Economics Working Paper Series

2020/014

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Equilibrium Design by Coarse Correlation in Quadratic Games*

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23rd July 2020

Abstract

In a public good provision or a public bad abatement situation, the non-cooperative interplay of the participants typically results in low levels of provision or abatement. In the familiar class of n-person quadratic games, we show that Coarse Correlated equilibria (CCEs) - simple mediated communication devices that do not alter the strategic structure of the game - can significantly outperform the Nash equilibrium in terms of the policy objective above.

Keywords: Quadratic game, Coarse correlated equilibrium, Abatement level, Efficiency gain.

JEL Classification Numbers: C72, Q52.

*This paper supersedes the working paper titled "Improving Abatement Levels and Welfare by Coarse Correlation in an Environmental Game" by Dokka, Moulin, Ray and SenGupta (Dokka, Moulin, Ray, and SenGupta (2019)). Moulin's research was supported by the NSF under Grant CCF 1101202 while Ray's research was supported by the British Academy Small Grant on "Mediation in Environmental Games". We wish to thank all seminar and conference participants at Birmingham, Cardiff, ISI New Delhi, Royal Holloway, São Paulo and Surrey and in particular, Sourav Bhattacharya, Matthew Cole, Colin Rowat, Herakles Polemarchakis and Aurélie Slechten for many helpful comments.

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1 INTRODUCTION

Mediated communication is a true and tested way to achieve incentive-compatible coordination on efficient outcomes in many non-cooperative games; such schemes generate concepts like *correlated equilibrium* (CE), as proposed by Aumann (Aumann (1974), Aumann (1987)) and *coarse correlated equilibrium* (CCE) (Moulin and Vial (1978))[1].

In non-cooperative games, the main motivation behind these notions has been to improve upon Nash, it terms of (expected) utility of the players themselves. CCEs obtain better utility compared to Nash outcomes even when CEs fail to do so (for example, when the game is a *potential game*, as shown in (Neyman (1997)). Moulin, Ray, and SenGupta (2014) explicitly computed *utility-maximising CCEs* in a 2-player *quadratic game*.

There are many economic situations in which one may wish to improve Nash outcomes not (just) in terms of utility of the players, but rather with respect to an exogenous social objective like aggregate effort or output. Here we focus on two particular instances:

1. Public good provision game, with constant marginal cost of contribution and with the benefit from the public good being concave, quadratic in total contributions;
2. Emission abatement game (Barrett (1994)), where each country's benefit function is a function of the total abatement level chosen by all countries, and the cost function of each country is a convex quadratic function of its own abatement level.

In situations like above, a mediator (such as, a public tax authority or an environmental agency) might very well be more interested in equilibrium outcomes which can extract better quantity levels than that in a purely non-cooperative Nash outcome; for example, a mediator would prefer to choose a device which maximises total abatement levels achieved or the total contributions made to the public good provision.

In this paper, we analyse CCEs to find such desired levels in a class of games; we consider n-person symmetric quadratic games, with a concave potential function. Abatement game (Barrett (1994)), that can also be viewed as the public good game with quadratic costs, serves as our special case for this general analysis.

To illustrate our contribution, consider a baseline 2-person abatement game in which the utility of country i is given by the function $q - 2q^2 - q_i^2$, where q_i

[1] Moulin and Vial (1978) called this equilibrium concept a 'correlation scheme', while Young (2004) and Roughgarden (2009) called this 'coarse correlated equilibrium' and Forgo (2010) called it 'weak correlated equilibrium'.

is the choice of abatement level by country i and thus q $(= q_1 + q_2)$ is the total abatement level. One can show that for this game, the optimal CCE (total) utility is $\pi^{CCE} = \frac{23}{104} \approx 0.2211$, while the Nash (total) payoff is $\pi^N = \frac{11}{50} \approx 0.22$ and hence, $\frac{\pi^{CCE}}{\pi^N} \approx 1.0052$ (which implies an improvement over the Nash payoff by 0.5%, that may seem small but can be a significant amount if one thinks of real-life magnitudes); in this particular case, in terms of quantities, we get $\frac{q^{CCE}}{q^N} \approx 1.057$, an improvement over the Nash abatement by 5.7%. If one wishes to maximise just the abatement quantity level for this specific game, then the best possible scenario would have been associated with another similar CCE for which $\frac{q^{CCE}}{q^N} \approx 1.53$, i.e., 53% improvement over the Nash abatement level; however, the corresponding utility falls by about 35% below the Nash utility level. Of course, one may then naturally ask what if we wish to achieve at least the Nash utility while maximising the abatement level; the answer (for this particular baseline game) is that we can have a maximum of 11.5% improvement in the abatement maintaining such a restriction, with the utility just above (0.05% more than) the Nash utility.

Not surprisingly, as will be clear from our analysis below, there may be a tension between two equally important goals, utility and quantity (abatement) levels for such games. Our results illustrate the severity of the trade-off between the CCE maximising players' payoffs and the CCE maximising aggregate abatement.

