

SOLVING EQUATIONS IN DENSE SIDON SETS

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ABSTRACT. We offer an alternative proof of a result of Conlon, Fox, Sudakov and Zhao [CFSZ20] on solving translation-invariant linear equations in dense Sidon sets. Our proof generalises to equations in more than five variables and yields effective bounds.

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1. INTRODUCTION

A set S of integers is a *Sidon* set if the only solutions to the equation

$$x - x' = y - y' \quad (x, x', y, y' \in S) \quad (1.1)$$

are trivial, in the sense that $x = y$ or $x = x'$. Writing $E(S)$ for the number of tuples (x, x', y, y') solving (1.1), a finite set S is Sidon if and only if $E(S) \leq 2|S|^2 - |S|$.

One can show that if $S \subset [N]$ is Sidon then $|S| \leq (1 + o(1))N^{1/2}$, and there are constructions with $|S| \geq (1 - o(1))N^{1/2}$, see [O'B04]. Conlon, Fox, Sudakov and Zhao [CFSZ20] have shown that Sidon sets whose cardinality is within a constant of this range possess arithmetic structure, in that they contain a solution to any translation-invariant linear equation in five variables, with all variables distinct. Furthermore they are able to demonstrate that this structure is also possessed by *almost Sidon* sets, that is sets for which

$$E(S) \leq (2 + o(1))|S|^2.$$

Their results are deduced using a regularity lemma for graphs with few 4-cycles. We use a Fourier-analytic transference principle developed by Helfgott and de Roton [HdR11] to give an alternative proof of this result, generalising to translation-invariant equations in more variables and extracting bounds.

Theorem 1.1. *Let $a_1, \dots, a_s \in \mathbb{Z} \setminus \{0\}$ with $a_1 + \dots + a_s = 0$ and $s \geq 5$. Given $0 < \delta \leq 1$, suppose that $S \subset [N]$ satisfies*

$$|S| \geq \delta N^{1/2} \quad \text{and} \quad E(S) \leq (2 + \eta) |S|^2.$$

Then either

$$N \leq \exp \exp(O_{a_i}(1/\delta)), \quad \text{or} \quad \eta \geq \exp(-\exp(O_{a_i}(1/\delta)))$$

or

$$\sum_{a_1 x_1 + \dots + a_s x_s = 0} \prod_i 1_S(x_i) \geq \exp(-O_{a_i}(1/\delta)) N^{\frac{s}{2}-1}. \quad (1.2)$$

Corollary 1.2. *Let $a_1, \dots, a_s \in \mathbb{Z} \setminus \{0\}$ with $a_1 + \dots + a_s = 0$ and $s \geq 5$. Given $0 < \delta \leq 1$, suppose that $S \subset [N]$ satisfies*

$$|S| \geq \delta N^{1/2} \quad \text{and} \quad E(S) \leq (2 + \eta) |S|^2.$$

If S lacks solutions to the equation

$$a_1 x_1 + \dots + a_s x_s = 0 \quad (1.3)$$

with $x_1, \dots, x_s \in S$ all distinct, then

$$N \leq \exp \exp(O_{a_i}(1/\delta)) \quad \text{or} \quad \eta \geq \exp(-\exp(O_{a_i}(1/\delta))).$$

Corollary 1.3. *If $S \subset [N]$ is a Sidon set lacking solutions to (1.3) with distinct variables then, for $N \geq 3$, we have*

$$|S| = O_{a_i}(N^{1/2}(\log \log N)^{-1}). \quad (1.4)$$

That such results are obtainable is noted in [CFSZ20], along with a path to proving them. We depart from the use of weak arithmetic regularity suggested therein. Instead our argument takes advantage of the fact that a Sidon set behaves very nicely with respect to convolution, so that convolving its indicator function with a suitably chosen Bohr set yields a function whose L^1 and L^2 norms are both comparable to that of a dense set of integers (after appropriate renormalisation). Functions whose L^p -norms behave in this manner are similar enough to dense sets of integers for us to import results from the dense setting to sparse Sidon sets. This observation originates with Helfgott and de Roton [Hdr11].

