Lifting the Knapsack Cover Inequalities for the Knapsack Polytope

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Abstract

Valid inequalities for the knapsack polytope have proven to be very useful in exact algorithms for mixed-integer linear programming. In this paper, we focus on the *knapsack cover* inequalities, introduced in 2000 by Carr and co-authors. In general, these inequalities can be rather weak. To strengthen them, we use *lifting*. Since exact lifting can be time-consuming, we present two fast approximate lifting procedures. The first procedure is based on mixed-integer rounding, whereas the second uses superadditivity.

Keywords: knapsack problems; lifted cover inequalities; polyhedral combinatorics; mixed-integer linear programming

1 Introduction

A *knapsack constraint* is a linear constraint of the form $\sum_{i=1}^n a_i x_i \leq b$, where b and n are positive integers and $a \in \mathbb{Z}_+^n$. Any linear inequality involving binary variables can be converted into a knapsack constraint, by complementing variables with negative coefficients [23]. The polyhedron

$$\text{conv}\Big\{x \in \{0,1\}^n : \sum_{i=1}^n a_i x_i \leq b\Big\}$$

is called a *knapsack polytope* [2]. Valid inequalities for knapsack polytopes have proven to be very useful in exact algorithms for mixed-integer linear programming (e.g., [4, 8, 12–14, 16, 17]).

There are many papers on valid inequalities for knapsack polytopes. Most of these focus on *lifted cover* inequalities (e.g., [2, 3, 8, 12, 13, 15, 16, 18,

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23,26]), but there are a few papers on other families of inequalities. These include *weight* inequalities [22], *lifted pack* inequalities [1,16], *Chvátal-Gomory cuts* [17], *Fenchel cuts* [4], and the inequalities in [5], which are (somewhat confusingly) called *knapsack cover* inequalities. These last inequalities have received very little attention in the literature, and have not been analysed from a polyhedral point of view.

We will see that, in general, knapsack cover inequalities can be rather weak. To strengthen them, we use *lifting* (see [2,21,23]). Since *exact* lifting can be time-consuming, we present two fast (and sequence-independent) approximate lifting procedures. The first procedure, which runs in $O(n)$ time, is based on a simple mixed-integer rounding argument (see [11,19,20]). The second procedure is stronger, but is a bit more complicated and runs in $O(n \log n)$ time. It is based on the construction of a valid superadditive lifting function (see [13, 25]). Our examples show that it is possible for both procedures to generate new facet-defining inequalities for the knapack polytope.

The paper has a simple structure. The literature is reviewed in Section 2, the new lifting procedures are presented in Section 3, and some concluding remarks are made in Section 4. Throughout the paper, we let N denote $\{1, \ldots, n\}$.

2 Literature Review

For brevity, we review here only works of direct relevance. We recall cover inequalities in Subsection 2.1, knapsack cover inequalities in 2.2, lifting in Subsection 2.3, and mixed-integer rounding in Subsection 2.4.

2.1 Cover inequalities

A set $C \subseteq N$ such that $\sum_{i \in C} a_i > b$ is called a *cover*. If C is a cover, then the *cover inequality* $\sum_{i \in C} x_i \leq |C| - 1$ is valid for the knapsack polytope [9]. A cover C is *minimal* if $\sum_{i \in C \setminus \{k\}} a_i \leq b$ for all $k \in C$. The minimal cover inequalities dominate all others [2, 23]. Although they are not guaranteed to define facets of the knapsack polytope, they can be strengthened to make them facet-defining (see Subsection 2.3).

2.2 Knapsack cover inequalities

Now consider a knapsack constraint of the form $\sum_{i \in N} a_i x_i \geq d$, where d and n are positive integers and $a \in \mathbb{Z}^n_+$. Crowder *et al.* [8] noted that such a constraint can be strengthened simply by replacing each a_i with $\min\{a_i, d\}$. Carr *et al.* [5] generalised this as follows. Consider any $S \subset N$, possibly empty, such that $\sum_{j \in S} a_j < d$. The inequality

$$\sum_{i \in N \setminus S} a_i x_i \geq d - \sum_{j \in S} a_j$$

is trivially valid, and it can be strengthened to yield:

$$\sum_{i \in N \setminus S} \min \left\{ a_i, d - \sum_{j \in S} a_j \right\} x_i \geq d - \sum_{j \in S} a_j. \tag{1}$$

Rather confusingly, Carr *et al.* call the inequalities (1) *knapsack cover* inequalities. We will therefore refer to them as KCIs. The standard cover inequalities, mentioned in the previous subsection, will be called CIs.