There have been quite a few applications of CE and CCE to economic modeling in the literature in the recent past. The randomisation device in any CE or a CCE has a natural interpretation in many economic situations (see Arce (1995), Arce (1997)) and can be seen as mediating institutions like government agencies, international bodies or trade associations like European Union, WTO (World Trade Organisation), United Nations Framework Convention on Climate Change (UNFCCC), etc. For instance, in the context of climate change negotiation, an independent agency[2] (European commission for EU Emissions Trading Scheme) can provide recommendations to all the signatories towards the ultimate goal of global emission reduction (see Forgo, Fulop, and Prill (2005), Forgo (2010), Forgo (2011)). CCE in one-shot games are shown to approximate Nash equilibria in repeated games (see for example, Awaya and Krishna (2019), Awaya and Krishna (2020)). CCE structure can be seen in other games of economic significance with expending effort (Fleckinger (2012), Deb, Li, and Mukherjee (2016)), gathering information (Gromb and Martimort (2007), public good provision[3], duopoly models (Ray and SenGupta (2013), Gerard-Varet

[2]Although we can not point to a precise example in real life, our abstract mediator embodies in spirit the kind of commitment shown in the 1992 UNFCCC that several authors have analysed (see for example Slechten (2020)).

[3]See Reischmann and Oechssler (2018) for some experimental results in implementing CCE outcome in a repeated public good game. Also see Georgalos, Ray, and SenGupta (2020) for some experimental results on CCE.

and Moulin (1978)) etc; a number of studies (Forgo et al. (2005), Forgo (2010), Roughgarden (2009)) relate CCE to no-regret learners.

Our paper builds on the above 2-player example and proves the following general results:

1. we study quantity-maximising CCEs for any n-person symmetric quadratic games and find an analytical algorithm to compute the optimal quantity-maximising CCEs for such n-person games;

2. we analyse an n-person abatement game showing that utility and quantity levels are two conflicting aspects in such a scenario;

3. we formally characterise CCEs in n-person abatement games and show that the improvement in abatement over the Nash level increases depending on a single parameter r (increasing as r decreases); CCEs obtain higher quantity (abatement) levels even maintaining the same utility as in Nash equilibrium but this (relative) improvement diminishes with the number of players, n.

International and transnational environmental agreements have been the subject of many research papers in last two decades. Following Barrett's work on the feasibility of creating stable international environmental agreements (IEAs), a number of cooperative and non-cooperative game theoretic approaches have been explored including coalition formation and applications of coalitional form games, (see Finus (2008), Tulkens (1998), Barrett (2004), McGinty (2007), and the references therein). To the best of our knowledge CE and CCE have not been explored in this context. Our results provide theoretical underpinnings to the belief that mediation is instrumental in such discussions.

The contribution of this paper is two-fold. First as a theoretical exercise, our result is perhaps the first attempt of characterising the benefit from coarse correlation in this class of games. Second, as the importance of enforcing agreements is an important theme in the environmental literature, our characterisation for the abatement game suggests why and how a mediator (an independent agency) could be used for agreements and commitments in abatement games in practice.

Our results are in the same spirit as and complement the recent results in public good games extended to networks, see Pandit and Kulkarni (2018), Chadha and Kulkarni (2020), Bramoullé and Kranton (2007), where authors study effort and utility maximising Nash equilibria.

2 MODEL

2.1 Coarse Correlated Equilibrium (CCE)

Consider an n-person normal form game, $G = [Q_1, Q_2, \ldots, Q_n; u_1, u_2, \ldots, u_n]$, with $\mathbb{Q} = \prod_i Q_i$, where the strategy sets, Q_i's are closed real intervals and the payoff functions $u_i : \mathbb{Q} \to \mathbb{R}$, $i = 1, \ldots, n$, are continuous. We write $\mathbb{C}(\mathbb{Q})$ for the set of such continuous functions and similarly, $\mathbb{C}(Q_i)$ for the set of continuous functions on Q_i.

Let $\mathcal{L}(\mathbb{Q})$ with generic element L and $\mathcal{L}(Q_i)$ with generic element ℓ_i denote the sets of probability measures on $\mathbb{Q}$ respectively. Let the mean of $u_i(\mathbb{Q})$ with respect to L be denoted by $u_i(L)$.

The deterministic distribution at z is denoted by δ_z, and for product distributions such as $\delta_{q_1} \otimes \ell_2 \otimes \ldots \otimes \ell_n$ we write $u_1(\delta_{x_1} \otimes \ell_2 \otimes \ldots \otimes \ell_n)$ simply as $u_1(q_1, \ell_2, \ldots, \ell_n)$, and for short $u_1(q_1, \ell_{-1})$ or more generally for any player i we write $u_i(q_i, \ell_{-i})$. Given $L \in \mathcal{L}(\mathbb{Q})$, we write L^i for the marginal distribution of L on Q_i, defined as follows: $\forall f \in \mathbb{C}(Q_i)$, $f(L^i) = f^*(L)$, where $f^*(q_1, q_2, \ldots, q_n) = f(x_1)$ for all $(q_1, q_2, \ldots, q_n) \in \mathbb{Q}$.

Definition 1. *A coarse correlated equilibrium (CCE) of the game G is a lottery $L \in \mathcal{L}(\mathbb{Q})$ such that $u_i(L) \geq u_i(q_i, L^{-i})$ for all $(q_1, q_2, \ldots, q_n) \in \mathbb{Q}$.*

2.2 Quadratic Games and its CCEs

We consider the following symmetric n-player games that we call a *quadratic* game; in this game, the strategy sets $Q_i = \mathbb{R}_+$, for all i, and the payoffs are of the following general (quadratic) form:

$$u_i(q_1, q_2, \ldots, q_n) = \sum_{i=1}^{n} a q_i + \sum_{i=1}^{n} \sum_{j=1}^{n} b q_i q_j + c q_i^2, \tag{1}$$

where a,b, c are constants.

