates in sign and is uniformly bounded, by Theorem 2.4. In particular,

$$F(s_{2k}(\mu)) = \widetilde{F}(s_{2k}(\mu)) \quad \text{and} \quad F(s_{2k+1}(\mu)) = -\widetilde{F}(-s_{2k+1}(\mu)) \quad (k \geq 0).$$

Recalling the form of the Hankel matrix  $H_{\delta_{-1}}$ , it follows that

$$F[H_\mu] = H_{\delta_{-1}} \circ \widetilde{F}[H_{\delta_{-1}} \circ H_\mu] \tag{7.1}$$

where  $\circ$  denotes the entrywise matrix product. This shows (2) because  $F[-]$  is the composite of two operations: the map  $\tilde{F}[-]$ , which sends  $\mathcal{M}^\rho([0, 1])$  into  $\mathcal{M}([0, 1])$ , by Theorem 4.1, and entrywise multiplication by the matrix  $H_{\delta_{-1}}$ , which maps  $H_\mu$  for some measure  $\mu$  to the Hankel matrix of the reflection of  $\mu$  about the origin.

That (2)  $\implies$  (1) is immediate. We now prove (1)  $\implies$  (3). Suppose (1) holds. Since

$$F[H_{a\delta_0}] = (F(a) - F(0))H_{\delta_0} + F(0)H_{\delta_1} = H_{(F(a)-F(0))\delta_0 + F(0)\delta_1},$$

the uniqueness in Theorem 2.4 gives that  $F(0) = 0$ .

By considering only even rows and columns of Hankel matrices corresponding to moments in  $\mathcal{M}_4^\rho(\{-u\})$ ,  $\mathcal{M}_4^\rho(\{-1, 0\})$ , and  $\mathcal{M}^\rho(\{-1, -u_0\})$ , we have embeddings

$$\begin{aligned} \mathcal{M}_2^\rho(\{u^2\}) &\hookrightarrow \mathcal{M}_4^\rho(\{-u\}), \\ \mathcal{M}_2^\rho(\{0, 1\}) &\hookrightarrow \mathcal{M}_4^\rho(\{-1, 0\}), \\ \text{and } \mathcal{M}^\rho(\{1, u_0^2\}) &\hookrightarrow \mathcal{M}^\rho(\{-1, -u_0\}). \end{aligned}$$

Thus  $F[-]$  maps  $\mathcal{M}_2^\rho(\{u^2\})$  into  $\mathcal{M}_2(\mathbb{R})$ ,  $\mathcal{M}_2^\rho(\{0, 1\})$  into  $\mathcal{M}_2(\mathbb{R})$ , and  $\mathcal{M}^\rho(\{1, u_0^2\})$  into  $\mathcal{M}(\mathbb{R})$ . Theorem 4.4 now gives that  $F$  agrees with an absolutely monotonic entire function  $\tilde{F}$  on  $(0, \rho)$ .

Next, considering  $\mathcal{M}_2^\rho(\{-1\})$  gives that  $|F(-a)| \leq F(a)$  for any  $a \in (0, \rho)$ , whence  $F$  is locally bounded. In particular,  $F$  maps  $\mathcal{M}^\rho(\{-1\})$  into  $\mathcal{M}([-1, 0])$ , by Theorem 2.4.

We conclude by showing that  $F$  is odd. Let  $\mu = a\delta_{-1}$  for some  $a \in (0, \rho)$  and note that  $p_n(t) = (-t)^n(1+t)$  is non-negative on  $[-1, 0]$  for any non-negative integer  $n$ . If  $F[\mathbf{s}(\mu)] = \mathbf{s}(\sigma)$ , then, by applying (4.1),

$$\begin{aligned} 0 &\leq \int_{-1}^0 p_n(t) d\sigma = (-1)^n(F(s_n(a\delta_{-1})) + F(s_{n+1}(a\delta_{-1}))) \\ &= (-1)^n(F((-1)^n a) + F((-1)^{n+1} a)). \end{aligned}$$

Taking  $n = 0$  and  $1$  gives that  $0 \leq F(a) + F(-a) \leq 0$ , and the final claim follows.  $\square$

Theorem 7.1 has the following consequence.

**Corollary 7.2.** *Define a checkerboard matrix to be any real matrix  $A = (a_{ij})$  such that  $(-1)^{i+j}a_{ij} > 0$  for all  $i, j$ . Given a function  $F : \mathbb{R} \rightarrow \mathbb{R}$ , the following are equivalent.*

- (1) *Applied entrywise,  $F$  maps the set of positive Hankel checkerboard matrices of all dimensions into itself.*
- (2) *Applied entrywise,  $F$  maps the set of positive checkerboard matrices of all dimensions into itself.*
- (3)  *$F$  is odd and agrees on  $(0, \infty)$  with an absolutely monotonic entire function.*

We conclude this section with an even analogue of Theorem 7.1.

**Theorem 7.3.** *Given  $u_0 \in (0, 1)$  and  $F : (-\rho, \rho) \rightarrow \mathbb{R}$ , where  $0 < \rho \leq \infty$ , the following are equivalent.*

- (1)  *$F[-]$  maps  $\mathcal{M}^\rho(\{-1, -u_0\})$  into  $\mathcal{M}([0, \infty))$  and*

$$\mathcal{M}_4^\rho(\{-1, 0\}) \cup \bigcup_{u \in (0, 1)} \mathcal{M}_4^\rho(\{-u\})$$

*into  $\mathcal{M}_4([0, \infty))$ .*

- (2)  $F[-]$  sends  $\mathcal{M}^\rho([-1, 0])$  to  $\mathcal{M}([0, 1])$ .  
(3) There exists an absolutely monotonic entire function  $\tilde{F}$  such that

$$F(x) = \begin{cases} \tilde{F}(x) & \text{if } x \in (0, \rho), \\ \tilde{F}(-x) & \text{if } x \in (-\rho, 0). \end{cases}$$

Moreover,  $0 \leq F(0) \leq \lim_{\epsilon \rightarrow 0} F(\epsilon)$ .

*Proof.* This is similar to the proof of Theorem 7.1; to show that (1)  $\implies$  (3), one may use the polynomials  $p_n(t) = t^n(1-t)$ . We omit further details.  $\square$

## 8. TRANSFORMERS WITH COMPACT DOMAIN

The goal of this section is to show how results in the previous sections can be refined when the moments are contained in a compact domain. Indeed, when the domain of  $F$  is a compact interval  $I$ , the situation is more complex; absolute monotonicity, or even continuity of  $F$ , does not extend automatically from the interior of  $I$  to its end points. This was already observed by Rudin via specific counterexamples; see Remark (a) at the end of [38]. To the best of our knowledge, characterization results in this setting are not known.

We now take a closer look at this phenomenon. We begin by characterizing the functions preserving positivity of Hankel matrices in  $\mathcal{P}_N(I)$  for all  $N$ , where  $I = [0, \rho]$  and  $0 < \rho < \infty$ .

**Proposition 8.1.** *Suppose  $F : I \rightarrow \mathbb{R}$ , where  $I = [0, \rho]$  and  $0 < \rho < \infty$ . The following are equivalent.*

- (1)  $F[-]$  preserves positivity on all positive Hankel matrices with entries in  $I$ .
- (2)  $F$  is absolutely monotonic on  $[0, \rho)$  and  $F(\rho) \geq \lim_{x \rightarrow \rho^-} F(x)$ .
- (3)  $F[-]$  preserves positivity on all positive matrices with entries in  $I$ .

If, instead,  $I = [0, \rho]$  where  $0 < \rho \leq \infty$ , then the same equivalences hold, with (2) replaced by the requirement that  $F$  is absolutely monotonic on  $[0, \rho)$ .

Note the contrast with Theorem 4.1: if  $F[-]$  is required only to preserve positive Hankel matrices arising from moment sequences, then  $F$  may be discontinuous at 0, but this cannot occur here.

*Proof.* Clearly (3)  $\implies$  (1). Next, suppose (1) holds and note that  $F$  is absolutely monotonic on  $(0, \rho)$ , by Theorem 4.1. Consider the positive Hankel matrices

$$H_a := \begin{pmatrix} a & 0 & a \\ 0 & a & a \\ a & a & 2a \end{pmatrix}, \quad \text{where } a \in [0, \rho/2).$$

As  $F[H_a]$  is positive, so  $0 \leq F(0) \leq F^+(0) := \lim_{a \rightarrow 0^+} F(a)$ . Furthermore,

$$0 \leq \lim_{a \rightarrow 0^+} \det F[H_a] = -F^+(0)(F(0) - F^+(0))^2,$$

whence  $F(0) = F^+(0)$ , and  $F$  is right continuous at the origin. Finally, considering the first two leading principal minors of the Hankel matrix for the measure  $(\rho - a)\delta_1 + a\delta_0$ , where  $a \rightarrow \rho^-$ , gives that  $F(\rho) \geq \lim_{a \rightarrow \rho^-} F(a)$ . Hence (1)  $\implies$  (2).

Finally, suppose (2) holds. We first claim that if  $A \in \mathcal{P}_N((-\infty, \rho])$  then the entries of  $A$  equalling  $\rho$  form a (possibly empty) block diagonal submatrix, upon suitably relabelling the indices. Indeed,

$$0 \leq \det \begin{pmatrix} \rho & \rho & a \\ \rho & \rho & \rho \\ a & \rho & \rho \end{pmatrix} = -\rho(\rho - a)^2 \implies a = \rho. \quad (8.1)$$

Now let  $B_A$  be the block-diagonal matrix with  $(i, j)$ th entry equal to 1 if  $a_{ij} = \rho$  and 0 otherwise. If  $g$  is the continuous extension of  $F|_{[0, \rho]}$  to  $\rho$ , then

$$F[A] = g[A] + (F(\rho) - g(\rho))B_A \geq 0,$$

since both matrices are positive semidefinite. Hence (2)  $\implies$  (3).

Finally, when  $I = [0, \rho]$ , that (2)  $\implies$  (3)  $\implies$  (1) is immediate, and (1)  $\implies$  (2) is shown as above.  $\square$

**Remark 8.2.** A similar argument to Proposition 8.1 reveals that  $F[-]$  preserves positivity on the set  $\{\mathbf{s}(\mu) \in \mathcal{M}([0, 1]) : s_0(\mu) \in [0, \rho]\}$  if and only if  $F$  is absolutely monotonic on  $(0, \rho)$  and such that  $0 \leq F(0) \leq \lim_{\epsilon \rightarrow 0^+} F(\epsilon)$  and  $\lim_{x \rightarrow \rho^-} F(x) \leq F(\rho)$ .