2.3 Lifting

We now recall the basics of *lifting* [21, 24], focusing on 0-1 linear programs (0-1 LPs). Let $P \subset [0,1]^n$ be the convex hull of feasible solutions to a 0-1 LP, let S be a proper subset of N, and let $P(S)$ be the face of P obtained by setting x_i to 0 for all $i \in S$. Suppose we know that $\dim\big(P(S)\big) = \dim(P) - |S|$, and that the inequality

$$\sum_{i \in N \setminus S} \alpha_i x_i \leq \beta$$

defines a facet of $P(S)$. Then, there exists at least one inequality of the form

$$\sum_{i \in N \setminus S} \alpha_i x_i + \sum_{i \in S} \gamma_i x_i \leq \beta,$$

that defines a facet of P. (In particular, any minimal cover inequality can be strengthened to make it facet-defining for the knapsack polytope.)

The process of computing the γ_i is called *lifting*. Lifting is usually done *sequentially*, i.e., one variable at a time. To compute each lifting coefficient, one has to solve an auxiliary 0-1 LP, which may be time-consuming. Fortunately, fast exact and approximate algorithms are available for sequentially lifting CIs [2, 3, 8, 12, 16, 23, 26]. For other kinds of inequalities, Wolsey [24] suggests solving the LP relaxations of the auxiliary 0-1 LPs.

There can sometimes exist facet-defining lifted inequalities that cannot be obtained by sequential lifting [23]. To obtain such inequalities, one must lift several variables *simultaneously*. Unfortunately, simultaneous lifting is very complicated, even for CIs [3, 15]. Wolsey [25] devised an elegant way to perform simultaneous lifting *approximately*, based on superadditive functions. This approach, sometimes called *sequence-independent* lifting, has been used to good effect in, e.g., [1, 13, 18]. However, the resulting inequality is not guaranteed to define a facet of P. For brevity, we omit the details.

2.4 Mixed-integer rounding

Finally, we recall some results from cutting-plane theory. Let $P \subset \mathbb{R}^n_+$ be a polyhedron, and suppose that the inequality $\alpha^T x \leq \beta$, with $\beta \notin \mathbb{Z}$, is valid for P. It is well known that the inequality $\sum_{i \in N} \lfloor \alpha_i \rfloor x_i \leq \lfloor \beta \rfloor$ is satisfied by all integer points in P [7, 10]. Less well known is that one can derive a stronger inequality as follows [11, 20]. Given a real number r, let $\phi(r)$ denote $r - \lfloor r \rfloor$, the so-called *fractional part* of r. Also define the folllowing (continuous and non-decreasing) function

$$F_\beta(r) = \begin{cases} \lfloor r \rfloor, & \text{if } \phi(r) \leq \phi(\beta) \\ \lfloor r \rfloor + \frac{\phi(r) - \phi(\beta)}{1 - \phi(\beta)}, & \text{if } \phi(r) > \phi(\beta). \end{cases}$$

The strengthened inequality takes the form:

$$\sum_{i \in N} F_\beta(\alpha_i) x_i \leq \lfloor \beta \rfloor.$$

We follow [19, 20] in calling these inequalities *mixed-integer rounding* (MIR) inequalities.

3 Lifting Knapsack Cover Inequalities

In this section, we show how to strengthen the KCIs by lifting. In Subsection 3.1, we present some simple results and examples to motivate our study. In Subsection 3.2, we define lifted KCIs formally and give examples. In Subsections 3.3 and 3.4, we present our sequence-independent lifting procedures.

We remind the reader that there is one KCI (1) for every $S \subseteq N$ satisfying $\sum_{j \in S} a_j < d$. Throughout this section, we let d^- denote $d - \sum_{j \in S} a_j$, and we sometimes refer to the sets $L = \{i \in N \setminus S : a_i > d^-\}$ and $R = N \setminus (S \cup L)$. (The idea here is that L contains indices with "large" a_i value, and R contains the "remaining" indices.) With this notation, the KCIs can be written in the simpler form

$$\sum_{i \in R} a_i x_i + d^- \sum_{i \in L} x_i \geq d^-. \tag{2}$$

We let e denote the all-ones vector of length n. We also frequently refer to the following two polytopes:

$$\begin{aligned} P^{\geq} &= \operatorname{conv}\Big\{ x \in \{0,1\}^n : a^T x \geq d \Big\} \\ P^{\leq} &= \operatorname{conv}\Big\{ \bar{x} \in \{0,1\}^n : a^T \bar{x} \leq e^T a - d \Big\}. \end{aligned}$$

Note that these polytopes are congruent, via the trivial mapping $\bar{x}_i = 1 - x_i$ for $i \in N$.

