The Distributional Stress-Energy Quadrupole

Jonathan Gratus[1,2,3,*], Paolo Pinto[1,2,4], Spyridon Talaganis[1,5]

May 7, 2020

[1] Physics department, Lancaster University, Lancaster LA1 4YB,
[2] The Cockcroft Institute Daresbury Laboratory, Daresbury, Warrington WA4 4AD UK.
[3] j.gratus@lancaster.ac.uk
[4] p.pinto@lancaster.ac.uk
[5] s.talaganis@lancaster.ac.uk
[*] Corresponding author.

Abstract

We investigate stress-energy tensors constructed from the delta function on a worldline. We concentrate on the quadrupole which has up to two partial or derivatives of the delta function. Unlike the dipole, we show that the quadrupole has 20 free components which are not determined by the properties of the stress-energy tensor. These need to be derived from an underlying model and we give an example modelling a divergent-free dust. We show that the components corresponding to the partial derivatives representation of the quadrupole, have a gauge like freedom. We give the change of coordinate formula which involves a second derivative and two integrals. We also show how to define the quadrupole without reference to a coordinate systems or a metric. For the representation using covariant derivatives, we show how to split a quadrupole into a pure monopole, pure dipole and pure quadrupole in a coordinate free way.

Contents

1 Introduction

With the recent confirmed observation of gravitational waves, it is natural to look at possible sources of gravity, in particular stress-energy tensors in which we can model compact systems, which are small with respect the distance to an observer. gravitational wave astronomy shall give rise to major developments in gravitational physics and astrophysics. The LIGO and VIRGO detectors have observed relativistic gravitational two-body systems. The existing network of gravitational wave interferometers is expanding both on Earth (for instance, via KAGRA and LIGO-India) and in space. Compact binary systems are important sources of gravitational waves. Two-body systems such as pairs of black holes or neutron stars can emit vast amounts of energy in the form of gravitational waves as their orbits decay and the bodies coalesce.

In this article we model the compact source, using a distribution, in which all the mass is concentrated in one point in space and hence a worldline in spacetime, but has an extended structure encoded as a multipole expansion. The zeroth order is the monopole, followed by the dipole and then the quadrupole. Here we consider in detail this quadrupole order. It is well known [1] that gravitational radiation will be dominated by the quadrupole moment.

When considering sources of gravitational waves, there are multiple approaches. For simple orbiting masses, where relativistic effects can be ignored one can find analytic solutions. By contrast the final stages of coalescing black holes require detailed numerical simulations. Once the stress-energy tensor is constructed one can evaluate the corresponding perturbation of the metric and hence the predicted gravitational wave. Our approach is different. In this article we examine the dynamics of quadrupole sources. This has a major advantage that the dynamics are encoded as ODEs for the components, as opposed to the coupled nonlinear PDEs which one is required to solve to model a general relativistic source. The only constraints we put on the source is that it obeys the rules of a stress-energy tensor, namely symmetry of its indices and the divergenceless condition. For the monopole and the dipole it is well known that these conditions constrain the dynamics so much that they prescribe the ODEs: the geodesic equation for the monopole and the Matterson-Papapetrou-Tulczyjew-Dixon equations for the dipole. One may therefore ask if these two conditions also constrain the quadrupole sufficiently to prescribe the ODEs for the components. In this article we show that, whereas 40 of the components are prescribed by ODEs, a further 20 are arbitrary. For example a quadrupole can expand and contract as depicted in figure 1. Thus by itself this approach cannot completely prescribed the dynamics of a quadrupole and one must

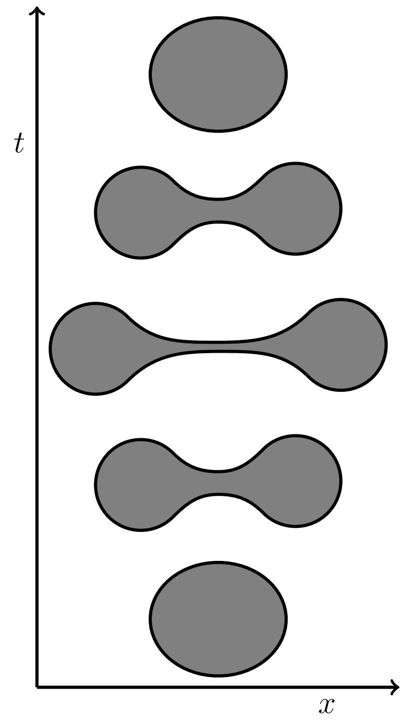

Figure 1: Schematic showing a blob of matter separating and then recombining. Such internal dynamics can take place solely within the free components, without affecting the divergencelessness of the stress-energy tensor (7).

add additional ODEs, or algebraic equations, which one can consider to be **constitutive relations** for quadrupole. These should arise from an underlying model of the source. I.e. coalescing black holes will have different constitutive relations to a rotating "rigid" body held by non gravitation, e.g. electromagnetic and quantum forces. Once the constitutive relations are decided on, the ODEs can be solved and compared to experiment.

Approximating a distribution of matter with an object at single point is a well established method in many branches of physics. Such approximations are valid if the size of the system is small compared to other distances involved. For example when considering coalescing Black Holes as a source of gravitational waves, the distance between the Black Holes is orders of magnitude smaller than there distance to earth. However there may be other objects in nature for which a multipole expansion may be a good model. For example, it is known that atomic nuclei and molecules have higher order moments. Although these objects are fundamentally quantum in nature, they may be modelled by a classical point particles with multipole structure. Knowing the dynamics of multipoles may also shed light on the problem of radiation reaction, in the case when it is the radiation reaction to the dipole or quadrupole dynamics.

There are many important articles which consider multipole expansions. These date back to at least the 1950s there Tulczyjew [2] considered a multipole expansions to derive the Matterson-Papapetrou-Tulczyjew-Dixon equations for the dipole. Then in the 1960s and 1970s Dixon [3, 4, 5] and Ellis [6] considered both charge and mass distributions using two different general formalism, which we compare here denoting them the Dixon and Ellis representations.

Recently Steinhoff and Puetzfeld [7, 8, 9] calculate the dynamic equation for the components of the quadrupole. In addition they consider the monopole-dipole and monopole-dipole-quadrupole system. In all cases the worldline of the multipole effects the dynamics of the components. However in the above the authors consider whether if and how the dynamics of the worldline is effected by the higher order moments. The conclude that one would need supplementary conditions in order to determine the worldline dynamics. We note that these supplementary conditions are distinct from the constitutive relations described here for the quadrupole. In this article, excluding the section on the monopole, the worldline is arbitrary but prescribed. Thus at the dipole order there are no supplementary conditions required. However as stated there are 20 constitutive relations required at the quadrupole order.

Let $\mathcal{M}$ be spacetime with metric $g_{\mu\nu}$ and the Levi-Civita[1] connection ∇_μ with Christoffel symbol $\Gamma^{\mu}_{\nu\rho}$. Here Greek indices run $\mu,\nu = 0,1,2,3$ and Latin indices $a,b = 1,2,3$. Let $C : \mathcal{I} \to \mathcal{M}$ where $\mathcal{I} \subset \mathbb{R}$ be the worldline of the source[2] with components $C^\mu(\sigma)$. At this point we do not assume that σ is proper time. Here we consider stress-energy tensors $T^{\mu\nu}$ which are non zero only on the worldline $C^\mu(\sigma)$, where it has a Dirac–δ like properties. Such stress-energy tensors are called **distributional**.

Being a non linear theory, one cannot simply apply the theory of distributions to general relativity. It is not meaningful to write Einstein's equations

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R = 8\pi\, T_{\mu\nu} \tag{1}$$

where the right hand side is a distribution. This contrasts with electromagnetism, which since it is a linear theory, one often uses distributional sources. For example an arbitrary moving point charge which gives rise to the Liénard-Wiechard fields.

There are various interpretations to (1) which one can try when right hand side is distributional. One approach is to extend the theory of distributions to include products. The most successful being Colombeau algebra [10].

Another approach is to consider $T_{\mu\nu}$ as a source of linearised gravity. Perturbatively expanding the gravitational metric, $g_{\mu\nu}$, about a background $\bar{g}_{\mu\nu}$, $g_{\mu\nu} = \bar{g}_{\mu\nu} + \epsilon\, h^{(1)}_{\mu\nu} + \cdots$ where $\epsilon \ll 1$ is the perturbation parameter, and plugging the expansion into the Einstein equation (1) one has

$$G_{\mu\nu} = \bar{G}_{\mu\nu} + \epsilon G^{(1)}_{\mu\nu} + \ldots \qquad \text{and} \qquad T_{\mu\nu} = \bar{T}_{\mu\nu} + \epsilon T^{(1)}_{\mu\nu} + \ldots \tag{2}$$

Hence the background metric $\bar{g}_{\mu\nu}$ satisfies $\bar{G}_{\mu\nu} = 8\pi\, \bar{T}_{\mu\nu}$. The linearised equations are then given by

$$G^{(1)}_{\mu\nu} = 8\pi\, T^{(1)}_{\mu\nu} \tag{3}$$

Setting $\mathcal{H}^{1}_{\mu\nu} = h^{1}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu} h^{1}$ and using the Lorenz gauge ($\bar{\nabla}^\mu \mathcal{H}^1_{\mu\nu} = 0$), (3) becomes

$$\bar{\Box}\mathcal{H}^{(1)}_{\mu\nu} = -16\pi T^{(1)}_{\mu\nu}. \tag{4}$$

where $\bar{\Box} = \bar{g}^{\mu\nu}\bar{\nabla}_\mu\bar{\nabla}_\nu$ is the covariant d'Alembertian operator and is constructed purely out of the background spacetime metric $\bar{g}_{\mu\nu}$. In the case where the background $\bar{g}_{\mu\nu}$ is the Minkowski metric, then $\bar{\Box} = \partial_\mu \partial^\mu$ and we can give $\mathcal{H}^{(1)}_{\mu\nu}$ in terms of an integral over the retarded Greens functions.

$$\mathcal{H}^{(1)}_{\mu\nu}(t,\vec{x}) = 4G \int \frac{T^{(1)}_{\mu\nu}(t - |\vec{x} - \vec{x}'|, \vec{x}')}{|\vec{x} - \vec{x}'|} d^3\vec{x}' \tag{5}$$

One should be careful as there is clearly a contradiction between the statement that the perturbation to the background stress-energy tensor is small, and the statement that it is distributional, and therefore infinite.

In this article we are concerned *only* with the structure of the distributional stress-energy, which we write as $T^{\mu\nu}$, and avoid questions of how it should be applied. Since $T^{\mu\nu}$ is a stress-energy tensor it has the symmetry

$$T^{\mu\nu} = T^{\nu\mu} \tag{6}$$

[1] It turns out that in most calculation the metric plays no roll and a arbitrary linear connection can be used. See section 6.

[2] Even using proper time in Minkowski space, one cannot assume that $\mathcal{I} = \mathbb{R}$ since it is possible to accelerate to lightlike infinity in finite proper time.

and is divergenceless, also known as covariantly conserved

$$\nabla_\mu T^{\mu\nu} = 0 \tag{7}$$

Observe that because $T^{\mu\nu}$ is a tensor density (7) becomes

$$0 = \nabla_\mu T^{\mu\nu} = \partial_\mu T^{\mu\nu} + \Gamma^\nu_{\mu\rho} T^{\mu\rho} \tag{8}$$

where $\Gamma^\nu_{\mu\rho}$ are the Christoffel symbols.

There are several ways of representing a multipole. However we consider multipoles to be distributions which are integrated with a symmetric test tensor $\phi_{\mu\nu} = \phi_{\nu\mu}$, so that

$$\int_{\mathcal{M}} T^{\mu\nu}\, \phi_{\mu\nu}\, d^4x \qquad \text{is a real number} \tag{9}$$

These all can be written as an integral over the worldline with a number of derivative of the Dirac δ-function. I.e. a multipole of order k is

$$T^{\mu\nu} = \sum_{r=0}^{k} \int_{\mathcal{I}} \zeta^{\mu\nu\cdots}(\sigma)\, \mathcal{D}^{(r)}_{\cdots}\, \delta^{(4)}\big(x - C(\sigma)\big)\, d\sigma \tag{10}$$

where there are r additional indices on $\zeta^{\mu\nu\cdots}$ and $\mathcal{D}^{(r)}_{\cdots}$. The subscript dots on $\mathcal{D}^{(r)}_{\cdots}$ contract with the superscript dots on $\zeta^{\mu\nu\cdots}$. Here $\mathcal{D}^{(r)}_{\cdots}$ represents r derivatives of the δ-function. The familiar cases are the **monopole** when $k = 0$, the **dipole** when $k = 1$ and the **quadrupole** when $k = 2$. As can be seen from (10) the general dipole contains the monopole term and the general quadrupole contains both the monopole and dipole terms. In general, it is not possible to extract the monopole and dipole terms from the quadrupole, without additional structure such as a preferred vector field or a coordinate system. For the monopole (6) and (7) lead to the geodesic equation. By contrasts, for the dipole and quadrupole there is no need to assume the worldline C is a geodesic. Therefore, unless otherwise stated, we present all the result for an arbitrary but prescribed worldline.

There are two main representations of multipoles. One uses the partial derivatives, which we call the **Ellis** representation. The other uses the covariant derivative and will be called the **Dixon** representation. Both have their advantages and disadvantages and these are outline in section 2 below. The Ellis formulation is greatly simplified when using a coordinate system (σ, z^1, z^2, z^3) which is adapted to the worldline, i.e. where

$$C^0(\sigma) = \sigma \qquad \text{and} \qquad C^a(\sigma) = 0 \tag{11}$$

for $a = 1, 2, 3$. In this coordinate system the integral in (10) can be removed. Observe that (11) implies $\dot{C}^0 = 1$ and $\dot{C}^a = 0$.

The monopole and dipole have been extensively studied in the literature, [11, 12, 13]. In this article we concentrate mainly on the quadrupole. This is particularly interesting. Not only is it the natural source of gravitational waves, but it has several unusual properties not seen in the case of the monopole or quadrupole. These include

- The quadrupole contains free components.
- In the Ellis representation, the components $\zeta^{\mu\nu\rho\kappa}$ to not transform as tensors but instead involve second derivatives and double integrals.
- There is no concept of mass. Instead one can only talk about the energy of a quadrupole and only really in the case where there is a timelike Killing symmetry.

The $\zeta^{\mu\nu\cdots} = \zeta^{\mu\nu\cdots}(\sigma)$ are called the components of $T^{\mu\nu}$ and are functions only of the position on the worldline C. Clearly from (6) they have the symmetry

$$\zeta^{\mu\nu\cdots} = \zeta^{\nu\mu\cdots} \tag{12}$$

depending on the representation, we may also choose to impose additional symmetries for uniqueness. We then apply the divergenceless condition (7) to establish further condition on the $\zeta^{\mu\nu\cdots}$. We can place the components $\zeta^{\mu\nu\cdots}$ into three categories.

	Electromagnetic		Gravitational	
	ODE	free	ODE	free
Monopole	1	0	1	0
Semi-dipole	1	3	7	0
full dipole	1	6	10	0
semi-quadrupole	1	12	22	6
full quadrupole	1	20	40	20

Table 1: List of the number of components which are determined by an ODE and the number which are free, for monopoles, dipoles and quadrupoles. The electromagnetic sources refer to a current $\mathcal{J}^\mu$ which is conserved and a source for Maxwell's equations. The gravitational source refers to a stress-energy tensor $T^{\mu\nu}$ which are sources for (linearised) Einstein's equations. Each order includes all the lower orders. That is the 10 components in the full stress-energy dipole includes 1 monopole component, while the $(40+20)$ components in the full quadrupole includes both dipole and monopole components. The definition of the semi-dipole and semi-quadrupole is given in section 6.6.

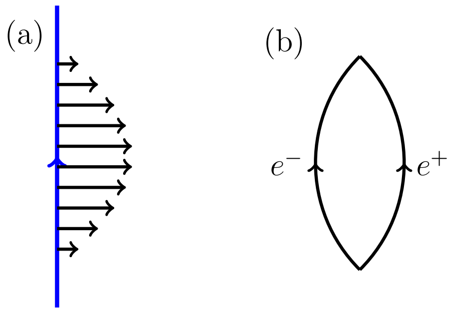

Figure 2: (a) An electric dipole appears for a finite period of time and then disappears. This does not break charge conservation. This corresponds to an electron-positron pair (b) appearing and then disappearing.

- Some components are algebraically related to other components and can therefore be removed.
- Some components can be are determined by a first order ODE. These are result of the differential equation (7). In order to specify these components it is only necessary to specify their initial value at some point along the worldline.
- This leaves the components we call **free**. These are not constrained by (6) and (7) and are allowed to take on any value. These free components can however influence the ODE components.

  In order to completely specify the dynamics of a quadrupole, these free components need to be replaced by constitutive equations. The choice of constitutive equations depends on a choice of a model for the material. For example the the quadrupole modelling an elastic material or a fluid with or without pressure, or something else. In section 5 we consider the dust stress-energy tensor and use it to suggest corresponding constitutive equations.

In table 1 the number of ODE and free components is given. This is compared to the electromagnetic dipoles and quadrupoles.

In addition, some components may have a **gauge freedom**. That is several $\zeta^{\mu\nu\cdots}$ correspond to the same stress-energy tensor. Equivalently a given stress-energy tensor does not completely specify the components $\zeta^{\mu\nu\cdots}$. Examples of this gauge freedom for the dipole and quadrupole are given in equations (45) and (64) below. This has similarities to other gauges freedoms in that it arises from integrating a physically observable tensor, although the components $\zeta^{\mu\nu\cdots}$ are not themselves tensors.

For the electromagnetic dipole there is one ODE component, which is simply the total charge and satisfies $dq/d\sigma = 0$, and there are six free components corresponding to the three electric and three magnetic components. These can be anything without braking charge conservation, as seen in figure 2. For the stress-energy tensor, the free components can correspond to the internal matter separating and coalescing, as in figure 1. In the electromagnetic current case, having free components was not so concerning as one would expect these components to be fixed by the internal dynamics of the charges. However the stress-energy tensor is supposed to contain all the dynamics, and one would like there not to be any free components. One therefore needs additional constitutive relations which encode the matter one is modelling. In this article we give an example of constitutive relations which corresponding to non divergent dust.

Given a regular stress-energy tensor $T^{\mu\nu}$ and a Killing vector field K^μ we can find a conserved quantity $T^{\mu\nu}K_\nu$ such that $\nabla_\mu(T^{\mu\nu}K_\nu) = 0$. The same is true for the distributional stress-energy tensor. Here K_μ gives rise to a conserved vector field $Q^\mu(\sigma)$ along the worldline C. If K^μ is a timelike Killing vector field it is natural to associate $Q^\mu(\sigma)$ as the conserved energy of the multipole. The relationship between the energy and mass is however subtle. In the monopole and dipole case there is a natural definition of the mass, the same is not true in the quadrupole case. Even when a mass can be defined, it is not conserved in general.

Outline of article

As stated above there are two established methods of representing the stress-energy distribution: one using partial derivatives in (10), which we call the **Ellis** representation, and the other using covariant derivatives, which we call the **Dixon** representation. The pro and cons of these two approaches is discussed in section 2 and summarised in table 2. In section 3 we summarise the key results of the monopole and dipole stress-energy tensors. We highlight the Ellis and Dixon representations of the dipole.

In section 4 we examine the quadrupole in detail. In this section we use the Ellis approach. We give the gauge freedom of the components and complicated change of coordinates which involve second derivatives and integrals over the worldline. We use the adapted coordinates (11) and give the differential equations arising from the symmetry (6) and divergencelessness (7) of $T^{\mu\nu}$. We can now identify which components are algebraic, which satisfy ODEs and which are free. In subsection 4.1 we give an example of the free components in Minkowski spacetime as depicted in figure 1. As stated above, if there is a Killing vector field, there exists a corresponding conserved quantity. These are given in section 4.2. This included a new interpretation of the conserved quantities corresponding the three Lorentz boosts.

In section 5 we use the limit of the dust stress-energy tensor as it is squeezed onto the worldline to construct a choice of constitutive relations to replace the free components with ODEs.

Although we have defined everything in terms of a coordinate system, it is useful to define the multipoles in a coordinate free manner. The advantage of such an approach is that complicated coordinate transformations are avoided. It is interesting to observe that, using deRham currents, multipoles can be defined without any additional structure on a manifold. I.e. it is not necessary to prescribe either a metric or a connection to define a general multipole. This is particularly useful if we wish to extend the notion of a general multipole tensor distributions to manifolds such as the tangent bundle which does not posses either metric or connection. However a connection is of course needed to define the covariantly conserved property (7). In section 6 we detail this approach. Having defined a multipole in a coordinate free manner, one can extract the components in the Ellis approach with respect a coordinate system. This is explicitly given in the case of an adapted coordinate system in section 6.4. By contrast to the Ellis approach, the Dixon approach contains more information about a multipole, namely how it splits into a monopole term, a dipole term, a quadrupole term and so on. This split, called here the **Dixon split**, is actually coordinate independent and the details are given in section 6.7.

As noted in [14], without a metric, connection or coordinate system, it is still possible to define and a pure electric dipole. In this article we call such a dipole a **semi-dipole**. We observe that the semi-dipole stress-energy consists of the displacement vector by not the spin. In section 6.6 we define the semi-dipole and semi-quadrupole stress-energy tensor.

We conclude, in section 7. Finally in the appendix we prove all the results in the body of the article.

Notation regarding derivatives

Given a coordinate system $(x^0,\dots,x^3)$ then Greek indices $\mu,\nu=0,\dots,3$. We write the partial derivatives

$$\partial_\mu = \frac{\partial}{\partial x^\mu} \tag{13}$$

In the case of the adapted coordinates (σ, z^1, z^2, z^3) obeying (11) we use both Greek indices $\mu,\nu = 0,\dots,3$ and Latin indices $a,b=1,2,3$. In this case we have

$$\partial_0 = \frac{\partial}{\partial \sigma} \qquad \text{and} \qquad \partial_a = \frac{\partial}{\partial z^a} \tag{14}$$

Thus, even if not stated explicitly, writing ∂_a implies we are referring to an adapted coordinates system.

Note that in both the adapted and non adapted case we use overdot to represent differentiation with respect to σ. In the non adapted coordinates this is only used for quantities, such as $C^\mu(\sigma)$ and $\dot{C}^\mu(\sigma)$ which are only defined on the worldline. In the adapted coordinate cases is it synonymous with ∂_0.

When we have two non adapted coordinate systems $(x^0,\dots,x^3)$ and $(\hat{x}^{\hat{0}},\dots,\hat{x}^{\hat{3}})$ we use the hat on the index to indicate the hatted coordinate system. Thus

$$\partial_{\hat{\mu}} = \frac{\partial}{\partial \hat{x}^{\hat{\mu}}} \tag{15}$$

Likewise for the adapted coordinate system $(\hat{\sigma}, \hat{z}^{\hat{1}}, \hat{z}^{\hat{2}}, \hat{z}^{\hat{3}})$ we have

$$\partial_{\hat{0}} = \frac{\partial}{\partial \hat{\sigma}} \qquad \text{and} \qquad \partial_{\hat{a}} = \frac{\partial}{\partial \hat{z}^{\hat{a}}} \tag{16}$$

2 Dixon's versus Ellis's approaches to multipoles

2.1 The Ellis approach

As stated in the introduction there are two standard approaches to writing down distributional multipoles.

