An introduction to spatial dispersion: revisiting the basic concepts

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I describe three ways that the spatial properties of a wave propagation medium can cause dispersion, and propose that they should form the basics for correctly understanding and *naming* phenomena described as "spatial dispersion". In particular, I emphasise the specific spatial properties which generate the resulting dispersive – i.e. *spatially dispersive* – behaviour. The properties are geometry, structure, and dynamics.

I. INTRODUCTION

Spatial dispersion is the non-local dependence of material properties on both direction and wavelength, and is of particular interest in electromagnetic systems and other fields where artificial functional materials (AFMs) are finding technological uses, such as in acoustic and elastic metamaterials. It has been long observable in crystals [1], but is also of particular importance in AFMs where the wavelength of radiation becomes comparable to the lattice parameters (see e.g. [2]). Further, it is also significant in the region of the material resonances utilized in metamaterial unit cells, where constitutive parameters such as the permittivity and permeability – and likewise their acoustic or elastic counterparts – are maximised or have near-zero *local* values [3] or where the response is non-reciprocal [4]. In some materials its presence is revealed by the shape of the equi-frequency surfaces of the dispersion relations – e.g. when they are non-symmetric, not elliptical or hyperbolic, or having multiple modes with the same direction and polarisation.

Spatial dispersion usually appears as a either spatially nonlocal effect that produces a wavevector dependence of the material parameters, or as a non-trival wavevector dependence for the dispersion relations. Although spatial dispersion can be modelled in an ad hoc manner to suit intuition or an empirical fit to some data, it is preferable to be more systematic.

However, the term "spatial dispersion" is often used rather loosely, and so sometimes the origin of the specific phenomenon being discussed is unclear. In an attempt to rectify this, here I describe three physically distinct ways that spatial dispersion can arise, briefly describing examples of each. Each of these mechanisms has (a) no time dependence beyond that required to support a wave, and (b) a specification of time-independent material properties. The lack of any (non-trivial) time dependence ensures that the phenomena treated here are entirely unrelated to the more commonly considered topic of temporal dispersion. The first two spatial dispersion mechanisms depend entirely on spatial properties, whereas the third derives from a coupling to a dynamic medium, leading to a changed wave velocity and/or an "effective mass" term.

The categorization here shows that instances of spatial dispersion can be caused by the geometric, structural, or dynamic properties of a system. This categorization is a result not based on existing (and often somewhat ad hoc) naming conventions or justifications for spatial dispersion, but is instead grounded in the following assertion:

Spatial dispersion refers to any dispersive behaviour that occurs (solely) as a result of spatial properties of the system.

This viewpoint is distinct from a widely used one (see e.g. [5]) where spatial dispersion is characterised as being due to (non-trivial) wavevector-dependence in the dispersion relations, and is then categorized into two cases 'weak' spatial dispersion, where the spatial effects have been approximated as a local effect; and 'strong' spatial dispersion, where non-local effects remain important [6, 7]. It is to be noted that so-called weak spatial dispersion is not necessarily small in any practical sense, despite the implications of 'weak'; and likewise so-called strong spatial dispersion need not be particularly dominant[1].

In particular, in the interest of both brevity and not repeating existing discussions in the literature, I do not:

1. discuss how homogenization schemes might reduce the description of a complicated medium into a simple one, (e.g.) perhaps one based on a series expansion in terms of the wavevector k;
2. consider dispersion relations that have been artificially constructed *without* reference to some specific physical system with spatial properties; e.g. putative dispersia based on suggested or "what-if" polynomials in frequency ω and wavevector k, some other specification of an ω, k interdependence, or a non-local convolution over spatial properties.