3.1 Motivation

In some preliminary experiments with the software package `PORTA` [6], we found that KCIs usually (though not always) define low-dimensional faces of $P^{\geq}$. A partial explanation is given by the following two lemmas:

Lemma 1 *If $\sum_{j \in R} a_j < d^-$, then the KCI (2) is equivalent to or dominated by the inequality $\sum_{i \in L} x_i \geq 1$. Note that this inequality is equivalent to a CI for $P^{\leq}$.*

Proof. By the definition of d^-, the stated condition can be written as $\sum_{j \in R \cup S} a_j < d$. Under this condition, the inequality $\sum_{i \in L} x_i \geq 1$ is valid for $P^{\geq}$. Writing this as $d^- \sum_{i \in L} x_i \geq d^-$, we see that it is at least as strong as the KCI. Writing it as $\sum_{i \in L} \bar{x}_i \leq |L| - 1$ instead, we see that it is equivalent to a CI for $P^{\leq}$. □

Lemma 2 *If $\sum_{j \in R} a_j \geq d^-$, but $\sum_{j \in R \setminus \{i\}} a_j < d^-$ for all $i \in R$, then the KCI (2) is dominated by the inequalities $\sum_{j \in L \cup \{i\}} x_j \geq 1 \, (i \in R)$. Note that these inequalities are equivalent to CIs for $P^{\leq}$.*

Proof. Suppose the stated conditions hold. If $x_i = 0$ for all $i \in L$, then we must set x_i to 1 for all $i \in R$. Thus, the following inequalities are valid for $P^{\geq}$:

$$\sum_{j \in L \cup \{i\}} x_j \geq 1 \qquad (i \in R). \tag{3}$$

Writing these inequalities in the form $\sum_{j \in L \cup \{i\}} \bar{x}_j \leq |L|$, we see that they are equivalent to CIs for $P^{\leq}$. Now, for each $i \in R$, multiply the inequality (3) by $a_i d^- / \sum_{j \in R} a_j$, and sum the resulting $|R|$ inequalities together, to yield:

$$\frac{d^-}{\sum_{j \in R} a_j} \sum_{i \in R} a_i x_i + d^- \sum_{i \in L} x_i \geq d^-.$$

Since $\sum_{j \in R} a_j \geq d^-$ by assumption, this last inequality is at least as strong as the KCI. □

When the conditions in Lemmas 1 and 2 do not hold, the KCI may or may not define a facet of $P^{\geq}$. This is shown in the following example.

Example 1: Let $n = d = 7$ and $a = (1, 2, 2, 2, 4, 4, 7)^T$. Taking $S = \{1\}$ yields the KCI $2(x_2 + x_3 + x_4) + 4(x_5 + x_6) + 6x_7 \geq 6$. We have $R = \{2, \ldots, 6\}$ and $\sum_{i \in R} a_i = 14$, so Lemmas 1 and 2 do not apply. One can check (either by hand or with the help of a package like `PORTA`) that this KCI defines a facet of $P^{\geq}$. On the other hand, if we take $S = \{2\}$, we get the KCI $x_1 + 2(x_3 + x_4) + 4(x_5 + x_6) + 5x_7 \geq 5$. We have $R = \{1, 3, 4, 5, 6\}$ and $\sum_{i \in R} a_i = 13$, so, again, the lemmas do not apply. Yet, this KCI does not define a facet, since every extreme point of $P^{\geq}$ that satisfies it at equality also satisfies $x_1 + x_7 = 1$. □

3.2 Lifted KCIs

We propose to lift KCIs, *regardless* of whether or not Lemma 1 or Lemma 2 applies. The idea is as follows. The KCI (2) is equivalent to the following valid inequality for $P^{\leq}$:

$$\sum_{i \in R} a_i \bar{x}_i + d^- \sum_{i \in L} \bar{x}_i \leq \sum_{i \in R} a_i + d^-(|L| - 1).$$

It is now apparent that we may be able to lift the variables in S, to obtain a valid inequality for $P^{\leq}$ of the form:

$$\sum_{i \in R} a_i \bar{x}_i + d^- \sum_{i \in L} \bar{x}_i + \sum_{i \in S} \gamma_i \bar{x}_i \leq \sum_{i \in R} a_i + d^-(|L| - 1).$$

The corresponding valid inequality for $P^{\geq}$ takes the form:

$$\sum_{i \in R} a_i x_i + d^- \sum_{i \in L} x_i \geq d^- + \sum_{i \in S} \gamma_i (1 - x_i). \tag{4}$$

We call (4) a *lifted knapsack cover inequality* or LKCI. In general, LKCIs are not guaranteed to define facets of $P^{\geq}$. On the other hand, the following example shows that LKCIs can define non-trivial facets even when Lemma 1 applies.