We now characterise the set of CCEs for the above game. If L is the distribution of a symmetric random variable $Z = (Z_1, \ldots, Z_n)$, consider respectively the expected values of Z_i, Z_i^2, and $Z_i \cdot Z_{j \neq i}$ and denote them as below; for $i = 1, \ldots, n$:

$$\begin{aligned} &\alpha = E_L[Z_i], \\ &\beta = E_L[Z_i^2] \text{ and} \\ &\gamma = E_L[Z_i \cdot Z_j] \text{ for any } j \neq i,\, j = 1, \ldots, n. \end{aligned}$$

We first show that the CCE participation constraint (as in Definition 1) for the n-player quadratic game can be completely expressed in terms of these three moments of L.

Lemma 1. *Any symmetric lottery $L \in \mathcal{L}^{sy}(\mathbb{R}_+^n)$ is a CCE of the quadratic game (1) if and only if*

$$max_{z \geq 0} \left\{ (a + b(n-1)\alpha) z + c z^2 \right\} \leq n a \alpha + b(n-1)\gamma + c\beta;$$

Proof of Lemma 1 above is a straightforward extension of the proof of Lemma 1 in Moulin et al. (2014), using Definition 1 above for our n-person quadratic game, and hence is omitted here. We now identify the range of the vector (α, β, γ) when $L \in \mathcal{L}^{sy}(\mathbb{R}^n_+)$ in the following lemma. Note the conditions given in Moulin et al., 2014 are special cases of the following derived set.

Lemma 2. *For any $L \in \mathcal{L}^{sy}(\mathbb{R}^n_+)$ and the corresponding n dimensional random variable $Z = (Z_i)$, we have*

$$\alpha, \gamma \geq 0;\ \beta \geq \gamma;\ \beta + (n-1)\gamma \geq n\alpha^2. \tag{2}$$

The proof of Lemma 2 is in the Appendix. Note that the conditions presented in Lemmata 1 and 2 of Moulin et al. (2014) are special cases of our results (Lemmata 1 and 2) above.

2.2.1 Quantity Maximisation

Note that the corresponding utility of any player i from the above CCE $\mathcal{L}^{sy}(\mathbb{R}^n_+)$ is:

$$na\alpha + \frac{n(n-1)}{2} b\gamma + c\beta.$$

As mentioned in the Introduction, in this paper we focus on achieving the maximum level of quantity (q_i's) in this quadratic game, instead of maximising the utility level using CCE. However, it is difficult to obtain any further explicit characterisation of the lotteries for the n-player case; so, we first provide an existence result of L for given parameters that satisfy (2).

First note that, given α, β and γ satisfying (2), it should be true that for some $\frac{k_1}{k_2} = \xi > 1$,

$$\beta = (k_1 + 1)\alpha^2 \tag{3}$$

$$\gamma = \frac{n-1-k_2}{n-1}\alpha^2 \tag{4}$$

Now observe that there exist distributions (for example, multivariate Gaussian) of the symmetric random variable $(Q_1, \ldots, Q_n)$ with the mean vector $(\mathbb{E}(Q_1) = \mathbb{E}(Q_n) = \alpha)$ and covariance matrix

$$\sum = k_1\alpha^2 \begin{bmatrix} 1 & \rho & \cdots & \rho \\ \rho & & & \rho \\ \vdots & & \ddots & \\ \rho & \rho & & 1 \end{bmatrix},$$

where $\rho = -\frac{1}{\xi(n-1)}$.

This argument leads to the following Lemma.

Lemma 3. *If α, β and γ meet the system (2), there exists a symmetric lottery L with precisely these parameters.*

Using symmetry, the quantity-maximising CCE can now be computed by solving the following convex quadratic programme:

$$\max_{\alpha,\beta,\gamma} \quad \alpha \tag{5}$$

$$\text{such that } \beta \geq \gamma \tag{6}$$

$$\beta + (n-1)\gamma \geq n\alpha^2 \tag{7}$$

$$(n-1)b\gamma + c\beta \geq max_{z\geq 0}\left\{(a + (n-1)b\alpha)z + cz^2\right\} - a\alpha \tag{8}$$

$$\alpha, \gamma \geq 0; \tag{9}$$

We are now ready to present the quantity-maximising CCE for our n-player quadratic game. Consider the following algorithm:

Algorithm 1. *Algorithm Quantity-CCE is given by the following steps:*

1. Fix α.
2. Find a feasible point in the polytope (5) - (9). If there exists such a point go to step 3; else, go to step 4.
3. Set $\alpha = \alpha + \epsilon$ and go to step 2.
4. Set $\epsilon = \frac{\epsilon}{2}$ and go to step 3.

Theorem 1. *Algorithm* 1 *(Algorithm Quantity-CCE) finds the quantity-maximising CCE for the n-person quadratic game (1).*

Theorem 1 can be proved easily (using Lemmata $1-3$). Since the objective of (5) - (9) is independent of β and γ, it is enough to find the largest value of α such that there exists at least one feasible solution to (6) - (9). Algorithm Quantity-CCE imitates binary search to find this maximum α, which proves our Theorem.

