

# The Distributional Stress-Energy Quadrupole

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## Abstract

We investigate stress-energy tensors constructed from the delta function on a worldline. We concentrate on quadrupoles as they make an excellent model for the dominant source of gravitational waves and have significant novel features. Unlike the dipole, we show that the quadrupole has 20 free components which are not determined by the properties of the stress-energy tensor. These need to be derived from an underlying model and we give an example motivated from a divergent-free dust. We show that the components corresponding to the partial derivatives representation of the quadrupole, have a gauge like freedom. We give the change of coordinate formula which involves second derivatives and two integrals. We also show how to define the quadrupole without reference to a coordinate systems or a metric. For the representation using covariant derivatives, we show how to split a quadrupole into a pure monopole, pure dipole and pure quadrupole in a coordinate free way.

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# 1 Introduction

Gravitational wave astronomy will give rise to major developments in gravitational physics and astrophysics. The LIGO and VIRGO detectors [1, 2] have detected relativistic gravitational two-body systems. The existing network of gravitational wave interferometers is expanding both on Earth (for instance, via KAGRA and LIGO-India [3, 4]) and in space.

In this article we model the compact source, using a distribution, in which all the mass is concentrated in one point in space and hence a worldline in spacetime, but has an extended structure encoded as a multipole expansion. The zeroth order is the monopole, followed by the dipole and then the quadrupole. Here we consider the quadrupole in detail. For a Minkowski background, it is well known [5, 6] that gravitational radiation is dominated by the quadrupole moment.

When considering sources of gravitational waves, there are multiple approaches. For a pair of simple orbiting masses, where relativistic effects can be ignored, one can find analytic solutions. By contrast the final stages of coalescing black holes require detailed numerical simulations. Once the stress-energy tensor is constructed one can evaluate the perturbation to the metric representing the gravitational waves. Between these two extremes the post Newtonian approximation [7] can be used.

Our approach is different. In this article we examine the dynamics of the moments of the distributional quadrupole stress-energy tensor. This has a major advantage that the dynamics are encoded as ODEs for the components, as opposed to the coupled nonlinear PDEs which one is required to solve to model a general relativistic source. In a Minkowski background, the gravitational waves can be directly calculated from these moments. The only constraints we put on the source is that it obeys the rules of a total stress-energy tensor, namely symmetry of its indices and the divergenceless condition. For the monopole and the dipole it is well known that these conditions constrain the dynamics so much that they prescribe the ODEs: the geodesic equation for the monopole and the Mathisson-Papapetrou-Tulczyjew-Dixon equations for the dipole [8, 9]. In the dipole case we assume that the worldline has been given. One may therefore ask if these two conditions also constrain the quadrupole sufficiently to prescribe the ODEs for its components. In this article we show that, whereas 40 of the components are prescribed by ODEs, a further 20 are arbitrary, for example, a quadrupole can expand and contract as depicted in figure 1. Thus by itself this approach cannot completely prescribe the dynamics of a quadrupole and one must add additional ODEs, or algebraic equations, which one can consider to be **constitutive relations** for the quadrupole. These should arise from an underlying model of the source, i.e. coalescing black holes will have different constitutive relations to a rotating “rigid” body held together by non gravitational, e.g. electromagnetic

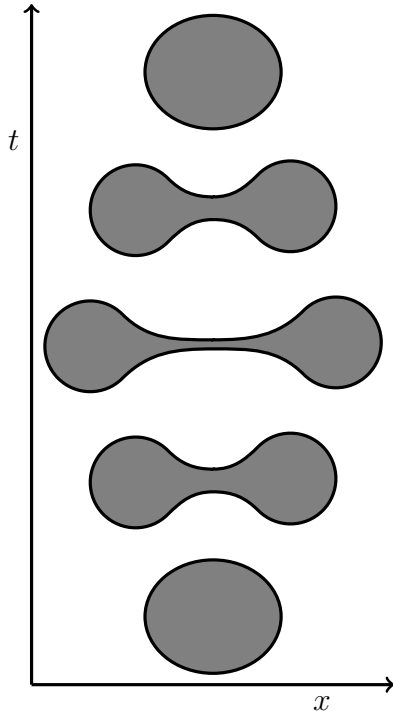


Figure 1: Schematic showing a blob of matter separating and then recombining. Such internal dynamics can take place solely within the free components, without affecting the divergencelessness of the stress-energy tensor (10).

or quantum forces. Once the constitutive relations are decided on, the ODEs can be solved and compared to numerical simulations or those derived from other models.

Approximating a distribution of matter with an object at a single point is a well established method in many branches of physics. Such approximations are valid if the size of the system is small compared to other distances involved. For example when considering coalescing black holes as a source of gravitational waves, the distance between the black holes is orders of magnitude smaller than their distance to Earth. Knowing the dynamics of multipoles may also shed light on the problem of radiation reaction in the context of dipole and quadrupole dynamics.

There are many important articles which consider multipole expansions. These date back to at least the 1950s where Tulczyjew [9] considered a multipole expansions to derive the Mathisson-Papapetrou-Tulczyjew-Dixon equations for the dipole. Then in the 1960s and 1970s Dixon [10, 11, 12, 13] and Ellis [14] considered both charge and mass distributions. They used two different general formalisms which we compare here, denoting them the Dixon and Ellis representations.

Recently Steinhoff and Puetzfeld [15, 16, 17] calculate the dynamic equation for the components of the quadrupole. In addition they consider the monopole-dipole and monopole-dipole-quadrupole system. In all cases the worldline of the multipole affects the dynamics of the components. However in the above the authors consider if and how the dynamics of the worldline is affected by the higher order moments. They conclude that one needs supplementary conditions in order to determine the worldline dynamics. We note that these supplementary conditions are distinct from the constitutive relations described here for the quadrupole. In this article, excluding section 3.1 on the monopole, the worldline is arbitrary but prescribed. Thus at the dipole order no supplementary conditions are required. However as stated there are 20 constitutive relations required at the quadrupole order.

Let  $\mathcal{M}$  be a spacetime with metric  $g_{\mu\nu}$  and the Levi-Civita<sup>1</sup> connection  $\nabla_\mu$  with Christoffel symbol  $\Gamma^\mu_{\nu\rho}$ . Here Greek indices run  $\mu, \nu = 0, 1, 2, 3$  and Latin indices  $a, b = 1, 2, 3$ . Let  $C : \mathcal{I} \rightarrow \mathcal{M}$  where

<sup>1</sup>It turns out that in most of our calculations in this article, the metric plays no role and an arbitrary linear connection can be used. See section 6.

$\mathcal{I} \subset \mathbb{R}$  is the worldline of the source<sup>2</sup> with components  $C^\mu(\sigma)$ . At this point we do not assume that  $\sigma$  is proper time. Here we consider stress-energy tensors  $T^{\mu\nu}$  which are non zero only on the worldline  $C^\mu(\sigma)$ , where it has Dirac- $\delta$  like properties. Such stress-energy tensors are called **distributional**.

Being a non linear theory, one cannot simply apply the theory of distributions to general relativity. It is not meaningful to write Einstein's equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi T_{\mu\nu}, \quad (1)$$

where the right hand side is a distribution. This contrasts with electromagnetism, which is a linear theory and so one often uses distributional sources. For example an arbitrary moving point charge gives rise to the Liénard-Wiechard fields.

There is a large quantity of literature discussing the nature of distributional stress-energy tensors. Often this links  $T^{\mu\nu}$  to one or a set of regular stress-energy tensors  $\mathcal{T}^{\mu\nu}$ , for which (1) is valid. Dixon [12] uses an exponential map, which connects points off the worldline to points on the worldline, to relate regular and distributional stress-energy tensors. Geroch and Weatherall [18] consider an infinite set of stress-energy tensors satisfying the dominate energy condition and find conditions such that there exists a sequence which tends to the monopole stress-energy tensor. In section 2.3 we show how the distributional stress-energy tensor  $T^{\mu\nu}$  is the weak limit of a set of regular tensors  $\mathcal{T}_\varepsilon^{\mu\nu}$ .

Another approach is to consider  $T_{\mu\nu}$  as a source within the context of linearised gravity. Perturbatively expanding the gravitational metric,  $g_{\mu\nu}$ , about a background  $\bar{g}_{\mu\nu}$ ,  $g_{\mu\nu} = \bar{g}_{\mu\nu} + \kappa h_{\mu\nu}^{(1)} + \dots$  where  $\kappa \ll 1$  is the perturbation parameter, and plugging the expansion into the Einstein equation (1) one has

$$G_{\mu\nu} = \bar{G}_{\mu\nu} + \kappa G_{\mu\nu}^{(1)} + \dots \quad \text{and} \quad T_{\mu\nu} = \bar{T}_{\mu\nu} + \kappa T_{\mu\nu}^{(1)} + \dots. \quad (2)$$

Hence the background metric  $\bar{g}_{\mu\nu}$  satisfies  $\bar{G}_{\mu\nu} = 8\pi \bar{T}_{\mu\nu}$ . The linearised equations are then given by

$$G_{\mu\nu}^{(1)} = 8\pi T_{\mu\nu}^{(1)}. \quad (3)$$

In the case when the background metric  $\bar{g}_{\mu\mu}$  is the Minkowski metric  $\eta_{\mu\nu}$  then (3) becomes [6]

$$\square \mathcal{H}_{\mu\nu}^{(1)} = -16\pi T_{\mu\nu}^{(1)}, \quad (4)$$

where  $\mathcal{H}_{\mu\nu}^{(1)} = h_{\mu\nu}^{(1)} - \frac{1}{2}\eta_{\mu\nu}(h^{(1)})^\rho{}_\rho$ ,  $\square = \partial_\mu \partial^\mu$  and we have used the Lorenz gauge, also called the de Donder gauge, ( $\partial^\mu \mathcal{H}_{\mu\nu}^{(1)} = 0$ ). We can give  $\mathcal{H}_{\mu\nu}^{(1)}$  in terms of an integral over the retarded Greens functions.

$$\mathcal{H}_{\mu\nu}^{(1)}(t, \vec{x}) = 4 \int \frac{T_{\mu\nu}^{(1)}(t - |\vec{x} - \vec{x}'|, \vec{x}')}{|\vec{x} - \vec{x}'|} d^3 \vec{x}'. \quad (5)$$

One should be careful as there is clearly a contradiction between the statement that the perturbation to the background stress-energy tensor is small, and the statement that it is distributional, and therefore infinite. However as long as one is sufficiently far from the source, one can make meaningful statements. One result is that if  $\mathcal{T}_\varepsilon^{\mu\nu} \rightarrow (T^{(1)})^{\mu\nu}$  weakly then, for a point off the worldline, the gravitational waves emanating from  $(T^{(1)})^{\mu\nu}$  are the limit of the gravitational waves emanating from  $\mathcal{T}_\varepsilon^{\mu\nu}$ , i.e.

$$\lim_{\varepsilon \rightarrow 0} \int \frac{(\mathcal{T}_\varepsilon)_{\mu\nu}(t - |\vec{x} - \vec{x}'|, \vec{x}')}{|\vec{x} - \vec{x}'|} d^3 \vec{x}' = \int \frac{T_{\mu\nu}^{(1)}(t - |\vec{x} - \vec{x}'|, \vec{x}')}{|\vec{x} - \vec{x}'|} d^3 \vec{x}'. \quad (6)$$

This requires that the intersection of the backward light cone of the point  $(t, \vec{x})$  with the support of  $\mathcal{T}_\varepsilon^{\mu\nu}$  is compact. This is proved in the appendix, proof number 1.

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<sup>2</sup>Even using proper time in Minkowski space, one cannot assume that  $\mathcal{I} = \mathbb{R}$  since it is possible to accelerate to lightlike infinity in finite proper time.

For sources such as coalescing black holes, one cannot use a regular stress-energy tensor. In this case one can only interpret the perturbation expansion (2) away from the source. It may then be possible to reinterpret the gravitational waves as though they were arising from a distributional quadrupole source.

An alternative approach to interpreting (1) with distributional  $T^{\mu\nu}$  is to extend the theory of distributions to include products, for example by using Colombeau algebra [19].

In this article we are concerned *only* with the structure of the distributional stress-energy tensor, which we write as  $T^{\mu\nu}$ , and avoid questions of how it should be applied. Since we are dealing with distributions it is most convenient to consider  $T^{\mu\nu}$  as a tensor density of weight 1. Thus  $\omega^{-1}T^{\mu\nu}$  is a tensor, where

$$\omega = \sqrt{-\det(g_{\mu\nu})}. \quad (7)$$

The definition of the covariant derivative of a tensor  $Y^{\mu\nu\rho\cdots}$  density of weight 1 is given by

$$\nabla_\mu Y^{\nu\rho\cdots} = \omega \nabla_\mu (\omega^{-1} Y^{\nu\rho\cdots}) = -\Gamma_{\mu\kappa}^\kappa Y^{\nu\rho\cdots} + \partial_\mu Y^{\nu\rho\cdots} + \Gamma_{\mu\kappa}^\nu Y^{\kappa\rho\cdots} + \Gamma_{\mu\kappa}^\rho Y^{\nu\kappa\cdots} + \cdots. \quad (8)$$

where  $\Gamma_{\mu\rho}^\nu$  are the Christoffel symbols. In this article the term stress-energy tensor, always refers to a stress-energy tensor density of weight 1, even if not explicitly stated. In addition the symbol  $T^{\mu\nu}$  always refers to a distributional stress-energy tensor density of weight 1 over the worldline  $C$ .

Since  $T^{\mu\nu}$  is a total stress-energy tensor<sup>3</sup>, it satisfies

$$T^{\mu\nu} = T^{\nu\mu} \quad (9)$$

and is divergenceless, also known as covariantly conserved

$$\nabla_\mu T^{\mu\nu} = 0. \quad (10)$$

which from (8) becomes

$$0 = \nabla_\mu T^{\mu\nu} = \partial_\mu T^{\mu\nu} + \Gamma_{\mu\rho}^\nu T^{\mu\rho}. \quad (11)$$

There are several ways of representing a multipole. However we consider multipoles to be distributions which are integrated with a symmetric test tensor  $\phi_{\mu\nu} = \phi_{\nu\mu}$ , so that<sup>4</sup>

$$\int_{\mathcal{M}} T^{\mu\nu} \phi_{\mu\nu} d^4x \quad \text{is a real number.} \quad (12)$$

Equation (12) can be written as an integral over the worldline with a number of derivatives of the Dirac  $\delta$ -function. In other words a multipole of order  $k$  is

$$T^{\mu\nu} = \sum_{r=0}^k \int_{\mathcal{I}} \zeta^{\mu\nu\cdots}(\sigma) \mathcal{D}_{\dots}^{(r)} \delta^{(4)}(x - C(\sigma)) d\sigma, \quad (13)$$

where there are  $r$  additional indices on  $\zeta^{\mu\nu\cdots}$  and  $\mathcal{D}_{\dots}^{(r)}$ . The subscript dots on  $\mathcal{D}_{\dots}^{(r)}$  contract with the superscript dots on  $\zeta^{\mu\nu\cdots}$ . Here  $\mathcal{D}_{\dots}^{(r)}$  represents  $r$  derivatives of the  $\delta$ -function. The familiar cases are the **monopole** when  $k = 0$ , the **dipole** when  $k = 1$  and the **quadrupole** when  $k = 2$ . As can be seen from (13) the general dipole contains the monopole term and the general quadrupole contains

<sup>3</sup>Non-symmetric stress-energy tensors, such as the Minkowski electromagnetic stress-energy tensor, are only part of the total stress-energy tensor. However we assume that  $T^{\mu\nu}$  satisfies (9) and (10) because all of the fields are dynamical (there are no background fields[20]) and  $T^{\mu\nu}$  is the total stress-energy tensor.

<sup>4</sup>An integral over  $\mathcal{M}$  must contain the measure  $\omega$ . There is therefore the following choice: one can choose  $T^{\mu\nu}$  or  $\phi_{\mu\nu}$  to be a density of weight 1, or put  $\omega$  explicitly in the integrand. Here we have chosen to make  $T^{\mu\nu}$  a density.

both the monopole and dipole terms. In general, it is not possible to extract the monopole and dipole terms from the quadrupole, without additional structure such as a preferred vector field or a coordinate system. For the monopole (9) and (10) lead to the pre-geodesic equation. By contrast, for the dipole and quadrupole there is no need to assume the worldline  $C$  is a geodesic. Therefore, unless otherwise stated, we present all the results for an arbitrary but prescribed worldline.

There are two important, equivalent representations of multipoles. One uses the partial derivatives, which we call the **Ellis** representation. The other uses the covariant derivative and we call the **Dixon** representation. Both have their advantages and disadvantages and these are outlined in section 2 below. The Ellis formulation is greatly simplified when using a coordinate system  $(\sigma, z^1, z^2, z^3)$  which is adapted to the worldline, i.e. where

$$C^0(\sigma) = \sigma \quad \text{and} \quad C^a(\sigma) = 0, \quad (14)$$

for  $a = 1, 2, 3$ . In this coordinate system the integral in (13) can be evaluated. Observe that (14) implies  $\dot{C}^0 = 1$  and  $\dot{C}^a = 0$ .

The monopole and dipole have been extensively studied in the literature, [21, 22, 23]. In this article we concentrate mainly on the quadrupole because it has interesting properties that do not appear to have been emphasised previously. Not only is it the natural source of gravitational waves, but it has several unusual properties not seen in the case of the monopole or the dipole. These properties are given below.

- The quadrupole contains free components.
- In the Ellis representation, the components  $\zeta^{\mu\nu\rho\kappa}$  do not transform as tensors but instead involve second derivatives and double integrals.
- In the Ellis representation, there is no natural way to endow a quadrupole with mass. Instead one can only talk about the energy of a quadrupole and only in the case where there is a timelike Killing symmetry.

The  $\zeta^{\mu\nu\dots} = \zeta^{\mu\nu\dots}(\sigma)$  are called the components of  $T^{\mu\nu}$  and are functions only of the position on the worldline  $C$ . Clearly from (9) they have the symmetry

$$\zeta^{\mu\nu\dots} = \zeta^{\nu\mu\dots}. \quad (15)$$

Depending on the representation, we may also choose to impose additional symmetries for uniqueness. We then apply the divergenceless condition (10) to establish further condition on the  $\zeta^{\mu\nu\dots}$ . We can place the components  $\zeta^{\mu\nu\dots}$  into three categories.

- Some components are algebraically related to other components and can therefore be removed.
- Some components are determined by a first order ODE. These are the result of the differential equation (10). In order to specify these components it is only necessary to specify their initial value at some point along the worldline.
- This leaves the components we call **free**. These are not constrained by (9) and (10) and are allowed to take on any value<sup>5</sup>. These free components can however influence the ODE components.

In order to completely specify the dynamics of a quadrupole, these free components need to be determined by constitutive equations. The choice of constitutive equations depends on a choice of a model for the material. In section 5 we consider the dust stress-energy tensor density [26] and use it to suggest corresponding constitutive equations.

In table 1 the number of ODEs and free components is given, and compared to the electromagnetic dipoles and quadrupoles.

In addition, some components may have a freedom in that different  $\zeta^{\mu\nu\dots}$  correspond to the same stress-energy tensor. Equivalently a given stress-energy tensor does not completely specify the components  $\zeta^{\mu\nu\dots}$ . Examples of this freedom for the dipole and quadrupole are given in equations

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<sup>5</sup>These 20 free components are not the same as the 20 independent components of the (reduced) quadrupole stress-energy tensor as described by Dixon [13, Equation (1.37)], [24, 25]. See appendix B for a discussion.

	Electromagnetic		Gravitational	
	ODE	free	ODE	free
Monopole	1	0	1	0
Semi-dipole	1	3	7	0
full dipole	1	6	10	0
semi-quadrupole	1	12	22	6
full quadrupole	1	20	40	20

Table 1: List of the number of components which are determined by ODEs and the number which are free, for monopoles, dipoles and quadrupoles. The electromagnetic sources refer to a current  $\mathcal{J}^\mu$  which is conserved and a source for Maxwell's equations. The gravitational source refers to a stress-energy tensor  $T^{\mu\nu}$  which are sources for (linearised) Einstein's equations. Each order includes all the lower orders. The 10 components in the full stress-energy dipole includes 1 monopole component, while the (40+20) components in the full quadrupole includes both dipole and monopole components. The definitions of the semi-dipole and semi-quadrupole are given in section 6.6.

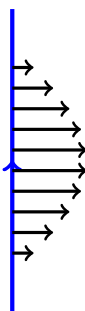


Figure 2: An electric dipole appears for a finite period of time and then disappears. This does not break charge conservation.

(52) and (71) below. In this article we will call this a **gauge-like freedom** since it is similar to other gauge freedoms in that it arises from integrating a physically observable tensor. In this case however, the components  $\zeta^{\mu\nu\dots}$  are not themselves tensors.

For comparison, the electromagnetic dipole has one ODE component, which is simply the total charge and satisfies  $dq/d\sigma = 0$ , and six free components corresponding to the three electric and three magnetic components [27]. These can be anything without breaking charge conservation, as seen in figure 1. For the stress-energy tensor, the free components can correspond to the internal matter separating and coalescing, as in figure 1. One therefore needs additional constitutive relations which encode the matter one is modelling. In this article we give an example of constitutive relations which correspond to non divergent dust.

Given a regular stress-energy tensor  $T^{\mu\nu}$  and a Killing vector field  $K^\mu$  we can find a conserved quantity  $T^{\mu\nu}K_\nu$  such that  $\nabla_\mu(T^{\mu\nu}K_\nu) = 0$ . The same is true for the distributional stress-energy tensor. Here  $K_\mu$  gives rise to a conserved scalar  $\mathcal{Q}_K$  along the worldline  $C$ . If  $K^\mu$  is a timelike Killing vector field it is natural to associate  $\mathcal{Q}_K$  as the conserved energy of the multipole. However, the relationship between the energy and mass is subtle. In the monopole and dipole case there is a natural definition of the mass, but the same is not true in the quadrupole case. Even when a mass can be defined, it is not conserved in general.

## Outline of article

As stated above there are two established methods of representing the stress-energy distribution: one using partial derivatives in (13), which we call the **Ellis** representation, and the other using covariant derivatives, which we call the **Dixon** representation. The pros and cons of these two approaches is discussed in section 2 and summarised in table 2. We show that the Ellis components with respect



to adapted coordinates are unique can be extracted by applying them to particular test tensors. In section 3 we summarise the key results of the monopole and dipole stress-energy tensors. We highlight the Ellis and Dixon representations of the dipole.

In section 4 we examine the quadrupole in detail. In this section we use the Ellis approach. We give the gauge-like freedom of the components and the complicated change of coordinates which involve second derivatives and integrals over the worldline, similar to [27]. We use the adapted coordinates (14) and give the differential equations arising from the symmetry (9) and divergencelessness (10) of  $T^{\mu\nu}$ . We can now identify which components are algebraic, which satisfy ODEs and which are free. In subsection 4.1 we give an example of the free components in Minkowski spacetime as depicted in figure 1. As stated above, if there is a Killing vector field, there exists a corresponding conserved quantity. These are given in section 4.2. This includes a new interpretation of the conserved quantities corresponding the three Lorentz boosts.

In section 5 we use the limit of the dust stress-energy tensor as it is squeezed onto the worldline to construct a choice of constitutive relations to replace the free components with ODEs.

It is interesting to observe that, using deRham currents, multipoles can be defined without any additional structure on a manifold. In other words, it is not necessary to prescribe either a metric or a connection to define a general multipole. This is particularly useful if we wish to extend the notion of a general multipole tensor distribution to manifolds such as the tangent bundle which does not possess either metric or connection. However a connection is of course needed to define the divergencelessness condition (10). Although standard general relativity only considers four dimensional manifolds with a Lorentzian metric and the Levi-Civita connection, there is much interest [28, 29, 30] in non-metric compatible connections. Since the approach in the section does not specify the metric, all the results apply to a non-metric compatible connection. Other circumstances where it is advantageous not to prescribe the metric include transformation optics where one has two metrics, the gravitational metric and the optical metric. Another case is when considering varying the metric to derive Einstein's equations where it is necessary to know precisely the dependency of the various object on the metric.

Up to section 5, we have defined everything in terms of a coordinate system. However, it is useful to define the multipoles in a coordinate free manner. When we refer to “coordinate free”, the goal is to define all the mathematical objects and give the statement in all the theorems, without reference to a coordinate system. Coordinates may, however, be used in proofs. Thus although a vector  $V^\mu$  is invariant under transformations of the coordinates, it is defined by how the components change under coordinate transformations. By contrast defining a vector by its action on scalar fields follows this coordinate free goal. There are a number of advantages of this approach. First, complicated coordinate transformations are avoided. Second, when needed, one can derive the coordinate transformations more easily. Third, it is easier to present metric free calculations. Four and most importantly, it makes manifest which objects are physical and which are merely “coordinate objects”. This is especially relevant in the case of moments where expressions such as  $\int_{\mathcal{M}} f(x) x^{\mu_1} \cdots x^{\mu_k} d^4x$  are so dependent on the coordinates that there is no coordinate transformation expression.

