CLASSIFYING THE CLOSED IDEALS OF BOUNDED OPERATORS ON TWO FAMILIES OF NON-SEPARABLE CLASSICAL BANACH SPACES

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Abstract. We classify the closed ideals of bounded operators acting on the Banach spaces $\big(\bigoplus_{n\in\mathbb{N}}\ell_2^n\big)_{c_0}\oplus c_0(\Gamma)$ and $\big(\bigoplus_{n\in\mathbb{N}}\ell_2^n\big)_{\ell_1}\oplus \ell_1(\Gamma)$ for every uncountable cardinal Γ.

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1. Introduction

Very few Banach spaces X are known for which the lattice of closed ideals of the Banach algebra $\mathscr{B}(X)$ of all bounded operators on X is fully understood. When X is finite-dimensional, $\mathscr{B}(X)$ is simple, meaning that it contains no non-zero, proper ideals, so we shall henceforth discuss infinite-dimensional Banach spaces only.

Our focus is on the "classical" case, that is, Banach spaces that can be defined by elementary means and/or were known to Banach and his contemporaries. There are currently only two families of Banach spaces of this kind whose lattices of closed operator ideals are fully understood:

(i) Daws [5, Theorem 7.4] has shown that for $X = c_0(\Gamma)$ or $X = \ell_p(\Gamma)$, where Γ is an infinite cardinal and $1 \leqslant p < \infty$, the lattice of closed ideals of $\mathscr{B}(X)$ is

$$\{0\} \subsetneq \mathscr{K}(X) \subsetneq \mathscr{K}_{\aleph_1}(X) \subsetneq \cdots \subsetneq \mathscr{K}_\kappa(X) \subsetneq \mathscr{K}_{\kappa^+}(X) \subsetneq \cdots$$
$$\cdots \subsetneq \mathscr{K}_\Gamma(X) \subsetneq \mathscr{K}_{\Gamma^+}(X) = \mathscr{B}(X)\ . \tag{1.1}$$

Here, $\mathscr{K}_\kappa(X)$ denotes the ideal of κ-compact operators for an uncountable cardinal κ (see page 4 for the precise definition), and κ^+ denotes the cardinal successor of κ. An alternative description of the closed ideals of $\mathscr{B}(X)$ is given in [12, Theorem 1.5]; see also [10, Theorem 3.7].

Daws' theorem generalises and unifies previous results of Calkin [3] for $X = \ell_2$, Gohberg, Markus and Feldman [8] for $X = c_0$ or $X = \ell_p$, $1 \leqslant p < \infty$, and Gramsch [9] and Luft [21] independently for $X = \ell_2(\Gamma)$, where Γ is an arbitrary infinite cardinal.

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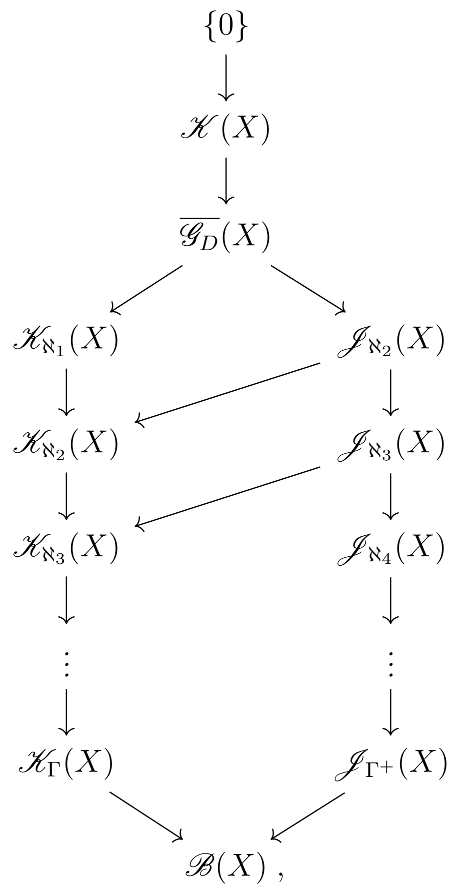
(ii) Let $E = \left(\bigoplus_{n\in\mathbb{N}} \ell_2^n\right)_D$ for $D = c_0$ or $D = \ell_1$. Then it was shown in [19] and [20], respectively, that the lattice of closed ideals of $\mathscr{B}(E)$ is

$$\{0\} \subsetneq \mathscr{K}(E) \subsetneq \overline{\mathscr{G}_D}(E) \subsetneq \mathscr{B}(E)\ , \tag{1.2}$$

where $\overline{\mathscr{G}_D}(E)$ denotes the closure of the ideal of operators on E that factor through the space D.

We shall combine the above results to obtain two new "hybrid" families of Banach spaces, namely $\left(\bigoplus_{n\in\mathbb{N}} \ell_2^n\right)_{c_0} \oplus c_0(\Gamma)$ and its dual space $\left(\bigoplus_{n\in\mathbb{N}} \ell_2^n\right)_{\ell_1} \oplus \ell_1(\Gamma)$, for any uncountable cardinal Γ, whose closed ideals of operators we classify. The precise statement is as follows.

