Raising Capital from Heterogeneous Investors[*]

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Abstract

A firm raises capital from multiple investors to fund a project. The project succeeds only if the capital raised exceeds a stochastic threshold, and the firm offers payments contingent on success. We study the firm's optimal unique-implementation scheme, namely the scheme that guarantees the firm the maximum payoff. This scheme treats investors differently based on size. We show that if the distribution of the investment threshold is log-concave, larger investors receive higher net returns than smaller investors. Moreover, higher dispersion in investor size increases the firm's payoff. Our analysis highlights strategic risk as an important potential driver of inequality.

JEL Codes: D86, G24, L24.

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1   Introduction

Firms often have worthwhile projects that require the participation of multiple in-
vestors. A key problem is that these investors face strategic risk: if not enough of
them choose to invest, the firm will not have enough capital to implement its project
and generate a return. As a result, there may be outcomes in which some or all the
investors choose not to invest because they expect that others will not invest. These
outcomes are bad for the firm and typically inefficient.

This paper studies the firm's optimal scheme that guarantees investment as the
unique outcome. In a world without contracting constraints, where payments can be
made contingent on third parties' choices, eliminating the possibility of bad outcomes
would impose no extra cost on the firm. But the real world is not unconstrained,
and as pointed out by the literature, bilateral contracts are often all a firm can rely
on.[1] Guaranteeing investment then requires compensating investors for their strategic
risk, a risk that depends on the amount of capital each investor pledges. A natural
question arises: how does heterogeneity in investor size affect the firm's scheme and the
returns yielded to the firm and the investors? In particular, does an optimal scheme
treat investors differently based on size, and, if so, which investors get better terms?
How does the distribution of capital among investors affect the firm's profits and the
feasibility of investment?

Our model consists of a firm and a set of agents. The firm owns a project that
generates a surplus if implemented, and each agent has an amount of capital to invest,
which varies across the agents. The firm's project can be implemented—i.e., the project
"succeeds"—only if the capital raised from the agents exceeds a stochastic, initially
unknown threshold.[2] The firm offers each agent two payments for investing, one if the
project succeeds and another if it fails. Each agent then chooses whether to invest with
the firm or put her capital in a safe asset that pays a fixed net return. We characterize
the firm's optimal unique-implementation scheme. This scheme specifies individual
capital amounts and the least-cost payments such that investing these amounts with
the firm is the unique Nash equilibrium outcome.[3]

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[1]See Section 2 as well as the discussions in Innes and Sexton (1994) and Segal (2003).

[2]This threshold captures common factors such as the project involving inputs whose prices are
random, or the firm having a stochastic source of external credit to use as additional funding. More
abstractly, our model simply assumes that the probability of project success is increasing in the amount
of capital invested.

[3]See Section 2 for details. Our unique implementation requirement is equivalent to having the firm

Our first main result addresses how the optimal returns that the firm offers vary with investor size, given a fixed set of investors. We show that if the project fails, the firm simply refunds the agents their capital, thus paying the same zero net return to each of them.[4] However, if the project succeeds, the firm pays the agents differential net returns depending on the size of their investments. Under a distributional condition on the investment threshold (which we discuss subsequently), we show that agents investing larger amounts of capital receive higher net returns (per unit of capital) than those investing smaller amounts. This pattern is consistent with evidence from private equity, where large limited partners are given preferential terms compared to small ones (see, e.g., Clayton, 2017). By showing that larger investors get more per unit invested, this result also yields important implications for dynamic capital markets: we identify a mechanism through which capital becomes dispersed, pointing to "winner-takes-all dynamics" such as those that arise in tournament theory and models of superstars (Lazear and Rosen, 1981; Rosen, 1981).

Our second main result concerns the optimal set of investors for the firm. Fixing the total amount of capital, we find that the firm benefits from dealing with agents whose capital is more unequal. Specifically, any increase in the dispersion of agents' capital (in the sense of majorization, i.e. concentrating capital in the hands of a small number of agents) reduces the firm's cost of raising any given level of capital. Higher dispersion in investor size therefore increases the firm's expected payoff from any given investment, as well as the range of investments that are feasible. Furthermore, as an implication, we find that the firm targets those agents with the largest endowments of capital, generating differences not only in agents' net investment returns but also in their access to investment opportunities.