The main focus here is on spatial dispersion in electromagnetic systems or artifical functional media such as metamaterials, but the distinctions apply equally well to waves in

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[1] In my view, terms such as 'weak' or 'strong' should be only used to to indicate the strength or significance of a phenomenon, not its origin or type, or whether it might be effectively local or non-local.

acoustic or elastic materials, or indeed potentially *any* material system which supports wave propagation of some kind. In Section II, I will define what I mean by dispersion (i.e. specifically non-trivial dispersion) by considering the relationships between the frequency ω of a wave and the wavevector k associated with that frequency. Following that the next three sections address each of geometric (III), structural (IV), and dynamic (V) spatial dispersions in turn. Note that the idea of spatial dispersion as being grounded in spatial properties as opposed to (spectral) wavevector ones is mirrored in the various treatments of temporal dispersion: although sometimes seen as a consequence of a dynamic time-domain process [8] particularly when implemented in FDTD algorithms [9], temporal dispersion is more often discussed in solely terms of a (spectral) frequency response [10, 11].

In what follows, I do not describe the example systems in a detailed manner. Only the minimum features necessary to make the point are presented, as such models are often worked through both exhaustively and frequently in other sources. Here their role is simply to provide examples that typify their role in creating a dispersive response as a result of their particular spatial features or structure. Nonlinear effects are not considered.

II. DISPERSION

The definition of dispersion used here is based on three criteria relating to how the frequency (f or ω) of a wave is related to its wavelength λ or wavevector k. Note that in discrete systems, analytic solutions typically depend on an integer index m (and possibly more indices as well). Since these indices usually indicate the number or spatial frequency of the oscillation in a given solution, they play the same role as the wavevector k, and contain essentially the same physical content. In simple cases, the wavevector k can be a straightforward multiple of m; in any case here the wavevector associated with an index m is k_m, and likewise the related angular frequency is ω_m.

The criteria are:

Linear – the relationship between ω and k (or between ω and m or k_m) is linear, i.e. it lies along a single straight line.

Origin – if $\omega = 0$ then $k = 0$ (or $m = 0$), and vice versa.

Continuous – the relationship is continuous, i.e. is in one piece, and has no gaps or jumps.

If all these criteria hold, then the wave is *dispersionless*, and the phase and group velocities are always the same. If any of the criteria do not hold, then the wave is dispersive, and an initial pulse shape will change over time as its different spectral components evolve at different rates. In general, this behaviour might be due to either the time-response of the medium (giving rise to temporal dispersion), or to the spatial properties of the medium (giving rise to spatial dispersion), or possibly both.

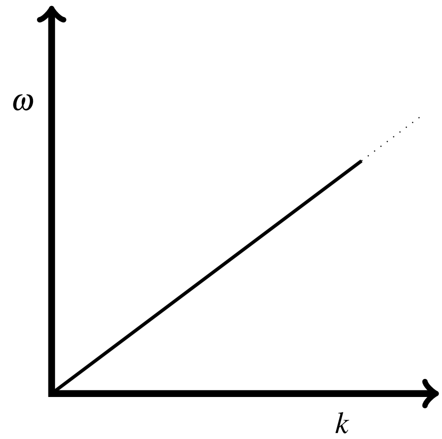

FIG. 1: Dispersionless case, where the dispersion relation is linear, passes through and includes the origin, and is continuous. The light dotted line indicates that the depicted dispersion behaviour continues in a similar manner for higher wavevectors.

Here I focus exclusively on dispersion that results solely from *spatial* properties. However, it is worth keeping in mind that typically the structural elements relied on in a physical device to create those spatial properties may in reality also be temporally dispersive – notably, reflecting walls will not be perfectly reflective at all frequencies, and different refractive indices exist primarily because of temporal dispersion; some of these issues are discussed in the Appendix. In what follows I ignore such complications because I wish to focus solely on spatial properties and their effects; but this is not to say that such time-domain responses may always be ignored in practical situations.

A dispersionless wave travelling along the z–axis with propagation speed c follows the wave equation

$$\partial_t^2 E - \partial_z c^2 \partial_z E = 0, \tag{1}$$

where $\partial_t \equiv d/dt$, and $\partial_z \equiv d/dz$. This equation can be Fourier transformed in both time and space, and the field strength terms cancelled, giving the "dispersionless" dispersion relation

$$\omega^2 - c^2 k^2 = 0. \tag{2}$$

A depiction of this relationship between ω and k can be seen on fig. 1.