Theorem 1.1. *Let $(D, D_\Gamma) = (c_0, c_0(\Gamma))$ or $(D, D_\Gamma) = (\ell_1, \ell_1(\Gamma))$ for an uncountable cardinal Γ, and set $E = \left(\bigoplus_{n\in\mathbb{N}} \ell_2^n\right)_D$ and $X = E \oplus D_\Gamma$. Then the lattice of closed ideals of $\mathscr{B}(X)$ is*



where

$$\mathscr{J}_\kappa(X) = \left\{ \begin{pmatrix} T_{1,1} & T_{1,2} \\ T_{2,1} & T_{2,2} \end{pmatrix} \in \mathscr{B}(X) : T_{1,1} \in \overline{\mathscr{G}_D}(E),\ T_{1,2} \in \mathscr{B}(D_\Gamma, E), \right.$$
$$\left. T_{2,1} \in \mathscr{B}(E, D_\Gamma),\ T_{2,2} \in \mathscr{K}_\kappa(D_\Gamma) \right\} \tag{1.3}$$

for each cardinal $\aleph_2 \leqslant \kappa \leqslant \Gamma^+$, *where an arrow from an ideal* $\mathscr{I}$ *pointing to an ideal* $\mathscr{J}$ *denotes that* $\mathscr{I} \subsetneq \mathscr{J}$ *and there are no closed ideals of* $\mathscr{B}(X)$ *strictly contained in between* $\mathscr{I}$ *and* $\mathscr{J}$.

Remark 1.2. In addition to the "classical" Banach spaces listed above, there are a number of "exotic", or purpose-built, Banach spaces whose closed ideals of operators can be classified. They occur in two classes:

- The first class contains the famous Banach space of Argyros and Haydon [1] solving the scalar-plus-compact problem, as well as several variants and descendants of it obtained in [26, 22, 16].
- The other class consists of the Banach space $C(K)$ of continuous, scalar-valued functions defined on Koszmider's Mrówka space K, as shown in [15, Theorem 5.5]. Koszmider's original construction of K in [17] assumed the Continuum Hypothesis. A construction within ZFC is given in [18].

A possible explanation for the scarcity of Banach spaces X whose closed ideals of operators have been classified, especially among classical spaces, is that recent research has shown that in many cases $\mathscr{B}(X)$ has $2^{\mathfrak{c}}$ closed ideals, where $\mathfrak{c}$ denotes the cardinality of the continuum. Note that this is the largest possible number of closed ideals for separable X. Spaces for which $\mathscr{B}(X)$ has $2^{\mathfrak{c}}$ closed ideals include $X = L_p[0,1]$ for $p \in (1,\infty) \setminus \{2\}$ (see [14]), $X = \ell_p \oplus \ell_q$ for $1 \leqslant p < q \leqslant \infty$ with $(p,q) \neq (1,\infty)$ and $X = \ell_p \oplus c_0$ for $1 < p < \infty$ (see [6, 7]).

For several other spaces X, it is known that $\mathscr{B}(X)$ contains at least continuum many closed ideals. This includes $X = L_1[0,1]$, $X = C[0,1]$ and $X = L_\infty[0,1]$ (see [13]; note that these results also cover $X = \ell_\infty$ because ℓ_∞ and $L_\infty[0,1]$ are isomorphic as Banach spaces by [23]), as well as the Tsirelson space and the Schreier space of order $n \in \mathbb{N}$ (see [2]). For $X = \ell_1 \oplus c_0$, the best known result is that $\mathscr{B}(X)$ has at least uncountably many closed ideals (see [25]).

2. Preliminaries

Below we explain this paper's most important notation, which is mostly standard. All of our results are valid for both real and complex Banach spaces; we write $\mathbb{K}$ for the scalar field. By an *operator* we always mean a bounded and linear map between normed spaces.

Operator ideals. Following Pietsch [24], an *operator ideal* is an assignment $\mathscr{I}$ which designates to each pair (X,Y) of Banach spaces a subspace $\mathscr{I}(X,Y)$ of the Banach space $\mathscr{B}(X,Y)$ of operators from X to Y such that:

(i) there exists a pair (X,Y) of Banach spaces for which $\mathscr{I}(X,Y) \neq \{0\}$;
(ii) for any quadruple (W,X,Y,Z) of Banach spaces and any operators $S \in \mathscr{B}(W,X)$, $T \in \mathscr{I}(X,Y)$, $U \in \mathscr{B}(Y,Z)$, we have that $UTS \in \mathscr{I}(W,Z)$.