One method [6] uses partial derivatives of the Dirac-δ function. Although Ellis principally defines it for the electric current $\mathcal{J}^\mu$ it is easy to extend this for the stress-energy tensor. So a multipole of order k is given by

$$T^{\mu\nu} = \frac{1}{k!}\int_{\mathcal{I}} \zeta^{\mu\nu\rho_1\dots\rho_k}(\sigma)\ \partial_{\rho_1}\cdots\partial_{\rho_k}\delta^{(4)}\big(x - C(\sigma)\big)\, d\sigma \tag{17}$$

where $\zeta^{\mu\nu\rho_1\dots\rho_k}(\sigma)$ are smooth functions of σ and ∂_{ρ_j} is given by (13). Thus when acting on the test tensor $\phi_{\mu\nu}$

$$\int_{\mathcal{M}} T^{\mu\nu}\ \phi_{\mu\nu}\ d^4x = (-1)^k \frac{1}{k!}\int_{\mathcal{I}} \zeta^{\mu\nu\rho_1\dots\rho_k}(\sigma)\ \big(\partial_{\rho_1}\cdots\partial_{\rho_k}\phi_{\mu\nu}\big)\big|_{C(\sigma)} \tag{18}$$

In this article we will refer to this representation of a multipole as the **Ellis** representation

The symmetry of $T^{\mu\nu}$ leads to

$$\zeta^{\mu\nu\rho_1\dots\rho_k} = \zeta^{\nu\mu\rho_1\dots\rho_k} \tag{19}$$

Ellis	Dixon
Can be defined using coordinates.	Can be defined using coordinates.
Components are unique for adapted coordinates. In a general coordinates they have a gauge freedom.	Components are unique.
For general coordinate transformation the components require higher derivatives and integrals.	Components transform as a tensor.
Do not require any additional structure. These can be defined without referring to a metric or additional vector field.	Requires the connection and the Dixon vector $N_\mu(\sigma)$ for the definition.
Contains all multipoles up to specific order.	Contains all multipoles up to specific order.
It is not possible to extract a multipole of a specific order without additional structure. For example an adapted coordinate system.	Easy to extract a multipoles of any order.
Can be easily defined in a coordinate free way using DeRham push forward.	The Dixon split can be defined in a coordinate free way, but this definition is complicated and requires the DeRham push forward plus a non intuitive additional axiom. This axiom is given in section 6.7 and encodes the orthogonality condition.
The dipole can be written in the Ellis representation, which is consistent with the Matterson-Papapetrou-Tulczyjew-Dixon equations.	The dipole can be written in the Dixon representation, which is consistent with the Matterson-Papapetrou-Tulczyjew-Dixon equations.
There is no concept of the mass of the multipole	The mass is given by the monopole term.
There is no orthogonality condition.	There is a complicated formula for the components with respect to different $N^\mu(\sigma)$. This will mix in multipoles of different orders.
The relationship between these moments and Fourier transforms is less clear than the Dixon representation.	There is a clear relationship between these moments and Fourier transform.
One can construct a regular tensor field whose moments, up to k, are the components of the distribution. The best method is using squeezed tensors that employ an adapted coordinate system. In principle is should be possible to do reconstruct an original distribution using the Fourier transform but this has not been investigated. This requires certain assumptions about analyticity of Fourier transform.	One can construct a tensor field whose moments, up to k, are the components of the distribution. This is by considering the fields on the transverse hyperspace constructed from the geodesic map of vectors orthogonal to $N^\mu(\sigma)$. If all the moments are know one can reconstruct an original distribution. This also requires certain assumptions about analyticity of Fourier transform.
There is a formula for extracting the components using test tensors, in adapted coordinate.	In principle the components can be extracted using test tensors.

Table 2: Comparison between the Ellis and Dixon representations.

In additional the partial derivatives commute it is natural to demand that the components of ζ are symmetric. Thus we set

$$\zeta^{\mu\nu\rho_1\dots\rho_k} = \zeta^{\mu\nu(\rho_1\dots\rho_k)} \tag{20}$$

where the round brackets mean the scaled sum over all permutations of the indices,

$$\zeta^{\mu\nu(\rho_1\dots\rho_k)} = \frac{1}{k!} \sum_{\substack{\text{All permutations} \\ i_1\dots i_k}} \zeta^{\mu\nu\rho_{i_1}\dots\rho_{i_k}} \tag{21}$$

One problem with the Ellis representation is that the $\zeta^{\mu\nu\rho_1\dots\rho_k}$ are not unique. Examples of the gauge freedom that these $\zeta^{\mu\nu\rho_1\dots\rho_k}$ have is given in (45) and (64). This contrasts with the case when one chooses and adapted coordinate system below.

2.2 Adapted coordinates

In general expressions for multipoles in the Ellis representation are complicated. They greatly simply if one chooses an adapted coordinate system as given by (11). In this coordinate system the integral over $\mathcal{I}$ is no longer necessary and we replace (17) with

$$T^{\mu\nu} = \sum_{r=0}^{k} \frac{1}{r!} \gamma^{\mu\nu a_1\dots a_r 0\dots 0}(\sigma)\, \partial_{a_1}\cdots\partial_{a_r}\, \delta^{(3)}(\boldsymbol{z}) \tag{22}$$

where $\boldsymbol{z} = (z^1, z^2, z^3)$. The component $\gamma^{\mu\nu a_1\dots a_r 0\dots 0}$ has $(k-r)$ zero indices, so that $\gamma^{\mu\nu a_1\dots a_r 0\dots 0}$ has $2+k$ indices. Observe we only differentiate $\delta^{(3)}(\boldsymbol{z})$ in the z^a direction. Thus when acting on a test tensor

$$\int_{\mathcal{M}} T^{\mu\nu}\, \phi_{\mu\nu}\, d^4x = \sum_{r=0}^{k} \frac{(-1)^r}{r!} \int_{\mathcal{I}} d\sigma\, \gamma^{\mu\nu a_1\dots a_r 0\dots 0}(\sigma)\, (\partial_{a_1}\cdots\partial_{a_r}\, \phi_{\mu\nu}) \tag{23}$$

See proof number 1 in the appendix.

We still impose the symmetry conditions (19) and (20) on the γ's so that

$$\gamma^{\mu\nu\rho_1\dots\rho_k} = \gamma^{\nu\mu\rho_1\dots\rho_k} = \gamma^{\mu\nu(\rho_1\dots\rho_k)} \tag{24}$$

The relationship between the $\gamma^{\mu\nu a_1\dots a_r 0\dots 0}$ and $\zeta^{\mu\nu\rho_1\dots\rho_k}$ is given by comparing (18) and (23) for an adapted coordinate system

$$\gamma^{\mu\nu a_1\dots a_r 0\dots 0} = \frac{1}{(k-r)!} \partial_0^{r-k} \zeta^{\mu\nu a_1\dots a_r 0\dots 0} \tag{25}$$

See proof number 2 in the appendix.

In an adapted coordinate system, the $\gamma^{\mu\nu a_1\dots a_r 0\dots 0}$ are uniquely determined by the distribution. The gauge freedom of the $\zeta^{\mu\nu a_1,\dots a_r 0\dots 0}$ in this case arising from the arbitrary constants when integrating (25) with respect to σ.

With respect to this coordinate system, one can partition the multipoles into a monopole, a pure dipole, a pure quadrupole and so on. However this is a coordinate dependent splitting and these terms will mix when changing the coordinate system. The coordinate transformation for quadrupoles is given in (76)-(78). Although they involve up to k derivatives of the coordinate transformation, they do not require any integrals.

2.3 Squeezed tensors

In an adapted coordinate system, one can construct a one parameter family of regular stress-energy tensors $\mathcal{T}_\varepsilon^{\mu\nu}$ from a given stress-energy tensor $\mathcal{T}^{\mu\nu}$ such that in the weak limit $\mathcal{T}_\varepsilon^{\mu\nu} \to T^{\mu\nu}$ at $\varepsilon \to 0$ to order k. Since we are using adapted coordinates, we write $(\sigma, \boldsymbol{z}) = (\sigma, z^1, z^2, z^3)$. We set

$$\mathcal{T}_\varepsilon^{\mu\nu}(\sigma, \boldsymbol{z}) = \frac{1}{\varepsilon^3}\, \mathcal{T}^{\mu\nu}\Big(\sigma, \frac{\boldsymbol{z}}{\varepsilon}\Big) \tag{26}$$

We assume that $\mathcal{T}^{\mu\nu}$ has compact support in the transverse planes. I.e. for each σ, there is a function $R(\sigma)$ such that

$$\mathcal{T}^{\mu\nu}(\sigma, \boldsymbol{z}) = 0 \qquad \text{for} \qquad g_{ab}\, z^a\, z^b > R(\sigma) \tag{27}$$

This guarantees that all the moments, are finite.

This leads to

$$\mathcal{T}_\varepsilon^{\mu\nu}(\sigma, \boldsymbol{z}) = \gamma^{\mu\nu 0\ldots 0}\; \delta^{(3)}(\boldsymbol{z}) + \varepsilon\, \gamma^{\mu\nu a 0\ldots 0}\, \partial_a \delta^{(3)}(\boldsymbol{z}) + \varepsilon^2\, \gamma^{\mu\nu ab0\ldots 0}\, \partial_a \partial_b \delta^{(3)}(\boldsymbol{z}) + \cdots \tag{28}$$

where

$$\begin{aligned} \gamma^{\mu\nu 0\ldots 0}(\sigma) &= \int_{\mathbb{R}^3} d^3\boldsymbol{z}\; \mathcal{T}^{\mu\nu}\big(\sigma, \boldsymbol{z}\big), \qquad \gamma^{\mu\nu a 0\ldots 0}(\sigma) = -\int_{\mathbb{R}^3} d^3\boldsymbol{z}\; z^a\, \mathcal{T}^{\mu\nu}\big(\sigma, \boldsymbol{z}\big), \\ \gamma^{\mu\nu ab0\ldots 0}(\sigma) &= \int_{\mathbb{R}^3} d^3\boldsymbol{z}\; z^a\, z^b\, \mathcal{T}^{\mu\nu}\big(\sigma, \boldsymbol{z}\big) \qquad \text{etc.} \end{aligned} \tag{29}$$

See proof number 3 in the appendix. Thus there is an intimate relationship between the components of a distribution and the moments of a regular stress-energy tensor. Here the zeroth order gives the monopole, the first order the dipole and so on. This split is with respect to the chosen adapted coordinate system and these will mix under a coordinate transformation.

2.4 The Dixon approach

The alternative approach, largely developed by Dixon [4] uses the covariant derivative and a choice of a vector field $N_\mu(\sigma)$ along the worldline C^μ. This we will call the **Dixon vector**. This vector is required to be not orthogonal to the worldline C^μ, i.e.

$$N_\mu\, \dot{C}^\mu \neq 0 \tag{30}$$

As long as the worldline C is timelike, a natural choice of the Dixon vector is $\dot{C}$, i.e. $N_\mu = g_{\mu\nu}\, \dot{C}^\mu$ but this is not the only choice. Having chosen N_μ, the **Dixon** representation of a multipole is defined by its action on the test tensor $\phi_{\mu\nu}$ as

$$\int_{\mathcal{M}} T^{\mu\nu}\; \phi_{\mu\nu}\; d^4x = \sum_{r=0}^{k} (-1)^r \frac{1}{r!} \int_{\mathcal{I}} \xi^{\mu\nu\rho_1\ldots\rho_r}(\sigma)\, \big(\nabla_{c_1} \cdots \nabla_{c_r} \phi_{\mu\nu}\big)\big|_{C(\sigma)}\; d\sigma \tag{31}$$

where we demand that the components $\xi^{\mu\nu\rho_1\ldots\rho_k}$ are orthogonal to the vector N^μ

$$N_{\rho_j}\; \xi^{\mu\nu\rho_1\ldots\rho_k} = 0 \tag{32}$$

for $j = 1, \ldots, k$. The covariant derivatives do not commute, as they give rise to curvature terms and lower the number of derivatives. Therefore we again assume that $\xi^{\mu\nu\rho_1\ldots\rho_k}$ are symmetric in the relevant indices.

$$\xi^{\mu\nu\rho_1\ldots\rho_k} = \xi^{\mu\nu(\rho_1\ldots\rho_k)} \tag{33}$$

speed of light	[1]
dx^μ	$[L]$
$g_{\mu\nu}$	[1]
$\dot{C}$	$[L^{-1}]$
$\dot{C}^\mu$	[1]
∂_μ	$[L^{-1}]$
$\delta^{(4)}\big(x - C(\sigma)\big)$	$[L^{-4}]$
mass m	$[M]$

$T^{\mu\nu}$	$[M\,L^{-3}]$
test tensor $\phi_{\mu\nu}$	$[L^{-1}]$
dipole displacement X^μ	$[ML]$
dipole 3–momentum P^μ	$[M]$
dipole spin $S^{\mu\nu}$	$[ML]$
$\zeta^{\mu\nu\rho_{i_1}\ldots\rho_{i_k}}$	$[M\,L^k]$
$\xi^{\mu\nu\rho_{i_1}\ldots\rho_{i_k}}$	$[M\,L^k]$
$\gamma^{\mu\nu a_{i_1}\ldots a_{i_k}0\ldots0}$	$[M\,L^k]$

Table 3: List of units for quantities, in terms of mass M and length L. The speed of light is 1.

Dixon [4, See equations (4.18), (7.4), (7.5)] writes the distribution for the electric current $\mathcal{J}^\mu$ in terms of the covariant derivatives of a distribution. We can extend this to the stress-energy tensor $T^{\mu\nu}$ via

$$T^{\mu\nu} = \sum_{r=0}^{k} \frac{1}{r!}\nabla_{\rho_1}\cdots\nabla_{\rho_r}\int_{\mathcal{I}} \xi^{\mu\nu\rho_1\ldots\rho_r}(\sigma)\,\delta^{(4)}\big(x - C(\sigma)\big)\,d\sigma \tag{34}$$

Since $T^{\mu\nu}$ is a tensor density this enables us to throw the covariant derivative over onto the test tensor. This follow since if v^μ is a vector density (of the correct weight) then $\nabla_\mu v^\mu = \partial_\mu v^\mu$.

From (34) we can use the Dixon vector to perform the **Dixon split** in order to take an arbitrary kth order multipole and split it into a monopole part, a dipole part and so on. Thus we set

$$T^{\mu\nu} = \sum_{r=0}^{k} T^{\mu\nu}_{(r)} \qquad \text{where} \qquad T^{\mu\nu}_{(r)} = \frac{1}{r!}\nabla_{\rho_1}\cdots\nabla_{\rho_r}\int_{\mathcal{I}} \xi^{\mu\nu\rho_1\ldots\rho_r}(\sigma)\,\delta^{(4)}\big(x - C(\sigma)\big)\,d\sigma \tag{35}$$

In section 6.7 we present a coordinate free approach to performing this split.

Both the Ellis and Dixon approaches have advantages and disadvantages and these are listed in table 2.

3 Summary of the monopole and dipole stress-energy tensors.

3.1 The monopole

From (17) with $k = 0$ we have the gravitational monopole

$$T^{\mu\nu} = \int_{\mathcal{I}} \zeta^{\mu\nu}\delta\big(x - C(\tau)\big)\,d\tau \tag{36}$$

The requirement to be a stress-energy tensor (6),(7) implies that C satisfies the pre-geodesic equation

$$\dot{C}^\nu\nabla_\nu\dot{C}^\mu = \kappa_{\text{pre}}(\sigma)\,\dot{C}^\mu \tag{37}$$

and

$$T^{\mu\nu} = \int_{\mathcal{I}} m_{\text{pre}}(\sigma)\,\dot{C}^\mu\,\dot{C}^\nu\,\delta\big(x - C(\sigma)\big)\;d\sigma \tag{38}$$

where

$$\dot{m}_{\text{pre}} + \kappa_{\text{pre}}\,m_{\text{pre}} = 0 \tag{39}$$

Here the overdot refers to differentiation with respect to differentiation with respect to σ. If σ is proper times so that

$$g_{\mu\nu}\,\dot{C}^\mu\,\dot{C}^\nu = -1 \tag{40}$$

then $\kappa_{\text{pre}} = 0$ and (37) gives the geodesic equation

$$\frac{D\dot{C}^\mu}{d\sigma} = 0 \tag{41}$$

where $\frac{D}{d\sigma}$ represents the covariant derivative along the worldline, i.e.

$$\frac{DX^\mu}{d\sigma} = \dot{X}^\mu + \Gamma^\mu_{\nu\rho}\,X^\nu\,\dot{C}^\rho \tag{42}$$

In this case we replace m_{pre} with m in (38). If $m > 0$ then we can associate it with the mass of the source. Thus (38) becomes

$$T^{\mu\nu} = m\int_{\mathcal{I}} \dot{C}^\mu\,\dot{C}^\nu\,\delta\big(x - C(\sigma)\big)\,d\sigma \tag{43}$$

Thus there remain just one ODE for the remaining component, namely $\dot{m} = 0$. There are no additional free components. See table 1. However as stated in the introduction, we do not impose the geodesic equation for the subsequent dipole and quadrupoles terms.

3.2 The dipole

Setting $k = 1$ in (17) gives the dipole

$$T^{\mu\nu} = \int_{\mathcal{I}} \zeta^{\mu\nu\rho}\,\partial_\rho\delta\big(x - C(\sigma)\big)\,d\sigma \tag{44}$$

where the symmetry condition (6) implies $\zeta^{\mu\nu\rho} = \zeta^{\nu\mu\rho}$. We observe that, whereas the components $\zeta^{\mu\nu\rho}$ uniquely specify $T^{\mu\nu}$, the contrast is not true. That is given $T^{\mu\nu}$ the gauge freedom in $\zeta^{\mu\nu\rho}$ given by

$$\zeta^{\mu\nu\rho} \to \zeta^{\mu\nu\rho} + M^{\mu\nu}\dot{C}^\rho \tag{45}$$

where $M^{\mu\nu} = M^{\nu\mu}$ are any set of constants. See proof number 4 in the appendix.

In addition the $\zeta^{\mu\nu\rho}$ are not tensorial quantities but have a coordinate transformation which involves a derivatives of the Jacobian matrix and an integral. Given two coordinate systems $(x^0, \ldots, x^3)$ and $(\hat{x}^0, \ldots, \hat{x}^3)$ then

$$\hat{\zeta}^{\hat{\mu}\hat{\nu}\hat{\rho}} = J^{\hat{\mu}}_\mu J^{\hat{\nu}}_\nu J^{\hat{\rho}}_\rho \zeta^{\mu\nu\rho} - \dot{\hat{C}}^{\hat{\rho}} \int^\sigma \partial_\rho (J^{\hat{\mu}}_\mu J^{\hat{\nu}}_\nu)\,\zeta^{\mu\nu\rho}\,d\sigma' \tag{46}$$

where

$$J^{\hat{\mu}}_\mu = \frac{\partial \hat{x}^{\hat{\mu}}}{\partial x^\mu} \tag{47}$$

Here the freedom to choose the arbitrary constant of integration in (46) is equivalent to the gauge freedom (45). In adapted coordinates (11) then (22) and (23) become

$$T^{\mu\nu} = \gamma^{\mu\nu 0}\delta^{(3)}(\boldsymbol{z}) + \gamma^{\mu\nu a}\,\partial_a\delta^{(3)}(\boldsymbol{z}) \quad \text{where} \quad \gamma^{\mu\nu 0} = \dot{\zeta}^{\mu\nu 0} \quad \text{and} \quad \gamma^{\mu\nu a} = \zeta^{\mu\nu a} \tag{48}$$

Fortunately for the dipole the requirements (6) and (7) restrict the components $\zeta^{\mu\nu\rho}$ so much that $T^{\mu\nu}$ can be written solely in terms of tensor quantities

$$T^{\mu\nu} = \int_{\mathcal{I}} \hat{P}^{(\mu}\,\dot{C}^{\nu)}\,\delta\big(x - C(\sigma)\big)d\sigma + \nabla_\rho \int_{\mathcal{I}} \hat{S}^{\rho(\mu}\,\dot{C}^{\nu)}\,\delta\big(x - C(\sigma)\big)d\sigma \tag{49}$$

where $\hat{P}^\mu$ and $\hat{S}^{\mu\nu} + \hat{S}^{\nu\mu} = 0$ satisfy the Matterson-Papapetrou-Tulczyjew-Dixon equations

$$\frac{D\hat{S}^{\mu\nu}}{d\sigma} = \hat{P}^\nu \dot{C}^\mu - \hat{P}^\mu \dot{C}^\nu \qquad \text{and} \qquad \frac{D\hat{P}^\mu}{d\sigma} = \tfrac{1}{2} R^\mu{}_{\nu\rho\kappa}\,\dot{C}^\nu\,\hat{S}^{\kappa\rho} \tag{50}$$

To interpret (49) as a Dixon representation of a Dipole requires we find a vector N^ρ such that $N_\rho\,\hat{S}^{\rho(\mu}\,\dot{C}^{\nu)} = 0$.

Clearly we can replace the covariant derivatives with partial derivatives and Christoffel symbols to give the representation of the dipole

$$T^{\mu\nu} = \int_{\mathcal{I}} \Big(\hat{P}^{(\mu}\,\dot{C}^{\nu)} + \hat{S}^{\rho(\nu}\,\Gamma^{\mu)}{}_{\rho\kappa}\,\dot{C}^\kappa\Big)\,\delta\big(x - C(\sigma)\big)d\sigma + \int_{\mathcal{I}} \hat{S}^{\rho(\mu}\,\dot{C}^{\nu)}\,\partial_\rho\,\delta\big(x - C(\sigma)\big)d\sigma \tag{51}$$

However this is not the Ellis representation which is given by (44) where

$$\zeta^{\mu\nu\rho} = \hat{S}^{\rho(\mu}\,\dot{C}^{\nu)} + \dot{C}^\rho \int^\sigma \Big(\hat{P}^{(\mu}\,\dot{C}^{\nu)} + \hat{S}^{\rho(\nu}\,\Gamma^{\mu)}{}_{\rho\kappa}\,\dot{C}^\kappa\Big)\,d\sigma' \tag{52}$$

So that in the adapted coordinates (48) we have

$$\gamma^{\mu\nu 0} = \hat{P}^{(\mu}\,\delta_0^{\nu)} + \hat{S}^{\rho(\nu}\,\Gamma^{\mu)}{}_{\rho 0} + \partial_0(\hat{S}^{0(\mu}\,\delta_0^{\nu)}) \qquad \text{and} \qquad \gamma^{\mu\nu a} = \hat{S}^{a(\mu}\,\delta_0^{\nu)} \tag{53}$$

Recall that K^μ is a Killing vector if

$$\nabla_\mu K_\nu + \nabla_\nu K_\mu = 0 \tag{54}$$

Then $\mathcal{Q}_K$ is a conserved quantity, where

$$\mathcal{Q}_K = \gamma^{\mu 00}\,K_\mu - \gamma^{\mu 0a}\,\partial_a\,K_\mu \tag{55}$$

See proof number 14 in the appendix. From (53) we have

$$\mathcal{Q}_K = \hat{P}^\mu\,K_\mu + \tfrac{1}{2}\hat{S}^{\mu\nu}\,\nabla_\nu\,K_\mu \tag{56}$$

See proofs numbers 6-7 in the appendix.

The situation is simplified in the case when C is a geodesic. In this case we can use the Dixon representation with $N^\mu = \dot{C}^\mu$.

$$T^{\mu\nu} = \int_{\mathcal{I}} \Big(m\dot{C}^\mu\,\dot{C}^\nu + P^{(\mu}\,\dot{C}^{\nu)}\Big)\,\delta\big(x - C(\sigma)\big)d\sigma + \nabla_\rho \int_{\mathcal{I}} \Big(X^\rho \dot{C}^\mu\,\dot{C}^\nu + S^{\rho(\mu}\,\dot{C}^{\nu)}\Big)\,\delta\big(x - C(\sigma)\big)d\sigma \tag{57}$$

where

$$\hat{S}^{\mu\nu} = S^{\mu\nu} - X^\mu \dot{C}^\nu + X^\nu \dot{C}^\mu \qquad \text{and} \qquad \hat{P}^\mu = P^\mu + m\dot{C}^\mu \tag{58}$$

See proof number 5 in the appendix. These quantities have intuitive meaning. See Table 3 for the units associated with each component.

- The rest mass m.
- A displacement vector X^μ with $X_\mu\,\dot{C}^\mu = 0$.
- The rate of change of the displacement vector P^μ with $P_\mu\,\dot{C}^\mu = 0$.

- A spin tensor $S^{\mu\nu}$ with $S^{\mu\nu}+S^{\nu\mu}=0$ and $\dot{C}_\mu\, S^{\mu\nu}=0$

These satisfy

$$\dot{m}=0\,,\qquad \frac{DX^\mu}{d\sigma}=P^\mu\,,\qquad \frac{DP^\mu}{d\sigma}=\tfrac{1}{2}R^\mu{}_{\nu\rho\kappa}\,\dot{C}^\nu\, S^{\kappa\rho}+R^\mu{}_{\nu\rho\kappa}\,\dot{C}^\nu\,\dot{C}^\rho\, X^\kappa\,,\qquad \frac{DS^{\mu\nu}}{d\sigma}=0 \tag{59}$$

Counting the number of components we see there are 10 ODEs, which completely specify the dynamics of the components of the dipoles. Thus there are no additional free components. The 10 components can be loosely counted as follows: One component is the rest mass. Three displacement vectors which specify the "centre of mass" from the position of the dipole and another three represent the velocity of the centre of mass. Finally three components are referred to as the spin. These statements make more sense if we assume that the spacetime has Killing symmetries.