Our last main result considers the relationship between the distribution of capital and the distribution of returns. One might be tempted to conclude from our previous results that not only larger investors are offered higher net returns, but also their return advantage is larger when the agents' investments are more unequal. We show that the opposite is true, in the following sense: higher capital dispersion reduces the difference in net returns between the largest and smallest investors. In fact, we find that this return difference can decline to the extent that even the difference in the investors'

maximize its expected payoff in its worst possible equilibrium outcome.

[4]This result applies to our benchmark setting with no initial firm capital. If the firm owns initial capital, a subset of the agents are paid a positive net return under failure; yet, as shown in Section 5, our qualitative conclusions are unchanged.

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final capital holdings declines when initial capital becomes more unequal.

To provide intuition for these results, we next describe a simple example. Consider a project that requires I units of capital to succeed, where the threshold I is uniformly distributed on the interval $[0, 30]$. If the project succeeds, it generates a fixed surplus $A > 0$ in addition to the initial investment. Suppose A is large enough that the firm wishes to guarantee full investment by two agents, where agent 1 has 10 units of capital and agent 2 has 20 units. The agents' outside option is to invest in a safe asset that pays a net return of 10%.

In this simple example, the project succeeds for sure if both agents invest with the firm. Hence, paying each agent a net return under success equal to the safe return of 10% would suffice to induce an equilibrium in which both agents invest. However, an equilibrium in which neither agent invests would also exist given this (or a slightly higher) return. To implement full investment as the unique equilibrium outcome, the firm must make it dominant for one of the agents to invest.

Consider first a scheme that makes investment dominant for agent 1. If only agent 1 invests, the project succeeds with probability 1/3, namely the probability that the investment threshold is $I \leq 10$. To ensure that agent 1 invests no matter what agent 2 does, it thus suffices to offer her a net return under success (slightly above) r satisfying $r/3 = 10\%$, i.e. $r = 30\%$. Given agent 1's participation, it then suffices to offer agent 2 a net return of 10% for her to also invest. It follows that the firm can guarantee full investment at a cost of $10(30\%) + 20(10\%) = 5$.

The alternative is to make investment dominant for agent 2. If only agent 2 invests, the project succeeds provided that $I \leq 20$, which occurs with probability 2/3. Thus, it suffices to offer agent 2 a net return under success of 15% to guarantee her participation. Since agent 1 will then invest as well if she is offered at least 10%, the firm's cost is now equal to $10(10\%) + 20(15\%) = 4$, which is lower than under the previous scheme. Intuitively, agent 2's larger investment provides her with more self-insurance compared to agent 1, and this allows the firm to pay a lower compensation for risk when agent 2's participation is made dominant. Consequently, the firm uses a scheme that pays a higher net return to the large investor compared to the small investor. This illustrates our first main result.

Consider next transferring capital from the small to the large investor. For example, suppose we transfer 4 units of capital from agent 1 to agent 2, so that the capital of agent 1 becomes 6 and that of agent 2 becomes 24. Following analogous steps to those

above, the firm's scheme in this case entails a net return under success of 12.5% for agent 2 and 10% for agent 1. The firm's cost is equal to $6(10\%) + 24(12.5\%) = 3.6$, which is lower than the cost of 4 prior to the transfer. Because the large investor becomes better self-insured when her capital is increased, the overall compensation for risk that the firm has to pay declines. We thus obtain that when the distribution of capital is more unequal, the firm's expected payoff is higher, and a lower surplus A from success suffices for the investment to be profitable. This illustrates our second main result. Our third main result is also clear in this exercise: the difference in the agents' net returns is smaller when their investments are more heterogeneous.

Our paper examines a general setting in which the number of agents and their capital levels are arbitrary, as is the distribution of the investment threshold I. We identify a condition on the distribution function of I under which our results hold for all capital distributions. The condition is that the reciprocal of the cumulative distribution function (cdf) be convex, a property implied by log-concavity of the cdf and thus satisfied by most commonly used distributions.[5] Our analysis elucidates the role of this condition and how our findings change if it is not satisfied. In the example above, the condition implies a risk premium per unit of capital which declines at a decreasing rate with the agents' investments, and this is why the firm minimizes costs by first guaranteeing the participation of the large investor.

We discuss different extensions and implications of our model. We show that our results apply to a setting in which the firm has some initial capital of its own. The firm in this case uses its capital to fully insure small investors, thus continuing to offer higher net returns to larger investors and delivering different levels of risk for investors of different size. Our results also apply to a general equilibrium setting in which the investors' outside options are endogenously determined. Moreover, while derived for a firm that maximizes its profits, our findings are also relevant to a social planner concerned with agents' welfare.