As usual, we write $\mathscr{I}(X)$ to abbreviate $\mathscr{I}(X,X)$. For any operator ideal $\mathscr{I}$, the map $\overline{\mathscr{I}}$ sending a pair of Banach spaces (X,Y) to the norm closure of $\mathscr{I}(X,Y)$ in $\mathscr{B}(X,Y)$ is also an operator ideal. If $\mathscr{I} = \overline{\mathscr{I}}$, then we call $\mathscr{I}$ a *closed operator ideal.*

We shall consider the following three operator ideals:

$\mathscr{K}$, the ideal of *compact operators.*

$\mathscr{K}_\kappa$, the ideal of κ-*compact operators,* defined for any infinite cardinal κ.
The precise definition is as follows. An operator $T \in \mathscr{B}(X,Y)$ is κ-*compact* if, for each $\epsilon > 0$, the closed unit ball B_X of X contains a subset X_ϵ with $|X_\epsilon| < \kappa$ such that

$$\inf\{\|T(x-y)\| : y \in X_\epsilon\} \leqslant \epsilon$$

for every $x \in B_X$. Writing $\mathscr{K}_\kappa(X,Y)$ for the set of κ-compact operators from X to Y, we obtain a closed operator ideal $\mathscr{K}_\kappa$.
Notice that $\mathscr{K}_{\aleph_0}(X,Y) = \mathscr{K}(X,Y)$, so the notion of κ-compactness is indeed a generalisation of compactness.

$\mathscr{G}_D$, the ideal of operators factoring through a certain Banach space D.
Here, we say that an operator $T \in \mathscr{B}(X,Y)$ *factors through* D if there are operators $U\colon X \to D$ and $V\colon D \to Y$ such that $T = VU$, and we write $\mathscr{G}_D(X,Y)$ for the set of operators factoring through D. This defines an operator ideal $\mathscr{G}_D$ provided that D contains a complemented subspace isomorphic to $D \oplus D$, which is true in the cases that we shall consider, namely $D = c_0$ and $D = \ell_p$ for some $p \in [1,\infty)$.

Operators on the direct sum of a pair of Banach spaces. Let X_1 and X_2 be Banach spaces, and endow their direct sum $X_1 \oplus X_2$ with any norm satisfying

$$\max\{\|x_1\|,\|x_2\|\} \leqslant \|(x_1,x_2)\| \leqslant \|x_1\| + \|x_2\| \qquad (x_1 \in X_1,\ x_2 \in X_2)\ .$$

For $i \in \{1,2\}$, we write $Q_i\colon X_1 \oplus X_2 \to X_i$ and $J_i\colon X_i \to X_1 \oplus X_2$ for the i^{th} coordinate projection and embedding, respectively. For $T \in \mathscr{B}(X_1 \oplus X_2)$ and $i,j \in \{1,2\}$, set $T_{i,j} = Q_iTJ_j \in \mathscr{B}(X_j,X_i)$. Then we have

$$T = \sum_{i,j=1}^{2} J_iT_{i,j}Q_j\ . \tag{2.1}$$

It follows that, for any operator ideal $\mathscr{I}$,

$$T \in \mathscr{I}(X_1 \oplus X_2) \qquad\Longleftrightarrow\qquad T_{i,j} \in \mathscr{I}(X_j,X_i) \text{ for } i,j \in \{1,2\}\ . \tag{2.2}$$

We shall identify the operator $T \in \mathscr{B}(X_1 \oplus X_2)$ with the matrix

$$T = \begin{pmatrix} T_{1,1} & T_{1,2} \\ T_{2,1} & T_{2,2} \end{pmatrix}\ .$$

Then the action of T on an element $(x_1,x_2) \in X_1 \oplus X_2$ is given by

$$\begin{pmatrix} T_{1,1} & T_{1,2} \\ T_{2,1} & T_{2,2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} T_{1,1}x_1 + T_{1,2}x_2 \\ T_{2,1}x_1 + T_{2,2}x_2 \end{pmatrix}\ .$$

Composition of operators is given by matrix multiplication, addition is entrywise, and the operator norm satisfies

$$\max_{i,j\in\{1,2\}} \|T_{i,j}\| \leqslant \|T\| \leqslant \sum_{i,j=1}^{2} \|T_{i,j}\|\ . \tag{2.3}$$

For a subset $\mathscr{I}$ of $\mathscr{B}(X_1 \oplus X_2)$ and $i, j \in \{1, 2\}$, we define the $(i, j)^{\text{th}}$ *quadrant* of $\mathscr{I}$ by

$$\mathscr{I}_{i,j} = \{Q_i T J_j : T \in \mathscr{I}\} \subseteq \mathscr{B}(X_j, X_i) \ . \tag{2.4}$$

On the other hand, given subsets $\mathscr{I}_{i,j}$ of $\mathscr{B}(X_j, X_i)$ for $i, j \in \{1, 2\}$, we define

$$\begin{pmatrix} \mathscr{I}_{1,1} & \mathscr{I}_{1,2} \\ \mathscr{I}_{2,1} & \mathscr{I}_{2,2} \end{pmatrix} = \left\{ \begin{pmatrix} T_{1,1} & T_{1,2} \\ T_{2,1} & T_{2,2} \end{pmatrix} : T_{i,j} \in \mathscr{I}_{i,j} \ (i, j \in \{1, 2\}) \right\} \subseteq \mathscr{B}(X_1 \oplus X_2) \ . \tag{2.5}$$

The first part of the following lemma can be seen as a generalisation of (2.2) to the case where the ideal $\mathscr{I}$ does not come from an operator ideal. It says that if we decompose $\mathscr{I}$ into its quadrants according to (2.4) and then reassemble the quadrants according to (2.5), we obtain $\mathscr{I}$.