Note that this is quadratic in both ω and k, so that either might have negative values and still help satisfy the dispersion relation. Any physical meaning(s) that is attributable to such negative spectral quantities is comprehensively discussed elsewhere [12], so for simplicity we will only consider positive values here. Typically the sign choices correspond to the different direction in which propagating waveforms will evolve [13–15].

III. GEOMETRIC SPATIAL DISPERSION

The *geometry* of a region in which the wave of interest is supported is perhaps the most primitive possible source of a spatial effects like dispersion.

For the purposes here, I consider a (purely) geometric system to be one where the properties of the propagation medium are homogeneous, isotropic, and are not temporally dispersive. This medium is present everywhere except at any boundaries, and to guarantee complete confinement within the propagation medium boundaries are assumed to be perfectly reflective. Note that this category also includes systems without boundaries, such as those confined on (e.g.) a closed surface such as a torus or a sphere.

In such systems, especially if there is sufficient symmetry, we can often find analytic solutions for its frequency eigenmodes. In such a case, we have an index m rather than a Fourier-transform based wavevector, but the role is the same: m indicates the number of spatial oscillations over some relevant distance interval; typically it is closely related to the number of nodes in its eigenfunction.

A. Cavity

Although any shape of wave-confining, empty cavity would a candidate for supporting spatial dispersion, the simplest would be a 1D perfectly reflective cavity with a length L. In such a case, where the wave amplitude is zero on the boundary, the dispersion relation is

$$\omega = \frac{m+1}{2} \frac{2\pi c}{L}, \tag{3}$$

where m is a non-negative integer, and the equivalent wavevector is $k_m = \pi(m+1)/L$. The dispersion relation for this system is depicted schematically on fig. 2.

By the criteria set in Section II, this *is* spatially dispersive since although the relationship between ω and k is linear, with $\omega = ck$, there is no supported wave with $\omega = 0$ and $k = 0$, and only a discrete spectrum of waves exist.

The primary effect of spatial dispersion here is simply the imposition of a discrete spectrum, although the removal of any $(\omega, k) = (0,0)$ solution is also important. Other cavity shapes, such as cylindrical or spherical, are widely covered in undergraduate textbooks on electromagnetism so I do not present them here, but they also will (at least) have a discrete spectrum and no $(0,0)$ solution.

B. Topology: torus

A wave-supporting space without boundaries is a candidate for supporting spatial dispersion (only) if it is also finite, so that it's size provides an intrinsic length scale. The simplest situation is probably a toroidal space, which is essentially the same as the case of an infinite and periodic lattice, or indeed of periodic boundary conditions. In 1D, the torus is a simple

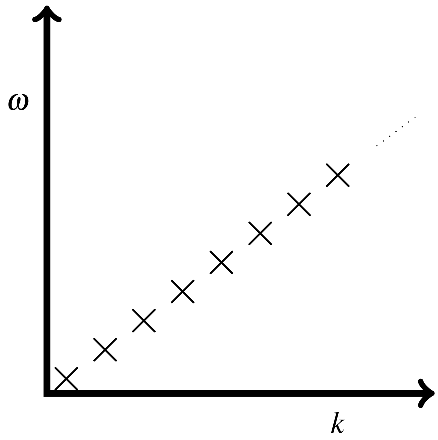

FIG. 2: Geometric spatial dispersion in a 1D cavity – the dispersion relation in this case is a series of points, as indicated by the crosses. It is linear, does not include the origin, and is not continuous.

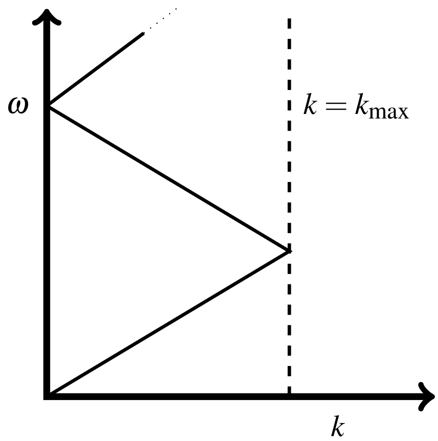

FIG. 3: Geometric spatial dispersion on a loop (a 1D torus) – the dispersion relation is piece-wise linear but not strictly linear, does include the origin, and is continuous.

