Limits of weighted graphs

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Declaration

Hereby I declare that the present thesis was prepared by me and none of its contents was obtained by means that are against the law.

I also declare that the present thesis is a part of a PhD Programme at Lancaster University. The thesis has never before been a subject of any procedure of obtaining an academic degree.

The thesis contains research carried out jointly with my supervisor, Professor Gábor Elek.

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Abstract

This thesis concerns the actions of countable groups and associated Schreier graphs. In Chapters 1 and 2 we give the motivation and overview for the research presented in this thesis and we establish the basics regarding group actions, especially about Schreier graphs and amenability. Furthermore, we recall the idea of equationally compact actions of groups defined by Banaschewski. Finally, we show two results about equationally compact subgroups of infinite groups which answer two questions of Rajani and Prest.

We start off Chapter 3 with recalling the construction of a space of rooted Schreier graphs which are associated with the actions of a group. A crucial notion related to the space of rooted Schreier graphs is that of a Benjamini-Schramm convergence of sequences of sparse graphs, which has connections with measure-preserving actions of groups. We are, however, particularly interested in actions which only preserve the measure class, i.e. the non-singular actions of groups. Let us notice that for such actions the classical theorem of Radon-Nikodym can be applied, which equips the graph structure on the space with an additional function on the edges which forms a cocycle. Thus, drawing inspiration from the space of rooted Schreier graphs, we construct a space of rooted Schreier cocycles of a group. Similarly as in the measure-preserving case, we obtain a correspondence between the space of cocycles and non-singular actions of groups.

In the final chapter of the thesis the central notion is that of hyperfiniteness, which has strong ties to amenability. The definition of hyperfiniteness varies between the settings of sequences of graphs, graphings of the actions of groups and for equivalence

relations. Broadly speaking, an object is hyperfinite if it is in some sense close to being finite. Thus, a sequence of graphs is hyperfinite if we can remove sets of arbitrarily small size relative to the size of graphs in such a way that the resulting objects have components of bounded size. On the other hand, a measure preserving group action yields an associated structure of a graphing on the space that it acts upon. If we can remove an arbitrarily small set from the probability space in such a way that the resulting graphing has bounded components then we call the action hyperfinite. In fact, these two notions of hyperfiniteness are strongly connected: by the theorem of Schramm, a measure preserving action is hyperfinite if and only if a sequence of graphs convergent to it is hyperfinite.

We consider a weighted version of hyperfiniteness, one which is suitable for this setting and we obtain a similar result to that of Schramm's in Chapter 4, namely that a limit action of a hyperfinite sequence of cocycles is hyperfinite. Finally, we find continuous actions which are isomorphic to a given Borel action and have the same Radon-Nikodym cocycle and we obtain examples of free continuous actions of exact groups with continuous Radon-Nikodym derivatives.

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Chapter 1

Introduction

In this thesis we study graphs with weighted vertices and their connections with actions of finitely generated groups on standard Borel spaces. Our main goal is to define a notion of convergence of sequences of weighted graphs which is analogous to the local convergence of graphs (without weights) introduced by Benjamini and Schramm [3]. Furthermore, we aim to show that the proposed notion of convergence has interesting properties, particularly for hyperfinite sequences of graphs.

Recall that an uncountable standard Borel space $(X, \mathcal{B})$ (where $\mathcal{B}$ denotes the Borel σ-algebra on X) is one which is Borel isomorphic to the Cantor set $\mathcal{C}$ or to the interval $[0, 1]$ with the euclidean topology (Chapter 3 in [35]). We are particularly interested in actions of finitely generated groups on a standard Borel space $(X, \mathcal{B})$ which preserve the Borel σ-algebra. Each such action induces an orbit equivalence relation on X which is Borel, i.e. it is a Borel subset of the product $X \times X$. Conversely, the Feldman-Moore theorem [15] states that each countable Borel equivalence relation E on X arises as an orbit equivalence relation of a Borel action of some countable discrete group.

Later we will also consider Borel measures on X. Observe that for a Borel space $(X, \mathcal{B})$ and a Borel measure μ on X, any Borel actions on X are also μ-measurable.

1.1 Orbits, stabilizers and the space of subgroups

Given a countably infinite group Γ, we consider the set 2^{Γ} consisting of all 0-1-valued functions on Γ endowed with the topology of pointwise convergence. Notice that 2^{Γ} is homeomorphic to the Cantor set $\mathcal{C}$. There exists a bijection h between $\mathcal{P}(\Gamma)$ – the power set of Γ – and 2^{Γ}, defined as $h(A) = \chi_A$, where χ_A denotes the characteristic function of A. The map h induces a topology on $\mathcal{P}(\Gamma)$ which makes h a homeomorphism. This topology can be metrized with the Hausdorff metric. We define $\mathrm{Sub}(\Gamma) \subseteq \mathcal{P}(\Gamma)$ to be the space of all subgroups of Γ endowed with the subspace topology.

Proposition 1.1.1. *The space* $\mathrm{Sub}(\Gamma)$ *is a closed subset of* $\mathcal{P}(\Gamma)$ *and so it is a compact space.*

Proof. Let $\{G_n\}_{n=1}^{\infty}$ be a convergent sequence of subgroups of Γ and suppose its limit is a subset G of Γ. We will show that G is a subgroup of Γ. By the definition of the topology on $\mathcal{P}(\Gamma)$, a point $x \in \Gamma$ is an element of G if and only if it lies in all but finitely many G_n's. Hence $e_{\Gamma} \in G$ as all G_n are groups. Now, if $g, h \in G$ then for all but finitely many n we have $g, h, h^{-1} \in G_n$ so $gh^{-1} \in G$ as well. $\square$

Let us notice that a group Γ has a natural left action on the space of its subgroups by conjugations, $H^g = g^{-1}Hg$. Now let the countable group Γ act on an arbitrary set X. For a point $x \in X$, its orbit $O(x)$ is the union of the images of x under the action of Γ. Furthermore, we may consider the map Stab:$X \to \mathrm{Sub}(\Gamma)$ which maps each point to its stabilizer, that is $\mathrm{Stab}(x) = \{\gamma \in \Gamma : \gamma \cdot x = x\}$. Notice that the map Stab is Γ-equivariant, that is for any $\gamma \in \Gamma$ and an $x \in X$ we have $\mathrm{Stab}(\gamma \cdot x) = \mathrm{Stab}(x)^{\gamma}$.

In Section 2.2 we study the notion of equational compactness of an action of a group which was introduced by Banaschewski in [2].

Definition 1.1.2. An action $\alpha : \Gamma \curvearrowright X$ is **equationally compact** if, for any set $S \subseteq \Gamma$ the following condition is satisfied: if for every finite subset T of S there is a

point fixed by all elements of T then there is a point fixed by every element of S.
A subgroup $H \subseteq \Gamma$ is called equationally compact if the action of Γ on the set Γ/H of left cosets of H on Γ is equationally compact.

Rajani and Prest in [31] asked two questions regarding equationally compact subgroups (which they call the subgroups with pure-injective property), to which we provide answers. Firstly, we show that in the group of finitary permutations of $\mathbb{N}$, denoted S^0_∞, the only equationally compact subgroups are the finite ones and the alternating subgroup A^0_∞ (Section 2.2.3). Secondly, we give examples of equationally compact actions of a countable group such that none of the stabilizers of points are equationally compact subgroups.

1.2 Schreier coset graphs

Let us fix a generating system Σ of Γ and let α be an action of Γ on a set X. Then we can put the structure of a directed graph G_α on X in the following way. We put the set of vertices of G_α to be X and the set of edges to be consisting of pairs $(x, s \cdot x)$ for $x \in X$ and $s \in \Sigma$. The graph G_α is connected if and only if the action α is transitive, i.e. if for every $x, y \in X$ there is a $\gamma \in \Gamma$ such that $\alpha(\gamma)(x) = y$.
For a subset $A \subseteq \Gamma$, we define A^{-1} to be the set of the inverses of elements from A. Throughout this thesis, we will assume that the generating system Σ of Γ is **symmetric**, that is $\Sigma = \Sigma^{-1}$. Note that in the case where Σ is symmetric, for each edge (x, y), (y, x) is also an edge. Thus, the graph G_α can also be viewed as an undirected graph.
Now we can define the Schreier coset graphs, which are central to this thesis. Let us fix a symmetric generating system Σ of Γ and a subgroup $H < \Gamma$. Then the **Schreier coset graph** associated to Γ and H with respect to Σ is defined as $S(\Gamma, H, \Sigma) := (V, E, l)$ with V, E, l defined as follows:

- V, the set of vertices, is the set Γ/H of left cosets of H in Γ.

- We note that Γ acts on Γ/H by left multiplication. Similarly as before we obtain a structure of a graph on Γ/H with respect to the generating system Σ. E is the set of edges of this graph.

- Furthermore, in these graphs we also wish to associate each edge e with a label $l(e)$ which 'remembers the generator of the edge'. This means that $l : E \to \mathcal{P}(\Sigma)$ is a function assigning to each edge (aH, bH) the set of those generators $s \in \Sigma$ for which $saH = bH$ (for a set Σ, $\mathcal{P}(\Sigma)$ denotes its power set).

We denote the set of Schreier coset graphs of Γ by $\Gamma\mathcal{G}'$. For shortness, we will write $S(H)$ when it is clear what group and its generating system we consider.
The notion of a Schreier graph was introduced in 1927 and can be viewed as a generalization of a Cayley graph of a group Γ with respect to a generating system Σ. Indeed, the Cayley graph $\mathrm{Cay}(\Gamma, \Sigma)$ arises as the Schreier graph $S(\Gamma, \{e\}, \Sigma)$, where e is the neutral element of Γ. Although we may consider the Schreier graphs of arbitrary finitely generated groups Γ, we show in Section 3.1 that it is often enough for to discuss Schreier graphs of the free groups of finite rank r, denoted $\mathbb{F}_r$.
Now, let α be an action of Γ on a set X. Then, for any element $x \in X$ we may consider its stabilizer $\mathrm{Stab}(x) = \{\gamma \in \Gamma : \alpha(\gamma)(x) = x\}$. Since stabilizers of points are subgroups of Γ, each point x has an associated Schreier coset graph $S(\mathrm{Stab}(x))$. Then for any point $x \in X$, the graph $S(\mathrm{Stab}(x))$ is isomorphic to the component of G_α which contains x.

1.2.1 Convergence of Schreier graphs

In this section we recall the definition of the local convergence of Schreier graphs. This notion of convergence was introduced in 2001 by Benjamini and Schramm [3] for the sparse graph sequences. These are such graph sequences $\{G_n\}_{n=1}^{\infty}, G_n = (V_n, E_n)$, for which there exists a constant C such that

$$\frac{|E_n|}{|V_n|} \leq C.$$

In the case of Schreier graphs of finitely generated groups with respect to a finite generating system Σ, the degree of each vertex of a graph from $\Gamma\mathcal{G}'$ is bounded by $|\Sigma|$. Thus, for any graph $G = (V, E, l) \in \Gamma\mathcal{G}'$ we have that $|E| \leq \frac{|\Sigma|}{2}|V|$, so any sequence of graphs in $\Gamma\mathcal{G}'$ is sparse.
In order to define convergence of sequences of graphs, first we need to consider rooted graphs.

Definition 1.2.1. A **rooted Schreier graph** is a pair (G, v) where G is a Schreier graph and v is a distinguished vertex of G, also called the root.

The set of all rooted Schreier graphs can be endowed with a metric which induces a totally disconnected compact topology. We denote this metric space by $\Gamma\mathcal{G}$; we discuss it in more detail in Section 3.1.
Let us fix a sequence $\{G_n\}_{n=1}^{\infty}, G_n = (V_n, E_n, l_n)$ of finite Γ-Schreier graphs such that $|V_n| \to \infty$ and consider any finite rooted Γ-Schreier graph H. We set p_n^H to be the probability that if we uniformly randomly pick a vertex v in G_n, then there is a rooted isomorphic copy of H in G_n with root v. Hence, for any rooted Schreier graph H of Γ we have defined a sequence of real numbers $\{p_n^H\}_{n=1}^{\infty}$. If for every H we have that $\{p_n^H\}_{n=1}^{\infty}$ is convergent, then we say that $\{G_n\}_{n=1}^{\infty}$ is **Benjamini-Schramm convergent** or **locally convergent** [3]. From now on we will only consider sequences $\{G_n\}_{n=1}^{\infty}$ for which $|V_n| \to \infty$, as if $|V_n|$ is bounded and the sequence $\{G_n\}_{n=1}^{\infty}$ is convergent, then from some point on it is constant.
It is not immediately clear how a limit object of a convergent sequence of graphs can

be represented. In order to establish that, we will think about the Schreier graphs a bit differently. If G is a finite Schreier graph of Γ, then it induces a probability measure μ_G on $\Gamma\mathcal{G}$ by

$$\mu_G = \frac{1}{|G|} \sum_{v \in V(G)} \delta_{(G,v)},$$

where $\delta_{(G,v)}$ denotes the Dirac's delta measure supported on the rooted graph $(G, v) \in \Gamma\mathcal{G}$. Therefore, any sequence of finite Γ-Schreier graphs G_n induces a sequence of measures μ_n in the space of Borel probability measures $\mathrm{Prob}(\Gamma\mathcal{G})$. If the sequence $\{G_n\}_{n=1}^{\infty}$ converges, then also $\{\mu_n\}_{n=1}^{\infty}$ weakly converges to some measure μ as $n \to \infty$. Furthermore, any sequence $\{H_n\}_{n=1}^{\infty}$ which converges to a different limit (i.e. such that the mixed sequence does not converge) induces a sequence of probability measures $\{\mu_n'\}_{n=1}^{\infty}$ which do not weakly converge to μ. Hence the weak limit μ of the probability measures μ_n represents the limit of the sequence of graphs $\{G_n\}_{n=1}^{\infty}$.

In Section 3.1.1 we consider Schreier graphs with vertices colored with elements of some space Q. Furthermore, we introduce a notion of conergence on the set of such colored graphs, denoted by $\Gamma\mathcal{G}^Q$. Later, we show that for any locally convergent sequence of Schreier graphs and any positive integer R we can find such a number Q and a Q-coloring of the sequence that in the resulting colored graphs, any two vertices which are at distance at most R from each other have different colors. Furthermore, the colored sequence is locally convergent (Section 4.1).

1.3 Amenability and hyperfiniteness

In 1929 John von Neumann introduced the concept which would later be known as amenability. A finitely generated group Γ is **amenable** if it admits a finitely additive probability measure which is invariant under left multiplication (see Chapter 5 in [22]). Furthermore, an arbitrary countable group is amenable if all of its finitely generated subgroups are amenable. Examples include all finite, abelian and solvable groups and groups of subexponential growth. Classic non-examples are the free groups $\mathbb{F}_r$ of rank $r > 1$ and infinite groups with Kazhdan's property (T). The von Neumann conjecture stated that any non-amenable group necessarily contains a copy of $\mathbb{F}_2$ as a subgroup. The conjecture was first disproved by Ol'šanskiĭ in 1980 [28]: he constructed a Tarski monster group which does not contain $\mathbb{F}_2$. Multiple other examples were found since then.

The notion of amenability arose from von Neumann's studies of the Banach-Tarski paradox. It states that a three dimensional unit ball can be divided into finitely many parts which then can be reassembled into two identical unit balls. The proof of this counterintuitive statement relies on the Axiom of Choice and the fact that the free group $\mathbb{F}_2$ is paradoxical. Following Kechris and Miller [22], we say that a group is **paradoxical** provided that there exist two disjoint subsets A, B of $\mathbb{F}_2$, a positive integer n, partitions $A_1, \ldots, A_n$, $B_1, \ldots, B_n$ and $C_1, \ldots, C_n$ of A, B, Γ respectively and elements $\gamma_1, \ldots, \gamma_n, \delta_1, \ldots, \delta_n$ of $\mathbb{F}_2$ such that for every $k \in \{1, \ldots, \}, \gamma_k A_k = C_k = \delta_k B_k$. In fact, Tarski later proved that a group is paradoxical if and only if it is non-amenable.

There is a number of other properties of groups which were shown to be equivalent to the definition of amenability given by von Neumann (see e.g. [22], Chapter 5). The Følner condition, which links amenability of a group to its (left) Cayley graph, is the most relevant to this thesis. Having fixed a finite generating system Σ for a group Γ, we will say that for a set $F \subseteq \Gamma$ its (left) **boundary** is defined as $\partial F = \{\gamma \in F :$ there exists an $s \in \Sigma$ such that $s\gamma \notin F\}$. The set ∂F may be viewed as the boundary of the set F in the Cayley graph of Γ with respect to Σ.

Definition 1.3.1. We say that Γ **satisfies the Følner condition** if there exists a sequence $\{F_n\}_{n=1}^{\infty}$ of finite subsets of Γ such that

$$\frac{|\partial F_n|}{|F_n|} \xrightarrow{n\to\infty} 0.$$

The sequence $\{F_n\}_{n=1}^{\infty}$ is called a **Følner sequence** of Γ.

Note that the Cayley graph of Γ depends on the generating system Σ. Thus, it is a priori possible that the existence of Følner sequences depend on the generating system as well. However, since any two Cayley graphs of Γ are quasi-isometric, we have that any Følner sequence for a group Γ does not depend on the choice of the generating system.
Furthermore, since the Følner condition depends only on the Cayley graphs of the groups, it can be extended to any graph. Thus, it is justified to say that an arbitrary graph is amenable if it posesses a Følner sequence.

1.3.1 Amenable equivalence relations

We are particularly interested in actions of groups on probability measure spaces. Firstly, we should recall another condition equivalent to amenability of a group. A finitely generated group Γ satisfies the Reiter's condition if it is possible to define a sequence of approximate invariant means on Γ. In relation to that, an equivalence relation E on an uncountable standard probability space (X, μ) is called amenable if it is possible to choose a sequence of approximate E-invariant means on each class of E in "a measurable way" ([22], Chapter 9). We shall say that a Γ action on a Borel probability space (X, μ) is amenable if the associated orbit equivalence relation is amenable. It is known that every measure preserving action of an amenable group is amenable, while non-amenable groups have actions which are amenable as well as ones which are not. In particular, a free measure preserving action of a non-amenable group is non-amenable.

On any Borel equivalence relation E we may put a structure of a Borel graph. For example, such a graph structure may come from the action of a group which is given by the Feldman-Moore theorem. It is known that for an amenable equivalence relation E and any graph structure G on E the connected components of G are amenable graphs. Kaimanovich ([21]) gave examples of non-amenable equivalence relations with measurable graph structures such that on each orbit the associated countable graph satisfies the Følner condition. In [20], he provided a condition which has to be satisfied by any Borel and bounded graph structure G generating E which is equivalent to the amenability of E. Here, G is a bounded graph structure if the vertices of G have bounded degrees and the Radon-Nikodym derivatives associated with the edges of G are bounded as well. In Section 4.2.2 we formally state this condition and we provide an alternative proof of its equivalence with the amenability of E.

1.3.2 Hyperfiniteness

One of the central notions in this thesis is hyperfiniteness. It can be defined in various settings, e.g. for Borel equivalence relations or for Γ-graphings. We will define hyperfinitess in the latter setting but first we should recall the notion of a Γ-graphing. For a measure preserving action α of the group Γ on a standard probability space (X, μ), we can define a graph structure G_α with respect to a generating system Σ of Γ as in Section 1.2. Then G_α is the graphing of the action α and **Γ-graphing** is any such G_α. The difference between a graphing and a graph on uncountably many vertices is that a graphing preserves the measure on the space of vertices. Thus, in order to distinguish graphs from graphings, we shall denote the latter by $\mathcal{G}, \mathcal{H}$ as opposed to G, H.

Definition 1.3.2. For a Γ-graphing $\mathcal{G}$ on a Borel probability measure space (X, μ) we say that $\mathcal{G}$ is **hyperfinite** if and only if for any $\epsilon > 0$ there exist a positive integer K and a μ-measurable set $Z \subseteq X$ such that

- $\mu(Z) \leq \epsilon$,
- if we denote $\mathcal{G}'$ to be the subgraphing of $\mathcal{G}$ induced on the set $X \setminus Z$, then every component of $\mathcal{G}'$ has at most K vertices.

Hyperfinite Γ-graphings are strictly related to amenable actions of Γ. The theorem of Connes-Feldman-Weiss (Theorem 10.1 in [22]) implies that any hyperfinite Γ-graphing is amenable.

1.3.3 Hyperfinite families of graphs

Elek introduced hyperfiniteness in the case of families of finite graphs in [9]. We shall recall this definition now.

Definition 1.3.3. Let us consider a group Γ with a finite symmetric generating system Σ. A set of graphs $\mathcal{A} \subseteq \Gamma\mathcal{G}'$ is **hyperfinite** if for every $\epsilon > 0$ there exists a positive integer K such that for every graph $G \in \mathcal{A}$ there is a set $Z \subseteq V(G)$ such that $\frac{|Z|}{|G|} < \epsilon$ and every component of $G - Z$ has at most K vertices.