Beyond capital raising, we discuss how our model may be applied to other contracting problems with externalities. These include a monopolist offering exclusive dealing contracts to buyers to deter market entry; a firm rewarding workers to complete a joint task; and a bank offering interest and collateral to depositors to prevent a run. Heterogeneity is common in these situations, and our results can be useful to understand

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[5]These include the exponential, gamma, log-normal, Pareto, and uniform distributions (see Bagnoli and Bergstrom, 2005).

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its implications.

A broad insight from our analysis is that strategic risk may be a driver of inequality. A profit-maximizing mechanism favors certain agents in order to pin down their choices and reduce the strategic risk on the part of other agents. We show that under a plausible condition, the more favorable terms are given to those agents who are already in a more favorable position. The mechanism therefore exacerbates initial differences among the agents, and it also extracts increased revenues from these differences. Inequality being undesirable for a number of reasons that we do not study, our paper uncovers important economic forces that may be behind it. We discuss policy implications for a social planner in Section 6.

Related literature. Our model is one of multi-agent contracting, similar to models used in the literature on contracting with externalities pioneered by Segal (1999, 2003) (see also Bernstein and Winter, 2012).[6] These are abstract models with externalities among the agents which are exogenously given. In contrast, we consider an applied problem in which the externalities among the agents are endogenously determined by the firm's contract offers.

Our main departure from the literature is that we study agents who are heterogeneous in their endowments. Our analysis of course also has implications for the case in which the agents are homogeneous: we find that the firm's optimal scheme gives investors differential net returns even if they all have the same amount of capital. This is analogous to the results in Winter (2004), where an optimal team incentive scheme is shown to discriminate among identical workers. Similar results appear in Segal (2003) and Eliaz and Spiegler (2015), as well as in Inostroza and Pavan (2018) in the context of persuasion. Given that an optimal scheme creates heterogeneity among homogenous agents, our paper examines the natural question of how the scheme deals with heterogeneous agents, and in doing so it reveals implications for inequality.

Two related papers that analyze heterogeneity are Bernstein and Winter (2012) and Sákovics and Steiner (2012). Unlike our model, neither of these feature contingent payments: the principal offers fixed subsidies for the agents to participate in the mechanism, and agents' benefits from participating and their externalities are exogenous. Bernstein and Winter (2012) study how asymmetries in the agents' bilateral externalities affect the principal's scheme and revenue. Instead, we look directly at

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[6]Most such models focus on unique implementation like we do; see Section 2.3.

differences in agents' attributes, whose effects on the matrix of externalities may be complex and endogenous.[7] Sákovics and Steiner (2012) consider a global game with incomplete information, where agents differ in their influence over the aggregate action, their benefit from project success, and their cost of investment. Importantly, these papers are silent about our main object of study, namely the per-dollar returns on investment. These returns and their dependence on investors' wealth are the crux of the persistent inequality that we show is generated by the firm's capital raising.

In this regard, the most closely related paper to ours is Akerlof and Holden (2019).[8] In independent work, the authors consider a principal who has access to a production technology and faces a set of investors. The production function achieves a global maximum at a high level of investment, has a local maximum at a low level, and is such that for an intermediate range the output would not suffice to compensate investors for their outside options. The paper first analyzes a setting with many identical small investors. If the principal seeks to raise the high investment level while offering investors their outside option, there is an equilibrium in which investors invest less and the principal gets zero payoff. As a result, the principal seeks to raise the low level of investment. The authors then contrast this setting with one in which there is a large investor in addition to the small investors. If the principal offers the large investor a junior debt claim and promises her a fraction of the surplus generated by the high investment, then both the large and small investors invest and high investment is the unique equilibrium outcome.

Our paper differs from Akerlof and Holden (2019) in a number of aspects. First, they show that by facilitating investment, a large investor can earn a high return if he has some bargaining power vis-a-vis the principal. If the large investor is instead a price-taker as the small ones, then all investors get the same net return regardless of size. In contrast, in our model all the bargaining power is in the hands of the firm, and the firm offers higher net returns to larger investors to guarantee itself a maximum payoff. Put differently, we take a mechanism design approach to solve for an optimal unique-implementation scheme, whereas Akerlof and Holden (2019) provide

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[7]In fact, while in our model the magnitude of an investor's externality is related to size, we find that the relationship between size and contract terms depends on a distributional condition, so a higher externality does not necessarily imply more favorable terms as in Bernstein and Winter. The difference arises primarily from the fact that here agents' externalities are neither bilateral nor additive. In our framework, the externality that an agent exerts on another agent's gains depends on who else is in the pool of investors.