loop, with the length L giving the periodicity scale and setting an effective maximum wavevector $k_{\max} = 2\pi/L$. In a dispersion plot, this gives rise to band folding [16] – i.e. although a non-periodic dispersion relation would normally extend to both high wavevector and high frequency, those parts of the dispersion at too-high wavevector are "folded" back to lower wavevector; so that the periodic case is restricted to a finite k range, as can be seen on fig. 3. This means that at low wavevectors and lowest frequencies, the system appears dispersionless, but once band folding occurs for the higher frequencies, even though the dispersion might be linear and continuous, the folded dispersion can no longer be extrapolated from every point to pass through $(0,0)$.

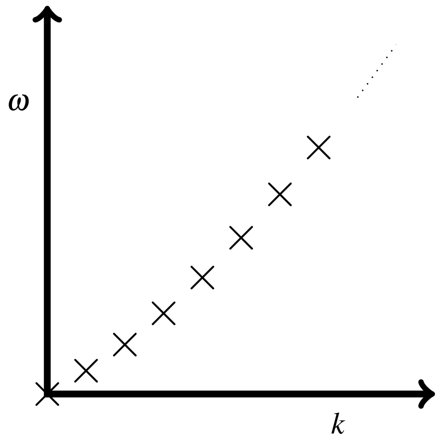

FIG. 4: Geometric spatial dispersion on a sphere – the dispersion relation in this case is a series of points, as indicated by the crosses. It is nonlinear, includes the origin, but is not continuous.

C. Topology: sphere

A more interesting case than those above is a resonator where the waves are confined on a spherical surface, which is equivalent to a Maxwell's fisheye lens [17]. The supported modes or this are derived from the Legendre polynomials[2] $P_m(\xi)$. These mode functions P_m provide a complete and countable orthonormal basis set for all possible radial field patterns in the resonator, with the argument $\xi = (r^2-1)/(r^2+1)$ being derived from the radial displacement r from some choice of preferred origin. Of course there can also be an angular dependence to field patterns, which can easily be included using the usual angular mode functions [18], but I omit those details here in the interests of brevity. Each discrete mode has a frequency ω_m determining its physical properties as determined from its index m,

$$\omega_m^2 \propto m\,(m+1) \tag{4}$$

with modes of larger m (or ω_m) having more spatial oscillations. A depiction of the spatial dispersion properties on a sphere can be seen on fig. 4.

Here the effect of spatial dispersion is significant, because two of the criteria are violated – the relationship is not linear, and the spectrum is discrete.

IV. STRUCTURAL SPATIAL DISPERSION

Structural spatial dispersion is distinct from the geometric spatial dispersion above, in that it is due to material inhomogeneity: i,e, that there are two or more types of material supporting the wave field that are arranged in a structure. This type is the origin of most instances of spatial dispersion that are considered. As we see in the slab waveguide example below, even very simple types of inhomogeneity can generate spatial dispersion; however, the necessary calculations are usually non-trivial.

A. Slab waveguide

An electromagnetic slab waveguide is one of the simplest structures that might be considered when looking for an example of structural spatial dispersion in a non-periodic system. It consists of a planar slab of one type of material sandwiched on either side by half-infinite regions of an alternative material, where the difference in material properties allows modes to exists that are localised in a way centred on the slab.

Electromagnetic slab waveguides are treated in a wide range of textbooks (see e.g. [19], or the abbreviated summary in [14] with a discussion of dispersion handling). Here the waveguide is taken to have thickness d in the perpendicular x direction, propagation in the z direction, and with core and cladding permittivities ε_1 and ε_2. To solve this system, we take Maxwell's equations in component form, assume plane-wave like behaviour in orientations parallel to the slab, and, at the boundaries, match the sin() or cos() functional form in the core with decaying exponentials in the cladding. Even for this simple slab design, the boundary conditions give the dispersion relation a non trivial form. Notably, for the bound modes it is given by the solution to a transcendental equation [19]. Typically, this is written in a way implying we want to calculate k_x and $\beta = k_z$ from a specified ω; although a compelling argument can be made [14] that it is better to calculate ω from a provided k_z.