We next examine the case where the domain of  $F$  is a symmetric compact interval  $[-\rho, \rho]$ . The functions preserving positivity of Hankel matrices when applied entrywise are completely characterized as follows.

**Proposition 8.3.** *Suppose  $F : I \rightarrow \mathbb{R}$ , where  $I = [-\rho, \rho]$  and  $0 < \rho \leq \infty$ . The following are equivalent.*

- (1)  $F[-]$  preserves positivity on all positive Hankel matrices with entries in  $I$ .
- (2)  $F[-]$  preserves positivity on all positive Hankel matrices with entries in  $I$  that arise from moment sequences.
- (3)  $F$  is real analytic on  $(-\rho, \rho)$ , absolutely monotonic on  $(0, \rho)$ , and such that

$$F(\rho) \geq \lim_{x \rightarrow \rho^-} F(x) \quad \text{and} \quad |F(-\rho)| \leq F(\rho).$$

*Proof.* That (1)  $\implies$  (2) is immediate, while (2)  $\implies$  (3) follows from the extension of Theorem 3.1 given by Theorem 6.1, and the proofs of Proposition 8.1 and Lemma 6.3. Finally, if (3) holds, then (1) follows by Proposition 8.1, the Schur product theorem, and the following claim.

*The only Hankel matrix in  $\mathcal{P}_{N+1}([-\rho, \rho])$  with an entry  $-\rho$  is the checkerboard matrix with  $(i, j)$ th entry  $(-1)^{i+j}\rho$ .*

To prove the claim, let the rows and columns of the positive Hankel matrix  $A$  be labelled by  $0, \dots, N$ , and suppose  $a_{ij} = -\rho$ . Then  $i + j$  is odd and  $a_{ll} = a_{l+1, l+1} = \rho$ , where  $2l + 1 = i + j$ . Repeatedly considering principal  $2 \times 2$  minors shows that  $a_{pq} = \rho$  if  $p + q$  is even. Now let  $m, n \in [0, N]$  be odd, with  $m < n$ , and denote by  $C$  the principal  $3 \times 3$  minor of  $A$  corresponding to the labels  $0, m$ , and  $n$ . Writing

$$C = \begin{pmatrix} \rho & a_{0m} & \rho \\ a_{0m} & \rho & a_{0n} \\ \rho & a_{0n} & \rho \end{pmatrix},$$

we have that  $0 \leq \det C = -\rho(a_{0m} - a_{0n})^2$ , whence  $a_{0m} = a_{0n}$ . Taking  $m$  or  $n$  to equal  $i + j$  shows that these entries equal  $-\rho$ , which gives the claim.  $\square$

We end this section by considering functions preserving positivity for all matrices in  $\bigcup_{N \geq 1} \mathcal{P}_N([- \rho, \rho])$ . Theorem 6.1 implies that every such function  $F$  is real analytic when restricted to  $(- \rho, \rho)$ , and absolutely monotonic on  $(0, \rho)$ . The following result provides a sufficient condition for  $F$  to preserve positivity, which is also necessary if the analytic restriction is odd or even.

**Proposition 8.4.** *Given  $\rho \in (0, \infty)$ , let  $I = [- \rho, \rho]$  and suppose  $F : I \rightarrow \mathbb{R}$  is real analytic on  $(- \rho, \rho)$ , absolutely monotonic on  $(0, \rho)$ , and such that the limits  $\lim_{x \rightarrow \rho^-} F(\pm x)$  both exist and are finite. If*

$$\left| F(-\rho) - \lim_{x \rightarrow -\rho^+} F(x) \right| \leq F(\rho) - \lim_{x \rightarrow \rho^-} F(x), \quad (8.2)$$

then  $F[-]$  preserves positivity on the space of positive matrices with entries in  $I$ . The converse holds if  $F|_{(-\rho, \rho)}$  is either odd or even.

The inequality (8.2) says that any jump in  $F$  at  $-\rho$  is bounded above by the jump at  $\rho$ , which is non-negative.

*Proof.* Let  $g$  denote the continuous function on  $[- \rho, \rho]$  which agrees with  $F$  on  $(- \rho, \rho)$ , and let the jumps  $\Delta_{\pm} := F(\pm \rho) - g(\pm \rho)$ . Then (8.2) is equivalent to  $|\Delta_-| \leq \Delta_+$ .

By the Schur product theorem and Proposition 8.1,  $F[-]$  preserves positivity on  $\mathcal{P}_N((-\rho, \rho))$  for all  $N$ . Now suppose  $A \in \mathcal{P}_N([- \rho, \rho])$  has some entry equal to  $-\rho$ , where  $N \geq 1$ . Then the entries of  $A$  with modulus  $\rho$  form a block diagonal submatrix upon suitable relabelling of indices. This follows from the argument given in the proof of Proposition 8.1, applied to the  $\rho^2$ -entries of  $A \circ A$ . Given this, and after further relabelling of indices, each block submatrix is of the form

$$\begin{pmatrix} \rho \mathbf{1}_{n_j \times n_j} & -\rho \mathbf{1}_{n_j \times m_j} \\ -\rho \mathbf{1}_{m_j \times n_j} & \rho \mathbf{1}_{m_j \times m_j} \end{pmatrix},$$

by the main result in [24], where  $j = 1, \dots, r$ . Then

$$F[A] = g[A] + B', \quad \text{where } B' = \bigoplus_{j=1}^k \begin{pmatrix} \Delta_+ \cdot \mathbf{1}_{n_j \times n_j} & \Delta_- \cdot \mathbf{1}_{n_j \times m_j} \\ \Delta_- \cdot \mathbf{1}_{m_j \times n_j} & \Delta_+ \cdot \mathbf{1}_{m_j \times m_j} \end{pmatrix},$$

and this is positive semidefinite, by (8.2). Thus  $F[-]$  preserves  $\bigcup_{N \geq 1} \mathcal{P}_N([- \rho, \rho])$ .

For the converse, we show that (8.2) holds if  $F[-]$  preserves positivity on just the set  $\mathcal{P}_3([- \rho, \rho])$  and  $F|_{(-\rho, \rho)}$  is odd or even. Note first that  $\Delta_+ \geq 0$ , working with  $2 \times 2$  matrices as above. Next, consider the positive matrix

$$A := \begin{pmatrix} a^2/\rho & -a & a \\ -a & \rho & -\rho \\ a & -\rho & \rho \end{pmatrix},$$

and note that

$$0 \leq \lim_{a \rightarrow \rho^-} \det F[A] = \begin{vmatrix} g(\rho) & g(-\rho) & g(\rho) \\ g(-\rho) & F(\rho) & F(-\rho) \\ g(\rho) & F(-\rho) & F(\rho) \end{vmatrix} = \Delta_+(g(\rho)F(\rho) - g(-\rho)^2) - g(\rho)\Delta_-^2.$$

It follows that  $\Delta_-^2 g(\rho) \leq \Delta_+^2 g(\rho)$  if  $g(\rho^2) = g(-\rho)^2$ , so if  $g = F|_{(-\rho, \rho)}$  is odd or even. This gives the result.  $\square$

**Remark 8.5.** Propositions 8.3 and 8.4 indicate the existence of functions discontinuous at  $\pm\rho$  which preserve positivity for Hankel matrices, but not all matrices, in contrast to the one-sided setting of Proposition 8.1.

Indeed, if  $g$  is an odd or even function which is continuous on  $[-\rho, \rho]$  and absolutely monotonic on  $(0, \rho)$ , define  $F$  to be equal to  $g$  on  $(-\rho, \rho]$ , and take  $F(-\rho)$  to be any element of  $(-F(\rho), F(\rho)]$ . Then  $F$  preserves positivity on all Hankel matrices with entries in  $[-\rho, \rho]$ , but does not preserve positivity on  $\bigcup_{N \geq 1} \mathcal{P}_N([-\rho, \rho])$ .

## 9. MULTIVARIABLE GENERALIZATIONS

In this section we classify the preservers of moments arising from admissible measures in higher-dimensional Euclidean space, both in their totality and by considering their marginals.

**9.1. Transformers of multivariable moment sequences.** The initial generalization to higher dimensions of our characterization of moment-preserving functions raises no complications. However, the failure of Hamburger's theorem in higher dimensions, that is, the lack of a characterization of moment sequences by positivity of an associated Hankel-type kernel, means some extra work is required. Below, we isolate this additional challenge and provide a generalization of our main result.

Let  $\mu$  be a non-negative measure on  $\mathbb{R}^d$ , where  $d \geq 1$ , which has moments of all orders; as before, such measures will be termed *admissible*. The multi-index notation

$$\mathbf{x}^\alpha = x_1^{\alpha_1} \dots x_d^{\alpha_d} \quad (\mathbf{x} \in \mathbb{R}^d)$$

allows us to define the moment family

$$s_\alpha(\mu) = \int \mathbf{x}^\alpha d\mu(\mathbf{x}) \quad (\alpha \in \mathbb{Z}_+^d),$$

where  $\mathbb{Z}_+$  denotes the set  $\{0, 1, 2, \dots\}$  of non-negative integers. As before, we focus on measures with uniformly bounded moments, so that

$$\sup_{\alpha \in \mathbb{Z}_+^d} |s_\alpha(\mu)| < \infty,$$

or, equivalently,  $\text{supp}(\mu) \subset [-1, 1]^d$ . In line with above, we let  $\mathcal{M}(K)$  denote the set of all moment families of admissible measures supported on  $K \subset \mathbb{R}^d$ .