Example 2: Let $n = 5$, $d = 8$ and $a = (2, 2, 2, 5, 5)^T$. Taking $S = \{1, 2\}$ yields the KCI $2x_3 + 4(x_4 + x_5) \geq 4$. We have $R = \{3\}$ and $\sum_{i \in R} a_i = 2 < d^- = 4$. Thus, Lemma 1 applies. One can check however that the LKCI $2x_3 + 4(x_4 + x_5) \geq 4 + 2(1 - x_1) + 2(1 - x_2)$ is valid and facet-defining for $P^{\leq}$. Moreover, if we write the LKCI in the form $\bar{x}_1 + \bar{x}_2 + \bar{x}_3 + 2(\bar{x}_4 + \bar{x}_5) \leq 3$, we see that it is not equivalent to a lifted CI. □

Now recall that lifting can be done sequentially or simultaneously. If one wishes to lift a KCI sequentially, one must solve an auxiliary knapsack problem (KP) to compute each lifting coefficient [21, 24]. This is likely to be too time-consuming to be useful in practical computation. Following Wolsey [25], one could compute approximate lifting coefficients sequentiallly, by solving the continuous relaxations of the KPs. We prefer however to use sequence-independent lifting, as described in the following two subsections.

3.3 Lifting via mixed-integer rounding

It turns out that one can lift KCIs using mixed-integer rounding. This can be done in four steps, as follows.

1. Write the constraint $a^T x \geq d$ in the form

$$\sum_{i \in S} a_i \bar{x}_i - \sum_{i \in N \setminus S} a_i x_i \leq -d^-. \tag{5}$$

2. Let $a^+ = \max_{i \in N \setminus S}\{a_i\}$, and assume that $a^+ > d^-$ (since, if not, the KCI is redundant.) Divide (5) by a^+ to obtain

$$\sum_{i \in S} \left(\frac{a_i}{a^+}\right) \bar{x}_i + \sum_{i \in N \setminus S} \left(\frac{-a_i}{a^+}\right) x_i \leq -\frac{d^-}{a^+}.$$

3. We now set $-d^-/a^+$ to β, and apply mixed-integer rounding to get

$$\sum_{i \in S} F_\beta\big(a_i/a^+\big)\bar{x}_i + \sum_{i \in N \setminus S} F_\beta(-a_i/a^+)x_i \leq \lfloor \beta \rfloor.$$

We can simplify this inequality, as follows. Since $a^+ > d^-$, we have $\lfloor \beta \rfloor = -1$ and $\phi(\beta) = 1 - d^-/a^+$. For $i \in R$, we have that $a_i \leq a^+$ (by the definition of a^+) and $a_i \leq d^-$ (by the definition of R). Hence, $\phi(-a_i/a^+) = 1 - a_i/a^+ \geq \phi(\beta)$ and $F_\beta(-a_i/a^+) = -a_i/d^-$. For $i \in L$, we have that $a_i \leq a^+$ (by the definition of a^+) and $a_i > d^-$ (by the definition of L). Hence, $\phi(-a_i/a^+) = 1 - a_i/a^+ < \phi(\beta)$ and $F_\beta(-a_i/a^+) = \lfloor -a_i/a^+ \rfloor = -1$. So, we get

$$\sum_{i \in S} F_\beta\big(a_i/a^+\big)\bar{x}_i - \sum_{i \in R} \frac{a_i}{d^-} x_i - \sum_{i \in L} x_i \leq -1.$$

4. Multiplying the MIR inequality by d^- and re-arranging, we obtain:

$$\sum_{i \in R} a_i x_i + d^- \sum_{i \in L} x_i \geq d^- + d^- \sum_{i \in S} F_\beta\big(a_i/a^+\big)(1 - x_i). \qquad (6)$$

This is the desired LKCI.

The following example shows that the above MIR procedure can yield non-trivial facet-defining LKCIs.

Example 3: Let $n = 7$, $d = 17$ and $a = (3, 3, 3, 4, 7, 7, 7)^T$. Taking $S = \{4, 5\}$ yields the KCI $3(x_1 + x_2 + x_3) + 6(x_6 + x_7) \geq 6$. We have $d^- = 6$ and $a^+ = 7$, which gives $\beta = -6/7$. We have $F_\beta\big(a_4/a^+\big) = F_{1/7}(4/7) = 1/2$ and $F_\beta\big(a_5/a^+\big) = F_{1/7}(1) = 1$. The resulting LKCI is therefore $3(x_1 + x_2 + x_3) + 6(x_6 + x_7) \geq 6 + 3(1 - x_4) + 6(1 - x_5)$. One can check (either by hand or with the help of a package like `PORTA`) that this LKCI defines a facet of $P^{\geq}$. Moreover, if we write the LKCI in the form $\bar{x}_1 + \bar{x}_2 + \bar{x}_3 + \bar{x}_4 + 2(\bar{x}_5 + \bar{x}_6 + \bar{x}_6) \leq 5$, we see that it is not equivalent to a lifted CI. □

For the purpose of what follows, we will find it helpful to express the lifting coefficients in the LKCI (6) in a more explicit form.