For the transverse electric (TE) field modes, the traditional presentation shows that in a slab waveguide we have

$$T(k_x d) = k_x^{-1}\sqrt{\omega^2\mu_0\left(\varepsilon_1-\varepsilon_2\right)-k_x^2}, \tag{5}$$

$$\text{where}\quad \beta^2(\omega) = \omega^2\mu_0\varepsilon_1 - k_x^2, \tag{6}$$

and $T(k_x d)$ is either $+\tan(k_x d)$ or $-\cot(k_x d)$.

Fig. 5 indicates the appearance of the dispersion relations for the first few bound modes of such a waveguide. This slab waveguide case is a somewhat similar problem to finding the modes of an optical fibre, although the cylindrical symmetry of a fibre means that the solutions involve matching Bessel functions across the slab boundaries.

As a passing note, the spatial dispersion due to the structural configuration of an optical fibre but instead temporal in origin. This is because a majority of optical pulse propagation techniques are propagated along a spatial axis [12, 13, 20–24], which means that the dispersion computations are more convenient in ω than they would be in the k available in temporally propagated techniques [14, 15].

B. Bragg mirror

A Bragg mirror is composed of multiple thin layers of dielectric material, with the layers designed so that the device is

[2] See http://dlmf.nist.gov/14

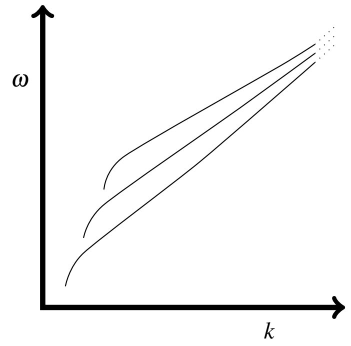

FIG. 5: Structural spatial dispersion in a slab waveguide – the dispersion relation is nonlinear, does not include the origin, and is not continuous. This schematic depiction shows exaggerated dispersion curves for the first few bound modes.

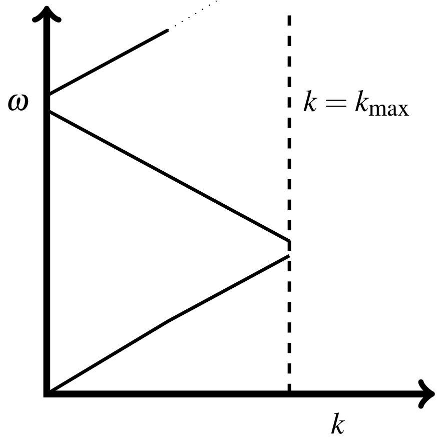

FIG. 6: Structural spatial dispersion in a Bragg mirror. The periodic structure gives it similarities with the torus case of fig. 3, but in the maximum and minumum k regions, the two branches bend apart to create bandgaps, where certain (ω, k) combinations do not propagate. The dispersion relation is not linear, does include the origin, and is not continuous.

highly reflective at some wavelength, or in some wavelength range. Simple versions consist of stacks of alternating high and low refractive material, with thicknesses chosen so that the path-length differences for internal reflections are integer multiples of the design wavelength. Here, a unit cell for the stack consists of just the two contrasting layers.

In the model calculation of Horsley et al. [25], an infinite periodic stack is considered. The analysis shows that the relationship between structure and eigenvalue – i.e. its dispersion relation is controlled by the expression

$$\lambda = z(\omega) \pm \sqrt{z(\omega)^2 - 1} = \exp\left[\imath\kappa(a+b)\right]. \qquad (7)$$

Here κ is the unit-cell Bloch wavevector, and the layer thicknesses are a and b. The effect of the material refractive indices n_a and n_b are subsumed into the real valued $z(\omega)$, which is calculated from the unit-cell's transfer matrix.

For our purposes it is sufficient to note that in cases where $|z| > 1$, κ becomes complex and $|\lambda|$ is no longer unity. These represent diverging (non-periodic) solutions which are physically prohibited and so do not form part of the dispersion relation. The resulting dispersion relation is nonlinear, and is discontinuous across the (reflective) bandgap, although continuous elsewhere; it is depicted in 6.