One may now wish to compare the maximised quantity levels from the optimal CCE found by Algorithm 1 with that from the Nash equilibrium of a quadratic game. However, for any arbitrary quadratic game (1), it involves too many possible cases, depending on the parameter values. Thus, we focus on a specific quadratic game from the literature (the abatement game as discussed in the Introduction) to provide some new insights.

3 ABATEMENT GAMES

We present below a specific quadratic game, proposed in Barrett (1994) as a model of pollution-abatement game played by n countries. The payoff function of a country $i = 1, 2, ..., n$ is a function of the abatement levels (q_i) chosen by the countries, with the total abatement as $Q = \sum_{i=1}^{n} q_i$. The benefit function[4] of country i is

$$B_i(Q) = \frac{B}{n}(AQ - \frac{Q^2}{2}),$$

while the cost function of each country is a convex quadratic function of its own abatement level q_i and is

$$C_i(q_i) = \frac{Cq_i^2}{2}.$$

The payoff function of country i is thus given by:

$$u_i(q_i, q_{j\neq i}) = \frac{AB}{n}(Q) - \frac{B}{2n}(Q)^2 - \frac{C}{2}q_i^2, \text{ where } A,\ B \text{ and } C \text{ are all positive.}$$

We now set $a = \frac{AB}{n}$, $b = \frac{B}{2n}$, $c = \frac{C}{2}$ for simplicity and rewrite the above payoff function in the following form:

$$u_i(q_i, q_{j\neq i}) = a(\sum_{i=1}^{n} q_i) - b(\sum_{i=1}^{n} q_i)^2 - cq_i^2. \tag{10}$$

We call the above game an *abatement game*. The (symmetric) Nash equilibrium level of abatement (q_i^{Neq}) and the corresponding payoff π_i^{Neq} are given by

$$q_i^{Neq} = \frac{a}{2(nb+c)};\ \pi_i^{Neq} = \frac{a^2[n^2b + (2n-1)c]}{4(nb+c)^2}.$$

The abatement game is clearly a potential game with the potential function

$$P(q_i) = a\left(\sum_{i=1}^{n} q_i\right) - b\left(\sum_{i=1}^{n}\sum_{j=1}^{n} q_i q_j\right) - c\sum_{i=1}^{n} q_i^2,$$

which is smooth and concave. Therefore, the only CE is the Nash equilibrium q^{Neq} (Neyman (1997)). The following Proposition characterises the CCE of this game.

Proposition 1. *A symmetric lottery $L \in \mathcal{L}^{sy}(\mathbb{R}_+^n)$ is a CCE of the abatement game if and only if*

$$\max_{z\geq 0}\{[a - 2(n-1)b\alpha]z - (b+c)z^2\} \leq a\alpha - (b+c)\beta - 2b(n-1)\gamma \tag{11}$$

and the corresponding utility (for country i) is

$$u_i(L) = na\alpha - (nb+c)\beta - n(n-1)b\gamma.$$

[4]Note that the benefit function in the published version of Barrett (1994) has a typo that we have corrected here.

Proof of Proposition 1 is a straightforward extension of the proof of Proposition 1 in Dokka et al. (2019), for our n-person abatement game, and hence is omitted here. The following Corollary is now immediate.

Corollary 1. *The lottery $L(\alpha, \beta, \gamma)$ is a CCE of the abatement game if and only if*

$$\text{either, } \alpha > \frac{a}{2b(n-1)} \text{ and } a\alpha \geq (b+c)\beta + 2(n-1)b\gamma, \tag{12}$$

$$\text{or, } \alpha \leq \frac{a}{2b(n-1)} \text{ and } a\alpha \geq (b+c)\beta + 2(n-1)b\gamma + \frac{[a - 2(n-1)b\alpha)]^2}{4(b+c)}. \tag{13}$$

We now present two further important observations out of the above characterisation (proofs of which are in the Appendix).

Claim 1. *When $n = 2$, the case $\alpha > \frac{a}{2b(n-1)}$ is impossible.*

Claim 1 shows that an abatement game with three or more players requires separate analysis as there are more CCEs possible resulting in different outcomes of the game.

Our next observation relates to the benefit and cost parameters of the game. Let us denote $r = \frac{c}{b}$.

Claim 2. *When $r = \frac{c}{b} > 1$, the only CCE of the abatement game coincides with the Nash equilibrium of the game.*

Henceforth, we will only consider the case when $r = \frac{c}{b} < 1$.

3.1 Abatement Maximisation

Using the techniques used in Algorithm 1, we can precisely characterise the CCE for the abatement game that maximises the total abatement level, $Q = \sum_{i=1}^{n} q_i$ and thereby compare it with the Nash equilibrium abatement level.

Proposition 2. *For a fixed $r = \frac{c}{b} < 1$,*

1. *when $(1-r) \geq \frac{2}{n}$, the optimal values of the three moments of the abatement-maximising CCE for the abatement game are $\alpha = \frac{a}{n(b+c)}$, $\beta = n\alpha^2$ and $\gamma = 0$,*
2. *when $(1-r) < \frac{2}{n}$, the optimal values of the three moments of the abatement-maximising CCE for the abatement game are $\alpha = \frac{a}{2b[(n+r)-\sqrt{(n-1)(1-r^2)}]}$, $\beta = n\alpha^2$ and $\gamma = 0$.*

Proposition 2 suggests that by having negatively correlated strategies ($\gamma = 0$), CCE could achieve better average abatement compared to a fully non-cooperative Nash outcome. The lottery in Proposition 2 has the following interpretation: if a particular country is not able to abate due to certain economic or other shocks, overall abatement is still achievable via correlation.