**Theorem 9.1.** *The map  $F[-] : \mathbb{R} \rightarrow \mathbb{R}$  maps  $\mathcal{M}([-1, 1]^d)$  into itself if and only if  $F$  is absolutely monotonic and entire.*

*Proof.* Any admissible measure  $\mu$  on  $[-1, 1]$  pushes forward to an admissible measure  $\tilde{\mu}$  on  $[-1, 1]^d$  via the canonical embedding onto the first coordinate. If  $F$  maps  $\mathcal{M}([-1, 1]^d)$  to itself then there exists an admissible measure  $\tilde{\sigma}$  on  $[-1, 1]^d$  such that  $F(s_\alpha(\tilde{\mu})) = s_\alpha(\tilde{\sigma})$  for all  $\alpha \in \mathbb{Z}_+^d$ , and a short calculation shows that  $F[s_n(\mu)] = s_n(\sigma)$  for all  $n \in \mathbb{Z}_+$ , where  $\sigma$  is the pushforward of  $\tilde{\sigma}$  under the projection onto the first coordinate. Theorem 6.1 now gives that  $F$  is as claimed.

To prove the converse, we need to explore the structure of the set  $\mathcal{M}([-1, 1]^d)$ . Denote the generators of the semigroup  $\mathbb{Z}_+^d$  by setting

$$\mathbf{1}_j := (0, \dots, 0, 1, 0, \dots, 0),$$

with 1 in the  $j$ th position. A multisequence of real numbers  $(s_\alpha)_{\alpha \in \mathbb{Z}_+^d}$  is the moment sequence of an admissible measure supported on  $[-1, 1]^d$  if and only if the weighted Hankel-type kernels

$$(s_{\alpha+\beta}), (s_{\alpha+\beta} - s_{\alpha+\beta+2\mathbf{1}_j}), \quad 1 \leq j \leq d,$$

indexed over  $\alpha, \beta \in \mathbb{Z}_+^d$  are positive semidefinite [36].

Now suppose  $F$  is absolutely monotonic and entire; given a multisequence  $s_\alpha$  subject to these positivity constraints, we have to check that the multisequence  $F(s_\alpha)$  satisfies the same conditions.

As  $F$  is absolutely monotonic, Schoenberg's Theorem 2.10 gives that the kernels  $(\alpha, \beta) \mapsto F(s_{\alpha+\beta})$  and  $(\alpha, \beta) \mapsto F(s_{\alpha+\beta+2\mathbf{1}_j})$  are positive semidefinite. It remains to prove that the kernel

$$(\alpha, \beta) \mapsto F(s_{\alpha+\beta}) - F(s_{\alpha+\beta+2\mathbf{1}_j})$$

is positive semidefinite, for  $1 \leq j \leq d$ . However, as  $F$  has the Taylor expansion  $F(x) = \sum_{n=0}^{\infty} c_n x^n$ , with  $c_n \geq 0$  for all  $n \in \mathbb{Z}_+$ , it is sufficient to check that the kernel

$$(\alpha, \beta) \mapsto (s_{\alpha+\beta})^{on} - (s_{\alpha+\beta+2\mathbf{1}_j})^{on}$$

is positive semidefinite for any  $n \in \mathbb{Z}_+$ . This follows from a repeated application of the Schur product theorem: if matrices  $A$  and  $B$  are such that  $A \geq B \geq 0$ , then

$$A^{on} \geq A^{\circ(n-1)} \circ B \geq A^{\circ(n-2)} \circ B^{\circ 2} \geq \dots \geq B^{on}. \quad \square$$

This proof also shows that the transformers of  $\mathcal{M}([-1, 1]^d)$  into  $\mathcal{M}(\mathbb{R}^d)$  are the same absolutely monotonic entire functions. On the other hand, we will see in Section 10 that, in general, a mapping  $F$  as in Theorem 9.1 does not preserve the semi-algebraic supports of the underlying measures.

## 9.2. Transformers of moment-sequence tuples: the positive orthant case.

Our next objective is to characterize functions  $F : \mathbb{R}^m \rightarrow \mathbb{R}$  which map tuples of moments  $(s_k(\mu_1), \dots, s_k(\mu_m))$  arising from admissible measures on  $\mathbb{R}$ , to a moment sequence  $s_k(\sigma)$  for some admissible measure  $\sigma$  on  $\mathbb{R}$ . This is a multivariable generalization of Schoenberg's theorem which we will demonstrate under significantly relaxed hypotheses.

More precisely, we will study the preservers  $F : I^m \rightarrow \mathbb{R}$ , where  $m \geq 1$  is a fixed integer, and

$$I = (0, \rho) \text{ or } [0, \rho) \text{ or } (-\rho, \rho), \quad \text{where } 0 < \rho \leq \infty. \quad (9.1)$$

Note that  $F : I^m \rightarrow \mathbb{R}$  acts entrywise on any  $m$ -tuple of  $N \times N$  matrices with entries in  $I$ , so that

$$F[-] : \mathcal{P}_N(I)^m \rightarrow \mathbb{R}^{N \times N}, \quad F[A_1, \dots, A_m]_{ij} := F(a_{1,ij}, \dots, a_{m,ij}). \quad (9.2)$$

By the Schur product theorem, every real analytic function  $F$  of the form

$$F(\mathbf{x}) = \sum_{\alpha \in \mathbb{Z}_+^m} c_\alpha \mathbf{x}^\alpha \quad (\mathbf{x} \in I^m) \quad (9.3)$$

preserves positivity on  $\mathcal{P}_N(I)^m$  if  $c_\alpha \geq 0$  for all  $\alpha \in \mathbb{Z}_+^m$ . The reverse implication was shown by FitzGerald, Micchelli, and Pinkus in [16] for  $\rho = \infty$ , and can be thought of as a multivariable version of Schoenberg's theorem. We now explain how results on several real and complex variables can be used to generalize the work in previous

sections to this multivariable setting, including over bounded domains in the original spirit of Schoenberg and Rudin. Namely, we characterize functions mapping tuples of positive Hankel matrices into themselves. Of course, this is equivalent to mapping tuples of moment sequences of admissible measures into the same set.

First we need some notation and terminology. Given  $I$  as in (9.1), suppose the sets  $K_1, \dots, K_m \subset \mathbb{R}$  are such that all sequences in  $\mathcal{M}^\rho(K_j)$  have entries in  $I$ , for  $j = 1, \dots, m$ . A function  $F : I^m \rightarrow \mathbb{R}$  acts on  $m$ -tuples of moment sequences of measures  $\mathcal{M}^\rho(K_1) \times \dots \times \mathcal{M}^\rho(K_m)$  to produce real sequences, so that

$$F[\mathbf{s}(\mu_1), \dots, \mathbf{s}(\mu_m)]_k := F(s_k(\mu_1), \dots, s_k(\mu_m)) \quad (k \in \mathbb{Z}_+). \quad (9.4)$$

Given  $I' \subset \mathbb{R}^m$ , a function  $F : I' \rightarrow \mathbb{R}$  is *absolutely monotonic* if  $F$  is continuous on  $I'$ , and for any interior point  $\mathbf{x} \in I'$  and  $\alpha \in \mathbb{Z}_+^m$ , the mixed partial derivative  $D^\alpha F(\mathbf{x})$  exists and is non-negative. As usual, for a tuple  $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{Z}_+^m$ , we set

$$D^\alpha F(\mathbf{x}) := \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_m^{\alpha_m}} F(x_1, \dots, x_m), \quad \text{where } |\alpha| := \alpha_1 + \dots + \alpha_m.$$

The analogue of Bernstein's Theorem for the multivariable case is proved and put in its proper context in Bochner's book; see [10, Theorem 4.2.2].

Our first observation is the connection between functions acting on tuples of moment sequences and on the corresponding Hankel matrices. Given admissible measures  $\mu_1, \dots, \mu_m$  and  $\sigma$  supported on the real line, it is clear that

$$F[\mathbf{s}(\mu_1), \dots, \mathbf{s}(\mu_m)] = \mathbf{s}(\sigma) \quad \iff \quad F[H_{\mu_1}, \dots, H_{\mu_m}] = H_\sigma.$$

In particular, equality holds at each finite truncation, that is, for the corresponding leading principal  $N \times N$  submatrices, for any  $N \geq 1$ . We will henceforth switch between moment sequences and positive Hankel matrices without further comment.

We begin by considering the case of matrices with positive entries, arising from tuples of sequences in  $\mathcal{M}^\rho([0, 1])$ . To state and prove the main result in this subsection, we require a preliminary technical result.

**Lemma 9.2.** *Given an integer  $m \geq 1$ , let  $\mathcal{Y}_m$  denote the set of all  $\mathbf{y} = (y_1, \dots, y_m)^T \in (0, 1)^m$  such that the scalars*

$$\mathbf{y}^\alpha := \prod_{l=1}^m y_l^{\alpha_l}$$

*are distinct for all  $\alpha \in \mathbb{Z}_+^m$ . Then the complement of  $\mathcal{Y}_m$  in  $(0, 1)^m$  has zero  $m$ -dimensional Lebesgue measure.*

*Proof.* Let

$$\mathcal{X} := \{\mathbf{x} = \log \mathbf{y} \in (-\infty, 0)^m : \mathbf{x} \not\prec \alpha \text{ for all } \alpha \in \mathbb{Z}^m \setminus \{(0, \dots, 0)\}\}.$$

The complement of  $\mathcal{X}$  in  $(-\infty, 0)^m$  is a countable union of hyperplanes, and so has measure zero. The result now follows since  $\mathcal{Y}_m$  is the image of  $\mathcal{X}$  under a smooth map.  $\square$

The new notion of a *facewise absolutely monotonic function* on  $[0, \rho]^m$  plays an important role in our next result. In order to define it, recall that the truncated orthant  $[0, \rho]^m$  is the truncation of a convex polyhedron, and as such, is the disjoint

union of the relative interiors of its faces. These faces are in bijection with subsets of  $[m] := \{1, \dots, m\}$  via the mapping

$$J \mapsto [0, \rho]^J := \{(x_1, \dots, x_m) \in [0, \rho]^m : x_l = 0 \text{ for all } l \notin J\}, \quad (9.5)$$

and this face has relative interior  $(0, \rho)^J \times \{0\}^{[m] \setminus J}$ .

**Definition 9.3.** A function  $F : [0, \rho]^m \rightarrow \mathbb{R}$ , where  $m \geq 1$  and  $0 < \rho \leq \infty$ , is *facewise absolutely monotonic* if, for each set of indices  $J \subset [m]$ , the function  $F$  agrees on  $(0, \rho)^J \times \{0\}^{[m] \setminus J}$  with an absolutely monotonic function  $g_J$  on  $(0, \rho)^J$ . Here and henceforth, we identify without further comment  $(0, \rho)^J$  and  $(0, \rho)^J \times \{0\}^{[m] \setminus J}$ .