In section 6 we detail this metric and coordinate free approach. By contrast to the Ellis approach, the Dixon approach contains more information about a multipole, namely how it splits into a monopole term, a dipole term, a quadrupole term and so on. This split, called here the **Dixon split**, is actually metric and coordinate independent and the details are given in section 6.7.

As noted in [27], without a metric, connection or coordinate system, it is still possible to define a pure electric dipole. In this article we call such a dipole a **semi-dipole**. We observe that the semi-dipole stress-energy consists of the displacement vector but not the spin. In section 6.6 we define the semi-dipole and semi-quadrupole stress-energy tensor.

We conclude, in section 7, and give some of the longer proofs in the appendix.



## Notation regarding derivatives

Given a coordinate system  $(x^0, \dots, x^3)$  then Greek indices range over the values  $\mu, \nu = 0, \dots, 3$ . We write the partial derivatives

$$\partial_\mu = \frac{\partial}{\partial x^\mu}. \quad (16)$$

In the case of the adapted coordinates  $(\sigma, z^1, z^2, z^3)$  obeying (14) we use both Greek indices  $\mu, \nu = 0, \dots, 3$  and Latin indices  $a, b = 1, 2, 3$ . In this case we have

$$\partial_0 = \frac{\partial}{\partial \sigma} \quad \text{and} \quad \partial_a = \frac{\partial}{\partial z^a}. \quad (17)$$

Thus, even if not stated explicitly, writing  $\partial_a$  implies we are referring to an adapted coordinates system.

Note that in both the adapted and non adapted case we use overdot to represent differentiation with respect to  $\sigma$ . In the non adapted coordinates this is only used for quantities, such as  $C^\mu(\sigma)$  and  $\dot{C}^\mu(\sigma)$  which are only defined on the worldline of the multipole. In the adapted coordinate cases it is synonymous with  $\partial_0$ .

When we have two non adapted coordinate systems  $(x^0, \dots, x^3)$  and  $(\hat{x}^0, \dots, \hat{x}^3)$  we use the hat on the index to indicate the hatted coordinate system. Thus

$$\partial_{\hat{\mu}} = \frac{\partial}{\partial \hat{x}^{\hat{\mu}}}. \quad (18)$$

Likewise for the adapted coordinate system  $(\hat{\sigma}, \hat{z}^1, \hat{z}^2, \hat{z}^3)$  we have

$$\partial_{\hat{0}} = \frac{\partial}{\partial \hat{\sigma}} \quad \text{and} \quad \partial_{\hat{a}} = \frac{\partial}{\partial \hat{z}^{\hat{a}}}. \quad (19)$$

## 2 Dixon's versus Ellis's approaches to multipoles

As stated in the introduction there are two standard approaches to writing down distributional multipoles, the Ellis and Dixon representations. These are equivalent as any multipoles of order  $k$  can be represented in both notations. The proof that a Dixon multipole is also an Ellis multipole is given in section 2.4. The converse is more difficult since it is necessary to extract the Dixon components. This involves the Dixon split and is given, for quadrupoles, in section 6.7.

Both the Ellis and Dixon approaches have advantages and disadvantages and these are listed in table 2.

### 2.1 The Ellis approach

One method [14] uses partial derivatives of the Dirac- $\delta$  function. Although Ellis principally defines it for the electric current  $\mathcal{J}^\mu$  it is easy to extend this for the stress-energy tensor. So a multipole of order  $k$  is given by

$$T^{\mu\nu} = \frac{1}{k!} \int_{\mathcal{I}} \zeta^{\mu\nu\rho_1 \dots \rho_k}(\sigma) \partial_{\rho_1} \dots \partial_{\rho_k} \delta^{(4)}(x - C(\sigma)) d\sigma, \quad (20)$$

where  $\zeta^{\mu\nu\rho_1 \dots \rho_k}(\sigma)$  are smooth functions of  $\sigma$ . Thus when acting on the test tensor  $\phi_{\mu\nu}$

$$\int_{\mathcal{M}} T^{\mu\nu} \phi_{\mu\nu} d^4x = (-1)^k \frac{1}{k!} \int_{\mathcal{I}} \zeta^{\mu\nu\rho_1 \dots \rho_k}(\sigma) (\partial_{\rho_1} \dots \partial_{\rho_k} \phi_{\mu\nu}) \Big|_{C(\sigma)} d\sigma. \quad (21)$$

In this article we will refer to this representation of a multipole as the **Ellis** representation.

<b>Ellis</b>	<b>Dixon</b>
Can be defined using coordinates.	Can be defined using coordinates.
Components are unique for adapted coordinates. In a general coordinate system they have a gauge-like freedom.	Components are unique.
For general coordinate transformation the components require higher derivatives and integrals.	Components transform as a tensor.
Do not require any additional structure. These can be defined without referring to a metric or additional vector field.	Requires the connection and the Dixon vector $N_\mu(\sigma)$ for the definition.
Contains all multipoles up to specific order.	Contains all multipoles up to specific order.
It is not possible to extract a multipole of a specific order without additional structure, for example an adapted coordinate system.	Easy to extract a multipoles of any order.
Can be easily defined in a coordinate free way using DeRham push forward.	The Dixon split can be defined in a coordinate free way, but this definition is complicated. It requires the DeRham push forward plus a non intuitive additional axiom, given in section 6.7.
The dipole can be written in the Ellis representation, which is consistent with the Mathisson-Papapetrou-Tulczyjew-Dixon equations.	The dipole can be written in the Dixon representation, which is consistent with the Mathisson-Papapetrou-Tulczyjew-Dixon equations.
There is no concept of the mass of the multipole	The monopole term may be used to define the mass, but in general it is not conserved.
There is no need of a Dixon vector.	There is a complicated formula for the components with respect to different $N^\mu(\sigma)$ . This will mix in multipoles of different orders.
One can construct a regular tensor field whose moments, up to $k$ , are the components of the distribution. The best method is using squeezed tensors that employ an adapted coordinate system. In principle it should be possible to reconstruct the original regular tensor using the Fourier transform but this has not been investigated.	One can construct a tensor field whose moments, up to $k$ , are the components of the distribution. This is by considering the fields on the transverse hyperspace constructed from the geodesic map of vectors orthogonal to $N^\mu(\sigma)$ . If all the moments are known then one can reconstruct an original distribution. This also requires certain assumptions about analyticity of Fourier transform.
There is a formula for extracting the components using test tensors in adapted coordinate.	In principle the components can be extracted using test tensors.

Table 2: Comparison between the Ellis and Dixon representations.

The symmetry of  $T^{\mu\nu}$  leads to

$$\zeta^{\mu\nu\rho_1\dots\rho_k} = \zeta^{\nu\mu\rho_1\dots\rho_k}. \quad (22)$$

In addition the partial derivatives commute and it is natural to demand that the components  $\zeta^{\nu\mu\rho_1\dots\rho_k}$  are symmetric. Thus we set

$$\zeta^{\mu\nu\rho_1\dots\rho_k} = \zeta^{\mu\nu(\rho_1\dots\rho_k)} \quad (23)$$

where the round brackets mean the symmetrisation of the indices,

$$\zeta^{\mu\nu(\rho_1\dots\rho_k)} = \frac{1}{k!} \sum_{\substack{\text{All permutations} \\ i_1\dots i_k}} \zeta^{\mu\nu\rho_{i_1}\dots\rho_{i_k}}. \quad (24)$$

We observe that every multipole of order  $k$  is also a multipole of order  $k+1$ . This follows since

$$\int_{\mathcal{I}} \zeta^{\mu\nu\rho_1\dots\rho_k} (\partial_{\rho_1} \dots \partial_{\rho_k} \phi_{\mu\nu})|_{C(\sigma)} d\sigma = \int_{\mathcal{I}} \zeta_{k+1}^{\mu\nu\rho_1\dots\rho_k} \dot{C}^\kappa (\partial_\kappa \partial_{\rho_1} \dots \partial_{\rho_k} \phi_{\mu\nu})|_{C(\sigma)} d\sigma \quad (25)$$

where

$$\dot{\zeta}_{k+1}^{\mu\nu\rho_1\dots\rho_k} = -\zeta^{\mu\nu\rho_1\dots\rho_k}. \quad (26)$$

One problem with the Ellis representation is that the  $\zeta^{\mu\nu\rho_1\dots\rho_k}$  are not unique. Examples of the gauge-like freedom possessed by  $\zeta^{\mu\nu\rho_1\dots\rho_k}$  are given in (52) and (71). This contrasts with the case when one chooses an adapted coordinate system below.

## 2.2 Adapted coordinates

In general expressions for multipoles in the Ellis representation are complicated. They greatly simplify if one chooses an adapted coordinate system as given by (14). In this coordinate system the integral over  $\mathcal{I}$  is evaluated and (20) becomes

$$T^{\mu\nu} = \sum_{r=0}^k \frac{1}{r!} \gamma^{\mu\nu a_1 \dots a_r 0 \dots 0}(\sigma) \partial_{a_1} \dots \partial_{a_r} \delta^{(3)}(\mathbf{z}), \quad (27)$$

where  $\mathbf{z} = (z^1, z^2, z^3)$ . The component  $\gamma^{\mu\nu a_1 \dots a_r 0 \dots 0}$  has  $(k-r)$  zero indices, so that  $\gamma^{\mu\nu a_1 \dots a_r 0 \dots 0}$  has  $2+k$  indices. The proof of (27) is given below after (30) once we have calculated  $T^{\mu\nu}$  on a test function and given the relationship between the  $\gamma^{\mu\nu a_1 \dots a_r 0 \dots 0}$  and  $\zeta^{\mu\nu\rho_1\dots\rho_k}$ . Since we only differentiate  $\delta^{(3)}(\mathbf{z})$  in the  $z^a$  direction, when acting on a test tensor

$$\int_{\mathcal{M}} T^{\mu\nu} \phi_{\mu\nu} d^4x = \sum_{r=0}^k \frac{(-1)^r}{r!} \int_{\mathcal{I}} d\sigma \gamma^{\mu\nu a_1 \dots a_r 0 \dots 0}(\sigma) (\partial_{a_1} \dots \partial_{a_r} \phi_{\mu\nu}). \quad (28)$$

*Proof.*

$$\begin{aligned} \int_{\mathcal{M}} T^{\mu\nu} \phi_{\mu\nu} d^4x &= \int_{\mathcal{I}} d\sigma \int_{\text{space}} d^3\mathbf{z} \left( \sum_{r=0}^k \frac{1}{r!} \gamma^{\mu\nu a_1 \dots a_r 0 \dots 0} \partial_{a_1} \dots \partial_{a_r} \delta^{(3)}(\mathbf{z}) \right) \phi_{\mu\nu} \\ &= \sum_{r=0}^k \frac{(-1)^r}{r!} \int_{\mathcal{I}} d\sigma \int_{\text{space}} d^3\mathbf{z} \gamma^{\mu\nu a_1 \dots a_r 0 \dots 0} (\partial_{a_1} \dots \partial_{a_r} \phi_{\mu\nu}) \delta^{(3)}(\mathbf{z}) \\ &= \sum_{r=0}^k \frac{(-1)^r}{r!} \int_{\mathcal{I}} d\sigma \gamma^{\mu\nu a_1 \dots a_r 0 \dots 0} (\partial_{a_1} \dots \partial_{a_r} \phi_{\mu\nu}). \end{aligned}$$

□

We still impose the symmetry conditions (22) and (23) on the  $\gamma$ 's so that

$$\gamma^{\mu\nu\rho_1\dots\rho_k} = \gamma^{\nu\mu\rho_1\dots\rho_k} = \gamma^{\mu\nu(\rho_1\dots\rho_k)}. \quad (29)$$

The relationship between the  $\gamma^{\mu\nu a_1\dots a_r 0\dots 0}$  and  $\zeta^{\mu\nu\rho_1\dots\rho_k}$  is given by comparing (21) and (28) for an adapted coordinate system

$$\gamma^{\mu\nu a_1\dots a_r 0\dots 0} = \frac{1}{(k-r)!} \partial_0^{k-r} \zeta^{\mu\nu a_1\dots a_r 0\dots 0}. \quad (30)$$

*Proof of (27) and (30).*

$$\begin{aligned} \int_{\mathcal{M}} T^{\mu\nu} \phi_{\mu\nu} d^4x &= (-1)^k \frac{1}{k!} \int_{\mathcal{I}} \zeta^{\mu\nu\rho_1\dots\rho_k} (\partial_{\rho_1} \dots \partial_{\rho_k} \phi_{\mu\nu}) \\ &= \sum_{r=0}^k (-1)^k \frac{1}{k!} \frac{k!}{r!(k-r)!} \int_{\mathcal{I}} \zeta^{\mu\nu a_1\dots a_r 0\dots 0} (\partial_{a_1} \dots \partial_{a_r} \partial_0^{k-r} \phi_{\mu\nu}) \\ &= \sum_{r=0}^k (-1)^r \frac{1}{r!(k-r)!} \int_{\mathcal{I}} (\partial_0^{k-r} \zeta^{\mu\nu a_1\dots a_r 0\dots 0}) (\partial_{a_1} \dots \partial_{a_r} \phi_{\mu\nu}). \end{aligned}$$

Hence comparing with (28) gives (30). From (28) we have (27).  $\square$

In an adapted coordinate system, the  $\gamma^{\mu\nu a_1\dots a_r 0\dots 0}$  are uniquely determined by the distribution. This is because we can extract  $\gamma^{\mu\nu a_1\dots a_r 0\dots 0}$  by acting on particular test functions

$$\gamma^{\mu\nu a_1\dots a_r 0\dots 0} = (-1)^r \lim_{\epsilon \rightarrow 0} \int_{\mathcal{M}} T^{\mu\nu} z^{a_1} \dots z^{a_r} \psi_{\epsilon,\sigma}(\sigma', \mathbf{z}) d\sigma' d^3\mathbf{z}, \quad (31)$$

where

$$\psi_{\epsilon,\sigma}(\sigma', \mathbf{z}) = \epsilon^{-1} \psi_1((\sigma - \sigma')/\epsilon) \psi_1((z^1)^2 + (z^2)^2 + (z^3)^2) \quad (32)$$

and  $\psi_1 : \mathbb{R} \rightarrow \mathbb{R}$  is a test function with  $\psi_1(0) = 1$ , is flat about 0 and  $\int \psi_1(\sigma) d\sigma = 1$ .

*Proof.* From (28) we have

$$\begin{aligned} (-1)^r \lim_{\epsilon \rightarrow 0} \int_{\mathcal{M}} T^{\mu\nu} z^{a_1} \dots z^{a_r} \psi_{\epsilon,\sigma}(\sigma', \mathbf{z}) d\sigma' d^3\mathbf{z} \\ &= \lim_{\epsilon \rightarrow 0} \sum_{s=0}^k \frac{(-1)^{s+r}}{s!} \int_{\mathcal{I}} d\sigma' \gamma^{\mu\nu b_1\dots b_s 0\dots 0}(\sigma') \partial_{b_1} \dots \partial_{b_s} \left( z^{a_1} \dots z^{a_r} \psi_{\epsilon,\sigma}(\sigma', \mathbf{z}) \right) \\ &= \lim_{\epsilon \rightarrow 0} \sum_{s=0}^k \frac{(-1)^{s+r}}{s!} \int_{\mathcal{I}} d\sigma' \gamma^{\mu\nu b_1\dots b_s 0\dots 0}(\sigma') (s! \delta_s^r) \psi_{\epsilon,\sigma}(\sigma', \mathbf{z}) \lim_{\epsilon \rightarrow 0} \\ &= \frac{1}{\epsilon} \lim_{\epsilon \rightarrow 0} \int_{\mathcal{I}} d\sigma' \gamma^{\mu\nu b_1\dots b_r 0\dots 0}(\sigma') \psi_1((\sigma - \sigma')/\epsilon) \psi_1(0) = \lim_{\epsilon \rightarrow 0} \int_{\mathcal{I}} d\sigma'' \gamma^{\mu\nu b_1\dots b_r 0\dots 0}(\sigma - \epsilon\sigma'') \psi_1(\sigma'') \\ &= \gamma^{\mu\nu b_1\dots b_r 0\dots 0}(\sigma) \int_{\mathcal{I}} d\sigma'' \psi_1(\sigma'') = \gamma^{\mu\nu b_1\dots b_r 0\dots 0}(\sigma). \end{aligned}$$

$\square$

The gauge-like freedom of the  $\zeta^{\mu\nu a_1\dots a_r 0\dots 0}$  in this case arises from the arbitrary constants when integrating (30) with respect to  $\sigma$ .

With respect to this coordinate system, one can partition the multipoles into a monopole, a pure dipole, a pure quadrupole and so on. However this is a coordinate dependent splitting and these terms will mix when changing the coordinate system. The coordinate transformation for quadrupoles is given in (83)-(85). Although they involve up to  $k$  derivatives of the coordinate transformation, they do not require any integrals.

## 2.3 Squeezed tensors

In an adapted coordinate system, one can construct a one parameter family of regular stress-energy tensor densities  $\mathcal{T}_\varepsilon^{\mu\nu}$  from a given stress-energy tensor  $\mathcal{T}^{\mu\nu}$ , such that in the weak limit  $\mathcal{T}_\varepsilon^{\mu\nu} \rightarrow T^{\mu\nu}$  at  $\varepsilon \rightarrow 0$  to order  $k$ . Since we are using adapted coordinates, we write  $(\sigma, \mathbf{z}) = (\sigma, z^1, z^2, z^3)$ . We set

$$\mathcal{T}_\varepsilon^{\mu\nu}(\sigma, \mathbf{z}) = \frac{1}{\varepsilon^3} \mathcal{T}^{\mu\nu}\left(\sigma, \frac{\mathbf{z}}{\varepsilon}\right). \quad (33)$$

We assume that  $\mathcal{T}^{\mu\nu}$  has compact support in the transverse planes, i.e. for each  $\sigma$ , there is a function  $R(\sigma)$  such that

$$\mathcal{T}^{\mu\nu}(\sigma, \mathbf{z}) = 0 \quad \text{for} \quad g_{ab} z^a z^b > R(\sigma). \quad (34)$$

This guarantees that all the moments, are finite.

This leads to

$$\mathcal{T}_\varepsilon^{\mu\nu}(\sigma, \mathbf{z}) = \gamma^{\mu\nu 0\dots 0} \delta^{(3)}(\mathbf{z}) + \varepsilon \gamma^{\mu\nu a 0\dots 0} \partial_a \delta^{(3)}(\mathbf{z}) + \varepsilon^2 \gamma^{\mu\nu ab 0\dots 0} \partial_a \partial_b \delta^{(3)}(\mathbf{z}) + \dots \quad (35)$$

where

$$\begin{aligned} \gamma^{\mu\nu 0\dots 0}(\sigma) &= \int_{\mathbb{R}^3} d^3 \mathbf{z} \mathcal{T}^{\mu\nu}(\sigma, \mathbf{z}), & \gamma^{\mu\nu a 0\dots 0}(\sigma) &= - \int_{\mathbb{R}^3} d^3 \mathbf{z} z^a \mathcal{T}^{\mu\nu}(\sigma, \mathbf{z}), \\ \gamma^{\mu\nu ab 0\dots 0}(\sigma) &= \int_{\mathbb{R}^3} d^3 \mathbf{z} z^a z^b \mathcal{T}^{\mu\nu}(\sigma, \mathbf{z}) \quad \text{etc.} \end{aligned} \quad (36)$$

*Proof.* This follows from setting  $w^a = z^a/\varepsilon$  and Taylor expanding around  $\varepsilon = 0$  we have

$$\begin{aligned} & \int_{\mathbb{R}^4} \mathcal{T}_\varepsilon^{\mu\nu}(\sigma, \mathbf{z}) \phi_{\mu\nu}(\sigma, \mathbf{z}) d\sigma d^3 z \\ &= \int_{\mathbb{R}} d\sigma \int_{\mathbb{R}^3} d^3 z \mathcal{T}_\varepsilon^{\mu\nu}(\sigma, \mathbf{z}) \phi_{\mu\nu}(\sigma, \mathbf{z}) \\ &= \int_{\mathbb{R}} d\sigma \int_{\mathbb{R}^3} d^3 z \frac{1}{\varepsilon^3} \mathcal{T}^{\mu\nu}\left(\sigma, \frac{\mathbf{z}}{\varepsilon}\right) \phi_{\mu\nu}(\sigma, \mathbf{z}) \\ &= \int_{\mathbb{R}} d\sigma \int_{\mathbb{R}^3} d^3 \mathbf{w} \mathcal{T}^{\mu\nu}(\sigma, \mathbf{w}) \phi_{\mu\nu}(\sigma, \varepsilon \mathbf{w}) \\ &= \int_{\mathbb{R}} d\sigma \int_{\mathbb{R}^3} d^3 \mathbf{w} \mathcal{T}^{\mu\nu}(\sigma, \mathbf{w}) \phi_{\mu\nu}(\sigma, \mathbf{0}) + \varepsilon \int_{\mathbb{R}} d\sigma \int_{\mathbb{R}^3} d^3 \mathbf{w} \mathcal{T}^{\mu\nu}(\sigma, \mathbf{w}) w^a (\partial_a \phi_{\mu\nu})(\sigma, \mathbf{0}) \\ &\quad + \varepsilon^2 \int_{\mathbb{R}} d\sigma \int_{\mathbb{R}^3} d^3 \mathbf{w} \mathcal{T}^{\mu\nu}(\sigma, \mathbf{w}) w^a w^b (\partial_a \partial_b \phi_{\mu\nu})(\sigma, \mathbf{0}) + \dots \\ &= \int_{\mathbb{R}} d\sigma \gamma^{\mu\nu 0\dots 0} \phi_{\mu\nu}|_{C(\sigma)} - \varepsilon \int_{\mathbb{R}} \gamma^{\mu\nu a 0\dots 0} d\sigma (\partial_a \phi_{\mu\nu})|_{C(\sigma)} + \varepsilon^2 \int_{\mathbb{R}} \gamma^{\mu\nu ab 0\dots 0} d\sigma (\partial_a \phi_{\mu\nu})|_{C(\sigma)} + \dots \end{aligned}$$

□

Thus there is an intimate relationship between the components of a distribution and the moments of a regular stress-energy tensor density. Here the zeroth order gives the monopole, the first order the dipole and so on. Again, the split between the different orders is with respect to the chosen adapted coordinate system and the different order terms will mix under a coordinate transformation.

## 2.4 The Dixon approach

The alternative approach, largely developed by Dixon uses the covariant derivative and a choice of a vector field  $N^\mu(\sigma)$  along the worldline  $C$ . This we will call the **Dixon vector**. This vector is required to be not orthogonal to the worldline  $C$ , i.e.

$$N_\mu \dot{C}^\mu \neq 0. \quad (37)$$

speed of light	[1]	$T^{\mu\nu}$	$[M L^{-3}]$
$dx^\mu$	$[L]$	test tensor $\phi_{\mu\nu}$	$[L^{-1}]$
$g_{\mu\nu}$	[1]	dipole displacement $X^\mu$	$[ML]$
$\dot{C}$	$[L^{-1}]$	dipole 3-momentum $P^\mu$	$[M]$
$\dot{C}^\mu$	[1]	dipole spin $S^{\mu\nu}$	$[ML]$
$\partial_\mu$	$[L^{-1}]$	$\xi^{\mu\nu\rho_1\dots\rho_k}$	$[M L^k]$
$\delta^{(4)}(x - C(\sigma))$	$[L^{-4}]$	$\xi^{\mu\nu\rho_1\dots\rho_k}$	$[M L^k]$
mass $m$	$[M]$	$\gamma^{\mu\nu a_1\dots a_k 0\dots 0}$	$[M L^k]$

Table 3: List of units for quantities, in terms of mass  $M$  and length  $L$ . The speed of light is 1.

As long as the worldline  $C$  is timelike, a natural choice of the Dixon vector is  $\dot{C}^\mu$ , i.e.  $N_\mu = g_{\mu\nu} \dot{C}^\mu$  but this is not the only choice. Having chosen  $N_\mu$ , the **Dixon** representation of a multipole is given [11, Equation (1.9)][12, Equation (4.18), (7.4), (7.5)] by

$$T^{\mu\nu} = \sum_{r=0}^k \frac{1}{r!} \nabla_{\rho_1} \dots \nabla_{\rho_r} \int_{\mathcal{I}} \xi^{\mu\nu\rho_1\dots\rho_r}(\sigma) \delta^{(4)}(x - C(\sigma)) d\sigma, \quad (38)$$

where we demand that the components  $\xi^{\mu\nu\rho_1\dots\rho_k}$  are orthogonal to the vector  $N^\mu$ ,

$$N_{\rho_j} \xi^{\mu\nu\rho_1\dots\rho_k} = 0 \quad (39)$$

for  $j = 1, \dots, k$ . The covariant derivatives do not commute. However, they give rise to curvature terms and lower the number of derivatives. We therefore make the minimal choice and impose  $\xi^{\mu\nu\rho_1\dots\rho_k}$  are symmetric in the relevant indices.

$$\xi^{\mu\nu\rho_1\dots\rho_k} = \xi^{\mu\nu(\rho_1\dots\rho_k)}. \quad (40)$$

Since  $T^{\mu\nu}$  is a tensor density this enables us to throw the covariant derivative over onto the test tensor  $\phi_{\mu\nu}$ , giving

$$\int_{\mathcal{M}} T^{\mu\nu} \phi_{\mu\nu} d^4x = \sum_{r=0}^k (-1)^r \frac{1}{r!} \int_{\mathcal{I}} \xi^{\mu\nu\rho_1\dots\rho_r}(\sigma) (\nabla_{\rho_1} \dots \nabla_{\rho_r} \phi_{\mu\nu})|_{C(\sigma)} d\sigma, \quad (41)$$

This follows since if  $v^\mu$  is a vector density of weight 1 then from (8)  $\nabla_\mu v^\mu = \partial_\mu v^\mu$ .