We will say that a sequence $\{G_n\}_{n=1}^{\infty}$ is hyperfinite if and only if the set $\{G_n : n = 1, 2, \ldots\}$ is hyperfinite. Examples of such families are the families of graphs with excluded minors (e.g. planar graphs) and families with a fixed polynomial growth. Non-examples include expander families. A family $\mathcal{A}$ of sparse graphs is an expander family if there exists an $\epsilon > 0$ such that for any $G \in \mathcal{A}$ and any subset $A \subseteq V(G)$ with $|V(G)| \leq \frac{1}{2}|A|$ we have that $\frac{|\partial A|}{|A|} \geq \epsilon$. Such families were constructed e.g. by Lubotzky, Phillips and Sarnak [26] and Margulis [27].

It was proved by Schramm in 2008 ([34]) that the definitions of hyperfiniteness of graphings and of families of graphs are linked, which is illustrated in the following theorem.

Theorem 1 (Schramm, 2008)**.** *Let $\mathcal{G}$ be a Γ-graphing with respect to a fixed generating system Σ and let $\{G_n\}_{n=1}^{\infty}$ be a sequence of finite Γ-Schreier graphs with respect to Σ, which is locally convergent to $\mathcal{G}$. Then $\mathcal{G}$ is a hyperfinite graphing if and only if the sequence $\{G_n\}_{n=1}^{\infty}$ is hyperfinite.*

An alternate proof of that theorem was given by Elek in [10]. Let us note that both of Elek and Schramm actually proved Theorem 1 for sequences of simple graphs, which is a more general case than the Schreier graphs considered in this thesis.

1.4 Weighted graphs and non-singular actions

1.4.1 Multiplicative cocycles and weighted graphs

A weighted graph is a pair (G, w) consisting of a Schreier graph $G = (V, E, l)$ together with a weight function $w : V \to (0, \infty)$. In Section 3.2 we define the local convergence of weighted graphs. However, in order to do that we first introduce the Schreier cocycles.

Definition 1.4.1. For a group Γ with a finite generating system Σ, a **Schreier cocycle** C is a pair (G, r), where $G = (V, E, l) \in \Gamma\mathcal{G}'$ and $r : E \to (0, \infty)$ is a function satisfying the following condition:

$$\text{if } \overrightarrow{e_1}, \overrightarrow{e_2}, \ldots, \overrightarrow{e_n} \text{ is a directed cycle in } G, \text{ then } r(\overrightarrow{e_1}) \cdot \ldots \cdot r(\overrightarrow{e_n}) = 1. \tag{1.1}$$

The condition 1.1 is called the **cocycle identity**.

The cocycles considered here are multiplicative, as they will turn out to be associated to cocycles arising from Radon-Nikodym derivatives, which are known to be multiplicative. In particular, the cocycle identity implies that

- for any loop (x, x), $r(x, x) = 1$,
- for any edge (x, y), $r(y, x) = r(x, y)^{-1}$.

We say that a weighted graph (G, w) is normal if $\sum_{v \in V} w(v) = 1$. Observe that there is a correspondence between Schreier cocycles and normal weighted Schreier graphs. In order to obtain a structure of a cocycle on a weighted graph (G, w), we put $r_w(x, y) := \frac{w(y)}{w(x)}$. Clearly, r defined in this way satisfies the cocycle identity. On the other hand, given a function r on the edges of G which satisfies the cocycle identity, we can retrieve a normal weight function w on the vertices of G in the following way. We start by picking any vertex v of G and we inductively define a function $w' : V(G) \to (0, \infty)$ by setting

- $w'(v) = 1$,
- for any vertex t of G, if t has a neighbor t', for which $w'(t')$ is already defined, then we set $w'(t) = w'(t')r(\overrightarrow{t't})$.

The cocycle identity implies that the function w' is well-defined. Then we put w to be the normalization of w', i.e. we put $w(x) = \frac{w'(x)}{\sum_{v \in V(G)} w'(v)}$ for every vertex x, in order to obtain $\sum_{v \in V(G)} w(v) = 1$.

We denote the set of rooted Schreier cocycles of a group Γ by $C\Gamma\mathcal{G}$. In Section 3.1.2 we endow the set $C\Gamma\mathcal{G}$ with a locally compact topology, similar to the topology on $\Gamma\mathcal{G}$. Now, for a normal weighted graph (G, w), the function w allows us to introduce a probability measure $\mu_{(G,w)}$ on the space $C\Gamma\mathcal{G}$. We formally describe this measure in Section 3.2 and we define the convergence of a sequence of weighted graphs to be the convergence of their associated measures. Woever, unlike in the case of the local convergence of graphs, the underlying space of rooted cocycles is not totally disconnected. Because of this, the convergence of weighted graphs cannot be simply defined in terms of local statistics; an additional condition is required (see Section 3.2.4).

Furthermore, we discuss hyperfiniteness for cocycles. Given a sequence $\{(G_n, r_n)\}_{n=1}^{\infty}$ of finite Schreier cocycles with associated weight functions w_n, we call $\{(G_n, r_n)\}_{n=1}^{\infty}$ hyperfinite if for any $\epsilon > 0$ there exists a positive integer K and sets Z_n of vertices of G_n such that

- the total weight $w(Z_n) := \sum_{v \in Z_n} w(v)$ is less than ϵ,
- the components of $G_n - Z_n$ have at most K vertices.

1.4.2 Non-singular actions of groups

It is known that actions of groups which preserve a given measure are associated with limits of graphs. However, we will be interested in group actions which do not necessarily preserve the measure itself but rather the measure class.

Definition 1.4.2. An action α of a group Γ on a Borel probability space (X, μ) is called **non-singular** if for any element $\gamma \in \Gamma$, the measures μ and $\alpha(\gamma)_*\mu$ (the pushforward of μ under the map induced by γ) have the same families of null-sets. In such a case, measure μ is called **quasi-invariant** under the action α.

The definition 1.4.2 implies that for the pushforward measures indued by non-singular actions the assumptions of the Radon-Nikodym theorem hold (Theorem 32.2 in [5]). Therefore, for any $\gamma \in \Gamma$ there exists a positive Radon-Nikodym derivative $\frac{d\alpha(\gamma)_*\mu}{d\mu}$. It turns out that the Radon-Nikodym derivatives define multiplicative cocycles on the orbits of the action (Section 2.1.3). We obtain that any non-singular action α of

Γ on a Borel probability measure space (X, μ) induces a map M_α from X to the space of rooted Γ cocycles (see 3.2.1). Any measure ν which is a pushforward of a quasi-invariant measure μ under the map M_α is called a Quasi-invariant Radon-Nikodym Cocycle (QRC for short).

Moreover, we define hyperfiniteness for quasi-invariant group and we then prove a result which is a non-singular analogue of one of the implications in Schramm's theorem (Theorem 1). Namely, in Section 4.2 we show that when we are given a tight sequence of Weighted Generalized Schreier Graphs (see Section 3.2.3) which converges to a non-singular action α, then the weighted hyperfiniteness of the sequence implies the hyperfiniteness of the limit.

Then, in Section 4.3, we prove a non-singular version of a recent theorem of Lovász ([25]). The result that we present asserts that for any non-singular Borel action α of a finitely generated group Γ on a probability space X with a bounded Radon-Nikodym cocycle, we can find a continuous action of Γ on a totally disconnected compact space K and an equivariant map of X into K which induces a measure on K with the same cocycle as α. Furthermore, the cocycle on K is continuous. In the final part of the thesis we show that any exact group admits a free non-singular action on the Cantor set such that the Radon-Nikodym derivatives of this action are continuous.

Chapter 2

Preliminaries

2.1 Actions of groups and their graphings

2.1.1 Borel actions and hyperfiniteness

Let Γ be a countable group acting on the uncountable standard Borel space X with an action α. Since the closed interval $[0,1]$ is Borel isomorphic to X, we can assume $X = [0,1]$. Moreover, if A, B are separable complete metric spaces and $U \subset A$, $V \subset B$ are uncountable open subsets, then there exists a Borel isomorphism $\varphi : U \to V$ (see e.g., Chapter 3 in [35]). An action $\alpha : \Gamma \curvearrowright X$ is Borel if for any $\gamma \in \Gamma$ and Borel set $U \subset X$, the subset $\alpha(\gamma)[U]$ is Borel as well. It is not hard to see that the map $\mathrm{Stab}_\alpha : X \to \mathrm{Sub}(\Gamma)$, assigning to each point in X its stabilizer, is always a Borel map. We define the **orbit equivalence relation of** α, denoted E_α, by $xE_\alpha y$ if there exists $\gamma \in \Gamma$ such that $\alpha(\gamma)(x) = y$. Note that if Γ is a countable group, then each class of E_α is countable. We call such an equivalence relation countable. Furthermore, if α is a Borel action, then E_α is a **Borel equivalence relation**, that is, the pairs (x, y) for which $xE_\alpha y$ form a Borel subset of the set $X \times X$. Conversely, let $E \subset X \times X$ be a countable Borel equivalence relation. Then, by the Feldman-Moore theorem there

exists a countable group Γ and a Borel action $\alpha : \Gamma \curvearrowright X$ such that $E = E_\alpha$ ([15]).
We call a Borel equivalence relation E **finite** if all the classes of E are finite. A countable Borel equivalence relation E is called **Borel hyperfinite** if there exist finite Borel equivalence relations $E_1 \subset E_2 \subset \ldots$ such that $\cup_{n=1}^{\infty} E_n = E$. It is known that if Γ is a countable nilpotent group and $\alpha : \Gamma \curvearrowright X$ is a Borel action, then E_α is hyperfinite. This follows from the Jackson-Kechris-Louveau theorem (Theorem 11.1 in [22]), since nilpotent groups have polynomial growth. It is conjectured that if Γ is an amenable group (see Section 1.3) and $\beta : \Gamma \curvearrowright X$ is a Borel action, then E_β is always hyperfinite ([36]).

2.1.2 Measurable actions

Let Γ be a countable group and $\alpha : \Gamma \curvearrowright (X, \mu)$ be a Borel action of Γ on a standard probability space (X, μ) preserving the measure μ. Then, we call α a **measurable action** and μ an **invariant measure** under the action α. Now, let E be a countable equivalence relation on the Borel space X. We say that a Borel isomorphism T of X is full if for any $x \in X$ there exists $\gamma \in \Gamma$ such that $T(x) = \alpha(\gamma)(x)$. Furthermore, for a Borel equivalence relation E on X, a Borel probability measure μ on X is E**-invariant** if for any full isomorphism T and Borel set $U \subset X$ we have $\mu(T[U]) = \mu(U)$. If $E = E_\alpha$ for some action α, then μ is E-invariant if and only if μ is invariant under the action α. In this case we call E a measurable countable equivalence relation (or shortly a measurable equivalence relation, as all groups under consideration are countable).
We say that a measurable equivalence relation E is μ**-hyperfinite** (also hyperfinite μ-almost everywhere, see Chapter 6 in [22]) if there exists a Borel subset $Y \subset X$ such that

- Y is a union of some E-equivalence classes of X,
- $\mu(X \backslash Y) = 0$,

- $E \cap (Y \times Y)$ is a hyperfinite Borel equivalence relation.

It follows from the Ornstein-Weiss theorem ([29], also see [22]) that if $\alpha : \Gamma \curvearrowright (X, \mu)$ is an action of an amenable group Γ preserving the probability measure μ, then E_α is always μ-hyperfinite.

2.1.3 Nonsingular actions and the Radon-Nikodym cocycle

Let Γ be a countable group and $\alpha : \Gamma \curvearrowright X$ be a Borel action on the standard Borel space X. Let μ be a probability measure and suppose that for any $\gamma \in \Gamma$ and set $Y \subset X$ of measure zero, the set $\alpha(\gamma)[Y]$ has measure zero as well. Then we say that the action α is **nonsingular** or that α **preserves the measure class of** μ. In this case, the measure μ is called **quasi-invariant under** α. Note that for any element $\gamma \in \Gamma$ we have the push-forward measure $\gamma_* \mu$ on X defined by

$$\gamma_* \mu(A) = \mu(\alpha(\gamma)^{-1}[A])$$

for all Borel subsets $A \subseteq X$.

For a measure space (X, μ) let $\mathcal{N}(\mu)$ denote the family of all nullsets of the measure μ. Recall that if μ, ν are measures on a set X and $\mathcal{N}(\mu) \subseteq \mathcal{N}(\nu)$, then μ is said to be absolutely continuous with respect to ν (denoted by $\nu \ll \mu$). The following is a classic theorem in measure theory which will be crucial to us.

Theorem 2 (Radon-Nikodym, see e.g. [5])**.** *Let $(X, \mathcal{B})$ be a Borel space and let μ, ν be σ-finite Borel measures on X such that $\mu \ll \nu$. Then there exists a function $f : X \to (0, \infty)$ such that for any set A which is measurable with respect to μ the following holds:*

$$\mu(A) = \int_A f d\nu.$$

The function f in the theorem is called the **Radon-Nikodym derivative of** μ **with respect to** ν (denoted by $\frac{d\mu}{d\nu}$) and is determined uniquely up to a nullset. Furthermore, the Radon-Nikodym theorem implies that for any μ-integrable function $g : X \to \mathbb{R}$ we have

$$\int_X g d\mu = \int_X g \cdot \frac{d\mu}{d\nu} d\nu.$$

Notice that in the case of a Borel action α of Γ on X, the Radon-Nikodym derivatives

$$\frac{d(\gamma_* \mu)}{d\mu}$$

exist for all $\gamma \in \Gamma$ if and only if μ is quasi-invariant. We call them the Radon-Nikodym derivatives of the action α. Observe that $\frac{d(\gamma_* \mu)}{d\mu} = 1$ holds for all $\gamma \in \Gamma$ if and only if α preserves the probability measure μ.

If $\alpha : \Gamma \curvearrowright (X, \mu)$ is nonsingular, then we have a Γ-invariant Borel set $X_0 \subset X$ and for any $\gamma \in \Gamma$ we have Borel functions (the Radon-Nikodym derivatives of the action of γ with respect to μ) $\frac{d\gamma_* \mu}{d\mu}$ satisfying the following conditions:

- $\mu(X \backslash X_0) = 0$;
- for any Borel set $A \subset X_0$

$$\mu(\gamma[A]) = \int_A \frac{d\gamma_* \mu}{d\mu}(z) d\mu(z);$$

- for all pairs $\gamma, \delta \in \Gamma$ and $x \in X$ we have the identity

$$\frac{d(\gamma\delta)_*(\mu)}{d\mu}(x) = \frac{d\gamma_* \mu}{d\mu}(\alpha(\delta)(x)) \frac{d\delta_* \mu}{d\mu}(x).$$

The final condition above implies that the function $R(\gamma, x) = \frac{d\gamma_* \mu}{d\mu}(x)$ forms a Borel cocycle (the Radon-Nikodym cocycle of the action, see e.g. Chapter 8 in [22]).

2.1.4 Graphings and subgraphings

Assume that Γ is a finitely generated group with a finite symmetric generating set Σ and let $\alpha : \Gamma \curvearrowright (X, \mu)$ be a nonsingular action. Similarly as in Section 1.3.2, for each $x \in X$, we can consider the orbit graph $O_G(x) = (V, E)$, where

- V, the set of vertices, is the orbit of x;
- $(x, y) \in E$ if $\alpha(\sigma)(x) = y$ for some generator σ.

Note that $O_G(x)$ is an undirected graph (later we will consider directed orbit graphs as well). The union of all the orbit graphs is called the **graphing** of α and denoted by $\mathcal{G}_\alpha$. Let $Y \subset X$ be a Borel subset. Then, the restriction of $\mathcal{G}_\alpha$ to Y, denoted $\mathcal{G}_\alpha^Y$, is called a subgraphing.

We can equivalently define the μ-hyperfiniteness of the action using its graphing as well.

Definition 2.1.1. The action α is **hyperfinite** (or its graphing $\mathcal{G}_\alpha$ is hyperfinite) if for any $\epsilon > 0$ there exists $K > 0$ and a Borel set $Z \subset X$ such that $\mu(Z) < \epsilon$ and all the components of the subgraphing $\mathcal{G}_\alpha^{X \setminus Z}$ are of size at most K (see e.g. [10]).

2.2 Equationally compact subgroups

The results in this section appeared in a joint paper with Gábor Elek in the form of a preprint [12].

2.2.1 Continuous actions

Let K be a compact metric space and Γ be a countable group. An action $\alpha : \Gamma \curvearrowright K$ is a **continuous action** if for any $\gamma \in \Gamma$, $\alpha(\gamma)$ is a homeomorphism. Hence, any continuous action is also a Borel action. If Γ is amenable, then all continuous actions $\alpha : \Gamma \curvearrowright K$

admit an invariant measure. If Γ is nonamenable, then there exist continuous actions of Γ that do not admit invariant measures.

We may ask the following question: if K is a compact metric space equipped with a metric d and $\alpha : \Gamma \curvearrowright K$ is a continuous action, then is the map $\mathrm{Stab}_\alpha : K \to \mathrm{Sub}(\Gamma)$ continuous? It turns out that the answer is not always positive, which is pictured in the following example.

Example 1. Consider $\mathbb{Z}$ acting on $2^{\mathbb{Z}}$ by the shift, i.e. for $\sigma \in 2^{\mathbb{Z}}, k, n \in \mathbb{Z}, n \cdot \sigma(k) = \sigma(k-n)$, where $2^{\mathbb{Z}}$ is equipped with the canonical metric d defined as

$$d(\sigma, \tau) = \sum_{n \in \mathbb{Z}} 2^{-n} |\sigma(n) - \tau(n)|.$$

For this action the stabilizer map is not continuous. To see this, consider the sequence $\{\sigma_n : n \in \mathbb{N}\}$ defined as

$$\sigma_n(k) = 1 \Leftrightarrow n = k.$$

Then clearly we have that for all n, $\mathrm{Stab}(\sigma_n) = \{0\}$ but $\sigma_n \to \overline{0}$ (the constant sequence which takes value 0 everywhere) and $\mathrm{Stab}(\overline{0}) = \mathbb{Z}$.

This example can be extended to show that the stabilizer map is not continuous for the Bernoulli shift of any infinite group. The following proposition classifies when the stabilizer map is continuous.

Proposition 2.2.1. *Suppose a countable group $\Gamma = \{\gamma_i : i \in \mathbb{N}\}$ acts on a compact metric space (K, d). Then the following conditions are equivalent.*

1. *The map* Stab: $K \to \mathrm{Sub}(\Gamma)$ *is continuous.*

2. *For any $x \in K$ and $\gamma \in \Gamma$ if $\gamma \cdot x = x$ then there is a neighborhood U of x such that for any $y \in U$ we have $\gamma \cdot y = y$.*

Proof. 1. $\Leftarrow$ 2.: Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in K converging to some $x \in K$. For any i by (2) there exists some $N_i \in \mathbb{N}$ such that for $n > N_i$ we have that $\gamma_i \in \mathrm{Stab}(x_n)$ if and only if $\gamma_i \in \mathrm{Stab}(x)$. We may also assume that the sequence $\{N_i : i \in \mathbb{N}\}$ is increasing. Hence the sequence $\{\mathrm{Stab}(x_n)\}_{n=1}^{\infty}$ is convergent and its limit is $\mathrm{Stab}(x)$.
1. $\Rightarrow$ 2.: Suppose that there is an $x \in K$, a sequence $\{x_n\}_{n=1}^{\infty}$ in K tending to x and $\gamma \in \Gamma$ such that $\gamma \cdot x = x$ but for all $n \in \mathbb{N}, \gamma \cdot x_n \neq x_n$. Then $\mathrm{Stab}(x_n)$ does not converge to $\mathrm{Stab}(x)$ and so the map Stab is not continuous. $\square$

The actions that satisfy the property 2. in Proposition 2.2.1 are called **stable**. Furthermore, an action of Γ on K is called **free** if for all $x \in K$ the stabiliser $\mathrm{Stab}(x)$ is trivial.

2.2.2 Equational compactness

Let Γ be a countable group acting on a set X by permutations. We denote the fixed point set of $\gamma \in \Gamma$ by $\mathrm{Fix}(\gamma)$. Following Banaschewski [2], we say that a Γ-action is equationally compact if for any subset S of Γ and for any finite subset T of S, $\cap_{s \in T} \mathrm{Fix}(s) \neq \emptyset$, then $\cap_{s \in S} \mathrm{Fix}(s) \neq \emptyset$. A subgroup H of Γ is equationally compact (or PIP, [31]) if the left action of Γ on Γ/H by multiplication is equationally compact. Observe that this is equivalent to Definition 1.1.2.
For a countable group Γ, let $\{0,1\}^{\Gamma}$ be the set of characteristic functions of all subsets of Γ with the product topology. Then the set of all subgroups of Γ forms a closed subspace of $\{0,1\}^{\Gamma}$ invariant under conjugation. If $\alpha : \Gamma \curvearrowright X$, then the set $T_\alpha := \{\mathrm{Stab}(x) : x \in X\}$ is an invariant subspace of $\mathrm{Sub}(\Gamma)$. Then, we have the following proposition.