In other words, a facewise absolutely monotonic function is piecewise absolutely monotonic, with the pieces being the relative interiors of the faces of the truncated polyhedral cone  $[0, \rho]^m$ . The following example illustrates this in the case  $m = 2$ .

**Example 9.4.** Let

$$F(x_1, x_2) := \begin{cases} x_1^2 + x_2^2 + 1 & \text{if } x_1, x_2 > 0, \\ 2x_1 & \text{if } x_1 > 0, x_2 = 0, \\ x_2^2 + 1 & \text{if } x_1 = 0, x_2 > 0, \\ 0 & \text{if } x_1 = x_2 = 0. \end{cases}$$

Then  $F$  is facewise absolutely monotonic, with

$$g_\emptyset = 0, \quad g_{\{1\}}(x_1) = 2x_1, \quad g_{\{2\}}(x_2) = x_2^2 + 1, \quad \text{and} \quad g_{\{1,2\}}(x_1, x_2) = x_1^2 + x_2^2 + 1.$$

In this example, and, in fact, for every facewise absolutely monotonic function, the function  $g_J$  extends to an absolutely monotonic function on the closure  $[0, \rho]^J$  of its domain, for all  $J \subset [m]$ . We denote this extension by  $\tilde{g}_J$ .

Furthermore, for Example 9.4, the functions  $\tilde{g}_J$  satisfy a form of monotonicity that is compatible with the partial order on their labels:

$$K \subset J \subset [m] \implies 0 \leq \tilde{g}_K \leq \tilde{g}_J \quad \text{on } [0, \rho]^K. \quad (9.6)$$

A word of caution: while  $\tilde{g}_{\{1\}}(x_1) \leq \tilde{g}_{\{1,2\}}(x_1, 0)$  for all  $x_1 \geq 0$ , it is not true that the difference of these functions is absolutely monotonic on  $[0, \rho]$ .

With this definition and example in hand, together with Lemma 9.2, we can now characterize the preservers of tuples of moment sequences in  $\mathcal{M}^\rho([0, 1])$ .

**Theorem 9.5.** *Let  $F : [0, \rho]^m \rightarrow \mathbb{R}$ , where  $m \geq 1$  and  $0 < \rho \leq \infty$ , and fix  $\mathbf{y} = (y_1, \dots, y_m)^T \in \mathcal{Y}_m$ , as in Lemma 9.2. The following are equivalent.*

- (1)  $F[-]$  maps  $\mathcal{M}^\rho(\{1, y_1\}) \times \dots \times \mathcal{M}^\rho(\{1, y_m\}) \cup \mathcal{M}^\rho(\{0, 1\})^m$  into  $\mathcal{M}(\mathbb{R})$ , and

$$F((a_1, \dots, a_m))F((b_1, \dots, b_m)) \geq F\left(\left(\sqrt{a_1 b_1}, \dots, \sqrt{a_m b_m}\right)\right)^2$$

for all  $a_1, \dots, a_m, b_1, \dots, b_m \in [0, \rho]$ .

- (2)  $F[-]$  maps  $\mathcal{M}^\rho([0, 1])^m$  into  $\mathcal{M}([0, 1])$ .

- (3)  $F$  is facewise absolutely monotonic, and the functions  $\{g_J : J \subset [m]\}$  satisfy the monotonicity condition (9.6).

Reformulating this result, as in the one-dimensional case above, it suffices to work only with Hankel matrices of rank at most two. Moreover, Theorem 4.1 is precisely Theorem 9.5 when  $m = 1$ . The proof builds on Theorem 4.1; however, the higher dimensionality introduces several additional challenges.

A large part of Theorem 9.5 can be deduced from the following reformulation on the open cell in the positive orthant.

**Theorem 9.6.** *Fix  $\rho \in (0, \infty]$ , an integer  $m \geq 1$  and a point  $\mathbf{y} = (y_1, \dots, y_m)^T \in \mathcal{Y}_m$ , as in Lemma 9.2. For  $1 \leq l \leq m$  and  $N \geq 1$ , let*

$$\mathbf{u}_{l,N} := (1, y_l, \dots, y_l^{N-1})^T,$$

$$\text{and } \mathbf{H}_l^+(N) := \{a\mathbf{1}_{N \times N} + b\mathbf{u}_{l,N}\mathbf{u}_{l,N}^T : a \in (0, \rho), b \in [0, \rho - a]\}.$$

*If the function  $F : (0, \rho)^m \rightarrow \mathbb{R}$  is such that  $F[-]$  preserves positivity on  $\mathcal{P}_2((0, \rho))^m$  and on  $\mathbf{H}_1^+(N) \times \dots \times \mathbf{H}_m^+(N)$  for all  $N \geq 1$ , then  $F$  is absolutely monotonic and is the restriction of an analytic function on  $D(0, \rho)^m$ .*

**Remark 9.7.** As noted in Remark 4.3 for the one-variable case, the proof of Theorem 9.6 goes through under a weaker hypothesis, with the test sets replaced by the set of rank-one  $m$ -tuples  $\mathcal{P}_2^1((0, \rho))^m$  and the set

$$\left\{ \left( \begin{pmatrix} a_1 & b_1 \\ b_1 & b_1 \end{pmatrix}, \dots, \begin{pmatrix} a_m & b_m \\ b_m & b_m \end{pmatrix} \right) : 0 < b_l < a_l < \rho, 1 \leq l \leq m \right\}. \quad (9.7)$$

The matrices in  $\mathbf{H}_l^+(N)$  and (9.7) are precisely the truncated moment matrices of admissible measures supported on  $\{1, y_l\}$  and on  $\{0, 1\}$ , respectively.

*Proof of Theorem 9.6.* We begin by recording a few basic properties of  $F$ . First, either  $F$  is identically zero, or it is everywhere positive on  $(0, \rho)^m$ ; this may be shown similarly to the proof of Theorem 4.2. Moreover, using only tuples from  $\mathcal{P}_2^1((0, \rho))$  and (9.7), as well as the hypotheses, one can argue as in the proof of Theorem 4.2, and show that  $F$  is continuous on  $(0, \rho)^m$ .

Next, given  $\mathbf{c} = (c_1, \dots, c_m)^T \in (0, \rho)^m$ , the function  $g$  such that

$$g(\mathbf{x}) := F(\mathbf{x} + \mathbf{c}) \quad \text{for all } \mathbf{x} \in (0, \rho - c_1) \times \dots \times (0, \rho - c_m)$$

satisfies the same hypotheses as  $F$ , but with  $\rho$  replaced by  $\rho - c_l$  in each  $\mathbf{H}_l^+(N)$ , and with  $\mathcal{P}_2^1((0, \rho))^m$  replaced by  $\mathcal{P}_2^1((0, \rho - c_1)) \times \dots \times \mathcal{P}_2^1((0, \rho - c_m))$ . Therefore, as in the proof of [16, Theorem 2.1], a mollifier argument reduces the problem to considering only smooth  $F$ . We now follow the proof of [16, Proposition 2.5], but with suitable modifications imposed by the weaker hypotheses.

Given  $r \geq 0$ , we take  $N \geq \binom{r+m}{m}$ , and let  $\mathbf{y}'_l := (1, y_l, \dots, y_l^{N-1})^T$  for  $1 \leq l \leq m$ . Fix some  $\mathbf{c} \in (0, \rho)^m$ , choose  $b_l \in (0, \rho - c_l)$  for all  $l$  and let

$$A_l := c_l \mathbf{1}_{N \times N} + b_l \mathbf{y}'_l \mathbf{y}'_l{}^T \in \mathbf{H}_l^+(N),$$

so that  $F[A_1, \dots, A_m] \in \mathcal{P}_N(\mathbb{R})$ . We now use Lemma 9.2: since  $\mathbf{y} \in \mathcal{Y}_m$  and  $N \geq \binom{r+m}{m}$  by assumption, for each  $\beta \in \mathbb{Z}_+^m$  with  $|\beta| \leq r$  we can choose  $\mathbf{v}_\beta \in \mathbb{R}^N$  such that

$$\mathbf{v}_\beta \perp (1, \mathbf{y}^\alpha, \mathbf{y}^{2\alpha}, \dots, \mathbf{y}^{(N-1)\alpha})^T \quad \text{for all } \alpha \in \mathbb{Z}_+^m \setminus \{\beta\} \text{ with } |\alpha| \leq r,$$

and  $(1, \mathbf{y}^\beta, \dots, \mathbf{y}^{(N-1)\beta}) \mathbf{v}_\beta = 1$ . An application of Taylor's theorem (similar to its use in Proposition 4.10 or [16, Proposition 2.5]) now gives that the derivative  $D^\beta F(\mathbf{c}) \geq 0$ .

Thus  $F$  is absolutely monotonic on  $(0, \rho)^m$ , and Schoenberg's observation [41, Theorem 5.2] implies that  $F$  is the restriction to  $(0, \rho)^m$  of an analytic function on  $D(0, \rho)^m$ .  $\square$

With this result in hand, we can now proceed.

*Proof of Theorem 9.5.* Clearly, (2)  $\implies$  (1).

We will show (1)  $\implies$  (3) by induction on  $m$ . As noted above, the case  $m = 1$  is precisely Theorem 4.1. For the induction step, we first restrict  $F$  to the relative interior of any face of the truncated polyhedron  $[0, \rho]^m$ , say  $(0, \rho)^J$  for some  $J \subset [m]$ . The induction hypothesis and Theorem 9.6 give that  $F$  is facewise absolutely monotonic, so  $F \equiv g_J$  on  $(0, \rho)^J$ , with  $g_J$  absolutely monotonic. To see that (9.6) holds, we claim that, for any pair of subsets  $L \subset K \subsetneq J \subset [m]$ ,

$$\tilde{g}_K(\mathbf{x}) \leq \tilde{g}_J(\mathbf{x}) \quad \text{whenever } \mathbf{x} \in (0, \rho)^L \subset [0, \rho]^m.$$

For ease of exposition, we show this for an illustrative example; the general case follows with minimal modification. Suppose  $J = \{1, 2, 3\}$ ,  $K = \{1, 2\}$ , and  $L = \{1\}$ . For any  $(x_1, 0, 0) \in (0, \rho)^L$ , we set

$$(a_1, a_2, a_3) := (x_1, x_2, x_3) \quad \text{and} \quad (b_1, b_2, b_3) := (x_1, x_2, 0),$$

where  $x_2 > 0$  and  $x_3 > 0$ . By hypothesis (1), it follows that

$$\tilde{g}_J(x_1, x_2, x_3) \tilde{g}_K(x_1, x_2, 0) \geq \tilde{g}_K(x_1, x_2, 0)^2,$$

and taking limits as  $x_2 = x_{K \setminus L} \rightarrow 0^+$  and  $x_3 = x_{J \setminus K} \rightarrow 0^+$ , we have that

$$\tilde{g}_J(x_1, 0, 0) \tilde{g}_K(x_1, 0, 0) \geq \tilde{g}_K(x_1, 0, 0)^2,$$

and so (9.6) holds as required.

