

UNIFORM HYPERFINITENESS

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ABSTRACT. Almost forty years ago, Connes, Feldman and Weiss proved that for measurable equivalence relations the notions of amenability and hyperfiniteness coincide. In this paper we define the uniform version of amenability and hyperfiniteness for measurable graphed equivalence relations of bounded vertex degrees and prove that these two notions coincide as well. Roughly speaking, a measured graph \mathcal{G} is uniformly hyperfinite if for any $\varepsilon > 0$ there exists $K \geq 1$ such that not only \mathcal{G} , but all of its subgraphs of positive measure are (ε, K) -hyperfiniteness. We also show that this condition is equivalent to weighted hyperfiniteness and a strong version of fractional hyperfiniteness, a notion recently introduced by Lovász. As a corollary, we obtain a characterization of exactness of finitely generated groups via uniform hyperfiniteness.

Keywords. uniform hyperfiniteness, uniform amenability, exact groups

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CONTENTS

1. Introduction	3
2. Further motivation and examples	6
3. Uniform amenability implies uniform local hyperfiniteness	9
4. Uniform local hyperfiniteness implies uniform hyperfiniteness	11
5. Uniform hyperfiniteness implies weighted hyperfiniteness	12
6. Weighted hyperfiniteness implies approximate strong hyperfiniteness	14
7. Fractional partitions	16
8. Approximate strong hyperfiniteness implies strong fractional hyperfiniteness	17
9. Strong fractional hyperfiniteness implies uniform amenability	18
10. Free actions	19
References	20

1. INTRODUCTION

First, let us recall the notion of amenability and hyperfiniteness in the context of Borel/measurable/continuous combinatorics.

Let X be a standard Borel space. A **Borel graph** $\mathcal{G} \subset X \times X$ is a Borel set such that

- for any $x \in X$, $(x, x) \notin \mathcal{G}$,
- if $(x, y) \in \mathcal{G}$, then $(y, x) \in \mathcal{G}$ as well, that is, \mathcal{G} is indeed a graph. (see [6] for details)

In this paper we always assume that the degrees of a Borel graph is countable. The components of a Borel graph \mathcal{G} are called **orbits**. The shortest path metric on the orbits will be denoted by $d_{\mathcal{G}}$. Now, let Γ be a countable group with symmetric generating system Σ and let $\alpha : \Gamma \curvearrowright X$ be a Borel action. An associated Borel graph $\alpha_{\mathcal{G}}^{\Gamma, \Sigma}$ is defined in the following way. We have $(x, y) \in \alpha_{\mathcal{G}}^{\Gamma, \Sigma}$ for $x \neq y$ if and only if there is a generator $\sigma \in \Sigma$ such that $y \in \alpha(\sigma)(x)$. By the Kechris-Solecki-Todorcevic Theorem [7], for any Borel graph \mathcal{G} , there exists (Γ, Σ) and an action $\alpha : \Gamma \curvearrowright X$ such that $\alpha_{\mathcal{G}}^{\Gamma, \Sigma} = \mathcal{G}$. Also, if \mathcal{G} is of bounded vertex degrees, then one can assume that Γ is finitely generated and Σ is a finite generating system.

A **Borel equivalence relation** $E \subset X \times X$ is called countable resp. finite, if the equivalence classes are countable resp. finite. If \mathcal{G} is a Borel graph, then the associated Borel equivalence relation $E_{\mathcal{G}}$ is defined in the following way. We have $x \equiv_{E_{\mathcal{G}}} y$ if x and y are vertices of the same orbit.

We call the Borel equivalence relation E **hyperfinit**e if there exist finite Borel equivalence relations $E_1 \subset E_2 \subset \dots$ such that $\cup_{n=1}^{\infty} E_n = E$.

We call the Borel equivalence relation E **amenable** if there exist Borel functions (the Reiter functions) $p_n : E \rightarrow [0, 1]$ such that

- for any $x \in X$ and $n \geq 1$, $\sum_{z, z \equiv_E x} p_n(x, z) = 1$,
- for any pair $x \equiv_E y$,

$$\lim_{n \rightarrow \infty} \sum_{z, z \equiv_E x} |p_n(x, z) - p_n(y, z)| = 0.$$

It is not hard to see that hyperfiniteness implies amenability. However, the converse statement is one of the classical conjectures in Borel combinatorics.

Conjecture 1.1 ([6]). *Every countable amenable Borel equivalence relation is hyperfinite.*

Now let us turn to the measurable case (for this part, see also [6] for further details). Let (\mathcal{G}, X) be a Borel graph and μ be a Borel probability measure on X . We say that μ is an **invariant measure** on \mathcal{G} if there exists a group

action $\alpha : \Gamma \curvearrowright X$, $\alpha_G^{\Gamma, \Sigma} = \mathcal{G}$ such that α preserves the measure μ . Note that if $\beta : \Gamma' \curvearrowright X$, $\beta_G^{\Gamma', \Sigma'} = \mathcal{G}$ is another action, then β preserves the measure as well. Also, if (\mathcal{H}, X) is another Borel graph such that $E_{\mathcal{H}} = E_{\mathcal{G}}$, then μ is invariant measure with respect to \mathcal{H} as well.

Similarly, μ is called a **quasi-invariant** measure on \mathcal{G} if the above action α preserves only the measure-class of μ , that is, if $\mu(A) = 0$ for some Borel set $A \subset X$, then $\mu(\alpha(\gamma)(A)) = 0$ for every $\gamma \in \Gamma$.

Now, let (\mathcal{G}, X, μ) be a **measured graph**, that is, a Borel graph with a quasi-invariant measure. Then, we call (\mathcal{G}, X, μ) μ -amenable resp. μ -hyperfinite, if there exists a Borel set $Y \subseteq X$ such that

- Y is a union of equivalence classes of $E_{\mathcal{G}}$ (that is, Y is an **invariant subset**),
- $\mu(Y) = 1$ (that is Y has **full measure**),
- the induced graph on the set Y , \mathcal{G}_Y is amenable resp. hyperfinite.

Then we have the celebrated theorem of Connes, Feldman and Weiss [2].

Theorem 1. *A measured graph (\mathcal{G}, X, μ) is μ -amenable if and only if it is μ -hyperfinite.*

Finally, let (\mathcal{G}, X, μ) be a bounded degree Borel graph with a quasi-invariant measure. For $\varepsilon > 0$ and $K \geq 1$, we call the measured graph (\mathcal{G}, X, μ) (ε, K) -hyperfinite if there exists some Borel subset $T \subset X$, $\mu(T) < \varepsilon$ such that all the components of the induced graph $\mathcal{G}_{X \setminus T}$ are of size at most K . Then, (\mathcal{G}, X, μ) is μ -hyperfinite if for all $\varepsilon > 0$ there exists $K \geq 1$ such that (\mathcal{G}, X, μ) is (ε, K) -hyperfinite.