As we see below, the same situation does not occur for the quadrupoles. The conditions (6) and (7) do not completely determine the dynamics of all the components, it is not possible to write all the components in terms of tensors, and there is no concept of mass.

A particular case of the dipole is when $S^{\mu\nu}=0$, which is compatible with its dynamic equation (59). We call this case a **semi-dipole**. The notion of semi-dipoles and semi-quadrupoles is purely geometric and is addressed in section 6.6.

4 The quadrupole stress-energy tensor.

Setting $k=2$ in (17) gives the formula for a quadrupole.

$$T^{\mu\nu}=\frac{1}{2}\int_{\mathcal{I}}\zeta^{\mu\nu\rho\kappa}(\sigma)\,\partial_\rho\partial_\kappa\delta\big(x-C(\sigma)\big)\,d\sigma \tag{60}$$

so that the action on the test tensor $\phi_{\mu\nu}$ is given by

$$\int_{\mathbb{R}^4}T^{\mu\nu}\,\phi_{\mu\nu}\,d^4x=\frac{1}{2}\int_{\mathcal{I}}\zeta^{\mu\nu\rho\kappa}(\sigma)\,\big(\partial_\rho\partial_\kappa\phi_{\mu\nu}\big)\big|_{C(\sigma)}\,d\sigma \tag{61}$$

From (6) we impose

$$\zeta^{\mu\nu\rho\kappa}=\zeta^{\nu\mu\rho\kappa} \tag{62}$$

and due to the commutation of partial derivatives we also set

$$\zeta^{\mu\nu\rho\kappa}=\zeta^{\mu\nu\kappa\rho} \tag{63}$$

Like the $\zeta^{\mu\nu\rho}$, the $\zeta^{\mu\nu\rho\kappa}$ are not uniquely specified by the $T^{\mu\nu}$, with the gauge freedom

$$\zeta^{\mu\nu\rho\kappa}\to\zeta^{\mu\nu\rho\kappa}+M^{\nu\mu}\,\dot{C}^{(\rho}\,C^{\kappa)}+\hat{M}^{\mu\nu(\rho}\,\dot{C}^{\kappa)} \tag{64}$$

where $M^{\nu\mu}$ and $\hat{M}^{\mu\nu\rho}$ are arbitrary constants. See proof number 8 in the appendix.

As in [14], under change of coordinate $(x^0,\ldots,x^3)$ to $(\hat{x}^{\hat{0}},\ldots,\hat{x}^{\hat{3}})$ we have have a complicated transformation involving derivatives and integrals

$$\begin{aligned}\hat{\zeta}^{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\kappa}}=&\ \zeta^{\mu\nu\rho\kappa}\,\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu}\,J^{\hat{\rho}}_{\rho}\,J^{\hat{\kappa}}_{\kappa}-\tfrac{1}{2}\dot{\hat{C}}^{\hat{\rho}}\int^{\sigma}\zeta^{\mu\nu\rho\kappa}\left(\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu}(\partial_\rho\,J^{\hat{\kappa}}_{\kappa})+2\,\partial_\rho\,(\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu})\,J^{\hat{\kappa}}_{\kappa}\right)d\sigma'\\&-\tfrac{1}{2}\dot{\hat{C}}^{\hat{\kappa}}\int^{\sigma}\zeta^{\mu\nu\rho\kappa}\left(\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu}(\partial_\rho\,J^{\hat{\rho}}_{\kappa})+2\,\partial_\rho\,(\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu})\,J^{\hat{\rho}}_{\kappa}\right)d\sigma'\\&+\tfrac{1}{2}\dot{\hat{C}}^{\hat{\kappa}}\int^{\sigma}\dot{\hat{C}}^{\hat{\rho}}\int^{\sigma'}\zeta^{\mu\nu\rho\kappa}\,\partial_\rho\,\partial_\kappa\,(\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu})\,d\sigma''\,d\sigma'+\tfrac{1}{2}\dot{\hat{C}}^{\hat{\rho}}\int^{\sigma}\dot{\hat{C}}^{\hat{\kappa}}\int^{\sigma'}\zeta^{\mu\nu\rho\kappa}\,\partial_\rho\,\partial_\kappa\,(\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu})\,d\sigma''\,d\sigma'\end{aligned} \tag{65}$$

where $J^{\hat\mu}_{\mu}$ is given by

$$J^{\hat\mu}_{\mu} = \frac{\partial \hat{x}^{\hat\mu}}{\partial x^{\mu}} \tag{66}$$

and

$$\mathcal{J}^{\hat\mu\hat\nu}_{\mu\nu} = J^{\hat\mu}_{\mu}\, J^{\hat\nu}_{\nu} \tag{67}$$

See proof number 9 in the appendix. It is not necessary to give the lower limits of the integrals at these are incorporate in gauge freedom (64). However we do need to enforce the symmetry on the indices $\hat\rho\hat\kappa \leftrightarrow \hat\kappa\hat\rho$. It is necessary to check that (67) is consistent with the gauge freedom (64). See proof number 10 in the appendix.

As stated in the introduction the quadrupole is greatly simplified if we choose adapted coordinates given in (11), so that $\dot{C}^{\mu} = \delta^{\mu}_{0}$. Equation (60) can now be written in terms of components $\gamma^{\mu\nu\rho\kappa}$

$$T^{\mu\nu}(\sigma, \boldsymbol{z}) = \gamma^{\mu\nu00}(\sigma)\, \delta^{(3)}(\boldsymbol{z}) + \gamma^{\mu\nu0a}(\sigma)\, \partial_a \delta^{(3)}(\boldsymbol{z}) + \tfrac{1}{2}\gamma^{\mu\nu ab}(\sigma)\, \partial_a\partial_b \delta^{(3)}(\boldsymbol{z}) \tag{68}$$

so that from (23) becomes

$$\int_{\mathcal{M}} T^{\mu\nu}\, \phi_{\mu\nu}\, d^4x = \int_{\mathcal{I}} \Big(\gamma^{\mu\nu00}\, \phi_{\mu\nu} - \gamma^{\mu\nu0a}\, (\partial_a \phi_{\mu\nu}) + \tfrac{1}{2}\gamma^{\mu\nu ab}(\partial_a\partial_b\, \phi_{\mu\nu})\Big)\, d\sigma \tag{69}$$

Here again we impose

$$\gamma^{\mu\nu\rho\kappa} = \gamma^{\nu\mu\rho\kappa} \qquad \text{and} \qquad \gamma^{\mu\nu\rho\kappa} = \gamma^{\mu\nu\kappa\rho} \tag{70}$$

In adapted coordinates, the components $\gamma^{\mu\nu\rho\kappa}$ are uniquely determined from $T^{\mu\nu}$, so there is no gauge freedom, as in (64). In this coordinate system we can still express $T^{\mu\nu}$ in terms of (60), and the relationship between $\gamma^{\mu\nu\rho\kappa}$ and $\zeta^{\mu\nu\rho\kappa}$ is given by

$$\gamma^{\mu\nu00} = \tfrac{1}{2}\ddot{\zeta}^{\mu\nu00}, \quad \gamma^{\mu\nu a0} = \dot{\zeta}^{\mu\nu a0} \quad \text{and} \quad \gamma^{\mu\nu ab} = \zeta^{\mu\nu ab} \tag{71}$$

which is consistent with (64). This follows from (25).

It is now much easier to express the differential and algebraic equations on the components arising from the divergenceless conditions (7).

$$\dot{\gamma}^{\mu000} = -\Gamma^{\mu}{}_{\nu\rho}\, \gamma^{\rho\nu00} + (\partial_a \Gamma^{0}{}_{\nu\rho})\, \gamma^{\rho\nu0a} - \tfrac{1}{2}\big(\partial_b\partial_a \Gamma^{0}_{\nu\rho}\big)\gamma^{\rho\nu ab} \tag{72}$$
$$\dot{\gamma}^{\mu00a} = -\gamma^{\mu a00} - \Gamma^{\mu}_{\nu\rho}\, \gamma^{\rho\nu0a} + (\partial_b \Gamma^{\mu}_{\nu\rho})\, \gamma^{\rho\nu ba} \tag{73}$$
$$\dot{\gamma}^{\mu0ab} = -2\gamma^{\mu(ba)0} - \Gamma^{\mu}_{\nu\rho}\, \gamma^{cbab} \tag{74}$$

together with the algebraic equation

$$\gamma^{\mu(abc)} = 0 \tag{75}$$

See proof number 11 in the appendix.

We can now count the number of components of the quadrupole. From (72)-(74) we have 40 first order ODEs. However not all the components are determined by these ODEs. From (70) we start with 100 components. The algebraic equation (75) gives 40 independent equations so that there are 60 independent components. Thus 40 are determined by ODEs and the remaining 20 are free components. As stated in the introduction these free components need to be replaced by constitutive equations. However the choice of constitutive equations depends on a choice of a model for the material. An example of such constitutive equations is given in section 5 below.

Under change of adapted coordinate (σ, z^1, z^2, z^3) to $(\hat\sigma, \hat z^1, \hat z^2, \hat z^3)$ we have

$$\hat\gamma^{\hat\mu\hat\nu\hat a\hat b} = \mathcal{J}^{\hat\mu\hat\nu}_{\mu\nu}\, J^{\hat a}_a\, J^{\hat b}_b\, \gamma^{\mu\nu ab} \tag{76}$$

$$\hat\gamma^{\hat\mu\hat\nu\hat a\hat 0} = \mathcal{J}^{\hat\mu\hat\nu}_{\mu\nu}\, J^{\hat a}_a\, \gamma^{\mu\nu a0} + \big(\mathcal{J}^{\hat\mu\hat\nu}_{\mu\nu}\, J^{\hat 0}_b\, J^{\hat a}_a\, \gamma^{\mu\nu ab}\big)^{\cdot} - \tfrac12\big(J^{\hat a}_{ab}\,\mathcal{J}^{\hat\mu\hat\nu}_{\mu\nu} + J^{\hat a}_a\,\partial_b\mathcal{J}^{\hat\mu\hat\nu}_{\mu\nu} + J^{\hat a}_b\,\partial_a\mathcal{J}^{\hat\mu\hat\nu}_{\mu\nu}\big)\gamma^{\mu\nu ab} \tag{77}$$

$$\begin{aligned}\hat\gamma^{\hat\mu\hat\nu\hat 0\hat 0} &= \mathcal{J}^{\hat\mu\hat\nu}_{\mu\nu}\,\gamma^{\mu\nu00} + \mathcal{J}^{\hat\mu\hat\nu}_{\mu\nu}\, J^{\hat 0}_c\,\dot\gamma^{\mu\nu a0} + \big((\mathcal{J}^{\hat\mu\hat\nu}_{\mu\nu}\; J^{\hat 0}_c)^{\cdot} - \partial_c\mathcal{J}^{\hat\mu\hat\nu}_{\mu\nu}\big)\gamma^{\mu\nu c0}\\ &\quad + \tfrac12\big((\mathcal{J}^{\hat\mu\hat\nu}_{\mu\nu}\; J^{\hat 0}_d\, J^{\hat 0}_c)\,\gamma^{\mu\nu cd}\big)^{\cdot\cdot} - \big((\tfrac12 J^{\hat 0}_{cd}\,\mathcal{J}^{\hat\mu\hat\nu}_{\mu\nu} + J^{\hat 0}_d\,\partial_c\mathcal{J}^{\hat\mu\hat\nu}_{\mu\nu})\gamma^{\mu\nu cd}\big)^{\cdot} + \big(\tfrac12\partial_c\partial_d\mathcal{J}^{\hat\mu\hat\nu}_{\mu\nu}\big)\,\gamma^{\mu\nu cd}\end{aligned} \tag{78}$$

where

$$J^{\hat\mu}_{\nu\rho} = \partial_\nu J^{\hat\mu}_\rho = \frac{\partial^2 \hat x^{\hat\mu}}{\partial x^\nu \partial x^\rho} \tag{79}$$

See proof number 12 in the appendix. Although this may be considered more complicated than (65) it does not involve any integrals. We have assumed that σ and $\hat\sigma$ parameterise the same points on the worldline C. Thus on the worldline $J^{\hat\mu}_0 = \delta^{\hat\mu}_0$. However this does not imply $J^{\hat\mu}_{\nu 0} = 0$.

4.1 The static semi-quadrupole and the free components

To get an intuition about the free components, consider to dynamic equations (72)-(75) on a flat Minkowski background with Cartesian coordinates $(t = z^0, z^1, z^2, z^3) = (t, \boldsymbol{z})$ and with the worldline at $\boldsymbol{z} = \boldsymbol{0}$. Thus we can set $t = \sigma$ so that $C^0(t) = t$ and $C^a(t) = 0$. The dynamic equations (72)-(75) become

$$\dot\gamma^{\mu000} = 0 \tag{80}$$

$$\dot\gamma^{\mu0a0} = -\gamma^{\mu a00} \tag{81}$$

$$\dot\gamma^{\mu0ba} = -2\gamma^{\mu(ab)0} \tag{82}$$

$$\gamma^{\mu(ab\rho)} = 0 \tag{83}$$

As a further simplification, consider only the semi-quadrupole. This is when

$$\gamma^{\mu ab\rho} = 0 \tag{84}$$

According to table 1 there should be 22 ODE components and 6 free components. This arises since (84) implies $\gamma^{a0b\rho} = 0$ which kills all but 6 of the ODEs in (82). See proof number 13 in the appendix. Also see section 6.6 for full details.

The general solution is given by

$$\begin{gathered}\gamma^{0000} = m, \quad \gamma^{a000} = P^a, \quad \gamma^{00a0} = X^a - t\,P^a,\\ \gamma^{00ba} = \kappa^{ba}(t), \quad \gamma^{b0a0} = S^{ba} - \tfrac12\dot\kappa^{ba}(t), \quad \gamma^{ba00} = \tfrac12\ddot\kappa^{ba}(t), \quad \gamma^{\rho ba0} = 0\end{gathered} \tag{85}$$

where the 10 constants m, P^a, X^a, S^{ab} with $S^{ab} + S^{ba} = 0$ and the six free components $\kappa^{ba}(t)$ with $\kappa^{ba}(t) = \kappa^{ab}(t)$. Here we interpret m as the total mass, P^a as the momentum and S^{ba} as the spin. The six free components $\kappa^{ab}(t)$ are the moments of inertia. Since there are 22 ODEs there should be 22 constants of integration. As well as the 10 already given, the remaining 12 are the six initial conditions for $\kappa^{ab}(0)$ and for $\dot\kappa^{ab}(0)$.

Consider the components of $T^{\mu\nu}$ as arising from squeezing a regular stress-energy tensor $\mathcal{T}^{\mu\nu}(t, \boldsymbol{z})$ as in section 2.3. Thus

$$\gamma^{\mu\nu00} = \int_{\mathbb{R}^3} \mathcal{T}^{\mu\nu}(t,\boldsymbol{z})d^3\boldsymbol{z}, \qquad \gamma^{\mu\nu a0} = \int_{\mathbb{R}^3} \mathcal{T}^{\mu\nu}(t,\boldsymbol{z})\, z^a\; d^3\boldsymbol{z}, \qquad \gamma^{\mu\nu ab} = \int_{\mathbb{R}^3} \mathcal{T}^{\mu\nu}(t,\boldsymbol{z})\, z^a\, z^b\; d^3\boldsymbol{z}, \tag{86}$$

Comparing (85) and (86) we see

$$m = \int_{\mathbb{R}^3} \mathcal{T}^{00}(t,\boldsymbol{z}) d^3\boldsymbol{z}, \qquad P^a = \int_{\mathbb{R}^3} \mathcal{T}^{a0}(t,\boldsymbol{z}) d^3\boldsymbol{z}, \qquad X^a = t\,P^a + \int_{\mathbb{R}^3} \mathcal{T}^{00}(t,\boldsymbol{z}) z^a\, d^3\boldsymbol{z},$$
$$S^{ba} = \int_{\mathbb{R}^3} z^{[a}\,\mathcal{T}^{b]0}(t,\boldsymbol{z}) d^3\boldsymbol{z}, \qquad \kappa^{ab} = \int_{\mathbb{R}^3} z^a\, z^b\, \mathcal{T}^{00}(t,\boldsymbol{z}) d^3\boldsymbol{z}, \tag{87}$$

For example let $P^a = 0$ and $S^{ab} = 0$ then

$$m = \int_{\mathbb{R}^3} \mathcal{T}^{00}(t,\boldsymbol{z})\, d^3\boldsymbol{z}, \qquad \kappa^{ab}(t) = \int_{\mathbb{R}^3} z^a\, z^b\, \mathcal{T}^{00}(t,\boldsymbol{z})\, d^3\boldsymbol{z} \tag{88}$$

Since $\kappa^{ab}(t)$ are free components we can choose any $\mathcal{T}^{\mu\nu}(t,\boldsymbol{z})$ we like so long as its total integral is m and they are sufficiently symmetric that $P^a = 0$ and $S^{ab} = 0$ hold. For example if $\mathcal{T}^{\mu\nu}(t,\boldsymbol{z})$ is symmetric about the three directions z^a. This explains why we can choose to have a distribution of matter which separates and then coalesces as in figure 1.

4.2 Conserved quantities

Recall that a killing vector (54), leads to a conserved quantity in the dipole case. The same is true for quadrupole. In an adapted coordinate system (σ, z^1, z^2, z^3) then the conserved quantity $\mathcal{Q}_K$ is given by

$$\mathcal{Q}_K = \gamma^{\mu 000} K_\mu - \gamma^{\mu 0a0} \partial_a K_\mu + \tfrac{1}{2} \gamma^{\mu 0ab} \partial_a \partial_b K_\mu \tag{89}$$

See proof number 14 in the appendix.

It is worth exploring the conserved quantities on the static semi-quadrupole given by (85). In Minkowski spacetime there are 10 Killing vectors.

- Mass or Energy: for $K_0 = 1$, $K_a = 0$ we have $\mathcal{Q}_K = m$.
- Momentum: for $K_0 = 0$, and for some a, $K_a = 1$ and $K_b = 0$ for $b \neq a$ then $\mathcal{Q}_K = p_a$.
- Angular momentum and spin: let $K_0 = 0$, $K_1 = z^2$, $K_2 = -z^1$ and $K_3 = 0$. We have

$$\begin{aligned}\mathcal{Q}_K &= \gamma^{1000} K_1 + \gamma^{2000} K_2 + \gamma^{2010} \partial_1 K_2 + \gamma^{1020} \partial_2 K_1 \\ &= p^1\, z^2 - p^2\, z^1 + \big(S^{12} - \dot{\kappa}^{12}(t)\big) - \big(S^{21} - \dot{\kappa}^{21}(t)\big) = S^{12}\end{aligned}$$

- Boost: Let $K_0 = z^1$, $K_1 = t + t_0$, $K_2 = 0$ and $K_3 = 0$ for some fixed t_0. Then

$$\mathcal{Q}_K = \gamma^{0000} K_0 + \gamma^{1000} K_1 + \gamma^{0010} \partial_1 K_0 = m\, z^1 + P^1\,(t + t_0) + (X^1 - t\,P^1) = X^1 + t_0\, P^1$$

Thus the 10 Killing symmetries of Minkowski spacetime correspond directly to the 10 constant of the solution to static semi-quadrupole. This also gives a new interpretation to the three somewhat obscure conserved quantities corresponding to the three boosts. Namely for the boost about the point $\boldsymbol{z} = \boldsymbol{0}$ and $t = t_0$ then $\mathcal{Q}_K$ is the displacement vector at the time t_0.

5 Non-divergent dust model of a quadrupole and the corresponding constitutive relations.

The familiar dust model is given in terms of a scalar density ϱ and a vector field U^μ with $g_{\mu\nu}\, U^\mu\, U^\nu = -1$. The stress-energy tensor density is given by

$$\mathcal{T}^{\mu\nu} = \varrho\, U^\mu\, U^\nu\, \mu \tag{90}$$

where $\mu = \sqrt{-\det(g_{\mu\nu})}$. Then the divergenceless condition implies that the U^μ are geodesics

$$U^\mu \nabla_\mu U^\nu = 0 \tag{91}$$

and the flow ϱ is conserved

$$U^\mu(\partial_\mu \varrho) = 0 \tag{92}$$

Furthermore let us assume that the dust is non divergent, so that it preserves the measure, i.e.

$$U^\mu \, \partial_\mu \mu = 0 \tag{93}$$

In order to create a squeezed tensor $\mathcal{T}_\varepsilon^{\mu\nu}$ from $\mathcal{T}^{\mu\nu}$ we need a choose a coordinate system. It is natural to choose the coordinate adapted to U^μ so that $U^\mu = \delta_0^\mu$. This gives $\dot{\varrho} = 0$ so that we can write $\varrho = \varrho(\boldsymbol{z})$. Likewise we have $a = a(\boldsymbol{z})$. Hence

$$\mathcal{T}^{\mu\nu}(\sigma, \boldsymbol{z}) = \varrho(\boldsymbol{z}) \, \delta_0^\mu \, \delta_0^\nu \, a(\boldsymbol{z}) \tag{94}$$

We require that $\varrho(\boldsymbol{z}) = 0$ for large $\boldsymbol{z}$. From (29) we see

$$\begin{aligned}
\gamma^{\mu\nu 00}(\sigma) &= \delta_0^\mu \, \delta_0^\nu \int_{\mathbb{R}^3} d^3\boldsymbol{z} \;\; \varrho(\boldsymbol{z}) \, a(\boldsymbol{z}), \\
\gamma^{\mu\nu a0}(\sigma) &= -\delta_0^\mu \, \delta_0^\nu \int_{\mathbb{R}^3} d^3\boldsymbol{z} \;\; z^a \, \varrho(\boldsymbol{z}) \, a(\boldsymbol{z}), \\
\gamma^{\mu\nu ab}(\sigma) &= \delta_0^\mu \, \delta_0^\nu \int_{\mathbb{R}^3} d^3\boldsymbol{z} \;\; z^a \, z^b \, \varrho(\boldsymbol{z}) \, a(\boldsymbol{z})
\end{aligned} \tag{95}$$

Since both ϱ and a are independent of σ we have the dynamic equations

$$\dot{\gamma}^{\mu\nu 00} = 0, \quad \dot{\gamma}^{\mu\nu a0} = 0 \quad \text{and} \quad \dot{\gamma}^{\mu\nu ab} = 0 \tag{96}$$

These are consistent with the dynamic equations (72)-(74) since in the adapted coordinate system the geodesics equation becomes $\Gamma_{00}^\mu = 0$.

Equation (96) completely defines the dynamics. However, our goal is use use (96) to inspires the constitutive relations in the case when we are not modelling a non-divergent dust, so that the (72)-(74) hold. One option is to require that some of the free components are in fact constants. This is challenging because we need to be consistent with (72)-(74).

As a simple example, consider the static semi-quadrupole given by (85). The non-divergent dust constitutive relations would make $\kappa^{ab}(t)$ a constant. It would also make $P^a = 0$. This replaces (85) with

$$\begin{gathered}
\gamma^{0000} = m, \quad \gamma^{a000} = 0, \quad \gamma^{00a0} = X^a, \\
\gamma^{00ba} = \kappa^{ba}, \quad \gamma^{b0a0} = S^{ba}, \quad \gamma^{ba00} = 0, \quad \gamma^{cba0} = 0
\end{gathered} \tag{97}$$

6 The coordinate free and metric free approach to quadrupoles.

In [14] the authors present a coordinate free definition of submanifold distributions, also known as deRham currents, in terms of the deRham push forward and then actions of the standard operations.