Finally, to show that (3)  $\implies$  (2), given positive Hankel matrices  $A_1, \dots, A_m$  arising from moment sequences in  $\mathcal{M}^\rho([0, 1])$ , let

$$J := \{l \in [m] : a_{l,11} > 0\} \quad \text{and} \quad K := \{l \in [m] : a_{l,22} > 0\}.$$

Note that  $K \subset J \subset [m]$ . Recalling that the only Hankel matrices arising from  $\mathcal{M}^\rho([0, 1])$  and having zero entries are of the form  $H_{a\delta_0}$  for some  $a \in [0, \rho)$ , we may write

$$F[A_1, \dots, A_m] = (g_J(a_{l,11} : l \in J) - g_K(a_{l,11} : l \in K)) H_{\delta_0} + g_K[A_l : l \in K]. \quad (9.8)$$

For example, given  $a, b, c, d > 0$ , we have that

$$\begin{aligned} F \left[ \begin{pmatrix} a & b \\ b & c \end{pmatrix}, \begin{pmatrix} d & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right] &= \begin{pmatrix} g_{\{1,2\}}(a, d) & g_{\{1\}}(b) \\ g_{\{1\}}(b) & g_{\{1\}}(c) \end{pmatrix} \\ &= (g_{\{1,2\}}(a, d) - g_{\{1\}}(a)) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + g_{\{1\}} \left[ \begin{pmatrix} a & b \\ b & c \end{pmatrix} \right]. \end{aligned}$$

The proof concludes by observing that both terms in the right-hand side of (9.8) are positive semidefinite, by the Schur product theorem and hypothesis (3):

$$\begin{aligned} g_J(a_{l,11} : l \in J) &\geq \lim_{a_{l,11} \rightarrow 0^+} \prod_{l \in J \setminus K} g_J(a_{l,11} : l \in J) = \tilde{g}_J(a_{l,11} : l \in K) \\ &\geq g_K(a_{l,11} : l \in K). \end{aligned} \quad \square$$

As Theorem 9.5 shows, the notion of facewise absolutely monotone maps on  $[0, \rho]^m$  is a refinement of absolute monotonicity, emerging from the study of positivity preservers of tuples of moment sequences, or, rather, of the Hankel matrices arising from them. If, instead, one studies maps preserving positivity on tuples of all positive semidefinite matrices, or even all Hankel matrices, then this richer class of maps does not arise.

**Proposition 9.8.** *Suppose  $\rho \in (0, \infty]$  and  $F : I^m \rightarrow \mathbb{R}$ , where  $I = [0, \rho]$ . The following are equivalent.*

- (1)  $F[-]$  preserves positivity on the space of  $m$ -tuples of positive Hankel matrices with entries in  $I$ .
- (2)  $F$  is absolutely monotonic on  $I^m$ .
- (3)  $F[-]$  preserves positivity on the space of  $m$ -tuples of all positive matrices with entries in  $I$ .

*Proof.* Clearly (2)  $\implies$  (3)  $\implies$  (1). Now suppose (1) holds. By Theorem 9.6,  $F$  is absolutely monotonic on the domain  $(0, \rho)^m$ , and agrees there with an analytic function  $g : D(0, \rho)^m \rightarrow \mathbb{C}$ . We now claim  $F \equiv g$  on  $I^m$ . The proof is by induction on  $m$ , with the  $m = 1$  case shown in Proposition 8.1.

Suppose  $m > 1$ , and let  $\mathbf{c} = (c_1, \dots, c_m) \in I^m \setminus (0, \rho)^m$ . Note that at least one coordinate of  $\mathbf{c}$  is zero. We choose  $\mathbf{u}_n = (u_{1,n}, \dots, u_{m,n}) \in (0, \rho)^m$  such that  $\mathbf{u}_n \rightarrow \mathbf{c}$ , and we wish to show that  $F(\mathbf{u}_n) = g(\mathbf{u}_n) \rightarrow F(\mathbf{c})$ . Let

$$H := \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix} \quad \text{and} \quad A_{l,n} := \begin{cases} u_{l,n} \mathbf{1}_{3 \times 3} & \text{if } c_l > 0, \\ u_{l,n} H & \text{if } c_l = 0. \end{cases}$$

Using (1) and the induction hypothesis for the (1, 2) and (2, 1) entries, it follows that

$$\lim_{n \rightarrow \infty} F[A_{1,n}, \dots, A_{m,n}] = \begin{pmatrix} g(\mathbf{c}) & F(\mathbf{c}) & g(\mathbf{c}) \\ F(\mathbf{c}) & g(\mathbf{c}) & g(\mathbf{c}) \\ g(\mathbf{c}) & g(\mathbf{c}) & g(\mathbf{c}) \end{pmatrix} \in \mathcal{P}_3.$$

Computing the determinants of the leading principal minors gives

$$g(\mathbf{c}) \geq 0, \quad g(\mathbf{c}) \geq |F(\mathbf{c})|, \quad \text{and} \quad -g(\mathbf{c})(g(\mathbf{c}) - F(\mathbf{c}))^2 \geq 0.$$

Hence  $F(\mathbf{c}) = g(\mathbf{c})$ , and the proof is complete.  $\square$

**9.3. Transformers of moment-sequence tuples: the general case.** Having resolved the characterization problem for functions defined on the positive orthant, we now work over the whole of  $\mathbb{R}^m$ . This requires us to consider admissible measures which may have support outside  $[-1, 1]$ . For such measures, the mass no longer dominates all moments, and so we include in our test sets truncations of the corresponding moment sequences, whereas for measures supported in  $[-1, 1]$ , the full moment sequence lies in the test set. More precisely, we have the following definition.

**Definition 9.9.** Given  $K \subset \mathbb{R}$  and  $\rho \in (0, \infty]$ , let  $\widetilde{\mathcal{M}}^\rho(K)$  be the collection of possibly truncated moment sequences for all measures  $\mu \in \text{Meas}^+(K)$ , where each sequence is truncated prior to the first moment of  $\mu$  that lies outside  $(-\rho, \rho)$ , or is not truncated if no such moment exists.

Clearly, if  $\rho = \infty$ , then  $\widetilde{\mathcal{M}}^\rho(K) = \mathcal{M}^\rho(K)$ , while if  $\rho$  is finite and  $K \subset [-1, 1]$ , then  $\widetilde{\mathcal{M}}^\rho(K) = \mathcal{M}^\rho(K)$ . Next is a more complex example, which occurs in the following theorem; see particularly Step 5 of its proof.

**Example 9.10.** If  $K = \{-1, v, 1\}$  for  $v > 1$ , and  $\rho < \infty$ , then  $\widetilde{\mathcal{M}}^\rho(K)$  consists of  $\mathcal{M}^\rho(\{-1, 1\})$  together with truncated moment sequences of measures with positive mass at  $v$ . If  $\mu = a\delta_{-1} + b\delta_1 + c\delta_v$  with  $a, b \geq 0$  and  $c > 0$ , then the moments of  $\mu$  are unbounded, and  $\widetilde{\mathcal{M}}^\rho(K)$  contains the truncated moment sequence  $(s_0(\mu), \dots, s_{n-1}(\mu))$ , where  $n$  is the smallest positive integer such that  $|a(-1)^n + b + cv^n| \geq \rho$ .

We can now state our final main result in this section.

**Theorem 9.11.** *Suppose  $F : I^m \rightarrow \mathbb{R}$ , where  $m \geq 1$  and  $I = (-\rho, \rho)$  with  $0 < \rho \leq \infty$ . The following are equivalent.*

- (1) *For some  $\delta > 0$ , the function  $F$ , when applied entrywise, maps  $\widetilde{\mathcal{M}}^\rho([-1, 1+\delta])^m$  into the set of possibly truncated moment sequences of measures on  $\mathbb{R}$ .*
- (2) *For every  $\delta > 0$ , the function  $F$ , when applied entrywise, maps  $\widetilde{\mathcal{M}}^\rho([-1, 1+\delta])^m$  into the set of possibly truncated moment sequences of measures on  $\mathbb{R}$ .*
- (3) *Applied entrywise, the function  $F$  maps  $\mathcal{P}_N(I)^m$  into  $\mathcal{P}_N(\mathbb{R})$  for any  $N \geq 1$ .*
- (4) *The function  $F$  is absolutely monotonic on  $[0, \rho]^m$  and agrees on  $I^m$  with an analytic function.*

In particular, analogously to the one-variable case, Theorem 9.11 strengthens the multivariable analogue of Schoenberg's theorem in [16] by using only Hankel matrices arising from tuples of moment sequences. Moreover, akin to the  $m = 1$  case, the proof reveals that one only requires Hankel matrices of rank at most 3.

Given any  $v > 0$ , we let  $\mathcal{M}_v^\rho := \widetilde{\mathcal{M}}^\rho(\{-1, v, 1\})$  and

$$\mathcal{M}_{[v]}^\rho := \bigcup_{s_1 \in \{-1, 0, 1\}, s_2 \in \{-v, 0, v\}} \mathcal{M}^\rho(\{s_1, s_2\}).$$

**Corollary 9.12.** *The hypotheses in Theorem 9.11 are also equivalent to the following.*

- (5) *There exist  $\epsilon > 0$  and  $u_0 \in (0, 1)$  such that  $F[-]$  maps*

$$(\mathcal{M}_{[u_0]}^\rho)^m \cup \bigcup_{v_1, \dots, v_m \in (0, 1+\epsilon)} \mathcal{M}_{v_1}^\rho \times \dots \times \mathcal{M}_{v_m}^\rho$$

*into the set of possibly truncated moment sequences of measures on  $\mathbb{R}$ .*

As the reader will observe, hypothesis (5) is stronger, even in the one-dimensional case, than the corresponding hypothesis in Theorem 6.1. As the proof shows, these extra assumptions are required to obtain continuity on every orthant and on 'walls' between orthants, as well as real analyticity on one-parameter curves.