Using Proposition 2, we can immediately measure the relative improvement in the abatement level from the abatement-maximising CCE over that of the Nash equilibrium abatement level, given by $\frac{q^{CCE}}{q^N}$.

Corollary 2. *For a fixed $r = \frac{c}{b} < 1$,*

1. *when $(1-r) \geq \frac{2}{n}$, $\frac{q^{CCE}}{q^N} = \frac{2(n+r)}{n(1+r)}$,*
2. *when $(1-r) < \frac{2}{n}$, $\frac{q^{CCE}}{q^N} = \frac{n+r}{(n+r)-\sqrt{(n-1)(1-r^2)}}$.*

Corollary 2 requires no detailed proof. From Corollary 2, we note that the total abatement from the abatement-maximising CCE over the Nash abatement increases as r decreases. Also, notice that for larger n, the condition $(1-r) \geq \frac{2}{n}$ is more likely to be satisfied.

We illustrate these features by plotting the maximum abatement gain by CCE over Nash, that is $\frac{q^{CCE}}{q^N}$, with respect to r, for three different values of n in Figure 1 (the blue curve is for $n = 3$, the green one is for $n = 10$ while is red is for $n = 50$). The coloured vertical lines in Figure 1 provide the cutoffs in Corollary 2; case 1 lies to the left of the corresponding line (blue for $n = 3$, green for $n = 10$ and red for $n = 50$) while case 2 lies to the right.

3.2 Utility Maximisation

One could also characterise the utility-maximising CCE for the abatement game. The expressions for α, β and γ associated with the utility-maximising CCE for the n-player case are messy and are difficult to interpret. Instead, here we present the simpler case of a 2-player abatement game.

Proposition 3. *If $r = \frac{c}{b} < 1$, the optimal values of the three moments of the utility-maximising L in a 2-player abatement game are given by $(\widetilde{\alpha}, \widetilde{\beta}, \widetilde{\gamma})$:*

$$\widetilde{\alpha} = \frac{a}{b}\frac{2+2r-r^2}{2(4+5r)},$$

$$\widetilde{\beta} = \frac{a^2}{b^2}\frac{4+8r+r^2-4r^3}{4(4+5r)^2} \text{ and } \widetilde{\gamma} = \frac{a^2}{b^2}\frac{4+8r-r^2-4r^3+2r^4}{4(4+5r)^2};$$

while the optimal CCE is $\widetilde{L} = \frac{1}{2}\delta_{(z,z')} + \frac{1}{2}\delta_{(z',z)}$, with

$$z, z' = \frac{a}{b}\frac{2+2r-r^2 \pm r\sqrt{1-r^2}}{2(4+5r)}.$$

Proof of Proposition 3 has been postponed to the Appendix. Recall also from Claim 1 that the CCE in the 2-player case can be very different from that of the n-player game.

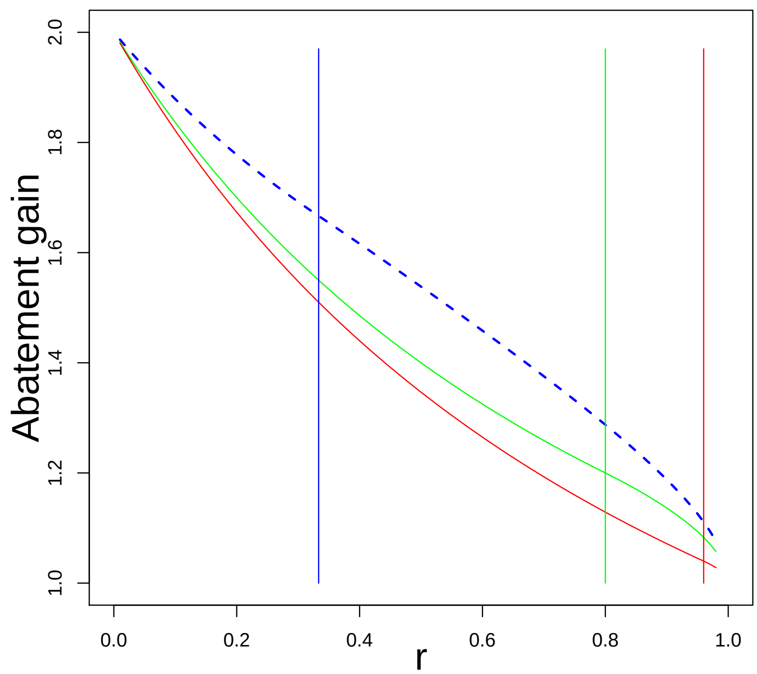

Figure 1: $\frac{q^{CCE}}{q^N}$ vs r for $n = 3$(blue), $n = 10$(green) and $n = 50$ (red)