An attentive reader will observe that our argument gives a superior exponent of $\log \log N$ than that stated in Corollary 1.3. Furthermore the improved exponent grows as the number of variables in (1.3) increases. This is due to our use of a result of Bloom [Blo12] which counts the number of solutions to a translation-invariant equation in a dense set of integers. For equations in four or more variables, there is a more effective density bound due to Schoen and Sisask [SS16]. As is indicated in Sanders' *Mathematical Review*¹ of this paper, one may adapt the argument² to improve Bloom's

¹MR3482282.

²Replace the sum set $A + A$ with a suitable set of popular sums. Thanks to Thomas Bloom and Olof Sisask for pointing this out.

counting result. This then improves (1.4) to

$$|S| = O_{a_i} \left(\frac{N^{1/2}}{\exp\left((\log \log N)^{\Omega(1)}\right)} \right).$$

The author would be very interested in any proof which yields a polylogarithmic bound in Corollary 1.3. For dense sets of integers, all polylogarithmic bounds require some kind of localisation from the interval $[N]$ to a sparser substructure, such as a subprogression or Bohr set. When dealing with sparse sets of integers like Sidon sets, such localisation is lossy, because the sparse set can be even sparser on the substructure. An example to bear in mind is that a subset of $[N]$ of cardinality \sqrt{N} may intersect each subinterval of length \sqrt{N} in at most one point. The author believes that obtaining a polylogarithmic bound in Corollary 1.3 may be a model problem for improving bounds in Roth's theorem in the primes [Gre05, HdR11, Nas15].

Paper Organisation. We prove Theorem 1.1 in §2, assuming three key lemmas. Proving these lemmas occupies §§3–5. We deduce Corollaries 1.2 and 1.3 in §6

Acknowledgements. The author thanks Jonathan Chapman for corrections, Sam Chow for numerous useful conversations, and Yufei Zhao for an inspiring talk in the (online) Stanford Combinatorics Seminar.

Notation.

Standard conventions. We use $[N]$ to denote the interval of integers $\{1, 2, \dots, N\}$. We use counting measure on \mathbb{Z} , so that for $f, g : \mathbb{Z} \rightarrow \mathbb{C}$, we have

$$\|f\|_p := \left(\sum_x |f(x)|^p \right)^{\frac{1}{p}} \quad \text{and} \quad (f * g)(x) := \sum_y f(y)g(x - y).$$

Any sum of the form \sum_x is to be interpreted as a sum over \mathbb{Z} . The *support* of f is the set $\text{supp}(f) := \{x \in \mathbb{Z} : f(x) \neq 0\}$.

We use Haar probability measure on $\mathbb{T} := \mathbb{R}/\mathbb{Z}$, so that for integrable $F : \mathbb{T} \rightarrow \mathbb{C}$, we have

$$\|F\|_p := \left(\int_{\mathbb{T}} |F(\alpha)|^p d\alpha \right)^{\frac{1}{p}} = \left(\int_0^1 |F(\alpha)|^p d\alpha \right)^{\frac{1}{p}}$$

and

$$\|F\|_{\infty} := \sup_{\alpha \in \mathbb{T}} |F(\alpha)|.$$

Write $\|\alpha\|_{\mathbb{T}}$ for the distance from $\alpha \in \mathbb{R}$ to the nearest integer $\min_{n \in \mathbb{Z}} |\alpha - n|$. This remains well-defined on \mathbb{T} .

Definition 1.4 (Fourier transform). For $f : \mathbb{Z} \rightarrow \mathbb{C}$ with finite support define $\hat{f} : \mathbb{T} \rightarrow \mathbb{C}$ by

$$\hat{f}(\alpha) := \sum_{n \in \mathbb{Z}} f(n)e(\alpha n).$$

Here $e(\beta)$ stands for $e^{2\pi i \beta}$.