**Remark 9.13.** Theorem 9.11 is the only instance when we deviate from Table 1.2 in the Introduction, but it should not come as a surprise that stronger conditions are required to guarantee real analyticity in several variables.

*Proof of Theorem 9.11 and Corollary 9.12.* Clearly (4)  $\implies$  (3)  $\implies$  (2)  $\implies$  (1)  $\implies$  (5) by the Schur product theorem. Thus, we will assume (5) and obtain (4). By Theorem 9.6, the function  $F$  is absolutely monotonic on the open positive orthant

$(0, \rho)^m$ , and equals the restriction to  $(0, \rho)^m$  of an analytic function  $g : D(0, \rho)^m \rightarrow \mathbb{C}$ . We now show that  $F \equiv g$  on all of  $(-\rho, \rho)^m$ . The proof follows the  $m = 1$  case in Section 6; for ease of exposition, we break it up into steps.

**Step 1.** We first prove  $F$  is locally bounded. This follows by using  $\mathcal{M}_2^\rho(\{-1, 1\})^m$ , as in the proof of Lemma 6.3. As above, this gives that

$$F[-] : (\mathcal{M}_{[u_0]}^\rho)^m \cup \mathcal{M}_{v_1}^\rho \times \cdots \times \mathcal{M}_{v_m}^\rho \rightarrow \mathcal{M}([-1, 1]) \quad \text{for all } v_1, \dots, v_m \in (-1, 1). \quad (9.9)$$

**Step 2.** Next, we show that  $F$  is continuous on  $(-\rho, \rho)^m$ . The first objective is to show continuity of  $F$  inside each open orthant of  $(-\rho, \rho)^m$ . Given non-zero scalars  $c_1, \dots, c_m$  with  $|c_1|, \dots, |c_m| < \rho$ , and any sequence  $\{(v_{1,n}, \dots, v_{m,n}) : n \geq 1\} \subset \mathbb{R}^m$  converging to the origin, let

$$a_{l,n} := |c_l| + \frac{\operatorname{sgn}(c_l)u_0}{u_0 - u_0^3}v_{l,n} \quad \text{and} \quad \mu_{l,n} := a_{l,n}\delta_{\operatorname{sgn}(c_l)} + \frac{|v_{l,n}|}{u_0 - u_0^3}\delta_{-\operatorname{sgn}(v_{l,n})u_0} \quad (9.10)$$

for  $l = 1, \dots, m$ . Note that, for all sufficiently large  $n$ , the sequence  $\mathbf{s}(\mu_{l,n}) \in \mathcal{M}_{[u_0]}^\rho$ .

We now follow the proof of Proposition 6.4. Suppose that  $F[H_{\mu_{1,n}}, \dots, H_{\mu_{m,n}}] = H_{\sigma_n}$  for some admissible measure  $\sigma_n \in \operatorname{Meas}^+([-1, 1])$ , for every  $n \geq 1$ . The polynomials  $p_\pm(t) := (1 \pm t)(1 - t^2)$  are non-negative on  $[-1, 1]$ , so, by (4.1),

$$\begin{aligned} \int_{-1}^1 p_\pm(t) d\sigma_n &\geq 0 \\ \implies F(s_0(\mu_{l,n})_{l=1}^m) - F(s_2(\mu_{l,n})_{l=1}^m) &\geq |F(s_1(\mu_{l,n})_{l=1}^m) - F(s_3(\mu_{l,n})_{l=1}^m)|. \end{aligned} \quad (9.11)$$

Computing the moments of  $\mu_{l,n}$  gives the following:

$$\begin{aligned} s_0(\mu_{l,n}) &= |c_l| + \frac{\operatorname{sgn}(c_l)u_0 + \operatorname{sgn}(v_{l,n})}{u_0 - u_0^3}v_{l,n}, & s_1(\mu_{l,n}) &= c_l, \\ s_2(\mu_{l,n}) &= |c_l| + \frac{\operatorname{sgn}(c_l)u_0 + \operatorname{sgn}(v_{l,n})u_0^2}{u_0 - u_0^3}v_{l,n}, & s_3(\mu_{l,n}) &= c_l + v_{l,n}. \end{aligned} \quad (9.12)$$

As  $n \rightarrow \infty$ , by the continuity of  $F$  in  $(0, \rho)^m$ , the left-hand side of (9.11) goes to zero, whence so does the right-hand side, which is  $|F(c_1, \dots, c_m) - F(c_1 + v_{1,n}, \dots, c_m + v_{m,n})|$ . This proves the continuity of  $F$  at  $(c_1, \dots, c_m)$ , so in every open orthant of  $(-\rho, \rho)^m$ .

To conclude this step, we show  $F$  is continuous on the boundary of the orthants, that is, on the union of the coordinate hyperplanes:

$$\mathcal{Z} := \{(x_1, \dots, x_m) \in (-\rho, \rho)^m : x_1 \cdots x_m = 0\}.$$

The proof is by induction on  $m$ , with the case  $m = 1$  shown in Proposition 6.4. For general  $m \geq 2$ , by the induction hypothesis  $F$  is continuous when restricted to  $\mathcal{Z}$ . It remains to prove  $F$  is continuous at a point  $\mathbf{c} = (c_1, \dots, c_m) \in \mathcal{Z}$  when approached along a sequence  $\{(c_1 + v_{1,n}, \dots, c_m + v_{m,n}) : n \geq 1\}$  which lies in the interior of some orthant in  $(-\rho, \rho)^m$ . Repeating the computations for (9.12), with the same sequences  $a_{l,n}$  and  $\mu_{l,n}$ , and the polynomials  $p_\pm(t)$ , we note that if  $c_l \neq 0$  then  $s_0(\mu_{l,n}) > 0$  and  $s_2(\mu_{l,n}) > 0$  for all sufficiently large  $n$ , while if  $c_l = 0$  then  $s_0(\mu_{l,n}) > 0$  and  $s_2(\mu_{l,n}) > 0$  for all  $n$ , since  $c_l + v_{l,n} \neq 0$  by assumption. Therefore, in all cases, the left-hand side of (9.11) eventually equals  $F(\mathbf{u}_n) - F(\mathbf{u}'_n)$ , with  $\mathbf{u}_n$  and  $\mathbf{u}'_n$  in the positive open

orthant  $(0, \rho)^m$ , and both converging to  $|\mathbf{c}| := (|c_1|, \dots, |c_m|)$ . Since  $F \equiv g$  on  $(0, \rho)^m$  for some analytic function  $g$  on  $D(0, \rho)^m$ , so (9.11) gives that

$$\lim_{n \rightarrow \infty} |F(\mathbf{c}) - F(c_1 + v_{1,n}, \dots, c_m + v_{m,n})| \leq \lim_{n \rightarrow \infty} F(\mathbf{u}_n) - F(\mathbf{u}'_n) = g(|\mathbf{c}|) - g(|\mathbf{c}|) = 0.$$

It follows that  $F$  is continuous at all  $\mathbf{c} \in \mathcal{Z}$ , and hence on all of  $(-\rho, \rho)^m$ , as claimed.

**Step 3.** The next step in the proof is to show that it suffices to consider  $F$  to be smooth. This is achieved using a mollifier argument, exactly as in the one-variable situation.

**Step 4.** Henceforth we assume  $F$  is smooth on  $(-\rho, \rho)^m$ ; akin to the one-variable case, we will show that  $F$  is in fact real analytic. The proof extends across multiple steps below. The first step is encoded into the following technical lemma, for convenience.

**Lemma 9.14.** *Fix  $\rho \in (0, \infty]$  and a non-zero vector  $\mathbf{v} \in \mathbb{R}^m$ . For any  $\mathbf{c} \in (-\rho, \rho)^m$ , let*

$$\eta_{\mathbf{v}, \mathbf{c}} := \begin{cases} e^{-\|\mathbf{v}\|_\infty} & \text{if } \rho = \infty, \\ e^{-\|\mathbf{v}\|_\infty(\rho - \|\mathbf{c}\|_\infty)} & \text{if } \rho < \infty. \end{cases} \quad (9.13)$$

*Then, for any  $\mathbf{w} \in (-\rho, \rho)^m$ , there exists  $\mathbf{c} \in (-\rho, \rho)^m$  such that  $\mathbf{w} = \mathbf{c} + \eta_{\mathbf{v}, \mathbf{c}} \mathbf{1}$ , where the vector  $\mathbf{1} := (1, \dots, 1)$ .*

*Proof.* The assertion is immediate if  $\rho = \infty$ , so we suppose henceforth that  $\rho$  is finite. Let

$$g(t) := \|\mathbf{w} - t\mathbf{1}\|_\infty - (\rho - te^{\|\mathbf{v}\|_\infty}) \quad (t \geq 0).$$

Clearly  $g(0) < 0 < g(\rho)$ , so  $g$  has a root  $t_0 \in (0, \rho)$ . Now the vector  $\mathbf{c} := \mathbf{w} - t_0\mathbf{1}$  is as required (and  $t_0 = \eta_{\mathbf{v}, \mathbf{c}}$ ).  $\square$

**Step 5.** We now claim that for every  $\mathbf{c} \in (-\rho, \rho)^m$  and every unit direction vector  $\mathbf{v} = (v_1, \dots, v_m) \in S^{m-1}$ , the function  $F$  is real analytic in the one-parameter space

$$\{\mathbf{c} + \eta_{\mathbf{v}, \mathbf{c}} e^{-x\mathbf{v}} : x \in (-1, 1)\} \subset (-\rho, \rho)^m,$$

at the point  $x = 0$ , i.e., at  $\mathbf{w} = \mathbf{c} + \eta_{\mathbf{v}, \mathbf{c}} \mathbf{1}$ . Here  $\eta_{\mathbf{v}, \mathbf{c}}$  and  $\mathbf{1}$  are as in Lemma 9.14, and we also use the notation

$$e^{-x\mathbf{v}} := (e^{-xv_1}, \dots, e^{-xv_m}).$$

Notice moreover that the  $l$ th coordinate of  $\mathbf{c} + \eta_{\mathbf{v}, \mathbf{c}} e^{-x\mathbf{v}}$  is strictly bounded above in absolute value by  $\|\mathbf{c}\|_\infty + \eta_{\mathbf{v}, \mathbf{c}} e^{\|\mathbf{v}\|_\infty}$ , which is no more than  $\rho$ .