3.3 Abatement and Utility

We observe that in the 2-player abatement game, the utility at the abatement-maximising CCE is always lower than the Nash utility level. One can thus naturally ask how much improvement in the abatement level we can achieve keeping the utility level at least that of Nash. The analytical answer to this question can be found for the 2-player case; the maximum improvement in the abatement level over Nash, keeping the utility level held at least at the Nash outcome, is at most 25%. (Proposition 4 in the working paper by Dokka et al. (2019))

We here performed a simulation for this analysis with $n > 2$ and observe that even after imposing the utility constraint, we still achieve improvement over the Nash abatement level, however this diminishes with n. In Table 1, we illustrate our simulation results for different value of n ($n = 3, 4, 5, 10, 50, 100$) for the parameter values $a = b = 1$ and $c = 0.78$.

n	α (with utility constraint)	Nash quantity
3	0.143	0.132275132
4	0.112	0.10460251
5	0.09	0.08650519
10	0.0474	0.046382189
50	0.009893	0.009846396
100	0.004965	0.004961302

Table 1: Abatement level at the optimal CCE with utility constraint

The simulation results presented in Table 1 clearly shows that CCEs (maintaining the same utility as in Nash equilibrium) do obtain higher abatement levels than the Nash levels; however, this improvement decreases with n.

4 REMARKS

We have analysed coarse correlated equilibria to find the quantity-maximising outcome for n-person symmetric quadratic games, such as abatement games. As an example of this class of games, we have characterised the abatement-maximising CCEs for the n-person abatement game. Such a computation is the first of its kind for coarse correlated equilibria for the abatement game and, this is why we regard this exercise as an interesting first step towards more sophisticated computations to understand mediation in general for such games. In addition, we explicitly characterise the utility-maximising CCE for a 2-player abatement game. We also contrast the abatement maximising levels with maintaining the utility level at the Nash outcome for any n-person abatement game.

There is a huge recent literature in the algorithmic game theory that focuses on the popular ratios, known as the price of anarchy (PoA) and price of stability (PoS) in similar framework. While the analysis of both PoA and PoS do apply to the situation we study here, the questions we consider in this paper are different. The existing literature focuses on measuring the loss of efficiency with respect to one measure only; PoA with respect to one measure (say, utility) is not studied conditional on PoA on another measure (say, quantity levels). To the best of our knowledge, we are the first to provide such an analysis in a small but economically relevant class of games.

There are clearly a few limitations of our results. We have used a quadratic payoff function, and not any general differentiable concave function; hence, subsequently, we considered symmetric lotteries. We would like to emphasise that even though our results are obtained in a symmetric game, they provide important basis for argument for mediation and correlation in this context. This is not just because it enables us to use the techniques identified in Moulin et al. (2014); this choice has been justified in the literature (such as the RICE model in Nordhaus and Boyer (2000) that tries to set up abatement cost functions fitting real data). Quadratic approximation is indeed a natural choice for payoffs as shown in the models by Bosetti et al. (2009), Finus, Altamirano-Cabrera, and Ierland (2005), Klepper and Peterson (2006).

As a natural extension, asymmetric version of our abatement game could be studied with asymmetry in costs, i.e., considering $C_i(q_i) = \frac{C_i q_i^2}{2}$. In this case, the Nash equilibrium quantities will be $q_i^{Neq-asym} = \frac{a}{2c_i(1+b\sum_j \frac{1}{c_j})}$. An analytical study of CCEs employing general asymmetric lotteries seems hard and closed-form expressions for lottery parameters seems unlikely. It is possible to extend our results to this version of the game by considering symmetric lotteries even though the game itself is asymmetric. Another extension that can be considered for future work is to consider asymmetry arising by introducing player-specific benefit functions, which leads to an entirely different game compared to the games we considered here.

5 APPENDIX: PROOFS

Proof of Lemma 2 For L to be feasible, it should be true that the variance-covariance matrix $M_L(Z)$ is positive semi-definite (PSD). Omitting the subscript L for ease of notation, let $Y_i = Z_i - \alpha$ for all i:

$$Var(Z_i) = E[Y_i^2] = \beta - \alpha^2 = \beta^*,$$
$$Cov(Z_i, Z_j) = E[Y_i Y_j] = \gamma - \alpha^2 = \gamma^*.$$

So, we need to express a matrix with β^* on the diagonal and γ^* on the off-diagonal is PSD. This means that we have for all $x \in \mathbb{R}^n$

$$\beta^*(\sum_1^n x_i^2) + 2\gamma^*(\sum_{1\leq i\leq j\leq n} x_i x_j) \geq 0. \tag{14}$$

Standard techniques show that this holds if and only if

$$\beta^* \geq \gamma^* \text{ and } \beta^* + (n-1)\gamma^* \geq 0, \tag{15}$$

where $\beta^* \geq 0$ but γ^* can be positive or negative. Note that β^* is necessary, and if $\beta^* = 0$ then we need $\gamma^* = 0$ as well. Assume now $\beta^* > 0$.

Case 1: $\gamma^* \geq 0$.
In this case, we can write (14) as

$$(1 - \frac{\gamma^*}{\beta^*})(\sum_1^n x_i^2) + \frac{\gamma^*}{\beta^*}(\sum_1^n x_i)^2 \geq 0, \tag{16}$$

which holds if and only if $\beta^* \geq \gamma^*$.

Case 2: $\gamma^* < 0$.