All Dixon multipoles of order  $k$  are also Ellis multipoles of order  $k$ .

*Proof.* Expanding out the left hand side of (41) replaces the covariant derivatives with partial derivatives and Christoffel symbols. The resulting expression is the integral of sum of up to  $k$  partial derivatives of  $\phi_{\mu\nu}$ . Using (25) and (26) one can express all the terms to have precisely  $k$  derivatives, and hence one has (21) for the appropriate  $\xi^{\mu\nu\rho_1\dots\rho_k}$ .  $\square$

From (38) we can use the Dixon vector to perform the **Dixon split** in order to take an arbitrary  $k$ th order multipole and split it into a monopole part, a dipole part and so on. Thus we set

$$T^{\mu\nu} = \sum_{r=0}^k T_{(r)}^{\mu\nu} \quad \text{where} \quad T_{(r)}^{\mu\nu} = \frac{1}{r!} \nabla_{\rho_1} \dots \nabla_{\rho_r} \int_{\mathcal{I}} \xi^{\mu\nu\rho_1\dots\rho_r}(\sigma) \delta^{(4)}(x - C(\sigma)) d\sigma. \quad (42)$$

In section 6.7 we present a coordinate free approach to performing this split. This is presented for quadrupoles, although the procedure can be extended. This is necessary to show that all Ellis multipole are also Dixon multipoles. It also gives a method to derive the relationship between the Dixon components with respect to different Dixon vectors.



### 3 Summary of the monopole and dipole stress-energy tensors.

#### 3.1 The monopole

From (20) with  $k = 0$  we have the gravitational monopole

$$T^{\mu\nu} = \int_{\mathcal{I}} \zeta^{\mu\nu} \delta(x - C(\sigma)) d\sigma. \quad (43)$$

The requirement to be a stress-energy tensor (9),(10) implies that  $C$  satisfies the pre-geodesic equation

$$\frac{D\dot{C}^\mu}{d\sigma} = \kappa_{\text{pre}}(\sigma) \dot{C}^\mu \quad (44)$$

and

$$T^{\mu\nu} = \int_{\mathcal{I}} m_{\text{pre}}(\sigma) \dot{C}^\mu \dot{C}^\nu \delta(x - C(\sigma)) d\sigma \quad (45)$$

where

$$\dot{m}_{\text{pre}} + \kappa_{\text{pre}} m_{\text{pre}} = 0. \quad (46)$$

Here  $\frac{D}{d\sigma}$  represents the covariant derivative along the worldline, i.e.

$$\frac{DX^\mu}{d\sigma} = \dot{X}^\mu + \Gamma_{\nu\rho}^\mu X^\nu \dot{C}^\rho \quad (47)$$

and the overdot refers to differentiation with respect to  $\sigma$ . If  $\sigma$  is proper times so that

$$g_{\mu\nu} \dot{C}^\mu \dot{C}^\nu = -1. \quad (48)$$

then  $\kappa_{\text{pre}} = 0$  and (44) gives the geodesic equation

$$\frac{D\dot{C}^\mu}{d\sigma} = 0. \quad (49)$$

In this case we replace  $m_{\text{pre}}$  with  $m$  in (45). If  $m > 0$  then we can associate it with the mass of the source. Thus (45) becomes

$$T^{\mu\nu} = m \int_{\mathcal{I}} \dot{C}^\mu \dot{C}^\nu \delta(x - C(\sigma)) d\sigma. \quad (50)$$

Thus there remains just one ODE for the remaining component, namely  $\dot{m} = 0$ . There are no additional free components. See table 1. However as stated in the introduction, we do not impose the geodesic equation for the subsequent analysis of the dipole and quadrupoles stress-energy tensors.

#### 3.2 The dipole

Setting  $k = 1$  in (20) gives the dipole

$$T^{\mu\nu} = \int_{\mathcal{I}} \zeta^{\mu\nu\rho} \partial_\rho \delta(x - C(\sigma)) d\sigma, \quad (51)$$

where the symmetry condition (9) implies  $\zeta^{\mu\nu\rho} = \zeta^{\nu\mu\rho}$ . We observe that, whereas the components  $\zeta^{\mu\nu\rho}$  uniquely specify  $T^{\mu\nu}$ , the converse is not true. That is, given  $T^{\mu\nu}$  the gauge-like freedom in  $\zeta^{\mu\nu\rho}$  is given by

$$\zeta^{\mu\nu\rho} \rightarrow \zeta^{\mu\nu\rho} + M^{\mu\nu} \dot{C}^\rho, \quad (52)$$

where  $M^{\mu\nu} = M^{\nu\mu}$  are any set of constants, i.e. independent of  $\sigma$ .

*Proof.* Substituting (52) into (51) we have

$$\begin{aligned} T^{\mu\nu} &\rightarrow T^{\mu\nu} + \int_{\mathcal{I}} M^{\mu\nu} \dot{C}^\rho \partial_\rho \delta(x - C(\sigma)) d\sigma = T^{\mu\nu} + \int_{\mathcal{I}} M^{\mu\nu} \frac{d}{d\sigma} \delta(x - C(\sigma)) d\sigma \\ &= T^{\mu\nu} + \int_{\mathcal{I}} \frac{d}{d\sigma} (M^{\mu\nu} \delta(x - C(\sigma))) d\sigma = T^{\mu\nu}. \end{aligned}$$

Thus (52) is a gauge-like freedom. To show it is the maximum freedom, we work in adaptive coordinates. It is clear that the freedom (52) is precisely equivalent to the freedom to choose  $\zeta^{\mu\nu 0}$  given  $\gamma^{\mu\nu 0}$ . For details of why this is the maximum gauge-like freedom see proofs number 2 and 3 in the appendix about the gauge-like freedom of the quadrupole (71).  $\square$

In addition the  $\zeta^{\mu\nu\rho}$  are not tensorial quantities but have a coordinate transformation which involves derivatives of the Jacobian matrix and an integral. Given two coordinate systems  $(x^0, \dots, x^3)$  and  $(\hat{x}^0, \dots, \hat{x}^3)$  then

$$\hat{\zeta}^{\hat{\mu}\hat{\nu}\hat{\rho}} = J_\mu^{\hat{\mu}} J_\nu^{\hat{\nu}} J_\rho^{\hat{\rho}} \zeta^{\mu\nu\rho} - \hat{C}^{\hat{\rho}} \int^\sigma \partial_\rho (J_\mu^{\hat{\mu}} J_\nu^{\hat{\nu}}) \zeta^{\mu\nu\rho} d\sigma', \quad (53)$$

where

$$J_\mu^{\hat{\mu}} = \frac{\partial \hat{x}^{\hat{\mu}}}{\partial x^\mu}. \quad (54)$$

*Proof of (53).* Observe that

$$\begin{aligned} \int_{\mathcal{I}} \zeta^{\mu\nu\rho} \partial_\rho (J_\mu^{\hat{\mu}} J_\nu^{\hat{\nu}}) \hat{\phi}_{\hat{\mu}\hat{\nu}} d\sigma &= \int_{\mathcal{I}} \zeta^{\mu\nu\rho} \partial_\rho (J_\mu^{\hat{\mu}} J_\nu^{\hat{\nu}}) \left( \int^\sigma \frac{d\hat{\phi}_{\hat{\mu}\hat{\nu}}}{d\sigma'} d\sigma' \right) d\sigma \\ &= \int_{\mathcal{I}} \zeta^{\mu\nu\rho} \partial_\rho (J_\mu^{\hat{\mu}} J_\nu^{\hat{\nu}}) \left( \int^\sigma \hat{C}^{\hat{\rho}} \partial_{\hat{\rho}} \hat{\phi}_{\hat{\mu}\hat{\nu}} d\sigma' \right) d\sigma = \int_{\mathcal{I}} \left( \int^\sigma \zeta^{\mu\nu\rho} \partial_\rho (J_\mu^{\hat{\mu}} J_\nu^{\hat{\nu}}) d\sigma' \right) \hat{C}^{\hat{\rho}} \partial_{\hat{\rho}} \hat{\phi}_{\hat{\mu}\hat{\nu}} d\sigma. \end{aligned}$$

Hence using (51) we have

$$\begin{aligned} \int_{\mathcal{I}} \hat{\zeta}^{\hat{\mu}\hat{\nu}\hat{\rho}} (\partial_{\hat{\rho}} \hat{\phi}_{\hat{\mu}\hat{\nu}}) d\sigma &= \int_{\mathbb{R}^4} \hat{T}^{\hat{\mu}\hat{\nu}} \hat{\phi}_{\hat{\mu}\hat{\nu}} d^4 \hat{x} = \int_{\mathbb{R}^4} T^{\mu\nu} \phi_{\mu\nu} d^4 x = \int_{\mathcal{I}} \zeta^{\mu\nu\rho} (\partial_\rho \phi_{\mu\nu}) d\sigma \\ &= \int_{\mathcal{I}} \zeta^{\mu\nu\rho} \partial_\rho (J_\mu^{\hat{\mu}} J_\nu^{\hat{\nu}} \hat{\phi}_{\hat{\mu}\hat{\nu}}) d\sigma = \int_{\mathcal{I}} \zeta^{\mu\nu\rho} \left( \partial_\rho (J_\mu^{\hat{\mu}} J_\nu^{\hat{\nu}}) \hat{\phi}_{\hat{\mu}\hat{\nu}} + J_\mu^{\hat{\mu}} J_\nu^{\hat{\nu}} \partial_\rho \hat{\phi}_{\hat{\mu}\hat{\nu}} \right) d\sigma \\ &= \int_{\mathcal{I}} \left( \left( \hat{C}^{\hat{\rho}} \int^\sigma \zeta^{\mu\nu\rho} \partial_\rho (J_\mu^{\hat{\mu}} J_\nu^{\hat{\nu}}) d\sigma' \right) + J_\mu^{\hat{\mu}} J_\nu^{\hat{\nu}} J_\rho^{\hat{\rho}} \zeta^{\mu\nu\rho} \right) \partial_{\hat{\rho}} \hat{\phi}_{\hat{\mu}\hat{\nu}} d\sigma. \end{aligned}$$

Hence (53).  $\square$

Here the freedom to choose the arbitrary constant of integration in (53) is equivalent to the gauge-like freedom (52).

*Proof.* Consider the cases where the limits of the integral in (53) are  $\int_{\sigma_0}^\sigma$  and  $\int_{\sigma_1}^\sigma$ . Then the difference between two expressions for  $\hat{\zeta}^{\hat{\mu}\hat{\nu}\hat{\rho}}$  is

$$\hat{\zeta}^{\hat{\mu}\hat{\nu}\hat{\rho}} \rightarrow \hat{\zeta}^{\hat{\mu}\hat{\nu}\hat{\rho}} + M_3^{\hat{\mu}\hat{\nu}} \hat{C}^{\hat{\rho}} \quad \text{where} \quad M_3^{\hat{\mu}\hat{\nu}} = \int_{\sigma_1}^{\sigma_0} \partial_\rho (J_\mu^{\hat{\mu}} J_\nu^{\hat{\nu}}) \zeta^{\mu\nu\rho} d\sigma',$$

hence the gauge-like freedom in (52).  $\square$

In adapted coordinates (14) then (27) and (28) become

$$T^{\mu\nu} = \gamma^{\mu\nu 0} \delta^{(3)}(\mathbf{z}) + \gamma^{\mu\nu a} \partial_a \delta^{(3)}(\mathbf{z}) \quad \text{where} \quad \gamma^{\mu\nu 0} = \dot{\zeta}^{\mu\nu 0} \quad \text{and} \quad \gamma^{\mu\nu a} = \zeta^{\mu\nu a}. \quad (55)$$

Fortunately for the dipole the requirements (9) and (10) restrict the components  $\zeta^{\mu\nu\rho}$  so much that  $T^{\mu\nu}$  can be written solely in terms of tensor quantities

$$T^{\mu\nu} = \int_{\mathcal{I}} \hat{P}^{(\mu} \dot{C}^{\nu)} \delta(x - C(\sigma)) d\sigma + \nabla_\rho \int_{\mathcal{I}} \hat{S}^{\rho(\mu} \dot{C}^{\nu)} \delta(x - C(\sigma)) d\sigma, \quad (56)$$

where  $\hat{P}^\mu$  and  $\hat{S}^{\mu\nu} + \hat{S}^{\nu\mu} = 0$  satisfy the Mathisson-Papapetrou-Tulczyjew-Dixon equations

$$\frac{D\hat{S}^{\mu\nu}}{d\sigma} = \hat{P}^\nu \dot{C}^\mu - \hat{P}^\mu \dot{C}^\nu \quad \text{and} \quad \frac{D\hat{P}^\mu}{d\sigma} = \frac{1}{2} R^\mu{}_{\nu\rho\kappa} \dot{C}^\nu \hat{S}^{\kappa\rho}. \quad (57)$$

Given a vector  $N^\rho$  such that  $N_\rho \hat{S}^{\rho(\mu} \dot{C}^{\nu)} = 0$ , then from (39) we can interpret (56) as the Dixon representation of a dipole with Dixon vector  $N^\rho$ .

Clearly we can replace the covariant derivatives with partial derivatives and Christoffel symbols to give the representation of the dipole

$$T^{\mu\nu} = \int_{\mathcal{I}} \left( \hat{P}^{(\mu} \dot{C}^{\nu)} + \hat{S}^{\rho(\nu} \Gamma^\mu{}_{\rho\kappa} \dot{C}^{\kappa)} \right) \delta(x - C(\sigma)) d\sigma + \int_{\mathcal{I}} \hat{S}^{\rho(\mu} \dot{C}^{\nu)} \partial_\rho \delta(x - C(\sigma)) d\sigma. \quad (58)$$

However this is not the Ellis representation. To translate (58) into the Ellis representation (51) we set

$$\zeta^{\mu\nu\rho} = \hat{S}^{\rho(\mu} \dot{C}^{\nu)} + \dot{C}^\rho \int^\sigma \left( \hat{P}^{(\mu} \dot{C}^{\nu)} + \hat{S}^{\alpha(\nu} \Gamma^\mu{}_{\alpha\kappa} \dot{C}^{\kappa)} \right) d\sigma'. \quad (59)$$

In the adapted coordinates (55) we have

$$\gamma^{\mu\nu 0} = \hat{P}^{(\mu} \delta_0^{\nu)} + \hat{S}^{\rho(\nu} \Gamma^\mu{}_{\rho 0} + \partial_0 (\hat{S}^{0(\mu} \delta_0^{\nu)}) \quad \text{and} \quad \gamma^{\mu\nu a} = \hat{S}^{a(\mu} \delta_0^{\nu)}. \quad (60)$$

Let  $K^\mu$  be Killing vector

$$\nabla_\mu K_\nu + \nabla_\nu K_\mu = 0 \quad (61)$$

and let

$$\mathcal{Q}_K = \gamma^{\mu 00} K_\mu - \gamma^{\mu 0a} \partial_a K_\mu. \quad (62)$$

We show below that  $\mathcal{Q}_K$  is a conserved quantity. From (60) we have

$$\mathcal{Q}_K = \hat{P}^\mu K_\mu + \frac{1}{2} \hat{S}^{\mu\nu} \nabla_\nu K_\mu. \quad (63)$$

*Proof that (62) and (63) are equivalent.* From (57) we have

$$\begin{aligned} \partial_0 \hat{S}^{\mu\nu} &= \frac{D\hat{S}^{\mu\nu}}{d\sigma} - \Gamma_{0\rho}^\mu \hat{S}^{\rho\nu} - \Gamma_{0\rho}^\nu \hat{S}^{\mu\rho} = \hat{P}^\nu \dot{C}^\mu - \hat{P}^\mu \dot{C}^\nu - \Gamma_{0\rho}^\mu \hat{S}^{\rho\nu} - \Gamma_{0\rho}^\nu \hat{S}^{\mu\rho} \\ &= \hat{P}^\nu \delta_0^\mu - \hat{P}^\mu \delta_0^\nu - \Gamma_{0\rho}^\mu \hat{S}^{\rho\nu} - \Gamma_{0\rho}^\nu \hat{S}^{\mu\rho} \end{aligned}$$

so

$$\partial_0 \hat{S}^{0\mu} = \hat{P}^\mu - \hat{P}^0 \delta_0^\mu - \Gamma_{0\rho}^0 \hat{S}^{\rho\mu} - \Gamma_{0\rho}^\mu \hat{S}^{0\rho}.$$

From (60) we have

$$\begin{aligned}
\gamma^{\mu 00} &= \hat{P}^{(\mu} \dot{C}^{0)} + \hat{S}^{\rho(0} \Gamma^{\mu)}_{\rho\kappa} \dot{C}^\kappa + \partial_0(\hat{S}^{0(\mu} \dot{C}^{0)}) \\
&= \frac{1}{2}(\hat{P}^\mu + \hat{P}^0 \delta_0^\mu + \hat{S}^{\rho 0} \Gamma^\mu_{\rho 0} + \hat{S}^{\rho\mu} \Gamma^0_{\rho 0} + \partial_0(\hat{S}^{0\mu})) \\
&= \frac{1}{2}(\hat{P}^\mu + \hat{P}^0 \delta_0^\mu + \hat{S}^{\rho 0} \Gamma^\mu_{\rho 0} + \hat{S}^{\rho\mu} \Gamma^0_{\rho 0} + \hat{P}^\mu - \hat{P}^0 \delta_0^\mu - \Gamma^0_{0\rho} \hat{S}^{\rho\mu} - \Gamma^0_{0\rho} \hat{S}^{0\rho}) \\
&= \hat{P}^\mu + \hat{S}^{\rho 0} \Gamma^\mu_{\rho 0}
\end{aligned}$$

and

$$\gamma^{\mu 0a} = \frac{1}{2} \hat{S}^{a\mu} + \frac{1}{2} \hat{S}^{a0} \delta_0^\mu.$$

From (61) we have

$$0 = \nabla_a K_0 + \nabla_0 K_a = \partial_a K_0 + \partial_0 K_a - 2\Gamma_{a0}^\mu K_\mu.$$

Hence from (62) we have

$$\begin{aligned}
\mathcal{Q}_K &= \gamma^{\mu 00} K_\mu - \gamma^{\mu 0a} \partial_a K_\mu = (\hat{P}^\mu + \hat{S}^{\rho 0} \Gamma^\mu_{\rho 0}) K_\mu - \frac{1}{2}(\hat{S}^{a\mu} + \hat{S}^{a0} \delta_0^\mu) \partial_a K_\mu \\
&= \hat{P}^\mu K_\mu + \hat{S}^{\rho 0} \Gamma^\mu_{\rho 0} K_\mu - \frac{1}{2} \hat{S}^{a\mu} \partial_a K_\mu - \frac{1}{2} \hat{S}^{a0} \partial_a K_0 \\
&= \hat{P}^\mu K_\mu + \hat{S}^{\rho 0} \Gamma^\mu_{\rho 0} K_\mu - \frac{1}{2} \hat{S}^{a\mu} \partial_a K_\mu + \frac{1}{2} \hat{S}^{a0} \partial_0 K_a - \hat{S}^{a0} \Gamma_{a0}^\mu K_\mu \\
&= \hat{P}^\mu K_\mu + \frac{1}{2} \hat{S}^{\mu a} \partial_a K_\mu + \frac{1}{2} \hat{S}^{\mu 0} \partial_0 K_\mu = \hat{P}^\mu K_\mu + \frac{1}{2} \hat{S}^{\mu\nu} \partial_\nu K_\mu = \hat{P}^\mu K_\mu + \frac{1}{2} \hat{S}^{\mu\nu} \nabla_\nu K_\mu.
\end{aligned}$$

□

*Proof that  $\mathcal{Q}_K$  in (63) is conserved.* Since  $K_\mu$  is Killing we have

$$\nabla_\mu \nabla_\nu K_\rho = R^\kappa_{\mu\nu\rho} K_\kappa.$$

From (63) and (57) we have

$$\begin{aligned}
\dot{\mathcal{Q}}_K &= \frac{D\mathcal{Q}_K}{d\sigma} = \frac{D\hat{P}^\mu}{d\sigma} K_\mu + \hat{P}^\mu \dot{C}^\nu \nabla_\nu K_\mu + \frac{1}{2} \frac{D\hat{S}^{\mu\nu}}{d\sigma} \nabla_\nu K_\mu + \frac{1}{2} \hat{S}^{\mu\nu} \dot{C}^\rho \nabla_\rho \nabla_\nu K_\mu \\
&= \frac{1}{2} R^\mu_{\nu\rho\kappa} \dot{C}^\nu \hat{S}^{\kappa\rho} K_\mu + \hat{P}^\mu \dot{C}^\nu \nabla_\nu K_\mu + \frac{1}{2} (\hat{P}^\nu \dot{C}^\mu - \hat{P}^\mu \dot{C}^\nu) \nabla_\nu K_\mu + \frac{1}{2} \hat{S}^{\mu\nu} \dot{C}^\rho \nabla_\rho \nabla_\nu K_\mu \\
&= \frac{1}{2} R^\mu_{\nu\rho\kappa} \dot{C}^\nu \hat{S}^{\kappa\rho} K_\mu + \frac{1}{2} \hat{S}^{\mu\nu} \dot{C}^\rho R^\kappa_{\rho\nu\mu} K_\kappa = 0.
\end{aligned}$$

□

The situation is simplified in the case when  $C$  is a geodesic. In this case we can use the Dixon representation with  $N^\mu = \dot{C}^\mu$ .

$$T^{\mu\nu} = \int_{\mathcal{I}} \left( m \dot{C}^\mu \dot{C}^\nu + P^{(\mu} \dot{C}^{\nu)} \right) \delta(x - C(\sigma)) d\sigma + \nabla_\rho \int_{\mathcal{I}} \left( X^\rho \dot{C}^\mu \dot{C}^\nu + S^{\rho(\mu} \dot{C}^{\nu)} \right) \delta(x - C(\sigma)) d\sigma, \quad (64)$$

where

$$\hat{S}^{\mu\nu} = S^{\mu\nu} - X^\mu \dot{C}^\nu + X^\nu \dot{C}^\mu \quad \text{and} \quad \hat{P}^\mu = P^\mu + m \dot{C}^\mu. \quad (65)$$

*Proof of the relationship between (66) and (57).* In this proof we refer to the two equations in (57) as (57.1) and (57.2) and likewise for (66.1) to (66.4). From (65) and (49) we have

$$\begin{aligned}
&\frac{D\hat{S}^{\mu\nu}}{d\sigma} - \hat{P}^\nu \dot{C}^\mu + \hat{P}^\mu \dot{C}^\nu \\
&= \frac{DS^{\mu\nu}}{d\sigma} - \frac{DX^\mu}{d\sigma} \dot{C}^\nu - X^\mu \frac{D\dot{C}^\nu}{d\sigma} + \frac{DX^\nu}{d\sigma} \dot{C}^\mu + X^\nu \frac{D\dot{C}^\mu}{d\sigma} - (P^\nu + m \dot{C}^\nu) \dot{C}^\mu + (P^\mu + m \dot{C}^\mu) \dot{C}^\nu \\
&= \frac{DS^{\mu\nu}}{d\sigma} - \left( \frac{DX^\mu}{d\sigma} - P^\mu \right) \dot{C}^\nu + \left( \frac{DX^\nu}{d\sigma} - P^\nu \right) \dot{C}^\mu.
\end{aligned}$$

Hence (66.2) and (66.4) imply (57.1). By contrast from (57.1) we can project out (66.2) and (66.4) using  $\dot{C}_\mu$ .

Likewise from (65) we have

$$\begin{aligned} \frac{D\hat{P}^\mu}{d\sigma} - \frac{1}{2}R^\mu{}_{\nu\rho\kappa}\dot{C}^\nu\hat{S}^{\kappa\rho} &= \frac{DP^\mu}{d\sigma} + \frac{Dm}{d\sigma}\dot{C}^\mu + m\frac{D\dot{C}^\mu}{d\sigma} - \frac{1}{2}R^\mu{}_{\nu\rho\kappa}\dot{C}^\nu(S^{\kappa\rho} - X^\kappa\dot{C}^\rho + X^\rho\dot{C}^\kappa) \\ &= \frac{DP^\mu}{d\sigma} + \dot{m}\dot{C}^\mu - \frac{1}{2}R^\mu{}_{\nu\rho\kappa}\dot{C}^\nu S^{\kappa\rho} - R^\mu{}_{\nu\rho\kappa}\dot{C}^\nu X^\kappa\dot{C}^\rho. \end{aligned}$$

Thus (66.1) and (66.3) imply (57.2). By contrast from (57.2) we can project out (66.1) and (66.3) using  $\dot{C}_\mu$ .  $\square$

These quantities have intuitive meaning. See Table 3 for the units associated with each component.