Proposition 2.2.2. *Let Γ be a countable group and let X be a compact metric space.*

1. *An action $\alpha : \Gamma \curvearrowright X$ is equationally compact if and only if for any subgroup K in the closure of T_α, there exists a subgroup $L \in T_\alpha$ such that $K \subseteq L$.*

2. *Any subgroup $H < \Gamma$ is equationally compact if for any K in the orbit closure of H, there exists $L < \Gamma$ conjugate to H such that $K \subseteq L$.*

Proof. 1. Let us assume that α is an equationally compact action. Let $K = \{\gamma_1, \gamma_2, \dots\}$ be a subgroup of Γ in the closure of T_α. For sets $H_k = \{\gamma_1, \dots, \gamma_k\}, k = 1, 2, \dots$ we have that there is some $x_k \in X$ such that $H_k \subseteq \mathrm{Stab}(x_k)$. By equational compactness of α there exists an $x \in X$ fixed by some $H \supseteq K$, $H \in T_\alpha$.
Now, assume that for any K in the closure of T_α, there exists $H \in T_\alpha$ containing K. Let $S = \{\gamma_1, \gamma_2, \dots\} \subseteq \Gamma$ be such that for any $k \geq 1$, we have $x_k \in \mathrm{Fix}(\gamma_i)$ provided that $i \leq k$. Then, let K be in the set of limit points of $\{\mathrm{Stab}(x_k)\}_{k=1}^{\infty}$. By our assumption, there exists $x \in X$ such that $K \subset \mathrm{Stab}(x)$. Since $S \subseteq K$, $S, \subseteq \mathrm{Stab}(x)$. Therefore, the action is equationally compact.
2. Observe that for any subgroup $H < \Gamma$ and element $\gamma \in \Gamma$, $\mathrm{Stab}(\gamma H) = \gamma H \gamma^{-1}$. This part of the Proposition follows immediately from this observation and part 1. □

The previous result immediately gives rise to multiple examples of equationally compact subgroups. This is pictured in the following corollary.

Corollary 2.2.3. *A subgroup $H \subset \Gamma$ is equationally compact if any of the following three conditions hold:*

- *H is a finite extension of a normal subgroup (in particular, if H is finite or normal);*
- *the normalizer subgroup of H has finite index in Γ;*
- *H is malnormal, i.e. for all $\gamma, \delta \in \Gamma$ either $\gamma H \gamma^{-1} = \delta H \delta^{-1}$ or the intersection of $\gamma H \gamma^{-1}$ and $\delta H \delta^{-1}$ is trivial.*

On the other hand, Banaschewski proved that the free group $\mathbb{F}_\infty$ on countably infinitely many generators has non-equationally compact subgroups ([2], Proposition 6.).

We will answer two queries of Prest and Rajani ([31]) concerning equational compactness by proving the following two theorems.

Theorem 3. *The only equationally compact subgroups of the finitary symmetric group* S^0_∞ *on* $\mathbb{N}$ *are the finite subgroups and the group of even permutations* A^0_∞.

Theorem 4. *There exists a countable group* Γ *acting on a set* X*, such that the action is equationally compact, but for any* $x \in X$ $\mathrm{Stab}(x)$ *is not an equationally compact subgroup of* Γ.

2.2.3 Equationally compact subgroups of the finitary symmetric group

Let $S^0_\infty = \cup_{n=1}^\infty S_n$ be the finitary symmetric group on the natural numbers. That is, S^0_∞ is the group of permutations on $\mathbb{N}$ fixing all but finitely many elements. The goal of this subsection is to prove Theorem 3 by showing that the list of equationally compact subgroups of S^0_∞ contains only the set of finite groups and the alternating subgroup A^0_∞ consisting of even permutations. Before getting into the proof let us fix some notations. Let $S^0_{[l,\infty]} \subset S^0_\infty$ be the subgroup consisting of permutations fixing the set $\{1, 2, \dots, l-1\}$. Let $A^0_{[l,\infty]} = S^0_{[l,\infty]} \cap A^0_\infty$. For a permutation $\gamma \in S^0_\infty$, we define $s(\gamma)$ as the maximum of k's for which $\gamma(k) \neq k$.

Proposition 2.2.4. *Let* H *be an equationally compact subgroup of* S^0_∞*. Then one of the following two conditions is satisfied.*

1. *There exists* $l \geq 0$ *such that* $H \cap S^0_{[l,\infty]} = \{e\}$.
2. *There exists* $l \geq 0$ *such that* $A^0_{[l,\infty]} \subset H$.

Proof. Let $H \subset S^0_\infty$ be a subgroup such that neither of the two conditions above are satisfied and set $\kappa_1 H \kappa_1^{-1}, \kappa_2 H \kappa_2^{-1}, \dots$ to be an enumeration of the conjugates of H. Recursively, we will define sequences $\{\gamma_n\}_{n=1}^\infty, \{\delta_n\}_{n=1}^\infty$ in S^0_∞ such that

- $\{\delta_1, \delta_2, \ldots, \delta_n\} \subset \gamma_n H \gamma_n^{-1}$.

- $\delta_n \notin \kappa_n H \kappa_n^{-1}$.

The existence of such sequences implies that the subgroup H cannot be equationally compact. Suppose that $\{\gamma_i\}_{i=1}^n, \{\delta_i\}_{i=1}^n$ has already been constructed and for any $1 \leq i \leq n$

- $\{\delta_1, \delta_2, \ldots, \delta_i\} \subset \gamma_i H \gamma_i^{-1}$.

- $\delta_i \notin \kappa_i H \kappa_i^{-1}$.

Let

$$l = \max(\max_{1 \leq i \leq n} s(\gamma_i), \max_{1 \leq i \leq n} s(\delta_i), s(\kappa_{n+1})) + 1\,.$$

Since a conjugacy class always generates a normal subgroup, there exists a non-unit conjugacy class C of $S^0_{[l,\infty]}$ such that $H \cap C$ is a proper subset of C. Let $\delta_{n+1} \in C \backslash H, \rho_{n+1} \in H \cap C$. Then, we have $\gamma \in S^0_{[l,\infty]}$ such that $\gamma \rho_{n+1} \gamma^{-1} = \delta_{n+1}$. Set $\gamma_{n+1} = \gamma\gamma_n$. By the definition of l, we have that γ commutes with $\{\gamma_i\}_{i=1}^n, \{\delta_i\}_{i=1}^n$ and κ_{n+1}, also, δ_{n+1} commutes with κ_{n+1} hence

- $\delta_{n+1} \notin \kappa_{n+1} H \kappa_{n+1}^{-1}$,

- $\delta_i \in \gamma_{n+1} H \gamma_{n+1}^{-1}$, whenever $1 \leq i \leq n+1$.

Therefore, H is not equationally compact. □

Lemma 2.2.5. *If H is equationally compact and contains $A^0_{[l,\infty]}$ for some $l > 0$, then either $H = S^0_\infty$ or $H = A^0_\infty$.*

Proof. If $A^0_{[l,\infty]} \subseteq H$, then for any $k \geq 1$ there exists a conjugate of H, $\gamma H \gamma^{-1}$ such that the subgroup A_k is contained in $\gamma H \gamma^{-1}$. Hence, if H is equationally compact, then some conjugate K of H must contain the whole group $A^0_\infty = \cup_{k=1}^\infty A_k$. Since A^0_∞

is a normal subgroup of S^0_∞, any conjugate of K contains A^0_∞. In particular, $H \supseteq A^0_\infty$. Therefore, $H = S^0_\infty$ or $H = A^0_\infty$. $\square$

Before we complete the proof of Theorem 3, let us first make some general observations about the infinite subgroups of S^0_∞. In the following two lemmas we assume that $H < S^0_\infty$ is infinite.

Lemma 2.2.6. *There exists an infinite subset $\{\gamma_n : n = 1, 2, \dots\} \subseteq H$ such that for any $n \geq 1$, $\gamma_n(1) = 1$.*

Proof. First, let us suppose that there exists $k \geq 1$ and an infinite subset $\{\delta_n\}_{n=1}^\infty \subseteq H$ such that $\delta_n(1) = k$. Let $\gamma_n = \delta_n^{-1}\delta_1$. Then for any $n \geq 1$, $\gamma_n(1) = 1$.
If such k does not exist, then we have a strictly increasing sequence of positive integers $\{k_n\}_{n=1}^\infty$ and an infinite subset $\{\gamma_n\}_{n=1}^\infty \subseteq H$ such that

- $\gamma_n(1) = k_n$,
- $k_n > s(\gamma_i)$, whenever $1 \leq i \leq n - 1$.

Then for any $n \geq 1$ and $1 \leq i \leq n - 1$, $\gamma_n^{-1}\gamma_i\gamma_n(1) = 1$, hence our lemma follows. $\square$

Now we prove a generalization of the previous lemma.

Lemma 2.2.7. *For any $s \geq 1$, there exists an infinite subset $\{\gamma_n : n = 1, 2, \dots\} \subseteq H$ such that $\gamma_n(j) = j$, if $1 \leq j \leq s + 1$.*

Proof. We prove this statement by induction on s. The basis of the induction is the previous lemma.
Suppose that the lemma holds for some $s \in \mathbb{N}$ but it does not hold for $s + 1$. Again, if there exists $k \geq 1$ and an infinite subset $\{\rho_n\}_{n=1}^\infty \subseteq H$ such that

- $\rho_n(j) = j$, if $1 \leq j \leq s$,
- $\rho_n(s + 1) = k$,

then the set $\{\gamma_n = \rho_n^{-1}\rho_1\}_{n=1}^{\infty}$ will satisfy the conditions of our lemma. On the other hand, if such k does not exist then we have an increasing sequence of positive integers $\{k_n\}_{n=1}^{\infty}$ and an infinite subset $\{\delta_n\}_{n=1}^{\infty} \subseteq H$ such that

- $\delta_n(j) = j$, if $1 \leq j \leq s$,
- $\delta_n(s+1) = k_n$,
- $k_n \geq s(\delta_i)$ if $1 \leq i \leq n-1$.

Hence $\delta_n^{-1}\delta_i\delta_n(j) = j$, if $1 \leq i \leq n$, $1 \leq j \leq s+1$, in contradiction with our assumption. □

The above lemmas lead to the following corollary.

Corollary 2.2.8. *Let H be a subgroup of S_∞^0. If there exists $l \geq 1$ such that $H \cap S_{[l,\infty]}^0 = \{e\}$, then H is finite.*

Proof. Suppose that H is an infinite subgroup of S_∞^0. By Lemma 2.2.7 if H is infinite, then $H \cap S_{[l,\infty]}^0$ is infinite for all $l \geq 1$. □

Proposition 2.2.4, Lemma 2.2.5 and Corollary 2.2.8 complete the proof of Theorem 3. □

2.2.4 Minimal actions and the proof of Theorem 4

Recall that a crucial notion in the theory of measure preserving actions of groups is that of ergodicity. We say that a transformation $T : (X, \mu) \to (X, \mu)$ is ergodic if for any measurable set such that $T(A) = A$ we have that either $\mu(A) = 0$ or 1. An action $\alpha : \Gamma \curvearrowright (X, \mu)$ with a generating system Σ is called ergodic if for every $\sigma \in \Sigma$ the transformation $\alpha(\sigma)$ is ergodic. Ergodic actions are, in a sense, indecomposable, as we cannot split the space (X, μ) into two parts of positive measure which would be invariant under the action. The role of indecomposability in the continuous setting is filled by the minimal actions.

Definition 2.2.9. A continuous action of the group Γ on a compact space X is called **minimal** if for every $x \in X$ we have that $\overline{O(x)} = X$.

An action which is key to us is that of the Bernoulli shift which is defined as follows. For a set $\{a, b, c\}$, let us consider the set $\{a, b, c\}^\Gamma$ be the set of all $\{a, b, c\}$-valued functions σ on the integers with the natural $\mathbb{Z}$-action

$$t_n(\sigma)(a) = \sigma(a - n)\,.$$

A subshift of the Bernoulli shift is a subspace Π of $\{a, b, c\}^\mathbb{Z}$ which is invariant under this action.

Definition 2.2.10. A **minimal subshift** is a closed, invariant subspace $\Pi \subset \{a, b, c\}^\mathbb{Z}$ such that the orbit closure of any $\sigma \in \Pi$ is Π itself.

The simplest examples of minimal subshifts are those generated by a cyclic sequence σ, i.e. such that for some $n > 0$ we have that $\sigma(k) = \sigma(n + k)$ for any $k \in \mathbb{Z}$. Let $w = (q_k, q_{k-1}, \dots, q_1) \in \Gamma = \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$ and $\sigma \in \{a, b, c\}^\mathbb{Z}$. We say that $n \in \mathbb{Z}$ **sees** w **in** σ if

$$\sigma(n - i) = q_i \quad \text{for any } 1 \le i \le k.$$

The following proposition gives a useful criterion for finding minimal subshifts (it also appeared in e.g. [13]).

Proposition 2.2.11. *The orbit closure of $\sigma \in \{a, b, c\}^\mathbb{Z}$ is a minimal subshift if and only if for any $w \in \Gamma$ that is seen by some integer n, there exists $m_w > 0$ such that the longest interval in $\mathbb{Z}$ without elements that see w in σ is shorter than m_w.*

Proof. Clearly, the orbit closure $\overline{O(\sigma)}$ of any σ is invariant under the shift action. Take any $\rho, \tau \in \overline{O(\sigma)}$. We aim to show that there exists a sequence $\{n_k\}_{n=1}^{\infty}$ such that $t_{n_k}(\rho)$ converges to τ.

Let $w_k = (\tau(-k), \tau(-k+1), \ldots, \tau(k))$ and let $\{l_k\}_{k=1}^{\infty}$ be a sequence such that $t_{l_k}(\sigma)$ converges to ρ. We set j_k to be large enough that for any $M > j_k$ we have that $t_M(\sigma)(i) = t_{j_k}(\sigma)(i)$ for all $i = 0, \ldots, m_{w_k}$. Then by our assumption, some n_k sees w_k in $t_{j_k}(\sigma)$. Since $t_{j_k}(\sigma)$ converges to ρ, n_k sees w_k in ρ. Thus, $t_{n_k}(\rho)$ converges to τ.

Conversely, let us suppose that there is a word w seen by some n in σ and a sequence of numbers $\{a_k\}_{k=1}^{\infty}$ such that no element of $\{a_k - k, a_k - k + 1, \ldots, a_k + k\}$ sees w. Then the sequence $\{t_{a_k}(\sigma)\}_{k=1}^{\infty}$ has a subsequence convergent to some $\pi \in \overline{O(\sigma)}$. However, no n sees w in π, so no sequence in the orbit of π may converge to σ. Thus, $\overline{O(\sigma)}$ is not minimal. $\square$

A σ which satisfies the condition in the above proposition is called a minimal sequence. A **good** minimal sequence is a minimal sequence that does not contain the same letter consecutively. It is well-known that good minimal sequences exist for which the associated subshift has the cardinality of the continuum.

Proof of Theorem 4. Let us consider the bi-infinite path graph L on $\mathbb{Z}$. That is, $a, b \in \mathbb{Z}$ are connected if and only if $|a - b| = 1$. Let σ be a good minimal sequence which is not cyclic and let Π be the minimal subshift generated by σ. Color the edge $(n, n+1)$ of L by $\sigma(n)$. Then we obtain the Γ-action $L_\sigma : \Gamma \curvearrowright \mathbb{Z}$ in the following way. We take the generating system of Γ to be $\Sigma = \{a, b, c\}$. We put

$$L_\sigma(a)(k) = \begin{cases} k+1 & \text{if } \sigma(k) = a; \\ k-1 & \text{if } \sigma(k-1) = a; \\ k & \text{otherwise;} \end{cases}$$

on the generator a and we proceed in the same fashion for b, c. This extends to an action of the whole Γ on $\mathbb{Z}$ in a unique way.

One can see immediately that if $\tau \in \Pi$ and $n \in \mathbb{Z}$, then $\mathrm{Stab}_{L_\tau}(n)$ is in the orbit closure of $\mathrm{Stab}_{L_\sigma}(0)$ in the space of subgroups $\mathrm{Sub}(\Gamma)$. Conversely, any element of the orbit closure of $\mathrm{Stab}_{L_\sigma}(0)$ is in the form of $\mathrm{Stab}_{L_\tau}(n)$, for some τ and n. That is,

$$\{\mathrm{Stab}_{L_\tau}(n)\}_{\tau\in\Pi, n\in\mathbb{Z}}$$

is a minimal Γ-system in $\mathrm{Sub}(\Gamma)$. Such a system is called a uniformly recurrent subgroup (URS in [18]). Since there are continuum many minimal subshifts in $\{a, b, c\}^{\mathbb{Z}}$, in this way we obtain continuum many URS's in $\mathrm{Sub}(\Gamma)$ (see Theorem 5.1 [18]). Note that if Π and Π' represents the same URS, then either $\Pi = \Pi'$ or $\Pi' = \Pi^{\mathrm{flip}}$, where Π^{flip} denotes the subshift consisting of elements $\sigma^{\mathrm{flip}}(n) := \sigma(-n)$, for any $\sigma \in \Pi$. We need the following lemma in order to proceed with the proof.

Lemma 2.2.12. *Let σ, τ be good elements of a minimal subshift Π. Suppose that τ is neither a $\mathbb{Z}$-translate of σ nor a $\mathbb{Z}$-translate of σ^{flip}. Then for any $n \geq 1$, $Stab_{L_\sigma}(0) \not\subset Stab_{L_\tau}(n)$.*

Proof. Let $n \geq 1$ and let $w = (q_k, q_{k-1}, \ldots, q_1)$ be the longest word such that σ sees w at 0 and one of the following two conditions hold:

1. $\sigma(-i) = \tau(n - i)$ for any $1 \leq i \leq k$, or

2. $\sigma(-i) = \tau(n + i)$ for any $1 \leq i \leq k$.

Without loss of generality we can suppose that the first condition holds, $\sigma(-k) = c$, $\sigma(-k - 1) = a$, $\tau(n - k - 1) = b$. Then clearly, $w^{-1}bw \in \mathrm{Stab}_{L_\sigma}(0)$. On the other hand, $w^{-1}bw \notin \mathrm{Stab}_{L_\tau}(n)$. Indeed, $bw(n) = n - k - 1$, hence $w^{-1}bw(n) < n$. $\square$

Note that a URS is an equationally compact set in $\mathrm{Sub}(\Gamma)$. This observation and the lemma above concludes the proof of Theorem 4. $\square$

Chapter 3

Space of cocycles

The objective of this chapter is to define the convergence of weighted graphs. In order to do that, we shall first recall the constructions of the space of the rooted Schreier graphs as well as the spaces of colored rooted Schreier graphs (see e.g. [10]). Then, we will introduce the notion of Schreier cocycles and we will endow the set of rooted cocycles with a topology analogous to that on the space of rooted Schreier graphs. Finally, we will establish the local convergence of cocycles.
Throughout this chapter, Γ denotes a finitely generated group equipped with a finite symmetric generating system Σ.

3.1 Schreier graphs and Schreier cocycles

Assume that Γ acts transitively on a countable set X with $\alpha : \Gamma \curvearrowright X$. The Schreier graph of the action α is formally defined as $\mathrm{Sch}(\alpha) := (V, E, l)$. We think of it as a directed edge-labeled graph where:

- $V = X$ is the set of vertices of $\mathrm{Sch}(\alpha)$;

- E is the set of edges of $\mathrm{Sch}(\alpha)$, where $\overrightarrow{(p,q)} \in V \times V$ is an edge provided that for some $\sigma \in \Sigma$ such that $\alpha(\sigma)(x) = y$;

- $l : E \to \mathcal{P}(\Sigma)$ is the edge labeling with the generators $\sigma \in \Sigma$ such that $\sigma \in l(\overrightarrow{(p,q)})$ provided that $\alpha(\sigma)(p) = q$.

It is possible that for some distinct $\sigma \in \Sigma$ and some $x \in X$, $\sigma(x) = x$, so $\mathrm{Sch}(\alpha)$ may have loops. We will denote by $\overline{\mathrm{Sch}(\alpha)}$ the corresponding loopless undirected simple connected graph without labels. Let us observe that the Schreier coset graphs defined in Section 1.2 are equivalent to the graphs defined above. From now on, given an action α, an element $\gamma \in \Gamma$ and $x \in V(S)$, for the sake of simplicity we will denote $\alpha(\gamma)(x)$ by $\gamma \cdot x$.