Asymptotic notation. For a complex-valued function f and positive-valued function g , write $f \lesssim g$ or $f = O(g)$ if there exists a constant C such that $|f(x)| \leq Cg(x)$ for all x . We write $f = \Omega(g)$ if $f \gtrsim g$. The notation $f \asymp g$ means that $f \lesssim g$ and $f \gtrsim g$. We subscript these symbols if the implicit constant depends on additional parameters.

We write $f = o(g)$ if for any $\varepsilon > 0$ there exists $X \in \mathbb{R}$ such that for all $x \geq X$ we have $|f(x)| \leq \varepsilon g(x)$.

Local conventions. As indicated in the introduction, we define the additive energy of a finitely supported function $f : \mathbb{Z} \rightarrow \mathbb{R}$ to be the quantity

$$E(f) := \sum_{x-x'=y-y'} f(x)f(x')f(y)f(y').$$

When $f = 1_S$ is the characteristic function of a finite set $S \subset \mathbb{Z}$ we write $E(S)$. Notice that

$$E(S) = \sum_n r_S(n)^2$$

where

$$r_S(n) := \sum_{n_1 - n_2 = n} 1_S(n_1)1_S(n_2)$$

is the number of representation of n as a difference of elements of S . In the literature this notation is sometimes used for the number of representations as a *sum* of two elements of S .

2. THE TRANSFERENCE ARGUMENT

In this section we prove Theorem 1.1 assuming the following three ingredients.

Lemma 2.1 (L^2 -Bloom). *Let $a_1, \dots, a_s \in \mathbb{Z} \setminus \{0\}$ with $s \geq 5$ and $a_1 + \dots + a_s = 0$. Let $f : I \rightarrow [0, \infty)$ be a function defined on an interval $I \subset \mathbb{Z}$ of length N . If $\sum_n f(n) \geq \delta N$ and $\sum_n f(n)^2 \leq N$ then we have the lower bound*

$$\sum_{a_1 x_1 + \dots + a_s x_s = 0} f(x_1) \cdots f(x_s) \geq \exp(-O_{a_i}(1/\delta)) N^{s-1}.$$

We deduce this from a theorem of Bloom [Blo12] in §3.

Lemma 2.2 (Counting lemma for bounded energy functions). *Let $s \geq 5$ and $a_1, \dots, a_s \in \mathbb{Z} \setminus \{0\}$. Let $\nu : I \rightarrow [0, \infty)$ be a function defined on an interval $I \subset \mathbb{Z}$ of length N . Suppose that*

$$\sum_n \nu(n) \leq N \quad \text{and} \quad E(\nu) \leq N^3.$$

Then for any $|f_i| \leq \nu$ we have

$$\left| \sum_{a_1 x_1 + \dots + a_s x_s = 0} f_1(x_1) \cdots f_s(x_s) \right| \leq N^{s-1} \frac{\min_i \|\hat{f}_i\|_\infty}{\|\hat{1}_{[N]}\|_\infty}.$$

This is proved in §4.

Lemma 2.3 (Dense model for almost-Sidon sets). *Let $0 \leq \eta \leq 1$ and suppose that $S \subset [N]$ satisfies*

$$|S| \geq \delta N^{1/2} \quad \text{and} \quad E(S) \leq (2 + \eta) |S|^2.$$

Then for any $0 < \varepsilon \leq \min\{\frac{1}{2}, \delta\}$ there exists $f : (-\varepsilon N, (1 + \varepsilon)N] \rightarrow [0, \infty)$ such that all of the following hold

- $\sum_n f(n) = N^{1/2} |S|$;
- $\|\hat{f} - N^{1/2} \hat{1}_S\|_\infty \leq \varepsilon N$;
- $\sum_n f(n)^2 \leq N [1 + (\eta + N^{-1/2}) (1 - \eta)^{-1} \exp(\varepsilon^{-O(1)})]$.