C. Wire media

Wire media are a class of metamaterials consisting of a regular (rectangular) array of parallel wires or rods. Thus the system is periodic along both the transverse axes, and (usually) uniform along the longitudinal one. This is a hard problem to solve in the general case, but results can be found when the wires have a radius is small compared to their spacing [26–28]. Conveniently, it turns out that under approximation, such media have a quadratic spatio-temporal dispersion relation. This behaviour for the light in such a structure mimics that for light propagating in a plasma. The approximate spatial dispersion relation is

$$\omega^2 - \beta^2 k^2 = c^2 K^2, \qquad (8)$$

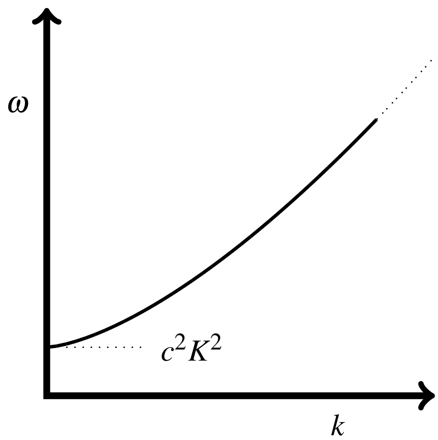

FIG. 7: Spatial dispersion in an approximated wire medium. The cut-off c^2K^2 gives a minimum allowed frequency, but in the limit as k becomes large the curve approaches the line $\omega = \beta k$.

which has a cut-off frequency c^2K^2. The resulting dispersion curve is depicted on fig. 7.

This is an example where the effect of inhomogeneity under approximation induce a structural spatial dispersion that mimics a mass-like term $\propto K^2$ in the dispersion (see e.g. [29]). After back Fourier transforming this relation for the electric field E from ω, k into t, x we get the wave equation

$$\partial_t^2 E - \beta^2 \partial_x^2 E + c^2 K^2 E = 0. \qquad (9)$$

It is interesting to compare this wave behaviour, and dispersion, to the dynamic case discussed next: despite their very different origins, the qualitative behaviour is the same.

V. DYNAMIC SPATIAL DISPERSION

Dynamic spatial dispersion is a result of non-trivial propagation properties of the material, and how that affects the wave of interest that propagates through it. For the spatial dispersion induced in wire media, we have already seen above that spatial properties can alter the effective dynamics of the light propagation. Partly because of this, and partly because of existing usage, I categorise such propagation effects as spatial mechanism here because it results from excitations (spatially) moving through the medium[3]. The origin of the resulting dispersion is a direct result of the material properties, where the propagating wave of interests drives excitations in the medium, and those excitations then affect the propagation of the orginal wave.

One might consider quite a wide variety of wave models when treating dynamic spatial dispersion; for example, in acoustic or elastic media, we might think that any of the somewhat eclectic selection in [15] could be suitable. Indeed, if used as a priori mechanisms, they might be, but it is worth noting that such wave models result from applying approximations to an underlying, more complicated microscopic description. As such, the physical effects treated are more like side-effects of structural spatial dispersion as treated in section IV, rather than native dynamic spatial dispersion.

Because of such complications, here I will consider only one model of dynamic spatial dispersion, the hydrodynamic model for plasmonics (HMP) [30]. As a simple wave equation, which is (only and exactly) second order in both space and time it avoids issues of causality as long as its built-in speed parameter stays less than that of light.

A. Plasmons

The HMP has recently found widespread popularity in the field of plasmonics and electromagnetism. One of its key features is that unlike simpler plasmonics approaches based on the Drude model for the dielectric permittivity, it incorporates spatial derivatives terms which represent the dynamics of the charge distribution. Although these are often called "non-local" effects, they are more usefully called propagation effects, since they are not non-local in any sense that violates relativistic (signalling) constraints on the physics. However, although much of its usage in plasmonics is recent, the basic model itself dates back to the 1970's [31] and has been used in a number of other contexts [32, 33].

One point regarding naming conventions that needs careful attention here is the use of "plasmon", which can be applied in various ways, just as "polariton" can refer to a range of phenomena. Here I use plasmon to mean the medium excitation only, i.e. the propagating disturbance in the electron gas (or dielectric polarization); I do not mean the coupled electromagnetic – electron-gas system which exhibits spatial dispersion[4].