Lemma 2.1. *Let $\mathscr{I}$ be an ideal of $\mathscr{B}(X_1 \oplus X_2)$ for some Banach spaces X_1 and X_2. Then*

$$\mathscr{I} = \begin{pmatrix} \mathscr{I}_{1,1} & \mathscr{I}_{1,2} \\ \mathscr{I}_{2,1} & \mathscr{I}_{2,2} \end{pmatrix}, \tag{2.6}$$

and $\mathscr{I}_{i,i}$ is an ideal of $\mathscr{B}(X_i)$ for $i \in \{1, 2\}$. Moreover, $\mathscr{I}_{i,j}$ is closed in $\mathscr{B}(X_j, X_i)$ for each $i, j \in \{1, 2\}$ if and only if $\mathscr{I}$ is closed in $\mathscr{B}(X_1 \oplus X_2)$.

Proof. The inclusion $\subseteq$ in (2.6) holds true by the definitions.

Conversely, suppose that $T = (T_{i,j})_{i,j=1}^2$ with $T_{i,j} \in \mathscr{I}_{i,j}$ for each $i, j \in \{1, 2\}$, say $T_{i,j} = Q_i S^{i,j} J_j$, where $S^{i,j} \in \mathscr{I}$. Then, by (2.1), we have

$$T = \sum_{i,j=1}^{2} J_i T_{i,j} Q_j = \sum_{i,j=1}^{2} (J_i Q_i) S^{i,j} (J_j Q_j) \in \mathscr{I}$$

because $\mathscr{I}$ is an ideal of $\mathscr{B}(X_1 \oplus X_2)$ and $J_k Q_k \in \mathscr{B}(X_1 \oplus X_2)$ for $k \in \{1, 2\}$.

Next, we verify that $\mathscr{I}_{i,i}$ is an ideal of $\mathscr{B}(X_i)$ for $i \in \{1, 2\}$. It is clear that $\mathscr{I}_{i,i}$ is a subspace. Suppose that $S \in \mathscr{I}_{i,i}$ and $T \in \mathscr{B}(X_i)$, say $S = U_{i,i}$, where $U \in \mathscr{I}$. Then $U J_i T Q_i \in \mathscr{I}$ because $\mathscr{I}$ is an ideal of $\mathscr{B}(X_1 \oplus X_2)$ and $J_i T Q_i \in \mathscr{B}(X_1 \oplus X_2)$, and hence

$$\mathscr{I}_{i,i} \ni (U J_i T Q_i)_{i,i} = Q_i U J_i T Q_i J_i = ST \ .$$

The proof that $TS \in \mathscr{I}_{i,i}$ is similar.

The final clause follows easily from (2.3) and (2.6). $\square$

Long sequence spaces. For a non-empty set Γ and $p \in [1, \infty)$, we consider the Banach spaces

$$c_0(\Gamma) = \big\{ x \in \mathbb{K}^\Gamma : \{\gamma \in \Gamma : |x(\gamma)| > \epsilon\} \text{ is finite for every } \epsilon > 0 \big\}$$

and

$$\ell_p(\Gamma) = \left\{ x \in \mathbb{K}^\Gamma : \sum_{\gamma \in \Gamma} |x(\gamma)|^p < \infty \right\} .$$

The norm of $c_0(\Gamma)$ is the supremum norm, and the norm of $\ell_p(\Gamma)$ is $\|x\| = \big(\sum_{\gamma \in \Gamma} |x(\gamma)|^p\big)^{\frac{1}{p}}$. As usual, we write c_0 and ℓ_p instead of $c_0(\mathbb{N})$ and $\ell_p(\mathbb{N})$, respectively. For notational convenience, we use the convention that D_Γ will denote either $c_0(\Gamma)$ or $\ell_p(\Gamma)$ for some $p \in [1,\infty)$, unless otherwise specified. Only the cardinality of the index set Γ matters, in the sense that D_Γ is isometrically isomorphic to $D_{|\Gamma|}$, and D_Γ is not isomorphic to D_Δ for any index set Δ of cardinality other than $|\Gamma|$.

The *support* of an element $x \in D_\Gamma$ is $\operatorname{supp} x = \{\gamma \in \Gamma : x(\gamma) \neq 0\}$, which is always a countable set.