Proposition 1 *For any $r \geq 0$, let $f(r)$ denote $d^- F_\beta\big(r/a^+\big)$. We have*

$$f(r) = \begin{cases} d^- \lfloor r/a^+ \rfloor & \text{if } r \bmod a^+ \leq a^+ - d^- \\ d^- \lceil r/a^+ \rceil - a^+ + r \bmod a^+ & \text{if } r \bmod a^+ > a^+ - d^-. \end{cases}$$

Proof. Write r as $ka^+ + \epsilon$, where $k = \lfloor r/a^+ \rfloor$ and $\epsilon = r \bmod a^+$. If $\epsilon \leq a^+ - d^-$, we have $\phi(r/a^+) = \epsilon/a^+ \leq 1 - d^-/a^+ = \phi(\beta)$, and therefore

$$F_\beta\big(r/a^+\big) = k.$$

On the other hand, if $\epsilon > a^+ - d^-$, we have $\phi(r/a^+) > \phi(\beta)$, and therefore

$$\begin{aligned} F_\beta\big(r/a^+\big) &= k + \tfrac{\phi\left(r/a^+\right) - \left(1 - d^-/a^+\right)}{d^-/a^+} \\ &= k + \tfrac{a^+\phi\left(r/a^+\right) - a^+ + d^-}{d^-} \\ &= k + 1 - \tfrac{a^+ - \epsilon}{d^-}. \end{aligned}$$

We have established that:

$$F_\beta\Big(\frac{r}{a^+}\Big) = \begin{cases} \lfloor r/a^+ \rfloor & \text{if } r \bmod a^+ \leq a^+ - d^- \\ \lceil r/a^+ \rceil - \frac{a^+ - \epsilon}{d^-} & \text{if } r \bmod a^+ > -d^-. \end{cases}$$

Multiplying by d^- yields the result. □

We remark that the MIR function F_β is superadditive for any β (see [20]). Thus, the function f is superadditive for any d^- and a^+.

3.4 Lifting via superadditivity

Consider again the LKCI (4), and suppose that we wish to lift x_k first, for some $k \in S$. Let r denote a_k. Following [21,24], we can compute the largest possible value of γ_k by computing

$$\begin{aligned} z(r) = \min \quad & \textstyle\sum_{i \in R} a_i x_i + d^- \sum_{i \in L} x_i && (7) \\ \text{s.t.} \quad & \textstyle\sum_{i \in L \cup R} a_i x_i \geq d^- + r && (8) \\ & x_i \in \{0,1\} \qquad (i \in L \cup R), && (9) \end{aligned}$$

and then setting γ_k to $z(r) - d^-$. The function $g(r) = z(r) - d^-$ is called the *exact lifting function.* Note that the domain of g is $\big[0, \sum_{j \in L \cup R} a_j - d^-\big]$ since, if r exceeds $\sum_{j \in L \cup R} a_j - d^-$, the above 0-1 LP becomes infeasible.

It follows from the main result in [25] that, if g is superadditive, then we can use it to lift all variables in S simultaneously. Unfortunately, g is not superadditive in general. This is demonstrated in the following example.

Example 4: Let $n = 7$, $a = (3, 3, 3, 7, 8, 9, 17)$, $d = 23$ and $S = \{7\}$. The

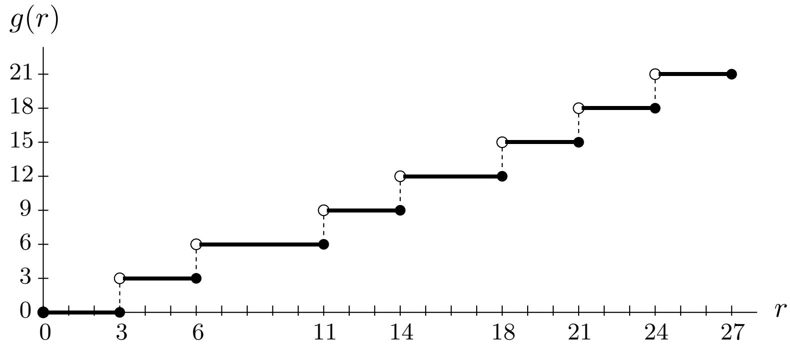

Figure 1: The exact lifting function $g(r)$ for Example 4.