Definition 3.1.1. The $\mathbb{F}_d$**-Schreier graphs** are the Schreier graphs of $\mathbb{F}_d$-actions with respect to the standard symmetric generating system $\Sigma_d := \{a_1, a_2, \dots, a_d, a_1^{-1}, a_2^{-1}, \dots, a_d^{-1}\}$ of $\mathbb{F}_d$. A **rooted** $\mathbb{F}_d$**-Schreier graph** is an $\mathbb{F}_d$-Schreier graph with a distinguished vertex.

Suppose that the generating system of Γ can be written as $\Sigma = \{\sigma_1, \dots, \sigma_d\}$. Then, by the universal property of the free group $\mathbb{F}_d$ there exists a surjective homomorphism $\pi : \mathbb{F}_d \to \Gamma$ such that $\pi(a_i) = \sigma_i$ for $1 \leq i \leq d$. If $\alpha : \Gamma \curvearrowright Y$ is a transitive action of Γ, then we can define an action $\alpha_\pi : \mathbb{F}_d \curvearrowright Y$ by setting $\alpha_\pi(\gamma) = \alpha(\pi(\gamma))$. Hence, the underlying Schreier graphs $\overline{\mathrm{Sch}(\alpha_\pi)}$ and $\overline{\mathrm{Sch}(\alpha)}$ are isomorphic. Thus, we can view the Schreier graphs of Γ-actions as special cases of $\mathbb{F}_d$-actions. Therefore, from now on, we will mainly focus on actions of $\mathbb{F}_d$, the free group of rank d.
Let us write $\mathbb{F}_d\mathcal{G}$ to denote the set of all rooted $\mathbb{F}_d$-Schreier graphs up to isomorphism. On $\mathbb{F}_d\mathcal{G}$ we put the metric d_S, where

$$d_S((\mathrm{Sch}(\alpha_1), x), (\mathrm{Sch}(\alpha_2), y)) = 2^{-n},$$

if for all $i < n$, the balls of radius i, $B_i(\mathrm{Sch}(\alpha_1), x)$ and $B_i(\mathrm{Sch}(\alpha_2), y)$, are rooted-labeled isomorphic graphs, however the balls $B_n(\mathrm{Sch}(\alpha_1), x)$ and $B_n(\mathrm{Sch}(\alpha_2), y)$ are not rooted-labeled isomorphic. Note that since the Schreier graphs arise from transitive actions of the group, they are connected. Hence, the metric d_S is well-defined.

An important action of $\mathbb{F}_d$ on $\mathbb{F}_d\mathcal{G}$ is given by **moving the roots** of Schreier graphs and is defined as follows. For a $\gamma \in \Gamma$ and a rooted Schreier graph $(S, x) \in \mathbb{F}_d\mathcal{G}$, we put $\gamma \cdot (S, x) = (S, \gamma \cdot x)$. Observe that this action is continuous. Let $\alpha : \mathbb{F}_d \curvearrowright X$ be a transitive action, $p \in X$ and $(\mathrm{Sch}(\alpha), p)$ be the associated rooted $\mathbb{F}_d$-Schreier graph. Then, $\mathrm{Stab}_\alpha(p) \in \mathrm{Sub}(\mathbb{F}_d)$ defines a bijection $\mathrm{Stab} : \mathbb{F}_d\mathcal{G} \to \mathrm{Sub}(\mathbb{F}_d)$. It is not hard to see that Stab is actually a homeomorphism.

3.1.1 Spaces of colored rooted Schreier graphs

Let Q be a set and $\mathbb{F}_d\mathcal{G}^Q$ be the set of all rooted $\mathbb{F}_d$-Schreier graphs (S, p) equipped with a Q-coloring $\varphi : V(S) \to Q$. In the case when Q is finite, we put a metric d_Q on $\mathbb{F}_d\mathcal{G}^Q$, where

$$d_Q((S_1, x_1, \varphi_1), (S_2, x_2, \varphi_2)) = 2^{-n}$$

if for any $i < n$ the balls of radius i, $B_i(S_1, x_1, \varphi_1)$ and $B_i(S_2, x_2, \varphi_2)$ are rooted-colored-labeled-isomorphic graphs, however the balls $B_n(S_1, x_1, \varphi_1)$ and $B_n(S_2, x_2, \varphi_2)$ are not rooted-colored-labeled-isomorphic. Note that $(\mathbb{F}_d\mathcal{G}^Q, d_Q)$ is a compact space. Again, the group $\mathbb{F}_d$ acts on $\mathbb{F}_d\mathcal{G}^Q$ by moving the roots and this action is continuous.

Furthermore, we can define $\mathbb{F}_d\mathcal{G}^\mathcal{C}$, the set of rooted $\mathbb{F}_d$-graphs (S, p) equipped with a $\mathcal{C}$-coloring $\varphi : V(S) \to \mathcal{C}$, where $\mathcal{C} = \{0, 1\}^\omega$ is the standard Cantor set. Let $\pi_k : \{0, 1\}^\omega \to \{0, 1\}^k$ be the projection onto the first k coordinates. The space $\mathbb{F}_d\mathcal{G}^\mathcal{C}$ is compact with respect to the topology with a base given by sets $A((S, p, \varphi), n, k)$ for a $\mathcal{C}$-colored rooted Schreier graph (S, p, φ) and positive integers n, k. $A((S, p, \varphi), n, k)$ is the set of those $\mathcal{C}$-colored rooted Schreier graphs (S', p', φ') which satisfy

1. $B_n(S,p)$ is rooted-isomorphic to $B_n(S',p')$ with an isomorphism τ,

2. $\pi_k \circ \varphi | V(B_n(S,p)) = \pi_k \circ \varphi' | \tau(V(B_n(S',p')))$.

In this topology $\{(S_n, p_n, \varphi_n)\}_{n=1}^{\infty} \subset \mathbb{F}_d\mathcal{G}^{\mathcal{C}}$ is a convergent sequence if and only if for any $k \geq 1$, $\{(S_n, p_n, \pi_k \circ \varphi_n)\}_{n=1}^{\infty}$ is convergent. Note that if a sequence $\{(S_n, p_n, \varphi_n)\}_{n=1}^{\infty} \subset \mathbb{F}_d\mathcal{G}^{\mathcal{C}}$ is convergent, then the underlying sequence $\{(S_n, p_n)\}_{n=1}^{\infty} \subset \mathbb{F}_d\mathcal{G}$ is convergent as well.

3.1.2 The space of cocycles

Let $I := [0,\infty]$ be the two-point compactification of the set of positive real numbers. A **rooted I-cocycle** (S,p,F) is defined in the following way. Let $S \in \mathbb{F}_d\mathcal{G}$ be a rooted $\mathbb{F}_d$-Schreier graph with root p. Let $F : E(S) \to I$ be an I-labeling of the directed edges of S satisfying the following conditions:

1. if $F(\overrightarrow{(a,b)}) = r \in \mathbb{R}^+$ then $F(\overrightarrow{(b,a)}) = \frac{1}{r}$;

2. if $F(\overrightarrow{(a,b)}) = 0$ then $F(\overrightarrow{(b,a)}) = \infty$;

3. if $F(\overrightarrow{(a,b)}) = \infty$ then $F(\overrightarrow{(b,a)}) = 0$;

4. for any loop $\overrightarrow{(a,a)}$, $F(\overrightarrow{(a,a)}) = 1$;

5. for any cycle $\overrightarrow{(a_1,a_2)}, \overrightarrow{(a_2,a_3)} \dots \overrightarrow{(a_n,a_1)}$ we have that $F(\overrightarrow{(a_1,a_2)})F(\overrightarrow{(a_2,a_3)}) \dots F(\overrightarrow{(a_n,a_1)}) = 1$, provided that all the terms are positive real numbers.

The set of all rooted I-cocycles will be denoted by $C\mathbb{F}_d\mathcal{G}$. One can define the compact topology of pointwise convergence on $C\mathbb{F}_d\mathcal{G}$ using the following open base. For a rooted I-cocycle $(S,p,F) \in C\mathbb{F}_d\mathcal{G}$, an integer $n > 0$ and a real number $\epsilon > 0$, let $B_{n,\epsilon}(S,p,F)$ be the set of rooted I-cocycles (T,q,G) such that the following conditions hold:

- the rooted balls $B_n(S,p)$ and $B_n(T,q)$ are isomorphic as Σ_d-labeled rooted directed graphs;

- for any edge $\overrightarrow{e} \in E(B_n(S,p))$, $|F(\overrightarrow{e}) - G(\overrightarrow{e})| < \epsilon$ provided that $F(\overrightarrow{e}) < \infty$. Otherwise $G(\overrightarrow{e}) > \frac{1}{\epsilon}$.

Again, we can define an $\mathbb{F}_d$-action on the compact metric space $C\mathbb{F}_d\mathcal{G}$ by moving the roots. We call an I-cocycle (S,p,F) **regular** if $F(\overrightarrow{e}) \notin \{0,\infty\}$, for all $\overrightarrow{e} \in E(S)$. We shall denote the Borel set of regular I-cocycles by $\mathrm{Reg}(C\mathbb{F}_d\mathcal{G})$.

Let us make a remark on the use of the word "cocycle". Assume that (S,p,F) is a regular I-cocycle. Then, we can define the function $C_F : \mathbb{F}_d \times V(S) \to \mathbb{R}^+$ by putting

$$C_F(\gamma, x) := \prod_{i=0}^{n} F(\overrightarrow{(x_i, \sigma_i \cdot x_i)}),$$

where $x = x_0$, $\sigma_0 \cdot x_0 = x_1, \sigma_1 \cdot x_1 = x_2, \ldots, \sigma_n \cdot x_n = \gamma \cdot x$. We obtain that C_F satisfies the identity

$$C_F(\gamma\delta, x) = C_F(\gamma, \delta \cdot x) C_F(\delta, x)\,,$$

which we also call the **Cocycle Identity**. So, $C : \mathbb{F}_d \times \mathrm{Reg}(C\mathbb{F}_d\mathcal{G}) \to \mathbb{R}^+$ defined by

$$C(\gamma, (S,p,F)) := C_F(\gamma, p)$$

is a real-valued Borel $\mathbb{F}_d$-cocycle on the regular cocycles in $C\mathbb{F}_d\mathcal{G}$.

Similarly as in Section 3.1, we can construct the compact spaces $C\mathbb{F}_d\mathcal{G}^Q$ and $C\mathbb{F}_d\mathcal{G}^{\mathcal{C}}$. These denote the spaces of cocycles with vertex colorings by the finite set Q and the Cantor set $\mathcal{C}$, respectively. Furthermore, we define the maps $\mathrm{Forg} : C\mathbb{F}_d\mathcal{G}^C \to C\mathbb{F}_d\mathcal{G}$, $\mathrm{Forg} : \mathbb{F}_d\mathcal{G}^C \to \mathbb{F}_d\mathcal{G}$ that are forgetting the colors; they will be relevant in Section 4.3.2. Let us note that these maps are continuous.

3.1.3 Cocycles of arbitrary groups

Later, it will be important for us to define the space of Γ-Schreier graphs as a closed subspace of $\mathbb{F}_d\mathcal{G}$. Let Γ be a group with a symmetric system of generators $\Sigma = \{\sigma_1, \ldots, \sigma_d, \sigma_1^{-1}, \ldots, \sigma_d^{-1}\}$. Take $\rho : \mathbb{F}_d \to \Gamma$ to be a surjective homomorphism such that $\rho(a_i) = \sigma_i$ for any $1 \leq i \leq d$, where $a_i \in \Sigma_d$. Let $\alpha : \Gamma \curvearrowright Y$ be a transitive action and let $\mathrm{Sch}(\alpha)$ be the associated Γ-Schreier graph with respect to Σ. Then we have the action $\alpha_\rho : \mathbb{F}_d \curvearrowright Y$, where $\alpha_\rho(\gamma) = \alpha(\rho(\gamma))$. Also, let $S^\alpha_\rho \in \mathbb{F}_d\mathcal{G}$ be the associated $\mathbb{F}_d$-Schreier graphs. The set of all rooted Schreier graphs of the form (S^α_ρ, p) is a compact subset of $\mathbb{F}_d\mathcal{G}$ and we will denote it by $\Gamma\mathcal{G}$. Note that $\Gamma\mathcal{G}$ is homeomorphic to the space of subgroups $\mathrm{Sub}(\Gamma)$. In the same way as in Sections 3.1 and 3.1.1, we can define the space of Γ-cocycles $C\Gamma\mathcal{G}$ and the colored versions $\Gamma\mathcal{G}^Q, \Gamma\mathcal{G}^\mathcal{C}, C\Gamma\mathcal{G}^Q$ and $C\Gamma\mathcal{G}^\mathcal{C}$.

3.2 Weighted convergence

3.2.1 Quasi-invariant Random Cocycles and the Canonical Map

We call a probability measure ν on the compact Γ-space $C\Gamma\mathcal{G}$ a **Quasi-Invariant Random Radon-Nikodym Cocycle** (QRC for short) if ν is supported on the $\mathrm{Reg}(C\Gamma\mathcal{G})$ and for any Borel set $Z \subset C\Gamma\mathcal{G}$ and $\gamma \in \Gamma$ we have that

$$\nu(\gamma \cdot Z) = \int_Z C(\gamma, s)\, d\nu(s)\,.$$

Recall that $C(\gamma, s) := C_F(\gamma, p)$, where p is the root of the rooted Schreier cocycle s. That is, C can be viewed as the Radon-Nikodym cocycle of the Γ-action on $C\Gamma\mathcal{G}$. Note that if ν is concentrated on cocycles (S, p, F) such that $C_F(\gamma, q) = 1$ for all $\gamma \in \Gamma$ and

$q \in V(S)$, then ν is an Invariant Random Subgroup [1]. Hence, we regard QRC's as the nonsingular analogues of the Invariant Random Subgroups.

Recall that if $\alpha : \Gamma \curvearrowright (X, \mu)$ is a probability measure-preserving Borel action, then the push-forward measure $\mathrm{Stab}^*_\alpha(\mu)$ defines an Invariant Random Subgroup. Now, let $\alpha : \Gamma \curvearrowright (X, \mu)$ be a nonsingular action of the group Γ. We know that there exists a conull set $X_0 \subseteq X$ on which the Radon-Nikodym derivatives of the action α are strictly positive. Then, we have an equivariant Borel map (which we call **the canonical map**) $M_\alpha : X_0 \to \mathrm{Reg}(C\Gamma\mathcal{G})$ which assigns to each point $x \in X_0$ the Schreier cocycle induced by α on the orbit $O(x)$ with x itself as the root. Let $\mu_\alpha = (M_\alpha)_*(\mu)$.

Proposition 3.2.1. *μ_α is a QRC.*

Proof. Let $Z \subset C\Gamma\mathcal{G}$ be a Borel set. We need to show that for all $\gamma \in \Gamma$ we have

$$\mu_\alpha(\gamma \cdot Z) = \int_Z C(\gamma, s) d\mu_\alpha(s)\,. \tag{3.1}$$

By definition of the push-forward measure we have that

$$\mu_\alpha(\gamma \cdot Z) = \mu(\gamma \cdot (M_\alpha)^{-1}[Z]) = \int_{(M_\alpha)^{-1}[Z]} \frac{d\gamma_*\mu}{d\mu}(x) d\mu(x)\,.$$

Observe that for any $x \in X_0$ $\frac{d\gamma_*\mu}{d\mu}(x) = C(\gamma, M_\alpha(x))$. Hence,

$$\int_{(M^\mu_\alpha)^{-1}[Z]} \frac{d\gamma_*\mu}{d\mu}(x) d\mu(x) = \int_Z C(\gamma, s) d\mu_\alpha(s)\,.$$

Therefore (3.1) holds. □

We will denote the set of all Γ-QRC's by QRC_Γ. Notice that we can view QRC_Γ as a closed subset of $\mathrm{QRC}_{\mathbb{F}_d}$.

Abért, Glasner and Virág showed in [1] that all Invariant Random Subgroups are push-forwards of invariant probability measures. Using the same method, it can be proved that all QRC's are push-forwards of quasi-invariant probability measures.

Proposition 3.2.2. *Let ν be an $\mathbb{F}_d$-QRC. Then there is a Borel probability measure μ on $C\mathbb{F}_d\mathcal{G}^{\mathcal{C}}$ such that the action $\alpha : \mathbb{F}_d \curvearrowright (C\mathbb{F}_d\mathcal{G}^{\mathcal{C}}, \mu)$ by moving the root is nonsingular and $\nu = \mu_\alpha$.*

Proof. Let us fix a cocycle $s \in C\mathbb{F}_d\mathcal{G}$. There is a canonical bijection between $\mathrm{Forg}^{-1}[\{s\}]$ and a product of Cantor sets $\prod_{v\in V(s)} \mathcal{C}$. Thus, we define μ_s to be the measure on $C\mathbb{F}_d\mathcal{G}^{\mathcal{C}}$ supported on $\mathrm{Forg}^{-1}[\{s\}]$ which is the product of the standard measures on the Cantor set $\mathcal{C}$. Then for any set A in the standard basis of $C\mathbb{F}_d\mathcal{G}^{\mathcal{C}}$ we define

$$\hat{\mu}(A) = \int \mu_s(A)\, d\nu(s).$$

Clearly, this is well-defined for any such A. Thus, by Theorem 452B in [16] $\hat{\mu}$ extends to a complete Borel probability measure μ on $C\mathbb{F}_d\mathcal{G}^{\mathcal{C}}$ such that for any Borel subset $B \subseteq C\mathbb{F}_d\mathcal{G}^{\mathcal{C}}$ we have

$$\mu(B) = \int \mu_s(B)\, d\nu(s).$$

Let us notice that the measure μ is quasi-invariant under moving the root.
We need to show that $\mu_\alpha = \nu$. Observe that the set of cocycles in $C\mathbb{F}_d\mathcal{G}^{\mathcal{C}}$ whose vertex labels are all different is μ-conull. Therefore, for μ-almost every cocycle $s \in C\mathbb{F}_d\mathcal{G}^{\mathcal{C}}$ we have that $\mathrm{Forg}(s) = M_\alpha(s)$. Then, for any μ_α-measurable set B we have that

$$\mu_\alpha(B) = \mu(M_\alpha^{-1}[B]) = \int \mu_s(\mathrm{Forg}^{-1}[B])\, d\nu(s) = \int \chi_B(s)\, d\nu(s) = \nu(B).$$

This completes the proof. $\square$

3.2.2 The Tightness Condition

For $M > 1$, let $T_M \subset C\mathbb{F}_d\mathcal{G}$ be defined as the set of regular cocycles (S, p, F) such that for any $\sigma \in \Sigma$, $\frac{1}{M} \leq F(\overrightarrow{p, \sigma \cdot p}) \leq M$. A subset $\mathcal{Q} \subset \mathrm{QRC}_{\mathbb{F}_d}$ is called **tight** if for any $\epsilon > 0$ there exists $M_\epsilon > 0$ such that for any $\nu \in \mathcal{Q}$, $\nu(C\mathbb{F}_d\mathcal{G} \backslash T_{M_\epsilon}) < \epsilon$.

Lemma 3.2.3. *Let $\{\nu_n\}_{n=1}^{\infty} \subset \mathrm{QRC}_{\mathbb{F}_d}$ be a tight sequence of QRC's weakly convergent to a probability measure ν. Then, ν is a QRC as well.*

Proof. Clearly, the probability measure ν is concentrated on the Borel set $\mathrm{Reg}(C\mathbb{F}_d\mathcal{G})$. Indeed, let $P : C\mathbb{F}_d\mathcal{G} \to \mathbb{R}^+$ be a continuous function such that $|P(s)| \leq 1$ for all $s \in C\mathbb{F}_d\mathcal{G}$ and $P\,|_{T_{M_\epsilon}} = 0$. Then, $\int_{C\mathbb{F}_d\mathcal{G}} P(s) d\nu_n(s) \leq \epsilon$ holds for all $n \geq 1$. Hence

$$\int_{C\mathbb{F}_d\mathcal{G}} P(s) d\nu(s) \leq \epsilon. \tag{3.2}$$

Since (3.2) holds for all continuous functions satisfying the two conditions above, we have that $\nu(T_{M_\epsilon}) > 1 - \epsilon$. Therefore, ν is concentrated on $\mathrm{Reg}(C\mathbb{F}_d\mathcal{G})$.
Now we prove the Radon-Nikodym condition for ν. Let $R : C\mathbb{F}_d\mathcal{G} \to \mathbb{R}$ be a continuous function. Then,

$$\int_{C\mathbb{F}_d\mathcal{G}} R(\gamma^{-1} \cdot s) d\nu_n(s) = \int_{C\mathbb{F}_d\mathcal{G}} R(s) C(\gamma, s) d\nu_n(s)\,.$$

holds for all $n \geq 1$, since ν_n's are QRC's. Therefore,

$$\int_{C\mathbb{F}_d\mathcal{G}} R(\gamma^{-1} \cdot s) d\nu(s) = \int_{C\mathbb{F}_d\mathcal{G}} R(s) C(\gamma, s) d\nu(s)\,.$$

holds as well. Consequently, ν is a QRC as well. $\square$

3.2.3 Weighted Benjamini-Schramm Convergence

Consider a finite set X with a system of permutations $\rho = \{\rho_i : i = 1, \ldots, d\}$ and a positive, real-valued function $w : X \to \mathbb{R}^+$ such that $\sum_{x \in X} w(x) = 1$. Then, $a_i \cdot x = \rho_i(x)$ defines a (not necessarily transitive) action of the free group $\mathbb{F}_d$ on the set X. Then, we can consider the **weighted generalized Schreier graph** (shortly a WGS-graph) $S = S_{X,\rho,w}$ and the cocycle function $F : E(S_{X,\rho,w}) \to \mathbb{R}^+$ defined in the following way:

$$F(\overrightarrow{(p, \sigma_i \cdot p)}) := \frac{w(\sigma_i \cdot p)}{w(p)}\,.$$

Furthermore, we can define an element ν_S of $\mathrm{QRC}_{\mathbb{F}_d}$ by

$$\nu_S((S, p, F)) = w(p)\,.$$

Observe that ν_S is supported on a finite set, hence there is a bijective correspondence between the set WGS_d of finite WGS-graphs and the set of elements in $\mathrm{QRC}_{\mathbb{F}_d}$ which are supported on a finite set, denoted by $\mathcal{F}\mathrm{QRC}_{\mathbb{F}_d}$. We say that a family $\mathcal{Q}$ of finite WGS graphs is tight if the corresponding family of QRC's is tight.