[8]See also Akerlof and Holden (2016).

an equilibrium analysis that shows how outcomes vary with the environment. Second, while their paper focuses on the role of large investors in improving overall investment, we study how the firm and investors' payoffs depend on the distribution of capital. In fact, unlike in Akerlof and Holden (2019), the firm in our model benefits from targeting larger investors, and offers them a higher net return compared to smaller investors, even if the overall investment is kept unchanged.

Finally, our paper is also related to Andreoni (1998), which studies the role of seed money in charitable contributions. Since the success of such fundraising relies on a minimum threshold of funds, contributions from seed donors increase the incentives of other donors to also contribute.

2 Model

2.1 Setup

A firm owns a project which yields a fixed surplus $A > 0$ if implemented. The firm can implement the project only if the capital invested in it exceeds an initially unknown, stochastic threshold I. We assume that I has a twice differentiable distribution function F with support $[0, \overline{I}]$, for some $\overline{I} > 0$.[9] Hence, if capital x is invested, with probability $F(x)$ the threshold satisfies $I \leq x$ and the project is implemented, yielding final capital $x + A$.[10] With the remaining probability $1 - F(x)$ the threshold is $I > x$ and the project is not implemented, so the final capital is x. We will refer to project implementation as success and to no implementation as failure.

We begin by assuming that the firm has no capital of its own, deferring the study of how the firm would use any initial capital to Section 5. The firm raises capital from a set of $N > 1$ heterogeneous agents, indexed by $n \in S = \{1, \ldots, N\}$. Each agent n has a capital endowment $\overline{x}_n > 0$. Instead of investing with the firm, agents can invest their capital in a safe asset that pays a net return $\theta > 0$.[11] (All returns are net percentage

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[9]Setting the lower bound of the support to zero simplifies the exposition. As will be clear in the next sections, our results are unchanged so long as this bound is smaller than the largest investor's amount of capital, and our problem is moot otherwise.

[10]In Section 6, we show that our results also apply if the firm's surplus from implementing the project is proportional to the capital invested instead of a constant amount.

[11]We thus model the loss from project failure as an opportunity cost, reflecting the fact that, in practice, there is a lag between the capital raising and the actual investment decision. This formulation is mathematically equivalent to one in which, instead of the agents forgoing an outside option, a failure

returns, meaning that if agent n invests $\overline{x}_n$ in the safe asset, her payoff is $(1+\theta)\,\overline{x}_n$.)
All of this is common knowledge.

The order of moves is as follows. First, the firm offers each agent a contract specifying payments in the events of project success and failure, as we describe in the next subsection. These are publicly observable contracts to which the firm commits. Second, the agents decide simultaneously whether to invest with the firm or put their capital in the safe asset. Finally, the investment threshold I is realized, the project is implemented if and only if the capital raised by the firm exceeds I, and payments are made.

2.2 Firm's problem

The firm wishes to guarantee its maximum possible payoff. Its problem is to choose a payoff-maximizing scheme subject to satisfying its budget constraint and to inducing a unique equilibrium outcome.

As further discussed in Section 2.3, we focus on contracts that are bilateral and simple. For each $n \in S$, the firm specifies an amount of capital $x_n \in [0, \overline{x}_n]$ and returns (r_n, k_n) conditional on agent n investing x_n in the firm's project. The return r_n is the net return that agent n receives if the project succeeds; the return k_n is the agent's net return in the case of failure.

Given a scheme specifying investments $(x_n)_{n \in S}$ and returns $(r_n, k_n)_{n \in S}$, denote agent n's decision by $y_n \in \{0, 1\}$, where $y_n = 1$ means invest x_n with the firm and $y_n = 0$ means invest x_n in the safe asset. The firm's budget constraint requires that the total payments offered to the agents do not exceed the firm's final capital, regardless of the set of agents who invest in the project and whether or not the project is implemented. That is, for all profiles of choices $\mathcal{Y} = (y_1, \ldots, y_N)$, the firm's scheme must satisfy[12]

$$\sum_{n=1}^{N} r_n y_n x_n \leq A \quad \text{and} \quad \sum_{n=1}^{N} k_n y_n x_n \leq 0. \tag{BC}$$

In addition, the firm's scheme must implement the agents' investments in a unique outcome. The firm's problem can be decomposed in two steps:

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corresponds to an unsuccessful investment that depletes a portion of the invested capital.