Before introducing our new notion, let us recall the definition of the **Radon-Nikodym cocycle**. Let (\mathcal{G}, X, μ) be a measured graph with a quasi-invariant measure and $\alpha : \Gamma \curvearrowright X$ be a Borel action such that $\alpha_G^{\Gamma, \Sigma} = \mathcal{G}$. Then, for any $\gamma \in \Gamma$ we have a Borel function $R_\gamma : X \rightarrow (0, \infty)$, the Radon-Nikodym derivative, which is unique up to zero-measure perturbation, such that for any Borel set $A \subset X$

- $\mu(\alpha(\gamma)(A)) = \int_A R_\gamma(x) d\mu(x)$,
- for any $\gamma, \delta \in \Gamma$ and $x \in X$

$$R_{\gamma\delta}(x) = R_\gamma(\alpha(\delta))(x)R_\delta(x).$$

Hence, we have a Borel function $R : E \rightarrow (0, \infty)$, the Radon-Nikodym cocycle.

Definition 1.1. If for any $\gamma \in \Gamma$ the function R_γ is bounded and \mathcal{G} is of bounded vertex degrees, then we call (\mathcal{G}, X, μ) a measured graph of **bounded type**.

Note that if μ is an invariant measure, then all the R_γ 's can be chosen as constant 1, hence bounded degree graphs with invariant measures are always

of bounded type. By the inequality in Section 3.2 of [8], for any random walk of a finitely generated group with finitely supported transition measure induces a bounded type structure on the Furstenberg boundary. It is important to note that if $\vec{e} = (x, y)$ is an oriented edge of \mathcal{G} , we have a well-defined Radon-Nikodym derivative $R_{\vec{e}}$ corresponding to the edge. If the measured graph is of bounded type, then the function $R_{\vec{e}}$ is bounded.

Before presenting our main definitions, let us recall the notion of topological amenability for free continuous actions. Let Γ be a finitely generated group with symmetric generating set Σ and let $\alpha : \Gamma \curvearrowright K$ a free continuous action of Γ on a compact Hausdorff space K . Following [1], we call α **topologically amenable** if for any $n \geq 1$, there exists $R_n \geq 1$ and a continuous function $p_n : K \rightarrow \text{Prob}(K)$ such that

- for any $x \in K$, $\text{Supp}(p_n(x)) \subset B_{R_n}(x, \alpha_G^{\Gamma, \Sigma})$ (the ball of radius R_n centered at x),
- for all $\alpha_G^{\Gamma, \Sigma}$ -adjacent pairs $x, y \in K$ we have

$$\|p_n(x) - p_n(y)\|_1 \leq \frac{1}{n}.$$

Clearly, the equivalence relation associated to a topologically amenable action is amenable. Hence, for any quasi-invariant measure μ , the measured graph $(\alpha_G^{\Gamma, \Sigma}, K, \mu)$ is μ -hyperfinite. By [1], we have the converse if a the equivalence relation associated to a free continuous action α of a finitely generated group is μ -hyperfinite for all quasi-invariant measure μ , then α must be topologically amenable.

Let (\mathcal{G}, X, μ) and (\mathcal{H}, Y, ν) be measured graphs of bounded vertex degrees and let $\Phi : X \rightarrow Y$ be a measure preserving map preserving almost all the orbits. Also, let us assume that there exists a constant $L > 1$ such that for μ -almost all x and every $y \equiv_{E_{\mathcal{G}}} x$

$$(1) \quad \frac{1}{L} d_{\mathcal{G}}(x, y) < d_{\mathcal{H}}(\Phi(x), \Phi(y)) < L d_{\mathcal{G}}(x, y).$$

Then, we say that \mathcal{G} and \mathcal{H} are **coarsely equivalent**. Note that coarse equivalence is much stronger than orbit equivalence and \mathcal{G} is of bounded type if and only if \mathcal{H} is of bounded type. In our paper we introduce a strengthening of the notion of μ -hyperfiniteness for measured graphs of bounded type.

Let (\mathcal{G}, X, μ) be a measured graph of bounded type. Let $A \subset X$ be a Borel subset of positive measure. Then, one can consider the measured graph $(\mathcal{G}_A, A, \mu_A)$ induced on A , where for any Borel set $B \subset A$,

$$\mu_A(B) = \frac{\mu(B)}{\mu(A)}.$$

Now we present the key definitions of our paper.

Definition 1.2. Let (\mathcal{G}, X, μ) be a measured graph of bounded vertex degrees, $\varepsilon > 0$ and $K \geq 1$. Then, (\mathcal{G}, X, μ) is (ε, K) -uniformly hyperfinite if for all subset $A \subset X$ of positive measure $(\mathcal{G}_A, A, \mu_A)$ is (ε, K) -hyperfinite.

We call the measured graph (\mathcal{G}, X, μ) **μ -uniformly hyperfinite** if for any $\varepsilon > 0$ there exists $K \geq 1$ such that (\mathcal{G}, X, μ) is (ε, K) -uniformly hyperfinite.

Definition 1.3. Let (\mathcal{G}, X, μ) be a measured graph of bounded vertex degrees, $\varepsilon > 0$ and $R \geq 1$. Then, (\mathcal{G}, X, μ) is (ε, R) -uniformly amenable, if there exists an invariant set $Y \subset X$ of full measure, a Borel function $p : Y \rightarrow \text{Prob}(Y)$ such that

- for all $x \in Y$

$$\text{Supp}(p(x)) \subset B_R(x, \mathcal{G}),$$

- and

$$\sum_{x \sim_{\mathcal{G}} y} \|p(x) - p(y)\|_1 \leq \varepsilon.$$

We call the measured graph (\mathcal{G}, X, μ) of bounded vertex degrees **μ -uniformly amenable** if for any $\varepsilon > 0$ there exists $R \geq 1$ such that (\mathcal{G}, X, μ) is (ε, R) -uniformly hyperfinite. The main result of the paper is the following theorem.

Theorem 2. *A measured graph (\mathcal{G}, X, μ) of bounded type is μ -uniformly amenable if and only if it is μ -uniformly hyperfinite.*

It will be clear from the proof that for a measured graph of bounded degrees (without the bounded type condition) uniform amenability implies uniform hyperfiniteness. The proof of the theorem will be given by proving the equivalence of six properties: uniform amenability, uniform local hyperfiniteness, uniform hyperfiniteness, weighted hyperfiniteness, approximate strong hyperfiniteness and strong fractional hyperfiniteness.

In Section 2, we give examples of hyperfinite, but not uniformly hyperfinite measured graphs. Also, we show that there exist measured graphs of unbounded type that are uniformly hyperfinite, but not uniformly amenable.

Finally, in the last section we prove a trichotomy in terms of uniform hyperfiniteness, characterizing exact non-amenable groups.

2. FURTHER MOTIVATION AND EXAMPLES

It is not very hard to show that μ -hyperfiniteness implies μ -amenability. The converse entails some significant work in the Connes-Feldman-Weiss Theorem. In the case of our Theorem 2, the more involved part of the proof is to show that μ -uniform hyperfiniteness implies μ -uniform amenability. In the course of the proof we will show that μ -hyperfiniteness is equivalent with a series of other notions. Let us describe briefly the motivation for this approach. Instead of measured graphs, let us consider infinite connected graphs

of bounded vertex degrees. The analogue of μ -uniform amenability is Property A. This important notion was introduced by Yu [13] in the context of the Baum-Connes Conjecture.