The above constitutes the main idea in our approach and is proved in §5.

Proof of Theorem 1.1. We may assume that $\eta \leq 1/2$, for the second possible conclusion of the theorem is that η is large. Let us apply Lemma 2.3, with ε to be chosen. This gives $f : (-\varepsilon N, (1 + \varepsilon)N] \rightarrow [0, \infty)$ satisfying $\sum_n f(n) \geq \delta N$, $\|\hat{f} - N^{1/2} \hat{1}_S\|_\infty \leq \varepsilon N$ and

$$\sum_n f(n)^2 \leq N [1 + (\eta + N^{-1/2}) \exp(\varepsilon^{-O(1)})]. \quad (2.1)$$

By (2.1), either $\sum_n f(n)^2 \leq 2N$ or one of the following two possibilities holds

$$\eta \geq \exp(-\varepsilon^{-O(1)}) \quad \text{or} \quad N \leq \exp(\varepsilon^{-O(1)}). \quad (2.2)$$

Notice that $(-\varepsilon N, (1 + \varepsilon)N]$ is an interval of length at most $2N$. Hence, assuming that neither option in (2.2) holds, Lemma 2.1 gives that

$$\sum_{a_1 x_1 + \dots + a_s x_s = 0} f(x_1) \cdots f(x_s) \geq \exp(-O_{a_i}(1/\delta)) N^{s-1}.$$

Define $\nu := f + N^{1/2} \mathbf{1}_S$. We claim that, provided we divide through by a suitable absolute constant, the function ν satisfies the hypotheses of

Lemma 2.2 on the interval $I = (-\varepsilon N, (1 + \varepsilon)N]$. By the triangle inequality in L^4 , and the Fourier-analytic interpretation of energy, we have

$$\begin{aligned} E(\nu)^{1/4} = \|\hat{\nu}\|_4 &\leq \|\hat{f}\|_4 + N^{1/2} \|\hat{1}_S\|_4 \leq \|f\|_1^{1/2} \|\hat{f}\|_2^{1/2} + N^{1/2} E(S)^{1/4} \\ &\lesssim N^{1/4} \|f\|_2 + N^{1/2} N^{1/4} \lesssim N^{3/4}. \end{aligned}$$

Assuming that neither option in (2.2) holds, we compare Fourier coefficients at zero to deduce that

$$\begin{aligned} \sum_n \nu(n) = \hat{f}(0) + N^{1/2} \hat{1}_S(0) &\leq 2\hat{f}(0) + \varepsilon N \\ &\leq 2(2N)^{1/2} \left(\sum_n f(n)^2 \right)^{1/2} + \varepsilon N \lesssim N. \end{aligned}$$

We may therefore apply Lemma 2.2 together with a telescoping identity to deduce that

$$\left| \sum_{a_1 x_1 + \dots + a_s x_s = 0} \left(\prod_i f(x_i) - \prod_i N^{1/2} 1_S(x_i) \right) \right| \lesssim_s \varepsilon N^{s-1}.$$

Hence either we deduce (1.2), or one of the following holds

- $\varepsilon \geq \exp(-O_{a_i}(1/\delta))$;
- $\eta \geq \exp(-\varepsilon^{-O(1)})$;
- $N \leq \exp(\varepsilon^{-O(1)})$.

We obtain the result on taking ε sufficiently small to preclude the first possibility. \square

3. RESULTS ON DENSE SETS OF INTEGERS

Theorem 3.1 (Bloom [Blo12]). *Let $a_1, \dots, a_s \in \mathbb{Z} \setminus \{0\}$ with $s \geq 5$ and $a_1 + \dots + a_s = 0$. Then for any $A \subset [N]$ with $|A| \geq \delta N$ we have the lower bound*

$$\sum_{a_1 x_1 + \dots + a_s x_s = 0} 1_A(x_1) \cdots 1_A(x_s) \geq \exp\left(-O_{a_i, \varepsilon}\left(\delta^{-\frac{1}{s-2}-\varepsilon}\right)\right) N^{s-1}.$$