Since we are using coordinate free notation we write a vector field as $V \in \Gamma T\mathcal{M}$. Here $T\mathcal{M}$ is the tangent bundle of spacetime and $\Gamma T\mathcal{M}$ refers to sections of the tangent bundle. A vector a point $p \in \mathcal{M}$ we write $V \in T_p\mathcal{M}$. A vector field and vectors at a point are differential operators and we write the action of a vector on a scalar field as $V\langle f \rangle$. The bundle of p–forms is written $\Lambda^p\mathcal{M}$ so a p–form field is written $\alpha \in \Gamma\Lambda^p\mathcal{M}$.

Given a coordinate system $(x^0, \ldots, x^3)$ then we write $V = V^\mu \partial_\mu$. Here ∂_μ are basis vectors and V^μ are indexed scalar fields. For 1–forms $\alpha \in \Gamma\Lambda^1\mathcal{M}$ we can write $\alpha = \alpha_\mu \, dx^\mu$ where again α_μ are indexed scalar fields.

6.1 The two types of ∇

In the literature on general relativity and differential geometry, there are two convention used when referring to the covariant derivative. One is typically used when using index tensor notation, the other when one is using coordinate free notation. Usually one has simply to choose one convention present all the results in that. We have done this up to know using index notation. However in this section we wish to present a coordinate free definition of all the objects. As a result it is necessary to use both definitions of the covariant derivatives, sometimes in the same expression. So to avoid confusion, from now one we introduce two different symbols.

The covariant derivative which we have used up to this point and which "knows" about the index of an object we write $\overline{\nabla}_\mu$. Acting on the indexed scalar fields V^μ then

$$\overline{\nabla}_\mu V^\nu = \partial_\mu(V^\nu) + V^\rho\,\Gamma^\nu_{ac} \tag{98}$$

I.e. the Christoffel symbols are tied to the indexes. By contrast the coordinate free covariant derivative is written $\boldsymbol{\nabla}_V$ where $V \in \Gamma T\mathcal{M}$. In this case the Christoffel symbol satisfies

$$\Gamma^\mu_{\nu\rho}\,\partial_\mu = \boldsymbol{\nabla}_{\partial_\nu}\partial_\rho \tag{99}$$

This covariant derivative knows about the tensor structure, but not the indexes. Thus

$$\boldsymbol{\nabla}_U V^\mu = U\langle V^\mu\rangle \tag{100}$$

The two covariant derivatives are related via the following

$$\boldsymbol{\nabla}_U(V) = U^\nu(\overline{\nabla}_\nu V^\mu)\partial_\mu \tag{101}$$

since

$$\begin{aligned}\boldsymbol{\nabla}_U(V) &= \boldsymbol{\nabla}_U(V^\mu\,\partial_\mu) = U\langle V^\mu\rangle\partial_\mu + U^\nu\,V^\mu\boldsymbol{\nabla}_{\partial_\nu}\partial^\mu = U\langle V^\mu\rangle\partial_\mu + U^\nu\,V^\mu\Gamma^\rho_{\mu\nu}\partial_\rho\\ &= U^\nu\big(\partial_\nu\langle V^\mu\rangle + V^\rho\Gamma^\mu_{\nu\rho}\big)\partial_\mu = U^\nu(\overline{\nabla}_\nu V^\mu)\partial_\mu\end{aligned}$$

In the coordinate definition of the Dixon quadrupole, setting $k = 2$ in (34), we see there is an operator $\overline{\nabla}_\mu\overline{\nabla}_\nu$. This is tensorial with respect to the indices μ and ν. To give coordinate free definition we define for any tensor S,

$$\boldsymbol{\nabla}^2_{U,V}S = \boldsymbol{\nabla}_U\,\boldsymbol{\nabla}_V S - \boldsymbol{\nabla}_{\boldsymbol{\nabla}_U V}S \tag{102}$$

This definition can be extended to arbitrary order. This is clearly tensorial in U, but is also tensorial (also known as f-linear) with respect to V. Thus

$$\boldsymbol{\nabla}^2_{(fU),V}S = \boldsymbol{\nabla}^2_{U,(fV)}S = f\,\boldsymbol{\nabla}^2_{U,V}S \tag{103}$$

See proof number 15 in the appendix.

The relationship between $\boldsymbol{\nabla}^2_{U,V}$ and $\overline{\nabla}_\mu\overline{\nabla}_\nu$ is given by

$$\boldsymbol{\nabla}^2_{U,V}W = U^\nu\,V^\rho\left(\overline{\nabla}_\nu\,\overline{\nabla}_\rho W^\mu\right)\partial_\mu \tag{104}$$

for any vector W^μ. See proof number 16 in the appendix.

6.2 Defining distributional forms

Following Schwartz, we define a distributional p–form by how acts on a test $(4-p)$–form $\varphi \in \Gamma\Lambda^{4-p}M$, i.e. with $(4-p)$–form with compact support [14]. Given $\alpha \in \Gamma\Lambda^p M$ is a smooth p−form, we construct a regular distribution α^D via

$$\alpha^D[\varphi] = \int_M \varphi\wedge\alpha \tag{105}$$

The definition of the wedge product, Lie derivatives, internal contraction and exterior derivatives on distributions are defined to be consistent with (105). Thus for a distribution Ψ we set

$$\begin{gathered}(\Psi_1+\Psi_2)[\varphi]=\Psi_1[\varphi]+\Psi_2[\varphi]\,,\quad (\beta\wedge\Psi)[\varphi]=\Psi[\varphi\wedge\beta]\,,\quad (d\Psi)[\varphi]=(-1)^{(3-p)}\Psi[d\varphi]\,,\\ (i_v\Psi)[\varphi]=(-1)^{(3-p)}\Psi[i_v\varphi]\quad\text{and}\quad (L_v\Psi)[\varphi]=-\Psi[L_v\varphi]\end{gathered}\tag{106}$$

for $v\in\Gamma T\mathcal{M}$. Given $C:\mathcal{I}\to\mathcal{M}$, is a closed embedding. The push forward with respect to C of a p–form, $\alpha\in\Gamma\Lambda^p\mathcal{I}$ is given by the distribution

$$\big(C_\varsigma(\alpha)\big)[\varphi]=\int_{\mathcal{I}}C^\star(\varphi)\wedge\alpha\tag{107}$$

where φ is a test form of degree 0 or 1. This has degree $\deg\big(C_\varsigma(\alpha)\big)=3+\alpha$. A general form distribution is then given by acting (106) on $C_\varsigma(\alpha)$.

The **order of a multipole** is defined as follows. If

$$\Psi[\lambda^{k+1}\varphi]=0\quad\text{for all}\quad\lambda\in\Gamma\Lambda^0M\text{ and }\varphi\in\Gamma_0\Lambda^1M\quad\text{such that}\quad C^\star(\lambda)=0\tag{108}$$

then we say that the order of Ψ is at most k. Since we impose that λ vanishes on the image of C, this implies that we need to differentiate the argument $\lambda^{k+1}\varphi$ at least $k+1$ times for $\Psi[\lambda^{k+1}\varphi]\neq0$. We say dipoles have order at most one and quadrupoles have order at most two. Therefore the terms in a dipole have at most one derivative, and those in a quadrupole at most two. This is consistent with the fact that the set of quadrupoles include all dipoles.

The deRham push forward is compatible with the exterior derivative

$$d\,C_\varsigma(\alpha)=C_\varsigma(d\alpha)\tag{109}$$

and the internal contraction for tangential fields

$$i_w\,C_\varsigma(\alpha)=C_\varsigma(i_v\,\alpha)\qquad\text{where}\qquad w\in\Gamma T\mathcal{M},\ v\in\Gamma T\mathcal{I},\ C_\star(v|_\sigma)=w|_{C(\sigma)}\quad\text{for all}\quad\sigma\in\mathcal{I}\tag{110}$$

These enable one to manipulate distributions, for example by finding the change of coordinates, without having to act on the test tensors.

6.3 The stress-energy 3–forms

In this section, we exploit the fact the although the stress-energy forms are 3–forms and have a similar structure to the electromagnetic current 3–form.

The stress-energy form τ is a map which takes a 1–form $\alpha\in\Gamma\Lambda^1\mathcal{M}$ and gives a deRham current 3–form τ_α over the worldline C.

$$\alpha\mapsto\tau_\alpha\tag{111}$$

The map (111) is not tensorial but does satisfy

$$\tau_{(\alpha+\beta)}=\tau_\alpha+\tau_\beta\qquad\text{and}\qquad\tau_{(f\alpha)}[\theta]=\tau_\alpha[f\theta]\tag{112}$$

for any test 1–form θ.

Observe that the stress-energy 3–forms take a 1–form α to give a 3–form. This is contrary to the usual definition where we take a vector v to give the 3–form τ_v. The advantage of (111) is that we do not need a metric to defined the stress-energy 3–forms or the symmetry and divergenceless conditions (114) and (115) below. This is useful if we wish to consider connections which are not metric compatible.

Using τ_α we define a tensor valued distribution τ which takes a tensor of type (0,2) as an argument. This is defined as

$$\tau[\theta\otimes\alpha]=\tau_\alpha[\theta]\tag{113}$$

The stress-energy tensor is symmetric (6) and divergenceless (7). The symmetry condition is given by

$$\tau[\beta \otimes \alpha] = \tau[\alpha \otimes \beta] \tag{114}$$

and the divergenceless condition is given by

$$D\tau = 0 \tag{115}$$

where

$$(D\tau)[\theta] = -\tau[D\theta] \tag{116}$$

and

$$(D\theta)(U, V) = (\boldsymbol{\nabla}_V \theta)(U) \tag{117}$$

Using a coordinate system, we can convert the map (111) into indexed 3–forms via

$$\tau^\mu = \tau_{dx^\mu} \tag{118}$$

The relationship between the stress-energy forms and the tensor density $T^{\mu\nu}$ is given by

$$\int_{\mathcal{I}} T^{\mu\nu}\,\phi_{\mu\nu}\,d^4x = \tau^\mu[\phi_{\mu\nu}\,dx^\nu] \tag{119}$$

Using this coordinate system, (114) becomes

$$dx^\mu \wedge \tau^\nu = dx^\nu \wedge \tau^\mu \tag{120}$$

and (115) becomes

$$d\tau^\mu + \Gamma^\mu_{\nu\rho}\,dx^\rho \wedge \tau^\nu = 0 \tag{121}$$

See proof number 17 in the appendix.

6.4 Killing forms and conservation

Killing forms (54) can be written in a coordinate free way. The 1–form $\alpha \in \Gamma\Lambda^1\mathcal{M}$ is Killing if

$$(\boldsymbol{\nabla}_V \alpha)(V) = 0 \tag{122}$$

for all vectors $V \in \Gamma T\mathcal{M}$. From (121) and (120) we have

$$\begin{aligned} d\tau_\alpha &= d(\alpha_\mu\,\tau^\mu) = d\alpha_\mu \wedge \tau^\mu + \alpha_\mu \wedge d\tau^\mu = (\partial_\rho \alpha_\mu)\,dx^\rho \wedge \tau^\mu - \Gamma^\mu_{\nu\rho}\alpha_\mu\,dx^\rho \wedge d\tau^\nu = \overline{\nabla}_\rho \alpha_\nu\,dx^\rho \wedge d\tau^\nu \\ &= \tfrac{1}{2}(\overline{\nabla}_\rho \alpha_\nu - \overline{\nabla}_\nu \alpha_\rho)\,dx^\rho \wedge d\tau^\nu \end{aligned}$$

Hence if $\alpha \in \Gamma\Lambda^1\mathcal{M}$ is a Killing 1–form then from (54) $d\tau_\alpha = 0$. This gives an alternative method of proving (89).

6.5 Defining and extraction of components

Using (60) and (119) we deduce in an arbitrary coordinate system

$$\tau^\mu = \tfrac{1}{2}\,i_\nu\,L_\rho\,L_\kappa C_\varsigma(\zeta^{\mu\nu\rho\kappa} d\sigma) \tag{123}$$

where $i_\nu = i_{\partial_\nu}$ and $L_\rho = L_{\partial_\rho}$. See proof number 18 in the appendix. In an adapted coordinate system (11) then (68) implies

$$\tau^\mu = i_\nu\, C_\varsigma(\gamma^{\mu\nu 00}\, d\sigma) + i_\nu\, L_a\, C_\varsigma(\gamma^{\mu\nu 0a}\, d\sigma) + \tfrac{1}{2} i_\nu\, L_a\, L_b\, C_\varsigma(\gamma^{\mu\nu ab}\, d\sigma) \tag{124}$$

See proof number 19 in the appendix. As stated the advantage of using an adapted coordinate system is that the $\gamma^{\mu\nu\rho\kappa}$ are unique. We can extract the values of the $\gamma^{\mu\nu\rho\kappa}$ by acting on test forms.

$$\gamma^{\mu\nu 00}(\sigma) = \lim_{\epsilon\to 0} \tau^\mu[dx^\nu\, \psi_{\epsilon,\sigma}]\,, \quad \gamma^{\mu\nu 0a}(\sigma) = \lim_{\epsilon\to 0} \tau^\mu[z^a\, dx^\nu\, \psi_{\epsilon,\sigma}]\,, \quad \gamma^{\mu\nu ab}(\sigma) = \lim_{\epsilon\to 0} \tau^\mu[z^a\, z^b\, dx^\nu\, \psi_{\epsilon,\sigma}] \tag{125}$$

where

$$\psi_{\epsilon,\sigma}(\sigma', z) = \epsilon^{-1}\, \psi_1((\sigma-\sigma')/\epsilon)\, \psi_1\big((z^1)^2 + (z^2)^2 + (z^3)^2\big)$$

and $\psi_1 : \mathbb{R} \to \mathbb{R}$ is a bump function. I.e. a test function with which is flat about 0.

6.6 Semi-dipoles and semi-quadrupoles

Having defined the quadrupoles in a coordinate free manner, one can identify properties which can be defined without reference to a coordinate system. In [14] we defined the semi-dipole and semi-quadrupole electromagnetic 3–form. The semi-dipole corresponded to the purely electric quadrupole. One can likewise define the semi-dipole and semi-quadrupole stress-energy distributions. In this case we say that τ_α is an semi-multipole of order at most ℓ if

$$\tau_\alpha[\lambda^\ell d\mu] = 0 \quad \text{for all} \quad \lambda, \mu \in \Gamma\Lambda^0 M \quad \text{such that} \quad C^\star(\lambda) = C^\star(\mu) = 0 \tag{126}$$

We observe that the semi-dipole ($\ell = 1$) corresponds to the case when the spin tensor is $S^{ba} = 0$. The semi-quadrupole ($\ell = 2$), does not have a natural interpretation, but is used as a quadrupole with fewer components.

When we apply this to the quadrupole (124) we see that the semi-quadrupole is given by

$$\tau^\mu = i_\nu\, C_\varsigma(\gamma^{\mu\nu 00}\, d\sigma) + i_\nu\, L_a\, C_\varsigma(\gamma^{\mu\nu 0a}\, d\sigma) + \tfrac{1}{2} L_a\, L_b\, C_\varsigma(\gamma^{\mu 0ab}) \tag{127}$$

This gives 22 ODE components and 6 free components as indicated in table 1. We presented the general solution for the static semi-quadrupole in section 4.1.

6.7 The coordinate free definition of the Dixon split only using N and the connection

We have defined the stress-energy distribution without reference to a coordinate system. When writing this in terms of coordinates (123) and (29) we see that this corresponds directly to the Ellis representation of the multipoles. Here we show how to perform the Dixon split (35) which separate the multipoles into different orders with respect to a 1–form N along the curve. We show this by separating the quadrupole into a pure Dixon quadrupole term, a pure Dixon dipole term and a monopole term. The pattern however is clear. The Dixon split (35) requires defining $\tau_{(0)}$, $\tau_{(1)}$ and $\tau_{(2)}$ such that an arbitrary quadrupole

$$\tau = \tau_{(0)} + \tau_{(1)} + \tau_{(2)} \tag{128}$$

Using (119) to convert these into $T^{\mu\nu}_{(r)}$ so that $T^{\mu\nu} = T^{\mu\nu}_{(0)} + T^{\mu\nu}_{(1)} + T^{\mu\nu}_{(2)}$ where

$$\tau_{(0)}[\phi] = \int_{\mathcal{M}} T^{\mu\nu}_{(0)}\, \phi_{\mu\nu}\, d^4x = \int_{\mathcal{I}} \xi^{\mu\nu}(\sigma)\, \phi_{\mu\nu}(\sigma)\, d\sigma\,, \tag{129}$$

$$\tau_{(1)}[\phi] = \int_{\mathcal{M}} T^{\mu\nu}_{(1)}\, \phi_{\mu\nu}\, d^4x = \int_{\mathcal{I}} \xi^{\mu\nu\rho}(\sigma)\, (\overline{\nabla}_\rho \phi_{\mu\nu})|_{C(\sigma)}\, d\sigma\,, \tag{130}$$

$$\tau_{(2)}[\phi] = \int_{\mathcal{M}} T^{\mu\nu}_{(2)}\, \phi_{\mu\nu}\, d^4x = \int_{\mathcal{I}} \xi^{\mu\nu\rho\kappa}(\sigma)\, (\overline{\nabla}_\rho \overline{\nabla}_\kappa \phi_{\mu\nu})|_{C(\sigma)}\, d\sigma \tag{131}$$

The Dixon split is with respect to a 1–form, as opposed to a vector along C. This is in order to avoid requiring the metric. The one requirement is that the 1–form N combined with the vector $\dot{C}$ is nowhere zero. I.e.

$$N(\dot{C}) \neq 0 \tag{132}$$

In order to perform the Dixon split, it is necessary to define a radial vector fields. We say that $R \in \Gamma TM$ is **Radial** (2 second order) with respect to C and $N \in \Gamma_\rho \Lambda^1 M$ if for all $p = C(\sigma)$

$$R|_p = 0, \qquad (\boldsymbol{\nabla}_V R)|_p = V|_p \qquad \text{and} \qquad \left(\boldsymbol{\nabla}^2_{U,V} R\right)\big|_p = 0 \tag{133}$$

for all vectors $U, V \in T\mathcal{M}$ such that $N(V) = N(U) = 0$. In appendix A.5 we express the components of R with respect to a coordinate system, which is adapted both for C and N.

Using this radial vector, the Dixon split (128) is given by

$$\tau_{(0)}[\phi] = \tau[\phi - \nabla_R \phi + \tfrac{1}{2}\nabla^2_{R,R}\phi] \tag{134}$$

$$\tau_{(1)}[\phi] = \tau[\nabla_R \phi - \nabla^2_{R,R}\phi] \tag{135}$$

$$\tau_{(2)}[\phi] = \tau[\tfrac{1}{2}\nabla^2_{R,R}\phi] \tag{136}$$

where ϕ is an type (0,2) test tensor. See proofs numbers 22-24 in the appendix. The advantage of this definition is that one can now show how the Dixon components mix when one changes N.

7 Discussion and outlook.

We have derived a number of key results about the distributional quadrupole stress-energy tensor, in particular the existence of the free components, which require additional constitutive relations to prescribe. An example of these constitutive relations is given. We have also given the coordinate transformation of the quadrupole components, the conserved quantities in the presence of a Killing vector, a definition of semi-quadrupoles and a coordinate free definition of the Dixon split.

The understanding of the quadrupole stress-energy tensor distribution is important for the study of gravitational wave sources, as well as being interesting in its own right. Many features arise at the quadrupole level, which were not present at the dipole level. In particular the non tensorial nature of the components and the existence of free components. These free components imply that it is not possible to know everything about a quadrupole simply from the initial conditions. There is clearly much research that needs to be done to find appropriate constitutive relations to replace the free components with ODEs or algebraic relations. One would expect different constitutive relations for a gravitationally bound object such as two orbiting point masses, a non gravitationally bound object such a rotating asteroid and an object where both gravitational and non gravitational forces are important such as a star. In section 5 we present only a very simple constitutive relation corresponding to a dust model. As presented this is only valid for a semi-quadrupole in Minkowski spacetime. With increasing sensitivity of gravitational wave astronomy one can hope to test the different constitutive relations using experimental data.

Although the observation of the need for constitutive relations for the quadrupole on a prescribe worldline is new, there are other cases where the need for constitutive relations has been observed. For example [7], they are needed to determine how dipoles or quadrupoles effect the worldline. There are other situations where one can expect constitutive relations will be needed. In future work we intend to look at the dynamics of charged multipoles in an electromagnetic field. One would expect in this case that constitutive relations are also needed, especially since a dipole has nine components, but the electromagnetic current, which provides the force and torque, has only six components. These constitutive relations describe the differences between the charge distribution and the mass distribution in the dipole. The situation has an additional challenge in that the electromagnetic field blows up on the worldline. This poses another question that has been tackled by many authors: how does a dipole respond to its own electromagnetic field [15, 16, 17].

We have detailed the nature of the Ellis representation of the quadrupole. As well as the differential equations, we have given the gauge freedom, the change of coordinates, the adapted coordinates and the change of coordinates for adapted coordinates. It is natural to ask what new features will arise for sextipoles. One will expect that the gauge freedom for sextipoles will include a term with $\dot{C}^\mu\, C^\nu \int^\sigma C^\rho \, d\sigma'$.

Having definitions which are coordinate free can be very useful. They make it clear which objects are coordinate dependent and which are truly geometric. Ironically, one principle use is to make it easier to derive the correct coordinate transformation. Although the Ellis representation of multipoles is easy to define in a coordinate free manner, here we have derived a coordinate free define of the Dixon split (134)-(136).

Although spacetime is endowed with both a metric and a connection, there is much research into which objects can be defined without such structures. In some cases this is a philosophical question, posing whether the electromagnetic field is more fundamental than the gravitational field [18]. In other cases it is useful for asking how does an object depend on a metric or a connection. This is necessary when doing variations with respect to the metric. It is important therefore that a general multipole does not require any additional structure beyond that defining a general manifold for its definition. This means that one can define multipoles on other manifolds such as the tangent bundle or jet bundles. Such an approach may also give an insight into prescribing constitutive relations, say for a plasma. Of course a connection is required to demand the stress-energy distribution is covariantly conserved, but there is no requirement to demand that such a connection is Levi-Civita. All the coordinate free presentation from section 6 does not require a metric, so one can choose a metric compatible or a non metric compatible connection. We have demanded that the connection is torsion free. On the whole this is to simplify the equations so that we do not have to write down all the torsion components and their derivatives. One can reproduce the results with these extra terms.

Acknowledgement

JG is grateful for the support provided by STFC (the Cockcroft Institute ST/P002056/1) and EPSRC (the Alpha-X project EP/N028694/1). ST and PP would like to thank the Faculty of Science and Technology, Lancaster University for their support. JG would like to thank the anonymous referee of [14] for suggesting applying the technique to gravity.