An infinite connected graph G of bounded vertex degrees has **Property A**, if for any $\varepsilon > 0$, there exists $R_\varepsilon > 0$ and a function $p : V(G) \rightarrow \text{Prob}(G)$ such that for each $x \in V(G)$,

- $\text{Supp}(p(x)) \subset B_{R_\varepsilon}(x, G)$,
- $\sum_{y, x \sim y} \|p(x) - p(y)\|_1 \leq \varepsilon$.

For sequences of graphs hyperfiniteness is well-defined and closely related to μ -hyperfiniteness via the Benjamini-Schramm convergence [3]. However, there seems to be no sensible way to define hyperfiniteness for infinite connected graphs. In [5], the author and Timár introduced the notion of weighted hyperfiniteness. An infinite connected graph G of bounded vertex degrees is weighted hyperfinite if for any $\varepsilon > 0$ there exists $K \geq 1$ such that for any probability measure $p : V(G) \rightarrow [0, 1]$, there exists a subset $A \subset V(G)$ such that

- $p(A) \leq \varepsilon p(V(G))$,
- the induced graph on $V(G) \setminus A$ has components of size at most K .

Sako [10] proved that Property A is, in fact, equivalent to weighted hyperfiniteness. Although hyperfiniteness cannot be defined on a countably infinite graph, one can define a related notion (this is strongly motivated by the work in [9]) strong hyperfiniteness. First, recall that if G is an infinite connected graph G of bounded vertex degrees, then a subset $A \subset V(G)$ is a K -separator if the induced graph on $V(G) \setminus A$ has components of size at most K .

Definition 2.1. An infinite connected graph G of bounded vertex degrees is strongly hyperfinite if for any $\varepsilon > 0$ there exists $K \geq 1$ such that we have a probability measure ν on the compact set of K -separators satisfying the following condition. For any $v \in V(G)$, the measure of separators containing v is not greater than ε .

One can prove [4] that strong hyperfiniteness is also equivalent to Property A. The main idea of the proof of Theorem 2 is to show that μ -uniform hyperfiniteness is equivalent to a measured version of weighted hyperfiniteness, which, in turn, is equivalent to some measured versions of strong hyperfiniteness and finally, they all are equivalent to μ -uniform amenability. The steps of the proof are motivated by the proofs of their combinatorial counterparts.

Before starting the proof of our main theorem, let us present two examples that are intended to demonstrate the subtlety of the notion of uniform hyperfiniteness.

Example 1. There exists a measured ergodic graph $(\hat{\mathcal{H}}, \hat{X}, \hat{\mu})$ of bounded degrees which is $\hat{\mu}$ -hyperfinite, but not $\hat{\mu}$ -uniformly hyperfinite.

Let $\alpha : \mathbb{Z} \curvearrowright X$ the irrational rotation action on the unit circle preserving the Lebesgue probability measure μ . We consider the standard generating system $\Sigma = \{1, -1\}$ and the corresponding measured graph $\mathcal{H} = \alpha_G^{\mathbb{Z}, \Sigma}$. Now, let T_1, T_2, T_3, \dots be an expander sequence of finite 3-regular graphs such that $|V(T_1)| < |V(T_2)| < |V(T_3)| < \dots$. Let $\{Y_n\}_{n=1}^\infty$ be a sequence of Borel subsets of X satisfying the following conditions.

- For any $n \geq 1$, there exists an integer $C_n > 0$ such that for all $x \in X$ there exists $y \in Y_n$ so that $d_{\mathcal{H}}(x, y) \leq C_n$,
- $\mu(Y_n) \leq \frac{1}{2^n |V(T_n)|}$.

The existence of such marker sets is well-known (see e.g. [6]). Now, we construct a new measured graph $(\hat{\mathcal{H}}, \hat{X}, \hat{\mu})$ in the following way. First we set

$$\hat{X} = X \cup (Y_1 \times V(T_1)) \cup (Y_2 \times V(T_2)) \cup \dots$$

Now we define a Borel measure ν on \hat{H} in the following way.

- $\nu(A) := \mu(A)$, if $A \subset X$ is a Borel set.
- $\nu(B \times \{p\}) := \mu(B)$, if $B \subset Y_n$ is a Borel set and $p \in V(T_n)$ for some $n \geq 1$.

By our assumption, $\nu(\hat{\mathcal{H}}) \leq 2$. Now, let $\hat{\mu}$ be the normalized probability measure associated to ν , that is, for a Borel set $C \subset \hat{X}$, $\hat{\mu}(C) := \frac{\nu(C)}{\nu(\hat{X})}$.

Finally, we define a Borel graph structure on \hat{X} . For each $n \geq 1$, fix a vertex $t_n \in V(T_n)$ and for each $s \in Y_n$, let us connect s and $s \times t_n$ by an edge e_s . Also, let the induced graph on $s \times V(T_n)$ be T_n . We denote the resulting Borel graph by $\hat{\mathcal{H}}$. Clearly, $(\hat{\mathcal{H}}, \hat{X}, \hat{\mu})$ is a measured graph with an invariant probability measure and the corresponding orbit equivalence relation is ergodic.

Lemma 2.1. *The measured graph $(\hat{\mathcal{H}}, \hat{X}, \hat{\mu})$ is $\hat{\mu}$ -hyperfinite, but it is not $\hat{\mu}$ -uniformly hyperfinite.*

Proof. Let $\varepsilon > 0$. Since \mathcal{H} is μ -hyperfinite, we have a subset $Z \subset X$ and an integer $K \geq 1$ such that $\mu(Z) < \frac{\varepsilon}{2}$ and all the components of $\mathcal{H}_{X \setminus Z}$ have size not greater than K .

Now, let $q > 0$ be an integer such that

$$Z' = \sum_{n=q+1}^{\infty} \hat{\mu}(Y_n \times V(T_n)) < \frac{\varepsilon}{2}.$$

Then, $\hat{\mu}(Z \cup Z') < \varepsilon$ and the size of all the components in the graph $\hat{\mathcal{H}}_{\hat{X} \setminus (Z \cup Z')}$ is not greater than $K + K|V(T_q)|$. Hence, $(\hat{\mathcal{H}}, \hat{X}, \hat{\mu})$ is $\hat{\mu}$ -hyperfinite. By the expander condition, for any $l \geq 1$, we have $n_l > 0$ such that the graph T_{n_l} is not (ε, n_l) -hyperfinite. Consequently, $(\hat{\mathcal{H}}, \hat{X}, \hat{\mu})$ is not $\hat{\mu}$ -uniformly hyperfinite. So, we have ergodic, invariant hyperfinite, but non-uniformly hyperfinite measured graphs.

Example 2. Our second example is a measured graph $(\hat{H}, \hat{X}, \hat{\nu})$ which is

- $\hat{\nu}$ -uniformly hyperfinite,
- not $\hat{\nu}$ -uniformly amenable,
- not of bounded type.

We start with the measured graph $(\hat{H}, \hat{X}, \hat{\mu})$ constructed in Example 1 and substitute the measure $\hat{\mu}$ with a quasi-invariant probability measure $\hat{\nu}$ in the same measure class. Let T_1, T_2, \dots be the graphs in Example 1. and for $n \geq 1$, let $t_n \in V(T_n)$ be the distinguished vertex. Also, for $n \geq 1$, let $t_n = s_1^n, s_2^n, \dots, s_{|V(T_n)|}^n$ be an enumeration of the vertices of the graph T_n . Finally, we define a probability measure w_n on each vertex set $V(T_n)$ in the following way.