- The rest mass  $m$ .
- A displacement vector  $X^\mu$  with  $X_\mu\dot{C}^\mu = 0$ .
- The rate of change of the displacement vector  $P^\mu$  with  $P_\mu\dot{C}^\mu = 0$ .
- A spin tensor  $S^{\mu\nu}$  with  $S^{\mu\nu} + S^{\nu\mu} = 0$  and  $\dot{C}_\mu S^{\mu\nu} = 0$ .

These satisfy

$$\dot{m} = 0, \quad \frac{DX^\mu}{d\sigma} = P^\mu, \quad \frac{DP^\mu}{d\sigma} = \frac{1}{2}R^\mu{}_{\nu\rho\kappa}\dot{C}^\nu S^{\kappa\rho} + R^\mu{}_{\nu\rho\kappa}\dot{C}^\nu\dot{C}^\rho X^\kappa, \quad \frac{DS^{\mu\nu}}{d\sigma} = 0. \quad (66)$$

Counting the number of components we see there are 10 ODEs, which completely determine the dynamical evolution of the dipole components on the prescribed worldline. There are no additional free components.

As we see below, the same situation does not occur for the quadrupoles. The conditions (9) and (10) do not completely determine the dynamics of all the components, it is not possible to write all the components in terms of tensors, and it is not possible to associate the concept of mass with the quadrupole.

A particular case of the dipole is when  $S^{\mu\nu} = 0$ , which is compatible with its dynamic equation (66). We call this case a **semi-dipole**. The notion of semi-dipoles and semi-quadrupoles is purely geometric and is addressed in section 6.6.

## 4 The quadrupole stress-energy tensor.

Setting  $k = 2$  in (20) gives the formula for a quadrupole,

$$T^{\mu\nu} = \frac{1}{2} \int_{\mathcal{I}} \zeta^{\mu\nu\rho\kappa}(\sigma) \partial_\rho \partial_\kappa \delta(x - C(\sigma)) d\sigma, \quad (67)$$

so that the action on the test tensor  $\phi_{\mu\nu}$  is given by

$$\int_{\mathbb{R}^4} T^{\mu\nu} \phi_{\mu\nu} d^4x = \frac{1}{2} \int_{\mathcal{I}} \zeta^{\mu\nu\rho\kappa}(\sigma) (\partial_\rho \partial_\kappa \phi_{\mu\nu})|_{C(\sigma)} d\sigma. \quad (68)$$

From (9) we impose

$$\zeta^{\mu\nu\rho\kappa} = \zeta^{\nu\mu\rho\kappa}, \quad (69)$$

and due to the commutation of partial derivatives we also set

$$\zeta^{\mu\nu\rho\kappa} = \zeta^{\mu\nu\kappa\rho}. \quad (70)$$

As in the dipole case, the  $\zeta^{\mu\nu\rho\kappa}$  are not uniquely specified by the  $T^{\mu\nu}$ , with the gauge-like freedom

$$\zeta^{\mu\nu\rho\kappa} \rightarrow \zeta^{\mu\nu\rho\kappa} + M_2^{\nu\mu} \dot{C}^{(\rho} C^{\kappa)} + M_3^{\mu\nu(\rho} \dot{C}^{\kappa)} \quad (71)$$

where  $M_2^{\nu\mu}$  and  $M_3^{\mu\nu\rho}$  are arbitrary constants.

*Proof.* Similar to the proof of (52), we have

$$\begin{aligned}
\int_{\mathcal{I}} M_2^{\mu\nu} \dot{C}^{(\rho} C^{\kappa)} \partial_\rho \partial_\kappa \delta(x - C(\sigma)) d\sigma &= \int_{\mathcal{I}} M_2^{\mu\nu} C^\kappa \dot{C}^\rho \partial_\rho \partial_\kappa \delta(x - C(\sigma)) d\sigma \\
&= M_2^{\mu\nu} \int_{\mathcal{I}} C^\kappa \frac{d}{d\sigma} \left( \partial_\kappa \delta(x - C(\sigma)) \right) d\sigma \\
&= M_2^{\mu\nu} \int_{\mathcal{I}} \frac{d}{d\sigma} \left( C^\kappa \partial_\kappa \delta(x - C(\sigma)) \right) d\sigma - M_2^{\mu\nu} \int_{\mathcal{I}} \dot{C}^\kappa \partial_\kappa \delta(x - C(\sigma)) d\sigma \\
&= -M_2^{\mu\nu} \int_{\mathcal{I}} \frac{d}{d\sigma} \delta(x - C(\sigma)) d\sigma = 0.
\end{aligned}$$

and

$$\begin{aligned}
\int_{\mathcal{I}} M_3^{\mu\nu(\kappa} \dot{C}^{\rho)} \partial_\rho \partial_\kappa \delta(x - C(\sigma)) d\sigma &= \int_{\mathcal{I}} M_3^{\mu\nu\kappa} \dot{C}^\rho \partial_\rho \partial_\kappa \delta(x - C(\sigma)) d\sigma \\
&= M_3^{\mu\nu\kappa} \int_{\mathcal{I}} \frac{d}{d\sigma} \left( \partial_\kappa \delta(x - C(\sigma)) \right) d\sigma = 0.
\end{aligned}$$

To see why this incorporates all the gauge-like freedom we use the adapted coordinates system. Therefore this proof (proof number 4) is given in the appendix.  $\square$

As in [27], under change of coordinate  $(x^0, \dots, x^3)$  to  $(\hat{x}^0, \dots, \hat{x}^3)$  we have have a complicated transformation involving derivatives and integrals

$$\begin{aligned}
\hat{\zeta}^{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\kappa}} &= \zeta^{\mu\nu\rho\kappa} \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} J_\rho^{\hat{\rho}} J_\kappa^{\hat{\kappa}} - \frac{1}{2} \hat{C}^{\hat{\rho}} \int^\sigma \zeta^{\mu\nu\rho\kappa} \left( \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} (\partial_\rho J_\kappa^{\hat{\kappa}}) + 2 \partial_\rho (\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}}) J_\kappa^{\hat{\kappa}} \right) d\sigma' \\
&\quad - \frac{1}{2} \hat{C}^{\hat{\kappa}} \int^\sigma \zeta^{\mu\nu\rho\kappa} \left( \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} (\partial_\rho J_\kappa^{\hat{\rho}}) + 2 \partial_\rho (\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}}) J_\kappa^{\hat{\rho}} \right) d\sigma' \\
&\quad + \frac{1}{2} \hat{C}^{\hat{\kappa}} \int^\sigma \hat{C}^{\hat{\rho}} \int^{\sigma'} \zeta^{\mu\nu\rho\kappa} \partial_\rho \partial_\kappa (\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}}) d\sigma'' d\sigma' + \frac{1}{2} \hat{C}^{\hat{\rho}} \int^\sigma \hat{C}^{\hat{\kappa}} \int^{\sigma'} \zeta^{\mu\nu\rho\kappa} \partial_\rho \partial_\kappa (\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}}) d\sigma'' d\sigma',
\end{aligned} \tag{72}$$

where  $J_\mu^{\hat{\mu}}$  is given by

$$J_\mu^{\hat{\mu}} = \frac{\partial \hat{x}^{\hat{\mu}}}{\partial x^\mu} \tag{73}$$

and

$$\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} = J_\mu^{\hat{\mu}} J_\nu^{\hat{\nu}}. \tag{74}$$

This is proved in the appendix, proof number 2. It is not necessary to give the lower limits of the integrals as these are incorporate in gauge-like freedom (71).

*Proof.* Similar to the proof following (54), consider  $\int_{\sigma_0}^\sigma$  and  $\int_{\sigma_1}^\sigma$ . Let  $\hat{A}^{\hat{\mu}\hat{\nu}}(\sigma) = \zeta^{\mu\nu\rho\kappa} \partial_\rho \partial_\kappa (\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}})$  then taking the last term

$$\begin{aligned}
\int_{\sigma_0}^\sigma \hat{C}^{\hat{\kappa}} \int_{\sigma_0}^{\sigma'} \hat{A}^{\hat{\mu}\hat{\nu}}(\sigma'') d\sigma'' d\sigma' &- \int_{\sigma_1}^\sigma \hat{C}^{\hat{\kappa}} \int_{\sigma_1}^{\sigma'} \hat{A}^{\hat{\mu}\hat{\nu}}(\sigma'') d\sigma'' d\sigma' \\
&= \int_{\sigma_0}^\sigma \hat{C}^{\hat{\kappa}} \int_{\sigma_0}^{\sigma'} \hat{A}^{\hat{\mu}\hat{\nu}} d\sigma'' d\sigma' - \int_{\sigma_1}^\sigma \hat{C}^{\hat{\kappa}} \int_{\sigma_0}^{\sigma'} \hat{A}^{\hat{\mu}\hat{\nu}} d\sigma'' d\sigma' - \int_{\sigma_0}^\sigma \hat{C}^{\hat{\kappa}} \int_{\sigma_1}^{\sigma_0} \hat{A}^{\hat{\mu}\hat{\nu}} d\sigma'' d\sigma' \\
&= \int_{\sigma_0}^{\sigma_1} \hat{C}^{\hat{\kappa}} \int_{\sigma_0}^{\sigma'} \hat{A}^{\hat{\mu}\hat{\nu}} d\sigma'' d\sigma' - \hat{C}^{\hat{\kappa}} \int_{\sigma_1}^{\sigma_0} \hat{A}^{\hat{\mu}\hat{\nu}} d\sigma'' + \hat{C}^{\hat{\kappa}}(\sigma_0) \int_{\sigma_1}^{\sigma_0} \hat{A}^{\hat{\mu}\hat{\nu}} d\sigma''.
\end{aligned}$$



Hence the difference between two expressions for  $\hat{\zeta}^{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\kappa}}$  is  $\hat{\zeta}^{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\kappa}} \rightarrow \hat{\zeta}^{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\kappa}} + M_2^{\hat{\nu}\hat{\mu}} \hat{C}^{(\hat{\rho}} \hat{C}^{\hat{\kappa})} + M_3^{\hat{\mu}\hat{\nu}(\hat{\rho}} \hat{C}^{\hat{\kappa})}$  where

$$M_2^{\hat{\nu}\hat{\mu}} = \int_{\sigma_0}^{\sigma_1} \hat{A}^{\hat{\mu}\hat{\nu}} d\sigma$$

and

$$M_3^{\hat{\mu}\hat{\nu}\hat{\rho}} = \int_{\sigma_0}^{\sigma_1} \hat{C}^{\hat{\kappa}} \int_{\sigma_0}^{\sigma'} \hat{A}^{\hat{\mu}\hat{\nu}} d\sigma'' d\sigma' + \hat{C}^{\hat{\kappa}}(\sigma_0) \int_{\sigma_1}^{\sigma_0} \hat{A}^{\hat{\mu}\hat{\nu}} d\sigma'' - \int_{\sigma_0}^{\sigma_1} \zeta^{\mu\nu\rho\kappa} \left( \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} (\partial_\rho J_\kappa^{\hat{\rho}}) + 2 \partial_\rho (\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}}) J_\kappa^{\hat{\rho}} \right) d\sigma'.$$

□

It also is necessary to check that (72) is consistent with the gauge-like freedom (71). This is given in proof number 3 in the appendix.

As stated in the introduction the quadrupole is greatly simplified if we choose adapted coordinates given in (14), so that  $\dot{C}^\mu = \delta^\mu$ . Equation (67) can now be written in terms of components  $\gamma^{\mu\nu\rho\kappa}$

$$T^{\mu\nu}(\sigma, \mathbf{z}) = \gamma^{\mu\nu 00}(\sigma) \delta^{(3)}(\mathbf{z}) + \gamma^{\mu\nu 0a}(\sigma) \partial_a \delta^{(3)}(\mathbf{z}) + \frac{1}{2} \gamma^{\mu\nu ab}(\sigma) \partial_a \partial_b \delta^{(3)}(\mathbf{z}) \quad (75)$$

so that from (28) becomes

$$\int_{\mathcal{M}} T^{\mu\nu} \phi_{\mu\nu} d^4x = \int_{\mathcal{I}} \left( \gamma^{\mu\nu 00} \phi_{\mu\nu} - \gamma^{\mu\nu 0a} (\partial_a \phi_{\mu\nu}) + \frac{1}{2} \gamma^{\mu\nu ab} (\partial_a \partial_b \phi_{\mu\nu}) \right) d\sigma. \quad (76)$$

Here again we impose

$$\gamma^{\mu\nu\rho\kappa} = \gamma^{\nu\mu\rho\kappa} \quad \text{and} \quad \gamma^{\mu\nu\rho\kappa} = \gamma^{\mu\nu\kappa\rho}. \quad (77)$$

In adapted coordinates, the components  $\gamma^{\mu\nu\rho\kappa}$  are uniquely determined from  $T^{\mu\nu}$ , so there is no gauge-like freedom, as in (71). In this coordinate system we can still express  $T^{\mu\nu}$  in terms of (67), and the relationship between  $\gamma^{\mu\nu\rho\kappa}$  and  $\zeta^{\mu\nu\rho\kappa}$  is given by

$$\gamma^{\mu\nu 00} = \frac{1}{2} \ddot{\zeta}^{\mu\nu 00}, \quad \gamma^{\mu\nu a0} = \dot{\zeta}^{\mu\nu a0} \quad \text{and} \quad \gamma^{\mu\nu ab} = \zeta^{\mu\nu ab} \quad (78)$$

which is consistent with (71). This follows from (30).

It is now much easier to express the differential and algebraic equations on the components arising from the divergenceless conditions (10), as proved below.

$$\dot{\gamma}^{\mu 000} = -\Gamma_{\nu\rho}^\mu \gamma^{\rho\nu 00} + (\partial_a \Gamma_{\nu\rho}^0) \gamma^{\rho\nu 0a} - \frac{1}{2} (\partial_b \partial_a \Gamma_{\nu\rho}^0) \gamma^{\rho\nu ab}, \quad (79)$$

$$\dot{\gamma}^{\mu 00a} = -\gamma^{\mu a 00} - \Gamma_{\nu\rho}^\mu \gamma^{\rho\nu 0a} + (\partial_b \Gamma_{\nu\rho}^\mu) \gamma^{\rho\nu ba}, \quad (80)$$

$$\dot{\gamma}^{\mu 0ab} = -2\gamma^{\mu(ba)0} - \Gamma_{\nu\rho}^\mu \gamma^{\rho\nu ab} \quad (81)$$

together with the algebraic equation

$$\gamma^{\mu(abc)} = 0. \quad (82)$$

Proof of (79)-(82). From (11) we have for any test vector  $\theta^\nu$

$$\begin{aligned}
0 &= \int_{\mathcal{M}} (\nabla_\mu T^{\mu\nu}) \theta_\nu d^4x = \int_{\mathcal{M}} (\partial_\mu T^{\mu\nu} + \Gamma_{\mu\rho}^\nu T^{\mu\rho}) \theta_\nu d^4x = \int_{\mathcal{M}} T^{\mu\nu} (\Gamma_{\mu\nu}^\rho \theta_\rho - \partial_\mu \theta_\nu) d^4x \\
&= \int_{\mathcal{M}} \left( \gamma^{\mu\nu 00} \delta^{(3)}(\mathbf{z}) + \gamma^{\mu\nu 0a} \partial_a \delta^{(3)}(\mathbf{z}) + \frac{1}{2} \gamma^{\mu\nu ab} \partial_a \partial_b \delta^{(3)}(\mathbf{z}) \right) (\Gamma_{\mu\nu}^\rho \theta_\rho - \partial_\mu \theta_\nu) d^4x \\
&= \int_{\mathcal{I}} d\sigma \left( \gamma^{\mu\nu 00} (\Gamma_{\mu\nu}^\rho \theta_\rho - \partial_\mu \theta_\nu) - \gamma^{\mu\nu 0a} \partial_a (\Gamma_{\mu\nu}^\rho \theta_\rho - \partial_\mu \theta_\nu) + \frac{1}{2} \gamma^{\mu\nu ab} \partial_a \partial_b (\Gamma_{\mu\nu}^\rho \theta_\rho - \partial_\mu \theta_\nu) \right) \\
&= \int_{\mathcal{I}} d\sigma \left( \gamma^{\mu\nu 00} \Gamma_{\mu\nu}^\rho \theta_\rho - \gamma^{\mu\nu 00} \partial_a \theta_\nu + \dot{\gamma}^{0\nu 00} \theta_\nu \right. \\
&\quad \left. - \gamma^{\mu\nu 0a} \partial_a (\Gamma_{\mu\nu}^\rho \theta_\rho) + \gamma^{b\nu 0a} \partial_a \partial_b \theta_\nu - \dot{\gamma}^{0\nu 0a} \partial_a \theta_\nu \right. \\
&\quad \left. + \frac{1}{2} \gamma^{\mu\nu ab} \partial_a \partial_b (\Gamma_{\mu\nu}^\rho \theta_\rho) - \frac{1}{2} \gamma^{c\nu ab} \partial_a \partial_b \partial_c \theta_\nu + \frac{1}{2} \dot{\gamma}^{0\nu ab} \partial_a \partial_b \theta_\nu \right) \\
&= \int_{\mathcal{I}} d\sigma \left( \gamma^{\mu\nu 00} \Gamma_{\mu\nu}^\rho \theta_\rho - \gamma^{\mu\nu 00} \partial_a \theta_\nu + \dot{\gamma}^{0\rho 00} \theta_\rho \right. \\
&\quad \left. - \gamma^{\mu\nu 0a} (\partial_a \Gamma_{\mu\nu}^\rho) \theta_\rho - \gamma^{\mu\nu 0a} \Gamma_{\mu\nu}^\rho \partial_a \theta_\rho + \gamma^{b\nu 0a} \partial_a \partial_b \theta_\nu - \dot{\gamma}^{0\nu 0a} \partial_a \theta_\nu \right. \\
&\quad \left. + \frac{1}{2} \gamma^{\mu\nu ab} (\partial_a \partial_b \Gamma_{\mu\nu}^\rho) \theta_\rho + \gamma^{\mu\nu ab} (\partial_a \Gamma_{\mu\nu}^\rho) (\partial_b \theta_\rho) + \frac{1}{2} \gamma^{\mu\nu ab} \Gamma_{\mu\nu}^\rho \partial_a \partial_b \theta_\rho \right. \\
&\quad \left. - \frac{1}{2} \gamma^{c\nu ab} \partial_a \partial_b \partial_c \theta_\nu + \frac{1}{2} \dot{\gamma}^{0\nu ab} \partial_a \partial_b \theta_\nu \right) \\
&= \int_{\mathcal{I}} d\sigma \left( \theta_\rho \left( \gamma^{\mu\nu 00} \Gamma_{\mu\nu}^\rho + \dot{\gamma}^{0\rho 00} - \gamma^{\mu\nu 0a} (\partial_a \Gamma_{\mu\nu}^\rho) + \frac{1}{2} \gamma^{\mu\nu ab} (\partial_a \partial_b \Gamma_{\mu\nu}^\rho) \right) \right. \\
&\quad \left. - \partial_a \theta_\rho \left( \gamma^{a\rho 00} + \gamma^{\mu\nu 0a} \Gamma_{\mu\nu}^\rho + \dot{\gamma}^{0\rho 0a} - \gamma^{\mu\nu ba} (\partial_b \Gamma_{\mu\nu}^\rho) \right) \right. \\
&\quad \left. + \partial_a \partial_b \theta_\rho \left( \gamma^{b\rho 0a} + \frac{1}{2} \gamma^{\mu\nu ab} \Gamma_{\mu\nu}^\rho + \frac{1}{2} \dot{\gamma}^{0\rho ab} \right) - \frac{1}{2} \gamma^{c\nu ab} \partial_a \partial_b \partial_c \theta_\nu \right).
\end{aligned}$$

The terms with  $\theta_\rho$ ,  $\partial_a \theta_\rho$ ,  $\partial_a \partial_b \theta_\rho$  and  $\partial_a \partial_b \partial_c \theta_\rho$  are independent. From this we get (79)-(82). Note we must take the symmetric part with respect to  $b, a$ .  $\square$

We can now count the number of components of the quadrupole. From (79)-(81) we have 40 first order ODEs. However not all the components are determined by these ODEs. From (77) we start with 100 components. The algebraic equation (82) gives 40 independent equations so that there are 60 independent components. Thus 40 are determined by ODEs and the remaining 20 are free components. As stated in the introduction these free components need to be replaced by constitutive equations. However the choice of constitutive equations depends on a choice of an underlying model for the stress-energy tensor. An example of such constitutive equations is given in section 5 below.

Under change of adapted coordinate  $(\sigma, z^1, z^2, z^3)$  to  $(\hat{\sigma}, \hat{z}^1, \hat{z}^2, \hat{z}^3)$  we have

$$\hat{\gamma}^{\hat{\mu}\hat{\nu}\hat{a}\hat{b}} = \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} J_a^{\hat{a}} J_b^{\hat{b}} \gamma^{\mu\nu ab}, \quad (83)$$

$$\hat{\gamma}^{\hat{\mu}\hat{\nu}\hat{a}\hat{0}} = \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} J_a^{\hat{a}} \gamma^{\mu\nu a0} + (\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} J_b^{\hat{0}} J_a^{\hat{a}} \gamma^{\mu\nu ab}) \cdot - \frac{1}{2} (\partial_a J_b^{\hat{a}} \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} + J_a^{\hat{a}} \partial_b \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} + J_b^{\hat{a}} \partial_a \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}}) \gamma^{\mu\nu ab}, \quad (84)$$

$$\begin{aligned}
\hat{\gamma}^{\hat{\mu}\hat{\nu}\hat{0}\hat{0}} &= \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} \gamma^{\mu\nu 00} + \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} J_c^{\hat{0}} \dot{\gamma}^{\mu\nu a0} + ((\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} J_c^{\hat{0}}) \cdot - \partial_c \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}}) \gamma^{\mu\nu c0} \\
&\quad + \frac{1}{2} ((\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} J_d^{\hat{0}} J_c^{\hat{0}}) \gamma^{\mu\nu cd}) \cdot - ((\frac{1}{2} \partial_c J_d^{\hat{0}} \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} + J_d^{\hat{0}} \partial_c \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}}) \gamma^{\mu\nu cd}) \cdot + (\frac{1}{2} \partial_c \partial_d \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}}) \gamma^{\mu\nu cd}
\end{aligned} \quad (85)$$

See proof number 5 in the appendix. Although these may be considered more complicated than (72) they does not involve any integrals. We have assumed that  $\sigma$  and  $\hat{\sigma}$  parameterise the same points on the worldline  $C$ . Thus on the worldline  $J_0^{\hat{\mu}} = \delta_0^{\hat{\mu}}$ . However this does not imply  $\partial_\nu J_0^{\hat{\mu}} = 0$ .

## 4.1 The static semi-quadrupole and the free components

To get an intuition about the free components, consider the dynamic equations (79)-(82) on a flat Minkowski background with Cartesian coordinates  $(t = z^0, z^1, z^2, z^3) = (t, \mathbf{z})$  and with the worldline

at  $\mathbf{z} = \mathbf{0}$ . Thus we can set  $t = \sigma$  so that  $C^0(t) = t$  and  $C^a(t) = 0$ . The dynamic equations (79)-(82) become

$$\dot{\gamma}^{\mu 000} = 0, \quad (86)$$

$$\dot{\gamma}^{\mu 0a0} = -\gamma^{\mu a00}, \quad (87)$$

$$\dot{\gamma}^{\mu 0ba} = -2\gamma^{\mu(ab)0}, \quad (88)$$

$$\gamma^{\mu(abc)} = 0. \quad (89)$$

As a further simplification, consider only the semi-quadrupole. This is when

$$\gamma^{\mu abc} = 0. \quad (90)$$

According to table 1, there should be 22 ODE components and 6 free components. This arises since (90) implies  $\gamma^{a0bc} = 0$  which eliminates all but 6 of the ODEs in (88). See section 6.6 below for full details.

The general solution is given by

$$\begin{aligned} \gamma^{0000} &= m, & \gamma^{a000} &= P^a, & \gamma^{00a0} &= X^a - t P^a, \\ \gamma^{00ba} &= \kappa^{ba}(t), & \gamma^{b0a0} &= S^{ba} - \frac{1}{2}\dot{\kappa}^{ba}(t), & \gamma^{ba00} &= \frac{1}{2}\ddot{\kappa}^{ba}(t), & \gamma^{cba0} &= 0 \end{aligned} \quad (91)$$

where the 10 quantities  $m, P^a, X^a, S^{ab}$  are constants,  $S^{ab}$  satisfies  $S^{ab} + S^{ba} = 0$  and the six free components  $\kappa^{ba}(t)$  satisfy  $\kappa^{ba}(t) = \kappa^{ab}(t)$ . Here we interpret  $m$  as the total mass,  $P^a$  as the momentum and  $S^{ba}$  as the spin. The six free components  $\kappa^{ab}(t)$  are the moments of inertia. Since there are 22 ODEs there should be 22 constants of integration. As well as the 10 already given, the remaining 12 are the six initial conditions for  $\kappa^{ab}(0)$  and for  $\dot{\kappa}^{ab}(0)$ .