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Appendix

A Details of the proofs

A.1 Proofs from introductory sections

Proof number 1: Proof of (23) *and* (25).

$$\begin{aligned}\int_{\mathcal{M}} T^{\mu\nu}\,\phi_{\mu\nu}\,d^4x &= \int_{\mathcal{I}} d\sigma \int_{\text{space}} d^3\boldsymbol{z}\Big(\sum_{r=0}^{k}\frac{1}{r!}\gamma^{\mu\nu a_1\ldots a_r 0\ldots 0}\,\partial_{a_1}\cdots\partial_{a_r}\,\delta^{(3)}(\boldsymbol{z})\Big)\phi_{\mu\nu}\\
&= \sum_{r=0}^{k}\frac{(-1)^r}{r!}\int_{\mathcal{I}} d\sigma\int_{\text{space}} d^3\boldsymbol{z}\;\gamma^{\mu\nu a_1\ldots a_r 0\ldots 0}\,(\partial_{a_1}\cdots\partial_{a_r}\,\phi_{\mu\nu})\,\delta^{(3)}(\boldsymbol{z})\\
&= \sum_{r=0}^{k}\frac{(-1)^r}{r!}\int_{\mathcal{I}} d\sigma\,\gamma^{\mu\nu a_1\ldots a_r 0\ldots 0}\,(\partial_{a_1}\cdots\partial_{a_r}\,\phi_{\mu\nu})\end{aligned}$$

□

Proof number 2: Proof of (25).

$$\int_{\mathcal{M}} T^{\mu\nu}\,\phi_{\mu\nu}\,d^4x = (-1)^k \frac{1}{k!}\int_{\mathcal{I}} \zeta^{\mu\nu\rho_1\ldots\rho_k}\left(\partial_{\rho_1}\cdots\partial_{\rho_k}\phi_{\mu\nu}\right)$$
$$= \sum_{r=0}^{k}(-1)^k \frac{1}{k!}\,\frac{k!}{r!(k-r)!}\int_{\mathcal{I}} \zeta^{\mu\nu a_1\ldots a_r 0\ldots 0}\left(\partial_{a_1}\cdots\partial_{a_r}\partial_0^{k-r}\phi_{\mu\nu}\right)$$
$$= \sum_{r=0}^{k}(-1)^r \frac{1}{r!(k-r)!}\int_{\mathcal{I}} (\partial_0^{k-r}\zeta^{\mu\nu a_1\ldots a_r,0\ldots 0})\left(\partial_{a_1}\cdots\partial_{a_r}\phi_{\mu\nu}\right)$$

Hence comparing with (23) gives (25). □

Proof number 3: Proof of (28) *and* (29). This follows from setting $w^a = z^a/\varepsilon$ and Taylor expanding around $\varepsilon = 0$ we have

$$\int_{\mathbb{R}^4} \mathcal{T}^{\mu\nu}_{\varepsilon}(\sigma,\boldsymbol{z})\,\phi_{\mu\nu}(\sigma,\boldsymbol{z})\,d\sigma\,d^3z$$
$$= \int_{\mathbb{R}} d\sigma \int_{\mathbb{R}^3} d^3z\;\mathcal{T}^{\mu\nu}_{\varepsilon}(\sigma,\boldsymbol{z})\,\phi_{\mu\nu}(\sigma,\boldsymbol{z})$$
$$= \int_{\mathbb{R}} d\sigma \int_{\mathbb{R}^3} d^3z\;\frac{1}{\varepsilon^3}\mathcal{T}^{\mu\nu}\Big(\sigma,\frac{\boldsymbol{z}}{\varepsilon}\Big)\,\phi_{\mu\nu}(\sigma,\boldsymbol{z})$$
$$= \int_{\mathbb{R}} d\sigma \int_{\mathbb{R}^3} d^3\boldsymbol{w}\;\mathcal{T}^{\mu\nu}\big(\sigma,\boldsymbol{w}\big)\,\phi_{\mu\nu}(\sigma,\varepsilon\boldsymbol{w})$$
$$= \int_{\mathbb{R}} d\sigma \int_{\mathbb{R}^3} d^3\boldsymbol{w}\;\mathcal{T}^{\mu\nu}\big(\sigma,\boldsymbol{w}\big)\,\phi_{\mu\nu}(\sigma,\boldsymbol{0}) + \varepsilon\int_{\mathbb{R}} d\sigma \int_{\mathbb{R}^3} d^3\boldsymbol{w}\;\mathcal{T}^{\mu\nu}\big(\sigma,\boldsymbol{w}\big)\,w^a\,\big(\partial_a\phi_{\mu\nu}\big)(\sigma,\boldsymbol{0})$$
$$+\,\varepsilon^2\int_{\mathbb{R}} d\sigma \int_{\mathbb{R}^3} d^3\boldsymbol{w}\;\mathcal{T}^{\mu\nu}\big(\sigma,\boldsymbol{w}\big)\,w^a\,w^b\,\big(\partial_a\,\partial_b\phi_{\mu\nu}\big)(\sigma,\boldsymbol{0}) + \cdots$$
$$= \int_{\mathbb{R}} d\sigma\gamma^{\mu\nu 0\ldots 0}\,\phi_{\mu\nu}|_{C(\sigma)} - \varepsilon\int_{\mathbb{R}} \gamma^{\mu\nu a0\ldots 0}d\sigma\,\big(\partial_a\phi_{\mu\nu}\big)\big|_{C(\sigma)} + \varepsilon^2\int_{\mathbb{R}} \gamma^{\mu\nu ab0\ldots 0}d\sigma\,\big(\partial_a\phi_{\mu\nu}\big)\big|_{C(\sigma)} + \cdots$$

□

A.2 Proofs about the dipole

Proof number 4: Proof of (45). Substituting (45) into (44) we have

$$T^{\mu\nu} \to T^{\mu\nu} + \int_{\mathcal{I}} M^{\mu\nu}\dot{C}^{\rho}\partial_{\rho}\delta(x - C(\tau))\,d\sigma = T^{\mu\nu} + \int_{\mathcal{I}} M^{\mu\nu}\tfrac{d}{d\sigma}\delta(x - C(\tau))\,d\sigma$$
$$= T^{\mu\nu} + \int_{\mathcal{I}} \tfrac{d}{d\sigma}\big(M^{\mu\nu}\delta(x - C(\tau))\big)\,d\sigma = T^{\mu\nu}$$

Thus (45) is a gauge freedom. To show it is the maximum freedom consider working in adaptive coordinates. It is clear that the freedom (45) is precisely equivalent to the freedom to choose $\zeta^{\mu\nu 0}$ given $\gamma^{\mu\nu 0}$. For details of why this is the maximum gauge freedom see proofs number 9 and 10. □

Proof number 5: Relationship between (59) *and* (50). In this proof we refer to the two equations in (50) as (50.1) and (50.2) and likewise for (59.1) to (59.4). From (58) and (41) we have

$$\frac{D\hat{S}^{\mu\nu}}{d\sigma} - \hat{P}^{\nu}\dot{C}^{\mu} + \hat{P}^{\mu}\dot{C}^{\nu}$$
$$= \frac{DS^{\mu\nu}}{d\sigma} - \frac{DX^{\mu}}{d\sigma}\dot{C}^{\nu} - X^{\mu}\frac{D\dot{C}^{\nu}}{d\sigma} + \frac{DX^{\nu}}{d\sigma}\dot{C}^{\mu} + X^{\nu}\frac{D\dot{C}^{\mu}}{d\sigma} - (P^{\nu} + m\dot{C}^{\nu})\dot{C}^{\mu} + (P^{\mu} + m\dot{C}^{\mu})\dot{C}^{\nu}$$
$$= \frac{DS^{\mu\nu}}{d\sigma} - \Big(\frac{DX^{\mu}}{d\sigma} - P^{\mu}\Big)\dot{C}^{\nu} + \Big(\frac{DX^{\nu}}{d\sigma} - P^{\nu}\Big)\dot{C}^{\mu}$$

Hence (59.2) and (59.4) imply (50.1). By contrast from (50.1) we can project out (59.2) and (59.4) using $\dot{C}_\mu$.

Likewise from (58) we have

$$\begin{aligned}\frac{D\hat{P}^\mu}{d\sigma} - \tfrac{1}{2}R^\mu{}_{\nu\rho\kappa}\,\dot{C}^\nu\,\hat{S}^{\kappa\rho} &= \frac{DP^\mu}{d\sigma} + \frac{Dm}{d\sigma}\dot{C}^\mu + m\frac{D\dot{C}^\mu}{d\sigma} - \tfrac{1}{2}R^\mu{}_{\nu\rho\kappa}\,\dot{C}^\nu\left(S^{\kappa\rho} - X^\kappa\dot{C}^\rho + X^\rho\dot{C}^\kappa\right)\\ &= \frac{DP^\mu}{d\sigma} + \dot{m}\dot{C}^\mu - \tfrac{1}{2}R^\mu{}_{\nu\rho\kappa}\,\dot{C}^\nu\,S^{\kappa\rho} - R^\mu{}_{\nu\rho\kappa}\,\dot{C}^\nu\,X^\kappa\dot{C}^\rho\end{aligned}$$

Thus (59.1) and (59.3) imply (50.2). By contrast from (50.2) we can project out (59.1) and (59.3) using $\dot{C}_\mu$.

□

Proof number 6: Proof of (56)*.* From (50) we have

$$\begin{aligned}\partial_0\hat{S}^{\mu\nu} &= \frac{D\hat{S}^{\mu\nu}}{d\sigma} - \Gamma^\mu_{0\rho}\,\hat{S}^{\rho\nu} - \Gamma^\nu_{0\rho}\,\hat{S}^{\mu\rho} = \hat{P}^\nu\dot{C}^\mu - \hat{P}^\mu\dot{C}^\nu - \Gamma^\mu_{0\rho}\,\hat{S}^{\rho\nu} - \Gamma^\nu_{0\rho}\,\hat{S}^{\mu\rho}\\ &= \hat{P}^\nu\delta^\mu_0 - \hat{P}^\mu\delta^\nu_0 - \Gamma^\mu_{0\rho}\,\hat{S}^{\rho\nu} - \Gamma^\nu_{0\rho}\,\hat{S}^{\mu\rho}\end{aligned}$$

so

$$\partial_0\hat{S}^{0\mu} = \hat{P}^\mu - \hat{P}^0\delta^\mu_0 - \Gamma^0_{0\rho}\,\hat{S}^{\rho\mu} - \Gamma^\mu_{0\rho}\,\hat{S}^{0\rho}$$

From (53) we have

$$\begin{aligned}\gamma^{\mu00} &= \hat{P}^{(\mu}\,\dot{C}^{0)} + \hat{S}^{\rho(0}\,\Gamma^{\mu)}{}_{\rho\kappa}\,\dot{C}^\kappa + \partial_0(\hat{S}^{0(\mu}\,\dot{C}^{0)})\\ &= \tfrac{1}{2}\big(\hat{P}^\mu + \hat{P}^0\delta^\mu_0 + \hat{S}^{\rho0}\,\Gamma^\mu{}_{\rho0} + \hat{S}^{\rho\mu}\,\Gamma^0{}_{\rho0} + \partial_0(\hat{S}^{0\mu})\big)\\ &= \tfrac{1}{2}\big(\hat{P}^\mu + \hat{P}^0\delta^\mu_0 + \hat{S}^{\rho0}\,\Gamma^\mu{}_{\rho0} + \hat{S}^{\rho\mu}\,\Gamma^0{}_{\rho0} + \hat{P}^\mu - \hat{P}^0\delta^\mu_0 - \Gamma^0_{0\rho}\,\hat{S}^{\rho\mu} - \Gamma^\mu_{0\rho}\,\hat{S}^{0\rho}\big)\\ &= \hat{P}^\mu + \hat{S}^{\rho0}\,\Gamma^\mu{}_{\rho0}\end{aligned}$$

and

$$\gamma^{\mu0a} = \tfrac{1}{2}\hat{S}^{a\mu} + \tfrac{1}{2}\hat{S}^{a0}\delta^\mu_0$$

From (54) we have

$$0 = \nabla_a K_0 + \nabla_0 K_a = \partial_a K_0 + \partial_0 K_a - 2\Gamma^\mu_{a0}K_\mu$$

Hence from (55) we have

$$\begin{aligned}\mathcal{Q}_K &= \gamma^{\mu00}\,K_\mu - \gamma^{\mu0a}\,\partial_a\,K_\mu = \big(\hat{P}^\mu + \hat{S}^{\rho0}\,\Gamma^\mu{}_{\rho0}\big)K_\mu - \tfrac{1}{2}\big(\hat{S}^{a\mu} + \hat{S}^{a0}\delta^\mu_0\big)\,\partial_a\,K_\mu\\ &= \hat{P}^\mu K_\mu + \hat{S}^{\rho0}\,\Gamma^\mu{}_{\rho0}K_\mu - \tfrac{1}{2}\hat{S}^{a\mu}\,\partial_a K_\mu - \tfrac{1}{2}\hat{S}^{a0}\,\partial_a K_0\\ &= \hat{P}^\mu K_\mu + \hat{S}^{\rho0}\,\Gamma^\mu{}_{\rho0}K_\mu - \tfrac{1}{2}\hat{S}^{a\mu}\,\partial_a K_\mu + \tfrac{1}{2}\hat{S}^{a0}\,\partial_0 K_a - \hat{S}^{a0}\Gamma^\mu_{a0}K_\mu\\ &= \hat{P}^\mu K_\mu + \tfrac{1}{2}\hat{S}^{\mu a}\,\partial_a K_\mu + \tfrac{1}{2}\hat{S}^{\mu0}\,\partial_0 K_\mu = \hat{P}^\mu K_\mu + \tfrac{1}{2}\hat{S}^{\mu\nu}\,\partial_\nu K_\mu = \hat{P}^\mu K_\mu + \tfrac{1}{2}\hat{S}^{\mu\nu}\,\nabla_\nu K_\mu\end{aligned}$$

□

Proof number 7: Proof that $\mathcal{Q}_K$ in (56) *is conserved.* Since K_μ is killing we have

$$\nabla_\mu\nabla_\nu K_\rho = R^\kappa{}_{\mu\nu\rho}K_\kappa$$

From (56) and (50) we have

$$\begin{aligned}\dot{\mathcal{Q}}_K = \frac{D\mathcal{Q}_K}{d\sigma} &= \frac{D\hat{P}^\mu}{d\sigma}\,K_\mu + \hat{P}^\mu\,\dot{C}^\nu\nabla_\nu K_\mu + \tfrac{1}{2}\frac{D\hat{S}^{\mu\nu}}{d\sigma}\,\nabla_\nu\;K_\mu + \tfrac{1}{2}\hat{S}^{\mu\nu}\,\dot{C}^\rho\nabla_\rho\nabla_\nu\;K_\mu\\ &= \tfrac{1}{2}R^\mu{}_{\nu\rho\kappa}\,\dot{C}^\nu\,\hat{S}^{\kappa\rho}\,K_\mu + \hat{P}^\mu\,\dot{C}^\nu\nabla_\nu K_\mu + \tfrac{1}{2}\Big(\hat{P}^\nu\dot{C}^\mu - \hat{P}^\mu\dot{C}^\nu\Big)\,\nabla_\nu K_\mu + \tfrac{1}{2}\hat{S}^{\mu\nu}\,\dot{C}^\rho\nabla_\rho\nabla_\nu K_\mu\\ &= \tfrac{1}{2}R^\mu{}_{\nu\rho\kappa}\,\dot{C}^\nu\,\hat{S}^{\kappa\rho}\,K_\mu + \tfrac{1}{2}\hat{S}^{\mu\nu}\,\dot{C}^\rho\,R^\kappa{}_{\rho\nu\mu}K_\kappa = 0\end{aligned}$$

□

A.3 Proofs about the quadrupole

Proof number 8: Proof of (64). Similarly to the proof of (45), we have

$$\begin{aligned}
\int_{\mathcal{I}} M^{\mu\nu}\, \dot{C}^{(\rho}\, C^{\kappa)} \partial_\rho \partial_\kappa \delta\big(x - C(\sigma)\big)\, d\sigma &= \int_{\mathcal{I}} M^{\mu\nu}\, C^{\kappa} \dot{C}^{\rho}\, \partial_\rho \partial_\kappa \delta\big(x - C(\sigma)\big)\, d\sigma \\
&= M^{\mu\nu} \int_{\mathcal{I}} C^{\kappa} \frac{d}{d\sigma}\Big(\partial_\kappa \delta\big(x - C(\sigma)\big)\Big)\, d\sigma \\
&= M^{\mu\nu} \int_{\mathcal{I}} \frac{d}{d\sigma}\Big(C^{\kappa} \partial_\kappa \delta\big(x - C(\sigma)\big)\Big)\, d\sigma - M^{\mu\nu} \int_{\mathcal{I}} \dot{C}^{\kappa} \partial_\kappa \delta\big(x - C(\sigma)\big)\, d\sigma \\
&= -M^{\mu\nu} \int_{\mathcal{I}} \frac{d}{d\sigma} \delta\big(x - C(\sigma)\big)\, d\sigma = 0
\end{aligned}$$

and

$$\begin{aligned}
\int_{\mathcal{I}} \hat{M}^{\mu\nu(\kappa} \dot{C}^{\rho)}\, \partial_\rho \partial_\kappa \delta\big(x - C(\sigma)\big)\, d\sigma &= \int_{\mathcal{I}} \hat{M}^{\mu\nu\kappa} \dot{C}^{\rho}\, \partial_\rho \partial_\kappa \delta\big(x - C(\sigma)\big)\, d\sigma \\
&= \hat{M}^{\mu\nu\kappa} \int_{\mathcal{I}} \frac{d}{d\sigma}\Big(\partial_\kappa \delta\big(x - C(\sigma)\big)\Big)\, d\sigma = 0
\end{aligned}$$

To see why this incorporates all the gauge freedom we use the adapted coordinates system. Assume $T^{\mu\nu}$ is given. From (125) we know that the components $\gamma^{\mu\nu\rho\kappa}$ are unique, i.e. have no gauge freedom. Integrating (71) we have

$$\zeta^{\mu\nu\rho\kappa} \to \zeta^{\mu\nu\rho\kappa} + \sigma M^{\nu\mu}\, \delta_0^{\rho}\, \delta_0^{\kappa} + \hat{M}^{\mu\nu(\kappa} \delta_0^{\rho)}$$

which is (64) in adapted coordinates. Hence (64) is incorporates all gauge freedom, in adapted coordinates. Now for a general coordinates system we use (65). We see in the proof 10 below, that (65) is consistent with the gauge freedom. Thus there are no additional gauge freedom in a general coordinate system. □

Proof number 9: Proof of (65). Using (61) we have

$$\begin{aligned}
\int_{\mathcal{I}} \hat{\zeta}^{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\kappa}} \left(\partial_{\hat{\rho}}\, \partial_{\hat{\kappa}}\, \hat{\phi}_{\hat{\mu}\hat{\nu}}\right)\big|_{C(\sigma)}\, d\sigma &= \int_{\mathbb{R}^4} \hat{T}^{\hat{\mu}\hat{\nu}}\, \hat{\phi}_{\hat{\mu}\hat{\nu}}\, d^4\hat{x} = \int_{\mathbb{R}^4} T^{\mu\nu}\, \phi_{\mu\nu}\, d^4x = \int_{\mathcal{I}} \zeta^{\mu\nu\rho\kappa} \left(\partial_\rho\, \partial_\kappa\, \phi_{\mu\nu}\right) d\sigma \\
&= \int_{\mathcal{I}} \zeta^{\mu\nu\rho\kappa}\, \partial_\rho\, \partial_\kappa \left(\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu}\, \hat{\phi}_{\hat{\mu}\hat{\nu}}\right)\, d\sigma \\
&= \int_{\mathcal{I}} \zeta^{\mu\nu\rho\kappa} \Big(\partial_\rho\, \partial_\kappa \left(\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu}\right) \hat{\phi}_{\hat{\mu}\hat{\nu}} + 2\, \partial_\rho \big(\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu}\big)\, \partial_\kappa\, \hat{\phi}_{\hat{\mu}\hat{\nu}} + \mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu}\ \partial_\rho\, \partial_\kappa\, \hat{\phi}_{\hat{\mu}\hat{\nu}}\Big)\, d\sigma
\end{aligned}$$

Take each of the terms in turn. For the third term we have

$$\begin{aligned}
\int_{\mathcal{I}} \zeta^{\mu\nu\rho\kappa}\, \mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu}\ \partial_\rho\, \partial_\kappa\, \hat{\phi}_{\hat{\mu}\hat{\nu}}\, d\sigma &= \int_{\mathcal{I}} \zeta^{\mu\nu\rho\kappa}\, \mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu}\ \partial_\rho \left(J^{\hat{\kappa}}_{\kappa}\, \partial_{\hat{\kappa}}\, \hat{\phi}_{\hat{\mu}\hat{\nu}}\right)\, d\sigma \\
&= \int_{\mathcal{I}} \zeta^{\mu\nu\rho\kappa}\, \mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu}\ \Big(\left(\partial_\rho\, J^{\hat{\kappa}}_{\kappa}\right) \partial_{\hat{\kappa}}\, \hat{\phi}_{\hat{\mu}\hat{\nu}} + J^{\hat{\kappa}}_{\kappa}\, \partial_\rho\, \partial_{\hat{\kappa}}\, \hat{\phi}_{\hat{\mu}\hat{\nu}}\Big)\, d\sigma \\
&= \int_{\mathcal{I}} \zeta^{\mu\nu\rho\kappa}\, \mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu}\ \left(\partial_\rho\, J^{\hat{\kappa}}_{\kappa}\right) \partial_{\hat{\kappa}}\, \hat{\phi}_{\hat{\mu}\hat{\nu}}\, d\sigma + \int_{\mathcal{I}} \zeta^{\mu\nu\rho\kappa}\, \mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu}\ J^{\hat{\rho}}_{\rho}\, J^{\hat{\kappa}}_{\kappa}\, \partial_{\hat{\rho}}\, \partial_{\hat{\kappa}}\, \hat{\phi}_{\hat{\mu}\hat{\nu}}\, d\sigma \\
&= \int_{\mathcal{I}} \zeta^{\mu\nu\rho\kappa}\, \mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu} \left(\partial_\rho\, J^{\hat{\kappa}}_{\kappa}\right) \left(\int^{\sigma} \dot{\hat{C}}^{\hat{\rho}}\ \partial_{\hat{\rho}}\, \partial_{\hat{\kappa}}\, \hat{\phi}_{\hat{\mu}\hat{\nu}}\, d\sigma'\right) d\sigma + \int_{\mathcal{I}} \zeta^{\mu\nu\rho\kappa}\, \mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu}\ J^{\hat{\rho}}_{\rho}\, J^{\hat{\kappa}}_{\kappa}\, \partial_{\hat{\rho}}\, \partial_{\hat{\kappa}}\, \hat{\phi}_{\hat{\mu}\hat{\nu}}\, d\sigma \\
&= -\int_{\mathcal{I}} \left(\int^{\sigma} \zeta^{\mu\nu\rho\kappa}\, \mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu} \left(\partial_\rho\, J^{\hat{\kappa}}_{\kappa}\right) d\sigma'\right) \dot{\hat{C}}^{\hat{\rho}}\ \partial_{\hat{\rho}}\, \partial_{\hat{\kappa}}\, \hat{\phi}_{\hat{\mu}\hat{\nu}}\ \, d\sigma + \int_{\mathcal{I}} \zeta^{\mu\nu\rho\kappa}\, \mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu}\ J^{\hat{\rho}}_{\rho}\, J^{\hat{\kappa}}_{\kappa}\, \partial_{\hat{\rho}}\, \partial_{\hat{\kappa}}\, \hat{\phi}_{\hat{\mu}\hat{\nu}}\, d\sigma
\end{aligned}$$

For the second term we have

$$\begin{aligned}
\int_{\mathcal{I}} \zeta^{\mu\nu\rho\kappa}\,\partial_\rho\big(\mathcal{J}^{\hat\mu\hat\nu}_{\mu\nu}\big)\,\partial_\kappa\,\hat\phi_{\hat\mu\hat\nu}\;d\sigma &= \int_{\mathcal{I}} \zeta^{\mu\nu\rho\kappa}\,\partial_\rho\big(\mathcal{J}^{\hat\mu\hat\nu}_{\mu\nu}\big)\,J^{\hat\kappa}_{\kappa}\,\partial_{\hat\kappa}\,\hat\phi_{\hat\mu\hat\nu}\;d\sigma \\
&= \int_{\mathcal{I}} \zeta^{\mu\nu\rho\kappa}\,\partial_\rho\big(\mathcal{J}^{\hat\mu\hat\nu}_{\mu\nu}\big)\,J^{\hat\kappa}_{\kappa}\left(\int^{\sigma}\dot{\hat C}^{\hat\rho}\,\partial_{\hat\rho}\,\partial_{\hat\kappa}\,\hat\phi_{\hat\mu\hat\nu}\,d\sigma'\right)d\sigma \\
&= -\int_{\mathcal{I}}\left(\int^{\sigma}\zeta^{\mu\nu\rho\kappa}\,\partial_\rho\big(\mathcal{J}^{\hat\mu\hat\nu}_{\mu\nu}\big)\,J^{\hat\kappa}_{\kappa}\,d\sigma'\right)\dot{\hat C}^{\hat\rho}\,\partial_{\hat\rho}\,\partial_{\hat\kappa}\,\hat\phi_{\hat\mu\hat\nu}\;d\sigma
\end{aligned}$$