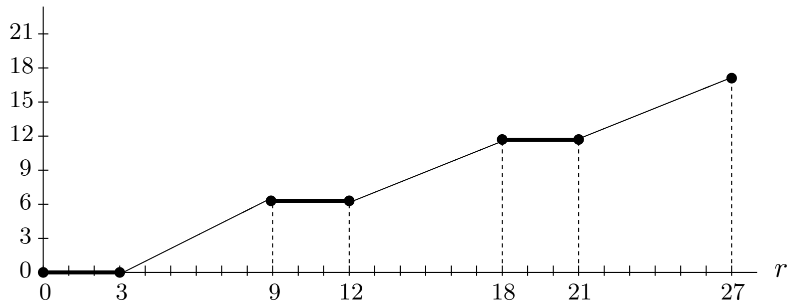

Figure 2: The lifting function $f(r)$ for Example 4.

reduced knapsack constraint is $3x_1 + 3x_2 + 3x_3 + 7x_4 + 8x_5 + 9x_6 \geq 6$. We have $d^- = 6$ and $a^+ = 9$. The KCI is $3(x_1 + x_2 + x_3) + 6(x_4 + x_5 + x_6) \geq 6$. The function g is shown in Figure 1. To see that g is not superadditive, note that, for example, $g(14) = 9 < 2g(7) = 12$. □

Following the approach in [13, 25], we are led to search for *superadditive valid lifting functions*, i.e., superadditive functions that do not exceed g. As, we remarked in subsection 3.3, the MIR lifting function, called f, is such a function. Figure 2 shows the function f for Example 4. Note that f is piecewise-linear, and each "piece" has a slope equal to either 0 or 1.

We now introduce a third lifting function, called h, which we will show to be superadditive and intermediate between f and g. Let $h(r) = \tilde{z}(r) - d^-$, where

$$\begin{aligned} \tilde{z}(r) = \min \quad & y + d^- \textstyle\sum_{i \in L} x_i && (10) \\ \text{s.t.} \quad & y + \textstyle\sum_{i \in L} a_i x_i \geq d^- + r && (11) \\ & x_i \in \{0, 1\} \qquad (i \in L) && (12) \\ & y \geq 0. && (13) \end{aligned}$$

Note that, due to the continuous variable y, $\tilde{z}(r)$ is feasible for all non-negative values of r. Hence, the domain of h is the whole of $\mathbb{R}_+$. In the

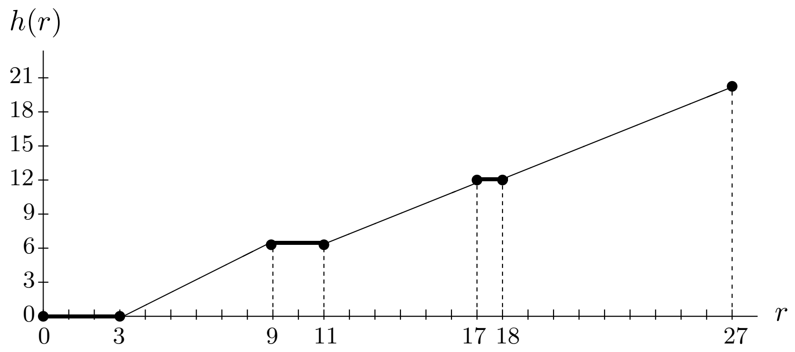

Figure 3: The lifting function $h(r)$ for Example 4.

following proposition, we give an explicit formula for h.

Proposition 2 *For $k = 1, \ldots, |L|$, let $S(k)$ be the sum of the k largest a_j values over the members of L. We trivially set $S(0) = 0$. Then, we can write $h(r)$ as follows*

$$h(r) = \begin{cases} kd^-, & \text{if } S(k) \leq r < S(k+1) - d^- \text{ for } k \in \{0, \ldots, |L|-1\} \\ r - (S(k) - d^-), & \text{if } S(k) - d^- \leq r < S(k) \text{ for } k \in \{1, \ldots, |L|-1\}. \\ r - (S(|L|) - d^-), & \text{if } r \geq S(|L|) - d^-. \end{cases}$$

Proof. Consider the mixed 0-1 knapsack problem (10)-(13). Observe that every time that we set a variable x_i to 1, we incur a cost d^-. This increases the LHS of the constraint (11) by a_i, which is greater than d^- by the definition of L. Hence, to find the optimal solution, it makes sense to keep switching on binary variables in decreasing order of a_i as long as the constraint (11) is violated. The continuous variable y will only take a positive value if the remaining violation of the constraint (11) is so "small" that it is not worth switching on an additional binary variable. □

To aid the reader, we show the lifting function h for Example 4.