[12]This budget constraint can be relaxed to only require budget balance on the equilibrium path without altering our results. See our discussion in Section 2.3.

(i) For fixed capital amounts $(x_n)_{n\in S}$, find the optimal return schedule $(r_n, k_n)_{n\in S}$ guaranteeing investments $(x_n)_{n\in S}$.

(ii) Given step (i), find the optimal capital amounts $(x_n)_{n\in S}$, where $x_n \in [0, \bar{x}_n]$ for each $n \in S$.

We address step (i) in Section 3 and step (ii) in Section 4. We next formalize step (i).

Fix capital amounts $(x_n)_{n\in S}$ where, to avoid trivialities and without loss, we take $x_n > 0$ for each $n \in S$. Given $(x_n)_{n\in S}$, say that a return schedule $(r_n, k_n)_{n\in S}$ is *incentive inducing (INI)* if $\mathcal{Y}^1 \equiv (1, \dots, 1)$ is the unique Nash equilibrium of the game induced by $(r_n, k_n)_{n\in S}$. An optimal return schedule for the firm is a least-cost INI schedule. A technical issue is that the set of INI schedules is open (since r_n and k_n take continuous values); we resolve this by assuming that agents invest with the firm when indifferent given their conjectures of others' behavior.[13] Formally, let $U_n(y_n, \mathcal{Y}_{-n})$ be agent n's expected return on x_n given net returns (r_n, k_n), investment choice y_n, and investment choices $\mathcal{Y}_{-n} = (y_1, \dots, y_{n-1}, y_{n+1}, \dots, y_N)$ of the other agents:

$$U_n(y_n, \mathcal{Y}_{-n}) = \left[F\left(\sum_{n'=1}^{N} y_{n'} x_{n'}\right) r_n + \left(1 - F\left(\sum_{n'=1}^{N} y_{n'} x_{n'}\right)\right) k_n \right] y_n x_n + \theta(1 - y_n) x_n.$$

Given our assumption on behavior under indifference, we define a Nash equilibrium as a profile $\mathcal{Y} = (y_1, \dots, y_N)$ such that, for each $n \in S$, $y_n = 1$ if $1 \in \mathrm{argmax}_{y \in \{0,1\}} U_n(y, \mathcal{Y}_{-n})$ and $y_n = 0$ otherwise. Denote by $E((r_n, k_n)_{n\in S})$ the set of Nash equilibrium profiles under schedule $(r_n, k_n)_{n\in S}$. Then the firm's unique implementation requirement is:

$$E((r_n, k_n)_{n\in S}) = \{\mathcal{Y}^1\}. \tag{U}$$

Let $X_N \equiv \sum_{n=1}^{N} x_n$. An optimal return schedule $(r_n^*, k_n^*)_{n\in S}$ guaranteeing investments $(x_n)_{n\in S}$ solves the following program:

$$V((x_n)_{n\in S}) = \max_{(r_n, k_n)_{n\in S}} \left\{ \left(A - \sum_{n=1}^{N} r_n x_n\right) F(X_N) - \sum_{n=1}^{N} k_n x_n (1 - F(X_N)) \right\} \tag{P}$$

subject to (BC) and (U).

[13]This assumption is equivalent to defining an optimal return schedule as a least-cost schedule $(r_n, k_n)_{n\in S}$ such that, for any $\varepsilon > 0$, raising r_n by ε for each $n \in S$ yields an INI schedule.

2.3 Discussion of assumptions

Before we solve the firm's problem, it is worth discussing our model assumptions.

Unique implementation. We have assumed that the firm cannot coordinate the agents to its preferred equilibrium when multiple equilibria exist. This assumption is what motivates the requirement of unique implementation, both in our paper and in related work (Segal, 2003; Winter, 2004; Bernstein and Winter, 2012). If the firm could "pick" the equilibrium to be played by the agents, then it would be able to extract the full surplus by specifying returns $(r_n, k_n) = (\theta/F(X_N), 0)$ for each $n \in S$ and some $(x_n)_{n \in S}$. Under such a scheme, there is an equilibrium that implements investments $(x_n)_{n \in S}$ and keeps all agents to their outside option, but equilibria with lower investment also exist. The presence of multiple equilibria gives rise to the possibility that agents may play a non-desirable one. Indeed, several experiments find that subjects are often trapped in bad equilibrium outcomes in environments with externalities (see, e.g., Devetag and Ortmann, 2007). This tendency bears on the fact that agents' first order optimism alone would not suffice to ensure the good equilibrium: even if an agent believes in the intentions of her peer to pitch in, it may be enough for her to suspect that the peer might be pessimistic about her for the good equilibrium to unravel.