*Proof of (91).* For the semi-quadrupole (90), then (89) is automatically satisfied. Equations (86)-(88) become

$$\dot{\gamma}^{0000} = 0, \quad \dot{\gamma}^{a000} = 0, \quad \dot{\gamma}^{00a0} = -\gamma^{0a00}, \quad \dot{\gamma}^{0ba0} = -\gamma^{ba00}, \quad \dot{\gamma}^{00ba} = -\gamma^{0(ab)0}, \quad 0 = \dot{\gamma}^{0cba} = -\gamma^{c(ab)0}.$$

It may appear that we have not stated anything about  $(\gamma^{cab0} - \gamma^{cba0})$ . However due to the symmetry of  $\gamma^{cab0}$  we have

$$\gamma^{cab0} - \gamma^{cba0} = \gamma^{acb0} - \gamma^{bca0} = -\gamma^{abc0} + \gamma^{bac0} = 0.$$

Thus from the last equation above we have  $\gamma^{cba0} = 0$ . Setting  $\gamma^{00ba} = \kappa^{ab}(\sigma)$  we have  $\gamma^{0(ba)0} = \dot{\kappa}^{ab}$  and  $\gamma^{ba00} = \ddot{\kappa}^{ab}$ . The remaining constants in (91) are then determined.  $\square$

Consider the components of  $T^{\mu\nu}$  as arising from squeezing a regular stress-energy tensor density  $\mathcal{T}^{\mu\nu}(t, \mathbf{z})$  as in section 2.3. Thus

$$\gamma^{\mu\nu 00} = \int_{\mathbb{R}^3} \mathcal{T}^{\mu\nu}(t, \mathbf{z}) d^3 \mathbf{z}, \quad \gamma^{\mu\nu a0} = \int_{\mathbb{R}^3} \mathcal{T}^{\mu\nu}(t, \mathbf{z}) z^a d^3 \mathbf{z}, \quad \gamma^{\mu\nu ab} = \int_{\mathbb{R}^3} \mathcal{T}^{\mu\nu}(t, \mathbf{z}) z^a z^b d^3 \mathbf{z}. \quad (92)$$

Comparing (91) and (92) we see

$$\begin{aligned} m &= \int_{\mathbb{R}^3} \mathcal{T}^{00}(t, \mathbf{z}) d^3 \mathbf{z}, & P^a &= \int_{\mathbb{R}^3} \mathcal{T}^{a0}(t, \mathbf{z}) d^3 \mathbf{z}, & X^a &= t P^a + \int_{\mathbb{R}^3} \mathcal{T}^{00}(t, \mathbf{z}) z^a d^3 \mathbf{z}, \\ S^{ba} &= \int_{\mathbb{R}^3} z^{[a} \mathcal{T}^{b]0}(t, \mathbf{z}) d^3 \mathbf{z} & \text{and} & & \kappa^{ab} &= \int_{\mathbb{R}^3} z^a z^b \mathcal{T}^{00}(t, \mathbf{z}) d^3 \mathbf{z}. \end{aligned} \quad (93)$$

For example let  $P^a = 0$  and  $S^{ab} = 0$  then

$$m = \int_{\mathbb{R}^3} \mathcal{T}^{00}(t, \mathbf{z}) d^3 \mathbf{z}, \quad \kappa^{ab}(t) = \int_{\mathbb{R}^3} z^a z^b \mathcal{T}^{00}(t, \mathbf{z}) d^3 \mathbf{z}. \quad (94)$$

Since  $\kappa^{ab}(t)$  are free components we can choose any  $\mathcal{T}^{\mu\nu}(t, \mathbf{z})$  we like so long as its total integral is  $m$  and they are sufficiently symmetric that  $P^a = 0$  and  $S^{ab} = 0$  hold. This can be achieved if, for example  $\mathcal{T}^{\mu\nu}(t, \mathbf{z})$  is symmetric about the three directions  $z^a$ . This explains why we can choose to have a distribution of matter which separates and then coalesces as in figure 1.

## 4.2 Conserved quantities

Recall that a Killing vector (61) leads to a conserved quantity in the dipole case. The same is true for quadrupole. In an adapted coordinate system  $(\sigma, z^1, z^2, z^3)$  the conserved quantity  $\mathcal{Q}_K$  is given by

$$\mathcal{Q}_K = \gamma^{\mu 000} K_\mu - \gamma^{\mu 0a0} \partial_a K_\mu + \frac{1}{2} \gamma^{\mu 0ab} \partial_a \partial_b K_\mu. \quad (95)$$

*Proof.* Let  $\varphi$  be a test function. Thus

$$\int_{\mathcal{M}} \nabla_\mu (T^{\mu\nu} K_\nu) \varphi d^4x = \int_{\mathcal{M}} (\nabla_\mu T^{\mu\nu} K_\nu + T^{\mu\nu} \nabla_\mu K_\nu \varphi) d^4x = 0.$$

from (9), (10) and (61). Since  $T^{\mu\nu}$  is a tensor density then so is  $T^{\mu\nu} K_\nu$ . Hence

$$\begin{aligned} 0 &= \int_{\mathcal{M}} \nabla_\mu (T^{\mu\nu} K_\nu) \varphi d^4x = \int_{\mathcal{M}} T^{\mu\nu} K_\nu \nabla_\mu \varphi d^4x = \int_{\mathcal{M}} T^{\mu\nu} K_\nu \partial_\mu \varphi d^4x \\ &= \int_{\mathcal{I}} \left( \gamma^{\mu\nu 00} K_\nu \partial_\mu \varphi - \gamma^{\mu\nu 0a} \partial_a (K_\nu \partial_\mu \varphi) + \frac{1}{2} \gamma^{\mu\nu ab} \partial_a \partial_b (K_\nu \partial_\mu \varphi) \right) d\sigma \\ &= \int_{\mathcal{I}} \left( \partial_\mu \varphi (\gamma^{\mu\nu 00} K_\nu - \gamma^{\mu\nu 0a} \partial_a K_\nu + \frac{1}{2} \gamma^{\mu\nu ab} \partial_a \partial_b K_\nu) \right) d\sigma + \text{higher derivatives of } \varphi. \\ &= \int_{\mathcal{I}} \left( \partial_0 \varphi (\gamma^{0\nu 00} K_\nu - \gamma^{0\nu 0a} \partial_a K_\nu + \frac{1}{2} \gamma^{0\nu ab} \partial_a \partial_b K_\nu) \right) d\sigma \\ &\quad + \int_{\mathcal{I}} \left( \partial_c \varphi (\gamma^{c\nu 00} K_\nu - \gamma^{c\nu 0a} \partial_a K_\nu + \frac{1}{2} \gamma^{c\nu ab} \partial_a \partial_b K_\nu) \right) d\sigma + \text{higher derivatives of } \varphi. \\ &= - \int_{\mathcal{I}} \left( \varphi \partial_0 (\gamma^{0\nu 00} K_\nu - \gamma^{0\nu 0a} \partial_a K_\nu + \frac{1}{2} \gamma^{0\nu ab} \partial_a \partial_b K_\nu) \right) d\sigma + \text{higher derivatives of } \varphi. \\ &= - \int_{\mathcal{I}} (\varphi \dot{\mathcal{Q}}_K) d\sigma + \text{higher derivatives of } \varphi. \end{aligned}$$

Thus since we can extract the different derivatives of  $\varphi$  we have  $\dot{\mathcal{Q}}_K = 0$ .  $\square$

It is worth exploring the conserved quantities on the static semi-quadrupole given by (91). In Minkowski spacetime there are 10 Killing vectors.

- Mass or Energy: for  $K_0 = 1$ ,  $K_a = 0$  we have  $\mathcal{Q}_K = m$ .
- Momentum: for  $K_0 = 0$ ,  $K_1 = 1$ ,  $K_2 = 0$  and  $K_3 = 0$  then  $\mathcal{Q}_K = p_1$ . Likewise for the other two cases.
- Angular momentum and spin: let  $K_0 = 0$ ,  $K_1 = z^2$ ,  $K_2 = -z^1$  and  $K_3 = 0$ . We have

$$\begin{aligned} \mathcal{Q}_K &= \gamma^{1000} K_1 + \gamma^{2000} K_2 + \gamma^{2010} \partial_1 K_2 + \gamma^{1020} \partial_2 K_1 \\ &= p^1 z^2 - p^2 z^1 + (S^{12} - \dot{\kappa}^{12}(t)) - (S^{21} - \dot{\kappa}^{21}(t)) = S^{12}. \end{aligned}$$

Likewise for the other two cases.

- Boost: Let  $K_0 = z^1$ ,  $K_1 = t + t_0$ ,  $K_2 = 0$  and  $K_3 = 0$  for some fixed  $t_0$ . Then

$$\mathcal{Q}_K = \gamma^{0000} K_0 + \gamma^{1000} K_1 + \gamma^{0010} \partial_1 K_0 = m z^1 + P^1 (t + t_0) + (X^1 - t P^1) = X^1 + t_0 P^1.$$

Likewise for the other two cases.

Thus the 10 Killing symmetries of Minkowski spacetime correspond directly to the 10 constants of the solution to static semi-quadrupole. This also gives a new interpretation to the three somewhat obscure conserved quantities corresponding to the three boosts.

## 5 Non-divergent dust model of a quadrupole and the corresponding constitutive relations.

The familiar dust model is given in terms of a scalar  $\varrho$  and a vector field  $U^\mu$  with  $g_{\mu\nu} U^\mu U^\nu = -1$ . The stress-energy tensor density is given by

$$\mathcal{T}^{\mu\nu} = \varrho U^\mu U^\nu \omega, \quad (96)$$

where  $\omega = \sqrt{-\det(g_{\mu\nu})}$ . Then the divergenceless condition implies that the  $U^\mu$  are geodesics

$$U^\mu \nabla_\mu U^\nu = 0 \quad (97)$$

and the flow  $\varrho$  is conserved

$$\nabla_\mu(\varrho U^\mu) = 0. \quad (98)$$

Furthermore let us assume that the dust is non divergent, so that it preserves the measure, i.e.

$$U^\mu \partial_\mu \omega = 0. \quad (99)$$

so that  $\partial_\mu(\varrho U^\mu) = 0$ .

In order to create a squeezed tensor  $\mathcal{T}_\varepsilon^{\mu\nu}$  from  $\mathcal{T}^{\mu\nu}$  we need to choose a coordinate system. It is natural to choose the coordinate adapted to  $U^\mu$  so that  $U^\mu = \delta_0^\mu$ . This gives  $\dot{\varrho} = 0$  so that we can write  $\varrho = \varrho(\mathbf{z})$ . Likewise we have  $a = a(\mathbf{z})$ . Hence

$$\mathcal{T}^{\mu\nu}(\sigma, \mathbf{z}) = \varrho(\mathbf{z}) \delta_0^\mu \delta_0^\nu a(\mathbf{z}). \quad (100)$$

We require that  $\varrho(\mathbf{z}) = 0$  for large  $\mathbf{z}$ . From (36) we see

$$\begin{aligned} \gamma^{\mu\nu 00}(\sigma) &= \delta_0^\mu \delta_0^\nu \int_{\mathbb{R}^3} d^3 \mathbf{z} \varrho(\mathbf{z}) a(\mathbf{z}), \\ \gamma^{\mu\nu a 0}(\sigma) &= -\delta_0^\mu \delta_0^\nu \int_{\mathbb{R}^3} d^3 \mathbf{z} z^a \varrho(\mathbf{z}) a(\mathbf{z}), \\ \gamma^{\mu\nu ab}(\sigma) &= \delta_0^\mu \delta_0^\nu \int_{\mathbb{R}^3} d^3 \mathbf{z} z^a z^b \varrho(\mathbf{z}) a(\mathbf{z}). \end{aligned} \quad (101)$$

Since both  $\varrho$  and  $a$  are independent of  $\sigma$  we have the dynamic equations

$$\dot{\gamma}^{\mu\nu 00} = 0, \quad \dot{\gamma}^{\mu\nu a 0} = 0 \quad \text{and} \quad \dot{\gamma}^{\mu\nu ab} = 0. \quad (102)$$

These are consistent with the dynamic equations (79)-(81) since in the adapted coordinate system the geodesic equation becomes  $\Gamma_{00}^\mu = 0$ .

Equation (102) completely defines the dynamics. However, our goal is to use (102) to inspire the constitutive relations in the case when we are not modelling a non-divergent dust, and (79)-(81) hold. One option is to require that some of the free components are in fact constants. This is challenging because we need to be consistent with (79)-(81).

As a simple example, consider the static semi-quadrupole in Minkowski spacetime, given by (91). The non-divergent dust constitutive relations would make  $\kappa^{ab}(t)$  a constant. It would also make  $P^a = 0$ . This replaces (91) with

$$\begin{aligned} \gamma^{0000} &= m, & \gamma^{a000} &= 0, & \gamma^{00a0} &= X^a, \\ \gamma^{00ba} &= \kappa^{ba}, & \gamma^{b0a0} &= S^{ba}, & \gamma^{ba00} &= 0, & \gamma^{cba0} &= 0. \end{aligned} \quad (103)$$

## 6 The coordinate free and metric free approach to quadrupoles.

As stated in the introduction we can construct the distributional stress-energy tensor density with only a connection. In this section we do not assume the connection is Levi-Civita connection. Indeed there is no mention of a metric at all. This, as stated, is particularly useful in the case of non-metric connections, or when there is no metric or multiple metrics. To simplify the mathematics we assume the metric is torsion free. However this too can be relaxed with the result of additional torsion terms in many expressions.

In [27] the authors present a coordinate free definition of submanifold distributions, also known as deRham currents, in terms of the deRham push forward [31] and standard operations.

Since we are using coordinate free notation we write a vector field as  $V \in \Gamma T\mathcal{M}$ . Here  $T\mathcal{M}$  is the tangent bundle of spacetime and  $\Gamma T\mathcal{M}$  refers to sections of the tangent bundle. A vector at a point  $p \in \mathcal{M}$  is written  $V \in T_p\mathcal{M}$ . A vector field and vectors at a point are differential operators and we write the action of a vector on a scalar field using angle brackets as  $V\langle f \rangle$ . The bundle of  $p$ -forms is written  $\Lambda^p\mathcal{M}$  so a  $p$ -form field is written  $\alpha \in \Gamma\Lambda^p\mathcal{M}$ .

Given a coordinate system  $(x^0, \dots, x^3)$  then we write  $V = V^\mu \partial_\mu$ . Here  $\partial_\mu$  are basis vectors and  $V^\mu$  are indexed scalar fields. Thus

$$V\langle f \rangle = V^\mu \partial_\mu f \quad \text{where} \quad V^\mu = V\langle x^\mu \rangle. \quad (104)$$

For 1-forms  $\alpha \in \Gamma\Lambda^1\mathcal{M}$  we can write  $\alpha = \alpha_\mu dx^\mu$  where again  $\alpha_\mu$  are indexed scalar fields.

### 6.1 The two types of $\nabla$

In the literature on general relativity and differential geometry, there are two conventions used when referring to the covariant derivative. One is typically used when using index tensor notation, the other when one is using coordinate free notation. Usually one has simply to choose one convention and present all the results using that. We have done this up to now using index notation. However in this section we wish to present a coordinate free definition of all the objects. As a result it is necessary to use both definitions of the covariant derivatives, sometimes in the same expression. So to avoid confusion, from now on we introduce two different symbols.

The covariant derivative which we have used up to this point and which “knows” about the index of an object we write  $\bar{\nabla}_\mu$ . The action on the indexed scalar fields  $V^\mu$  is then

$$\bar{\nabla}_\mu V^\nu = \partial_\mu(V^\nu) + V^\rho \Gamma_{\mu\rho}^\nu. \quad (105)$$

In other words, the Christoffel symbols are tied to the indices. By contrast the coordinate free covariant derivative is written  $\nabla_V$  where  $V \in \Gamma T\mathcal{M}$ . In this case the Christoffel symbol satisfies

$$\Gamma_{\nu\rho}^\mu \partial_\mu = \nabla_{\partial_\nu} \partial_\rho. \quad (106)$$

This covariant derivative knows about the tensor structure, but not the indices. Thus

$$\nabla_U V^\mu = U\langle V^\mu \rangle = U^\nu \partial_\nu V^\mu. \quad (107)$$

The two covariant derivatives are related via the following

$$\nabla_U(V) = U^\nu (\bar{\nabla}_\nu V^\mu) \partial_\mu, \quad (108)$$

since

$$\begin{aligned} \nabla_U(V) &= \nabla_U(V^\mu \partial_\mu) = U\langle V^\mu \rangle \partial_\mu + U^\nu V^\mu \nabla_{\partial_\nu} \partial^\mu = U\langle V^\mu \rangle \partial_\mu + U^\nu V^\mu \Gamma_{\nu\rho}^\mu \partial_\rho \\ &= U^\nu (\partial_\nu \langle V^\mu \rangle + V^\rho \Gamma_{\nu\rho}^\mu) \partial_\mu = U^\nu (\bar{\nabla}_\nu V^\mu) \partial_\mu. \end{aligned}$$



Setting  $k=2$  in (38) gives the Dixon quadrupole, and we see that it contains the operator  $\bar{\nabla}_\mu \bar{\nabla}_\nu$ . Thus (38) is tensorial with respect to the indices  $\mu$  and  $\nu$ . To give a coordinate free definition we define for any tensor  $S$ ,

$$\nabla_{U,V}^2 S = \nabla_U \nabla_V S - \nabla_{\nabla_U V} S. \quad (109)$$

This definition can be extended to arbitrary order. This is clearly tensorial in  $U$ , but is also tensorial (also known as  $f$ -linear) with respect to  $V$ . Thus

$$\nabla_{(fU),V}^2 S = \nabla_{U,(fV)}^2 S = f \nabla_{U,V}^2 S. \quad (110)$$

*Proof.*

$$\begin{aligned} \nabla_{U,fV}^2 S &= \nabla_U \nabla_{(fV)} S - \nabla_{\nabla_U (fV)} S = \nabla_U (f \nabla_V S) - \nabla_{(f \nabla_U V + U \langle f \rangle V)} S \\ &= f \nabla_U \nabla_V S + U \langle f \rangle \nabla_V S - f \nabla_{\nabla_U V} S - U \langle f \rangle \nabla_V S = f \nabla_{U,V}^2 S. \end{aligned}$$

□

The relationship between  $\nabla_{U,V}^2$  and  $\bar{\nabla}_\mu \bar{\nabla}_\nu$  is given by

$$\nabla_{U,V}^2 W = U^\nu V^\rho \left( \bar{\nabla}_\nu \bar{\nabla}_\rho W^\mu \right) \partial_\mu. \quad (111)$$

for any vector  $W^\mu$ .

*Proof.*

$$\begin{aligned} \nabla_{U,V}^2 W &= \nabla_U \nabla_V W - \nabla_{\nabla_U V} W = U^\nu \bar{\nabla}_\nu (\nabla_V W)^\mu \partial_\mu - (\nabla_U V)^\rho (\bar{\nabla}_\rho W^\mu) \partial_\mu \\ &= U^\nu \bar{\nabla}_\nu (V^\rho \bar{\nabla}_\rho W^\mu) \partial_\mu - U^\nu (\bar{\nabla}_\nu V^\rho) (\bar{\nabla}_\rho W^\mu) \partial_\mu \\ &= U^\nu \left( \bar{\nabla}_\nu (V^\rho \bar{\nabla}_\rho W^\mu) - (\bar{\nabla}_\nu V^\rho) (\bar{\nabla}_\rho W^\mu) \right) \partial_\mu \\ &= U^\nu V^\rho \left( \bar{\nabla}_\nu \bar{\nabla}_\rho W^\mu \right) \partial_\mu. \end{aligned}$$

□

## 6.2 Defining distributional forms

Following Schwartz, we define a distributional  $p$ -form by its action on a test  $(4-p)$ -form  $\varphi \in \Gamma \Lambda^{4-p} M$ , i.e. a  $(4-p)$ -form with compact support [27]. Given  $\alpha \in \Gamma \Lambda^p M$  is a smooth  $p$ -form, we construct a regular distribution  $\alpha^D$  via

$$\alpha^D[\varphi] = \int_M \varphi \wedge \alpha. \quad (112)$$

The definition of the wedge product, Lie derivatives, internal contraction and exterior derivatives on distributions are defined to be consistent with (112). Thus for a distribution  $\Psi$  we set

$$\begin{aligned} (\Psi_1 + \Psi_2)[\varphi] &= \Psi_1[\varphi] + \Psi_2[\varphi], \quad (\beta \wedge \Psi)[\varphi] = \Psi[\varphi \wedge \beta], \quad (d\Psi)[\varphi] = (-1)^{(3-p)} \Psi[d\varphi], \\ (i_v \Psi)[\varphi] &= (-1)^{(3-p)} \Psi[i_v \varphi] \quad \text{and} \quad (L_v \Psi)[\varphi] = -\Psi[L_v \varphi] \end{aligned} \quad (113)$$

for  $v \in \Gamma T\mathcal{M}$ . Given that  $C : \mathcal{I} \rightarrow \mathcal{M}$ , is a closed embedding, the DeRham push forward with respect to  $C$  of a  $p$ -form,  $\alpha \in \Gamma \Lambda^p \mathcal{I}$  is given by the distribution

$$(C_\#(\alpha))[\varphi] = \int_{\mathcal{I}} C^*(\varphi) \wedge \alpha. \quad (114)$$

where  $\varphi$  is a test form of degree 0 or 1 and  $C^*(\varphi)$  is the pullback of  $\varphi \in \Gamma\Lambda^q\mathcal{M}$  to  $\Gamma\Lambda^q\mathcal{I}$ . This has degree  $\deg(C_\zeta(\alpha)) = 3 + p$ . A general form distribution is then given by applying an arbitrary number of wedges, exterior derivatives, etc, to  $C_\zeta(\alpha)$  using the rules given in (113).

The **order of a multipole** is defined as follows. If

$$\Psi[\lambda^{k+1}\varphi] = 0 \quad \text{for all } \lambda \in \Gamma\Lambda^0 M \text{ and } \varphi \in \Gamma_0\Lambda^1 M \quad \text{such that } C^*(\lambda) = 0, \quad (115)$$

then we say that the order of  $\Psi$  is at most  $k$ . Since we impose that  $\lambda$  vanishes on the image of  $C$ , (115) implies that we need to differentiate the argument  $\lambda^{k+1}\varphi$  at least  $k+1$  times for  $\Psi[\lambda^{k+1}\varphi] \neq 0$ . We say dipoles have order at most one and quadrupoles have order at most two. Therefore the terms in a dipole have at most one derivative, and those in a quadrupole at most two. This is consistent with the fact that the set of quadrupoles include all dipoles.

The deRham push forward is compatible with the exterior derivative

$$dC_\zeta(\alpha) = C_\zeta(d\alpha), \quad (116)$$

and the internal contraction for fields tangent to  $C$

$$i_w C_\zeta(\alpha) = C_\zeta(i_w \alpha) \quad \text{where} \quad w \in \Gamma TM, \quad v \in \Gamma T\mathcal{I}, \quad C_*(v|_\sigma) = w|_{C(\sigma)} \quad \text{for all } \sigma \in \mathcal{I}. \quad (117)$$

### 6.3 The stress-energy 3-forms

Since we wish to work without a metric it is natural to use the stress-energy 3-forms [32]. The relationship to  $T^{\mu\nu}$  is given in (126) below. We exploit the fact that the stress-energy 3-forms have a similar structure to the electromagnetic current 3-form. This enables us to use the technology developed in [27].

We define the stress-energy form  $\tau$  as a map which takes a 1-form  $\alpha \in \Gamma\Lambda^1\mathcal{M}$  and gives a deRham current 3-form  $\tau_\alpha$  over the worldline  $C$ .

$$\alpha \mapsto \tau_\alpha. \quad (118)$$

The map (118) is not 'f'-linear but does satisfy

$$\tau_{(\alpha+\beta)} = \tau_\alpha + \tau_\beta \quad \text{and} \quad \tau_{(f\alpha)}[\theta] = \tau_\alpha[f\theta], \quad (119)$$

for any test 1-form  $\theta$  and scalar field  $f$ .