Proof of Lemma 2.1. Translating, we may assume that $I = [N]$. Define

$$A := \{x \in [N] : f(x) \geq \delta/2\}.$$

Then, employing the Cauchy–Schwarz inequality, we have

$$\begin{aligned} \delta N &\leq \sum_x f(x) = \sum_{x \notin A} f(x) + \sum_{x \in A} f(x) \\ &\leq \frac{1}{2} \delta N + |A|^{1/2} \left(\sum_x f(x)^2 \right)^{1/2} \\ &\leq \frac{1}{2} \delta N + (|A|N)^{1/2}. \end{aligned} \tag{3.1}$$

Therefore

$$|A| \geq \frac{\delta^2}{4} N. \quad (3.2)$$

Applying Theorem 3.1 we deduce that

$$\begin{aligned} \sum_{a_1 x_1 + \dots + a_s x_s = 0} f(x_1) \cdots f(x_s) &\geq (\delta/4)^s \sum_{a_1 x_1 + \dots + a_s x_s = 0} 1_A(x_1) \cdots 1_A(x_s) \\ &\gtrsim \delta^s \exp(-O_{a_i}(\delta^{-0.8})) N^{s-1}. \quad \square \end{aligned}$$

4. AN ALMOST-SIDON COUNTING LEMMA

Proof of Lemma 2.2. For any finitely supported $f : \mathbb{Z} \rightarrow \mathbb{C}$ and $a \in \mathbb{Z} \setminus \{0\}$ we have

$$\int_{\mathbb{T}} |\hat{f}(a\alpha)|^{s-1} d\alpha \leq \left(\sum_n |f(n)| \right)^{s-5} \int_{\mathbb{T}} |\hat{f}(a\alpha)|^4 d\alpha.$$

If $|f| \leq \nu$, then

$$\sum_n |f(n)| \leq \sum_n \nu(n) \leq N.$$

By orthogonality

$$\begin{aligned} \int_{\mathbb{T}} |\hat{f}(a\alpha)|^4 d\alpha &= \sum_{x-x'=y-y'} f(x) \overline{f(x')} \overline{f(y)} f(y') \\ &\leq \sum_{x-x'=y-y'} \nu(x) \nu(x') \nu(y) \nu(y') \leq N^3. \end{aligned}$$

Therefore

$$\int_{\mathbb{T}} |\hat{f}(a\alpha)|^{s-1} d\alpha \leq N^{s-2}.$$

Again by orthogonality, together with Hölder's inequality

$$\begin{aligned} \left| \sum_{a_1 x_1 + \dots + a_s x_s = 0} f_1(x_1) \cdots f_s(x_s) \right| &= \left| \int_{\mathbb{T}} \hat{f}_1(a_1 \alpha) \cdots \hat{f}_s(a_s \alpha) d\alpha \right| \\ &\leq \|\hat{f}_i\|_{\infty} \prod_{j \neq i} \left(\int_{\mathbb{T}} |\hat{f}_j(a_j \alpha)|^{s-1} d\alpha \right)^{\frac{1}{s-1}} \leq \|\hat{f}_i\|_{\infty} N^{s-2}. \quad \square \end{aligned}$$

5. A MODELLING LEMMA FOR ALMOST-SIDON SETS

We begin our proof of Lemma 2.3 with two subsidiary results on almost Sidon sets.