Our unique implementation requirement is equivalent to having the firm maximize its profits in the equilibrium outcome yielding the lowest profits for the firm.[14] While real-world firms and managers may not be directly worried about this worst-case scenario, they do aim to ensure a minimum payoff. The unique implementation solution provides insight into how these principals structure payments in order to avoid bad outcomes, without having to specify priors over different action profiles, which may be arguably difficult in practice. One can also view this worst-case focus as reflecting how the agents behave: if investors are reluctant to invest in the firm's project unless they are sufficiently compensated in every equilibrium outcome, then the firm's worst equilibrium outcome would indeed prevail.

Another desirable feature of our solution concept is that it permits an analysis free of strong assumptions on agents' ability to predict others' behavior. Specifically, we will show in Section 3 that our requirement of unique implementation in Nash equilibria yields a unique rationalizable outcome. The firm therefore only relies on agents being

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[14]See Segal (2003) for a general argument.

rational and this rationality being common knowledge, and not on agents being able to make correct conjectures of others agents' choices. This is in contrast with other approaches such as selecting equilibria based on the risk dominance criterion, which imposes strong demands on the ability of agents to predict how others will behave.

Timing of moves. We have assumed that the agents make their investment choices simultaneously, i.e. under imperfect information. This simultaneous game is a simple (and stark) way to capture the fact that investors in reality may have limited information about others' investment choices, and their decisions may not be sequential insofar as they can be revised.

Our analysis however also applies to a sequential game. Suppose that the firm approaches the agents sequentially, with each agent observing the investment decisions of her predecessors.[15] Naturally, by offering $r_n = \theta/F(X_N)$ and $k_n = 0$ to each agent $n \in S$, the firm can induce investments $(x_n)_{n \in S}$ as the unique subgame-perfect Nash equilibrium and extract the full surplus. But such a solution appears unrealistic, as it requires investors to believe that others who have not yet moved will choose to invest with the firm. Without these beliefs, guaranteeing investments $(x_n)_{n \in S}$ amounts to making each of these a dominant strategy in the underlying subgame of the sequential game, a solution concept that is used in Innes and Sexton (1994) among others. One can show such an approach yields the same results as our unique implementation requirement in the simultaneous game.

The sequential moves specification is of interest in itself. This specification imposes the weakest demands on agents' information and behavior: when making their investment decisions, agents are not required to know the structure of the remaining game, the contracts offered to other agents, or how much capital other agents own. In fact, agents need not even know whether other agents are rational, as they simply establish whether they want to invest with the firm given the capital already accumulated, no matter what happens next.[16] In terms of our application, the sequential moves setting offers additional predictions on the order in which the firm should approach investors to maximize profits. Our results imply that, under the condition of Proposition 2,

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[15]The discussion that follows is valid regardless of whether the firm commits to the rules of the game (i.e., the contracts and the order of moves) ex ante or not.

[16]Interestingly, since such a decision rule allows the investors to extract more surplus, it is to their benefit—and they would want to tell the firm when negotiating the terms—that they are reluctant to rely on speculations about the behavior of future investors.

it is optimal for the firm to first lock large investors and only then approach smaller investors.

Bilateral, simple contracts.   Following the literature (Segal, 2003; Winter, 2004), we have assumed that the firm can rely on bilateral contracts only. That is, contracts cannot directly condition on third parties' actions: the payment to an agent does not depend on other agents' investment decisions except insofar as these decisions affect whether the project gets implemented. The motivation for this restriction stems from the difficulty to verify in practice the capital pledged by third parties. If an agent sues for breach of contract, a court can require the agent to prove that she invested with the firm (or else she lacks standing to sue), and it can plausibly verify whether or not the firm implemented some large project. It is less clear whether the court can identify the firm's other investors and the amounts that they may or may not have invested. We focus on situations in which it cannot.[17]