- Let $w_n(s_1^n) = \frac{1}{2}, w_n(s_2^n) = \frac{1}{4}, \dots$
- Let $w_n(s_{|V(T_n)|-1}^n) = \frac{1}{2^{|V(T_n)|-1}}, w_n(s_{|V(T_n)|}^n) = \frac{1}{2^{|V(T_n)|-1}}$.

The new measure $\hat{\nu}$ will coincide with $\hat{\mu}$ on Borel subsets of X and for $n \geq 1$,

$$\hat{\nu}(Y_n \times V(T_n)) = \hat{\mu}(Y_n \times V(T_n)).$$

We redistribute the weights on each of the sets $(Y_n \times V(T_n))$ in the following way.

Let $n \geq 1, 1 \leq i \leq |V(T_n)|$. Now, for a Borel set $B_n \subset Y_n$, we define

$$\hat{\nu}(B_n \times \{s_i^n\}) = w_n(s_i^n) \hat{\mu}(B_n \times \{s_i^n\}).$$

Clearly, $\hat{\mu}$ and $\hat{\nu}$ are in the same measure class. So, the measured graph $(\hat{H}, \hat{X}, \hat{\nu})$ is still not $\hat{\nu}$ -uniformly amenable, since uniform amenability depends only on the measure class. Thus, we need to prove the following lemma.

Lemma 2.2. *The measured graph $(\hat{H}, \hat{X}, \hat{\nu})$ is $\hat{\nu}$ -uniformly hyperfinite.*

Proof. Let $\varepsilon > 0$ and $A \subset \hat{X}$ be a Borel set. Since, (\mathcal{G}, X, μ) is (ε, K) -uniformly hyperfinite, we have $Y \subseteq A \cap X$ and an integer $K \geq 1$ such that $\mu(Z) < \varepsilon \mu(A \cap X)$, consequently, $\hat{\nu}(Z) < \varepsilon \hat{\nu}(A \cap X)$ and all the components of $\mathcal{G}_{(A \cap X) \setminus Z}$ have size at most K . Now, let $j \geq 1$ be an integer such that

$$(2) \quad \sum_{i=1}^j \frac{1}{2^i} > 1 - \varepsilon.$$

Let Z' be the set of elements in $A \setminus X$ in the form of $y \times t$, where for some $n \geq 1, y \in Y_n$ and $t = s_k^n, k > j$. Then, by (2), $\hat{\nu}(Z) < \varepsilon \hat{\nu}(A \setminus X)$. Therefore, $\hat{\nu}(Z \cup Z') < \varepsilon \hat{\nu}(A)$ and the size of the components in $\hat{\mathcal{H}}_{A \setminus (Z \cup Z')}$ are bounded by $K + Kj$.

3. UNIFORM AMENABILITY IMPLIES UNIFORM LOCAL HYPERFINITENESS

In the course of our paper we introduce several notions equivalent to uniform amenability in the realm of measured graphs of bounded type. The first such

notion is uniform local hyperfiniteness. Let (\mathcal{G}, X, μ) be a measured graph of bounded vertex degrees. Let $A \subset B \subset X$ be Borel subsets. Then, the outer boundary set $\bar{\partial}_B(A) \subset X$ is defined as the set of vertices $x \in B \setminus A$ such that there exists $y \in A$, $x \sim_{\mathcal{G}} y$.

Definition 3.1. Let (\mathcal{G}, X, μ) be a measured graph of bounded vertex degrees, where μ is a quasi-invariant measure. Let $\varepsilon > 0$, $K \geq 1$ be positive constants. We say that (\mathcal{G}, X, μ) is (ε, K) -locally hyperfinite if for any Borel subset $B \subset X$ of positive measure, there exists a Borel subset $A \subset B$ of positive measure such that

- $\mu(\partial_B(A)) < \varepsilon\mu(A)$,
- each component of \mathcal{G}_A has size at most K .

We call the measured graph (\mathcal{G}, X, μ) μ -uniformly locally hyperfinite, if for any $\varepsilon > 0$ there exists $K \geq 1$ such that (\mathcal{G}, X, μ) is (ε, K) -locally hyperfinite. The main result of this section is the following proposition.

Proposition 3.1. *If the measured graph (\mathcal{G}, X, μ) is (ε, R) -uniformly amenable, then it is (ε, N_{2R}) -locally hyperfinite as well, where N_{2R} denotes the size of the largest ball of radius $2R$ in the graph \mathcal{G} .*

Proof. First, we need a lemma.

Lemma 3.1. *Let $B \subset Y$ be a Borel set, where Y is the invariant set of full measure in the definition of (ε, R) -uniform amenability. Then, there exists a Borel function $p_B : B \rightarrow \text{Prob}(B)$ such that*

- for each $x \in B$, we have

$$p_B(x) \subset B_{2R}(x, \mathcal{G}) \cap B,$$

- for any $x \in B$,

$$\sum_{x \sim_{\mathcal{G}} y} \|p_B(x) - p_B(y)\|_1 \leq \varepsilon.$$

Proof. If $y \in Y$ is a point that there exists $x \in B$ so that $x \equiv_{\mathcal{G}} y$, then we define $\tau(y) \in B$, $y \equiv_{\mathcal{G}} \tau(y)$ such that $d_{\mathcal{G}}(y, \tau(y)) = d_{\mathcal{G}}(y, B)$ clearly, the function τ can be defined in a Borel fashion.

Now, for $x \in B$ and $z \in B$ we define the probability measure $p_B(x)$ by setting

$$p_B(x)(z) = \sum_{t \in \tau^{-1}(z)} p(x)(t).$$

Then $\text{Supp}(p_B(x)) \subset B_{2R}(x, \mathcal{G}) \cap B$ and $\sum_{x \sim y} \|p_B(x) - p_B(y)\|_1 \leq \varepsilon$ holds, hence our lemma follows. \square

Now we follow the proof of Marks [11]. First, we need a version of Namioka's Trick (Lemma 5.1 [11]).

Lemma 3.2. *For $a > 0$, let $I_{a,\infty}$ be the characteristic function of the half-line (a, ∞) and let $f, g \in \text{Prob}(X)$ be finitely supported functions. Then,*

$$(3) \quad \int_0^\infty \|I_{a,\infty}(f) - I_{a,\infty}(g)\|_1 da = \|f - g\|_1.$$

Let $p_B : B \rightarrow \text{Prob}(B)$ be the function defined in Lemma 3.1. Then, we have that

$$\int_B \sum_{x \sim y} \int_0^\infty \|I_{a,\infty}(p_B(x)) - I_{a,\infty}(p_B(y))\|_1 da d\mu(x) \leq \varepsilon \int_B \|I_{a,\infty}(p_B(x))\|_1 d\mu(x).$$

That is, there exists $a \in (0, \infty)$ such that

$$(4) \quad \int_B \sum_{x \sim y} \|I_{a,\infty}(p_B(x)) - I_{a,\infty}(p_B(y))\|_1 d\mu(x) \leq \varepsilon \int_B \|I_{a,\infty}(p_B(x))\|_1 d\mu(x).$$

For $x \in B$, let $\Lambda_x = \{z \mid p_B(x)(z) > a\}$. Then, by (4),

$$(5) \quad \int_B \sum_{x \sim y} |\Lambda_x \Delta \Lambda_y| d\mu(x) \leq \varepsilon \int_B |\Lambda_x| d\mu(x).$$

By the classical result of Kechris, Solecki and Todorćevic [7], there exists a Borel coloring $\varphi : X \rightarrow Q$ such that

- Q is a finite set,
- $\varphi(x) \neq \varphi(y)$, provided that $d_{\mathcal{G}}(x, y) \leq 10R$.