Lemma 5.1. *Let $S \subset [N]$ satisfy*

$$E(S) := \sum_{x-x'=y-y'} 1_S(x) 1_S(x') 1_S(y) 1_S(y') \leq (2 + \eta) |S|^2.$$

Then, on writing

$$r_S(n) := \sum_{n_1 - n_2 = n} 1_S(n_1) 1_S(n_2),$$

we have

$$\sum_{\substack{r_S(n) > 1 \\ n \neq 0}} r_S(n) \leq \eta |S|^2 + |S|.$$

Proof. We observe that

$$\begin{aligned} \sum_{\substack{r_S(n) > 1 \\ n \neq 0}} r_S(n) &\leq \sum_{n \neq 0} r_S(n)(r_S(n) - 1) = \sum_{n \neq 0} r_S(n)^2 - \sum_{n \neq 0} r_S(n) \\ &\leq (1 + \eta) |S|^2 - (|S|^2 - |S|) = \eta |S|^2 + |S|. \quad \square \end{aligned}$$

Lemma 5.2. *Let $\eta \in [0, 1)$ and suppose that $S \subset [N]$ satisfies*

$$E(S) := \sum_{x-x'=y-y'} 1_S(x)1_S(x')1_S(y)1_S(y') \leq (2 + \eta) |S|^2.$$

Then

$$|S| \leq 2 \left(\frac{N}{1-\eta} \right)^{1/2}.$$

Proof. Using Lemma 5.1 we have

$$|S|^2 = \sum_{\substack{r_S(n) \leq 1 \\ n \neq 0}} r_S(n) + \sum_{\substack{r_S(n) > 1 \\ n \neq 0}} r_S(n) + |S| \leq 2N + \eta |S|^2 + 2|S|. \quad \square$$

We are now in a position to prove Lemma 2.3 in earnest.

Proof of Lemma 2.3. Define the large spectrum of S to be the set

$$\text{Spec}(S, \varepsilon) := \{ \alpha \in \mathbb{T} : |\hat{1}_S(\alpha)| \geq \varepsilon |S| \}.$$

Define the Bohr set

$$B := \{ n \in [-\varepsilon N, \varepsilon N] : \|n\alpha\|_{\mathbb{T}} \leq \varepsilon \quad \forall \alpha \in \text{Spec}(S, \varepsilon) \}. \quad (5.1)$$

Write μ_B for the normalised characteristic function of B , so that

$$\mu_B := |B|^{-1} 1_B.$$

Then we define

$$f := N^{1/2} 1_S * \mu_B, \quad (5.2)$$

where, for finitely supported f_i , we set

$$f_1 * f_2(n) := \sum_{m_1+m_2=n} f_1(m_1) f_2(m_2).$$

It is straightforward to check that f is supported on $(-\varepsilon N, (1 + \varepsilon)N]$ and that $\sum_n f(n) = N^{1/2} |S|$. Let us next estimate $|N^{1/2} \hat{1}_S - \hat{f}|$. The key identity is

$$\widehat{f_1 * f_2} = \hat{f}_1 \hat{f}_2,$$

so that $|N^{1/2} \hat{1}_S - \hat{f}| = N^{1/2} |\hat{1}_S| |1 - \hat{\mu}_B|$.

If $\alpha \notin \text{Spec}(S, \varepsilon)$ then we have

$$|N^{1/2} \hat{1}_S(\alpha) - \hat{f}(\alpha)| = N^{1/2} |\hat{1}_S(\alpha)| |1 - \hat{\mu}_B(\alpha)| \leq 2N^{1/2} \varepsilon |S|.$$

If $\alpha \in \text{Spec}(S, \varepsilon)$, then for each $n \in B$ we have $e(\alpha n) = 1 + O(\varepsilon)$. Hence $\hat{\mu}_B(\alpha) = 1 + O(\varepsilon)$, and consequently

$$|N^{1/2}\hat{1}_S(\alpha) - \hat{f}(\alpha)| = N^{1/2}|\hat{1}_S(\alpha)||1 - \hat{\mu}_B(\alpha)| \lesssim N^{1/2}|S|\varepsilon.$$

Combining both cases and Lemma 5.2 gives

$$\|N^{1/2}\hat{1}_S - \hat{f}\|_\infty \lesssim \varepsilon N. \quad (5.3)$$

We have

$$\sum_n f(n)^2 = N|B|^{-2} \sum_{n_1 - n_2 = m_1 - m_2} 1_S(n_1)1_S(n_2)1_B(m_1)1_B(m_2). \quad (5.4)$$