The equation for the HMP in a 1D case is [30]

$$\partial_t^2 \boldsymbol{P} + \gamma \partial_t \boldsymbol{P} - \partial_x \beta^2 \partial_x \boldsymbol{P} = \varepsilon_0 \omega_{\mathrm{P}}^2 \boldsymbol{E}, \tag{10}$$

where the polarization field is $\boldsymbol{P} \equiv \boldsymbol{P}(t,x)$ and the driving electric field is $\boldsymbol{E} \equiv \boldsymbol{E}(t,x)$. The speed of disturbances in the polarization field $\boldsymbol{E}$ is β, polarization losses are given by γ, and ω_{P}^2 is a measure of how strongly the electric field drives disturbances in the amplitude of $\boldsymbol{P}$. With respect to discussions about the utility and meaning of the HMP, we may consider the deconstructed plasmon model at [34], and also the variant current-based version as opposed to the usual polarization version as used here.

Taking the 1D case for simplicity, then after a double transform into ω and k, while also ignoring losses[5] we get

$$\omega^2 \boldsymbol{P} - k\beta^2 k \boldsymbol{P} = \varepsilon_0 \omega_{\mathrm{P}}^2 \boldsymbol{E}. \tag{11}$$

When linked to the standard EM wave equation we find that this gives a homogeneous effective permittivity of

$$\varepsilon(\omega,k) = \varepsilon_0 \left(\frac{\omega_{\mathrm{P}}^2}{\omega^2 - k^2\beta^2} \right), \tag{12}$$

and a dispersion relation

$$\omega^2 - \beta^2 k^2 - \omega_{\mathrm{P}}^2 = 0. \tag{13}$$

If rewritten as a wave equation for light–plasmon polaritons, and with the substitution $\omega_{\mathrm{P}} = ck_{\mathrm{P}}$, this is

$$\partial_t^2 E - \beta^2 \partial_z^2 E + c^2 k_{\mathrm{P}}^2 E = 0. \tag{14}$$

This polariton equation can be factorized into two first order pieces [34] in a way which emphasizes a spatial origin for the behaviour. In the freely propagating case without driving terms, and with the auxilliary field Q, this is

$$\partial_t E = (\beta \partial_z + ck_{\mathrm{P}}) Q \tag{15}$$

$$\partial_t Q = (\beta \partial_z - ck_{\mathrm{P}}) E. \tag{16}$$

The behaviour of this polariton model can be seen on figure 8, and shows the dynamic spatial dispersion, with the dispersion relation being non-linear and lacking a $(0,0)$ solution.

VI. SUMMARY

Here I have addressed the basic causes of spatial dispersion, and briefly summarized examples of each. However, since the term "spatial dispersion" is often used rather loosely,

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[3] Note that it is arguable that it should be instead regarded as a space-time effect, due to its wave-like underlying model. However, here I will retain it as a pure spatial effect for consistency with existing usage; also note that the ω^2 term is not modified from that in the usual vacuum case.

[4] I would regard this coupled system as a kind of polariton.

[5] Losses are ignored because they are solely a temporal effect, and these examples are intended to present purely spatial mechanisms.

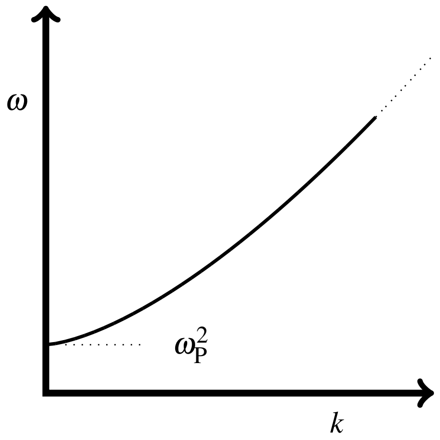

FIG. 8: Dynamic spatial dispersion – a typical dispersion curve based on how the EM field couples to the hydrodynamic model for plasmons. The cut off frequency ω_P gives a minimum allowed frequency, but in the limit as k becomes large the curve approaches the line $\omega = \beta k$.