Example 4 (cont.): Recall that $d^- = 6$ and $L = \{4, 5, 6\}$. Moreover, $S(0) = 0$, $S(1) = 9$, $S(2) = 17$ and $S(3) = 24$. Using these, one can compute the function h, which is shown in Figure 3. □

Note that, like f, h is piecewise-linear, and each "piece" has a slope equal to either 0 or 1. In the following proposition, we prove that h is intermediate between f and g.

Proposition 3 *For all $r \in \left[0, \sum_{j \in L \cup R} a_j - d^-\right]$, we have $f(r) \leq h(r) \leq g(r)$.*

Proof. We will first prove that $h(r) \leq g(r)$. Given that $g(r) = z(r) - d^-$ and $h(r) = \tilde{z}(r) - d^-$, it is sufficient to show that $\tilde{z}(r) \leq z(r)$. To see why this inequality holds, note that the mixed 0-1 knapsack problem (10)-(13) is a relaxation of the integer program (7)–(9), obtained by replacing the binary variables x_i for $i \in R$ with a single continuous variable y.

We will now prove that $h(r) \geq f(r)$. Recall that f and h are piecewise-linear. For the function h, the k-th "piece" with slope 0 has length $\ell_k^h = S(k+1) - S(k) - d^-$. For the function f, the k-th piece with slope 0 has length $\ell_k^f = a^+ - d^-$. By the definitions of S and a^+, we have $\ell_k^h \leq \ell_k^f$. Hence, $h(r) \geq f(r)$ for all $r \in \left[0, \sum_{j \in L \cup R} a_j - d^-\right]$. (The reader may find it helpful to compare Figures 2 and 3.) □

In the following proposition, we prove the h is superadditive, which immediately implies that we can use h for simultaneous lifting.

Proposition 4 *The function h is superadditive in its domain.*

Proof. We will prove that h is superadditive by contradiction. Suppose that h is not superadditive. Then, there are values r_1 and r_2 such that $h(r_1 + r_2) < h(r_1) + h(r_2)$. Recall that each "piece" of the function has a slope of either 0 or 1. We will call the points where h is non-differentiable "breakpoints", and the points where h is differentiable "interior" points.

Suppose that r_1 is either an interior point where the slope is 0, or a breakpoint where the slope on the left of r_1 is 0 and the slope on the right is 1. Then, there exists a small $\epsilon > 0$ such that $h(r_1 - \epsilon) = h(r_1)$. We therefore have $h(r_1 - \epsilon) + h(r_2) = h(r_1) + h(r_2) > h(r_1 + r_2) \geq h(r_1 + r_2 - \epsilon)$. This means that $r_1 - \epsilon$ and r_2 also form a counter-example. So, we can assume w.l.o.g. that r_1 is neither an interior point with slope 0 nor a breakpoint where the slope on the left is 0.

Now, suppose that r_1 is an interior point with slope 1. Then, there exists a small $\epsilon > 0$ such that $h(r_1 + \epsilon) = h(r_1) + \epsilon$. We therefore have $h(r_1 + \epsilon) + h(r_2) = h(r_1) + h(r_2) + \epsilon > h(r_1 + r_2) + \epsilon \geq h(r_1 + r_2 + \epsilon)$. This means that $r_1 + \epsilon$ and r_2 also form a counter-example. So, we can assume w.l.o.g. that r_1 is a breakpoint where the slope on the left is 1 and the slope on the right is 0.

The same argument enables us to assume that r_2 is also a breakpoint of the same type. Hence, to complete the proof, we have to show superadditivity for the case where both r_1 and r_2 are breakpoints of that type. Note these points are such that there exist positive integers k_1, k_2 such that $r_1 = S(k_1)$ and $r_2 = S(k_2)$. So, $r_1 + r_2 = S(k_1) + S(k_2) \geq S(k_1 + k_2)$ by the definition of S. The function h is increasing. So, $h(r_1 + r_2) \geq h(S(k_1 + k_2))$. By the definition of h, we have that $h(r_1) = h(S(k_1)) = k_1 d$, $h(r_2) = h(S(k_2)) = k_2 d$ and $h(S(k_1 + k_2)) = (k_1 + k_2)d^-$. This implies that $h(r_1 + r_2) \geq h(r_1) + h(r_2)$, which is a contradiction. □

We now revisit Example 4 to demonstrate the LKCIs that we get using f and h.