So, for every $x \in B$ and $q \in Q$ there exists at most one $z \in B$ so that

- $\varphi(z) = q$ and
- either $z \in \Lambda_x$ or $z \in \Lambda_y$ for some $y, x \sim y$.

Consequently by (5), there exists an $r \in Q$ such that

$$(6) \quad \int_B \sum_{x \sim y} |\{z \in \Lambda_x \Delta \Lambda_y, \varphi(z) = r\}| d\mu(x) \leq \varepsilon \int_B |\{z \in \Lambda_x, \varphi(z) = r\}| d\mu(x).$$

Let A be the set of the elements $x \in B$ for which there exists $z \in B$ such that $z \in \Lambda_x$ and $\varphi(z) = r$. Observe that the right hand side of (6) equals to $\varepsilon \mu(A)$. On the other hand, the left hand side of (6) is not greater than $\mu(\overline{\partial}_B(A))$. Hence, $\mu(\overline{\partial}_B(A)) \leq \varepsilon \mu(A)$. Also, all the components of \mathcal{G}_A has size at most N_{2R} . Thus, our proposition follows. \square

Corollary 3.1. *If the measured graph (\mathcal{G}, X, μ) of bounded vertex degrees is μ -uniformly amenable, then it is μ -uniformly locally hyperfinite, as well.*

4. UNIFORM LOCAL HYPERFINITENESS IMPLIES UNIFORM HYPERFINITENESS

The goal of this section is to prove the following proposition.

Proposition 4.1. *(ε, K) -local hyperfinite measured graphs (\mathcal{G}, X, μ) of bounded vertex degrees are (ε, K) -uniformly hyperfinite.*

Proof. Let $X_1 = X$. By definition, there exists a Borel set $A_1 \subset X_1$ such that

- $\mu(\partial_{X_1}(A_1)) \leq \varepsilon\mu(A_1)$,
- all the components of \mathcal{G}_{A_1} have size at most K .

Now, let $X_2 = X \setminus (A_1 \cup \partial_{X_1}(A_1))$ and if let $A_2 \subset X_2$ be a Borel set such that

- $\mu(\partial_X(A_2)) \leq \varepsilon\mu(A_2)$,
- all the components of \mathcal{G}_{A_2} have size at most K ,
- $\mu(A_2) > 0$, provided that $\mu(X_2) > 0$.

By transfinite induction, for each ordinal we can construct Borel sets $A_\alpha \subset X_\alpha$, such that

- if $\alpha_1 < \alpha_2$ then $X_{\alpha_1} \supset X_{\alpha_2}$,
- if $\alpha = \beta + 1$, then

$$X_\alpha = X_\beta \setminus (A_\beta \cup \partial_{X_\beta}(A_\beta)),$$

- if $\alpha = \lim_{\beta < \alpha} \beta$, then $X_\alpha = \bigcap_{\beta < \alpha} X_\beta$.
- $\mu(\partial_X(A_\alpha)) \leq \varepsilon\mu(A_\alpha)$,
- all the components of \mathcal{G}_{A_α} have size at most K ,
- $\mu(A_\alpha) > 0$, provided that $\mu(X_\alpha) > 0$.

By our positivity assumption, there exists a countable ordinal α for which $\mu(X_\alpha) = 0$. Now, let

$$T := \bigcup_{\beta \leq \alpha} \partial_{X_\beta}(A_\beta) \cup X_\alpha$$

and let $A = \bigcup_{\beta \leq \alpha} A_\beta$. Then,

- $X \setminus T = A$,
- $\mu(T) \leq \varepsilon\mu(X)$
- all the components of \mathcal{G}_A have size at most K .

Hence, our proposition follows. □

Corollary 4.1. *If the measured graph (\mathcal{G}, X, μ) of bounded vertex degrees is μ -uniformly locally hyperfinite, then it is μ -uniformly hyperfinite, as well.*

5. UNIFORM HYPERFINITENESS IMPLIES WEIGHTED HYPERFINITENESS

In Section 2, we recalled the notion of weighted hyperfiniteness for countably infinite graphs of bounded vertex degrees. Now, we introduce a related notion for measured graphs.

Definition 5.1. Let (\mathcal{G}, X, μ) be a measured graph of bounded vertex degrees, $\varepsilon > 0$ and $K \geq 1$. We say that (\mathcal{G}, X, μ) is (ε, K) -weighted hyperfinite, if for any μ -integrable function $W : X \rightarrow [0, \infty)$, there exists a Borel subset $A \subset X$ such that

- $\int_A W(x)d\mu(x) \leq \varepsilon \int_X W(x)d\mu(x)$,
- all the components of $\mathcal{G}_{X \setminus A}$ have size at most K .

We call (\mathcal{G}, X, μ) **μ -weighted hyperfinite** if for any $\varepsilon > 0$, there exists $K \geq 1$ such that (\mathcal{G}, X, μ) is (ε, K) -weighted hyperfinite. Clearly, weighted hyperfiniteness implies uniform hyperfiniteness. The goal of this section is to prove the converse statement. Note that the bounded type condition is crucial.

Proposition 5.1. *Let (\mathcal{G}, X, μ) be an (ε', K) -uniformly hyperfinite measured graph of bounded type, where d is the degree bound of \mathcal{G} , $M = \sup_{x,y,x \sim y} R_{x,y}$. $L = \lceil \frac{3}{\varepsilon} \rceil$, and $\varepsilon' = \frac{\varepsilon}{3} \left(\frac{3Md}{\varepsilon}\right)^{-L}$. Then, (\mathcal{G}, X, μ) is (ε, K) -weighted hyperfinite.*

Proof. We follow the combinatorial approach (used in the context of finite graphs) by Romero, Wrochna and Živný [9] up to the point, where the Radon-Nikodym cocycle enters the picture.

So, let (\mathcal{G}, X, μ) be an (ε', K) -uniformly hyperfinite measured graph of bounded type and $W : X \rightarrow [0, \infty)$ be an integrable Borel function. Set

$$B_i = \left\{ x \in X \mid \left(\frac{\varepsilon}{3Md}\right)^{i+1} \leq W(x) < \left(\frac{\varepsilon}{3Md}\right)^i \right\},$$

and for $j \in \{0, 1, \dots, L-1\}$ we define

$$B'_j = \cup_{i \in \mathbb{Z}} B_{j+iL}.$$

Hence, we must have $1 \leq j^* \leq L-1$ such that

$$W(B'_{j^*}) \leq \frac{1}{L}W(X) \leq \frac{\varepsilon}{3}W(X).$$

Now, set

$$C_i = B_{i+j^*+1} \cup B_{i+j^*+2} \cup \dots \cup B_{i+j^*+L-1}.$$

Observe that

$$(7) \quad \inf_{x \in C_i} W(x) \geq \left(\frac{\varepsilon}{3Md}\right)^L \sup_{x \in C_i} W(x).$$

Also, if $x \in C_j, y \in C_i$ and $i < j$, then

$$(8) \quad W(x) \leq \left(\frac{\varepsilon}{3Md}\right) W(y).$$

Now, let $F_i \subset X$ be defined as the set of points x in X such that $x \in C_j, j > i$ and $x \sim y$ for some $y \in C_i$.