For the first term we have

$$\begin{aligned}
\int_{\mathcal{I}} \zeta^{\mu\nu\rho\kappa}\,\partial_\rho\,\partial_\kappa\,\big(\mathcal{J}^{\hat\mu\hat\nu}_{\mu\nu}\big)\,\hat\phi_{\hat\mu\hat\nu}\;d\sigma &= \int_{\mathcal{I}} \zeta^{\mu\nu\rho\kappa}\,\partial_\rho\,\partial_\kappa\,\big(\mathcal{J}^{\hat\mu\hat\nu}_{\mu\nu}\big)\left(\int^{\sigma}\dot{\hat C}^{\hat\rho}\,\partial_{\hat\rho}\,\hat\phi_{\hat\mu\hat\nu}\,d\sigma'\right)d\sigma \\
&= -\int_{\mathcal{I}}\left(\int^{\sigma}\zeta^{\mu\nu\rho\kappa}\,\partial_\rho\,\partial_\kappa\,\big(\mathcal{J}^{\hat\mu\hat\nu}_{\mu\nu}\big)\,d\sigma'\right)\dot{\hat C}^{\hat\rho}\,\partial_{\hat\rho}\,\hat\phi_{\hat\mu\hat\nu}\;d\sigma \\
&= -\int_{\mathcal{I}}\left(\int^{\sigma}\zeta^{\mu\nu\rho\kappa}\,\partial_\rho\,\partial_\kappa\,\big(\mathcal{J}^{\hat\mu\hat\nu}_{\mu\nu}\big)\,d\sigma''\right)\dot{\hat C}^{\hat\rho}\left(\int^{\sigma}\partial_{\hat\rho}\,\dot{\hat C}^{\hat\kappa}\,\partial_{\hat\kappa}\,\hat\phi_{\hat\mu\hat\nu}\,d\sigma'\right)d\sigma \\
&= \int_{\mathcal{I}}\left(\int^{\sigma}\left(\int^{\sigma'}\zeta^{\mu\nu\rho\kappa}\,\partial_\rho\,\partial_\kappa\,\big(\mathcal{J}^{\hat\mu\hat\nu}_{\mu\nu}\big)\,d\sigma''\right)\dot{\hat C}^{\hat\rho}\,d\sigma'\right)\dot{\hat C}^{\hat\kappa}\,\partial_{\hat\rho}\,\partial_{\hat\kappa}\,\hat\phi_{\hat\mu\hat\nu}\;d\sigma \\
&= \int_{\mathcal{I}}\left(\dot{\hat C}^{\hat\kappa}\int^{\sigma}\left(\dot{\hat C}^{\hat\rho}\int^{\sigma'}\zeta^{\mu\nu\rho\kappa}\,\partial_\rho\,\partial_\kappa\,\big(\mathcal{J}^{\hat\mu\hat\nu}_{\mu\nu}\big)\,d\sigma''\right)d\sigma'\right)\partial_{\hat\rho}\,\partial_{\hat\kappa}\,\hat\phi_{\hat\mu\hat\nu}\;d\sigma
\end{aligned}$$

Thus adding these terms together we have

$$\begin{aligned}
\int_{\mathcal{I}} \hat\zeta^{\hat\mu\hat\nu\hat\rho\hat\kappa}\,\big(\partial_{\hat\rho}\,\partial_{\hat\kappa}\,\hat\phi_{\hat\mu\hat\nu}\big)\;d\sigma &= \int_{\mathcal{I}} \zeta^{\mu\nu\rho\kappa}\Big(\partial_\rho\,\partial_\kappa\,\big(\mathcal{J}^{\hat\mu\hat\nu}_{\mu\nu}\big)\,\hat\phi_{\hat\mu\hat\nu} + 2\,\partial_\rho\big(\mathcal{J}^{\hat\mu\hat\nu}_{\mu\nu}\big)\,\partial_\kappa\,\hat\phi_{\hat\mu\hat\nu} + \mathcal{J}^{\hat\mu\hat\nu}_{\mu\nu}\;\partial_\rho\,\partial_\kappa\,\hat\phi_{\hat\mu\hat\nu}\Big)\;d\sigma \\
= -\int_{\mathcal{I}}&\left(\int^{\sigma}\zeta^{\mu\nu\rho\kappa}\,\mathcal{J}^{\hat\mu\hat\nu}_{\mu\nu}(\partial_\rho\,J^{\hat\kappa}_{\kappa})\,d\sigma'\right)\dot{\hat C}^{\hat\rho}\,\partial_{\hat\rho}\,\partial_{\hat\kappa}\,\hat\phi_{\hat\mu\hat\nu}\;d\sigma + \int_{\mathcal{I}}\zeta^{\mu\nu\rho\kappa}\,\mathcal{J}^{\hat\mu\hat\nu}_{\mu\nu}\;J^{\hat\rho}_{\rho}\,J^{\hat\kappa}_{\kappa}\,\partial_{\hat\rho}\,\partial_{\hat\kappa}\,\hat\phi_{\hat\mu\hat\nu}\;d\sigma \\
&- 2\int_{\mathcal{I}}\left(\int^{\sigma}\zeta^{\mu\nu\rho\kappa}\,\partial_\rho\big(\mathcal{J}^{\hat\mu\hat\nu}_{\mu\nu}\big)\,J^{\hat\kappa}_{\kappa}\,d\sigma'\right)\dot{\hat C}^{\hat\rho}\,\partial_{\hat\rho}\,\partial_{\hat\kappa}\,\hat\phi_{\hat\mu\hat\nu}\;d\sigma \\
&+ \int_{\mathcal{I}}\left(\dot{\hat C}^{\hat\kappa}\int^{\sigma}\left(\dot{\hat C}^{\hat\rho}\int^{\sigma'}\zeta^{\mu\nu\rho\kappa}\,\partial_\rho\,\partial_\kappa\,\big(\mathcal{J}^{\hat\mu\hat\nu}_{\mu\nu}\big)\,d\sigma''\right)d\sigma'\right)\partial_{\hat\rho}\,\partial_{\hat\kappa}\,\hat\phi_{\hat\mu\hat\nu}\;d\sigma \\
= \int_{\mathcal{I}}&\Bigg(\zeta^{\mu\nu\rho\kappa}\,\mathcal{J}^{\hat\mu\hat\nu}_{\mu\nu}\;J^{\hat\rho}_{\rho}\,J^{\hat\kappa}_{\kappa} - \dot{\hat C}^{\hat\rho}\int^{\sigma}\zeta^{\mu\nu\rho\kappa}\,\mathcal{J}^{\hat\mu\hat\nu}_{\mu\nu}(\partial_\rho\,J^{\hat\kappa}_{\kappa})\,d\sigma' - 2\dot{\hat C}^{\hat\rho}\int^{\sigma}\zeta^{\mu\nu\rho\kappa}\,\partial_\rho\big(\mathcal{J}^{\hat\mu\hat\nu}_{\mu\nu}\big)\,J^{\hat\kappa}_{\kappa}\,d\sigma' \\
&\quad + \dot{\hat C}^{\hat\kappa}\int^{\sigma}\left(\dot{\hat C}^{\hat\rho}\int^{\sigma'}\zeta^{\mu\nu\rho\kappa}\,\partial_\rho\,\partial_\kappa\,\big(\mathcal{J}^{\hat\mu\hat\nu}_{\mu\nu}\big)\,d\sigma''\right)d\sigma'\Bigg)\partial_{\hat\rho}\,\partial_{\hat\kappa}\,\hat\phi_{\hat\mu\hat\nu}\;d\sigma
\end{aligned}$$

Hence (65) follows by symmetrising $\hat\rho$ and $\hat\kappa$. $\square$

Proof number 10: Proof that the change of coordinates (65) *is consistent with the gauge freedom* (64). First observe that the lower limits in (65) correspond to the to the gauge freedom (64) for $\hat\zeta^{\hat\mu\hat\nu\hat\rho\hat\kappa}$.

It is necessary to establish that, the Gauge freedom (64) for $\zeta^{\mu\nu\rho\kappa}$ when substituted into (65) does not effect the value of $\hat\zeta^{\hat\mu\hat\nu\hat\rho\hat\kappa}$. This is achieved by setting $\zeta^{\mu\nu\rho\kappa} = M^{\nu\mu}\,\dot C^{(\rho}\,C^{\kappa)} + \hat M^{\mu\nu(\rho}\,\dot C^{\kappa)}$, i.e. $\zeta^{\mu\nu\rho\kappa}$ is equivalent to zero, and checking that $\hat\zeta^{\hat\mu\hat\nu\hat\rho\hat\kappa} = 0$. As they are independent, we can consider the two terms $M^{\nu\mu}\,\dot C^{(\rho}\,C^{\kappa)}$ and $\hat M^{\mu\nu(\rho}\,\dot C^{\kappa)}$ separately.

For the case $\zeta^{\mu\nu\rho\kappa} = M^{\nu\mu}\,\dot{C}^{(\rho}\,C^{\kappa)}$ we have for the fifth term on the right hand side of (65)

$$\begin{aligned}
\int^{\sigma}\dot{C}^{\hat{\kappa}}\int^{\sigma'}\dot{C}^{(\rho}C^{\kappa)}\,\partial_\rho\,\partial_\kappa\left(\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu}\right)d\sigma''\,d\sigma' &= \int^{\sigma}\dot{C}^{\hat{\kappa}}\int^{\sigma'}\dot{C}^{\rho}C^{\kappa}\,\partial_\rho\,\partial_\kappa\left(\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu}\right)d\sigma''\,d\sigma' \\
&= \int^{\sigma}\dot{C}^{\hat{\kappa}}\int^{\sigma'}C^{\kappa}\,\frac{d}{d\sigma''}\Big(\partial_\kappa\,\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu}\Big)\,d\sigma''\,d\sigma' \\
&= \int^{\sigma}\dot{C}^{\hat{\kappa}}\int^{\sigma'}\frac{d}{d\sigma''}\Big(C^{\kappa}\partial_\kappa\,\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu}\Big)\,d\sigma''\,d\sigma' - \int^{\sigma}\dot{C}^{\hat{\kappa}}\int^{\sigma'}\dot{C}^{\kappa}\partial_\kappa\,\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu}\,d\sigma''\,d\sigma' \\
&= \int^{\sigma}\dot{C}^{\hat{\kappa}}\,C^{\kappa}\partial_\kappa\,\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu}\,d\sigma' - \int^{\sigma}\dot{C}^{\hat{\kappa}}\int^{\sigma'}\frac{d}{d\sigma''}\,\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu}\,d\sigma''\,d\sigma' \\
&= \int^{\sigma}\dot{C}^{\hat{\kappa}}\,C^{\kappa}\partial_\kappa\,\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu}\,d\sigma' - \int^{\sigma}\dot{C}^{\hat{\kappa}}\,\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu}\,d\sigma' \\
&= \int^{\sigma}\dot{C}^{\hat{\kappa}}\,C^{\kappa}\partial_\kappa\,\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu}\,d\sigma' - \int^{\sigma}\dot{C}^{\kappa}J^{\hat{\kappa}}_{\kappa}\,\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu}\,d\sigma' \\
&= \int^{\sigma}\dot{C}^{\kappa}J^{\hat{\kappa}}_{\kappa}C^{\rho}\partial_\rho\,\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu}\,d\sigma' - C^{\kappa}J^{\hat{\kappa}}_{\kappa}\,\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu} + \int^{\sigma}C^{\kappa}\frac{d}{d\sigma'}(J^{\hat{\kappa}}_{\kappa}\,\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu})\,d\sigma'
\end{aligned}$$

Since

$$\int^{\sigma}\dot{C}^{\kappa}C^{\rho}\,\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu}\partial_\rho\,J^{\hat{\kappa}}_{\kappa}\,d\sigma' = \int^{\sigma}\dot{C}^{\kappa}C^{\rho}\,\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu}\partial_\kappa\,J^{\hat{\kappa}}_{\rho}\,d\sigma' = \int^{\sigma}\dot{C}^{\rho}C^{\kappa}\,\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu}\partial_\rho\,J^{\hat{\kappa}}_{\kappa}\,d\sigma'$$

we have for the second term on the right hand side of (65)

$$\begin{aligned}
&\int^{\sigma}\dot{C}^{(\rho}C^{\kappa)}\left(\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu}(\partial_\rho\,J^{\hat{\kappa}}_{\kappa}) + 2\,\partial_\rho\,(\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu})\,J^{\hat{\kappa}}_{\kappa}\right)d\sigma' \\
&= \tfrac{1}{2}\int^{\sigma}\dot{C}^{\rho}C^{\kappa}\left(\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu}(\partial_\rho\,J^{\hat{\kappa}}_{\kappa}) + 2\,\partial_\rho\,(\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu})\,J^{\hat{\kappa}}_{\kappa}\right)d\sigma' + \tfrac{1}{2}\int^{\sigma}\dot{C}^{\kappa}C^{\rho}\left(\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu}(\partial_\rho\,J^{\hat{\kappa}}_{\kappa}) + 2\,\partial_\rho\,(\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu})\,J^{\hat{\kappa}}_{\kappa}\right)d\sigma' \\
&= \tfrac{1}{2}\int^{\sigma}\dot{C}^{\rho}C^{\kappa}\left(\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu}(\partial_\rho\,J^{\hat{\kappa}}_{\kappa}) + 2\,\partial_\rho\,(\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu})\,J^{\hat{\kappa}}_{\kappa}\right)d\sigma' + \tfrac{1}{2}\int^{\sigma}\dot{C}^{\kappa}C^{\rho}\,\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu}(\partial_\rho\,J^{\hat{\kappa}}_{\kappa})\,d\sigma' \\
&\qquad\qquad + \int^{\sigma}\dot{C}^{\kappa}C^{\rho}\,\partial_\rho\,(\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu})\,J^{\hat{\kappa}}_{\kappa}\,d\sigma' \\
&= \int^{\sigma}\dot{C}^{\rho}C^{\kappa}\left(\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu}(\partial_\rho\,J^{\hat{\kappa}}_{\kappa}) + \partial_\rho\,(\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu})\,J^{\hat{\kappa}}_{\kappa}\right)d\sigma' + \int^{\sigma}\dot{C}^{\kappa}C^{\rho}\,\partial_\rho\,(\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu})\,J^{\hat{\kappa}}_{\kappa}\,d\sigma' \\
&= \int^{\sigma}C^{\kappa}\dot{C}^{\rho}\,\partial_\rho\left(\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu}\,J^{\hat{\kappa}}_{\kappa}\right)d\sigma' + \int^{\sigma}\dot{C}^{\kappa}C^{\rho}\,\partial_\rho\,(\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu})\,J^{\hat{\kappa}}_{\kappa}\,d\sigma' \\
&= \int^{\sigma}C^{\kappa}\frac{d}{d\sigma'}(\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu}\,J^{\hat{\kappa}}_{\kappa})\,d\sigma' + \int^{\sigma}\dot{C}^{\kappa}C^{\rho}\,\partial_\rho\,(\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu})\,J^{\hat{\kappa}}_{\kappa}\,d\sigma'
\end{aligned}$$

Thus taking the difference between these two terms gives

$$\begin{aligned}
&\int^{\sigma}\dot{C}^{\hat{\kappa}}\int^{\sigma'}\dot{C}^{(\rho}C^{\kappa)}\,\partial_\rho\,\partial_\kappa\left(\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu}\right)d\sigma''\,d\sigma' - \int^{\sigma}\dot{C}^{(\rho}C^{\kappa)}\left(\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu}(\partial_\rho\,J^{\hat{\kappa}}_{\kappa}) + 2\,\partial_\rho\,(\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu})\,J^{\hat{\kappa}}_{\kappa}\right)d\sigma' \\
&\quad= \left(\int^{\sigma}\dot{C}^{\kappa}J^{\hat{\kappa}}_{\kappa}C^{\rho}\partial_\rho\,\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu}\,d\sigma' - C^{\kappa}J^{\hat{\kappa}}_{\kappa}\,\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu} + \int^{\sigma}C^{\kappa}\frac{d}{d\sigma'}(J^{\hat{\kappa}}_{\kappa}\,\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu})\,d\sigma'\right) \\
&\qquad\qquad - \left(\int^{\sigma}C^{\kappa}\frac{d}{d\sigma'}(\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu}\,J^{\hat{\kappa}}_{\kappa})\,d\sigma' + \int^{\sigma}\dot{C}^{\kappa}C^{\rho}\,\partial_\rho\,(\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu})\,J^{\hat{\kappa}}_{\kappa}\,d\sigma'\right) = -C^{\kappa}J^{\hat{\kappa}}_{\kappa}\,\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu}
\end{aligned}$$

Hence the sum of half the first term, with the second and fifth terms of (65) we have

$$\begin{aligned}
&\tfrac{1}{2}\big(M^{\nu\mu}\,\dot{C}^{(\rho}\,C^{\kappa)}\big)\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu}\,J^{\hat{\rho}}_{\rho}\,J^{\hat{\kappa}}_{\kappa} + \dot{C}^{\hat{\kappa}}\int^{\sigma'}\big(M^{\nu\mu}\,\dot{C}^{(\rho}\,C^{\kappa)}\big)\,\partial_\rho\,\partial_\kappa\left(\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu}\right)d\sigma''\,d\sigma' \\
&\qquad\qquad - \int^{\sigma}\big(M^{\nu\mu}\,\dot{C}^{(\rho}\,C^{\kappa)}\big)\left(\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu}(\partial_\rho\,J^{\hat{\kappa}}_{\kappa}) + 2\,\partial_\rho\,(\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu})\,J^{\hat{\kappa}}_{\kappa}\right)d\sigma' = 0
\end{aligned}$$

Likewise for the sum of half the first term, with the third and fourth terms of (65). Hence setting $\zeta^{\mu\nu\rho\kappa} = M^{\nu\mu}\dot{C}^{(\rho}\,C^{\kappa)}$ we have $\hat{\zeta}^{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\kappa}} = 0$.

Repeating for $\zeta^{\mu\nu\rho\kappa} = \hat{M}^{\mu\nu(\rho}\,\dot{C}^{\kappa)}$, we have for the fifth term in (65)

$$\int^{\sigma} \hat{\dot{C}}^{\hat{\kappa}} \int^{\sigma'} \left(\hat{M}^{\mu\nu(\rho}\,\dot{C}^{\kappa)}\right) \partial_\rho\,\partial_\kappa \left(\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu}\right) d\sigma''\,d\sigma' = \hat{M}^{\mu\nu\rho} \int^{\sigma} \hat{\dot{C}}^{\hat{\kappa}} \int^{\sigma'} \frac{d}{d\sigma''}\left(\partial_\kappa \mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu}\right) d\sigma''\,d\sigma'$$
$$= \hat{M}^{\mu\nu\rho} \int^{\sigma} \hat{\dot{C}}^{\hat{\kappa}} \left(\partial_\kappa \mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu}\right) d\sigma'$$

while for the second term in (65)

$$\begin{aligned}
&\int_{\mathcal{I}} \left(\hat{M}^{\mu\nu(\rho}\,\dot{C}^{\kappa)}\right) \left(\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu}(\partial_\rho\, J^{\hat{\kappa}}_{\kappa}) + 2\,\partial_\rho\,(\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu})\,J^{\hat{\kappa}}_{\kappa}\right) d\sigma' \\
&= \tfrac{1}{2}\hat{M}^{\mu\nu\rho} \int_{\mathcal{I}} \dot{C}^{\kappa}\left(\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu}(\partial_\rho\, J^{\hat{\kappa}}_{\kappa}) + 2\,\partial_\rho\,(\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu})\,J^{\hat{\kappa}}_{\kappa}\right) d\sigma' + \tfrac{1}{2}\hat{M}^{\mu\nu\kappa} \int_{\mathcal{I}} \dot{C}^{\rho}\left(\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu}(\partial_\rho\, J^{\hat{\kappa}}_{\kappa}) + 2\,\partial_\rho\,(\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu})\,J^{\hat{\kappa}}_{\kappa}\right) d\sigma' \\
&= \hat{M}^{\mu\nu\rho} \int_{\mathcal{I}} \dot{C}^{\kappa}\, J^{\hat{\kappa}}_{\kappa}\,(\partial_\rho\,\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu})\,d\sigma' + \hat{M}^{\mu\nu\kappa} \int_{\mathcal{I}} \dot{C}^{\rho}\left(\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu}(\partial_\rho\, J^{\hat{\kappa}}_{\kappa}) + \partial_\rho\,(\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu})\,J^{\hat{\kappa}}_{\kappa}\right) d\sigma' \\
&= \hat{M}^{\mu\nu\rho} \int_{\mathcal{I}} \hat{\dot{C}}^{\hat{\kappa}}\,(\partial_\rho\,\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu})\,d\sigma' + \hat{M}^{\mu\nu\kappa} \int_{\mathcal{I}} \frac{d}{d\sigma'}\left(\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu}\,J^{\hat{\kappa}}_{\kappa}\right) d\sigma' \\
&= \hat{M}^{\mu\nu\rho} \int_{\mathcal{I}} \hat{\dot{C}}^{\hat{\kappa}}\,(\partial_\rho\,\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu})\,d\sigma' + \hat{M}^{\mu\nu\kappa}\,\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu}\,J^{\hat{\kappa}}_{\kappa}
\end{aligned}$$

Hence when $\zeta^{\mu\nu\rho\kappa} = \hat{M}^{\mu\nu(\rho}\,\dot{C}^{\kappa)}$ then $\hat{\zeta}^{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\kappa}} = 0$. □

Proof number 11: Proof of (72)-(75). From (8) we have for any test vector θ_ν

$$\begin{aligned}
0 &= \int_{\mathcal{M}} (\nabla_\mu T^{\mu\nu})\,\theta_\nu\,d^4x = \int_{\mathcal{M}} \left(\partial_\mu T^{\mu\nu} + \Gamma^{\nu}_{\mu\rho} T^{\mu\rho}\right)\theta_\nu\,d^4x = \int_{\mathcal{M}} T^{\mu\nu}\left(\Gamma^{\rho}_{\mu\nu}\,\theta_\rho - \partial_\mu\theta_\nu\right) d^4x \\
&= \int_{\mathcal{M}} \left(\gamma^{\mu\nu 00}\,\delta^{(3)}(\boldsymbol{z}) + \gamma^{\mu\nu 0a}\,\partial_a\delta^{(3)}(\boldsymbol{z}) + \tfrac{1}{2}\gamma^{\mu\nu ab}\,\partial_a\partial_b\delta^{(3)}(\boldsymbol{z})\right)\left(\Gamma^{\rho}_{\mu\nu}\,\theta_\rho - \partial_\mu\theta_\nu\right) d^4x \\
&= \int_{\mathcal{I}} d\sigma\Big(\gamma^{\mu\nu 00}\left(\Gamma^{\rho}_{\mu\nu}\,\theta_\rho - \partial_\mu\theta_\nu\right) - \gamma^{\mu\nu 0a}\,\partial_a\left(\Gamma^{\rho}_{\mu\nu}\,\theta_\rho - \partial_\mu\theta_\nu\right) + \tfrac{1}{2}\gamma^{\mu\nu ab}\partial_a\partial_b\left(\Gamma^{\rho}_{\mu\nu}\,\theta_\rho - \partial_\mu\theta_\nu\right)\Big) \\
&= \int_{\mathcal{I}} d\sigma\Big(\gamma^{\mu\nu 00}\,\Gamma^{\rho}_{\mu\nu}\,\theta_\rho - \gamma^{a\nu 00}\,\partial_a\theta_\nu + \dot{\gamma}^{0\nu 00}\,\theta_\nu \\
&\qquad - \gamma^{\mu\nu 0a}\,\partial_a\left(\Gamma^{\rho}_{\mu\nu}\,\theta_\rho\right) + \gamma^{b\nu 0a}\,\partial_a\partial_b\theta_\nu - \dot{\gamma}^{0\nu 0a}\,\partial_a\theta_\nu \\
&\qquad + \tfrac{1}{2}\gamma^{\mu\nu ab}\partial_a\partial_b\left(\Gamma^{\rho}_{\mu\nu}\,\theta_\rho\right) - \tfrac{1}{2}\gamma^{c\nu ab}\partial_a\partial_b\partial_c\theta_\nu + \tfrac{1}{2}\dot{\gamma}^{0\nu ab}\partial_a\partial_b\theta_\nu\Big) \\
&= \int_{\mathcal{I}} d\sigma\Big(\gamma^{\mu\nu 00}\,\Gamma^{\rho}_{\mu\nu}\,\theta_\rho - \gamma^{a\nu 00}\,\partial_a\theta_\nu + \dot{\gamma}^{0\rho 00}\,\theta_\rho \\
&\qquad - \gamma^{\mu\nu 0a}\,(\partial_a\Gamma^{\rho}_{\mu\nu})\,\theta_\rho - \gamma^{\mu\nu 0a}\,\Gamma^{\rho}_{\mu\nu}\,\partial_a\theta_\rho + \gamma^{b\nu 0a}\,\partial_a\partial_b\theta_\nu - \dot{\gamma}^{0\nu 0a}\,\partial_a\theta_\nu \\
&\qquad + \tfrac{1}{2}\gamma^{\mu\nu ab}\left(\partial_a\partial_b\Gamma^{\rho}_{\mu\nu}\right)\theta_\rho + \gamma^{\mu\nu ab}\left(\partial_a\Gamma^{\rho}_{\mu\nu}\right)\left(\partial_b\theta_\rho\right) + \tfrac{1}{2}\gamma^{\mu\nu ab}\Gamma^{\rho}_{\mu\nu}\partial_a\partial_b\theta_\rho \\
&\qquad\quad - \tfrac{1}{2}\gamma^{c\nu ab}\partial_a\partial_b\partial_c\theta_\nu + \tfrac{1}{2}\dot{\gamma}^{0\nu ab}\partial_a\partial_b\theta_\nu\Big) \\
&= \int_{\mathcal{I}} d\sigma\bigg(\theta_\rho\Big(\gamma^{\mu\nu 00}\,\Gamma^{\rho}_{\mu\nu} + \dot{\gamma}^{0\rho 00} - \gamma^{\mu\nu 0a}\,(\partial_a\Gamma^{\rho}_{\mu\nu}) + \tfrac{1}{2}\gamma^{\mu\nu ab}\left(\partial_a\partial_b\Gamma^{\rho}_{\mu\nu}\right)\Big) \\
&\qquad - \partial_a\theta_\rho\Big(\gamma^{a\rho 00} + \gamma^{\mu\nu 0a}\,\Gamma^{\rho}_{\mu\nu} + \dot{\gamma}^{0\rho 0a} - \gamma^{\mu\nu ba}\left(\partial_b\Gamma^{\rho}_{\mu\nu}\right)\Big) \\
&\qquad + \partial_a\partial_b\theta_\rho\Big(\gamma^{b\rho 0a} + \tfrac{1}{2}\gamma^{\mu\nu ab}\Gamma^{\rho}_{\mu\nu} + \tfrac{1}{2}\dot{\gamma}^{0\rho ab}\Big) - \tfrac{1}{2}\gamma^{c\nu ab}\partial_a\partial_b\partial_c\theta_\nu\bigg)
\end{aligned}$$

The terms with θ_ρ, $\partial_a\theta_\rho$, $\partial_a\partial_b\theta_\rho$ and $\partial_a\partial_b\partial_\rho\theta_\rho$ are independent. In section 6.5 we give values of θ_μ which demonstrate this. From this we get (72)-(75). Note we must take the symmetric part with respect to b, a. □

Proof number 12: Proof of (76)-(78). This follows from substituting (71) into (65).