Example 4 (cont.): Recall that $d^- = 6$ and $L = \{4, 5, 6\}$. Using the MIR lifting function, we get the LKCI $3(x_1 + x_2 + x_3) + 6(x_4 + x_5 + x_6) \geq 6 + 11(1 - x_7)$. Using the lifting function h, we get the stronger LKCI $3(x_1+x_2+x_3)+6(x_4+x_5+x_6) \geq 6+12(1-x_7)$. One can check (either by hand or with the help of a package like `PORTA`) that the latter is facet-defining. Note that this inequality is not equivalent to a lifted cover inequality. □

4 Concluding Remarks

We have introduced two lifting procedures for knapsack cover inequalities. Our examples show that it is possible for these lifting procedures to yield non-trivial facet-defining inequalities. An interesting extension to our work would be the design and implementation of efficient separation heuristics for LKCIs. It would also be interesting to compare LKCIs with lifted cover inequalities.

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References

[1] A. Atamtürk (2005) Cover and pack inequalities for (mixed) integer programming. *Ann. Oper. Res.*, 139, 21–38.

[2] E. Balas (1975) Facets of the knapsack polytope. *Math. Program.*, 8, 146–164.

[3] E. Balas & E. Zemel (1978) Facets of the knapsack polytope from minimal covers. *SIAM J. Appl. Math.*, 34, 119–148.

[4] E.A. Boyd (1993) Generating Fenchel cutting planes for knapsack polyhedra. *SIAM J. Optim.*, 3, 734–750.

[5] R.D. Carr, L.K. Fleischer, V.J. Leung & C.A. Phillips (2000) Strengthening integrality gaps for capacitated network design and covering problems. In *Proceedings of SODA XI*, pp. 106–115. New York, NY: SIAM.

[6] T. Christof & A. Loebl, PORTA (polyhedron representation transformation algorithm). Software package, available for download at `http://comopt.ifi.uni-heidelberg.de/software/PORTA/`

[7] V. Chvátal (1973) Edmonds polytopes and a hierarchy of combinatorial problems. *Discr. Math.*, 4, 305–337.

[8] H. Crowder, E. Johnson & M. Padberg (1983) Solving large-scale 0-1 linear programming programs. *Oper. Res.*, 31, 803–834.

[9] F. Glover (1973) Unit-coefficient inequalities for zero-one programming. *Management Science Report 73-7*, University of Colorado, July 1973.

[10] R.E. Gomory (1958) Outline of an algorithm for integer solutions to linear programs. *Bull. Amer. Math. Soc.*, 64, 275–278.

[11] R.E. Gomory (1960) An algorithm for the mixed integer problem. Technical Report RAND-P-1885, Santa Monica, CA: Rand Corporation.

[12] Z. Gu, G.L. Nemhauser & M.W.P. Savelsbergh (1998) Lifted cover inequalities for 0-1 integer programs: computation. *INFORMS J. Comput.*, 10, 427–437.

[13] Z. Gu, G.L. Nemhauser & M.W.P. Savelsbergh (2000) Sequence-independent lifting in mixed integer programming. *J. Comb. Optim.*, 4, 109–129.

[14] K.L. Hoffman & M.W. Padberg (1991) Improving LP-representations of zero-one linear programs for branch-and-cut. *ORSA J. Comput.*, 3, 121–134.

[15] D. Hartvigsen & E. Zemel (1992) The complexity of lifted inequalities for the knapsack problem. *Discr. Appl. Math.*, 39, 113–123.

[16] K. Kaparis & A.N. Letchford (2010) Separation algorithms for 0-1 knapsack polytopes. *Math. Program.*, 124, 69–91.

[17] K.S. Lee (2012) Separation heuristic for the rank-1 Chvátal-Gomory inequalities for the binary knapsack problem. *J. Korean Inst. Indust. Eng.*, 38, 74–79.

[18] A.N. Letchford & G. Souli (2019) On lifted cover inequalities: a new lifting procedure with unusual properties. *Oper. Res. Lett.*, 47, 83–87.

[19] H. Marchand & L.A. Wolsey (2001) Aggregation and mixed integer rounding to solve MIPs. *Oper. Res.*, 49, 363–371.

[20] G.L. Nemhauser & L.A. Wolsey (1990) A recursive procedure to generate all cuts for 0–1 mixed integer programs. *Math. Program.*, 46, 379–390.

[21] M.W. Padberg (1975) A note on zero-one programming. *Oper. Res.*, 23, 833–837.

[22] R. Weismantel (1997) On the 0-1 knapsack polytope. *Math. Program.*, 77, 49–68.

[23] L.A. Wolsey (1975) Faces for a linear inequality in 0–1 variables. *Math. Program.*, 8, 165–178.

[24] L.A. Wolsey (1976) Facets and strong valid inequalities for integer programs. *Oper. Res.*, 24, 367–372.

[25] L.A. Wolsey (1977) Valid inequalities and superadditivity for 0-1 integer programs. *Math. Oper. Res.*, 2, 66–77.

[26] E. Zemel (1989) Easily computable facets of the knapsack polytope. *Math. Oper. Res.*, 14, 760–765.