Using  $\tau_\alpha$  we define a tensor valued distribution  $\tau$  which takes a tensor of type (0,2) as an argument. This is defined as

$$\tau[\theta \otimes \alpha] = \tau_\alpha[\theta]. \quad (120)$$

The stress-energy tensor is symmetric (9) and divergenceless (10). We show below that the symmetry condition is given by

$$\tau[\beta \otimes \alpha] = \tau[\alpha \otimes \beta], \quad (121)$$

and the divergenceless condition is given by

$$D\tau = 0, \quad (122)$$

where

$$(D\tau)[\theta] = -\tau[D\theta] \quad (123)$$

and

$$(D\theta)(U, V) = (\nabla_V \theta)(U). \quad (124)$$

Using a coordinate system, we can convert the map (118) into indexed 3-forms via

$$\tau^\mu = \tau_{dx^\mu}. \quad (125)$$

The relationship between the stress-energy forms and the tensor density  $T^{\mu\nu}$  is given by

$$\int_{\mathcal{I}} T^{\mu\nu} \phi_{\mu\nu} d^4x = \tau^\mu [\phi_{\mu\nu} dx^\nu] = \tau [\phi_{\mu\nu} dx^\nu \otimes dx^\mu]. \quad (126)$$

Using this coordinate system, (121) becomes

$$dx^\mu \wedge \tau^\nu = dx^\nu \wedge \tau^\mu, \quad (127)$$

and (122) becomes

$$d\tau^\mu + \Gamma_{\nu\rho}^\mu dx^\rho \wedge \tau^\nu = 0. \quad (128)$$

*Proof.* Let  $\theta$  be a test 1-form then

$$(D\theta)(U, V) = (\nabla_V \theta)(U) = U^\nu (\nabla_V \theta)_\nu = U^\nu V^\mu \bar{\nabla}_\mu \theta_\nu = (\bar{\nabla}_\nu \theta_\mu) (dx^\nu \otimes dx^\mu)(U, V),$$

hence

$$D\theta = (\bar{\nabla}_\nu \theta_\mu) (dx^\nu \otimes dx^\mu).$$

Thus

$$\begin{aligned} D\tau[\theta] &= -\tau[D\theta] = -\tau[(\bar{\nabla}_\nu \theta_\mu) (dx^\nu \otimes dx^\mu)] = -\tau^\mu [(\bar{\nabla}_\nu \theta_\mu) dx^\nu] = -\tau^\mu [(\partial_\nu \theta_\mu - \Gamma_{\nu\mu}^\rho \theta_\rho) dx^\nu] \\ &= -\tau^\mu [\partial_\nu \theta_\mu dx^\nu - \Gamma_{\nu\mu}^\rho \theta_\rho dx^\nu] = -\tau^\mu [\partial_\nu \theta_\mu dx^\nu] + \tau^\mu [\Gamma_{\nu\mu}^\rho \theta_\rho dx^\nu] = -\tau^\mu [d\theta_\mu] + \Gamma_{\nu\mu}^\rho dx^\nu \wedge \tau^\mu [\theta_\rho] \\ &= d\tau^\mu [\theta_\mu] + \Gamma_{\nu\mu}^\rho dx^\nu \wedge \tau^\mu [\theta_\rho] = (d\tau^\rho + \Gamma_{\nu\mu}^\rho dx^\nu \wedge \tau^\mu) [\theta_\rho]. \end{aligned}$$

□

## 6.4 Killing forms and conservation

Killing forms (61) can be written in a coordinate free way. The 1-form  $\alpha \in \Gamma\Lambda^1\mathcal{M}$  is Killing if

$$(\nabla_V \alpha)(V) = 0, \quad (129)$$

for all vectors  $V \in \Gamma T\mathcal{M}$ . If there is a metric  $g$ ,  $\nabla$  is metric compatible and  $\alpha$  is the metric dual of  $K$  then (129) is equivalent to  $K$  being Killing. This follows since  $(\nabla_V \alpha)(V) = g(\nabla_V K, V)$ . From (128) and (127) we have

$$\begin{aligned} d\tau_\alpha &= d(\alpha_\mu \tau^\mu) = d\alpha_\mu \wedge \tau^\mu + \alpha_\mu \wedge d\tau^\mu = (\partial_\rho \alpha_\mu) dx^\rho \wedge \tau^\mu - \Gamma_{\nu\rho}^\mu \alpha_\mu dx^\rho \wedge d\tau^\nu = \bar{\nabla}_\rho \alpha_\nu dx^\rho \wedge d\tau^\nu \\ &= \frac{1}{2} (\bar{\nabla}_\rho \alpha_\nu - \bar{\nabla}_\nu \alpha_\rho) dx^\rho \wedge d\tau^\nu. \end{aligned}$$

Hence if  $\alpha \in \Gamma\Lambda^1\mathcal{M}$  is a Killing 1-form then from (61)  $d\tau_\alpha = 0$ . This gives an alternative method of proving (95).

## 6.5 The definition of components

Using (67) and (126) we deduce in an arbitrary coordinate system

$$\tau^\mu = \frac{1}{2} i_\nu L_\rho L_\kappa C_\varsigma (\zeta^{\mu\nu\rho\kappa} d\sigma), \quad (130)$$

where  $i_\nu = i_{\partial_\nu}$  and  $L_\rho = L_{\partial_\rho}$ .

*Proof.* From (126) and (68) we have

$$\begin{aligned}
\tau^\mu[\phi_{\mu\nu} dx^\nu] &= \frac{1}{2} i_\alpha L_\rho L_\kappa C_\zeta(\zeta^{\mu\alpha\rho\kappa} d\sigma)[\phi_{\mu\nu} dx^\nu] = \frac{1}{2} L_\rho L_\kappa C_\zeta(\zeta^{\mu\alpha\rho\kappa} d\sigma)[i_\alpha \phi_{\mu\nu} dx^\nu] \\
&= \frac{1}{2} \delta_\alpha^\nu L_\rho L_\kappa C_\zeta(\zeta^{\mu\alpha\rho\kappa} d\sigma)[\phi_{\mu\nu}] = \frac{1}{2} L_\rho L_\kappa C_\zeta(\zeta^{\mu\nu\rho\kappa} d\sigma)[\phi_{\mu\nu}] \\
&= \frac{1}{2} C_\zeta(\zeta^{\mu\nu\rho\kappa} d\sigma)[\partial_\rho \partial_\kappa \phi_{\mu\nu}] = \frac{1}{2} \int_{\mathcal{I}} \zeta^{\mu\nu\rho\kappa} (\partial_\rho \partial_\kappa \phi_{\mu\nu}) d\sigma = \int_{\mathcal{I}} T^{\mu\nu} \phi_{\mu\nu} d^4x.
\end{aligned}$$

□

In an adapted coordinate system (14) equation (75) implies

$$\tau^\mu = i_\nu C_\zeta(\gamma^{\mu\nu 00} d\sigma) + i_\nu L_a C_\zeta(\gamma^{\mu\nu 0a} d\sigma) + \frac{1}{2} i_\nu L_a L_b C_\zeta(\gamma^{\mu\nu ab} d\sigma). \quad (131)$$

*Proof.*

$$\begin{aligned}
\tau^\mu[\phi_{\mu\nu} dx^\nu] &= i_\alpha C_\zeta(\gamma^{\mu\alpha 00} d\sigma)[\phi_{\mu\nu} dx^\nu] + i_\alpha L_a C_\zeta(\gamma^{\mu\alpha 0a} d\sigma)[\phi_{\mu\nu} dx^\nu] \\
&\quad + \frac{1}{2} i_\alpha L_a L_b C_\zeta(\gamma^{\mu\alpha ab} d\sigma)[\phi_{\mu\nu} dx^\nu] \\
&= C_\zeta(\gamma^{\mu\nu 00} d\sigma)[\phi_{\mu\nu}] + L_a C_\zeta(\gamma^{\mu\nu 0a} d\sigma)[\phi_{\mu\nu}] + \frac{1}{2} L_a L_b C_\zeta(\gamma^{\mu\nu ab} d\sigma)[\phi_{\mu\nu}] \\
&= C_\zeta(\gamma^{\mu\nu 00} d\sigma)[\phi_{\mu\nu}] - C_\zeta(\gamma^{\mu\nu 0a} d\sigma)[\partial_a \phi_{\mu\nu}] + \frac{1}{2} C_\zeta(\gamma^{\mu\nu ab} d\sigma)[\partial_a \partial_b \phi_{\mu\nu}] \\
&= \int_{\mathcal{I}} \gamma^{\mu\nu 00} \phi_{\mu\nu} d\sigma - \int_{\mathcal{I}} \gamma^{\mu\nu 0a} (\partial_a \phi_{\mu\nu}) d\sigma + \frac{1}{2} \int_{\mathcal{I}} \gamma^{\mu\nu ab} (\partial_a \partial_b \phi_{\mu\nu}) d\sigma \\
&= \int_{\mathcal{I}} \left( \gamma^{\mu\nu 00} \phi_{\mu\nu} - \gamma^{\mu\nu 0a} (\partial_a \phi_{\mu\nu}) + \frac{1}{2} \gamma^{\mu\nu ab} (\partial_a \partial_b \phi_{\mu\nu}) \right) d\sigma.
\end{aligned}$$

□

As stated the advantage of using an adapted coordinate system is that the  $\gamma^{\mu\nu\rho\kappa}$  are unique, as seen from (31), (32).

## 6.6 Semi-dipoles and semi-quadrupoles

Having defined the quadrupoles in a coordinate free manner, one can identify properties which can be defined without reference to a coordinate system. In [27] we defined the semi-dipole and semi-quadrupole electromagnetic 3-form. The semi-dipole corresponded to the purely electric dipole. One can likewise define the semi-dipole and semi-quadrupole stress-energy distributions. In this case we say that  $\tau_\alpha$  is an semi-multipole of order at most  $\ell$  if

$$\tau_\alpha[\lambda^\ell d\mu] = 0 \quad \text{for all } \lambda, \mu \in \Gamma\Lambda^0 M \quad \text{such that } C^*(\lambda) = C^*(\mu) = 0. \quad (132)$$

We observe that the semi-dipole ( $\ell = 1$ ) corresponds to the case when the spin tensor is  $S^{ba} = 0$ . The semi-quadrupole ( $\ell = 2$ ), does not have a natural interpretation, but is used as a quadrupole with fewer components.

When we apply this to the quadrupole (131), in an adapted coordinate system  $(\sigma, z^1, z^2, z^3)$ , we see that the semi-quadrupole is given by

$$\tau^\mu = i_\nu C_\zeta(\gamma^{\mu\nu 00} d\sigma) + i_\nu L_a C_\zeta(\gamma^{\mu\nu 0a} d\sigma) + \frac{1}{2} L_a L_b C_\zeta(\gamma^{\mu 0ab}). \quad (133)$$

This gives 22 ODE components and 6 free components as indicated in table 1. We presented the general solution for the static semi-quadrupole in section 4.1.

*Proof of (133) and Semi-quadrupole counting.* A simple application using  $\lambda = \lambda_1 + \lambda_2$  and  $\lambda = \lambda_1 - \lambda_2$  implies we can replace (132) for  $\ell = 2$  with (132) with  $\tau_\alpha[\lambda_1 \lambda_2 da] = 0$  where  $C^*(\lambda_1) = C^*(\lambda_2) = C^*(a) = 0$ . Hence

$$0 = \lim_{\epsilon \rightarrow 0} \tau^\mu[z^a z^b d(z^c \psi_{\epsilon, \sigma'})] = \gamma^{\mu cab}(\sigma')$$

and hence (30).

We can now count the number and type of components. The dynamic equation (79) and (80) remain unchanged but (81) becomes

$$\gamma^{cb0a} = -\Gamma^c_{00} \gamma^{00ab} \quad \text{and} \quad \dot{\gamma}^{00ab} = -2\gamma^{0(b0)a} - \Gamma^0_{00} \gamma^{00ab},$$

since the symmetry condition (77) implies  $\gamma^{c0ab} = \gamma^{0cab} = 0$ . Thus we have  $4+12+6=22$  ODEs.

Starting with the 100 components given after applying (77) we have  $9 \times 6 = 54$  constraints coming from  $\gamma^{\mu cab} = 0$  plus 18 constraints coming from the first equation above. This leaves 28 components. Of these 22 are given by the ODEs and 6 are free.  $\square$

## 6.7 The coordinate free definition of the Dixon split only using $N$ and the connection

We have defined the stress-energy distribution without reference to a coordinate system. When writing this in terms of coordinates (130) and (36) we see that this corresponds directly to the Ellis representation of the multipoles. Here we show how to perform the Dixon split (42) which separates the multipoles into different orders with respect to a 1-form  $N$  along the curve. Our procedure separates the quadrupole into a pure Dixon quadrupole term, a pure Dixon dipole term and a monopole term. The pattern however is clear. The Dixon split (42) requires defining  $\tau_{(0)}$ ,  $\tau_{(1)}$  and  $\tau_{(2)}$  such that an arbitrary quadrupole has the form

$$\tau = \tau_{(0)} + \tau_{(1)} + \tau_{(2)}. \quad (134)$$

Using (126) to convert these into  $T_{(r)}^{\mu\nu}$  we find that  $T^{\mu\nu} = T_{(0)}^{\mu\nu} + T_{(1)}^{\mu\nu} + T_{(2)}^{\mu\nu}$  where

$$\tau_{(0)}[\phi] = \int_{\mathcal{M}} T_{(0)}^{\mu\nu} \phi_{\mu\nu} d^4x = \int_{\mathcal{I}} \xi^{\mu\nu}(\sigma) \phi_{\mu\nu}(\sigma) d\sigma, \quad (135)$$

$$\tau_{(1)}[\phi] = \int_{\mathcal{M}} T_{(1)}^{\mu\nu} \phi_{\mu\nu} d^4x = \int_{\mathcal{I}} \xi^{\mu\nu\rho}(\sigma) (\bar{\nabla}_\rho \phi_{\mu\nu})|_{C(\sigma)} d\sigma, \quad (136)$$

$$\tau_{(2)}[\phi] = \int_{\mathcal{M}} T_{(2)}^{\mu\nu} \phi_{\mu\nu} d^4x = \int_{\mathcal{I}} \xi^{\mu\nu\rho\kappa}(\sigma) (\bar{\nabla}_\rho \bar{\nabla}_\kappa \phi_{\mu\nu})|_{C(\sigma)} d\sigma. \quad (137)$$

The Dixon split is with respect to a 1-form, as opposed to a vector along  $C$ , in order to avoid using the metric. The one requirement is that the 1-form  $N$  combined with the vector  $\dot{C}$  is nowhere zero, i.e.

$$N(\dot{C}) \neq 0. \quad (138)$$

In order to perform the Dixon split, it is necessary to define a radial vector fields. We say that  $R \in \Gamma TM$  is **Radial** (to second order) with respect to  $C$  and  $N$  if for all  $p = C(\sigma)$

$$R|_p = 0, \quad (\nabla_V R)|_p = V|_p \quad \text{and} \quad (\nabla_{U,V}^2 R)|_p = 0, \quad (139)$$

for all vectors  $U, V \in T\mathcal{M}$  such that  $N(V) = N(U) = 0$ . Below in (145) we express the components of  $R$  with respect to a coordinate system, which is adapted both for  $C$  and  $N$ .

Using this radial vector, the Dixon split (134) is given by

$$\tau_{(0)}[\phi] = \tau[\phi - \nabla_R \phi + \frac{1}{2} \nabla_{R,R}^2 \phi], \quad (140)$$

$$\tau_{(1)}[\phi] = \tau[\nabla_R \phi - \nabla_{R,R}^2 \phi], \quad (141)$$

$$\tau_{(2)}[\phi] = \tau[\frac{1}{2} \nabla_{R,R}^2 \phi]. \quad (142)$$

where  $\phi$  is an type (0,2) test tensor. The proof of (140)-(142) is given below. One advantage of using (140)-(142) is that one can now show how the Dixon components mix when one changes  $N$ , and that all Ellis multipoles are also Dixon Multipoles.

## Proofs of Dixon split

In this section we work in a coordinate system  $(\sigma, z^1, z^2, z^3)$ , which is adapted both for  $C$  and  $N$ , so that  $N = N_0 d\sigma$  with  $N_0 \neq 0$ . We see that if  $N(V) = 0$  then  $V^0 = 0$ . Likewise we can replace  $\xi^{\mu\nu\rho\kappa}$  with  $\xi^{\mu\nu ab}$  since  $\xi^{\mu\nu 0a} = \xi^{\mu\nu a0} = 0$ .

In this coordinate system a radial vector  $R$  has the properties

$$\begin{aligned} R^\mu|_p = 0, \quad \partial_\mu R^0|_p = 0, \quad \partial_\mu R^a|_p = \delta_\mu^a, \quad \partial_0 \partial_\mu R^\nu|_p = 0, \\ \partial_b \partial_c R^0|_p = -2\Gamma_{bc}^0 \quad \text{and} \quad \partial_b \partial_c R^a|_p = -\Gamma_{bc}^a, \end{aligned} \quad (143)$$

for any  $p = C(\sigma)$ . This can be expressed as

$$R^0 = -z^b z^c \Gamma_{bc}^0 \partial_0 + O(\mathbf{z}^3) \quad \text{and} \quad R^a = z^a - \frac{1}{2} z^b z^c \Gamma_{bc}^a + O(\mathbf{z}^3), \quad (144)$$

or alternatively as

$$R = z^a \partial_a - z^b z^c \Gamma_{bc}^0 \partial_0 - \frac{1}{2} z^b z^c \Gamma_{bc}^a \partial_a + O(\mathbf{z}^3). \quad (145)$$

where  $O(\mathbf{z}^3)$  is any function (or vector) of  $(\sigma, z^1, z^2, z^3)$  which is at least cubic in its  $z^a$  arguments.

*Proof of (143).* In the adapted coordinate system, assume first that  $R^\mu$  satisfies (143) and that  $U, V$  satisfy  $N(U) = N(V) = 0$ , so  $U^0 = V^0 = 0$ .

Clearly from either (139.1) or (143.1) we have  $R|_p = 0$ . Here (139.1) refers to the first equation in (139).

$$(\nabla_V R - V)^\mu|_p = (V^\nu \partial_\nu (R^\mu) + V^\nu R^\rho \Gamma_{\nu\rho}^\mu - V^\mu)|_p = (V^a (\partial_a (R^\mu) - \delta_a^\mu))|_p.$$

Thus (139.2) is equivalent to (143.2), (143.3). From (143.2) and (143.3) we have (143.4)

From (139.2) we have, (implicitly evaluating at  $p$ ),

$$\begin{aligned} \bar{\nabla}_b (\bar{\nabla}_c R^a) &= \partial_b (\bar{\nabla}_c R^a) + (\bar{\nabla}_c R^d) \Gamma_{bd}^a - (\bar{\nabla}_d R^a) \Gamma_{bc}^d \\ &= \partial_b \partial_c R^a + \partial_b (R^e \Gamma_{ce}^a) + (\partial_c R^d) \Gamma_{bd}^a + R^e \Gamma_{ce}^d \Gamma_{bd}^a - (\partial_d R^a) \Gamma_{bc}^d - R^e \Gamma_{de}^a \Gamma_{bc}^d \\ &= \partial_b \partial_c R^a + \delta_b^e \Gamma_{ce}^a + \delta_c^d \Gamma_{bd}^a - \delta_d^a \Gamma_{bc}^d = \partial_b \partial_c R^a + \Gamma_{cb}^a + \Gamma_{bc}^a - \Gamma_{bc}^a \\ &= \partial_b \partial_c R^a + \Gamma_{cb}^a \end{aligned}$$

and

$$\begin{aligned} \bar{\nabla}_b (\bar{\nabla}_c R^0) &= \partial_b (\bar{\nabla}_c R^0) + (\bar{\nabla}_c R^d) \Gamma_{bd}^0 - (\bar{\nabla}_d R^0) \Gamma_{bc}^d = \partial_b \partial_c R^0 + \partial_b (R^e \Gamma_{ce}^0) + \Gamma_{bc}^0 \\ &= \partial_b \partial_c R^0 + \partial_b (R^e) \Gamma_{ce}^0 + \Gamma_{bc}^0 = \partial_b \partial_c R^0 + 2\Gamma_{bc}^0. \end{aligned}$$

Thus

$$\begin{aligned} \nabla_{U,V}^2 R &= V^\mu U^\nu (\partial_\mu \partial_\nu R^a + \Gamma_{cb}^a) \partial_a + V^\mu U^\nu (\partial_\mu \partial_\nu R^0 + 2\Gamma_{cb}^a) \partial_0 \\ &= V^a U^b (\partial_b \partial_c R^a + \Gamma_{cb}^a) \partial_a + V^a U^b (\partial_b \partial_c R^0 + 2\Gamma_{cb}^a) \partial_0. \end{aligned}$$

Hence (139.3) holds if and only if (143.5) and (143.6) hold. □

*Proof of (142).* In the adapted coordinate system and evaluating at  $C(\sigma)$  we have

$$\xi^{\mu\nu} (R^\alpha R^\lambda \bar{\nabla}_\alpha \bar{\nabla}_\lambda \phi_{\mu\nu}) = 0.$$

Thus the monopole term (135) does not contribute to  $\tau_{(2)}$ . Likewise

$$\xi^{\mu\nu\rho\kappa} \bar{\nabla}_\rho (R^\alpha R^\lambda \bar{\nabla}_\alpha \bar{\nabla}_\lambda \phi_{\mu\nu}) = 0,$$

so the dipole term (136) does not contribute to  $\tau_{(2)}$ . Finally we have

$$\begin{aligned} \xi^{\mu\nu\rho\kappa} \bar{\nabla}_\rho \bar{\nabla}_\kappa (R^\alpha R^\lambda \bar{\nabla}_\alpha \bar{\nabla}_\lambda \phi_{\mu\nu}) &= \xi^{\mu\nu ab} \bar{\nabla}_a \bar{\nabla}_b (R^\alpha R^\lambda \bar{\nabla}_\alpha \bar{\nabla}_\lambda \phi_{\mu\nu}) = \xi^{\mu\nu ab} (\partial_a \partial_b (R^\alpha R^\lambda) \bar{\nabla}_\alpha \bar{\nabla}_\lambda \phi_{\mu\nu}) \\ &= \xi^{\mu\nu ab} (\delta_a^\alpha \delta_b^\lambda + \delta_a^\lambda \delta_b^\alpha) (\bar{\nabla}_\alpha \bar{\nabla}_\lambda \phi_{\mu\nu}) = 2\xi^{\mu\nu ab} (\bar{\nabla}_a \bar{\nabla}_b \phi_{\mu\nu}) = 2\xi^{\mu\nu\rho\kappa} (\bar{\nabla}_\rho \bar{\nabla}_\kappa \phi_{\mu\nu}). \end{aligned} \quad (146)$$

Thus  $\tau_{(2)}$  is given by (142). □

*Proof of (141).* Since

$$\xi^{\mu\nu}(R^\alpha \bar{\nabla}_\alpha \phi_{\mu\nu} - R^\alpha R^\lambda \bar{\nabla}_\alpha \bar{\nabla}_\lambda \phi_{\mu\nu}) = 0,$$

the monopole term does not contribute to  $\tau_{(1)}$ . Also

$$\begin{aligned} \bar{\nabla}_a \bar{\nabla}_b (R^\alpha \bar{\nabla}_\alpha \phi_{\mu\nu}) &= \bar{\nabla}_a ((\bar{\nabla}_b R^\alpha) \bar{\nabla}_\alpha \phi_{\mu\nu}) + \bar{\nabla}_a (R^\alpha \bar{\nabla}_b \bar{\nabla}_\alpha \phi_{\mu\nu}) \\ &= (\bar{\nabla}_a \bar{\nabla}_b R^\alpha) \bar{\nabla}_\alpha \phi_{\mu\nu} + (\bar{\nabla}_b R^\alpha) \bar{\nabla}_a \bar{\nabla}_\alpha \phi_{\mu\nu} + (\bar{\nabla}_a R^\alpha) \bar{\nabla}_b \bar{\nabla}_\alpha \phi_{\mu\nu} + R^\alpha \bar{\nabla}_a \bar{\nabla}_b \bar{\nabla}_\alpha \phi_{\mu\nu} \\ &= \delta_b^\alpha \bar{\nabla}_a \bar{\nabla}_\alpha \phi_{\mu\nu} + \delta_a^\alpha \bar{\nabla}_b \bar{\nabla}_\alpha \phi_{\mu\nu} = \bar{\nabla}_a \bar{\nabla}_b \phi_{\mu\nu} + \bar{\nabla}_b \bar{\nabla}_a \phi_{\mu\nu}. \end{aligned}$$

Hence

$$\begin{aligned} \xi^{\mu\nu\rho\kappa} \bar{\nabla}_\rho \bar{\nabla}_\kappa (R^\alpha \bar{\nabla}_\alpha \phi_{\mu\nu}) &= \xi^{\mu\nu ab} \bar{\nabla}_a \bar{\nabla}_b (R^\alpha \bar{\nabla}_\alpha \phi_{\mu\nu}) = \xi^{\mu\nu ab} (\bar{\nabla}_a \bar{\nabla}_b \phi_{\mu\nu} + \bar{\nabla}_b \bar{\nabla}_a \phi_{\mu\nu}) \\ &= 2\xi^{\mu\nu ab} (\bar{\nabla}_a \bar{\nabla}_b \phi_{\mu\nu}) = 2\xi^{\mu\nu\rho\kappa} (\bar{\nabla}_\rho \bar{\nabla}_\kappa \phi_{\mu\nu}). \end{aligned} \quad (147)$$

Thus using (146) we see

$$\xi^{\mu\nu\rho\kappa} \bar{\nabla}_\rho \bar{\nabla}_\kappa (R^\alpha \bar{\nabla}_\alpha \phi_{\mu\nu} - R^\alpha R^\lambda \bar{\nabla}_\alpha \bar{\nabla}_\lambda \phi_{\mu\nu}) = 0.$$

Thus the quadrupole term (137) does not contribute to  $\tau_{(1)}$ . Finally

$$\begin{aligned} \xi^{\mu\nu\rho} \bar{\nabla}_\rho (R^\alpha \bar{\nabla}_\alpha \phi_{\mu\nu}) &= \xi^{\mu\nu a} \bar{\nabla}_a (R^\alpha \bar{\nabla}_\alpha \phi_{\mu\nu}) = \xi^{\mu\nu a} (\bar{\nabla}_a R^\alpha) \bar{\nabla}_\alpha \phi_{\mu\nu} = \xi^{\mu\nu a} \delta_a^\alpha \bar{\nabla}_\alpha \phi_{\mu\nu} \\ &= \xi^{\mu\nu a} \bar{\nabla}_a \phi_{\mu\nu} = \xi^{\mu\nu\rho} \bar{\nabla}_\rho \phi_{\mu\nu}. \end{aligned} \quad (148)$$

Thus  $\tau_{(1)}$  is given by (141). □

*Proof of (140).* From (146) and (147) we have

$$\xi^{\mu\nu\rho\kappa} \bar{\nabla}_\rho \bar{\nabla}_\kappa (\phi_{\mu\nu} - R^\alpha \bar{\nabla}_\alpha \phi_{\mu\nu} + \frac{1}{2} R^\alpha R^\lambda \bar{\nabla}_\alpha \bar{\nabla}_\lambda \phi_{\mu\nu}) = 0.$$

Thus the quadrupole term (137) does not contribute to  $\tau_{(0)}$ . Using (148) we have

$$\xi^{\mu\nu\rho} \bar{\nabla}_\rho (\phi_{\mu\nu} - R^\alpha \bar{\nabla}_\alpha \phi_{\mu\nu} + \frac{1}{2} R^\alpha R^\lambda \bar{\nabla}_\alpha \bar{\nabla}_\lambda \phi_{\mu\nu}) = 0,$$

so the dipole term (136) does not contribute to  $\tau_{(0)}$ . Finally

$$\xi^{\mu\nu} (\phi_{\mu\nu} - R^\alpha \bar{\nabla}_\alpha \phi_{\mu\nu} + \frac{1}{2} R^\alpha R^\lambda \bar{\nabla}_\alpha \bar{\nabla}_\lambda \phi_{\mu\nu}) = \xi^{\mu\nu} \phi_{\mu\nu},$$

so  $\tau_{(0)}$  is given by (140). □

## 7 Discussion and outlook.

We have derived a number of key results about the distributional quadrupole stress-energy tensor, in particular the existence of the free components, which require additional constitutive relations to prescribe. An example of these constitutive relations is given. We have also given the coordinate transformation of the quadrupole components, the conserved quantities in the presence of a Killing vector and a definition of semi-quadrupoles. We presented a metric and coordinate free definition of the quadrupole and a way of separating the quadrupole into the monopole, dipole and quadrupole terms corresponding to the Dixon representation.