We set $(x^0,\dots,x^3)=(\sigma,z^1,z^2,z^3)$ and $(\hat{x}^{\hat{0}}\dots\hat{x}^{\hat{3}})=(\hat{\sigma},\hat{z}^{\hat{1}},\hat{z}^{\hat{2}},\hat{z}^{\hat{2}})$ into (65) and use the fact that $\dot{\hat{C}}^{\hat{\mu}}=\delta^{\hat{\mu}}_0$. Hence (76) follows directly.

For (77) we have from (65)

$$
\begin{aligned}
\hat{\zeta}^{\hat{\mu}\hat{\nu}\hat{c}\hat{0}} &= \zeta^{\mu\nu\rho\kappa}\,\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu}\,J^{\hat{c}}_{\rho}\,J^{\hat{0}}_{\kappa}-\tfrac12\dot{\hat{C}}^{\hat{c}}\int^{\sigma}\zeta^{\mu\nu\rho\kappa}\Big(\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu}(\partial_\rho\,J^{\hat{0}}_{\kappa})+2\,\partial_\rho\,(\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu})\,J^{\hat{0}}_{\kappa}\Big)\,d\sigma'\\
&\quad-\tfrac12\dot{\hat{C}}^{\hat{0}}\int^{\sigma}\zeta^{\mu\nu\rho\kappa}\Big(\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu}(\partial_\rho\,J^{\hat{c}}_{\kappa})+2\,\partial_\rho\,(\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu})\,J^{\hat{c}}_{\kappa}\Big)\,d\sigma'\\
&\quad+\tfrac12\dot{\hat{C}}^{\hat{0}}\int^{\sigma}\dot{\hat{C}}^{\hat{c}}\int^{\sigma'}\zeta^{\mu\nu\rho\kappa}\,\partial_\rho\,\partial_\kappa\,(\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu})\,d\sigma''\,d\sigma'+\tfrac12\dot{\hat{C}}^{\hat{c}}\int^{\sigma}\dot{\hat{C}}^{\hat{0}}\int^{\sigma'}\zeta^{\mu\nu\rho\kappa}\,\partial_\rho\,\partial_\kappa\,(\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu})\,d\sigma''\,d\sigma'\\
&= \zeta^{\mu\nu\rho\kappa}\,\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu}\,J^{\hat{c}}_{\rho}\,J^{\hat{0}}_{\kappa}-\tfrac12\int^{\sigma}\zeta^{\mu\nu\rho\kappa}\Big(\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu}(\partial_\rho\,J^{\hat{c}}_{\kappa})+2\,\partial_\rho\,(\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu})\,J^{\hat{c}}_{\kappa}\Big)\,d\sigma'
\end{aligned}
$$

Thus from (71)

$$
\begin{aligned}
\hat{\gamma}^{\hat{\mu}\hat{\nu}\hat{c}\hat{0}} &= \dot{\hat{\zeta}}^{\hat{\mu}\hat{\nu}\hat{c}\hat{0}} = (\zeta^{\mu\nu\rho\kappa}\,\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu}\,J^{\hat{c}}_{\rho}\,J^{\hat{0}}_{\kappa})^{\cdot}-\tfrac12\zeta^{\mu\nu\rho\kappa}\,(\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu}\,J^{\hat{c}}_{\rho\kappa}+2\,\partial_\rho\,(\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu})\,J^{\hat{c}}_{\kappa})\\
&= (\zeta^{\mu\nu00}\,\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu}\,J^{\hat{c}}_{0}\,J^{\hat{0}}_{0})^{\cdot}+(\zeta^{\mu\nu c0}\,\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu}\,J^{\hat{c}}_{c}\,J^{\hat{0}}_{0})^{\cdot}+(\zeta^{\mu\nu0c}\,\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu}\,J^{\hat{c}}_{0}\,J^{\hat{0}}_{c})^{\cdot}+(\zeta^{\mu\nu cd}\,\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu}\,J^{\hat{c}}_{c}\,J^{\hat{0}}_{d})^{\cdot}\\
&\quad-\tfrac12\zeta^{\mu\nu00}\,(\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu}\,J^{\hat{c}}_{00}+2\,\partial_0\,(\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu})\,J^{\hat{c}}_{0})-\tfrac12\zeta^{\mu\nu c0}\,(\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu}\,J^{\hat{c}}_{c0}+2\,\partial_c\,(\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu})\,J^{\hat{c}}_{0})\\
&\quad-\tfrac12\zeta^{\mu\nu c0}\,(\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu}\,J^{\hat{c}}_{c0}+2\,\partial_0\,(\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu})\,J^{\hat{c}}_{c})-\tfrac12\zeta^{\mu\nu cd}\,(\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu}\,J^{\hat{c}}_{cd}+2\,\partial_c\,(\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu})\,J^{\hat{c}}_{d})\\
&= (\zeta^{\mu\nu c0}\,\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu}\,J^{\hat{c}}_{c})^{\cdot}+(\zeta^{\mu\nu cd}\,\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu}\,J^{\hat{c}}_{c}\,J^{\hat{0}}_{d})^{\cdot}\\
&\quad-\zeta^{\mu\nu c0}\,(\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu}\,J^{\hat{c}}_{c0}+\dot{\mathcal{J}}^{\hat{\mu}\hat{\nu}}_{\mu\nu}\,J^{\hat{c}}_{c})-\tfrac12\zeta^{\mu\nu cd}\,(\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu}\,J^{\hat{c}}_{cd}+2\,\partial_c\,(\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu})\,J^{\hat{c}}_{d})\\
&= \dot{\zeta}^{\mu\nu c0}\,\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu}\,J^{\hat{c}}_{c}+(\zeta^{\mu\nu cd}\,\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu}\,J^{\hat{c}}_{c}\,J^{\hat{0}}_{d})^{\cdot}-\tfrac12\zeta^{\mu\nu cd}\,(\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu}\,J^{\hat{c}}_{cd}+2\,\partial_c\,(\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu})\,J^{\hat{c}}_{d})\\
&= \gamma^{\mu\nu c0}\,\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu}\,J^{\hat{c}}_{c}+(\gamma^{\mu\nu cd}\,\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu}\,J^{\hat{c}}_{c}\,J^{\hat{0}}_{d})^{\cdot}-\tfrac12\gamma^{\mu\nu cd}\,(\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu}\,J^{\hat{c}}_{cd}+2\,\partial_c\,(\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu})\,J^{\hat{c}}_{d})
\end{aligned}
$$

In order to show (78) we have from (65)

$$
\begin{aligned}
\hat{\zeta}^{\mu\nu\hat{0}\hat{0}} &= (\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu}\,J^{\hat{0}}_{\rho}\,J^{\hat{0}}_{\kappa})\,\zeta^{\mu\nu\rho\kappa}-\int^{\sigma}\Big((\partial_\kappa J^{\hat{0}}_{\rho})\,\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu}+J^{\hat{0}}_{\rho}\,\partial_\kappa\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu}+J^{\hat{0}}_{\kappa}\,\partial_\rho\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu}\Big)\zeta^{\mu\nu\rho\kappa}\,d\sigma'\\
&\quad+\int^{\sigma}d\sigma'\int^{\sigma'}\partial_{\rho\kappa}\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu}\,\zeta^{\mu\nu\rho\kappa}\,d\sigma''
\end{aligned}
$$

where $\partial_{\rho\kappa}=\partial_\rho\partial_\kappa$. Hence

$$
\begin{aligned}
\hat{\gamma}^{\hat{\mu}\hat{\nu}\hat{0}\hat{0}} &= \tfrac12\ddot{\hat{\zeta}}^{\hat{\mu}\hat{\nu}\hat{0}\hat{0}}\\
&= \tfrac12\Big(((\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu}\;J^{\hat{0}}_{\kappa}\,J^{\hat{0}}_{\rho})\,\zeta^{\mu\nu\rho\kappa})^{\cdot\cdot}-(((\partial_\kappa J^{\hat{0}}_{\rho})\,\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu}+J^{\hat{0}}_{\rho}\,\partial_\kappa\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu}+J^{\hat{0}}_{\kappa}\,\partial_\rho\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu})\zeta^{\mu\nu\rho\kappa})^{\cdot}+\partial_{\rho\kappa}\mathcal{J}^{\hat{\mu}\hat{\nu}}_{\mu\nu}\,\zeta^{\mu\nu\rho\kappa}\Big)
\end{aligned}
\tag{137}
$$

It is important to establish that all the $\zeta^{\mu\nu\rho\kappa}$ on the right hand side of (137) can be replaced by the corresponding $\gamma^{\mu\nu\rho\kappa}$ without using integrals. However since from (71) $\gamma^{\mu\nu00}=\frac12\ddot{\zeta}^{\mu\nu00}$ and $\gamma^{\mu\nu a0}=\dot{\zeta}^{\mu\nu a0}$ we need to expand (137) to confirm that no terms $\zeta^{\mu\nu00}$, $\dot{\zeta}^{\mu\nu00}$ or $\zeta^{\mu\nu a0}$ exist on the right hand side.

$$
\begin{aligned}
\hat{\gamma}^{\hat{\mu}\hat{\nu}\hat{0}\hat{0}} &= \tfrac{1}{2}\Big((\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}}\ J_{\kappa}^{\hat{0}}\,J_{\rho}^{\hat{0}})\,\zeta^{\mu\nu\rho\kappa}\Big)^{\cdot\cdot} - \Big(\big(\tfrac{1}{2}J_{\rho\kappa}^{\hat{0}}\,\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} + J_{\kappa}^{\hat{0}}\,\partial_{\rho}\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}}\big)\zeta^{\mu\nu\rho\kappa}\Big)^{\cdot} + \big(\tfrac{1}{2}\partial_{\rho\kappa}\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}}\big)\zeta^{\mu\nu\rho\kappa} \\
&= \tfrac{1}{2}\Big((\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}}\ J_{0}^{\hat{0}}\,J_{0}^{\hat{0}})\,\zeta^{\mu\nu00}\Big)^{\cdot\cdot} + \Big((\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}}\ J_{c}^{\hat{0}}\,J_{0}^{\hat{0}})\,\zeta^{\mu\nu c0}\Big)^{\cdot\cdot} + \tfrac{1}{2}\Big((\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}}\ J_{d}^{\hat{0}}\,J_{c}^{\hat{0}})\,\zeta^{\mu\nu cd}\Big)^{\cdot\cdot} \\
&\quad - \Big(\big(\tfrac{1}{2}J_{00}^{\hat{0}}\,\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} + J_{0}^{\hat{0}}\,\partial_{0}\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}}\big)\zeta^{\mu\nu00}\Big)^{\cdot} - \Big(\big(J_{0c}^{\hat{0}}\,\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} + J_{c}^{\hat{0}}\,\partial_{0}\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} + J_{0}^{\hat{0}}\,\partial_{c}\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}}\big)\zeta^{\mu\nu c0}\Big)^{\cdot} \\
&\quad - \Big(\big(\tfrac{1}{2}J_{cd}^{\hat{0}}\,\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} + J_{c}^{\hat{0}}\,\partial_{d}\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}}\big)\zeta^{\mu\nu cd}\Big)^{\cdot} \\
&\quad + \big(\tfrac{1}{2}\partial_{00}\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}}\big)\,\zeta^{\mu\nu00} + \big(\partial_{0c}\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}}\big)\,\zeta^{\mu\nu c0} + \big(\tfrac{1}{2}\partial_{cd}\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}}\big)\,\zeta^{\mu\nu cd} \\
&= \tfrac{1}{2}(\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}}\,\zeta^{\mu\nu00})^{\cdot\cdot} + \Big((\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}}\ J_{c}^{\hat{0}}\,\zeta^{\mu\nu c0}\Big)^{\cdot\cdot} + \tfrac{1}{2}\Big((\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}}\ J_{d}^{\hat{0}}\,J_{c}^{\hat{0}})\,\zeta^{\mu\nu cd}\Big)^{\cdot\cdot} - \Big(\dot{\mathcal{J}}_{\mu\nu}^{\hat{\mu}\hat{\nu}}\zeta^{\mu\nu00}\Big)^{\cdot} \\
&\quad - \Big(\big(J_{0c}^{\hat{0}}\,\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} + J_{c}^{\hat{0}}\,\dot{\mathcal{J}}_{\mu\nu}^{\hat{\mu}\hat{\nu}} + \partial_{c}\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}}\big)\zeta^{\mu\nu c0}\Big)^{\cdot} - \Big(\big(\tfrac{1}{2}J_{cd}^{\hat{0}}\,\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} + J_{d}^{\hat{0}}\,\partial_{c}\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}}\big)\zeta^{\mu\nu cd}\Big)^{\cdot} \\
&\quad + \tfrac{1}{2}\,\ddot{\mathcal{J}}_{\mu\nu}^{\hat{\mu}\hat{\nu}}\,\zeta^{\mu\nu00} + \big(\partial_{c}\dot{\mathcal{J}}_{\mu\nu}^{\hat{\mu}\hat{\nu}}\big)\,\zeta^{\mu\nu c0} + \big(\tfrac{1}{2}\partial_{cd}\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}}\big)\,\zeta^{\mu\nu cd} \\
&= \tfrac{1}{2}\ddot{\mathcal{J}}_{\mu\nu}^{\hat{\mu}\hat{\nu}}\,\zeta^{\mu\nu00} + \dot{\mathcal{J}}_{\mu\nu}^{\hat{\mu}\hat{\nu}}\,\dot{\zeta}^{\mu\nu00} + \tfrac{1}{2}\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}}\,\ddot{\zeta}^{\mu\nu00} \\
&\quad + \ddot{\mathcal{J}}_{\mu\nu}^{\hat{\mu}\hat{\nu}}\ J_{c}^{\hat{0}}\,\zeta^{\mu\nu c0} + \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}}\ J_{c00}^{\hat{0}}\,\zeta^{\mu\nu c0} + \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}}\ J_{c}^{\hat{0}}\,\ddot{\zeta}^{\mu\nu c0} + 2\dot{\mathcal{J}}_{\mu\nu}^{\hat{\mu}\hat{\nu}}\ J_{c0}^{\hat{0}}\,\zeta^{\mu\nu c0} + 2\dot{\mathcal{J}}_{\mu\nu}^{\hat{\mu}\hat{\nu}}\ J_{c}^{\hat{0}}\,\dot{\zeta}^{\mu\nu c0} + 2\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}}\ J_{c0}^{\hat{0}}\,\dot{\zeta}^{\mu\nu c0} \\
&\quad + \tfrac{1}{2}\Big((\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}}\ J_{d}^{\hat{0}}\,J_{c}^{\hat{0}})\,\zeta^{\mu\nu cd}\Big)^{\cdot\cdot} - \ddot{\mathcal{J}}_{\mu\nu}^{\hat{\mu}\hat{\nu}}\zeta^{\mu\nu00} - \dot{\mathcal{J}}_{\mu\nu}^{\hat{\mu}\hat{\nu}}\dot{\zeta}^{\mu\nu00} \\
&\quad - \big(J_{00c}^{\hat{0}}\,\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} + 2J_{0c}^{\hat{0}}\,\dot{\mathcal{J}}_{\mu\nu}^{\hat{\mu}\hat{\nu}} + J_{c}^{\hat{0}}\,\ddot{\mathcal{J}}_{\mu\nu}^{\hat{\mu}\hat{\nu}} + \partial_{0c}\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}}\big)\zeta^{\mu\nu c0} - \big(J_{0c}^{\hat{0}}\,\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} + J_{c}^{\hat{0}}\,\dot{\mathcal{J}}_{\mu\nu}^{\hat{\mu}\hat{\nu}} + \partial_{c}\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}}\big)\dot{\zeta}^{\mu\nu c0} \\
&\quad - \Big(\big(\tfrac{1}{2}J_{cd}^{\hat{0}}\,\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} + J_{d}^{\hat{0}}\,\partial_{c}\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}}\big)\zeta^{\mu\nu cd}\Big)^{\cdot} \\
&\quad + \tfrac{1}{2}\,\ddot{\mathcal{J}}_{\mu\nu}^{\hat{\mu}\hat{\nu}}\,\zeta^{\mu\nu00} + \big(\partial_{c}\dot{\mathcal{J}}_{\mu\nu}^{\hat{\mu}\hat{\nu}}\big)\,\zeta^{\mu\nu c0} + \big(\tfrac{1}{2}\partial_{cd}\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}}\big)\,\zeta^{\mu\nu cd} \\
&= \tfrac{1}{2}\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}}\,\ddot{\zeta}^{\mu\nu00} + \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}}\ J_{c}^{\hat{0}}\,\ddot{\zeta}^{\mu\nu c0} + 2\dot{\mathcal{J}}_{\mu\nu}^{\hat{\mu}\hat{\nu}}\ J_{c}^{\hat{0}}\,\dot{\zeta}^{\mu\nu c0} + 2\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}}\ J_{c0}^{\hat{0}}\,\dot{\zeta}^{\mu\nu c0} + \tfrac{1}{2}\Big((\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}}\ J_{d}^{\hat{0}}\,J_{c}^{\hat{0}})\,\zeta^{\mu\nu cd}\Big)^{\cdot\cdot} \\
&\quad - \big(J_{0c}^{\hat{0}}\,\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} + J_{c}^{\hat{0}}\,\dot{\mathcal{J}}_{\mu\nu}^{\hat{\mu}\hat{\nu}} + \partial_{c}\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}}\big)\dot{\zeta}^{\mu\nu c0} - \Big(\big(\tfrac{1}{2}J_{cd}^{\hat{0}}\,\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} + J_{d}^{\hat{0}}\,\partial_{c}\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}}\big)\zeta^{\mu\nu cd}\Big)^{\cdot} + \big(\tfrac{1}{2}\partial_{cd}\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}}\big)\,\zeta^{\mu\nu cd} \\
&= \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}}\,\gamma^{\mu\nu00} + \mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}}\ J_{c}^{\hat{0}}\,\dot{\gamma}^{\mu\nu a0} + \big((\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}}\ J_{c}^{\hat{0}})^{\cdot} - \partial_{c}\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}}\big)\gamma^{\mu\nu c0} \\
&\quad + \tfrac{1}{2}\big((\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}}\ J_{d}^{\hat{0}}\,J_{c}^{\hat{0}})\,\gamma^{\mu\nu cd}\big)^{\cdot\cdot} - \big(\big(\tfrac{1}{2}J_{cd}^{\hat{0}}\,\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}} + J_{d}^{\hat{0}}\,\partial_{c}\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}}\big)\gamma^{\mu\nu cd}\big)^{\cdot} + \big(\tfrac{1}{2}\partial_{cd}\mathcal{J}_{\mu\nu}^{\hat{\mu}\hat{\nu}}\big)\,\gamma^{\mu\nu cd}
\end{aligned}
$$

□

Proof number 13: Proof of (85). For the semi-quadrupole, (83) is automatically satisfied. Equations (80)-(82) become

$$
\dot{\gamma}^{0000} = 0, \quad \dot{\gamma}^{a000} = 0, \quad \dot{\gamma}^{00a0} = -\gamma^{0a00}, \quad \dot{\gamma}^{0(ba)0} = -\gamma^{ba00}, \quad \dot{\gamma}^{0[ba]0} = 0,
$$
$$
\dot{\gamma}^{00ba} = -\gamma^{0(ab)0}, \quad 0 = \dot{\gamma}^{0cba} = -\gamma^{c(ab)0}
$$

It may appear that we have not stated anything about $(\gamma^{cab0} - \gamma^{cba0})$. However due to the symmetry of γ^{cab0} we have

$$
\gamma^{cab0} - \gamma^{cba0} = \gamma^{acb0} - \gamma^{bca0} = -\gamma^{abc0} + \gamma^{bac0} = 0
$$

Thus from the last equation above we have $\gamma^{cba0} = 0$. Setting $\gamma^{00ba} = \kappa^{ab}(\sigma)$ we have $\gamma^{0(ba)0} = \dot{\kappa}^{ab}$ and $\gamma^{ba00} = \ddot{\kappa}^{ab}$. The remaining constants in (85) are then set.