I first set out to specify clearly what I meant by dispersion, and what kinds of *spatial* properties might generate it: notably by the geometric, structural, or dynamic properties of a system. Since my classification is not based on existing (and often somewhat ad hoc) naming conventions or justifications for phenomena called (or attributed to) "spatial dispersion", it may be challenging to adapt to or accept. However, it is a definition derived solely based on trying to answer the question: What different types of (solely) spatial properties can result in dispersive behaviour?

Acknowledgments

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Appendix: Materials and Metamaterials – a discussion

Material properties generally are a result of both spatial (structural) and temporal effects, and often these are coupled together. Even apparently simple properties such as the refractive index (which is usually simply a proxy for the material permittivity) are infact a complicated combination of the (many) individual temporal responses of the material's constituent atoms and molecules. In a crystal, the electronic structure of each atom gives that atom it's own independent temporal response, and the arrangement of those atoms in space specifies the material's spatial (structural) properties. The combination gives the crystal a combined spatio-temporal dispersion which potentially quite complicated behaviour.

These complications were deliberately not addressed above, because that would have obscured the goal of defining those spatial features which by themselves can induce dispersive behaviour. Of course, in many practical cases the fine details can be ignored. If we are happy with the approximation that "the refractive index of window glass is about 1.5", then we have ignored its true spatio-temporal dispersion, and any residual solely-spatial or solely-temporal dispersion, and settled for describing it in a non-dispersive way. Within the regime of such (effectively) non-dispersive media, we can add large-scale (i.e. non-microscopic) spatial properties, and look at solely spatially dispersive effects, as I have done above.

However, if we want to reproduce how light of different colours travels at different speeds in glass, then we must retain the temporally dispersive part of the glass behavior. Further, if we want to show how a glass prism can split white light into a rainbow, we also add in some of the spatial information, i.e. only the prism's shape, but not the spatial properties of the atoms and molecules within the glass. In such natural media it is typically the case that the scales on which microscopic and bulk properties act are distinct enough so that we can treat them additively.

In contrast to natural media, metamaterials tend to have dispersive behaviour which is harder to approximate. There are two significant reasons for this. Firstly, metamaterial design often relies on time dependent (dynamic) behavior to get strong responses, so since the impinging wave field is – by design – rather near the metamaterial resonance, we cannot easily ignore the temporal contributions to dispersion. Secondly, because metamaterials are themselves constructed of natural metarials, their unit cell scales are larger with respect to the impinging wavelengths than for natural materials where unit cells are atomic of molecular in size; thus we cannot easily ignore the spatial (structural) contributions to dispersion. Indeed, if we want to engineer a magnetic response, then we cannot make a metamaterial cell size "negligible" in any useful sense [35].

An important point to emphasize is that many types of metamaterials rely on both their spatial and temporal properties to work. For example, many (such as the split ring resonator (SRR)) are based on shaped metal structures, in which electric currents can be induced by any impinging electromagnetic fields. Once induced, such currents can then follow their own shape-dependent time-domain dynamics, whilst also being driven by, and emitting into, the field. Strictly, therefore, the true response of such metamaterials is due to interlinked spatial and temporal dispersions. However, in the appropriate (or convenient) limits, the spatial properties can often be homogenized away, leaving only the temporally dispersive properties as significant.

The extra work required to reduce a lattice of metamaterial cells is significant, and has lead to a great interest in how to measure, simplify, or otherwise describe their dispersive properties – leding to the topic known as *homogenization* [36]. This goal of getting approximate but still sufficiently accurate dispersion relations for uniform arrays of unit-cell structures has been widely examined, but except in rare cases [7, 26] the spatial properties are either ignored (as in e.g. the F-model for split-ring resonators (SRRs) [8, 37]) or poorly resolved. It is well beyond the intended scope of this paper to attempt to present the wide variety of possible spatial dispersia generated by metamaterials, but the interested reader might perhaps look at either the Special Issue of PNFA [36], or the more recent posibilities covered by the work of Mnasri et al. [7] and Khrabustovskyi et al. [6].