The understanding of the quadrupole stress-energy tensor distribution is important for the study of gravitational wave sources, as well as being interesting in its own right. The existence of free components imply that it is not possible to know everything about a quadrupole simply from the initial conditions. There is clearly much research that needs to be done to find appropriate constitutive relations to replace the free components with ODEs or algebraic relations. It may be possible to calculate these from underlying models, such as two orbiting black holes or neutron stars. In



section 5 we presented only a very simple constitutive relation corresponding to a dust model. With increasing sensitivity of gravitational wave astronomy one can hope to test the different constitutive relations using experimental data.

Although the observation of the need for constitutive relations for the quadrupole on a prescribe worldline is new, there are other cases where the need for constitutive relations has been observed. For example [15], they are needed to determine how dipoles or quadrupoles affect the worldline. There are other situations where one can expect constitutive relations will be needed. In future work we intend to look at the dynamics of charged multipoles in an electromagnetic field. One would expect in this case that constitutive relations are also needed, especially since a dipole has nine components, but the electromagnetic current, which provides the force and torque, has only six components. These constitutive relations describe the differences between the charge distribution and the mass distribution in the dipole. The situation has an additional challenge in that the electromagnetic field diverges on the worldline. This poses another question that has been tackled by many authors: how does a dipole respond to its own electromagnetic field [33, 34, 35].

We have investigated the representation of the quadrupole using partial derivatives (the Ellis representation). As well as the differential equations, we have given the gauge-like freedom, the change of coordinates, the adapted coordinates and the change of coordinates for adapted coordinates. It is natural to ask what new features will arise for sextupoles. One will expect that the gauge-like freedom for sextupoles will include a term with  $\dot{C}^\mu C^\nu \int^\sigma C^\rho d\sigma'$ .

Having definitions which are coordinate free can be very useful. They make it clear which objects are coordinate dependent and which are truly geometric. Although the Ellis representation of multipoles is easy to define in a coordinate free manner, here we have derived a coordinate free definition of Dixon's split of the quadrupole into the pure monopole, dipole and quadrupole terms. In future work, we intend to reproduce these results in coordinates. This will enable us to write the Dixon components  $\xi^{\mu\nu\dots}$  in terms of Ellis components  $\zeta^{\mu\nu\dots}$ , and also derive the complicated relationship between the Dixon components  $\xi^{\mu\nu\dots}$  for different  $N^\mu$ . The dynamical equations for the Ellis quadrupole components were derived (79)-(82). Using this split, one could translate these into dynamical equations for the Dixon components. This will enable a comparison between the 20 independent components of (reduced) quadrupole stress-energy tensor as described by Dixon, as discussed in appendix B.

Although spacetime is endowed with both a metric and a connection, there is much research into which objects can be defined without such structures. In some cases this questions the underlying physics, asking whether the electromagnetic field is more fundamental than the gravitational field [36]. In other cases it is useful for examining how an object depends on a metric or a connection. This is necessary when calculating the result of varying a Lagrangian with respect to the metric. It is useful therefore, that a general multipole does not require any additional structures beyond those in a general manifold. This means that one can define multipoles on other manifolds such as tangent bundles or jet bundles. Such an approach may also give an insight into prescribing constitutive relations. Of course a connection is required in order to demand the stress-energy distribution be divergenceless, but there is no requirement that such a connection be Levi-Civita. All the coordinate free presentation from section 6 does not require a metric, so one can choose a metric compatible or a non metric compatible connection. We have demanded that the connection is torsion free. On the whole this is to simplify the equations so that we do not have to write down all the torsion components and their derivatives. One can reproduce the results with these extra terms and it would be interesting to see how the results depend on the torsion.

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# Appendix

## A Details of the proofs

### A.1 Proof from introductory sections

*Proof number 1: Proof of (6): gravitational waves from a distribution.* Fix components  $\mu\nu$  with respect to the global Cartesian coordinate system and set  $\mathfrak{f}_\varepsilon$  and  $f$  to be the components  $\mathcal{T}_\varepsilon^{\mu\nu}$  and  $(T^{(1)})^{\mu\nu}$ . Then  $\mathfrak{f}_\varepsilon$  is a family of regular scalars with support in a region  $\mathcal{U}$  about the worldline  $C$  and  $f$  is a distribution on the worldline and  $\mathfrak{f}_\varepsilon \rightarrow f$  weakly. Given a point  $(t, \vec{x})$  not on the worldline and outside  $\mathcal{U}$ , let  $\mathcal{V}$  be the intersection of the backward light cone of  $(t, \vec{x})$  and  $\mathcal{U}$ . The assumption after (6) implies  $\mathcal{V}$  is compact.

Introduce an adapted coordinate system (27) which is adapted for both the worldline and the backward light cone of the point  $(t, \vec{x})$  and such that  $z^a = x^a - x'^a$ .

Let  $\chi(\mathbf{z})$  be a test function which coincides with  $|\mathbf{z}|^{-1}$  on  $\mathcal{V}$  and let  $\psi(\sigma)$  another test function such that the support of  $\psi(\sigma)\chi(\mathbf{z})$  does not include  $(t, \vec{x})$ . Then since  $\psi(\sigma)\chi(\mathbf{z})$  is a test function

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_{\mathcal{I}} d\sigma \psi(\sigma) \int \chi(\mathbf{z}) \mathfrak{f}_\varepsilon(\sigma, \mathbf{z}) d^3 \mathbf{z} &= \lim_{\varepsilon \rightarrow 0} \int_{\mathcal{I}} \int \chi(\mathbf{z}) \psi(\sigma) \mathfrak{f}_\varepsilon(\sigma, \mathbf{z}) d^3 \mathbf{z} d\sigma = \int_{\mathcal{I}} \int \chi(\mathbf{z}) \psi(\sigma) f(\sigma, \mathbf{z}) d^3 \mathbf{z} d\sigma \\ &= \int_{\mathcal{I}} \psi(\sigma) d\sigma \int \chi(\mathbf{z}) f(\sigma, \mathbf{z}) d^3 \mathbf{z}. \end{aligned}$$

Since this is true for all appropriate  $\psi(\sigma)$  we have

$$\lim_{\varepsilon \rightarrow 0} \int \frac{\mathfrak{f}_\varepsilon(\sigma, \mathbf{z})}{|\mathbf{z}|} d^3 \mathbf{z} = \lim_{\varepsilon \rightarrow 0} \int \chi(\mathbf{z}) \mathfrak{f}_\varepsilon(\sigma, \mathbf{z}) d^3 \mathbf{z} = \int \chi(\mathbf{z}) f(\sigma, \mathbf{z}) d^3 \mathbf{z} = \int \frac{f(\sigma, \mathbf{z})}{|\mathbf{z}|} d^3 \mathbf{z}.$$

Hence (6). □

### A.2 Proofs about the quadrupole

*Proof number 2: Proof of (72): change of coordinates for quadrupole.* This is similar to the proof of (53). Using (68) we have

$$\begin{aligned} \int_{\mathcal{I}} \hat{\zeta}^{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\kappa}} (\partial_{\hat{\rho}} \partial_{\hat{\kappa}} \hat{\phi}_{\hat{\mu}\hat{\nu}})|_{C(\sigma)} d\sigma &= \int_{\mathbb{R}^4} \hat{T}^{\hat{\mu}\hat{\nu}} \hat{\phi}_{\hat{\mu}\hat{\nu}} d^4 \hat{x} = \int_{\mathbb{R}^4} T^{\mu\nu} \phi_{\mu\nu} d^4 x = \int_{\mathcal{I}} \zeta^{\mu\nu\rho\kappa} (\partial_{\rho} \partial_{\kappa} \phi_{\mu\nu}) d\sigma \\ &= \int_{\mathcal{I}} \zeta^{\mu\nu\rho\kappa} \partial_{\rho} \partial_{\kappa} (\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} \hat{\phi}_{\hat{\mu}\hat{\nu}}) d\sigma \\ &= \int_{\mathcal{I}} \zeta^{\mu\nu\rho\kappa} \left( \partial_{\rho} \partial_{\kappa} (\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}}) \hat{\phi}_{\hat{\mu}\hat{\nu}} + 2 \partial_{\rho} (\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}}) \partial_{\kappa} \hat{\phi}_{\hat{\mu}\hat{\nu}} + \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} \partial_{\rho} \partial_{\kappa} \hat{\phi}_{\hat{\mu}\hat{\nu}} \right) d\sigma. \end{aligned}$$

Take each of the terms in turn. For the third term we have

$$\begin{aligned} \int_{\mathcal{I}} \zeta^{\mu\nu\rho\kappa} \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} \partial_{\rho} \partial_{\kappa} \hat{\phi}_{\hat{\mu}\hat{\nu}} d\sigma &= \int_{\mathcal{I}} \zeta^{\mu\nu\rho\kappa} \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} \partial_{\rho} (J_{\kappa}^{\hat{\kappa}} \partial_{\hat{\kappa}} \hat{\phi}_{\hat{\mu}\hat{\nu}}) d\sigma \\ &= \int_{\mathcal{I}} \zeta^{\mu\nu\rho\kappa} \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} \left( (\partial_{\rho} J_{\kappa}^{\hat{\kappa}}) \partial_{\hat{\kappa}} \hat{\phi}_{\hat{\mu}\hat{\nu}} + J_{\kappa}^{\hat{\kappa}} \partial_{\rho} \partial_{\hat{\kappa}} \hat{\phi}_{\hat{\mu}\hat{\nu}} \right) d\sigma \\ &= \int_{\mathcal{I}} \zeta^{\mu\nu\rho\kappa} \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} (\partial_{\rho} J_{\kappa}^{\hat{\kappa}}) \partial_{\hat{\kappa}} \hat{\phi}_{\hat{\mu}\hat{\nu}} d\sigma + \int_{\mathcal{I}} \zeta^{\mu\nu\rho\kappa} \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} J_{\rho}^{\hat{\rho}} J_{\kappa}^{\hat{\kappa}} \partial_{\hat{\rho}} \partial_{\hat{\kappa}} \hat{\phi}_{\hat{\mu}\hat{\nu}} d\sigma \\ &= \int_{\mathcal{I}} \zeta^{\mu\nu\rho\kappa} \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} (\partial_{\rho} J_{\kappa}^{\hat{\kappa}}) \left( \int^{\sigma} \hat{C}^{\hat{\rho}} \partial_{\hat{\rho}} \partial_{\hat{\kappa}} \hat{\phi}_{\hat{\mu}\hat{\nu}} d\sigma' \right) d\sigma + \int_{\mathcal{I}} \zeta^{\mu\nu\rho\kappa} \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} J_{\rho}^{\hat{\rho}} J_{\kappa}^{\hat{\kappa}} \partial_{\hat{\rho}} \partial_{\hat{\kappa}} \hat{\phi}_{\hat{\mu}\hat{\nu}} d\sigma \\ &= - \int_{\mathcal{I}} \left( \int^{\sigma} \zeta^{\mu\nu\rho\kappa} \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} (\partial_{\rho} J_{\kappa}^{\hat{\kappa}}) d\sigma' \right) \hat{C}^{\hat{\rho}} \partial_{\hat{\rho}} \partial_{\hat{\kappa}} \hat{\phi}_{\hat{\mu}\hat{\nu}} d\sigma + \int_{\mathcal{I}} \zeta^{\mu\nu\rho\kappa} \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} J_{\rho}^{\hat{\rho}} J_{\kappa}^{\hat{\kappa}} \partial_{\hat{\rho}} \partial_{\hat{\kappa}} \hat{\phi}_{\hat{\mu}\hat{\nu}} d\sigma. \end{aligned}$$

For the second term we have

$$\begin{aligned}
\int_{\mathcal{I}} \zeta^{\mu\nu\rho\kappa} \partial_\rho (\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}}) \partial_\kappa \hat{\phi}_{\hat{\mu}\hat{\nu}} d\sigma &= \int_{\mathcal{I}} \zeta^{\mu\nu\rho\kappa} \partial_\rho (\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}}) J_\kappa^{\hat{\kappa}} \partial_{\hat{\kappa}} \hat{\phi}_{\hat{\mu}\hat{\nu}} d\sigma \\
&= \int_{\mathcal{I}} \zeta^{\mu\nu\rho\kappa} \partial_\rho (\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}}) J_\kappa^{\hat{\kappa}} \left( \int^\sigma \hat{C}^{\hat{\rho}} \partial_{\hat{\rho}} \partial_{\hat{\kappa}} \hat{\phi}_{\hat{\mu}\hat{\nu}} d\sigma' \right) d\sigma \\
&= - \int_{\mathcal{I}} \left( \int^\sigma \zeta^{\mu\nu\rho\kappa} \partial_\rho (\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}}) J_\kappa^{\hat{\kappa}} d\sigma' \right) \hat{C}^{\hat{\rho}} \partial_{\hat{\rho}} \partial_{\hat{\kappa}} \hat{\phi}_{\hat{\mu}\hat{\nu}} d\sigma.
\end{aligned}$$

For the first term we have

$$\begin{aligned}
\int_{\mathcal{I}} \zeta^{\mu\nu\rho\kappa} \partial_\rho \partial_\kappa (\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}}) \hat{\phi}_{\hat{\mu}\hat{\nu}} d\sigma &= \int_{\mathcal{I}} \zeta^{\mu\nu\rho\kappa} \partial_\rho \partial_\kappa (\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}}) \left( \int^\sigma \hat{C}^{\hat{\rho}} \partial_{\hat{\rho}} \hat{\phi}_{\hat{\mu}\hat{\nu}} d\sigma' \right) d\sigma \\
&= - \int_{\mathcal{I}} \left( \int^\sigma \zeta^{\mu\nu\rho\kappa} \partial_\rho \partial_\kappa (\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}}) d\sigma' \right) \hat{C}^{\hat{\rho}} \partial_{\hat{\rho}} \hat{\phi}_{\hat{\mu}\hat{\nu}} d\sigma \\
&= - \int_{\mathcal{I}} \left( \int^\sigma \zeta^{\mu\nu\rho\kappa} \partial_\rho \partial_\kappa (\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}}) d\sigma'' \right) \hat{C}^{\hat{\rho}} \left( \int^\sigma \partial_{\hat{\rho}} \hat{C}^{\hat{\kappa}} \partial_{\hat{\kappa}} \hat{\phi}_{\hat{\mu}\hat{\nu}} d\sigma' \right) d\sigma \\
&= \int_{\mathcal{I}} \left( \int^\sigma \left( \int^{\sigma'} \zeta^{\mu\nu\rho\kappa} \partial_\rho \partial_\kappa (\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}}) d\sigma'' \right) \hat{C}^{\hat{\rho}} d\sigma' \right) \hat{C}^{\hat{\kappa}} \partial_{\hat{\rho}} \partial_{\hat{\kappa}} \hat{\phi}_{\hat{\mu}\hat{\nu}} d\sigma \\
&= \int_{\mathcal{I}} \left( \hat{C}^{\hat{\kappa}} \int^\sigma \left( \hat{C}^{\hat{\rho}} \int^{\sigma'} \zeta^{\mu\nu\rho\kappa} \partial_\rho \partial_\kappa (\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}}) d\sigma'' \right) d\sigma' \right) \partial_{\hat{\rho}} \partial_{\hat{\kappa}} \hat{\phi}_{\hat{\mu}\hat{\nu}} d\sigma.
\end{aligned}$$

Thus adding these terms together we have

$$\begin{aligned}
\int_{\mathcal{I}} \hat{\zeta}^{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\kappa}} (\partial_{\hat{\rho}} \partial_{\hat{\kappa}} \hat{\phi}_{\hat{\mu}\hat{\nu}}) d\sigma &= \int_{\mathcal{I}} \zeta^{\mu\nu\rho\kappa} \left( \partial_\rho \partial_\kappa (\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}}) \hat{\phi}_{\hat{\mu}\hat{\nu}} + 2 \partial_\rho (\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}}) \partial_\kappa \hat{\phi}_{\hat{\mu}\hat{\nu}} + \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} \partial_\rho \partial_\kappa \hat{\phi}_{\hat{\mu}\hat{\nu}} \right) d\sigma \\
&= - \int_{\mathcal{I}} \left( \int^\sigma \zeta^{\mu\nu\rho\kappa} \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} (\partial_\rho J_\kappa^{\hat{\kappa}}) d\sigma' \right) \hat{C}^{\hat{\rho}} \partial_{\hat{\rho}} \partial_{\hat{\kappa}} \hat{\phi}_{\hat{\mu}\hat{\nu}} d\sigma + \int_{\mathcal{I}} \zeta^{\mu\nu\rho\kappa} \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} J_\rho^{\hat{\rho}} J_\kappa^{\hat{\kappa}} \partial_{\hat{\rho}} \partial_{\hat{\kappa}} \hat{\phi}_{\hat{\mu}\hat{\nu}} d\sigma \\
&\quad - 2 \int_{\mathcal{I}} \left( \int^\sigma \zeta^{\mu\nu\rho\kappa} \partial_\rho (\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}}) J_\kappa^{\hat{\kappa}} d\sigma' \right) \hat{C}^{\hat{\rho}} \partial_{\hat{\rho}} \partial_{\hat{\kappa}} \hat{\phi}_{\hat{\mu}\hat{\nu}} d\sigma \\
&\quad + \int_{\mathcal{I}} \left( \hat{C}^{\hat{\kappa}} \int^\sigma \left( \hat{C}^{\hat{\rho}} \int^{\sigma'} \zeta^{\mu\nu\rho\kappa} \partial_\rho \partial_\kappa (\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}}) d\sigma'' \right) d\sigma' \right) \partial_{\hat{\rho}} \partial_{\hat{\kappa}} \hat{\phi}_{\hat{\mu}\hat{\nu}} d\sigma \\
&= \int_{\mathcal{I}} \left( \zeta^{\mu\nu\rho\kappa} \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} J_\rho^{\hat{\rho}} J_\kappa^{\hat{\kappa}} - \hat{C}^{\hat{\rho}} \int^\sigma \zeta^{\mu\nu\rho\kappa} \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} (\partial_\rho J_\kappa^{\hat{\kappa}}) d\sigma' - 2 \hat{C}^{\hat{\rho}} \int^\sigma \zeta^{\mu\nu\rho\kappa} \partial_\rho (\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}}) J_\kappa^{\hat{\kappa}} d\sigma' \right. \\
&\quad \left. + \hat{C}^{\hat{\kappa}} \int^\sigma \left( \hat{C}^{\hat{\rho}} \int^{\sigma'} \zeta^{\mu\nu\rho\kappa} \partial_\rho \partial_\kappa (\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}}) d\sigma'' \right) d\sigma' \right) \partial_{\hat{\rho}} \partial_{\hat{\kappa}} \hat{\phi}_{\hat{\mu}\hat{\nu}} d\sigma.
\end{aligned}$$

Hence (72) follows by symmetrising  $\hat{\rho}$  and  $\hat{\kappa}$ . □

*Proof number 3: Proof that the change of coordinates (72) is consistent with the gauge-like freedom (71).* First observe that the lower limits in (72) correspond to the gauge-like freedom (71) for  $\hat{\zeta}^{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\kappa}}$ .

It is necessary to establish that the gauge-like freedom (71) for  $\zeta^{\mu\nu\rho\kappa}$  when substituted into (72) does not affect the value of  $\hat{\zeta}^{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\kappa}}$ . This is achieved by setting  $\zeta^{\mu\nu\rho\kappa} = M_2^{\nu\mu} \dot{C}^{(\rho} C^{\kappa)} + M_3^{\mu\nu(\rho} \dot{C}^{\kappa)}$ , i.e.  $\zeta^{\mu\nu\rho\kappa}$  is equivalent to zero, and checking that  $\hat{\zeta}^{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\kappa}} = 0$ . As they are independent, we can consider the two terms  $M_2^{\nu\mu} \dot{C}^{(\rho} C^{\kappa)}$  and  $M_3^{\mu\nu(\rho} \dot{C}^{\kappa)}$  separately.