In this case, (14) is

$$(1 - \frac{-\gamma^*}{\beta^*})(\sum_1^n x_i^2) \geq \frac{-\gamma^*}{\beta^*}(\sum_1^n x_i)^2. \tag{17}$$

If we fix the sum $\sum_1^n x_i$, the minimum of the LHS above is achieved when all x_i are equal, so that the inequality holds for all x if and only if it holds for x on the diagonal, i.e.,

$$1 + \frac{-\gamma^*}{\beta^*} \geq n\frac{-\gamma^*}{\beta^*} \iff \beta^* + (n-1)\gamma^* \geq 0.$$

Combining both cases and switching back to β and γ, we get the result. ■

Proof of Claim 1 Fix α and consider the system of conditions on the vector (β, γ). The line $(b+c)\beta + 2(n-1)b\gamma = a\alpha$ is flatter than the line $\beta + \gamma = 2\alpha^2$; therefore, the two corresponding half spaces intersect in the positive orthant if and only if $\frac{a\alpha}{b+c} \geq 2\alpha^2$. But the latter contradicts $\alpha > \frac{a}{2b}$. ■

Proof of Claim 2 Consider the following polytope for a fixed $\Psi = \{(\beta, \gamma)|\beta \geq \gamma, \beta + (n-1)\gamma \geq n\alpha^2\}$ under the additional constraint 13. Note that Ψ is unbounded from above and bounded from below by the interval $[P, Q]$, where $P = (\alpha^2, \alpha^2)$ and $Q = (n\alpha^2, 0)$. The minimum in Ψ of $(b+c)\beta + 2(n-1)b\gamma$ is achieved at P. Therefore, if P meets 13 it is our optimal pair of $(\beta(\alpha), \gamma(\alpha))$; if not there is no CCE. P meets 13 if and only if $((2n-1)b + c)\alpha^2 \leq -\frac{b^2\alpha^2 - a(nb+c)\alpha + \frac{a^2}{4}}{b+c}$, which is $(a - 2(nb+c)\alpha)^2 \leq 0$. This is only possible when $\alpha = \frac{a}{2(nb+c)}$ which is nothing but Nash outcome. ■

Proof of Proposition 2 (First part). Increasing the value of α shrinks the feasible region of the polytope

$$\{(\beta, \gamma)|\beta \geq \gamma; \beta + (n-1)\gamma \geq n\alpha^2; (b+c)\beta + 2b(n-1)\gamma \leq a\alpha\}$$

eventually to a single point which is the intersection of the half-lines $(b+c)\beta + 2b(n-1)\gamma = a\alpha$ and $\beta + (n-1)\gamma = n\alpha^2$ on the β-axis. This point gives $\alpha = \frac{a}{n(b+c)}$. However, this is only valid when resulting $\alpha \geq \frac{a}{2b(n-1)}$, that is, we must have $(1-r) > \frac{2}{n}$. ■

Proof of Proposition 2 (Second part). For the second case, the intersection point of $\beta = n\alpha^2$ and $(b+c)\beta + 2b(n-1)\gamma = a\alpha - \frac{(a-2b(n-1)\alpha)^2}{4(b+c)}$ is the positive root of the following quadratic equation:

$$[(n-1)^2 + (1+r)^2 n]\alpha'^2 - (n+r)\alpha' + \frac{1}{4} = 0,$$

where $\alpha = \frac{a}{b}\alpha'$. ■

Proof of Proposition 3 First consider the equilibrium condition in Proposition . Note that if $a - 2b\alpha < 0 \iff \alpha > \frac{a}{2b}$, the LHS. of that inequality (the maximum over $z \geq 0$) is 0; therefore, the equilibrium condition in Proposition becomes

$$a\alpha \geq (b+c)\beta + 2b\gamma = b(\beta+\gamma) + c\beta + b\gamma > b(\beta+\gamma) \geq 2b\alpha^2,$$

which is a contradiction. So, we must have $\alpha \leq \frac{a}{2b}$; then, the LHS. of the equilibrium condition is $\frac{(a-2b\alpha)^2}{4(b+c)}$ and the condition is now

$$(b+c)\beta + 2b\gamma \leq a\alpha - \frac{(a-2b\alpha)^2}{4(b+c)} = -\frac{b^2\alpha^2 - a(2b+c)\alpha + \frac{a^2}{4}}{b+c}. \tag{18}$$

We now fix α and solve step 1 in Lemma 3: we must minimise $(2b+c)\beta + 2b\gamma$ in the polytope $\Psi = \{(\beta,\gamma) | \beta \geq \gamma, \beta+\gamma \geq 2\alpha^2\}$ under the additional constraint (18). Note that Ψ is unbounded from above and bounded from below by the interval $[P,Q]$, where $P = (\alpha^2, \alpha^2)$ and $Q = (2\alpha^2, 0)$. We distinguish two cases here.