To show the claim, we use the notation  $|\mathbf{c}| := (|c_1|, \dots, |c_m|)$  and also fix a scalar  $x \in (-1, 1)$ . We let  $p_{\pm, n}(t) := (1 \pm t)(1 - t^2)^n$  for  $n \geq 0$  and

$$\mu_{l, s} := |c_l| \delta_{\text{sgn}(c_l)} + \eta_{\mathbf{v}, \mathbf{c}} e^{-xv_l} \delta_{e^{-sv_l}} \quad \text{whenever } 0 < s < (1 - x)/(2n + 1),$$

where  $1 \leq l \leq m$ .

As  $p_{\pm, n}(t) \geq 0$  for all  $t \in [-1, 1]$  and all  $n \geq 0$ , applying (4.1) gives that

$$\left| \sum_{k=0}^n \binom{n}{k} (-1)^k F(|\mathbf{c}| + \eta_{\mathbf{v}, |\mathbf{c}|} e^{-(x+2ks)\mathbf{v}}) \right| \geq \left| \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} F(\mathbf{c} + \eta_{\mathbf{v}, \mathbf{c}} e^{-(x+(2k+1)s)\mathbf{v}}) \right|,$$

where we note that  $\eta_{\mathbf{v}, \mathbf{c}} = \eta_{\mathbf{v}, |\mathbf{c}|}$ , and that all arguments of  $F$  lie in  $(-\rho, \rho)^m$  by the restriction on  $s$ . Note that we use the fact that our test set contains  $\widetilde{\mathcal{M}}^\rho(\{-1, v, 1\})$  for  $v \in (1, 1 + \epsilon)$  here, and only here, in this proof.

Now setting  $H_{\mathbf{v},\mathbf{c}}(x) := F(\mathbf{c} + \eta_{\mathbf{v},\mathbf{c}}e^{-x\mathbf{v}})$ , dividing both sides of this inequality by  $s^n$ , and then taking  $s \rightarrow 0^+$ , it follows that

$$\left| \frac{d^n}{dx^n} H_{\mathbf{v},|\mathbf{c}|}(x) \right| \geq \left| \frac{d^n}{dx^n} H_{\mathbf{v},\mathbf{c}}(x) \right| \quad \text{whenever } x \in (-1, 1).$$

These estimates prove that the function  $F$  is real analytic at the point in the one-parameter space as claimed.

**Step 6.** We now complete the proof. The real-analytic local diffeomorphism

$$T : (u_1, \dots, u_m) \mapsto (e^{u_1} - 1, e^{u_2} - 1, \dots, e^{u_m} - 1)$$

maps the origin to itself and, by the previous step, the function

$$\mathbf{u} \mapsto F(\mathbf{c} + \eta_{\mathbf{v},\mathbf{c}}\mathbf{1} + \eta_{\mathbf{v},\mathbf{c}}T(-\mathbf{u}))$$

is smooth and real analytic in the unit ball along every straight line passing through the origin. Standard criteria for real analyticity (see [2, Theorem 5.5.33], for example) now give that  $F$  is real analytic at the point  $\mathbf{c} + \eta_{\mathbf{v},\mathbf{c}}\mathbf{1}$ , hence at every point  $\mathbf{w} \in (-\rho, \rho)^m$ , by Lemma 9.14.

Finally, recall that  $F$  agrees on  $(0, \rho)^m$  with an analytic function  $g : D(0, \rho)^m \rightarrow \mathbb{C}$ . As  $F : (-\rho, \rho)^m \rightarrow \mathbb{R}$  is real analytic, so  $F = g|_{(-\rho, \rho)^m}$  and the proof is complete.  $\square$

**Remark 9.15.** As Step 2 in the proof above shows, we may replace  $(\mathcal{M}_{[u_0]}^\rho)^m$  in hypothesis (5) of Corollary 9.12 by  $\mathcal{M}_{[u_1]}^\rho \times \dots \times \mathcal{M}_{[u_m]}^\rho$  for any  $u_1, \dots, u_m \in (0, 1)$ .

**Remark 9.16.** Akin to the one-dimensional case, one may now show that Theorems 9.5 and 9.11 hold more generally for tuples of measures with bounded mass. More precisely, one should fix  $\rho_1, \dots, \rho_m \in (0, \infty)$  and work with tuples of admissible measures  $(\mu_1, \dots, \mu_m)$  supported in  $[-1, 1]$  and such that  $s_0(\mu_l) < \rho_l$  for  $l = 1, \dots, m$ , whence  $s_k(\mu_l) < \rho_l$  for every  $k \geq 0$  and all such  $l$ . As discussed in the Introduction, this explains how our results unify and strengthen the Schoenberg–Rudin theorem and the FitzGerald–Micchelli–Pinkus result for positivity preservers.

To prove Theorem 9.5 for  $F : I_1 \times \dots \times I_m \rightarrow \mathbb{R}$ , where  $I_l = [0, \rho_l)$ , one should first define facewise absolutely monotonic maps on  $I_1 \times \dots \times I_m$  using the relative interiors of the faces cut out by the same functionals as for  $[0, \rho)^m$ . The existing proof for the case  $\rho_1 = \dots = \rho_m$  goes through with minimal modifications, including to Theorem 9.6. The same is true for proving Theorem 9.11 with the domain  $(-\rho_1, \rho_1) \times \dots \times (-\rho_m, \rho_m)$  in place of  $(-\rho, \rho)^m$ .

**Remark 9.17.** There is a simple and potentially very useful conditioning operation which can assist with numerical or computational entrywise manipulation of Hankel matrices or Hankel kernels arising from moments. Namely, the moments

$$s_\alpha = \int_K x^\alpha d\mu(x) \quad (\alpha \in \mathbb{Z}_+^m)$$

of a positive measure with compact support  $K$  can be rescaled,

$$s_\alpha \mapsto u_\alpha = t^{|\alpha|} s_\alpha,$$

by a factor  $t > 0$ , so that  $u_\alpha$  are the moments of a positive measure supported by the unit cube, or even by its interior. Of course, *a priori* information on the size of the support  $K$  is essential for this step, but in this way some of the complications outlined in Theorem 9.11 and its proof can be avoided.

## 10. LAPLACE-TRANSFORM INTERPRETATIONS

When speaking about completely monotonic or absolutely monotonic functions one cannot leave aside Laplace transforms. We briefly touch the subject below, in connection with our theme.

Let  $F$  be an absolutely monotonic function on  $(0, \infty)$ , and let  $\mu$  and  $\sigma$  be admissible measures supported on  $[0, 1]$  such that

$$F(s_k(\mu)) = s_k(\sigma) \quad \text{for all } k \geq 0. \quad (10.1)$$

By the change of variable  $x = e^{-t}$ , we can push forward the restriction of the measure  $\mu$  to  $(0, 1]$  to a measure  $\mu_1$  on  $[0, \infty)$ , and similarly for  $\sigma$ . Thus, with the possible loss of zeroth-order moments, we obtain

$$s_k(\mu) = \int_0^\infty e^{-kt} d\mu_1(t) \quad \text{and} \quad s_k(\sigma) = \int_0^\infty e^{-kt} d\sigma_1(t).$$

If  $\mathcal{L}$  denotes the Laplace transform, so that

$$\mathcal{L}\nu(z) = \int_0^\infty e^{-tz} d\nu(t),$$

then  $\mathcal{L}\nu$  is a complex-analytic function in the open half-plane  $\mathbb{C}^+ := \{z \in \mathbb{C} : \Re z > 0\}$ . Our assumption (10.1) becomes

$$F(\mathcal{L}\mu_1(k)) = \mathcal{L}\sigma_1(k) \quad \text{for all } k \geq 1,$$

and a classical observation due to Carlson [11] implies that

$$F(\mathcal{L}\mu_1(z)) = \mathcal{L}\sigma_1(z) \quad \text{for all } z \in \mathbb{C}^+.$$

More precisely, Carlson's Theorem asserts that a bounded analytic function in the right half-plane is identically zero if it vanishes at all positive integers. The proof relies on the Phragmén–Lindelöf principle [34]; see also [8] or [51, §5.8] for more details.

In this section, we will show some results from the interplay between the Laplace transform and functions which transform positive Hankel matrices.

For point masses, the situation is rather straightforward. If  $\mu = \delta_{e^{-a}}$  for some point  $a \in [0, \infty)$ , and  $F(x) = \sum_{n=0}^\infty c_n x^n$ , then

$$F(\mathcal{L}\mu(z)) = F(e^{-az}) = \sum_{n=0}^\infty c_n e^{-anz} = \mathcal{L}\sigma_1(z),$$

where

$$\sigma_1 = \sum_{n=0}^\infty c_n \delta_{an} \quad \text{and} \quad \sigma = \sum_{n=0}^\infty c_n \delta_{e^{-an}}.$$

More generally, if  $\mu$  has countable support, then the transform  $F[-]$  will yield a measure with countable support also. A strong converse to this is the following result.