Since there are 22 ODEs, one may expect 22 constants instead of 10. However the remaining 12 arise from the initial values of κ^{ab} and $\dot{\kappa}^{ab}$. □

Proof number 14: Proof of (55) *and* (89). Let φ be a test function. Thus

$$
\int_{\mathcal{M}} \nabla_{\mu}(T^{\mu\nu}\,K_{\nu})\,\varphi\,d^4x = \int_{\mathcal{M}} (\nabla_{\mu}T^{\mu\nu}\,K_{\nu} + T^{\mu\nu}\,\nabla_{\mu}\,K_{\nu}\,\varphi)d^4x = 0
$$

from (6), (7) and (54). Since $T^{\mu\nu}$ is a tensor density then so is $T^{\mu\nu}K_\nu$. Hence

$$
\begin{aligned}
0 &= \int_{\mathcal{M}} \nabla_\mu (T^{\mu\nu}\,K_\nu)\,\varphi d^4x = \int_{\mathcal{M}} T^{\mu\nu}\,K_\nu\,\nabla_\mu\varphi d^4x = \int_{\mathcal{M}} T^{\mu\nu}\,K_\nu\,\partial_\mu\varphi d^4x \\
&= \int_{\mathcal{I}} \Big(\gamma^{\mu\nu 00}\,K_\nu\partial_\mu\varphi - \gamma^{\mu\nu 0a}\,\partial_a(K_\nu\partial_\mu\varphi) + \tfrac{1}{2}\gamma^{\mu\nu ab}\partial_a\partial_b\,(K_\nu\partial_\mu\varphi)\Big)\,d\sigma \\
&= \int_{\mathcal{I}} \Big(\partial_\mu\varphi\big(\gamma^{\mu\nu 00}\,K_\nu - \gamma^{\mu\nu 0a}\,\partial_a K_\nu + \tfrac{1}{2}\gamma^{\mu\nu ab}\partial_a\partial_b\,K_\nu\big)\Big)\,d\sigma + \text{higher derivatives of }\varphi. \\
&= \int_{\mathcal{I}} \Big(\partial_0\varphi\big(\gamma^{0\nu 00}\,K_\nu - \gamma^{0\nu 0a}\,\partial_a K_\nu + \tfrac{1}{2}\gamma^{0\nu ab}\partial_a\partial_b\,K_\nu\big)\Big)\,d\sigma \\
&\qquad + \int_{\mathcal{I}} \Big(\partial_c\varphi\,\big(\gamma^{c\nu 00}\,K_\nu - \gamma^{c\nu 0a}\,\partial_a K_\nu + \tfrac{1}{2}\gamma^{c\nu ab}\partial_a\partial_b\,K_\nu\big)\Big)\,d\sigma + \text{higher derivatives of }\varphi. \\
&= -\int_{\mathcal{I}} \Big(\varphi\;\partial_0\big(\gamma^{0\nu 00}\,K_\nu - \gamma^{0\nu 0a}\,\partial_a K_\nu + \tfrac{1}{2}\gamma^{0\nu ab}\partial_a\partial_b\,K_\nu\big)\Big)\,d\sigma + \text{higher derivatives of }\varphi. \\
&= -\int_{\mathcal{I}} \big(\varphi\;\dot{\mathcal{Q}}_K\big)\,d\sigma + \text{higher derivatives of }\varphi.
\end{aligned}
$$

Thus since we can extract the different derivatives of φ we have $\dot{\mathcal{Q}}_K = 0$.

Clearly for dipoles we have $\gamma^{0\nu ab} = 0$ and have (55). □

A.4 Proofs for section 6

Proof number 15: Proof of (103).

$$
\begin{aligned}
\boldsymbol{\nabla}^2_{U,fV} S &= \boldsymbol{\nabla}_U\,\boldsymbol{\nabla}_{(fV)} S - \boldsymbol{\nabla}_{\boldsymbol{\nabla}_U (fV)} S = \boldsymbol{\nabla}_U\,(f\boldsymbol{\nabla}_V S) - \boldsymbol{\nabla}_{(f\boldsymbol{\nabla}_U V + U\langle f\rangle V)} S \\
&= f\,\boldsymbol{\nabla}_U\,\boldsymbol{\nabla}_V S + U\langle f\rangle\,\boldsymbol{\nabla}_V S - f\boldsymbol{\nabla}_{\boldsymbol{\nabla}_U V} S - U\langle f\rangle\,\boldsymbol{\nabla}_V S = f\boldsymbol{\nabla}^2_{U,V} S
\end{aligned}
$$

□

Proof number 16: Proof of (104).

$$
\begin{aligned}
\boldsymbol{\nabla}^2_{U,V} W &= \boldsymbol{\nabla}_U\,\boldsymbol{\nabla}_V W - \boldsymbol{\nabla}_{\boldsymbol{\nabla}_U V} W = U^\nu \overline{\nabla}_\nu\,(\boldsymbol{\nabla}_V W)^\mu \partial_\mu - (\boldsymbol{\nabla}_U V)^\rho (\overline{\nabla}_\rho W^\mu)\partial_\mu \\
&= U^\nu \overline{\nabla}_\nu\,(V^\rho \overline{\nabla}_\rho W^\mu)\partial_\mu - U^\nu (\overline{\nabla}_\nu V^\rho)(\overline{\nabla}_\rho W^\mu)\partial_\mu \\
&= U^\nu \Big(\overline{\nabla}_\nu\,(V^\rho \overline{\nabla}_\rho W^\mu) - (\overline{\nabla}_\nu V^\rho)(\overline{\nabla}_\rho W^\mu)\Big)\partial_\mu \\
&= U^\nu\,V^\rho\,\Big(\overline{\nabla}_\nu\,\overline{\nabla}_\rho W^\mu\Big)\partial_\mu
\end{aligned}
$$

□

Proof number 17: Proof of (121). Let θ be a test 1–form then

$$
(D\theta)(U,V) = (\boldsymbol{\nabla}_V \theta)(U) = U^\nu (\boldsymbol{\nabla}_V \theta)_\nu = U^\nu\,V^\mu\,\overline{\nabla}_\mu \theta_\nu = (\overline{\nabla}_\nu \theta_\mu)\;(dx^\nu \otimes dx^\mu)(U,V)
$$

hence

$$
D\theta = (\overline{\nabla}_\nu \theta_\mu)\;(dx^\nu \otimes dx^\mu)
$$

Thus

$$
\begin{aligned}
D\tau[\theta] &= -\tau[D\theta] = -\tau[(\overline{\nabla}_\nu \theta_\mu)\;(dx^\nu \otimes dx^\mu)] = -\tau^\mu[(\overline{\nabla}_\nu \theta_\mu)\;dx^\nu] = -\tau^\mu[(\partial_\nu\theta_\mu - \Gamma^\rho_{\nu\mu}\theta_\rho)\,dx^\nu] \\
&= -\tau^\mu[\partial_\nu\theta_\mu\,dx^\nu - \Gamma^\rho_{\nu\mu}\theta_\rho\,dx^\nu] = -\tau^\mu[\partial_\nu\theta_\mu\,dx^\nu] + \tau^\mu[\Gamma^\rho_{\nu\mu}\theta_\rho\,dx^\nu] = -\tau^\mu[d\theta_\mu] + \Gamma^\rho_{\nu\mu}\,dx^\nu \wedge \tau^\mu[\theta_\rho] \\
&= d\tau^\mu[\theta_\mu] + \Gamma^\rho_{\nu\mu}\,dx^\nu \wedge \tau^\mu[\theta_\rho] = \big(d\tau^\rho + \Gamma^\rho_{\nu\mu}\,dx^\nu \wedge \tau^\mu\big)[\theta_\rho]
\end{aligned}
$$

□

Proof number 18: Proof of (123). From (119) and (61) we have

$$\begin{aligned}\tau^\mu[\phi_{\mu\nu}\,dx^\nu] &= \tfrac{1}{2}\,i_\alpha\,L_\rho\,L_\kappa C_\varsigma(\zeta^{\mu\alpha\rho\kappa}d\sigma)[\phi_{\mu\nu}\,dx^\nu] = \tfrac{1}{2}\,L_\rho\,L_\kappa C_\varsigma(\zeta^{\mu\alpha\rho\kappa}d\sigma)[i_\alpha\,\phi_{\mu\nu}\,dx^\nu]\\ &= \tfrac{1}{2}\,\delta^\nu_\alpha L_\rho\,L_\kappa C_\varsigma(\zeta^{\mu\alpha\rho\kappa}d\sigma)[\phi_{\mu\nu}] = \tfrac{1}{2}\,L_\rho\,L_\kappa C_\varsigma(\zeta^{\mu\nu\rho\kappa}d\sigma)[\phi_{\mu\nu}]\\ &= \tfrac{1}{2}\,C_\varsigma(\zeta^{\mu\nu\rho\kappa}d\sigma)[\partial_\rho\,\partial_\kappa\phi_{\mu\nu}] = \tfrac{1}{2}\int_{\mathcal{I}}\zeta^{\mu\nu\rho\kappa}\,(\partial_\rho\,\partial_\kappa\phi_{\mu\nu})\,d\sigma = \int_{\mathcal{I}} T^{\mu\nu}\,\phi_{\mu\nu}\,d^4x\end{aligned}$$

from (61). □

Proof number 19: Proof of (124).

$$\begin{aligned}\tau^\mu[\phi_{\mu\nu}\,dx^\nu] &= i_\alpha\,C_\varsigma(\gamma^{\mu\alpha 00}\,d\sigma)[\phi_{\mu\nu}\,dx^\nu] + i_\alpha\,L_a\,C_\varsigma(\gamma^{\mu\alpha 0a}\,d\sigma)[\phi_{\mu\nu}\,dx^\nu]\\ &\qquad + \tfrac{1}{2}i_\alpha\,L_a\,L_b\,C_\varsigma(\gamma^{\mu\alpha ab}\,d\sigma)[\phi_{\mu\nu}\,dx^\nu]\\ &= C_\varsigma(\gamma^{\mu\nu 00}\,d\sigma)[\phi_{\mu\nu}] + L_a\,C_\varsigma(\gamma^{\mu\nu 0a}\,d\sigma)[\phi_{\mu\nu}] + \tfrac{1}{2}L_a\,L_b\,C_\varsigma(\gamma^{\mu\nu ab}\,d\sigma)[\phi_{\mu\nu}]\\ &= C_\varsigma(\gamma^{\mu\nu 00}\,d\sigma)[\phi_{\mu\nu}] - C_\varsigma(\gamma^{\mu\nu 0a}\,d\sigma)[\partial_a\phi_{\mu\nu}] + \tfrac{1}{2}C_\varsigma(\gamma^{\mu\nu ab}\,d\sigma)[\partial_a\partial_b\,\phi_{\mu\nu}]\\ &= \int_{\mathcal{I}}\gamma^{\mu\nu 00}\,\phi_{\mu\nu}\,d\sigma - \int_{\mathcal{I}}\gamma^{\mu\nu 0a}\,(\partial_a\phi_{\mu\nu})d\sigma + \tfrac{1}{2}\int_{\mathcal{I}}\gamma^{\mu\nu ab}(\partial_a\partial_b\,\phi_{\mu\nu})\,d\sigma\\ &= \int_{\mathcal{I}}\Big(\gamma^{\mu\nu 00}\,\phi_{\mu\nu} - \gamma^{\mu\nu 0a}\,(\partial_a\phi_{\mu\nu}) + \tfrac{1}{2}\gamma^{\mu\nu ab}(\partial_a\partial_b\,\phi_{\mu\nu})\Big)\,d\sigma\end{aligned}$$

□

Proof number 20: Proof of (127) *and Semi-quadrupole counting.* A simple application using $\lambda = \lambda_1 + \lambda_2$ and $\lambda = \lambda_1 - \lambda_2$ implies we can replace (126) for $\ell = 2$ with (126) with $\tau_\alpha[\lambda_1\lambda_2 da] = 0$ where $C^\star(\lambda_1) = C^\star(\lambda_2) = C^\star(a) = 0$.

In an adapted coordinate system (σ, z^1, z^2, z^3) apply this to the test form $z^a\,z^b\,dz^c$ in (124) we see that this leads to the equation

$$\gamma^{\mu cab} = 0 \tag{138}$$

and hence (25).

We can now count the number and type of components. The dynamic equation (72) and (73) remain unchanged but (74) becomes

$$\gamma^{cb0a} = -\Gamma^c{}_{00}\,\gamma^{00ab} \qquad \text{and} \qquad \dot\gamma^{00ab} = -2\gamma^{0(b0)a} - \Gamma^0{}_{00}\,\gamma^{00ab}$$

since the symmetry condition (70) implies $\gamma^{c0ab} = \gamma^{0cab} = 0$. Thus we have 4+12+6=22 ODEs.

Starting with the 100 components given after applying (70) we have $9\times 6 = 54$ constraints coming from (127) plus 18 constraint-es coming from the first equation above. This leaves 28 components left. Of these 22 are given by the ODEs and 6 are free. □

A.5 Lemmas and proofs associated with the Dixon split

In this section we work in a coordinate system (σ, z^1, z^2, z^3), which is adapted both for C and N, so that $N = N_0 d\sigma$ with $N_0 \neq 0$. We see that if $N(V) = 0$ then we can replace V^μ with V^a. Likewise we can replace $\xi^{\mu\nu\rho\kappa}$ with $\xi^{\mu\nu ab}$ since $\xi^{\mu\nu 0a} = \xi^{\mu\nu a0} = 0$.

In this coordinate system a radial vector R has the properties

$$\begin{gathered} R^\mu|_p = 0, \quad \partial_\mu R^0|_p = 0, \quad \partial_\mu R^a|_p = \delta^a_\mu, \quad \partial_0\partial_\mu R^\nu|_p = 0,\\ \partial_b\partial_c R^0|_p = -2\Gamma^a_{bc} \quad \text{and} \quad \partial_b\partial_c R^a|_p = -\Gamma^a_{bc}\end{gathered} \tag{139}$$

for any $p = C(\sigma)$. This can be expressed as

$$R^0 = -z^b z^c\,\Gamma^0_{bc}\,\partial_0 + O(\boldsymbol{z}^3) \qquad \text{and} \qquad R^a = z^a - \tfrac{1}{2} z^b z^c\,\Gamma^a_{bc} + O(\boldsymbol{z}^3) \tag{140}$$

or alternatively as

$$R = z^a\,\partial_a - z^b z^c\,\Gamma^0_{bc}\,\partial_0 - \tfrac{1}{2} z^b z^c\,\Gamma^a_{bc}\,\partial_a + O(\boldsymbol{z}^3) \tag{141}$$

where $O(\boldsymbol{z}^3)$ is any function (or vector) of (σ, z^1, z^2, z^3) which is at least cubic in its z^a arguments.

Proof number 21: Proof of (139). In the adapted coordinate system, assume first that R^μ satisfies (139) and that U, V satisfy $N(U) = N(V) = 0$, so $U^0 = V^0 = 0$.

Clearly from either (133.1) or (139.1) we have $R|_p = 0$. Here (133.1) refers to the first equation in (133).

$$\big(\boldsymbol{\nabla}_V R - V\big)^\mu\big|_p = \big(V^\nu \partial_\nu(R^\mu) + V^\nu R^\rho \Gamma^\mu_{\nu\rho} - V^\mu\big)\big|_p = \big(V^a(\partial_a(R^\mu) - \delta^\mu_a)\big)\big|_p$$

Thus (133.2) is equivalent to (139.2), (139.3). From (139.2) and (139.3) we have (139.4)

From (133.2) we have, (implicitly evaluating at p),

$$\begin{aligned}
\overline{\nabla}_b(\overline{\nabla}_c R^a) &= \partial_b(\overline{\nabla}_c R^a) + (\overline{\nabla}_c R^d)\Gamma^a_{bd} - (\overline{\nabla}_d R^a)\Gamma^d_{bc} \\
&= \partial_b\partial_c R^a + \partial_b(R^e\,\Gamma^a_{ce}) + (\partial_c R^d)\Gamma^a_{bd} + R^e\Gamma^d_{ce}\Gamma^a_{bd} - (\partial_d R^a)\Gamma^d_{bc} - R^e\Gamma^a_{de}\Gamma^d_{bc} \\
&= \partial_b\partial_c R^a + \delta^e_b\,\Gamma^a_{ce} + \delta^d_c\Gamma^a_{bd} - \delta^a_d\Gamma^d_{bc} = \partial_b\partial_c R^a + \Gamma^a_{cb} + \Gamma^a_{bc} - \Gamma^a_{bc} \\
&= \partial_b\partial_c R^a + \Gamma^a_{cb}
\end{aligned}$$

and

$$\begin{aligned}
\overline{\nabla}_b(\overline{\nabla}_c R^0) &= \partial_b(\overline{\nabla}_c R^0) + (\overline{\nabla}_c R^d)\Gamma^0_{bd} - (\overline{\nabla}_d R^0)\Gamma^d_{bc} = \partial_b\partial_c R^0 + \partial_b(R^e\,\Gamma^0_{ce}) + \Gamma^0_{bc} \\
&= \partial_b\partial_c R^0 + \partial_b(R^e)\,\Gamma^0_{ce} + \Gamma^0_{bc} = \partial_b\partial_c R^0 + 2\Gamma^0_{bc}
\end{aligned}$$

Thus

$$\begin{aligned}
\boldsymbol{\nabla}^2_{U,V} R &= V^\mu U^\nu(\partial_\mu\partial_\nu R^a + \Gamma^a_{cb})\partial_a + V^\mu U^\nu(\partial_\mu\partial_\nu R^0 + 2\Gamma^a_{cb})\partial_0 \\
&= V^a U^b(\partial_b\partial_c R^a + \Gamma^a_{cb})\partial_a + V^a U^b(\partial_b\partial_c R^0 + 2\Gamma^a_{cb})\partial_0
\end{aligned}$$

Hence (133.3) if and only if (139.5) and (139.6)

□

Proof number 22: Proof of (136). In the adapted coordinate system and evaluating at $C(\sigma)$ we have

$$\xi^{\mu\nu}(R^\alpha\,R^\lambda\,\overline{\nabla}_\alpha\overline{\nabla}_\lambda\phi_{\mu\nu}) = 0$$

Thus the monopole term (129) does not contribute to $\tau_{(2)}$. Likewise

$$\xi^{\mu\nu\rho\kappa}\overline{\nabla}_\rho(R^\alpha\,R^\lambda\,\overline{\nabla}_\alpha\overline{\nabla}_\lambda\phi_{\mu\nu}) = 0$$

so the dipole term (130) does not contribute to $\tau_{(2)}$. Finally we have

$$\begin{aligned}
\xi^{\mu\nu\rho\kappa}\overline{\nabla}_\rho\overline{\nabla}_\kappa(R^\alpha\,R^\lambda\,\overline{\nabla}_\alpha\overline{\nabla}_\lambda\phi_{\mu\nu}) &= \xi^{\mu\nu ab}\overline{\nabla}_a\overline{\nabla}_b(R^\alpha\,R^\lambda\,\overline{\nabla}_\alpha\overline{\nabla}_\lambda\phi_{\mu\nu}) = \xi^{\mu\nu ab}(\partial_a\partial_b(R^\alpha\,R^\lambda)\,\overline{\nabla}_\alpha\overline{\nabla}_\lambda\phi_{\mu\nu}) \\
&= \xi^{\mu\nu ab}(\delta^\alpha_a\delta^\lambda_b + \delta^\lambda_a\delta^\alpha_b)(\overline{\nabla}_\alpha\overline{\nabla}_\lambda\phi_{\mu\nu}) = 2\xi^{\mu\nu ab}(\overline{\nabla}_a\overline{\nabla}_b\phi_{\mu\nu}) = 2\xi^{\mu\nu\rho\kappa}(\overline{\nabla}_\rho\overline{\nabla}_\kappa\phi_{\mu\nu})
\end{aligned} \tag{142}$$

Thus $\tau_{(2)}$ is given by (136).

□

Proof number 23: Proof of (135). Since

$$\xi^{\mu\nu}(R^\alpha\,\overline{\nabla}_\alpha\,\phi_{\mu\nu} - R^\alpha\,R^\lambda\,\overline{\nabla}_\alpha\overline{\nabla}_\lambda\phi_{\mu\nu}) = 0$$

the monopole term does not contribute to $\tau_{(1)}$. Also

$$\begin{aligned}
\overline{\nabla}_a\overline{\nabla}_b\,(R^\alpha\,\overline{\nabla}_\alpha\,\phi_{\mu\nu}) &= \overline{\nabla}_a\big((\overline{\nabla}_b R^\alpha)\overline{\nabla}_\alpha\,\phi_{\mu\nu}\big) + \overline{\nabla}_a\big(R^\alpha\overline{\nabla}_b\overline{\nabla}_\alpha\,\phi_{\mu\nu}\big)\\
&= (\overline{\nabla}_a\overline{\nabla}_b R^\alpha)\overline{\nabla}_\alpha\,\phi_{\mu\nu} + (\overline{\nabla}_b R^\alpha)\overline{\nabla}_a\overline{\nabla}_\alpha\,\phi_{\mu\nu} + (\overline{\nabla}_a R^\alpha)\overline{\nabla}_b\overline{\nabla}_\alpha\,\phi_{\mu\nu} + R^\alpha\overline{\nabla}_a\overline{\nabla}_b\overline{\nabla}_\alpha\,\phi_{\mu\nu}\\
&= \delta^\alpha_b\overline{\nabla}_a\overline{\nabla}_\alpha\,\phi_{\mu\nu} + \delta^\alpha_a\overline{\nabla}_b\overline{\nabla}_\alpha\,\phi_{\mu\nu} = \overline{\nabla}_a\overline{\nabla}_b\,\phi_{\mu\nu} + \overline{\nabla}_b\overline{\nabla}_a\,\phi_{\mu\nu}
\end{aligned}$$

Hence

$$\begin{aligned}
\xi^{\mu\nu\rho\kappa}\,\overline{\nabla}_\rho\overline{\nabla}_\kappa\,(R^\alpha\,\overline{\nabla}_\alpha\,\phi_{\mu\nu}) &= \xi^{\mu\nu ab}\,\overline{\nabla}_a\overline{\nabla}_b\,(R^\alpha\,\overline{\nabla}_\alpha\,\phi_{\mu\nu}) = \xi^{\mu\nu ab}\,\big(\overline{\nabla}_a\overline{\nabla}_b\,\phi_{\mu\nu} + \overline{\nabla}_b\overline{\nabla}_a\,\phi_{\mu\nu}\big)\\
&= 2\xi^{\mu\nu ab}(\overline{\nabla}_a\overline{\nabla}_b\phi_{\mu\nu}) = 2\xi^{\mu\nu\rho\kappa}(\overline{\nabla}_\rho\overline{\nabla}_\kappa\phi_{\mu\nu})
\end{aligned} \tag{143}$$

Thus using (142) we see

$$\xi^{\mu\nu\rho\kappa}\,\overline{\nabla}_\rho\overline{\nabla}_\kappa\,(R^\alpha\,\overline{\nabla}_\alpha\,\phi_{\mu\nu} - R^\alpha\,R^\lambda\,\overline{\nabla}_\alpha\overline{\nabla}_\lambda\phi_{\mu\nu}) = 0$$

Thus the quadrupole term (131) does not contribute to $\tau_{(1)}$. Finally

$$\begin{aligned}
\xi^{\mu\nu\rho}\,\overline{\nabla}_\rho(R^\alpha\,\overline{\nabla}_\alpha\,\phi_{\mu\nu}) &= \xi^{\mu\nu a}\,\overline{\nabla}_a(R^\alpha\,\overline{\nabla}_\alpha\,\phi_{\mu\nu}) = \xi^{\mu\nu a}\,(\overline{\nabla}_a R^\alpha)\,\overline{\nabla}_\alpha\,\phi_{\mu\nu} = \xi^{\mu\nu a}\,\delta^\alpha_a\,\overline{\nabla}_\alpha\,\phi_{\mu\nu}\\
&= \xi^{\mu\nu a}\,\overline{\nabla}_a\,\phi_{\mu\nu} = \xi^{\mu\nu\rho}\,\overline{\nabla}_\rho\,\phi_{\mu\nu}
\end{aligned} \tag{144}$$

Thus $\tau_{(1)}$ is given by (135). □

Proof number 24: Proof of (134). From (142) and (143) we have

$$\xi^{\mu\nu\rho\kappa}\,\overline{\nabla}_\rho\overline{\nabla}_\kappa\,(\phi_{\mu\nu} - R^\alpha\,\overline{\nabla}_\alpha\,\phi_{\mu\nu} + \tfrac{1}{2}R^\alpha\,R^\lambda\,\overline{\nabla}_\alpha\overline{\nabla}_\lambda\phi_{\mu\nu}) = 0$$

Thus the quadrupole term (131) does not contribute to $\tau_{(0)}$. Using (144) we have

$$\xi^{\mu\nu\rho}\,\overline{\nabla}_\rho(\phi_{\mu\nu} - R^\alpha\,\overline{\nabla}_\alpha\,\phi_{\mu\nu} + \tfrac{1}{2}R^\alpha\,R^\lambda\,\overline{\nabla}_\alpha\overline{\nabla}_\lambda\phi_{\mu\nu}) = 0$$

so the dipole term (130) does not contribute to $\tau_{(0)}$. Finally

$$\xi^{\mu\nu}\,(\phi_{\mu\nu} - R^\alpha\,\overline{\nabla}_\alpha\,\phi_{\mu\nu} + \tfrac{1}{2}R^\alpha\,R^\lambda\,\overline{\nabla}_\alpha\overline{\nabla}_\lambda\phi_{\mu\nu}) = \xi^{\mu\nu}\,\phi_{\mu\nu}$$

so $\tau_{(0)}$ is given by (134). □