Here, the minimum of $(b+c)\beta + 2b\gamma$ in Ψ is achieved at Q; so, if Q fails to meet the constraint (18), this constraint does not satisfy anywhere in Ψ. Thus, we must choose α such that

$$2(b+c)\alpha^2 \leq -\frac{b^2\alpha^2 - a(2b+c)\alpha + \frac{a^2}{4}}{b+c} \tag{19}$$

$$\iff \Lambda(\alpha) = (3b^2 + 4bc + 2c^2)\alpha^2 - a(2b+c)\alpha + \frac{a^2}{4} \leq 0 \tag{20}$$

The discriminant of the right-hand polynomial $\Lambda(\alpha)$ is $a^2(b^2 - c^2)$; therefore, (19) restricts α to an interval $[\alpha_-, \alpha_+]$, between the two positive roots of $\Lambda(\alpha)$. For such a choice of α, the constraint (18) cuts a subinterval $[R,Q]$ of $[P,Q]$, where R meets (18) with an equality. Note that $R = P$ only if $\alpha = q_i^N$ (from Case 1 and the fact that $\Lambda(q_i^N) < 0$); otherwise $R \neq P$. Clearly, R is our optimal choice for $(\beta(\alpha), \gamma(\alpha))$ and it solves the system

$$\beta + \gamma = 2\alpha^2;\ (b+c)\beta + 2b\gamma = -\frac{b^2\alpha^2 - a(2b+c)\alpha + \frac{a^2}{4}}{b+c}.$$

Therefore,

$$\beta(\alpha) = \frac{1}{b^2 - c^2}\left[b(5b+4c)\alpha^2 - a(2b+c)\alpha + \frac{a^2}{4}\right] \text{ and}$$

$$\gamma(\alpha) = \frac{1}{b^2 - c^2}\left[-(3b^2 + 4bc + 2c^2)\alpha^2 + a(2b+c)\alpha - \frac{a^2}{4}\right].$$

Now in step 2 of Lemma 3, we must maximise $2a\alpha-(2b+c)\beta(\alpha)-2b\gamma(\alpha)$ under the constraints $\alpha \geq 0$ and $\Lambda(\alpha) \leq 0$. Developing this objective function yields the programme

$$\frac{1}{b^2-c^2}\max_{\alpha}\{-b^2(4b+5c)\alpha^2+a(2b^2+2bc-c^2)\alpha-\frac{a^2c}{4}\} \tag{21}$$

under the constraints

$$\alpha \geq 0 \text{ and } \Lambda(\alpha)=(3b^2+4bc+2c^2)\alpha^2-a(2b+c)\alpha+\frac{a^2}{4} \leq 0.$$

The unconstrained maximum of the objective function is achieved at $\widetilde{\alpha} = \frac{a(2b^2+2bc-c^2)}{2b^2(4b+5c)}$.

We now show that $\Lambda(\widetilde{\alpha}) \leq 0$. With the change of variable $r=\frac{c}{b}$, this amounts to

$$\frac{(3+4r+2r^2)(2+2r-r^2)^2}{4(4+5r)^2}-\frac{(2+r)(2+2r-r^2)}{2(4+5r)}+\frac{1}{4} \leq 0$$

$$\iff 4+8r-5r^2-12r^3+3r^4+4r^5-2r^6 \geq 0.$$

The above polynomial is 0 at $r=1$; it is also easy to check, numerically, that it is non-negative on $[0,1]$. The proof is now complete if we express $\widetilde{\alpha}$, $\widetilde{\beta}$ and $\widetilde{\gamma}$ in terms of r. This is indeed easy for $\widetilde{\alpha}$. One may also verify, using the expression for $\widetilde{\alpha}$ that

$$\begin{aligned}\widetilde{\beta} &= \beta(\widetilde{\alpha}) = \frac{1}{b^2-c^2}\left[b(5b+4c)\widetilde{\alpha}^2-a(2b+c)\widetilde{\alpha}+\frac{a^2}{4}\right] \\ &= \frac{a^2}{b^2}\frac{4+8r+r^2-4r^3}{4(4+5r)^2} \text{ and}\end{aligned}$$

$$\begin{aligned}\widetilde{\gamma} &= \gamma(\widetilde{\alpha}) = \frac{1}{b^2-c^2}\left[-(3b^2+4bc+2c^2)\widetilde{\alpha}^2+a(2b+c)\widetilde{\alpha}-\frac{a^2}{4}\right] \\ &= \frac{a^2}{b^2}\frac{4+8r-r^2-4r^3+2r^4}{4(4+5r)^2}.\end{aligned}$$

Finally, we construct the optimal CCE $\widetilde{L}$. For $n=2$, our Lemma 2 implies $\widetilde{\beta}+\widetilde{\gamma}=2\widetilde{\alpha}^2$; moreover, from Lemma 2*iii*) in Moulin et al. (2014), we see that $\widetilde{L}$ is an anti-diagonal lottery of the form $\widetilde{L}=\frac{1}{2}\delta_{(z,z')}+\frac{1}{2}\delta_{(z',z)}$, where z and z' are non-negative numbers such that $z+z'=2\widetilde{\alpha}$ and $z^2+z'^2=2\widetilde{\beta}$. This implies $2zz'=(2\widetilde{\alpha})^2-(2\widetilde{\beta})=2\widetilde{\gamma}$; hence, z, z' solve $Z^2-2\widetilde{\alpha}Z+\widetilde{\gamma}=0$. The discriminant is $\widetilde{\alpha}^2-\widetilde{\gamma}=\widetilde{\beta}-\widetilde{\alpha}^2=\frac{a^2}{b^2}\frac{r^2(1-r^2)}{4(4+5r)^2}$; thus, the expressions for z and z' follow. ■

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