For $\gamma \in \Gamma$, $e_\gamma \in D_\Gamma$ denotes the element given by $e_\gamma(\beta) = 1$ if $\beta = \gamma$ and $e_\gamma(\beta) = 0$ otherwise. The (transfinite) sequence $(e_\gamma)_{\gamma\in\Gamma}$ is the *(long) unit vector basis* for D_Γ. The only facts about it that we shall use are that $(e_\gamma)_{\gamma\in\Gamma}$ spans a dense subspace of D_Γ, and that the existence of such a basis implies that D_Γ has the approximation property.

For a subset Δ of Γ, $P_\Delta \in \mathscr{B}(D_\Gamma)$ denotes the *basis projection* given by $(P_\Delta x)(\gamma) = x(\gamma)$ if $\gamma \in \Delta$ and $(P_\Delta x)(\gamma) = 0$ otherwise, for every $x \in D_\Gamma$.

3. The proof of Theorem 1.1

To aid the presentation, we split the proof of Theorem 1.1 into a series of lemmas, some of which may essentially be known. However, to keep the the presentation as self-contained as possible, we include full proofs, except for references to the ideal classifications [5, 19] that we will ultimately need anyway. The proof of Theorem 1.1 itself requires results about the transfinite sequence spaces $c_0(\Gamma)$ and $\ell_1(\Gamma)$ only, not $\ell_p(\Gamma)$ for $1 < p < \infty$. However, our first few results hold true also for the latter spaces and with identical proofs, so we give these more general results.

Lemma 3.1. *Let $D_\Gamma = c_0(\Gamma)$ or $D_\Gamma = \ell_p(\Gamma)$ for some $p \in [1,\infty)$ and some set $\Gamma \neq \emptyset$.*

(i) *Every separable subspace of D_Γ is contained in the image of the basis projection P_Δ for some countable subset Δ of Γ.*

(ii) *Suppose that $D_\Gamma \neq \ell_1(\Gamma)$. Then, for every Banach space E and every operator $T\colon D_\Gamma \to E$ for which there exists an injective operator from the image of T into ℓ_∞, there is a countable subset Δ of Γ such that $T = TP_\Delta$.*

Proof. (i). Every separable subspace E of D_Γ has the form $E = \overline{W}$ for some countable subset W of D_Γ. Define $\Delta = \bigcup_{w\in W} \operatorname{supp} w$, which is a countable union of countable sets and is thus countable. The continuity of the projection P_Δ implies that $x = P_\Delta x$ for every $x \in E$. Hence the image of P_Δ contains E.

(ii). Let $U\colon T(D_\Gamma) \to \ell_\infty$ be an injective operator. Assume towards a contradiction that the set

$$\Delta_{k,m} = \Big\{\gamma \in \Gamma : |UTe_\gamma(m)| \geqslant \frac{1}{k}\Big\}$$

is infinite for some $k, m \in \mathbb{N}$, so that it contains an infinite sequence $(\gamma_n)_{n\in\mathbb{N}}$ of distinct elements. For each $n \in \mathbb{N}$, take a scalar σ_n of modulus one such that $\sigma_n \cdot (UTe_{\gamma_n})(m) \geqslant 1/k$. Since $D_\Gamma \neq \ell_1(\Gamma)$, we have $x = \sum_{n\in\mathbb{N}} \frac{\sigma_n}{n} e_{\gamma_n} \in D_\Gamma$, but

$$(UTx)(m) = \sum_{n\in\mathbb{N}} \frac{\sigma_n}{n}(UTe_{\gamma_n})(m) \geqslant \frac{1}{k}\sum_{n\in\mathbb{N}} \frac{1}{n} = \infty\ ,$$

a contradiction. Hence $\Delta_{k,m}$ is finite for each $k, m \in \mathbb{N}$, so the union $\Delta = \bigcup_{k,m\in\mathbb{N}} \Delta_{k,m}$ is countable. For each $\gamma \in \Gamma \setminus \Delta$, we have $UTe_\gamma = 0$, so $Te_\gamma = 0$ by the injectivity of U, and therefore $TP_\Delta = T$. □

Remark 3.2. (i) The case $D_\Gamma = \ell_1(\Gamma)$ must be excluded in Lemma 3.1(ii) because, for every non-zero Banach space E, there is an operator $T\colon \ell_1(\Gamma) \to E$ which has one-dimensional image and satisfies $Te_\gamma \neq 0$ for every $\gamma \in \Gamma$, namely the summation operator T given by $Tx = \sum_{\gamma\in\Gamma} x(\gamma)\, y$ for every $x \in \ell_1(\Gamma)$, where y is any fixed non-zero element of E.

(ii) We refer to [11] for a detailed discussion of the condition in Lemma 3.1(ii) that the image of T admits an injective operator into ℓ_∞.