Definition 3.2.4. A measure $\nu \in \mathrm{QRC}_{\mathbb{F}_d,\Sigma_d}$ is a **sofic** QRC if there exists a tight sequence of WGS graphs $\{S_n\}_{n=1}^{\infty} \subset \mathrm{WGS}_d$ such that $\{\mu_{S_n}\}_{n=1}^{\infty}$ is weakly convergent to ν.

In this case we say that the sequence $\{S_n\}_{n=1}^{\infty}$ is **convergent in the sense of Benjamini and Schramm** (or locally convergent) and the QRC ν is its **limit**.

Remark 3.2.5. One can consider the space $\mathbb{F}_d\mathcal{G}_{\mathbb{R}}$ of the rooted Schreier graphs from $\mathbb{F}_d\mathcal{G}$ together with an edge coloring with real numbers. We can give the topology on this space in the same way as in the case of $\mathbb{F}_d\mathcal{G}^{\mathcal{C}}$; the only difference is that we consider edge-colored graphs instead of vertex-colored ones. Then, similarly as in the case of

$\mathbb{F}_d\mathcal{G}^{\mathcal{C}}$, we could define the local convergence of Schreier graphs which are edge-colored by the real numbers.
Let us notice that the local convergence of WGS graphs is not equivalent to this notion. That is because in case of the WGS graphs, an associated measure on $C\mathbb{F}_d\mathcal{G}$ is dependent on the cocycle. However, for the convergence of graphs with edges colored by elements of $\mathbb{R}$ the associated measures on $\mathbb{F}_d\mathcal{G}_{\mathbb{R}}$ are independent of the coloring - each vertex could be the root of a graph with the same probability.

We will call an element $\nu \in \mathrm{QRC}_\Gamma$ sofic if it is sofic as an element of $\mathrm{QRC}_{\mathbb{F}_d}$. Note that if $\nu \in \mathrm{QRC}_\Gamma$ is supported on the elements (S, p, F) for which F takes only the value 1 (that is, ν is an IRS), then the soficity of ν coincides with the previously defined soficity of an IRS (see co-sofic IRS in [17]). If $\alpha : \Gamma \curvearrowright (Y, \mu)$ is a nonsingular action and its canonical QRC μ_α is the limit of the sequence $\{\mu_{S_n}\}_{n=1}^{\infty}$, we will also say that the action α is the limit of the WGS graphs $\{S_n\}_{n=1}^{\infty}$.

3.2.4 Ball Statistics and Weighted Convergence

In Section 1.2.1, we observed that for a sequence of finite $\mathbb{F}_d$-Schreier graphs $\{G_n\}_{n=1}^{\infty}$ the local convergence is equivalent to the weak convergence of measures μ_{G_n} on $\mathbb{F}_d\mathcal{G}$. Hence, it is natural to ask whether a similar equivalence could be established for the Benjamini-Schramm convergence of cocycles. That is, we wish to characterize the convergence of finite cocycles with the convergence of their local statistics. In this section we present how to obtain such a characterization. However, this is more complicated than in the case of unweighted graphs due to the fact that the space of rooted Schreier cocycles is not totally disconnected.
Let $\mathcal{B}^{r,d}$ be the set of all rooted balls of radius r in $\mathbb{F}_d$-Schreier graphs. Let $\lambda > 0$ and $\mathcal{B}^{r,d,\lambda}$ be the set of all rooted Σ_d labeled balls with extra edge labelings with rational pairs $\{a, b\}$ (which we call the Code of the edge), where $a < b$. Then, $\tilde{B} \in \mathcal{B}^{r,d,\lambda}$

defines an open subset $U_{\tilde{B}} \in C\mathbb{F}_d\mathcal{G}$ in the following way. Let $(S,p,F) \in C\mathbb{F}_d\mathcal{G}$. Then $(S,p,F) \in U_{\tilde{B}}$ provided that

- the underlying Σ_d-labeled ball of radius r around p is isomorphic to the underlying Σ_d-labeled ball of $\tilde{B}$,
- if $\overrightarrow{e}$ is an edge in $\tilde{B}$ and $\mathrm{Code}(\overrightarrow{e}) = \{a,b\}$, then $a\lambda < F(\overrightarrow{e}) < b\lambda$.

Note that the elements of $\mathcal{B}^{r,d,\lambda}$ do not depend on the variable λ; only the interpretation of the function Code does.

Clearly, for every $\lambda > 0$ the open subsets $U_{\tilde{B}}$ define a base for the topology of $C\mathbb{F}_d\mathcal{G}$. Now, suppose that $\{S_n, w_n\}_{n=1}^{\infty}$ is a tight, convergent sequence of WGS's and $\mu \in \mathrm{QRC}_{\mathbb{F}_d}$. We say that λ is a **generic value** for μ if the set of cocycles which contain an edge whose label is a rational multiple of λ is μ-null. Note that there exist only countably many nongeneric λ's for μ. The Portmanteau Theorem (Theorem 2.1. in [4]) implies that if λ is generic for μ, then $\{S_n, w_n\}_{n=1}^{\infty}$ converges to μ if and only if for any $r > 0$ and $\tilde{B} \in \mathcal{B}^{r,d,\lambda}$ we have

$$\lim_{n\to\infty} \mu_{w_n}(U_{\tilde{B}}) = \mu(U_{\tilde{B}}).$$

3.3 Weighted hyperfinite families

Let $\mathcal{A} \subset C\mathbb{F}_d\mathcal{G}$ be a tight family of finite Schreier cocycles. Similarly as in [14], we call $\mathcal{A}$ **weighted hyperfinite** if for any $\epsilon > 0$ there exists K_ϵ satisfying the following condition. For each $(G,w) \in \mathcal{A}$ there exists a subset $Y \subset V(G)$, $w(Y) \leq \epsilon$ such that if we remove Y and all the edges adjacent to Y, the remaining graph G' consists of components with at most K_ϵ vertices. Note that if the probability measure is uniform for each $G \in \mathcal{A}$, then weighted hyperfiniteness coincides with the notion of hyperfiniteness (see e.g. [8]).

Example 2. Planar graphs or graphs with uniform subexponential growth form hyperfinite families with respect to the uniform measure ([9], [24]).

The following example will be crucial in this thesis.

Example 3. Let Γ be a finitely generated nonamenable group with a symmetric generating system Σ where $|\Sigma| = 2d$. Assume $\varphi : \mathbb{F}_d \to \Gamma$ is surjective and Σ is the image of Σ_d, the standard generating system of $\mathbb{F}_d$. Then let $B_r = B_r(\mathrm{Cay}(\Gamma,\Sigma), e_\Gamma)$ be the ball of radius r around the unit element in the left Cayley graph $\mathrm{Cay}(\Gamma,\Sigma)$. Let $\kappa = \lim_{r\to\infty} \frac{\log|B_r|}{r}$ be the growth of the group Γ with respect to Σ. We define the probability measure $w_r : B_r \to \mathbb{R}^+$ by

$$w_r(p) = \frac{1}{|S_k|}\,,$$

where $S_k = B_k \backslash B_{k-1}$ and $k = d_{\mathrm{Cay}(\Gamma,\Sigma)}(p, e_\Gamma)$.

Lemma 3.3.1. *The sequence* $\{(B_r, w_r)\}_{r=1}^\infty$ *is tight for any system* (Γ,Σ).

Proof. First, observe that the sequence $\left\{\frac{|B_r|}{|S_r|}\right\}_{r=1}^\infty$ is bounded. Indeed by nonamenability, there exists $c > 0$ such that $|S_{r+1}| \geq c|B_r|$, also $|B_{r+1}| \leq 2d|B_r|$. Therefore, $|S_{r+1}| \geq \frac{c}{2d}|B_{r+1}|$. If $p, q \in B_r$ are adjacent vertices, and

$$k = d_{\mathrm{Cay}(\Gamma,\Sigma)}(p, e_\Gamma) = d_{\mathrm{Cay}(\Gamma,\Sigma)}(q, e_\Gamma) + 1\,,$$

then

$$\frac{w_r(p)}{w_r(q)} \geq \frac{|S_k|}{|S_{k+1}|} \geq \frac{|S_k|}{|B_{k+1}|} \geq \frac{c}{4d^2}\,.$$

Therefore the lemma follows. $\square$

Example 3 is particularly interesting in the case when Γ is an exact group. A countable group Γ is called **exact** if it admits an amenable action on a compact metric space. In particular, all amenable and word-hyperbolic groups are exact. It was shown that exactness is closed under taking subgroups, directed unions, extensions and free products [30]. The first non-exact group was constructed by Gromov in the early 2000's [19].

The notion of exactness interests us because it is linked to weighted hyperfinite graphs. This property was introduced by Elek and Timár in [14] and is defined as follows.

Definition 3.3.2. An infinite $\mathbb{F}_d$-Schreier graph is **weighted hyperfinite** if the family of all of its finite induced subgraphs taken with all possible roots and tight cocycles is weighted hyperfinite.

Sako showed that a finitely generated group Γ is exact if and only if its Cayley graph is weighted hyperfinite (Theorem 5.2 in [33]). The following proposition follows from this and Lemma 3.3.1.

Proposition 3.3.3. *If* Γ *is an exact group, then* $\{(B_r, w_r)\}_{r=1}^{\infty}$ *is a weighted hyperfinite system.*

Chapter 4

Quasi-invariant Random Cocycles

4.1 Coloring graph sequences

Let G be a graph of vertex degree bound $2d$. Then, for any $x \in V(G)$ we have that

$$|B_r(G,x)| < (2d)^{r+1}\,.$$

Hence, we can color the vertex set $V(G)$ with $(2d)^{r+1}$ colors $\{0,\dots,(2d)^{r+1}-1\}$ in such a way that if $x, y \in V(G)$ satisfy that $0 < d_G(x,y) \leq r$ then the color of x differs from the color of y. Indeed, we can use a greedy algorithm. Let us list the vertices of $V(G)$ with $x_1, x_2, \dots, x_t$. Firstly, color x_1 with the 0. Suppose we have colored x_i for all $i \leq k$, for some k. Then, let A_{k+1} be the set of those $j \leq k$, for which $d_G(x_{k+1}, x_j) \leq r$. We color x_{k+1} with the lowest number which doesn't color any $x_j, j \in A_{k+1}$.

In this section we should be particularly careful about distinguishing colored and uncolored graphs as well as which coloring of a given graph we consider. Thus, for a Schreier graph S we write S^c to denote the graph S together with a vertex coloring c. Fix $R > 0$ and pick an integer $Q \geq (2d)^{r+1}$. Throughout this chapter, we write Q to denote the set $\{0, 1, \dots, Q-1\}$, as in the definition of the natural numbers.

Definition 4.1.1. Suppose a coloring $\varphi : V(G) \to Q$ is given.

1. Let $x \in V(G)$. If for all $y \in V(G)$ such that $0 < d_G(x, y) \le r$ we have that $\varphi(x) \neq \varphi(y)$, then we say that the vertex x is (φ, r)**-good**. Otherwise we say that x is (φ, r)**-bad**.

2. We say that φ is a **good** (Q, r)**-coloring** if $\varphi(x) \neq \varphi(y)$ provided that $0 < d_G(x, y) \le r$.

The aim of this section is to provide an answer to the following problem.

Question. Given a convergent sequence of $\mathbb{F}_d$-Schreier graphs $\{S_n\}_{n=1}^{\infty}$ and a number $R > 0$, does there exist a $Q > 0$ and Q-colorings $c_n : S_n \to Q$ which satisfy the following properties:

- for any $n \ge 1$, c_n is a (Q, R)-good coloring,
- $\{S_n, c_n\}_{n=1}^{\infty}$ is convergent as a sequence of Q-colored Schreier graphs?

Our strategy for this construction can be broken down into the following steps.

1. Firstly, we take an $\epsilon > 0$ and a large enough Q_ϵ. We will then define the initial colorings c_n^0 to be uniformly random colorings of $V(G_n)$ with the colors from Q_ϵ.

2. Next, we randomly recolor the set of badly colored vertices in each graph G_n multiple times until we obtain that the relative size of each set of bad points tends to 0 as $n \to \infty$.

3. Finally, we can recolor each set of bad points using the greedy algorithm.

The main issue that we need to address in this construction is so show that in the end we obtain a convergent sequence of colored graphs. Proposition 4.1.2 shows that a randomly colored sequence of graphs is convergent with probability 1. In Propositions

4.1.6 and 4.1.7 we prove that after randomly recoloring the badly colored parts of each graph G_n, the resulting colored sequences are still convergent with probability 1. Here we state our theorem in a formal way.

Theorem 5. *Let R be a positive integer and let $\{S_n\}_{n=1}^{\infty}$ be a convergent sequence of $\mathbb{F}_d$-Schreier graphs. If $\liminf_{n\to\infty} \frac{|S_n|}{n} > 0$, then there exists a positive integer Q and good (Q,R)-colorings $c_n : S_n \to Q$ such that the sequence $\{(S_n, c_n)\}_{n=1}^{\infty}$ is convergent as a sequence of colored Schreier graphs.*

We start the proof with a proposition.

Proposition 4.1.2. *Take $\{S_n\}_{n=1}^{\infty}$ to be as in Theorem 5 and fix a positive $Q \in \mathbb{N}$. Let $c_n^0 : S_n \to Q$ be a uniformly random vertex coloring of S_n. Then, with probability 1, the sequence $\{(S_n, c_n^0)\}_{n=1}^{\infty}$ is convergent.*

Proof. Let $B \in \mathcal{B}^{r,d}$ be a ball type. By the definition of Benjamini-Schramm convergence, we know that

$$\lim_{n\to\infty} \frac{|S_n(B)|}{|S_n|}$$

exists, where $S_n(B)$ is the set of vertices in S_n such that $B_r(S_n, x)$ is rooted-isomorphic to B. Let $\hat{B}$ be some Q-colored copy of B. We need to prove that $\lim_{n\to\infty} \frac{|S_n^{c_n^0}(B)|}{|S_n|}$ exists. Without loss of generality, we assume that $\lim_{n\to\infty} \frac{|S_n(B)|}{|S_n|} \neq 0$. Observe that if k is large enough, then for any $n \geq 1$ we can partition $S_n(B)$ into sets $A_n^1, A_n^2, \ldots, A_n^k$ in such a way that for any $i = 1, \ldots, k$ and $v, w \in A_n^i$ we have that if $v \neq w$, then $d(v, w) > 2r$. We may also guarantee that there exists a constant $C > 0$ such that for every $i = 1, \ldots, S$, we have that $|A_n^i| \geq C|S_n|$. Let us consider the event that a given vertex $v \in V(G_n)$ is such that $B_r(S_n, c_n^0, v) \cong \hat{B}$ as a colored graph. We will denote this event by E_v. Then we have that

$$\Pr(E_v) = Q^{-|B|},$$

as the uniformly random coloring of the r-ball is a Bernoulli random variable. We will simply denote this probability by $P(B)$. Moreover, for $v \neq w \in A_n^i$, the events E_v and E_w are independent. Thus, by Hoeffding's inequality we have that

$$\Pr\left(\left|\frac{|S_n^{c_n^0}(\hat{B}) \cap A_n^i|}{|A_n^i|} - P(B)\right| > \epsilon\right) < 2\exp(-\epsilon^2|A_n^i|). \tag{4.1}$$

Therefore,

$$\begin{aligned}
&\Pr\left(\left|\frac{|S_n^{c_n^0}(\hat{B})|}{|S_n|} - P(B)\frac{|S_n(B)|}{|S_n|}\right| > \epsilon\right) \\
&= \Pr\left(\left|\sum_{i=1}^{k}\left(\frac{|S_n^{c_n^0}(\hat{B}) \cap A_n^i|}{|S_n|} - P(B)\frac{|S_n(B) \cap A_n^i|}{|S_n|}\right)\right| > \epsilon\right) \\
&\leq \Pr\left(\sum_{i=1}^{k}\left|\frac{|S_n^{c_n^0}(\hat{B}) \cap A_n^i|}{|S_n|} - P(B)\frac{|A_n^i|}{|S_n|}\right| > \epsilon\right) \\
&\leq \Pr\left(\bigvee_{i=1}^{k}\left(\left|\frac{|S_n^{c_n^0}(\hat{B}) \cap A_n^i|}{|S_n|} - P(B)\frac{|A_n^i|}{|S_n|}\right| > \epsilon\frac{|A_n^i|}{|S_n|}\right)\right) \\
&\leq \sum_{i=1}^{k}\Pr\left(\left|\frac{|S_n^{c_n^0}(\hat{B}) \cap A_n^i|}{|S_n|} - P(B)\frac{|A_n^i|}{|S_n|}\right| > \epsilon\frac{|A_n^i|}{|S_n|}\right) \\
&= \sum_{i=1}^{k}\Pr\left(\left|\frac{|S_n^{c_n^0}(\hat{B}) \cap A_n^i|}{|A_n^i|} - P(B)\right| > \epsilon\right).
\end{aligned} \tag{4.2}$$

In the above, the first inequality follows from the triangle inequality and the second one from the pigeonhole principle. (The symbol $\bigvee_{i=1}^{k}\varphi_i$ denotes the alternative of events, or the statement that at least one of the events φ_i occurs.)

Thus, from inequalities (4.1) and (4.2) we obtain

$$\Pr\left(\left|\frac{|S_n^{c_n^0}(\hat{B})|}{|S_n|} - P(B)\frac{|S_n(B)|}{|S_n|}\right| > \epsilon\right) \leq 2\sum_{i=1}^{k}\exp(-\epsilon^2|A_n^i|) \leq 2k\exp(-\epsilon^2 C|S_n|),$$

since we assumed $|A_n^i| \geq C|S_n|$. Now we notice that since $\liminf_{n\to\infty} \frac{|S_n|}{n} = D > 0$, it follows that

$$\sum_{n=1}^{\infty} \Pr\left(\left|\frac{|S_n^{c_n^0}(\hat{B})|}{|S_n|} - P(B)\frac{|S_n(B)|}{|S_n|}\right| > \epsilon\right) \leq \sum_{n=1}^{\infty} 2k\exp(-\epsilon^2 C|S_n|) < \infty.$$

Thus, by Borel-Cantelli Lemma,

$$\lim_{n\to\infty} \frac{|S_n^{c_n^0}(\hat{B})|}{|S_n|} = P(B) \lim_{n\to\infty} \frac{|S_n(B)|}{|S_n|}$$

with probability 1. □

From now on, we call a positive integer T (r,d)**-large** if T is larger than the size of any r-ball in a graph of vertex degree bound d. Now we shall give an upper bound on the probability that in the randomly Q-colored sequence of graphs the relative sizes of the sets of badly colored vertices are large.

Proposition 4.1.3. *Let $\{S_n\}_{n\in\mathbb{N}}$ be a sequence of graphs as above and let us fix an $\epsilon > 0$ and $R > 0$. Then there exists a $Q = Q_{\epsilon,R} > 0$ such that for the uniformly random Q-colorings c_n of S_n and for large enough n, we have that*

$$\Pr\left(\frac{|\mathcal{B}_n^R|}{|S_n|} > \epsilon\right) \leq \exp(-C\epsilon^2|S_n|),$$

where the constant C is only dependent on R. Here $\mathcal{B}_n^R$ denotes the set of (c_n, R)-bad vertices in (S_n, c_n).