**Proposition 10.1.** *Let  $a \in (0, 1)$  and suppose the function  $F : x \mapsto \sum_{n=0}^\infty c_n x^n$  is absolutely monotonic on  $(0, \infty)$ . The following are equivalent.*

- (1) *There exists an admissible measure  $\mu$  on  $[0, 1]$  such that*

$$F(s_k(\mu)) = a^k \quad \text{for all } k \geq 0.$$

- (2)  *$F(x) = x^N$  for some  $N \geq 1$ , and  $\mu = \delta_{a^{1/N}}$ .*

*Proof.* That (2)  $\implies$  (1) is clear. Now suppose (1) holds. Setting  $\psi(t) := -\log t$ ,

$$s_k(\mu) = \int_0^1 x^k d\mu(t) = \int_0^\infty e^{-kt} d\nu(t) = \mathcal{L}\nu(k) \quad \text{for all } k \geq 0,$$

where  $\nu := \psi_*\mu$  is the push-forward of  $\mu$  under  $\psi$ . If  $a = e^{-\lambda}$  for some  $\lambda > 0$ , then, by assumption,

$$F(\mathcal{L}\nu(k)) = e^{-\lambda k} \quad \text{for all } k \geq 1.$$

and, by Carlson's Theorem,

$$F(\mathcal{L}\nu(z)) = e^{-\lambda z} \quad \text{for all } z \in \mathbb{C}^+. \quad (10.2)$$

In view of Bernstein's theorem, Theorem 2.9, the function  $\mathcal{L}\nu$  is completely monotonic on  $[0, \infty)$ . Now, since the composition of an absolutely monotonic function and a completely monotonic function is completely monotonic, so

$$z \mapsto (\mathcal{L}\nu(z))^k = \left( \int_0^\infty e^{-zt} d\nu(t) \right)^k$$

is completely monotonic on  $[0, \infty)$  for all  $k \in \mathbb{Z}_+$ . Thus, by another application of Bernstein's theorem, there exists an admissible measure  $\nu_k$  on  $[0, \infty)$  such that

$$(\mathcal{L}\nu(z))^k = \int_0^\infty e^{-zt} d\nu_k(t) \quad \text{for all } z \in \mathbb{C}^+.$$

Using the above expression, we can rewrite (10.2) as

$$F(\mathcal{L}\nu(z)) = \sum_{n=0}^{\infty} c_n (\mathcal{L}\nu_n)(z) = \mathcal{L} \left( \sum_{n=0}^{\infty} c_n \nu_n \right) (z) = e^{-\lambda z} = (\mathcal{L}\delta_\lambda)(z),$$

and, by the uniqueness principle for Laplace transforms, we conclude that

$$\sum_{n=0}^{\infty} c_n \nu_n = \delta_\lambda.$$

Now, let  $A$  be any measurable subset of  $[0, \infty)$  that does not contain  $\lambda$ . Then,

$$\left( \sum_{n=0}^{\infty} c_n \nu_n \right) (A) = \delta_\lambda(A) = 0.$$

Since  $c_n \geq 0$ , it follows that  $c_n \nu_n(A) = 0$  for all measurable sets  $A$  not containing  $\lambda$ , and all  $n \in \mathbb{Z}_+$ . Hence, either  $c_n = 0$ , or  $\nu_n = \delta_\lambda$ . Moreover,  $\sum_{n=0}^{\infty} c_n = 1$ .

Now, suppose  $c_n \neq 0$  for some  $n$ . By the above argument, we must have  $\nu_n = \delta_\lambda$ . Thus,

$$\mathcal{L}\nu_n(z) = \left( \int_0^\infty e^{-zt} d\nu(t) \right)^n = e^{-\lambda z} \quad \text{for all } z \in \mathbb{C}^+.$$

Equivalently,

$$\int_0^\infty e^{-zt} d\nu(t) = e^{-\lambda z/n},$$

and applying the uniqueness principle for the Laplace transform one more time gives that  $\nu = \delta_{\lambda/n}$ . Hence  $c_n \neq 0$  for at most one  $n$ , say for  $n = N$ , so  $F(x) = x^N$  and  $\nu = \delta_{\lambda/N}$ . Finally, since  $\nu = \psi_*\mu$ , we conclude that  $\mu = \delta_{a^{1/N}}$ , as claimed.  $\square$

## APPENDIX A. TWO LEMMAS ON ADJUGATE MATRICES

In this appendix we prove two lemmas. These allow us to establish Equation (5.2), which is key to our proof of Theorem 5.8, and they may be of independent interest.

Let  $\mathbb{F}$  denote an arbitrary field. Given a matrix  $M \in \mathbb{F}^{N \times N}$ , where  $N \geq 1$ , and a function  $f : \mathbb{F} \rightarrow \mathbb{F}$ , we let  $\text{adj}(M)$  denote the adjugate matrix of  $M$  and  $f[M] \in \mathbb{F}^{N \times N}$  the matrix obtained by applying  $f$  to each entry of  $M$ .

**Lemma A.1.** *Given a polynomial  $f(x) = \alpha_0 + \alpha_1 x + \cdots + \alpha_n x^n + \cdots \in \mathbb{F}[x]$  and a matrix  $M \in \mathbb{F}^{N \times N}$ , the polynomial*

$$\det f[xM] = \alpha_0 \alpha_1^{N-1} \mathbf{1}_{1 \times N} \text{adj}(M) \mathbf{1}_{N \times 1} x^{N-1} + O(x^N).$$

*Proof.* Let  $M$  have columns  $\mathbf{m}_1, \dots, \mathbf{m}_N$ ; we write  $M = (\mathbf{m}_1 | \cdots | \mathbf{m}_N)$  to denote this. Using the multi-linearity of the determinant, we see that

$$\det f[xM] = \sum_{i_1, \dots, i_N=0}^{\infty} \alpha_{i_1} \cdots \alpha_{i_N} x^{i_1 + \cdots + i_N} \det(\mathbf{m}_1^{o_{i_1}} | \cdots | \mathbf{m}_N^{o_{i_N}}). \quad (\text{A.1})$$

Observe that the only way to obtain a term where  $x$  has degree less than  $N - 1$  is for at least two of the indices  $i_l$  to be 0. The corresponding determinants are all 0 since they contain two columns equal to  $\mathbf{1}_{N \times 1}$ .

For terms containing  $x^{N-1}$ , the only ones where the determinant does not contain two columns equal to  $\mathbf{1}_{N \times 1}$  sum to give

$$\alpha_0 \alpha_1^{N-1} x^{N-1} \sum_{l=1}^N \det(\mathbf{m}_1 | \cdots | \mathbf{m}_{l-1} | \mathbf{1}_{N \times 1} | \mathbf{m}_{l+1} | \cdots | \mathbf{m}_N).$$

By Cramer's Rule, this sum is precisely  $\mathbf{1}_{N \times 1}^T \text{adj}(M) \mathbf{1}_{N \times 1}$ .  $\square$

We also require the following result, which we believe to be folklore. We include a proof for completeness.

**Lemma A.2.** *Suppose  $M \in \mathbb{F}^{N \times N}$  has rank  $N - 1$ . If  $\mathbf{u}$  spans the null space of  $M^T$ , and  $\mathbf{v}$  spans the null space of  $M$ , then  $A = \text{adj } M$  is a non-zero scalar multiple of  $\mathbf{v} \mathbf{u}^T$ .*

*Proof.* That  $A \neq 0$  follows by considering the rank of  $M$ . Since  $\det M = 0$ , we have that  $AM = 0$  and  $MA = 0$ . After taking the transpose, the first identity implies that the rows of  $A$  are multiples of  $\mathbf{u}^T$ ; the second identity implies immediately that the columns of  $A$  are multiples of  $\mathbf{v}$ . This gives the result.  $\square$

We may now show that  $\det M_4 = -57168 \alpha_0 \alpha_1^2 \alpha_2 x^4 + O(x^5)$ , where

$$M_4 := \sum_{k=0}^4 \alpha_k x^k M^{\circ k} \quad \text{and} \quad M := \begin{pmatrix} 3 & 6 & 14 & 36 \\ 6 & 14 & 36 & 98 \\ 14 & 36 & 98 & 276 \\ 36 & 98 & 284 & 842 \end{pmatrix}.$$

Note that these matrices are totally non-negative, and would be Hankel but for one entry.

By Lemma A.1,  $\det M_4$  has no constant, linear, or quadratic term. Moreover, since the matrix  $M$  has rank 3 and the vectors

$$\mathbf{v} = (6, -11, 6, -1) \quad \text{and} \quad \mathbf{u} = (46, -59, 18, -1)$$

span the null spaces of  $M$  and  $M^T$ , respectively, Lemma A.2 gives that  $\text{adj}(M)$  is equal to  $c\mathbf{v}\mathbf{u}^T$  for some non-zero  $c \in \mathbb{R}$ . The cubic term in  $\det M_4$  equals  $c\mathbf{1}^T\mathbf{v}\mathbf{u}^T\mathbf{1}\alpha_0\alpha_1^3x^3$ , by Lemma A.1, and this vanishes because  $\mathbf{1}^T\mathbf{v} = 0$ .

Finally, we compute the coefficient of the quartic term; we need to examine all the terms in (A.1) that arise from quadruples  $(i_1, i_2, i_3, i_4)$  which sum to 4. Terms with indices of the form  $(4, 0, 0, 0)$ ,  $(3, 1, 0, 0)$ , and  $(2, 2, 0, 0)$ , and their permutations, are zero since the determinants contain two identical columns. We are therefore left with quadruples of the form  $(2, 1, 1, 0)$  and  $(1, 1, 1, 1)$ . The term corresponding to  $(1, 1, 1, 1)$  is zero since  $\det M = 0$ , as one can see as  $M$  does not have full rank. Thus the only non-zero quartic terms arise from one of the twelve permutations of the quadruple  $(2, 1, 1, 0)$ . Therefore  $\det M_4 = k\alpha_0\alpha_1^2\alpha_2x^4 + O(x^5)$ , and to find the constant  $k$ , we compute all twelve determinants.

$i_1, i_2, i_3, i_4$	$\det(\mathbf{m}_1^{o_{i_1}}   \mathbf{m}_2^{o_{i_2}}   \mathbf{m}_3^{o_{i_3}}   \mathbf{m}_4^{o_{i_4}})$	$i_1, i_2, i_3, i_4$	$\det(\mathbf{m}_1^{o_{i_1}}   \mathbf{m}_2^{o_{i_2}}   \mathbf{m}_3^{o_{i_3}}   \mathbf{m}_4^{o_{i_4}})$
0, 1, 1, 2	1398912	1, 1, 2, 0	-72224
0, 1, 2, 1	-138048	1, 2, 0, 1	-46224
0, 2, 1, 1	-96384	1, 2, 1, 0	21520
1, 0, 1, 2	-2431744	2, 0, 1, 1	14432
1, 0, 2, 1	598304	2, 1, 0, 1	-5208
1, 1, 0, 2	699552	2, 1, 1, 0	-56

The sum of these determinants is  $-57168$ , as claimed.  $\square$

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