Corollary 3.3. *Let $(D, D_\Gamma) = (c_0, c_0(\Gamma))$ or $(D, D_\Gamma) = (\ell_p, \ell_p(\Gamma))$ for some $p \in [1,\infty)$ and some uncountable set Γ, and let E be any separable Banach space. Then*

$$\mathscr{B}(D_\Gamma, E) = \mathscr{G}_D(D_\Gamma, E) \qquad \text{and} \qquad \mathscr{B}(E, D_\Gamma) = \mathscr{G}_D(E, D_\Gamma)\ .$$

Proof. The first identity for $D_\Gamma \neq \ell_1(\Gamma)$, and the second identity in full generality, both follow easily from Lemma 3.1 because the image of the projection P_Δ for Δ countable is either finite-dimensional or isomorphic to D, and E, being separable, embeds isometrically into ℓ_∞.

It remains to show that every operator $T\colon \ell_1(\Gamma) \to E$ factors through ℓ_1. We use the lifting property of ℓ_1 (see for instance [4, Theorem 5.1]) to verify this. Indeed, since E is separable, we can take a surjective operator $S\colon \ell_1 \to E$. By the open mapping theorem, there is a constant $c > 0$ such that, for every $y \in E$, there is $x \in \ell_1$ with $Sx = y$ and $\|x\| \leqslant c\|y\|$. Hence, for each $\gamma \in \Gamma$, we can find $x_\gamma \in \ell_1$ such that $Sx_\gamma = Te_\gamma$ and $\|x_\gamma\| \leqslant c\|Te_\gamma\| \leqslant c\|T\|$. It follows that we can define an operator $R\colon \ell_1(\Gamma) \to \ell_1$ by $Re_\gamma = x_\gamma$ for each $\gamma \in \Gamma$, and clearly $T = SR$. □

Lemma 3.4. *Let $D = c_0$ or $D = \ell_p$ for some $p \in [1,\infty)$, and let $(E_n)_{n\in\mathbb{N}}$ be a sequence of non-zero Banach spaces. Then there are operators $R\colon D \to \big(\bigoplus_{n\in\mathbb{N}} E_n\big)_D$ and $S\colon \big(\bigoplus_{n\in\mathbb{N}} E_n\big)_D \to D$ such that $SR = I_D$.*

Proof. For every $n \in \mathbb{N}$, choose $y_n \in E_n$ and $f_n \in E_n^*$ with $\|y_n\| = \|f_n\| = \langle y_n, f_n\rangle = 1$, and define $R\colon (\lambda_n) \mapsto (\lambda_n y_n)$ for $(\lambda_n) \in D$ and $S\colon (x_n) \mapsto (\langle x_n, f_n\rangle)$ for $(x_n) \in \big(\bigoplus_{n\in\mathbb{N}} E_n\big)_D$. □

Lemma 3.5. *Let $D = c_0$ or $D = \ell_1$. Then D contains a subspace which is isomorphic to $\big(\bigoplus_{n\in\mathbb{N}} \ell_2^n\big)_D$.*

Proof. This follows by combining the fact that D contains almost isometric copies of ℓ_2^n for every $n \in \mathbb{N}$ with the fact that D is isomorphic to the D-direct sum of countably many copies of itself. □

Lemma 3.6. *Let $(D, D_\Gamma) = (c_0, c_0(\Gamma))$ or $(D, D_\Gamma) = (\ell_1, \ell_1(\Gamma))$ for some infinite set Γ, and set $E = \big(\bigoplus_{n\in\mathbb{N}} \ell_2^n\big)_D$. Then the identity operator on D factors through every non-compact operator belonging to either $\mathscr{B}(E)$, $\mathscr{B}(D_\Gamma)$, $\mathscr{B}(E, D_\Gamma)$ or $\mathscr{B}(D_\Gamma, E)$.*

Proof. Let T be a non-compact operator. We examine each of the four cases separately:

(i) If $T \in \mathscr{B}(E)$, then I_D factors through T by [19, Corollary 3.8 and Example 3.9].

(ii) For $T \in \mathscr{B}(D_\Gamma)$, a careful examination of the proofs of [5, Proposition 4.3 and Theorems 6.2 and 7.3] shows that there are operators $R, S \in \mathscr{B}(D_\Gamma)$ such that $STR = P_\Delta$ for some infinite subset Δ of Γ. Choose an infinite sequence (γ_n) of distinct elements in Δ, and define operators $U\colon D \to D_\Gamma$ and $V\colon D_\Gamma \to D$ by $U(e_n) = e_{\gamma_n}$ and $V(e_{\gamma_n}) = e_n$ for each $n \in \mathbb{N}$, and $V(e_\gamma) = 0$ for $\gamma \in \Gamma \setminus \{\gamma_n : n \in \mathbb{N}\}$. Then we have $VSTRU = I_D$.