Proof. Let us pick a $Q > 0$ to be at least twice as large as the lowest $(R, 2d)$-large number. Moreover, let us fix a K which is $(3R, 2d)$-large. Similarly to the proof of Proposition 4.1.2, we find partitions of $V(S_n)$ into sets $A_n^i, i \in \{1, \dots, K\}$, such that

- whenever $v, w \in A_n^i, v \neq w$, we have that $d(v, w) > 2R$,

- there exists $\alpha > 0$ such that $\liminf_{n\to\infty} \frac{|A_n^i|}{|V(G_n)|} \geq \alpha$ for all $i \in \{1, \ldots, K\}$.

From now on, we assume that n is large enough that $\frac{|A_n^i|}{|S_n|} \geq \frac{\alpha}{2}$. Since $\{A_n^1, \ldots, A_n^K\}$ is a partition of $V(S_n)$, we obtain that

$$\begin{aligned}\Pr\left(|\mathcal{B}_n^R| > \epsilon|V(S_n)|\right) &= \Pr\left(\sum_{i=1}^K |\mathcal{B}_n^R \cap A_n^i| > \sum_{i=1}^K \epsilon|A_n^i|\right) \\ &\leq \Pr\left(\bigvee_{i=1}^K |\mathcal{B}_n^R \cap A_n^i| > \epsilon|A_n^i|\right) \leq \sum_{i=1}^K \Pr\left(|\mathcal{B}_n^R \cap A_n^i| > \epsilon|A_n^i|\right).\end{aligned}$$

For a vertex $v \in V(S_n)$, let F_v denote the event that v is a (c_n, R)-bad vertex. Notice that

$$\Pr(F_v) = 1 - \left(\frac{(Q-1)}{Q}\right)^{|V(B_R(S_n,v))|-1} \leq \frac{\epsilon}{2}.$$

Since for $v, w \in A_n^i$ the events F_v, F_w are independent, we have that

$$\mathbb{E}(|\mathcal{B}_n^R \cap A_n^i|) = \sum_{v \in A_n^i} \Pr(F_v) \leq \frac{\epsilon}{2}|A_n^i|.$$

Therefore,

$$\sum_{i=1}^K \Pr\left(|\mathcal{B}_n^R \cap A_n^i| - \frac{\epsilon}{2}|A_n^i| > \frac{\epsilon}{2}|A_n^i|\right) \leq \sum_{i=1}^K \Pr\left(|\mathcal{B}_n^R \cap A_n^i| - \mathbb{E}(|\mathcal{B}_n^R \cap A_n^i|) > \frac{\epsilon}{2}|A_n^i|\right).$$

Now we apply Hoeffding's inequality to obtain

$$\begin{aligned}\sum_{i=1}^K \Pr\left((|\mathcal{B}_n^R \cap A_n^i| - \mathbb{E}(|\mathcal{B}_n^R \cap A_n^i|) > \frac{\epsilon}{2}|A_n^i|\right) &\leq \sum_{i=1}^K \exp\left(-2\frac{\epsilon}{2}^2|A_n^i|\right) \\ &\leq K\exp\left(-\epsilon^2|S_n|\frac{\alpha}{4}\right) = \exp\left(-\epsilon^2|S_n|\frac{\alpha}{4} + \log(K)\right).\end{aligned}$$

We may pick n large enough that $\log K \leq \epsilon^2 |V(S_n)| \frac{\alpha}{8}$. Thus, we obtain that there exists a constant C dependent on R such that

$$\Pr\left(\frac{|\mathcal{B}_n^R|}{|S_n|} > \epsilon\right) \leq \exp(-\epsilon^2 |S_n| C)$$

holds for large enough n. $\square$

4.1.1 Recolorings

Given any $R \in \mathbb{N}$ we will define a sequence of colorings $\{c_n^m\}_{m\in\mathbb{N}}$ for each graph S_n in the sequence in the following way:

- c_n^0 is a uniformly random Q-coloring of the vertices of S_n with Q as in Proposition 4.1.3;

- having defined c_n^m, we set $\mathcal{B}_n^{R,m}$ to be the set of (c_n^m, R)-bad points. For a vertex $v \in V(S_n)$, set $Q_v^m = c_n^m[B_R(S_n, v)] \setminus \{c_n^m(v)\}$ - the set of those colors which appear in the R-neighborhood of v excluding the color of v. Then we let

$$c_n^{m+1}(v) = \begin{cases} \text{a random number from } Q \setminus Q_v^m & \text{if } v \in \mathcal{B}_n^{R,m}, \\ c_n^m(v) & \text{otherwise.} \end{cases}$$

Throughout, we will call the sequence $\{c_n^m\}_{m=1}^{\infty}$ the **recolorings** of S_n.
Let $\mathcal{B}^{d,R,Q}$ denote the set of all rooted R-balls in $\mathbb{F}_d\mathcal{G}^Q$. We can view the recolorings of S_n as being obtained in a local recoloring procedure that does not depend on the graph itself, but only on the neighborhoods of vertices. That is, the local recoloring procedure is a random variable $X : \mathcal{B}^{d,R,Q} \to \mathcal{B}^{d,R,Q}$ such that if $\hat{B} = (B, c, v) \in \mathcal{B}^{d,R,Q}$, where B is the underlying graph of $\hat{B}$, c is the coloring function and v is the root of $\hat{B}$, then

- if v is a (c, R)-good vertex, then $X(\hat{B}) = \hat{B}$;
- otherwise, $X(\hat{B}) = (B, c', v)$, where $c'(w) = c(w)$ for all vertices $w \neq v$ of B and $c'(v)$ is a uniformly randomly chosen element of $(Q \setminus c[B]) \cup \{c(v)\}$.

In this way, we view the coloring c_n^m as the local recoloring procedure applied to the colored graph (S_n, c_n^0) m times.

If H is a rooted graph (colored or uncolored) of radius r and $0 < l < r$, we will write $H{\restriction}l$ to denote the subgraph of H induced on the vertices whose distance from the root is at most l. Let us now fix any $r \in \mathbb{N}$. For the purpose of showing convergence of the sequence $\{S_n, c_n^m\}_{n=1}^{\infty}$ (with a fixed $m > 0$) we will be interested in the rate of occurence of each ball $\hat{B} \in \mathcal{B}^{d,r,Q}$. For a coloring c_n^m of $V(S_n)$, we will write $p_n^{B,m} := \frac{|S_n^{c_n^m}(B)|}{|S_n|}$. Let $B_r^m(v)$ denote the r-ball around a vertex $v \in V(G_n)$ with its coloring arising from the coloring c_n^m. Notice that for any $m \geq 1$ and vertex v of S_n, its neighborhood $B_r^m(v)$ depends on the neighborhood $B_{r+R}^{m-1}(v)$. Therefore, for every colored ball $D \in \mathcal{B}^{d,r+R,Q}$ there is a probability $p^{D,\hat{B}}$ that after applying the local recoloring procedure to each vertex of $D{\restriction}r$, the resulting graph D' satisfies $D'{\restriction}r \cong \hat{B}$ as colored graphs. For example, if $D \in \mathcal{B}^{d,r+R,Q}$ is an R-well colored graph and $D{\restriction}r \cong \hat{B}$ as colored graphs, then $p^{D,\hat{B}} = 1$. If, on the other hand, $D{\restriction}r \ncong \hat{B}$ and D is R-well colored, then $p^{D,\hat{B}} = 0$. Let us fix an $m \geq 0$ and consider the sequences $\{(S_n, c_n^m)\}_{n=1}^{\infty}$ and $\{(S_n, c_n^{m+1})\}_{n=1}^{\infty}$. One may ask, for a fixed $\hat{B} \in \mathcal{B}^{d,r,Q}$, how does the $p_n^{\hat{B},m+1}$ depend on $p_n^{D,m}$ for $D \in \bigcup_{r\in\mathbb{N}} \mathcal{B}^{d,R,Q}$? It is clear that $p_n^{\hat{B},m+1}$ in fact only depends on $p_n^{D,m}$ for $D \in \mathcal{B}^{d,r+R,Q}$. For a $D \in \mathcal{B}^{d,r+R,Q}$ and an $\hat{B} \in \mathcal{B}^{d,r,Q}$, we define $A_n^{D,\hat{B},t+1}$ to be the set of those vertices v of S_n for which

- $B_{r+R}^t(v) \cong D$,
- $B_r^{t+1}(v) \cong \hat{B}$,

as colored graphs. Then we note the following.

Remark 4.1.4. Let $r > 0, m > 0, \hat{B} \in \mathcal{B}^{d,r,Q}$, and $\{(S_n, c_n^m)\}_{n=1}^{\infty}$ be a sequence of colored graphs. Then

$$p_n^{\hat{B},m+1} = \sum_{D \in \mathcal{B}^{d,r+R,Q}} \frac{|A_n^{D,\hat{B},m}|}{|S_n|}.$$

When $A \subseteq \bigcup_{r=1}^{\infty} \mathcal{B}^{d,r,Q}$ and $\{(S_n, c_n)\}_{n=1}^{\infty}$ is a sequence of coloured Schreier graphs, we will write $p_n^A = \sum_{\hat{B} \in A} p_n^{\hat{B}}$ and $p^A = \sum_{\hat{B} \in A} p^{\hat{B}}$ provided that $p^{\hat{B}}$ exist. The following result shows that if we recolor each graph enough times, then the size of the badly colored parts of each S_n will tend to 0.

Proposition 4.1.5. *Suppose that $\{S_n\}_{n=1}^{\infty}$ is as in Theorem 5 and set $m_n = \lceil \log n \rceil$ for $n \geq 2$ and $m_n = 1$ for $n = 1$. Then, with probability* 1*,*

$$\lim_{n \to \infty} \frac{|\mathcal{B}_n^{R,m_n}|}{|S_n|} = 0.$$

Proof. First, let us recall that $0 < \epsilon < \frac{1}{2l}$, where l is an upper bound for the size of an R-ball in a graph with degree bound d. Notice that for an arbitrary $m > 1$

$$\Pr\left(\frac{|\mathcal{B}_n^{R,m}|}{|S_n|} \leq \frac{\epsilon}{2^m}\right) \geq$$
$$\Pr\left(\frac{|\mathcal{B}_n^{R,m}|}{|S_n|} \leq \frac{\epsilon}{2^m} \middle| \frac{|\mathcal{B}_n^{R,m-1}|}{|S_n|} \leq \frac{\epsilon}{2^{m-1}}\right) \cdot \Pr\left(\frac{|\mathcal{B}_n^{R,m-1}|}{|S_n|} \leq \frac{\epsilon}{2^{m-1}}\right).$$

We wish to bound $\Pr\left(\frac{|\mathcal{B}_n^{R,m}|}{|S_n|} \leq \frac{\epsilon}{2^m} \middle| \frac{|\mathcal{B}_n^{R,m-1}|}{|S_n|} \leq \frac{\epsilon}{2^{m-1}}\right)$ from below. We are considering the probability of an event that having recolored S_n m times, the size of the badly colored part is at most $\frac{\epsilon}{2^m}|S_n|$, on the condition that before the m-th round of recoloring of the graph S_n, the bad part is of size at most $\frac{\epsilon}{2^{m-1}}|S_n|$. When we recolor a part of a graph which has some k vertices, we certainly have at least some l available colors with l $(R, 2d)$-large because we picked Q to be twice the size of the smallest $(R, 2d)$-large number. Then, the event that the bad part after the m-th recoloring is of size at most $\epsilon 2^{-m}$ occurs with probability which is at least the same as the probability that a graph on $\lfloor \epsilon |S_n| \rfloor$ vertices which is randomly colored with at least l colors has at most $\frac{1}{2}\lfloor \epsilon |S_n| \rfloor$ badly colored vertices.

From this observation and by the bound on ϵ it follows that

$$\Pr\left(\frac{|\mathcal{B}_n^{R,m}|}{|S_n|} \leq \frac{\epsilon}{2^m} \middle| \frac{|\mathcal{B}_n^{R,m-1}|}{|S_n|} \leq \frac{\epsilon}{2^{m-1}}\right) \geq \Pr\left(\frac{|\mathcal{B}_n^{R,1}|}{\lfloor \epsilon |S_n| \rfloor} \leq \frac{1}{2}\right) \geq \Pr\left(\frac{|\mathcal{B}_n^{R,1}|}{|S_n| - 1} \leq \frac{\epsilon}{2}\right).$$

The above implies that

$$\Pr\left(\frac{|\mathcal{B}_n^{R,m}|}{|S_n|} \leq \frac{\epsilon}{2^m}\right) \geq \prod_{i=1}^{m} \Pr\left(\frac{|\mathcal{B}_n^{R,1}|}{2^{-1}(|S_n| - 1)} \leq \epsilon\right).$$

Moreover, notice that Proposition 4.1.3 holds for any $(R, 2d)$-large number; the Q was chosen with a large margin of safety for that Proposition. Thus, it follows that

$$\Pr\left(\frac{|\mathcal{B}_n^{R,1}|}{|S_n|} \leq \epsilon\right) \geq 1 - \exp(-C|S_n|\epsilon^2). \tag{4.3}$$

Hence we obtain that

$$\Pr\left(\frac{|\mathcal{B}_n^{R,m}|}{|S_n|} \leq \frac{\epsilon}{2^m}\right) \geq \prod_{i=1}^{m}(1-\exp(-C(|S_n|-1)2^{-1}\epsilon^2)) \geq (1-\exp(-C(|S_n|-1)2^{-1}\epsilon^2))^m.$$

Now we use the inequality $(1-a)^k \geq 1-ka$ which holds for any natural k and $a \in [0,1]$ to find that

$$\Pr\left(\frac{|\mathcal{B}_n^{R,m}|}{|S_n|} \leq \frac{\epsilon}{2^m}\right) \geq 1 - m\exp(-C(|S_n| - 1)2^{-1}\epsilon^2).$$

Notice that we set $m_n = \lceil \log n \rceil$ and $\liminf_{n\to\infty} \frac{|S_n|}{n} > 0$ and so there exists a constant p such that for large enough n we have the following:

- $|S_n| - 1 \geq pn$,
- $\log\lceil \log n \rceil \leq C\epsilon^2(|S_n| - 1)2^{-2}$.

Then for these n, k, p we have that

$$\Pr\left(\frac{|\mathcal{B}_n^{R,m_n}|}{|S_n|} \geq \frac{\epsilon}{2^{m_n}}\right) \leq \lceil \log n \rceil \exp(-C(|S_n|-1)2^{-1}\epsilon^2)$$
$$\leq \exp(-C(|S_n|-1)2^{-1}\epsilon^2 + \log\lceil \log n \rceil) \leq \exp\left(-C(|S_n|-1)\epsilon^2(2)^{-2}\right)$$

Thus we obtain that the series

$$\sum_{n=1}^{\infty} \Pr\left(\frac{|\mathcal{B}^{R,m_n}|}{|S_n|} \geq \frac{\epsilon}{2^{m_n}}\right)$$

is convergent. Hence by the Borel-Cantelli Lemma, with probability 1 only finitely many times we have that $|\mathcal{B}_n^{R,m_n}| \geq |S_n|\epsilon 2^{-m_n}$. As $m_n \to \infty$, the Proposition follows. $\square$

From now on we consider a sequence of colored graphs $\{(S_n, c_n')\}_{n\in\mathbb{N}}$ where $c_n' = c_n^{m_n}$ where m_n is as in Proposition 4.1.5. For an $\hat{B} \in \mathcal{B}^{d,r,Q}$ we will write $q_n^{\hat{B}} := \frac{|A_n^{\hat{B},m_n}|}{|S_n|}$. We aim to prove that the sequence $\{(S_n, c_n')\}_{n\in\mathbb{N}}$ is convergent. However, first we need to show the following statement.

Proposition 4.1.6. *For a fixed $t \in \mathbb{N}$, the sequence $\{(S_n, c_n^t)\}_{n=1}^{\infty}$ is convergent as a sequence of colored graphs.*

Proof. We shall proceed by induction. By Proposition 4.1.2, for $t = 0$ the result is true. Assume it is true for all $0 \leq u \leq t$ and we will show it holds for $t + 1$.
Let us fix a $\hat{B} \in \mathcal{B}^{d,r,Q}$ and a $D \in \mathcal{B}^{d,r+R,Q}$.

Claim 4.1.1. *There exists a $q \geq 0$ for which with probability 1 we have that $\frac{|A_n^{D,\hat{B},t+1}|}{|S_n|} \to q$ as $n \to \infty$.*

We prove the Claim similarly to the Proposition 4.1.2. Consider $p^{D,\hat{B}}$. If $p^{D,\hat{B}} = 0$, then by definition $A_n^{D,\hat{B},t+1} = \emptyset$ and so $q = 0$. We define $A_n^{D,t} \subseteq S_n$ to be the set of those vertices v of (S_n, c_n^t), for which $B_{r+R}(v) \cong D$. Then, if $\frac{|A_n^{D,t}|}{|S_n|} \to 0$, then similarly

$q=0$. Let us thus assume that $p^{D,\hat{B}}>0$ and $\frac{|A_n^{D,t}|}{|S_n|}\to s>0$.
We pick an S to be an $(3(R+r),d)$-large integer and we let $A_n^i, i=1,\dots,S$ to be a partition of $A_n^{D,t}$ which satisfies

1. if $v,w\in A_n^i, v\neq w$, then $d(v,w)>2(R+r)$,

2. there exists a constant $C>0$ such that for all $i=1,\dots,S$, $\liminf_{n\to\infty}\frac{|A_n^i|}{|S_n|}>C$.

Then, applying Hoeffding inequality, pigeonhole principle, triangle inequality and Borel-Cantelli lemma, as in Proposition 4.1.2, we obtain that with probability 1, $\frac{|A_n^{D,\hat{B},t+1}|}{|S_n|}$ is convergent as $n\to\infty$. We denote the limit by q.
Observe that there exist finitely many $D\in\mathcal{B}^{d,r+R,Q}$ and by Remark 4.1.4

$$p_n^{\hat{B},t+1}=\sum_{D\in\mathcal{B}^{d,r+R,Q}}\frac{|A_n^{D,\hat{B},t+1}|}{|S_n|},$$

and thus $\{p_n^{\hat{B},t+1}\}_{n=1}^\infty$ is convergent. □

Proposition 4.1.7. *The sequence $\{(S_n,c_n')\}_{n\in\mathbb{N}}$ is convergent as a sequence of colored graphs.*

Proof. First let us pick any $\epsilon>0$, an $\hat{B}\in\mathcal{B}^{d,r,Q}$ for a fixed $r\in\mathbb{N}$ with Q given in Proposition 4.1.3. We aim to show that the sequence $\{q_n^{\hat{B}}\}_{n=1}^\infty$ is Cauchy. To this end, let us fix a positive integer M which satisfies $\sum_{i=M}^\infty\frac{1}{2^i}\leq\frac{\epsilon}{4}$. By Proposition 4.1.6 the sequence $\{p_n^{\hat{B},M}\}_{n=1}^\infty$ is convergent with probability 1, so we pick an $N>0$ such that for all $n,m>N$ it is true that $|p_n^{\hat{B},M}-p_m^{\hat{B},M}|<\frac{\epsilon}{2}$. Inequality (4.3) in the proof of Proposition 4.1.5 implies that

$$\Pr\left(\frac{|\mathcal{B}_n^{r,m}|}{|S_n|}\right)>\Pr\left(\frac{|\mathcal{B}_n^{r,1}|}{|S_n|}>\epsilon\right)\leq\exp(-C|S_n|\epsilon^2).$$

Let us fix some $n, m > N$. Then we have that with probability 1

$$|q_n^{\hat{B}} - q_m^{\hat{B}}| \leq |q_n^{\hat{B}} - p_n^{\hat{B},M}| + |q_m^{\hat{B}} - p_m^{\hat{B},M}| + |p_m^{\hat{B},M} - q_n^{\hat{B},M}| \leq 2\sum_{i=M}^{\infty} 2^{-i} + \frac{\epsilon}{2} \leq \epsilon.$$

Hence the sequence $\{q_n^{\hat{B}}\}_{n=1}^{\infty}$ is Cauchy and therefore convergent. $\square$

Let B_n denote the set of (c'_n, R)-badly colored vertices in S_n. We have constructed colorings c'_n of S_n for which the sequence $\{S_n, c'_n\}_{n=1}^{\infty}$ is convergent and $\frac{|B_n|}{|S_n|} \to 0$ with probability 1. To finish the proof of Theorem 5, we recolor the sets B_n in $\{S_n, c'_n\}_{n=1}^{\infty}$ using the greedy algorithm in order to obtain the good R-colorings of S_n's. We call these colorings c_n. Since the colorings c_n differ from c'_n on vertices whose size relative to the size of S_n tends to 0 and $\{S_n, c'_n\}_{n=1}^{\infty}$ is convergent, we have that $\{S_n, c_n\}_{n=1}^{\infty}$ is convergent as well. $\square$

4.2 The Nonsingular Schramm Theorem

In [34] Oded Schramm proved (using a bit different language) that if a sequence $\{G_n\}_{n=1}^{\infty}$ of Schreier graphs converges to a measure preserving action $\alpha : \Gamma \curvearrowright (X, \mu)$ then α is hyperfinite if and only if $\{G_n\}_{n=1}^{\infty}$ is a hyperfinite sequence. The goal of this section is to prove the following theorem.