For the case  $\zeta^{\mu\nu\rho\kappa} = M_2^{\nu\mu} \dot{C}^{(\rho} C^{\kappa)}$  we have for the fifth term on the right hand side of (72)

$$\begin{aligned}
& \int^\sigma \hat{C}^{\hat{\kappa}} \int^{\sigma'} \dot{C}^{(\rho} C^{\kappa)} \partial_\rho \partial_\kappa (\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}}) d\sigma'' d\sigma' = \int^\sigma \hat{C}^{\hat{\kappa}} \int^{\sigma'} \dot{C}^\rho C^\kappa \partial_\rho \partial_\kappa (\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}}) d\sigma'' d\sigma' \\
& = \int^\sigma \hat{C}^{\hat{\kappa}} \int^{\sigma'} C^\kappa \frac{d}{d\sigma''} (\partial_\kappa \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}}) d\sigma'' d\sigma' \\
& = \int^\sigma \hat{C}^{\hat{\kappa}} \int^{\sigma'} \frac{d}{d\sigma''} (C^\kappa \partial_\kappa \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}}) d\sigma'' d\sigma' - \int^\sigma \hat{C}^{\hat{\kappa}} \int^{\sigma'} \dot{C}^\kappa \partial_\kappa \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} d\sigma'' d\sigma' \\
& = \int^\sigma \hat{C}^{\hat{\kappa}} C^\kappa \partial_\kappa \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} d\sigma' - \int^\sigma \hat{C}^{\hat{\kappa}} \int^{\sigma'} \frac{d}{d\sigma''} \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} d\sigma'' d\sigma' \\
& = \int^\sigma \hat{C}^{\hat{\kappa}} C^\kappa \partial_\kappa \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} d\sigma' - \int^\sigma \hat{C}^{\hat{\kappa}} \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} d\sigma' \\
& = \int^\sigma \hat{C}^{\hat{\kappa}} C^\kappa \partial_\kappa \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} d\sigma' - \int^\sigma \dot{C}^\kappa J_\kappa^{\hat{\kappa}} \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} d\sigma' \\
& = \int^\sigma \dot{C}^\kappa J_\kappa^{\hat{\kappa}} C^\rho \partial_\rho \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} d\sigma' - C^\kappa J_\kappa^{\hat{\kappa}} \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} + \int^\sigma C^\kappa \frac{d}{d\sigma'} (J_\kappa^{\hat{\kappa}} \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}}) d\sigma'.
\end{aligned}$$

Since

$$\int^\sigma \dot{C}^\kappa C^\rho \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} \partial_\rho J_\kappa^{\hat{\kappa}} d\sigma' = \int^\sigma \dot{C}^\kappa C^\rho \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} \partial_\kappa J_\rho^{\hat{\kappa}} d\sigma' = \int^\sigma \dot{C}^\rho C^\kappa \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} \partial_\rho J_\kappa^{\hat{\kappa}} d\sigma',$$

while for the second term in (72)

$$\begin{aligned}
& \int_{\mathcal{I}} (M_3^{\mu\nu(\rho} \dot{C}^{\kappa)}) \left( \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} (\partial_\rho J_\kappa^{\hat{\kappa}}) + 2 \partial_\rho (\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}}) J_\kappa^{\hat{\kappa}} \right) d\sigma' \\
& = \frac{1}{2} M_3^{\mu\nu\rho} \int_{\mathcal{I}} \dot{C}^\kappa \left( \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} (\partial_\rho J_\kappa^{\hat{\kappa}}) + 2 \partial_\rho (\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}}) J_\kappa^{\hat{\kappa}} \right) d\sigma' + \frac{1}{2} M_3^{\mu\nu\kappa} \int_{\mathcal{I}} \dot{C}^\rho \left( \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} (\partial_\rho J_\kappa^{\hat{\kappa}}) + 2 \partial_\rho (\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}}) J_\kappa^{\hat{\kappa}} \right) d\sigma' \\
& = M_3^{\mu\nu\rho} \int_{\mathcal{I}} \dot{C}^\kappa J_\kappa^{\hat{\kappa}} (\partial_\rho \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}}) d\sigma' + M_3^{\mu\nu\kappa} \int_{\mathcal{I}} \dot{C}^\rho \left( \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} (\partial_\rho J_\kappa^{\hat{\kappa}}) + \partial_\rho (\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}}) J_\kappa^{\hat{\kappa}} \right) d\sigma' \\
& = M_3^{\mu\nu\rho} \int_{\mathcal{I}} \hat{C}^{\hat{\kappa}} (\partial_\rho \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}}) d\sigma' + M_3^{\mu\nu\kappa} \int_{\mathcal{I}} \frac{d}{d\sigma'} \left( \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} J_\kappa^{\hat{\kappa}} \right) d\sigma' \\
& = M_3^{\mu\nu\rho} \int_{\mathcal{I}} \hat{C}^{\hat{\kappa}} (\partial_\rho \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}}) d\sigma' + M_3^{\mu\nu\kappa} \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} J_\kappa^{\hat{\kappa}}.
\end{aligned}$$

Hence when  $\zeta^{\mu\nu\rho\kappa} = M_3^{\mu\nu(\rho} \dot{C}^{\kappa)}$  then  $\hat{\zeta}^{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\kappa}} = 0$ . □

*Proof number 4: Proof that (71) incorporates all the gauge-like freedom.* Assume  $T^{\mu\nu}$  is given. From (31) we know that the components  $\gamma^{\mu\nu\rho\kappa}$  are unique, i.e. have no gauge-like freedom. Integrating (78) we have

$$\zeta^{\mu\nu\rho\kappa} \rightarrow \zeta^{\mu\nu\rho\kappa} + \sigma M_2^{\nu\mu} \delta_0^\rho \delta_0^\kappa + M_3^{\mu\nu(\kappa} \delta_0^{\rho)},$$

which is (71) in adapted coordinates. Hence (71) is incorporates all gauge-like freedom, in adapted coordinates. Now for a general coordinate system we use (72). We see in the proof 3 in the appendix, that (72) is consistent with the gauge-like freedom. Thus there is no additional gauge-like freedom in a general coordinate system. □

*Proof number 5: Proof of (83)-(85) The coordinate transformation for adapted coordinates.* This follows from substituting (78) into (72).

We set  $(x^0, \dots, x^3) = (\sigma, z^1, z^2, z^3)$  and  $(\hat{x}^0 \dots \hat{x}^3) = (\hat{\sigma}, \hat{z}^1, \hat{z}^2, \hat{z}^3)$  into (72) and use the fact that  $\hat{C}^{\hat{\mu}} = \delta_0^{\hat{\mu}}$ . Hence (83) follows directly.

For (84) we have from (72)

$$\begin{aligned}
\hat{\zeta}^{\hat{\mu}\hat{\nu}\hat{c}\hat{d}} &= \zeta^{\mu\nu\rho\kappa} \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} J_{\rho}^{\hat{c}} J_{\kappa}^{\hat{d}} - \frac{1}{2} \hat{C}^{\hat{c}} \int^{\sigma} \zeta^{\mu\nu\rho\kappa} \left( \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} (\partial_{\rho} J_{\kappa}^{\hat{d}}) + 2 \partial_{\rho} (\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}}) J_{\kappa}^{\hat{d}} \right) d\sigma' \\
&\quad - \frac{1}{2} \hat{C}^{\hat{d}} \int^{\sigma} \zeta^{\mu\nu\rho\kappa} \left( \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} (\partial_{\rho} J_{\kappa}^{\hat{c}}) + 2 \partial_{\rho} (\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}}) J_{\kappa}^{\hat{c}} \right) d\sigma' \\
&\quad + \frac{1}{2} \hat{C}^{\hat{c}} \int^{\sigma} \hat{C}^{\hat{d}} \int^{\sigma'} \zeta^{\mu\nu\rho\kappa} \partial_{\rho} \partial_{\kappa} (\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}}) d\sigma'' d\sigma' + \frac{1}{2} \hat{C}^{\hat{d}} \int^{\sigma} \hat{C}^{\hat{c}} \int^{\sigma'} \zeta^{\mu\nu\rho\kappa} \partial_{\rho} \partial_{\kappa} (\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}}) d\sigma'' d\sigma' \\
&= \zeta^{\mu\nu\rho\kappa} \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} J_{\rho}^{\hat{c}} J_{\kappa}^{\hat{d}} - \frac{1}{2} \int^{\sigma} \zeta^{\mu\nu\rho\kappa} \left( \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} (\partial_{\rho} J_{\kappa}^{\hat{c}}) + 2 \partial_{\rho} (\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}}) J_{\kappa}^{\hat{c}} \right) d\sigma'.
\end{aligned}$$

Thus from (78)

$$\begin{aligned}
\hat{\gamma}^{\hat{\mu}\hat{\nu}\hat{c}\hat{d}} &= \dot{\hat{\zeta}}^{\hat{\mu}\hat{\nu}\hat{c}\hat{d}} = (\zeta^{\mu\nu\rho\kappa} \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} J_{\rho}^{\hat{c}} J_{\kappa}^{\hat{d}}) \cdot - \frac{1}{2} \zeta^{\mu\nu\rho\kappa} (\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} J_{\rho\kappa}^{\hat{c}} + 2 \partial_{\rho} (\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}}) J_{\kappa}^{\hat{c}}) \\
&= (\zeta^{\mu\nu 00} \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} J_0^{\hat{c}} J_0^{\hat{d}}) \cdot + (\zeta^{\mu\nu c0} \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} J_c^{\hat{c}} J_0^{\hat{d}}) \cdot + (\zeta^{\mu\nu 0c} \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} J_0^{\hat{c}} J_c^{\hat{d}}) \cdot + (\zeta^{\mu\nu cd} \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} J_c^{\hat{c}} J_d^{\hat{d}}) \cdot \\
&\quad - \frac{1}{2} \zeta^{\mu\nu 00} (\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} J_{00}^{\hat{c}} + 2 \partial_0 (\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}}) J_0^{\hat{c}}) - \frac{1}{2} \zeta^{\mu\nu c0} (\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} J_{c0}^{\hat{c}} + 2 \partial_c (\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}}) J_0^{\hat{c}}) \\
&\quad - \frac{1}{2} \zeta^{\mu\nu c0} (\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} J_{c0}^{\hat{c}} + 2 \partial_0 (\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}}) J_c^{\hat{c}}) - \frac{1}{2} \zeta^{\mu\nu cd} (\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} J_{cd}^{\hat{c}} + 2 \partial_c (\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}}) J_d^{\hat{c}}) \\
&= (\zeta^{\mu\nu c0} \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} J_c^{\hat{c}}) \cdot + (\zeta^{\mu\nu cd} \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} J_c^{\hat{c}} J_d^{\hat{d}}) \cdot \\
&\quad - \zeta^{\mu\nu c0} (\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} J_{c0}^{\hat{c}} + \dot{\mathcal{J}}_{\mu\nu}^{\hat{\mu}\hat{\nu}} J_c^{\hat{c}}) - \frac{1}{2} \zeta^{\mu\nu cd} (\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} J_{cd}^{\hat{c}} + 2 \partial_c (\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}}) J_d^{\hat{c}}) \\
&= \dot{\zeta}^{\mu\nu c0} \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} J_c^{\hat{c}} + (\zeta^{\mu\nu cd} \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} J_c^{\hat{c}} J_d^{\hat{d}}) \cdot - \frac{1}{2} \zeta^{\mu\nu cd} (\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} J_{cd}^{\hat{c}} + 2 \partial_c (\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}}) J_d^{\hat{c}}) \\
&= \gamma^{\mu\nu c0} \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} J_c^{\hat{c}} + (\gamma^{\mu\nu cd} \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} J_c^{\hat{c}} J_d^{\hat{d}}) \cdot - \frac{1}{2} \gamma^{\mu\nu cd} (\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} J_{cd}^{\hat{c}} + 2 \partial_c (\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}}) J_d^{\hat{c}}),
\end{aligned}$$

where  $J_{\nu\rho}^{\hat{\mu}} = \partial_{\nu} J_{\rho}^{\hat{\mu}}$ .

In order to show (85) we have from (72)

$$\begin{aligned}
\hat{\zeta}^{\mu\nu\hat{0}\hat{0}} &= (\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} J_{\rho}^{\hat{0}} J_{\kappa}^{\hat{0}}) \zeta^{\mu\nu\rho\kappa} - \int^{\sigma} \left( (\partial_{\kappa} J_{\rho}^{\hat{0}}) J_{\mu\nu}^{\hat{\mu}\hat{\nu}} + J_{\rho}^{\hat{0}} \partial_{\kappa} J_{\mu\nu}^{\hat{\mu}\hat{\nu}} + J_{\kappa}^{\hat{0}} \partial_{\rho} J_{\mu\nu}^{\hat{\mu}\hat{\nu}} \right) \zeta^{\mu\nu\rho\kappa} d\sigma' \\
&\quad + \int^{\sigma} d\sigma' \int^{\sigma'} \partial_{\rho\kappa} \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} \zeta^{\mu\nu\rho\kappa} d\sigma'',
\end{aligned}$$

where  $\partial_{\rho\kappa} = \partial_{\rho} \partial_{\kappa}$ . Hence

$$\begin{aligned}
\hat{\gamma}^{\hat{\mu}\hat{\nu}\hat{0}\hat{0}} &= \frac{1}{2} \dot{\hat{\zeta}}^{\hat{\mu}\hat{\nu}\hat{0}\hat{0}} \\
&= \frac{1}{2} \left( ((\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} J_{\rho}^{\hat{0}} J_{\kappa}^{\hat{0}}) \zeta^{\mu\nu\rho\kappa}) \cdot - (((\partial_{\kappa} J_{\rho}^{\hat{0}}) \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} + J_{\rho}^{\hat{0}} \partial_{\kappa} \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} + J_{\kappa}^{\hat{0}} \partial_{\rho} \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}}) \zeta^{\mu\nu\rho\kappa}) \cdot + \partial_{\rho\kappa} \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} \zeta^{\mu\nu\rho\kappa} \right).
\end{aligned} \tag{149}$$

It is important to establish that all the  $\zeta^{\mu\nu\rho\kappa}$  on the right hand side of (149) can be replaced by the corresponding  $\gamma^{\mu\nu\rho\kappa}$  without using integrals. However since from (78)  $\gamma^{\mu\nu 00} = \frac{1}{2} \ddot{\zeta}^{\mu\nu 00}$  and  $\gamma^{\mu\nu a0} = \dot{\zeta}^{\mu\nu a0}$  we need to expand (149) to confirm that no terms  $\zeta^{\mu\nu 00}$ ,  $\dot{\zeta}^{\mu\nu 00}$  or  $\zeta^{\mu\nu a0}$  exist on the right hand side.

$$\begin{aligned}
\hat{\gamma}^{\hat{\mu}\hat{\nu}\hat{0}\hat{0}} &= \frac{1}{2} \left( (\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} J_{\kappa}^{\hat{0}} J_{\rho}^{\hat{0}}) \zeta^{\mu\nu\rho\kappa} \right) \cdot - \left( \left( \frac{1}{2} J_{\rho\kappa}^{\hat{0}} \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} + J_{\kappa}^{\hat{0}} \partial_{\rho} \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} \right) \zeta^{\mu\nu\rho\kappa} \right) \cdot + \left( \frac{1}{2} \partial_{\rho\kappa} \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} \right) \zeta^{\mu\nu\rho\kappa} \\
&= \frac{1}{2} \left( (\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} J_0^{\hat{0}} J_0^{\hat{0}}) \zeta^{\mu\nu 00} \right) \cdot + \left( (\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} J_c^{\hat{0}} J_d^{\hat{0}}) \zeta^{\mu\nu cd} \right) \cdot + \frac{1}{2} \left( (\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} J_d^{\hat{0}} J_c^{\hat{0}}) \zeta^{\mu\nu cd} \right) \cdot \\
&\quad - \left( \left( \frac{1}{2} J_{00}^{\hat{0}} \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} + J_0^{\hat{0}} \partial_0 \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} \right) \zeta^{\mu\nu 00} \right) \cdot - \left( \left( J_{0c}^{\hat{0}} \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} + J_c^{\hat{0}} \partial_0 \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} + J_0^{\hat{0}} \partial_c \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} \right) \zeta^{\mu\nu c0} \right) \cdot \\
&\quad - \left( \left( \frac{1}{2} J_{cd}^{\hat{0}} \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} + J_c^{\hat{0}} \partial_d \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} \right) \zeta^{\mu\nu cd} \right) \cdot \\
&\quad + \left( \frac{1}{2} \partial_{00} \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} \right) \zeta^{\mu\nu 00} + \left( \partial_{0c} \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} \right) \zeta^{\mu\nu c0} + \left( \frac{1}{2} \partial_{cd} \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} \right) \zeta^{\mu\nu cd} \\
&= \frac{1}{2} \left( \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} \zeta^{\mu\nu 00} \right) \cdot + \left( (\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} J_c^{\hat{0}}) \zeta^{\mu\nu c0} \right) \cdot + \frac{1}{2} \left( (\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} J_d^{\hat{0}} J_c^{\hat{0}}) \zeta^{\mu\nu cd} \right) \cdot - \left( \dot{\mathcal{J}}_{\mu\nu}^{\hat{\mu}\hat{\nu}} \zeta^{\mu\nu 00} \right) \cdot \\
&\quad - \left( \left( J_{0c}^{\hat{0}} \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} + J_c^{\hat{0}} \dot{\mathcal{J}}_{\mu\nu}^{\hat{\mu}\hat{\nu}} + \partial_c \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} \right) \zeta^{\mu\nu c0} \right) \cdot - \left( \left( \frac{1}{2} J_{cd}^{\hat{0}} \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} + J_d^{\hat{0}} \partial_c \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} \right) \zeta^{\mu\nu cd} \right) \cdot \\
&\quad + \frac{1}{2} \ddot{\mathcal{J}}_{\mu\nu}^{\hat{\mu}\hat{\nu}} \zeta^{\mu\nu 00} + \left( \partial_c \dot{\mathcal{J}}_{\mu\nu}^{\hat{\mu}\hat{\nu}} \right) \zeta^{\mu\nu c0} + \left( \frac{1}{2} \partial_{cd} \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} \right) \zeta^{\mu\nu cd} \\
&= \frac{1}{2} \ddot{\mathcal{J}}_{\mu\nu}^{\hat{\mu}\hat{\nu}} \zeta^{\mu\nu 00} + \dot{\mathcal{J}}_{\mu\nu}^{\hat{\mu}\hat{\nu}} \zeta^{\mu\nu c0} + \frac{1}{2} \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} \ddot{\zeta}^{\mu\nu 00} \\
&\quad + \ddot{\mathcal{J}}_{\mu\nu}^{\hat{\mu}\hat{\nu}} J_c^{\hat{0}} \zeta^{\mu\nu c0} + \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} J_{c0}^{\hat{0}} \zeta^{\mu\nu c0} + \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} J_c^{\hat{0}} \ddot{\zeta}^{\mu\nu c0} + 2 \dot{\mathcal{J}}_{\mu\nu}^{\hat{\mu}\hat{\nu}} J_{c0}^{\hat{0}} \zeta^{\mu\nu c0} + 2 \dot{\mathcal{J}}_{\mu\nu}^{\hat{\mu}\hat{\nu}} J_c^{\hat{0}} \dot{\zeta}^{\mu\nu c0} + 2 \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} J_{c0}^{\hat{0}} \dot{\zeta}^{\mu\nu c0} \\
&\quad + \frac{1}{2} \left( (\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} J_d^{\hat{0}} J_c^{\hat{0}}) \zeta^{\mu\nu cd} \right) \cdot - \ddot{\mathcal{J}}_{\mu\nu}^{\hat{\mu}\hat{\nu}} \zeta^{\mu\nu 00} - \dot{\mathcal{J}}_{\mu\nu}^{\hat{\mu}\hat{\nu}} \dot{\zeta}^{\mu\nu 00} \\
&\quad - \left( J_{0c}^{\hat{0}} \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} + 2 J_{0c}^{\hat{0}} \dot{\mathcal{J}}_{\mu\nu}^{\hat{\mu}\hat{\nu}} + J_c^{\hat{0}} \ddot{\mathcal{J}}_{\mu\nu}^{\hat{\mu}\hat{\nu}} + \partial_{0c} \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} \right) \zeta^{\mu\nu c0} - \left( J_{0c}^{\hat{0}} \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} + J_c^{\hat{0}} \dot{\mathcal{J}}_{\mu\nu}^{\hat{\mu}\hat{\nu}} + \partial_c \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} \right) \dot{\zeta}^{\mu\nu c0} \\
&\quad - \left( \left( \frac{1}{2} J_{cd}^{\hat{0}} \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} + J_d^{\hat{0}} \partial_c \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} \right) \zeta^{\mu\nu cd} \right) \cdot \\
&\quad + \frac{1}{2} \ddot{\mathcal{J}}_{\mu\nu}^{\hat{\mu}\hat{\nu}} \zeta^{\mu\nu 00} + \left( \partial_c \dot{\mathcal{J}}_{\mu\nu}^{\hat{\mu}\hat{\nu}} \right) \zeta^{\mu\nu c0} + \left( \frac{1}{2} \partial_{cd} \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} \right) \zeta^{\mu\nu cd} \\
&= \frac{1}{2} \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} \ddot{\zeta}^{\mu\nu 00} + \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} J_c^{\hat{0}} \ddot{\zeta}^{\mu\nu c0} + 2 \dot{\mathcal{J}}_{\mu\nu}^{\hat{\mu}\hat{\nu}} J_c^{\hat{0}} \dot{\zeta}^{\mu\nu c0} + 2 \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} J_{c0}^{\hat{0}} \dot{\zeta}^{\mu\nu c0} + \frac{1}{2} \left( (\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} J_d^{\hat{0}} J_c^{\hat{0}}) \zeta^{\mu\nu cd} \right) \cdot \\
&\quad - \left( J_{0c}^{\hat{0}} \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} + J_c^{\hat{0}} \dot{\mathcal{J}}_{\mu\nu}^{\hat{\mu}\hat{\nu}} + \partial_c \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} \right) \dot{\zeta}^{\mu\nu c0} - \left( \left( \frac{1}{2} J_{cd}^{\hat{0}} \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} + J_d^{\hat{0}} \partial_c \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} \right) \zeta^{\mu\nu cd} \right) \cdot + \left( \frac{1}{2} \partial_{cd} \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} \right) \zeta^{\mu\nu cd} \\
&= \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} \gamma^{\mu\nu 00} + \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} J_c^{\hat{0}} \dot{\gamma}^{\mu\nu c0} + \left( (\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} J_c^{\hat{0}}) \cdot - \partial_c \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} \right) \gamma^{\mu\nu c0} \\
&\quad + \frac{1}{2} \left( (\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} J_d^{\hat{0}} J_c^{\hat{0}}) \gamma^{\mu\nu cd} \right) \cdot - \left( \left( \frac{1}{2} J_{cd}^{\hat{0}} \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} + J_d^{\hat{0}} \partial_c \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} \right) \gamma^{\mu\nu cd} \right) \cdot + \left( \frac{1}{2} \partial_{cd} \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} \right) \gamma^{\mu\nu cd} .
\end{aligned}$$

□

## B Dixon's independent components

In the following we refer to [12] as [II] and [13] as [III].

It may appear that the 20 free components in this article directly correspond to the 20 independent components of  $J^{\mu\nu\rho\kappa}$  given in [III(1.37)]. This follows because as well as both having 20 quadrupole components, they both arise from the divergenceless condition (10).

We can relate the Dixon moments to our moments as follows. In [III(10.17)] we see the term  $I[\Phi_{\lambda\mu}]$  when we expand out the right hand side. To unpick this we use in turn [III(10.9)], [III(10.6)],

[II(4.5)], [II(7.4)] to give

$$\begin{aligned}
I[\Phi_{\lambda\mu}] &= (2\pi)^{-4} \int_{\mathcal{I}} d\sigma \int_{T_{C(\sigma)}} Dk \tilde{I}^{\lambda\mu}(\sigma, k) \tilde{\Phi}_{\lambda\mu}(C(\sigma), k) \\
&= (2\pi)^{-4} \int_{\mathcal{I}} d\sigma \int_{T_{C(\sigma)}} Dk \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} k_{\kappa_1} \cdots k_{\kappa_n} I^{\kappa_1 \cdots \kappa_n \lambda\mu}(\sigma) \tilde{\Phi}_{\lambda\mu}(C(\sigma), k) \\
&= (2\pi)^{-4} \sum_{n=0}^N \frac{1}{n!} \int_{\mathcal{I}} d\sigma I^{\kappa_1 \cdots \kappa_n \lambda\mu}(\sigma) \nabla_{(\kappa_1 \cdots \kappa_n)} \phi_{\alpha\beta} \\
&\quad + (2\pi)^{-4} \int_{\mathcal{I}} d\sigma \int_{T_{C(\sigma)}} Dk \sum_{n=N+1}^{\infty} \frac{(-i)^n}{n!} k_{\kappa_1} \cdots k_{\kappa_n} I^{\kappa_1 \cdots \kappa_n \lambda\mu}(\sigma) \tilde{\Phi}_{\lambda\mu}(C(\sigma), k),
\end{aligned}$$

since from [III(10.17)]  $\Phi_{\lambda\mu} = \text{Exp}^A \phi_{\alpha\beta}$ . Here the moments  $I^{\kappa_1 \cdots \kappa_n \lambda\mu}$  satisfy the symmetry conditions [III(10.3)], and orthogonality condition [III(10.4)]. Thus the first term in the last expression corresponds to the right hand side of (41) if we set  $\xi^{\lambda\mu\kappa_1 \cdots \kappa_n} = (-1)^n I^{\kappa_1 \cdots \kappa_n \lambda\mu}$ . Although the orthogonality condition does not completely correspond.

For the quadrupole Dixon [III(1.37)] constructs  $J^{\rho\mu\nu\kappa} = \frac{1}{4}(I^{\rho\mu\nu\kappa} - I^{\mu\rho\nu\kappa} - I^{\rho\mu\kappa\nu} + I^{\mu\rho\kappa\nu})$ . Since this automatically has the symmetries of the Riemann curvature tensor it has 20 independent components. Most of these symmetries are imposed because it is contracted with the Riemann curvature tensor [III(1.28), (1.29)].

The key difference is the imposition of the divergenceless condition. In [III] this is achieved by putting the divergence operator into the argument of  $I$  as seen in the term  $I[\frac{1}{2}\Lambda^k \nabla_{*\{\lambda} G_{\kappa\mu}\}]$  in [III(10.16)]. As stated as comment (vii) in [III page 109], this does not lead to any additional algebraic or differential equations for the  $I^{\kappa_1 \cdots \kappa_n \lambda\mu}$ . It then only affects the dynamics of the dipole [III(1.28),(1.29)]. By contrast in our treatment we apply the divergence operator directly to the distribution, and derive the ODEs and free components of the  $\gamma^{\mu\nu\rho\kappa}$ . In addition our free components do not have these symmetries.

Future work will be to convert the ODEs for  $\gamma^{\mu\nu\rho\kappa}$  into ODEs for the Dixon components  $\xi^{\mu\nu\rho\kappa}$ . The moments  $t^{\kappa_1 \cdots \kappa_n \lambda\mu}$  in [III(1.16)] can then be related to  $\xi^{\mu\nu\rho\kappa}$  via the squeeze tensors, but with the coordinates adapted to the hypersurfaces  $\Sigma(s)$  [III Just after (10.20)]. This will enable a direct comparison between [III] and the results in this article.