(iii) For $T \in \mathscr{B}(E, D_\Gamma)$, Lemma 3.1(i) implies that $T = P_\Delta T$ for some countable subset Δ of Γ. Note that Δ is infinite, as otherwise $P_\Delta T$ would be compact. Enumerate Δ as $\{\gamma_n : n \in \mathbb{N}\}$. Then, defining the operators $U\colon D \to D_\Gamma$ and $V\colon D_\Gamma \to D$ as in case (ii), we have $UV = P_\Delta$. Choose operators $R\colon D \to E$ and $S\colon E \to D$ as in Lemma 3.4, and observe that RVT is non-compact, as otherwise $(US)(RVT) = P_\Delta T = T$ would be compact. Now the conclusion follows by applying case (i) to the operator $RVT \in \mathscr{B}(E)$.

(iv) Finally, suppose that $T \in \mathscr{B}(D_\Gamma, E)$. By Lemma 3.5 and the fact that Γ is infinite, we can find an isomorphic embedding $U \in \mathscr{B}(E, D_\Gamma)$. Then UT is non-compact, and the conclusion follows by applying case (ii) to the operator $UT \in \mathscr{B}(D_\Gamma)$. □

Remark 3.7. Lemma 3.6 is also true for $(D, D_\Gamma) = (\ell_p, \ell_p(\Gamma))$ when $1 < p < \infty$. However, $E = \big(\bigoplus \ell_2^n\big)_{\ell_p}$ is isomorphic to ℓ_p in these cases, so only case (ii) above would be non-trivial.

Corollary 3.8. *Let* $X = \big(\bigoplus_{n\in\mathbb{N}} \ell_2^n\big)_D \oplus D_\Gamma$, *where* $(D, D_\Gamma) = (c_0, c_0(\Gamma))$ *or* $(D, D_\Gamma) = (\ell_1, \ell_1(\Gamma))$ *for some infinite set* Γ, *and let* $\mathscr{I}$ *be an ideal of* $\mathscr{B}(X)$. *Then either* $\mathscr{I} \subseteq \mathscr{K}(X)$ *or* $\mathscr{G}_D(X) \subseteq \mathscr{I}$.

Proof. For notational convenience, write $X = X_1 \oplus X_2$, where $X_1 = \big(\bigoplus_{n\in\mathbb{N}} \ell_2^n\big)_D$ and $X_2 = D_\Gamma$. Suppose that $\mathscr{I} \nsubseteq \mathscr{K}(X)$, and choose $T \in \mathscr{I} \setminus \mathscr{K}(X)$. Then (2.2) shows that $T_{i,j} \notin \mathscr{K}(X_j, X_i)$ for some $i, j \in \{1, 2\}$. Lemma 3.6 implies that there are operators $U\colon D \to X_j$ and $V\colon X_i \to D$ such that $VT_{i,j}U = I_D$. Hence, for each $S = R_2R_1 \in \mathscr{G}_D(X)$, where $R_1 \in \mathscr{B}(X, D)$ and $R_2 \in \mathscr{B}(D, X)$, we have

$$S = R_2VT_{i,j}UR_1 = (R_2VQ_i)T(J_jUR_1) \in \mathscr{I}$$

because $\mathscr{I}$ is an ideal of $\mathscr{B}(X)$. This shows that $\mathscr{G}_D(X) \subseteq \mathscr{I}$, as desired. □

For a Banach space X, define

$$\Xi(X) = \{\mathscr{I} : \mathscr{I} \text{ is a closed ideal of } \mathscr{B}(X) \text{ and } \mathscr{I} \supsetneq \mathscr{K}(X)\}\,,$$

and order $\Xi(X)$ by inclusion. For a pair of Banach spaces X_1 and X_2, we endow the set $\Xi(X_1) \times \Xi(X_2)$ with the product order; that is,

$$(\mathscr{I}_1, \mathscr{I}_2) \leqslant (\mathscr{J}_1, \mathscr{J}_2) \iff [\mathscr{I}_1 \subseteq \mathscr{J}_1] \wedge [\mathscr{I}_2 \subseteq \mathscr{J}_2]\,.$$

Proposition 3.9. *Let* $X = E \oplus D_\Gamma$, *where* $E = \big(\bigoplus_{n\in\mathbb{N}} \ell_2^n\big)_D$ *and either* $(D, D_\Gamma) = (c_0, c_0(\Gamma))$ *or* $(D, D_\Gamma) = (\ell_1, \ell_1(\Gamma))$ *for some infinite set* Γ. *The map*

$$\xi \,:\, \Xi(E) \times \Xi(D_\Gamma) \to \Xi(X)\ ,\ (\mathscr{I}, \mathscr{J}) \mapsto \begin{pmatrix} \mathscr{I} & \mathscr{B}(D_\Gamma, E) \\ \mathscr{B}(E, D_\Gamma) & \mathscr{J} \end{pmatrix}\ ,$$

is an order isomorphism.