Then, for any $i \in \mathbb{Z}$, $\mu(F_i) \leq Md\mu(C_i)$ and by (8),

$$\sup_{x \in F_i} W(x) \leq \frac{\varepsilon}{3Md} \inf_{y \in C_i} W(y).$$

That is,

$$W(F_i) = \int_{F_i} W(x)d\mu(x) \leq \frac{\varepsilon}{3} \int_{C_i} W(x)d\mu(x) = \frac{\varepsilon}{3}W(C_i).$$

Let $F = \cup_{i=1}^{\infty} F_i$. Then, $W(F) \leq \frac{\varepsilon}{3}W(X)$. Now, let $Z = F \cup B'_{j^*}$, so $W(Z) \leq \frac{2\varepsilon}{3}W(X)$ and consider the graph $\mathcal{G}_{X \setminus Z}$. By (7), if x and y are in the same component of $\mathcal{G}_{X \setminus Z}$, then we have

$$\left(\frac{\varepsilon}{3Md}\right)^L W(y) \leq W(x).$$

Since the measured graph (\mathcal{G}, X, μ) is (ε', K) -uniform hyperfinite, we have a set $Z' \subset X \setminus Z$, such that

- $\mu(Z') \leq \varepsilon'$ and
- all the components of $\mathcal{G}_{X \setminus (Z \cup Z')}$ have size at most K .

Therefore, we have that

$$W(Z') \leq \varepsilon' \left(\frac{3Md}{\varepsilon}\right)^L W(X) = \frac{\varepsilon}{3}W(X).$$

Hence, $W(Z \cup Z') \leq \varepsilon W(X)$. Therefore, the measured graph (\mathcal{G}, X, μ) is (ε, K) -weighted hyperfinite. \square

Corollary 5.1. *If the measured graph (\mathcal{G}, X, μ) of bounded type is μ -uniformly hyperfinite, then it is μ -weighted hyperfinite, as well.*

6. WEIGHTED HYPERFINITENESS IMPLIES APPROXIMATE STRONG HYPERFINITENESS

Let (\mathcal{G}, X, μ) be a measured graph of bounded vertex degrees and $K \geq 1$ be an integer. We say that $Y \subset X$ is a K -**separator** if all the components of $\mathcal{G}_{X \setminus Y}$ have components of size at most K . By the Banach-Alaoglu Theorem the unit ball B of $L^2(X, \mu)$ is a compact, convex metrizable space with respect to the weak topology. If $Y \subset X$, then the characteristic function of Y , c_Y is an element of B . In Section 2, we recalled the notion of strong hyperfiniteness for infinite graphs of bounded vertex degrees, now we define the obvious analogue of this notion for measured graphs of bounded vertex degrees.

Definition 6.1. The measured graph (\mathcal{G}, X, μ) of bounded vertex degrees is strongly hyperfinite if there exists a probability measure ν on the unit ball B such that

- ν is supported on the characteristic functions on K -separators.
- The barycenter $b_\nu := \int_B v d\nu(v)$ satisfies the inequality $b \leq \underline{\varepsilon}$ almost everywhere, where $\underline{\varepsilon}$ is the constant function taking the value ε .

We call (\mathcal{G}, X, μ) **strongly hyperfinite** if for any $\varepsilon > 0$ there exists $K \geq 1$ such that (\mathcal{G}, X, μ) is (ε, K) -strongly hyperfinite. Unfortunately, we are not able to prove that strong hyperfiniteness is equivalent to μ -uniform hyperfiniteness, due to the fact that the set of characteristic functions of K -separators is not closed (the statement might not even be true). In order to circumvent this difficulty, we introduce two very similar notions which are, in

fact, equivalent to μ -uniform hyperfiniteness: approximate strong hyperfiniteness and strong fractional hyperfiniteness.

Definition 6.2. Let (\mathcal{G}, X, μ) be a measured graph of bounded vertex degrees $\varepsilon > 0$ and $K \geq 1$. We say that (\mathcal{G}, X, μ) is (ε, K) -approximately strongly hyperfinite, if there exists a sequence of finitely supported probability measures

$$\{\nu_i = \sum_{j=1}^{t_n} x_j^n \delta_{c_{Y_j^n}}\}_{n=1}^\infty$$

non-negative bounded measurable functions $\{z^n\}_{n=1}^\infty$ such that

$$\text{wlim}_{n \rightarrow \infty} \sum_{i=1}^{t_n} (x_i^n c_{Y_i^n} + z^n) = \underline{\varepsilon},$$

where wlim stands for the weak limit.

Again, (\mathcal{G}, X, μ) is called μ -**approximately strongly hyperfinite** if for any $\varepsilon > 0$ there exists $K \geq 1$ such that (\mathcal{G}, X, μ) is approximately (ε, K) -strongly hyperfinite. So, finally we can state the main result of this section.

Proposition 6.1. *A measured graph (\mathcal{G}, X, μ) of bounded vertex degrees is (ε, K) -approximately strongly hyperfinite if it is (ε, K) -weighted hyperfinite.*

Proof. We closely follow the combinatorial proof of Lemma 4.1 in [4]. Let C be the set of elements $y \in L^2(X, \mu)$ which can be written in the form

$$y = \sum_{i=1}^n t_i c_{Y_i} + z,$$

where for all $i \geq 1$ $t_i \geq 0$, $\sum_{i=1}^n t_i = 1$ and z is a non-negative function.

The closure of C , \overline{C} is a closed, convex set in $L^2(X, \mu)$. We have two cases.

Case 1. $\underline{\varepsilon} \in \overline{C}$. Then, there exists a sequence of finitely supported measures

$$\{\nu_i = \sum_{j=1}^{t_n} x_j^n \delta_{c_{Y_j^n}}\}_{n=1}^\infty$$

together with non-negative bounded measurable functions $\{z_n\}_{n=1}^\infty$ such that

$$\text{wlim}_{n \rightarrow \infty} \sum_{i=1}^{t_n} x_i^n c_{Y_i^n} + z^n = \underline{\varepsilon}.$$

that is, (\mathcal{G}, X, μ) is (ε, K) -approximately strongly hyperfinite.

Case 2. $\underline{\varepsilon} \notin \overline{C}$. Then, by the Hahn-Banach Separation Theorem there exists a non-negative $W \in L^2(X, \mu)$ such that

$$(9) \quad \langle W, \underline{\varepsilon} \rangle < \langle W, c_Y + z \rangle$$

holds for all K -separators Y and bounded non-negative functions z . We can also assume that W is non-negative, otherwise choosing an appropriate z (9) would not hold. Thus, $\int_{c_Y} W(x) d\mu(x) > \varepsilon$ holds for all K -separators Y , therefore (\mathcal{G}, X, μ) is not (ε, K) -weighted hyperfinite. \square

Corollary 6.1. *If the measured graph (\mathcal{G}, X, μ) of bounded type is μ -weighted hyperfinite, then it is μ -approximately strongly hyperfinite, as well.*

7. FRACTIONAL PARTITIONS

Fractional partitions were recently introduced by Lovász [12]. The notion will be crucial in our proof of Theorem 2, since it will provide the right analogue for strong hyperfiniteness of infinite graphs of bounded vertex degrees.