Write

$$r_S(n) := \sum_{n_1 - n_2 = n} 1_S(n_1)1_S(n_2).$$

Then by Lemma 5.1, the inner sum in (5.4) is

$$\begin{aligned} \sum_n r_S(n)r_B(n) &\leq |B|^2 + |B| \sum_{\substack{r_S(n) > 1 \\ n \neq 0}} r_S(n) + |B||S| \\ &\leq |B|^2 + (\eta|S|^2 + 2|S|)|B|. \end{aligned}$$

Using the estimate $|S| \lesssim (1-\eta)^{-1/2}N^{1/2}$ afforded by Lemma 5.2, it remains to establish the lower bound

$$|B| \geq \exp(-\varepsilon^{O(1)})N. \quad (5.5)$$

Let $\alpha_1, \dots, \alpha_R$ be a maximal $(1/N)$ -separated subset of $\text{Spec}(S, \varepsilon)$. Since every element of $\text{Spec}(S, \varepsilon)$ is within $1/N$ of some α_i , one can check that

$$B \supset \{n \in [-\varepsilon N/2, \varepsilon N/2] : \|\alpha_i n\|_{\mathbb{T}} \leq \varepsilon/2 \quad \forall i = 1, \dots, R\}. \quad (5.6)$$

Hence the argument proving the standard lower bound for Bohr sets (e.g. [TV06, Lemma 4.2]) gives

$$|B| \geq \lceil 4/\varepsilon \rceil^{1+R} N.$$

By the large sieve inequality (e.g. [Vau97, Lemma 5.3]) we have

$$R\varepsilon^4|S|^4 \leq \sum_{i=1}^R |\hat{1}_S(\alpha_i)|^4 \lesssim N \sum_n r_S(n)^2 \lesssim N(2+\eta)|S|^2.$$

Hence $R \lesssim \delta^{-2}\varepsilon^{-4}$. □

6. PROOF OF COROLLARIES

Proof of Corollary 1.2. Let us first obtain an upper bound for the number of solutions in S to the equation

$$a_1x_1 + \dots + a_sx_s = 0.$$

By our hypotheses, all such solutions should have $x_i = x_j$ for some $i \neq j$. At the cost of a factor of $\binom{s}{2}$, we may assume that $x_{s-1} = x_s$. Writing

$n := a_4x_4 + \dots + a_sx_s$, the number of choices for the remaining three variables is at most

$$\begin{aligned} \sum_{a_1x_1+a_2x_2+a_3x_3=-n} 1_S(x_1)1_S(x_2)1_S(x_3) &= \int_{\mathbb{T}} e(\alpha n) \prod_{i=1}^3 \hat{1}_S(a_i x_i) d\alpha \\ &\leq \prod_{i=1}^3 \left(\int_{\mathbb{T}} |\hat{1}_S(a_i \alpha_i)|^3 d\alpha \right)^{\frac{1}{3}} \leq \prod_{i=1}^3 \left(\int_{\mathbb{T}} |\hat{1}_S(a_i \alpha_i)|^4 d\alpha \right)^{\frac{1}{4}} = E(S)^{\frac{3}{4}}. \end{aligned}$$

We may assume that $\eta \leq 1/2$ (otherwise we are done). Using Lemma 5.2 we deduce that the number of choices for x_1, x_2, x_3 is $O(N^{3/4})$. Since there are $N^{\frac{s-4}{2}}$ choices for the remaining variables, we deduce that

$$\sum_{a_1x_1+\dots+a_sx_s=0} \prod_i 1_S(x_i) \lesssim_s N^{\frac{s}{2}-\frac{5}{4}}.$$

We deduce the result on comparing this with the lower bound given in Theorem 1.1 \square

We obtain Corollary 1.3 from Corollary 1.2 since, for a Sidon set S , we have $E(S) \leq (2 + \eta)|S|$ with $\eta = 0$.

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