Proof. Recall from Corollary 3.3 that $\mathscr{B}(E, D_\Gamma) = \mathscr{G}_D(E, D_\Gamma)$ and $\mathscr{B}(D_\Gamma, E) = \mathscr{G}_D(D_\Gamma, E)$, and that $\mathscr{G}_D(E) \subseteq \mathscr{I}$ and $\mathscr{G}_D(D_\Gamma) \subseteq \mathscr{J}$ for every $(\mathscr{I}, \mathscr{J}) \in \Xi(E) \times \Xi(D_\Gamma)$ by the ideal classifications (1.2) and (1.1), respectively. Using these facts, one can easily verify that $\xi(\mathscr{I}, \mathscr{J})$ is an ideal of $\mathscr{B}(X)$ with $\mathscr{K}(X) \subsetneq \xi(\mathscr{I}, \mathscr{J})$. Moreover, $\xi(\mathscr{I}, \mathscr{J})$ is closed by Lemma 2.1, so it belongs to $\Xi(X)$.

To see that ξ is surjective, let $\mathscr{L} \in \Xi(X)$. Lemma 2.1 shows that

$$\mathscr{L} = \begin{pmatrix} \mathscr{L}_{1,1} & \mathscr{L}_{1,2} \\ \mathscr{L}_{2,1} & \mathscr{L}_{2,2} \end{pmatrix},$$

where $\mathscr{L}_{1,1}$ and $\mathscr{L}_{2,2}$ are closed ideals of $\mathscr{B}(E)$ and $\mathscr{B}(D_\Gamma)$, respectively. Moreover, Corollary 3.8 implies that $\mathscr{G}_D(X) \subseteq \mathscr{L}$, so by (2.2), we have:

- $\mathscr{L}_{1,1} \supseteq \mathscr{G}_D(E)$, so $\mathscr{L}_{1,1} \in \Xi(E)$;
- $\mathscr{L}_{1,2} \supseteq \mathscr{G}_D(D_\Gamma, E) = \mathscr{B}(D_\Gamma, E)$, so $\mathscr{L}_{1,2} = \mathscr{B}(D_\Gamma, E)$, and similarly $\mathscr{L}_{2,1} = \mathscr{B}(E, D_\Gamma)$;
- $\mathscr{L}_{2,2} \supseteq \mathscr{G}_D(D_\Gamma)$, so $\mathscr{L}_{2,2} \in \Xi(D_\Gamma)$.

This verifies that $\mathscr{L} = \xi(\mathscr{L}_{1,1}, \mathscr{L}_{2,2})$.

Finally, working straight from the definitions, we see that $(\mathscr{I}_1, \mathscr{J}_1) \leqslant (\mathscr{I}_2, \mathscr{J}_2)$ if and only if $\xi(\mathscr{I}_1, \mathscr{J}_1) \subseteq \xi(\mathscr{I}_2, \mathscr{J}_2)$ for $(\mathscr{I}_1, \mathscr{J}_1), (\mathscr{I}_2, \mathscr{J}_2) \in \Xi(E) \times \Xi(D_\Gamma)$. This shows first that ξ is injective and thus a bijection, and secondly that both ξ and its inverse are order-preserving. □

We can now prove Theorem 1.1 easily.

Proof of Theorem 1.1. Both E and D_Γ have the approximation property, so the same is true for their direct sum X. Therefore $\mathscr{K}(X)$ is the smallest non-zero closed ideal of $\mathscr{B}(X)$. Proposition 3.9 shows that any other non-zero closed ideal $\mathscr{L}$ of $\mathscr{B}(X)$ has the form $\mathscr{L} = \xi(\mathscr{I}, \mathscr{J})$ for unique closed ideals $\mathscr{I} \in \Xi(E)$ and $\mathscr{J} \in \Xi(D_\Gamma)$. By the ideal classifications (1.2) and (1.1), either $\mathscr{I} = \overline{\mathscr{G}_D}(E)$ or $\mathscr{I} = \mathscr{B}(E)$, while $\mathscr{J} = \mathscr{K}_\kappa(D_\Gamma)$ for a unique cardinal $\aleph_1 \leqslant \kappa \leqslant \Gamma^+$.

Suppose first that $\mathscr{I} = \overline{\mathscr{G}_D}(E)$. If $\kappa = \aleph_1$, then $\mathscr{J} = \overline{\mathscr{G}_D}(D_\Gamma)$, so $\mathscr{L} = \overline{\mathscr{G}_D}(X)$. Otherwise $\kappa \geqslant \aleph_2$ and $\mathscr{L} = \mathscr{J}_\kappa(X)$ in the notation of (1.3).

Next, suppose that $\mathscr{I} = \mathscr{B}(E)$, which is equal to $\mathscr{K}_\kappa(E)$ because E has density $\aleph_0 < \kappa$. Hence we have $\mathscr{L} = \mathscr{K}_\kappa(X)$. (Note that this is equal to $\mathscr{B}(X)$ if $\kappa = \Gamma^+$.) □

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