Let (\mathcal{G}, X, μ) be a measured graph of bounded vertex degrees. For an integer $K \geq 1$, we call a subset $L \subset X$ a K -**subset** if $|L| \leq K$ and the induced graph \mathcal{G}_L is connected. Following Lovász let us consider the Borel space \mathcal{R}_K of all K -subsets. Note that we have a natural Borel measure μ_K on \mathcal{R}_K . Let $\mathcal{A} \subset \mathcal{R}_K$ be a Borel set, then the Borel function $\Lambda_{\mathcal{A}} : X \rightarrow \mathbb{Z}$ is defined in the following way.

$$\Lambda_{\mathcal{A}}(x) := |\{A \in \mathcal{A} \mid x \in A\}|.$$

Then,

$$\mu_K(\mathcal{A}) := \int_X \Lambda_{\mathcal{A}}(x) d\mu(x).$$

For a measurable function $\Phi : \mathcal{R}_K \rightarrow \mathbb{R}$ let $\Phi^* : X \rightarrow \mathbb{R}$ be defined by setting

$$\Phi^*(x) := \sum_{A \in \mathcal{R}_K, x \in A} \Phi(A).$$

Clearly, Φ^* is a measurable function as well. A measurable function $\Phi : \mathcal{R}_K \rightarrow \{0, 1\}$ is called a K -partition if for all $x \in X$, $\Phi^*(x) = 1$.

Definition 7.1. A non-negative measurable function $\Phi : X \rightarrow \mathbb{R}$ is a **fractional K -partition** if for almost all $x \in X$, $\Phi^*(x) = 1$.

Let $Y \subset X$ be a K -separator as in the previous section. Then, the associated K -partition Φ_Y is defined in the following way.

- If $y \in Y$, then $\Phi_Y(\bar{y}) = 1$, where $\bar{y} \in \mathcal{R}_K$ is the singleton containing y ,
- $\Phi_Y(A) = 1$, if A is a component of $\mathcal{G}_{X \setminus Y}$,
- otherwise, $\Phi_Y(A) = 0$.

Let $\underline{t} = \sum_{i=1}^n t_i \delta_{Y_i}$ be a finitely supported probability distribution on the set of K -separators. Then, $\Phi_{\underline{t}} := \sum_{i=1}^n t_i \Phi_{Y_i}$ is a fractional partition of X . Now, the measurable function $\partial\Phi : X \rightarrow \mathbb{R}$ is defined by setting

$$\partial\Phi(x) = \sum_{A \in \mathcal{R}_K, x \in \partial A} \Phi(A).$$

Note that ∂A (the inner boundary of A) denotes the set of vertices $x \in A$ such that there exists $y \notin A$, $x \sim y$.

For a probability distribution \underline{t} as above an $\varepsilon > 0$, the set $S_{\underline{t}, \varepsilon}$ is defined as the set of points $x \in X$ such that

$$\varepsilon \leq \sum_{i, x \in Y_i} t_i.$$

Also, if $\Phi : \mathcal{R}_K \rightarrow \mathcal{R}$ is a fractional K -partition, then $Q_{\Phi, \varepsilon}$ is defined as the set of points $x \in X$ such that

$$\varepsilon \leq \partial\Phi(x).$$

We end this section with two useful technical lemmas.

Lemma 7.1. *Let (\mathcal{G}, X, μ) be a measured graph of bounded vertex degrees, let \underline{t} is a probability distribution as above and $\varepsilon > 0$. Then*

$$\mu(Q_{\Phi_{\underline{t}, \varepsilon}) \leq (d+1)M\mu(S_{\underline{t}, \frac{\varepsilon}{d+1}}),$$

where d is the degree bound of \mathcal{G} and $M = \sup_{x \sim y \in X} R_{x,y}$.

Proof. If Y is a K -separator, then $x \in \partial\Phi_Y$ implies that either x or at least one of its neighbours is the element of Y . Hence, if $\varepsilon \leq \partial\Phi_{\underline{t}}$, then there exists y , $d_{\mathcal{G}}(x, y) \leq 1$ such that $\sum_{i, y \in Y_i} t_i \geq \frac{\varepsilon}{d+1}$. So, we have a measurable map $Z : Q_{\Phi_{\underline{t}, \varepsilon} \rightarrow S_{\underline{t}, \frac{\varepsilon}{d+1}}$ such that if $x \in Q_{\Phi_{\underline{t}, \varepsilon}$, then $d_{\mathcal{G}}(x, Z(x)) \leq 1$. Therefore, we have

$$\mu(Q_{\Phi_{\underline{t}, \varepsilon}) \leq (d+1)M\mu(S_{\underline{t}, \frac{\varepsilon}{d+1}}),$$

so our lemma follows. \square

Lemma 7.2. *Suppose that $\{\Phi_n\}_{n=1}^{\infty}$ are fractional K -partitions such that $\text{wlim}_{n \rightarrow \infty} \Phi_n = \Psi$ in the Hilbert space $L^2(\mathcal{R}_k, \mu_k)$. Then, Ψ is a fractional K -partition as well. Also,*

$$(10) \quad \text{wlim}_{n \rightarrow \infty} \partial\Phi_n = \partial\Psi.$$

Proof. The correspondence $\Phi \rightarrow \Phi^*$ defines a bounded linear map from $L^2(\mathcal{R}_k, \mu_k)$ onto $L^2(X, \mu)$. Hence, $\text{wlim}_{n \rightarrow \infty} \Phi_n = \Psi$ implies that

$$1 = \text{wlim}_{n \rightarrow \infty} \Phi_n^* = \Psi^*.$$

Thus, Ψ is a fractional K -partition. Similarly, the correspondence $\Psi \rightarrow \partial\Psi$ defines a bounded linear map from $L^2(\mathcal{R}_k, \mu_k)$ onto $L^2(X, \mu)$, therefore (10) holds as well. \square

8. APPROXIMATE STRONG HYPERFINITENESS IMPLIES STRONG FRACTIONAL HYPERFINITENESS

Definition 8.1. Let (\mathcal{G}, X, μ) be a measured graph of bounded type $\varepsilon > 0$, $K \geq 1$. Then, (\mathcal{G}, X, μ) is (ε, K) -strongly fractionally hyperfinite if there exists a fractional K -partition Φ such that for almost all $x \in X$, $\varepsilon \leq \partial\Phi(x)$.

We call (\mathcal{G}, X, μ) **strongly fractional μ -hyperfinite** if for every $\varepsilon > 0$ there exists $K \geq 1$ such that (\mathcal{G}, X, μ) is (ε, K) -strongly fractional hyperfinite. The goal of this section is to prove the following proposition.