Theorem 6. *Let $\{(S_n, w_n)\}_{n=1}^{\infty}$ be a tight sequence of WGS-graphs converging to a nonsingular action $\alpha : \mathbb{F}_d \curvearrowright (X, \mu)$ in the weighted Benjamini-Schramm sense. Then α is hyperfinite if $\{(S_n, w_n)\}_{n=1}^{\infty}$ is weighted hyperfinite.*

4.2.1 Construction of a hyperfinite limit action

The first step to prove Theorem 6 is to find a single hyperfinite action which is a limit of the sequence $\{(S_n, w_n)\}_{n=1}^{\infty}$.

Proposition 4.2.1. *Let* $\{(S_n, w_n)\}_{n=1}^{\infty}$ *be a convergent weighted hyperfinite sequence of WGS-graphs. Then there exists a hyperfinite nonsingular action* $\alpha : \mathbb{F}_d \curvearrowright (X, \mu)$ *which is the limit of the sequence* $\{(S_n, w_n)\}_{n=1}^{\infty}$.

Proof. Let $m \geq 1$. Since $\{(S_n, w_n)\}_{n=1}^{\infty}$ is hyperfinite, there exists $K_m > 0$ and subsets $Y_n^m \subset V(S_n)$ for which

- $w_n(Y_n^m) < \frac{1}{m}$;
- the size of the components of the graphs induced on $V(S_n) \backslash Y_n^m$ is at most K_m.

Let $Q_m > Q'_m > 0$ be such that for any $n \geq 1$ there exists a coloring $\varphi_n^m : V(S_n) \to Q_m$ satisfying the following conditions:

- if $0 < d_{S_n}(x, y) < K_m$ then $\varphi_n^m(x) \neq \varphi_n^m(y)$,
- if $x \in Y_n^m$ then $\varphi_n^m(x) \in Q'_m$,
- if $x \in V(S_n) \setminus Y_n^m$, then $\varphi_n^m(x) \notin Q^1_m$.

Since the vertex degrees of S_n are bounded by $2d$, such Q_m, Q'_m exist. Thus, we obtain a sequence of colorings

$$\Phi_n : V(S_n) \to \prod_{m=1}^{\infty} Q_m \, .$$

Since $\prod_{m=1}^{\infty} Q_m$ is homeomorphic to the Cantor set $\mathcal{C}$, we abuse notation slightly to say that for any $n \geq 1$ $\{(S_n, w_n, \Phi_n)\}_{n=1}^{\infty} \in C\mathbb{F}_d\mathcal{G}^{\mathcal{C}}$. By compactness, there exists a subsequence $\{(S_{n_i}, w_{n_i}, \Phi_{n_i})\}_{i=1}^{\infty}$ convergent to an element $\kappa \in \mathrm{QRC}_{\mathbb{F}_d}^{\mathcal{C}}$.

Let us consider the associated Γ-action $\alpha : \Gamma \curvearrowright (C\mathbb{F}_d\mathcal{G}^{\mathcal{C}}, \kappa)$. Observe that κ is concentrated on colored cocycles (S, p, F, Φ) for which Φ is injective. In other words, for κ-almost every $x \in C\mathbb{F}_d\mathcal{G}^{\mathcal{C}}$ the orbit graph of x coincides with the image of the canonical map $M_\alpha(x)$. By definition, on the space $C\mathbb{F}_d\mathcal{G}^{\mathcal{C}}$ the canonical map M_α and the forgetting map Forg coincide. Therefore, the QRC $M_\alpha(\kappa)$ is a weighted Benjamini-Schramm limit of the sequence $\{S_{n_i}, w_{n_i}\}_{i=1}^{\infty}$. Hence, α is a limit action of the sequence $\{S_n, w_n\}_{n=1}^{\infty}$.

Now, we need to prove that the action $\alpha : \Gamma \curvearrowright (C\mathbb{F}_d\mathcal{G}^{\mathcal{C}}, \kappa)$ is hyperfinite. Let $m \geq 1$ and let $Y^m \subseteq C\mathbb{F}_d\mathcal{G}^{\mathcal{C}}$ denote the set of those vertices $Y^m \subset C\mathbb{F}_d\mathcal{G}^{\mathcal{C}}$ whose m-th coordinate is in Q'_m. By convergence, $\kappa(Y^m) \leq \frac{1}{m}$. This is meaningful since we identified $\mathcal{C}$ with the product space $\prod_{m=1}^{\infty} Q_m$. Furthermore, for κ-almost every element $x \in C\mathbb{F}_d\mathcal{G}^{\mathcal{C}}$ the components of the subgraph induced on the vertices of the orbit graph of x which lie outside of Y^m are of size at most K_m. Therefore, the action α is indeed hyperfinite. $\square$

4.2.2 Kaimanovich's Theorem

We found a hyperfinite limit action α for a hyperfinite sequence of WGS-graphs. However, it is not clear whether any action β with the same QRC as α is hyperfinite. The first step to proving that this is indeed the case is Kaimanovich's theorem.

In [20], Kaimanovich formulated a theorem which links the hyperfiniteness of an action with the isoperimetric properties of its graphing. However, the original proof is somewhat sketchy, so we provide an alternative one. Elek provided the proof of the theorem of Kaimanovich for probability measure preserving group actions in [10]. In this section we present the theorem for non-singular actions of groups with a proof similar to that in [10].

Let $\alpha : \Gamma \curvearrowright (X, \mu)$ be a non-singular action and let $\mathcal{G}_\alpha$ denote its graphing. We say that $\mathcal{H}$ is a subgraphing of positive measure of $\mathcal{G}_\alpha$ if $\mu(V(\mathcal{H})) > 0$. Without losing generality, we can assume that the Radon-Nikodym cocycle exists on X (if not, then we remove a nullset from the set X where the cocycle is not well-defined). If $x, y \in X$ and $\gamma \in \Gamma$ are such that $\alpha(\gamma)(x) = y$, we denote by $R(x, y)$ the Radon-Nikodym derivative $\frac{d\alpha(\gamma)_*\mu}{\mu}(x)$. Thus, for any triple x, y, z lying in the same orbit of α, we have

$R(x,y)R(y,z) = R(x,z)$. For any $x \in X$ we can define a (possibly infinite) vertex measure $|\cdot|_x$ on the orbit of x, denoted $O_\alpha(x)$, by defining

$$|A|_x = \sum_{y \in V(A)} R(x,y)$$

for any subset $A \subseteq O_\alpha(x)$. Given any $x \in X$ let $N(x)$ denote the set of neighbors of x in the orbit graph of x and for any $A \subseteq O_\alpha(x)$ let $N(A) = \bigcup_{x \in A} N(x)$ be the set of neighbors of elements of A.

Given a point x and a finite subset F of $O_\alpha(x)$ we will say that the isoperimetric constant of F is

$$i_x(F) = \frac{|\partial(F)|_x}{|F|_x}$$

where $\partial(F) = \{x \in V(H) : N(x) \not\subseteq V(H)\}$ denotes the boundary of F. Note that since F is finite and the graphing of α is taken with respect to a finite generating system, $i_x(F)$ is well-defined. Furthermore, because R forms a cocycle, the isoperimetric constant of any given set does not depend on the choice of the point x. Thus, we can define $i(F) = i_x(F)$.

Now we define the isoperimetric constant of any connected (Γ, Σ)-Schreier graph G as

$$i(G) = \inf\{i(F) : F \subseteq V(G), |F| < \aleph_0\}.$$

In particular, if G is finite, then the isoperimetric constant of G is zero.

Definition 4.2.2. Let $\mathcal{G}_\alpha$ be a graphing of a non-singular action of Γ on a Borel probability measure space (X, μ) with respect to a finite symmetric generating system Σ. Then

- $\mathcal{G}_\alpha$ has **property (A)** if for every subgraphing $\mathcal{H} \subseteq \mathcal{G}_\alpha$ of positive measure, μ-almost every component of $\mathcal{H}$ has isoperimetric constant 0;

- $\mathcal{G}_\alpha$ has **property (B)** if for every $\epsilon > 0$ and any subgraphing $\mathcal{H} \subseteq \mathcal{G}_\alpha$ of positive measure, there is a subgraphing $\mathcal{S} \subseteq \mathcal{H}$ which intersects μ-almost every component of $\mathcal{H}$ such that each component C of $\mathcal{S}$ is finite and the isoperimetric constant of C in $\mathcal{H}$ is less than ϵ.

We now formally state the theorem of Kaimanovich.

Theorem 7 (Kaimanovich, [20]). *Let $\alpha : \Gamma \curvearrowright (X, \nu)$ be nonsingular action as above. Then the following conditions are equivalent:*

1. *the action α is hyperfinite (in this case we say that $\mathcal{G}_\alpha$ is hyperfinite);*

2. *$\mathcal{G}_\alpha$ has property (A).*

The following proposition is crucial in the proof of the theorem.

Proposition 4.2.3. *The properties (A) and (B) are equivalent.*

Proof. Clearly a graphing with property (B) has property (A), so we only need to prove the converse.

Let $\mathcal{H} \subseteq \mathcal{G}_\alpha$ be any subgraphing of positive measure such that almost all components of $\mathcal{H}$ have isoperimetric constant 0. For $x \in V(\mathcal{H})$ let $B_n(x)$ denote the ball of radius n around x in the component of $\mathcal{H}$ containing x. First, we pick an $\epsilon > 0$. We will construct a subgraphing $\mathcal{S}$ of $\mathcal{H}$ with the desired properties using induction.

Let $d_{\mathcal{G}_\alpha}$ denote the pseudometric on X given by

$$d_{\mathcal{G}_\alpha}(x, y) = \begin{cases} p_{xy} & \text{if } x, y \text{ lie in the same component of } \mathcal{G}_\alpha; \\ \infty & \text{otherwise,} \end{cases}$$

where p_{xy} denotes the length of the shortest path between x and y in the graphing $\mathcal{G}_\alpha$. By the result of Kechris, Solecki and Todorcevic [23] there exists some $r_1 \in \mathbb{N}$ and a Borel coloring c_1 of $V(\mathcal{H})$ with colors from r_1 which satisfies that if $d_{\mathcal{G}_\alpha}(x, y) < 6$ then

$c_1(x) \neq c_1(y)$. Now, let $A_i^1 = c_1^{-1}[\{i\}]$ be the i-th color class. Then, for any $x \in V(\mathcal{H})$ let $K_x^{1,1}$ be the (possibly empty) family of finite subsets of $B_2(x)$ with isoperimetric constant less than ϵ which contain x. We may assume that X is actually the interval $[0,1]$ (as there exists a measurable isomorphism between these spaces, see e.g. Chapter 3 in [35]) and we set the following linear order $\prec$ on $K_x^{1,1}$:

- if $|A| < |B|$ then let $A \prec B$,
- if $|A| = |B|$ and $\min(A \setminus B) < \min(B \setminus A)$, then $A \prec B$.

Then let $R_x^{1,1}$ be the $\prec$-smallest element of $K_x^{1,1}$. Clearly, $R^{1,1} := \bigcup_{x \in A_1^1} R_x^{1,1}$ is a measurable set.

We construct sets $R^{1,k}$, $1 \leq k \leq r_1$ inductively. Having defined $R^{1,i}$ for all $1 \leq i \leq k-1 \leq r_1$ take any $x \in A_k^1$ and define $K_x^{1,k}$ in the following way. If the component of x already contains some element in the form of $R_y^{1,i}$, where $1 \leq i \leq k-1$, then let $K_x^{1,k}$ be the empty set. Otherwise, let $K_x^{1,k}$ be the family of finite subsets of $B_2(x)$ which contain x and whose isoperimetric constant in $\mathcal{H}$ is less than ϵ. In the same way as before, we can define an order $\prec$ on $K_x^{1,k}$ and let $R_x^{1,k}$ be the $\prec$-smallest subset of $K_x^{1,k}$. We set $R^{1,k} = \bigcup_{x \in A_k^1} R_x^{1,k}$. The graphing $\mathcal{S}_1$ is defined as the subgraphing of $\mathcal{H}$ induced on $\bigcup_{k \leq r_1} R^{1,k}$.

Observe that $\mathcal{S}_1$ consists of finite components whose isoperimetric constants in $\mathcal{H}$ are less than ϵ. However, $\mathcal{S}_1$ might not intersect almost all the components of $\mathcal{H}$ so we continue by constructing a sequence of graphings $\{\mathcal{S}_n\}_{n=1}^{\infty}$ inductively in the following way.

Assume $\mathcal{S}_i$ has already been defined for all $1 \leq i \leq n$. Now, let c_{n+1} be a Borel coloring of $V(\mathcal{H})$ with colors from the large enough set $\{1, \ldots, r_{n+1}\}$ such that for any x, y we have that $c_{n+1}(x) \neq c_{n+1}(y)$ whenever $d_{\mathcal{G}}(x,y) < 2n+4$. Like before, set $A_i^{n+1} = c_{n+1}^{-1}(i)$ to be the i-th color class. For $x \in A_i^{n+1}$, we define $K_x^{n+1,1}$ in the following way. If the $\mathcal{H}$-component of x intersects $\mathcal{S}_i$ for $1 \leq i \leq n$, then let $K_x^{n+1,1}$ be the empty set. If the component of x does not intersect $\mathcal{S}_i$ then we set $K_x^{n+1,1}$ to be the family of finite sets F contained in $B_{n+1}(x)$ such that F has isoperimetric constant less than ϵ. Again, we define an ordering $\prec$ on $K_x^{n+1,1}$ in order to obtain the subset $R_x^{n+1,1}$. Continuing this process, we define the subgraphing $\mathcal{S}_{n+1}$.

It is clear from the construction that $\mathcal{S} = \bigcup_{n=1}^{\infty} \mathcal{S}_n$ is an induced subgraphing of $\mathcal{H}$ with finite components whose $\mathcal{H}$-isoperimetric constants are less than ϵ. It suffices to show that $\mathcal{S}$ as defined above intersects μ-almost all of the components of $\mathcal{H}$. Take an element $x \in X$ whose component in $\mathcal{H}$ has isoperimetric constant 0. Then, there exists a finite subset F in the orbit graph of F with isoperimetric constant less than ϵ. Therefore, for some $n \in \mathbb{N}$, $F \subset B_n(x)$. By the construction of $\mathcal{S}$, there is some $k \leq n$, such that $\mathcal{S}_k$ intersects the component of x. $\square$

Given a graphing $\mathcal{G}_\alpha$ on a standard probability measure space (X, μ) and a measurable $Z \subseteq X$ we write $\mathcal{G} - Z$ to denote the subgraphing of $\mathcal{G}$ induced on $X \setminus Z$. Now, we prove Kaimanovich's Theorem.

Proof of theorem 7. First, assume that $\mathcal{G}_\alpha$ is hyperfinite and take its subgraphing $\mathcal{H}$ of positive measure. Suppose that $\mu(V(\mathcal{H})) = a > 0$ and fix a real number $\epsilon > 0$. Let $Z \subseteq X$ be a set with $\mu(Z) < \frac{\epsilon^2}{a}$ and such that all components of $\mathcal{G}_\alpha - Z$ are of size at most K for some $K \in \mathbb{N}$. Then for any $x \in V(\mathcal{H}) \setminus Z$ we denote by F_x the set consisting of the vertices in the component C_x of x in $\mathcal{H} - Z$ together with the elements of Z which are adjacent (in $\mathcal{H}$) to elements of C_x. Before we continue the proof of Theorem 7, we make the following observation.

Lemma 4.2.4. *Let $\mathcal{H}_\epsilon$ be the set consisting of those $x \in V(\mathcal{H})$, for which the $\mathcal{H}$-isoperimetric constant of F_x is less than ϵ. Then $\mu(\mathcal{H}_\epsilon) \geq a - \frac{\epsilon}{a}$.*

Proof. Let $A \subseteq V(\mathcal{H})$ be the complement of the set of vertices of $\mathcal{H}_\epsilon$. By assumption, $\partial(A)$ is contained in Z. Therefore A satisfies the inequality

$$\mu(A) < \frac{\mu(Z)}{\epsilon} = \frac{\frac{\epsilon^2}{a}}{\epsilon} = \frac{\epsilon}{a}.$$

□

It follows that if we set $\epsilon_n = \frac{1}{n}$ then $\mu(\mathcal{H}_{\frac{1}{n}}) \geq a - \frac{1}{an}$. Therefore μ-almost every component of $\mathcal{H}$ has isoperimetric constant zero.

Now assume that $\mathcal{G}_\alpha$ has property (A). Pick any $\epsilon > 0$ and set $\mathcal{H}^0 = \mathcal{G}_\alpha$. By Proposition 4.2.3, there is a subgraphing $\mathcal{S}^0 \subseteq \mathcal{H}^0$ of positive measure which intersects μ-almost every component of $\mathcal{H}^0$, has finite components and the $\mathcal{H}$-isoperimetric constant of each component of $\mathcal{S}^0$ is less than ϵ. Then we have that $\mu(\partial V(\mathcal{S}^0)) < \epsilon\mu(V(\mathcal{S}^0))$ and we set $M^0 = \partial V(\mathcal{S}^0)$.

We proceed by transfinite induction. Having defined $\mathcal{H}^\beta, \mathcal{S}^\beta$ and M^β for an ordinal β let $\mathcal{H}^{\beta+1}$ be the subgraphing of $\mathcal{H}^\beta$ induced on the set $V(\mathcal{H}^\beta) \setminus V(\mathcal{S}^\beta)$. If $\mathcal{H}^{\beta+1}$ is of positive measure, then by Proposition 4.2.3 there exists a subgraphing $\mathcal{S}^{\beta+1}$ of $\mathcal{H}^\beta$ which is of positive measure, has finite components and each of these components has $\mathcal{H}$-isoperimetric constant smaller than ϵ. Now put $M^{\beta+1} = \partial V(\mathcal{S}^{\beta+1})$. For a limit ordinal λ, having defined $\mathcal{H}^\beta, \mathcal{S}^\beta$ and M^β for all $\beta < \lambda$ let $\mathcal{H}^\lambda$ be the subgraphing of $\mathcal{G}_\alpha$ induced on the set $V(\mathcal{G}) \setminus \bigcup_{\beta<\lambda} V(\mathcal{S}^\beta)$. If $\mathcal{H}^\lambda$ is of positive measure, then let $\mathcal{S}^\lambda$ be an induced subgraphing of $\mathcal{H}^\lambda$ of positive measure with finite components and such that the $\mathcal{H}$-isoperimetric constant of each of its components is less than ϵ. Moreover, we set $M^\lambda = \partial(\mathcal{S}^\lambda)$. Now, since $\mu(V(\mathcal{H}^\beta))$ is a strictly decreasing transfinite sequence of positive reals, there is a countable ordinal γ such that $\mu(V(\mathcal{H}^\gamma)) = 0$. Then, we let $Z = \bigcup_{\beta<\gamma} M^\beta \cup V(H^\gamma)$. We have that the graphing $\mathcal{G}_\alpha - Z$ has finite components. This is enough, as for any $\delta > 0$ there exists a K such that the set of vertices of

$G_\alpha - Z$ which lie in components of size greater than K has measure smaller than δ. Furthermore, since for any $\beta < \gamma$ $\mu(M^\beta) = \mu(\partial V(\mathcal{S}^\beta)) < \epsilon\mu(V(\mathcal{S}^\beta))$ we have that

$$\mu(Z) = \mu\left(\bigcup_{\beta<\gamma} M^\beta\right) \leq \epsilon \cdot \sum_{\beta<\gamma} \mu(V(\mathcal{S}^\beta)) < \epsilon.$$

It follows that we can pick such a set Z of measure less than ϵ such that all components of the subgraphing $\mathcal{G}_\alpha - Z$ have finite components. Therefore, $\mathcal{G}_\alpha$ is hyperfinite. □

4.2.3 The proof of Theorem 6

In order to complete the proof of Theorem 6 we need to show that if α and β are quasi-invariant actions of Γ, α is hyperfinite and β is not, then they cannot have the same QRC. The theorem of Kaimanovich implies that if α is a hyperfinite action then the QRC induced by α is concentrated on hyperfinite elements of $C\mathbb{F}_d\mathcal{G}$. However, it is possible that a nonamenable group Γ has a measure preserving non-hyperfinite action β for which all orbit graphs of $\mathcal{G}_\beta$ have isoperimetric constants 0 (e.g. Example 4 in [21]).