Proposition 8.1. *Let (\mathcal{G}, X, μ) be a measured graph of bounded vertex degrees. If (\mathcal{G}, X, μ) is (ε, K) -approximately strongly hyperfinite, then (\mathcal{G}, X, μ) is $(3(d+1)\varepsilon, K)$ -strongly fractional hyperfinite.*

Proof. Let $\{\underline{t}^n\}_{n=1}^\infty$ be finitely supported distributions on the space of K -separators such that

$$\text{wlim}_{n \rightarrow \infty} \sum_{i=1}^{s_n} t_i^n c_{Y_i^n} \leq \underline{\varepsilon}.$$

By definition,

$$\lim_{n \rightarrow \infty} \mu(S_{\underline{t}^n, 2\varepsilon}) = 0.$$

Hence, by Lemma 7.1,

$$(11) \quad \lim_{n \rightarrow \infty} \mu(Q_{\Phi_{\underline{t}^n, 2(d+1)\varepsilon}}) = 0.$$

Let Ψ be the weak limit of a subsequence $\{\Phi_{\underline{t}^{n_k}}\}_{k=1}^\infty$. By Lemma 7.2,

$$\text{wlim}_{k \rightarrow \infty} \partial\Phi_{\underline{t}^{n_k}} = \partial\Psi$$

and Ψ is a fractional K -partition. Now, let

$$A := \{x \mid \partial\Psi(x) \geq 3(d+1)\varepsilon\}.$$

Then by weak convergence,

$$2(d+1)\varepsilon\mu(A) \geq \lim_{k \rightarrow \infty} \int_A \partial\Psi_{\underline{t}^{n_k}}(x) d\mu(x) = \int_A \partial\Psi(x) d\mu(x) \geq 3(d+1)\varepsilon\mu(A).$$

Therefore, $\mu(A) = 0$ and thus (\mathcal{G}, X, μ) is $(3(d+1)\varepsilon, K)$ -strongly fractionally hyperfinite. \square

Corollary 8.1. *If the measured graph (\mathcal{G}, X, μ) of bounded vertex degrees is μ -approximately strongly hyperfinite, then it is μ -strongly fractionally hyperfinite as well.*

9. STRONG FRACTIONAL HYPERFINITENESS IMPLIES UNIFORM AMENABILITY

In this section we finish to proof of Theorem 2.

Proposition 9.1. *Let (\mathcal{G}, X, μ) be a measured graph of bounded vertex degrees, $\varepsilon > 0$, $K \geq 1$. If (\mathcal{G}, X, μ) is (ε, K) -strongly fractionally hyperfinite, then (\mathcal{G}, X, μ) is $(2\varepsilon d, K)$ -uniformly amenable, as well.*

Proof. Let Φ be a fractional K -partition such that for an invariant subset $Y \subset X$ of full measure $\varepsilon \leq \partial\Phi(x)$ holds provided that $x \in Y$. For an element

$x \in Y$ let $\Theta(x)$ denote the set of all K -subsets containing x . If $A \in \Theta(x)$, let $\varphi_A(x) := \frac{1}{|A|}c_A$, that is, $\sum_{y \in A} \varphi_A(y) = 1$. Define $p(x) \in \text{Prob}(Y)$ by setting

$$p(x) := \sum_{A \in \Theta(x)} \Phi(A) \varphi_A(x).$$

Clearly, $\text{Supp}(p(x)) \subset B_K(x, \mathcal{G})$.

Lemma 9.1. *If $x \sim_{\mathcal{G}} y$, then $\|p(x) - p(y)\|_1 \leq 2\varepsilon$.*

Proof. Observe that

$$\begin{aligned} \|p(x) - p(y)\|_1 &\leq \sum_{A, x \in A, y \notin A} \Phi(A) + \sum_{A, x \notin A, y \in A} \Phi(A) \leq \\ &\leq \sum_{A, x \in \partial A} \Phi(A) + \sum_{A, y \in \partial A} \Phi(A) \leq 2\varepsilon. \end{aligned}$$

Therefore,

$$\sum_{x \sim_{\mathcal{G}} y} \|p(x) - p(y)\|_1 \leq 2\varepsilon d.$$

Hence our Proposition follows. \square

Corollary 9.1. *If the measured graph (\mathcal{G}, X, μ) of bounded vertex degrees is μ -strongly fractionally hyperfinite, then it is μ -uniformly amenable, as well.*

Now Theorem 2 follows from Corollaries 3.1, 4.1, 5.1, 6.1, 8.1 and 9.1. \square

10. FREE ACTIONS

In this section we consider measure class preserving actions of finitely generated groups such that the Radon-Nikodym derivative of any element is bounded. We call these actions "actions of bounded type". Also, we call a measure class preserving action uniformly hyperfinite, if the associated measured graph is uniformly hyperfinite. The following proposition provides a trichotomy for finitely generated groups.

Proposition 10.1. *Let Γ be a finitely generated group.*

- *If Γ is amenable, then all free Γ -actions of bounded type are μ -uniformly hyperfinite,*
- *if Γ is non-exact then none of the free Γ -actions of bounded type are μ -uniformly hyperfinite,*
- *if Γ is an exact non-amenable group, then some of the free Γ -actions of bounded type are μ -uniformly hyperfinite, some of them are not.*

Proof. If Γ is amenable, then all free actions of Γ are μ -uniformly amenable, hence by our Theorem all free Γ -actions of bounded type are μ -uniformly hyperfinite. If Γ is non-exact, then none of the free actions of Γ are μ -uniformly amenable, hence none of the free Γ -actions of bounded type are μ -uniformly

hyperfinite. If Γ is a non-amenable exact group, then some of its measure-preserving actions are non-hyperfinite. So, we need to prove that each such Γ has at least one μ -uniformly amenable actions of bounded type.

Lemma 10.1. *Let Σ be symmetric generating system for Γ , then there exists a non-negative function $\rho : \Gamma \rightarrow \mathbb{R}$ and a positive integer M such that*

$$(12) \quad \sum_{\gamma \in \Gamma} \rho(\gamma) = 1.$$

for any $\sigma \in \Sigma$,

$$(13) \quad \frac{\rho(\gamma\sigma)}{\rho(\gamma)} < M.$$

Proof. Let $\text{Cay}(\Gamma, \Sigma)$ be the right Cayley graph of Γ with the usual length function $l(\gamma) := d_{\text{Cay}}(e, \gamma)$, For $r \geq 0$, let $S_r = \{\gamma \mid l(\gamma) = r\}$. Pick a constant λ such that $|S_r| \leq e^{\lambda r}$. Let

$$\rho(\gamma) = \frac{e^{-2\lambda l(\gamma)}}{\sum_{r=0}^{\infty} |S_r| e^{-2\lambda r}}.$$

Then, for large enough M both (12) and (13) hold. \square

Let $\alpha : \Gamma \curvearrowright \mathcal{C}$ be a free continuous topologically amenable action of Γ on the Cantor set (such action exists by definition). Let ν be the standard Cantor measure. The quasi-invariant measure μ is defined in the usual way. For a measurable set $A \subset \mathcal{C}$

$$\mu(A) := \sum_{\gamma \in \Gamma} \rho(\gamma) \nu(\alpha(\gamma)(A)).$$

Then, for any $\sigma \in \Sigma$ we have

$$\frac{\mu(\alpha(\sigma)(A))}{\mu(A)} < M.$$

Therefore the action is of bounded type. Consequently, $\alpha : \Gamma \curvearrowright (X, \mu)$ is a μ -uniformly hyperfinite action. \square

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