Definition 4.2.5. Let $\alpha : \Gamma \curvearrowright (X, \mu)$ and $\beta : \Gamma \curvearrowright (Y, \nu)$ be two actions. We say that β is a **proper factor** of α, if there exists $\pi : X \to Y$ such that

1. π is surjective and measure preserving;

2. the restriction of π onto any orbit set of X is bijective;

3. for μ-almost all $x \in X$ and for all $\gamma \in \Gamma$, $R_\alpha(\gamma, x) = R_\beta(\gamma, \pi(x))$, where $R_\alpha(\gamma, x)$ is the Radon-Nikodym cocycle of α.

In this case we call π a **proper factor map**.

Clearly, the canonical map M_α for an action α is not necessarily a proper factor map. For instance, for a free measure preserving action of a group Γ, the QRC induced by the action is the delta measure concentrated on the Cayley graph of Γ (with respect to the fixed generating system). Thus, M_α is not bijective on the orbits. Proper factor maps are of particular interest to us since they preserve hyperfiniteness of the actions.

Proposition 4.2.6. *Let β be a proper factor of α. Then α is hyperfinite if and only if β is hyperfinite.*

Proof. First, suppose that β is hyperfinite. Let $\epsilon > 0$ and $Z \subset Y$ such that the complement of Z has components of size at most K. Then, the complement of $\pi^{-1}[Z]$ has components of size at most K. Thus, α is hyperfinite.

Now, suppose that β is not hyperfinite. By Kaimanovich's Theorem, we have a positive measure subgraphing $\mathcal{S}$ of the graphing $\mathcal{G}_\beta$ such that it is not true that almost all components of $\mathcal{S}$ has isoperimetric constant 0. Since π is a proper factor map, we have that $\pi^{-1}[\mathcal{S}]$ is also a positive measure subgraphing of $\mathcal{G}_\alpha$ and not all components of $\pi^{-1}[\mathcal{S}]$ have isoperimetric constants 0. Therefore, α is not hyperfinite as well. $\square$

The next step in the proof of Theorem 6 is to construct a non-singular independent joining of actions. To this end, we use the classical theorem on disintegration of measure in order .

Theorem 8 (Disintegration of measure, see e.g. Section 452 in [16])**.** *Let X a compact metric space with a Borel probability measure μ and let us suppose that $\mathcal{D}$ is a sub-σ-algebra of the Borel σ-algebra. Then for μ-almost every element $x \in X$ there exists a measure μ_x such that for any μ-integrable function f on X the following conditions are satisfied:*

- *the function $g_f := \int_X f(y) d\mu_x(y)$ is $\mathcal{D}$-measurable,*
- $\int_X f(x) d\mu(x) = \int_X \left(\int_X f(y) d\mu_x(y)\right) d\mu(x)$.

Proof of Theorem 6. Let $\alpha : \Gamma \curvearrowright (X, \mu)$ be the hyperfinite limit action of the weighted Schreier graph sequence $\{(S_n, w_n)\}_{n=1}^{\infty}$ constructed in Proposition 4.2.1 and $\beta : \Gamma \curvearrowright$

(Y, ν) be another limit action. By Proposition 4.2.6, it is enough to prove that there exists an action $\delta : \Gamma \curvearrowright (Z, \tau)$ such that both α and β are proper factors of δ. Then, β must be hyperfinite as well. Our idea is similar to the one in [10], but uses a slightly different construction.

Let $M_\alpha : X \to C\Gamma\mathcal{G}$ and $M_\beta : Y \to C\Gamma\mathcal{G}$ be the canonical maps as in Section 3.2.1. Clearly, $(M_\alpha)_*\mu = (M_\beta)_*\nu$, since both α and β are limit actions of the same convergent sequence. We denote this probability measure on $C\Gamma\mathcal{G}$ by ρ. Now, we construct the nonsingular independent joining of α and β over $\gamma : \Gamma \curvearrowright (C\Gamma\mathcal{G}, \rho)$, where γ is the action on the space of cocycles by moving the root. We follow [7] Section 10. By the theorem of disintegration of measures, we have measurable maps

$$\rho_X : (C\Gamma\mathcal{G}, \rho) \to \mathrm{Prob}(X)$$

and

$$\rho_Y : (C\Gamma\mathcal{G}, \rho) \to \mathrm{Prob}(Y)$$

such that for any μ- measurable set $A \subseteq X$ $\int_{C\Gamma\mathcal{G}} \rho_X(x)(A)\, d\rho(x) = \mu$ and $\int_{C\Gamma\mathcal{G}} \rho_Y(y)(A)\, d\rho(y) = \nu$. Then, the nonsingular independent joining is defined as the natural action $\delta : \Gamma \curvearrowright (X \times Y, \mu \times_\rho \nu)$, where the measure $\mu \times_\rho \nu$ is defined by

$$\int_{C\Gamma\mathcal{G}} (\mu_X \times \mu_Y) d\rho\,.$$

Then, both α and β are proper factors of δ and our theorem follows. $\square$

4.3 The Cantor Model of a Nonsingular Action

Recently, László Lovász proved the following theorem.

Theorem 9 (Lovász [25])**.** *Let $\alpha : \Gamma \curvearrowright X$ be a Borel action of a finitely generated group Γ on the standard Borel space X. Then, there exists a stable (see Section 2.2.1) continuous action of Γ on a totally disconnected compact set $\beta : \Gamma \curvearrowright \mathcal{C}$ and a Borel embedding $\Phi : X \to \mathcal{C}$ such that $\Phi \circ \alpha = \beta \circ \Phi$.*

The goal of this section is to provide a nonsingular generalization of this result.

4.3.1 The Role of the Normalizer Subgroup

Let Γ be a finitely generated group, $H < \Gamma$ be a subgroup and $(S(\Gamma/H), H)$ be the associated rooted Schreier graph. Then we have two actions of Γ:

1. the action on the left coset space Γ/H by left multiplication;

2. the action on the orbit $O(S(\Gamma/H), H) \subset \Gamma\mathcal{G}$ by moving the root.

Recall that a normalizer of a subgroup H of Γ is the maximal subgroup of Γ containing H in which H is a normal subgroup. Note that the stabilizer of $(S(\Gamma/H), H)$ in the second action is the normalizer subgroup $N(H)$ of H. Indeed, $gHg^{-1} = H$ if and only if the rooted Schreier graphs $(S(\Gamma/H), H)$ and $(S(\Gamma/H), gH)$ are isomorphic. Furthermore the stabilizer of H in action 1. is H itself. Therefore the two actions above coincide if and only if $H = N(H)$.

Now, let $\varphi : \Gamma/H \to \mathcal{C}$ be a coloring function such that $\varphi(aH) = \varphi(bH)$ if and only if $aH = bH$. Then we have a third action of Γ:

3. the action on the orbit $O(S(\Gamma/H), H, \varphi) \subset \Gamma\mathcal{G}^{\mathcal{C}}$ by moving the root.

It is easy to see that the third action is always Γ-isomorphic to the first action, even if H is not its own stabilizer in action 2. This simple observation leads to the following useful lemma (see also [11]).

Lemma 4.3.1. *Let $H < \Gamma$ and $\varphi : \Gamma/H \to \mathcal{C}$ be a function which satisfies that for any $r \geq 1$ there exists $s_r > 0$ such that if*

$$0 < d_{S(\Gamma/H)}(x, y) \leq r$$

then

$$d_{\mathcal{C}}(\varphi(x), \varphi(y)) \geq s_r,$$

where $d_{\mathcal{C}}$ is a metric on $\mathcal{C}$ defining the standard topology on the Cantor set. Then, for any element $(S(\Gamma/H'), H', \varphi')$ in the orbit closure of $(S(\Gamma/H), H, \varphi)$ in $\Gamma\mathcal{G}^{\mathcal{C}}$, the action on the orbit of $(S(\Gamma/H'), H', \varphi')$ is isomorphic to the action on the coset space Γ/H'. Hence, the action on the orbit closure of $(S(\Gamma/H), H, \varphi)$ is stable.

Proof. Clearly, a coloring φ with the property in the Lemma is injective. Furthermore, $d_{\mathcal{C}}(\varphi'(x), \varphi'(y)) \geq s_r$ holds if $0 < d_{S(\Gamma/H')}(x, y) \leq r$, since $(S(\Gamma/H'), H', \varphi')$ is in the closure of the orbit of $(S(\Gamma/H), H, \varphi)$. Hence, the map $\varphi' : \Gamma/H' \to \mathcal{C}$ is also injective. Thus, the lemma follows from our earlier observation. $\square$

Immediately, we have the following corollary.

Corollary 4.3.2. *Let $\varphi : \Gamma \to \mathcal{C}$ be a function such that for any $r \geq 1$ there exists $s_r > 0$ so that if $0 < d_{\mathrm{Cay}(\Gamma)}(x, y) \leq r$, then $d_{\mathcal{C}}(\varphi(x), \varphi(y)) > s_r$. Then, the action of Γ on the orbit closure of $(\mathrm{Cay}(\Gamma), e, \varphi) \in \Gamma\mathcal{G}^{\mathcal{C}}$ is free.*

4.3.2 The Nonsingular Lovász Theorem

Let us now formally state our theorem.

Theorem 10. *Let $\alpha : \Gamma \curvearrowright X$ be a Borel action and $c : \Gamma \times X \to \mathbb{R}^+$ be a multiplicative Borel cocycle with respect to the action α such that for all $\gamma \in \Gamma$ the function $x \mapsto c(\gamma, x)$*

is bounded and Borel. Then, there exists a stable continuous action β of Γ on a totally disconnected compact set K, a continuous multiplicative cocycle $d : \Gamma \times K \to \mathbb{R}^+$ and a a Borel embedding $\Phi : X \to K$ such that

- $\Phi \circ \alpha = \beta \circ \Phi$*;*
- *for any $x \in X$ and $\gamma \in \Gamma$ we have that $c(\gamma, x) = d(\gamma, \Phi(x))$*

Proof. First, we define a pseudometric $d_\alpha : X \times X \to \mathbb{N} \cup \{\infty\}$ in the following way:

- if x, y are not on the same orbit of α, then $d_\alpha(x, y) := \infty$;
- if $x, y \in S$, where S is the Schreier graph of an orbit of α, then $d_\alpha(x, y) := d_S(x, y)$.

Let $\Omega : X \to \{0,1\}^\omega$ be an arbitrary Borel isomorphism. For $i > 1$, let Q_i be a large finite set and $\Psi_i : X \to Q_i$ be a Borel function such that $\Psi_i(x) \neq \Psi_i(y)$, if $0 < d_\alpha(x, y) \leq i$. We write $\tilde{\mathcal{C}} := \{0,1\} \times Q_1 \times \{0,1\} \times Q_2 \times \ldots$. Then, we have a Borel map $\tilde{\tau} : X \to \tilde{\mathcal{C}}$, defined by

$$\tilde{\tau}(x) := ((\Omega(x))_1, \Psi_1(x), (\Omega(x))_2, \Psi_2(x), \ldots)$$

Let us notice that for any $x \in X$, the restriction of $\tilde{\tau}$ to the orbit graph of x satisfies the property in Lemma 4.3.1. Now, for the standard Cantor set $\mathcal{C}$ there exists a continuous isomorphism $\iota : \tilde{\mathcal{C}} \to \mathcal{C}$ which preserves this property. Thus, we define a $\mathcal{C}$-coloring $\tau : X \to \mathcal{C}$ by $\tau := \iota \circ \tilde{\tau}$. We obtain a Borel function $\Phi : X \to C\Gamma\mathcal{G}^{\mathcal{C}}$ by mapping each $x \in X$ to its orbit Schreier graph colored with τ and with a cocycle structure d given by

$$d(\gamma p, \sigma\gamma p) = c(\sigma, \gamma x)$$

for any $\gamma, \sigma \in \Gamma$ and any vertex p. By our assumption Φ is a Borel embedding. Let us note that $\mathrm{Forg} \circ \Phi$ is just the canonical map M_α, where

$$\mathrm{Forg} : C\Gamma\mathcal{G}^{\mathcal{C}} \to C\Gamma\mathcal{G}$$

is the map that "forgets" the $\mathcal{C}$-colors (see 3.1.2).

Lemma 4.3.1 implies that the action on the closure of $\Phi[X]$ is stable. Also, the closure of $\Phi[X]$ is a totally disconnected compact set. Hence, our Theorem follows. $\square$

Let us suppose that the cocycle c in Theorem 10 is the Radon-Nikodym cocycle of a quasi-invariant measure μ. Then, d is the continuous Radon-Nikodym cocycle of the quasi-invariant measure $\Phi_*(\mu)$ on the closure of $\Phi[X]$. Therefore, we have the following corollary of Theorem 10.

Corollary 4.3.3. *Let $\alpha : \Gamma \curvearrowright (X, \mu)$ be a Borel action preserving the measure class of μ and let $R : \Gamma \times X \to \mathbb{R}^+$ be the Radon-Nikodym cocycle of the action α. If for each $\gamma \in \Gamma$, the function $x \mapsto R(\gamma, x)$ is bounded, then there exists a totally disconnected compact set K and a Borel embedding $\Phi : X \to K$ such that*

- *$\tilde{R} : \Gamma \times K \to \mathbb{R}^+$ defined by $\tilde{R}(\gamma, x) = R(\gamma, \Phi(x))$ is continuous,*
- *$\tilde{R}$ is the Radon-Nikodym cocycle of $\Phi_*(\mu)$.*

4.4 Continuous Radon-Nikodym Derivatives

In this section we consider the situation when the Radon-Nikodym cocycle of an action of a finitely generated group is continuous.

4.4.1 The Radon-Nikodym Problem

It is known that for a group action α which is preserves the measure class of a probability measure μ, the Radon-Nikodym derivatives form a cocycle. On may ask the converse: for a given multiplicative cocycle S on a set X, when does there exist a measure μ for which S is the Radon-Nikodym cocycle? This is called the Radon-Nikodym Problem and it was studied e.g. by Renault [32]. Here we consider a continuous version of the problem, stated as follows.

Question: Let $\alpha : \Gamma \curvearrowright X$ be a continuous action of a finitely generated group on a compact space and $S : \Gamma \times X \to \mathbb{R}^+$ be a continuous multiplicative cocycle. Under what circumstances does there exist a quasi-invariant measure μ on X such that the Radon-Nikodym cocycle of α with respect to μ equals to S?

In [6], Cuesta and Rechtman showed that an invariant measure μ can be found if the cocycle S admits a Følner sequence, as defined below.
Let us fix a symmetric generating set Σ for Γ. Let $A \subseteq X$ be a finite set. We say that $x \in \partial(A)$ if there exists $\sigma \in \Sigma$ such that $\alpha(\sigma)(x) \notin A$. Let $y \in A$. Then we can put a probability measure $F_A : A \to \mathbb{R}^+$ by

$$F_A(x) = \frac{S_y(x)}{\sum_{x \in A} S_y(x)} \,,$$

where $S_y(x) = S(\gamma, y)$, provided $\alpha(\gamma)(y) = x$. Since S is a multiplicative cocycle, S_y is well-defined and furthermore, F_A does not depend on the choice of y. We define the isoperimetric constant of A as

$$i_S(A) := \sum_{x \in \partial(A)} F_A(x).$$

We say that a sequence of finite sets $\{A_n\}_{n=1}^{\infty}$ is a Følner sequence in X if $\lim_{n\to\infty} i_S(A_n) = 0$. In this thesis, we provide an alternative proof to the result of Cuesta and Rechtman.

Proposition 4.4.1 (Cuesta-Rechtman, [6])**.** *If S admits a weighted Følner sequence then there exists a measure μ on X such that S is the Radon-Nikodym derivative of α with respect to a quasi-invariant measure μ.*

Proof. First, we construct a continuous (real) functional F on the Banach space $C(X)$ of continuous functions $f : X \to \mathbb{R}$ with the supremum metric. Let $\{A_n\}_{n=1}^{\infty}$ be a Følner sequence on X and $f : X \to \mathbb{R}^+$ be a continuous function. Define

$$T_n(f) := \sum_{x\in A_n} f(x)F_{A_n}(x)\,.$$

Then:

- T_n is a continuous positive linear functional on $C(X)$,
- $\|T_n\| = 1$,
- $T_n(\overline{1}) = 1$, where $\overline{1}$ denotes the constant function taking value 1 everywhere.

Now, we fix a non-principal ultrafilter ω on the natural numbers and set $T(f) := \lim_{\omega} T_n(f)$, where $\lim_{\omega}$ is the ultralimit associated to ω. Then T is a continuous positive linear functional such that $T(\overline{1}) = 1$. Therefore, there exists a probability measure μ on X such that for any $f \in C(X)$,

$$T(f) = \int_X f(x)d\mu(x)\,.$$

It is enough to prove that for any generator $\sigma \in \Sigma$,

$$T(f \circ \alpha(\sigma^{-1})) = T(f_\sigma)\,,$$

where $f_\sigma(x) = f(x)S(\sigma, x)$.

Observe that

$$T_n(f \circ \alpha(\sigma^{-1})) = \sum_{x \in A_n} f(\alpha(\sigma)(x))F_{A_n}(x)$$

and

$$T_n(f_\sigma) = \sum_{x \in A_n} f(x)S(\sigma, x)F_{A_n}(x)\,.$$

Furthermore, for any $y \in A_n$ we have that

$$\frac{F_{A_n}(\alpha(\sigma)(x))}{F_{A_n}(x)} = \frac{S_y(\alpha(\sigma)(x))}{S_y(x)} = S(\sigma, x)\,.$$

So, after cancellations, we obtain that

$$|T_n(f \circ \alpha(\sigma^{-1})) - T_n(f_\sigma)| \leq \sum_{x \in \partial(A)} 2KF_{A_n}(x)\,,$$

where $K = \sup_{x \in X} S(\sigma, x)\,.$ (K exists because the cocycle S is continuous on the compact space K.) Now, since A_n is a Følner sequence, we obtain that

$$\lim_{\omega} T_n(f \circ \alpha(\sigma^{-1})) = \lim_{\omega} T_n(f_\sigma)\,,$$

that is,

$$T(f \circ \alpha(\sigma^{-1})) = T(f_\sigma).$$

□

4.4.2 Actions of exact groups

The goal of this section is to prove the following theorem.

Theorem 11. *Any finitely generated exact group Γ has a free action on the Cantor set with a quasi-invariant probability measure μ such that all the Radon-Nikodym derivatives are continuous.*

Proof. Let Γ be an exact group and Σ be a symmetric generating system for Γ. We consider the weight system w_k on the balls B_k of the Cayley graph $\mathrm{Cay}(\Gamma, \Sigma)$ as in Example 3 in Section 3.3. Note that there exists $K \geq 1$ such that for all $k \geq 1$

$$\frac{1}{K} \leq \frac{w_k(p)}{w_k(q)} \leq K\,,$$

holds, provided that p, q are adjacent vertices in the ball B_k. For $k \geq 1$ and $p \in B_k$ let $v_k(p)$ be the largest integer power of 2, which is less than or equal to $w_k(p)$. This implies that we have the bounds

$$\frac{1}{4K} \leq \frac{v_k(p)}{v_k(q)} \leq 4K\,.$$

By Proposition 3.3.3, we know that $\{B_n, w_n\}_{n=1}^{\infty}$ is a weighted hyperfinite system. This, together with the bounds on the ratios $\frac{v_k(p)}{v_k(q)}$, implies that $\{B_n, v_n\}_{n=1}^{\infty}$ is weighted hyperfinite as well. Clearly, for any $k \geq 1$ and adjacent vertices $p, q \in B_k$

$$\frac{v_k(p)}{v_k(q)}$$

can take only finitely many values, since each such ratio is bounded and each is an integer power of 2. Let $a_1, a_2, \ldots, a_r$ be a listing of these values. Now, we use the same technique as in Section 4.3. We choose finite sets $\{Q_n\}_{n=1}^{\infty}$ and we pick colorings $\varphi_k : B_k \to \prod_{n=1} Q_n$ which satisfy that for each k, n and any $x, y \in B_k$ such that $d_{B_k}(x, y) \leq n$, we have have that $\varphi_k^n(x) \neq \varphi_k^n(y)$, where φ_k^n denotes the projection of

φ_k on the n-th coordinate. By compactness, there exists a $\rho \in \mathrm{QRC}_\Gamma$ which is a limit point of the sequence $\{B_k, v_k\}_{k=1}^{\infty}$. Then we have that:

1. as it was shown in Section 4.3, ρ is given by an action of Γ on $C\Gamma\mathcal{G}^{\mathcal{C}}$ by moving the root with a quasi-invariant measure μ. The action is essentially free and the measure μ is concentrated on cocycles with values $\{a_1, a_2, \ldots, a_r\}$ and with a proper $\mathcal{C}$-coloring. The set of such colored cocycles is homeomorphic to a compact subset of the Cantor set;

2. the Radon-Nikodym derivatives on the generators take values in $\{a_1, a_2, \ldots, a_r\}$ and all the derivatives are continuous;

3. by Theorem 6 the action is μ-hyperfinite.

Hence Theorem 11 follows. $\square$

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