Another assumption is that the firm uses "simple" contracts. Specifically, our analysis abstracts from menu contracts in which the firm offers an agent n different returns $(r_n(x'_n), k_n(x'_n))$ for different amounts x'_n that the agent may choose to invest. In a simple contract, the firm specifies an amount x_n and returns (r_n, k_n) conditional on the agent investing that amount (and zero returns otherwise). Naturally, only simple contracts are relevant if agents' decisions are binary, as is the case when there are indivisibilities in investment.[18] Moreover, even when investment is fully divisible, we provide conditions in the Online Appendix under which simple contracts are without loss of optimality.[19]

Budget constraint.   We have required that the firm satisfy its budget constraint both on and off the equilibrium path. That is, the firm must be able to follow through on its commitments to the agents regardless of which agents decide to invest in the project.[20] An alternative possibility would be to allow the firm to offer any returns $(r_n, k_n)_{n \in S}$ that satisfy its budget constraint on the equilibrium path (i.e. under the

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[17]If instead contracts can condition on third parties' choices, then it can be shown that the firm would be able to extract the full surplus.

[18]Indivisibilities are common in applications where capital takes the form of a specific resource or skill, or where the project requires a number of discrete investments. See Section 6 for some examples.

[19]See Segal (2003) for an analysis of menu contracts in a more general setting.

[20]This is analogous, for example, to the requirements in Holmström (1982).

investments $(x_n)_{n \in S}$), and each agent $n \in S$ would then assess the credibility of her offer (r_n, k_n) according to her conjecture of others' behavior. We show in the Online Appendix that, given our focus on unique implementation, both possibilities yield the same results. We regard the stronger budget-balance condition as more plausible, since the irrational behavior of some investors cannot serve the firm with an excuse for not fulfilling its contracts with other investors.

3 Return Schedule

In this section, we address step (i) of the firm's problem: for fixed capital amounts $(x_n)_{n \in S}$, we study the firm's optimal return schedule that guarantees these investments, namely the schedule that solves program (P). Without loss, we take $x_n > 0$ for each $n \in S$. We begin by restating constraint (U) in program (P) using the following equivalence:

Lemma 1. (U) *holds if and only if there exists a permutation* $\pi = (n_1, \ldots, n_N)$ *of the set of agents such that, for each* $i \in S$, *agent* n_i *is willing to invest with the firm if agents* $(n_1, \ldots, n_{i-1})$ *invest with the firm, no matter what the other agents do.*

An optimal return schedule makes it iteratively dominant for each agent to invest with the firm. To see why this follows from (U), note that uniqueness of the full-participation equilibrium $\mathcal{Y}^1$ implies that there is an agent n_1 who is willing to invest with the firm when no other agent does. Moreover, existence of this equilibrium implies that n_1 is also willing to invest when all other agents do. We show that as a result, n_1 is willing to invest with the firm no matter what the other agents do. The reason is that n_1's expected payoff from investing is a weighted average of her returns under success and under failure, where the weights are the probabilities of each event and thus achieve their highest and lowest values when all and none of the other agents invest. Having established this property for n_1, we then use an induction argument to complete the proof of the "only if" claim in Lemma 1.[21]

Given this result, an optimal schedule specifies some permutation $\pi = (n_1, \ldots, n_N)$ of the set of agents and returns (r_i, k_i) for each agent $n_i \in S$ satisfying the criterion in

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[21] The proof of Lemma 1 is general in that it does not rely on specific externalities between the agents. The result will also apply to the setting studied in Section 5 in which the firm owns some initial capital.

Lemma 1. We proceed by first characterizing the optimal returns $(r_i^*, k_i^*)_{i \in S}$ and then solving for an optimal permutation $\pi^* = (n_1^*, \ldots, n_N^*)$.

3.1 Optimal returns

Given a permutation $\pi = (n_1, \ldots, n_N)$, denote the aggregate capital of the first i agents in the permutation by $X_i \equiv \sum_{j=1}^{i} x_{n_j}$, where we omit the dependence on π to ease the exposition. (Note that, as previously defined, X_N corresponds to the total amount of capital.) We obtain:

Proposition 1. *Suppose that there exists an optimal return schedule guaranteeing investments $(x_n)_{n \in S}$. Any such schedule specifies some permutation $\pi = (n_1, \ldots, n_N)$ and returns $(r_i^*, k_i^*)_{i \in S}$ such that, for each $i \in S$, agent n_i is indifferent over investing with the firm if agents $(n_1, \ldots, n_{i-1})$ invest with the firm and agents $(n_{i+1}, \ldots, n_N)$ do not. Moreover, the following returns are optimal:*

$$r_i^* = \frac{\theta}{F(X_i)} \quad and \quad k_i^* = 0.$$

An optimal schedule implies a permutation $\pi = (n_1, \ldots, n_N)$ such that the first agent in the permutation is indifferent between investing and not when no other agent invests, the second agent is indifferent between investing and not when the first agent invests and the others do not, and so on. For intuition, recall that by Lemma 1, there is a permutation $\pi = (n_1, \ldots, n_N)$ in which each agent n_i is willing to invest when agents $(n_1, \ldots, n_{i-1})$ invest, no matter the rest. This implies that for each $i \in S$ and each $j \in \{i, \ldots, N\}$,

$$r_i^* F(X_j) + k_i^* (1 - F(X_j)) \geq \theta. \tag{1}$$

Now note that the firm's budget constraint (BC) requires $k_i \leq 0$ for each $i \in S$;[22] given no initial capital of its own, the firm cannot credibly commit to pay an agent a positive return under failure. Since the agents can obtain a net return $\theta > 0$ by investing in the safe asset, condition (1) then requires that the firm offer a strictly

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[22](BC) requires that the sum of net returns under failure be non-positive for all profiles of choices. Since any one agent being the only investor is a possible choice profile, this constraint in turn requires that each agent's net return under failure be non-positive.

positive net return $r_i > 0$ under success. It follows that for each $i \in S$,

$$r_i^* > 0 \geq k_i^*, \tag{2}$$

and thus the schedule induces strategic complementarities. That is, under an optimal return schedule, each agent n_i's expected payoff from investing with the firm is increasing in the other agents' investments.

The strategic complementarities in turn simplify the agents' participation constraints. Given the inequalities in (2), we obtain that condition (1) is satisfied for each $i \in S$ and each $j \in \{i, \dots, N\}$ if and only if it is satisfied for each $i \in S$ and $j = i$. Intuitively, the firm can induce agent n_i to participate no matter what agents $(n_{i+1}, \dots, n_N)$ do if it can induce agent n_i to participate when all such other agents do not. Furthermore, we show that by optimality, condition (1) must hold with equality for each $i \in S$ and $j = i$: otherwise, the firm could lower a return r_i and increase its payoff while preserving the agents' incentives to participate and relaxing its budget constraint. Therefore, we obtain

$$r_i^* F(X_i) + k_i^* (1 - F(X_i)) = \theta \tag{3}$$

for each $i \in S$. This yields the first part of Proposition 1, which, in the literature's jargon, shows that any optimal scheme is a "divide and conquer" scheme.[23]

The second part of Proposition 1 uses the binding participation constraints in (3) to derive optimal returns. We show that it is optimal to set $(r_i^*, k_i^*) = (\theta/F(X_i), 0)$ for each $i \in S$.[24] The idea is intuitive. The firm conditions on all agents $(n_1, \dots, n_N)$ investing in the project, whereas, as shown in (3), each agent n_i conditions on only agents $(n_1, \dots, n_i)$ investing. Hence, for all $i \in S$, the firm assigns a higher probability to success than agent n_i does, which means that the firm values r_i relative to k_i more than agent n_i. As a consequence, the firm benefits from reducing r_i, and thus increasing k_i, as much as it can, subject to its budget constraint (BC) and the participation constraints in (3).

Formally, we show that if a return schedule specifies $k_i < 0$ for some $i \in S$, we can

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[23]See Segal (2003). Divide and conquer strategies are also discussed in the literature on exclusionary contracts, including Rasmusen, Ramseyer and Wiley (1991), Innes and Sexton (1994), and Segal and Whinston (2000).

[24]Given (BC), if the agents were protected by limited liability, then the firm would be constrained to offer $k_i = 0$ for all $i \in S$. Here we obtain these same returns under failure but by optimality.

perform a perturbation in which we slightly increase k_i and reduce r_i so as to keep the left-hand side of (3) unchanged. The perturbed schedule satisfies the firm's budget constraint and preserves the agents' incentives to participate. Moreover, we show that the perturbation increases the firm's expected payoff in (P). It follows that it is optimal to set $k_i^* = 0$ and thus, by (3), $r_i^* = \theta/F(X_i)$ for each $i \in S$.[25]

Proposition 1 has important implications for the agents' payoffs. The proposition shows that the firm treats the agents symmetrically under failure: each agent is refunded her capital if the project is not implemented. However, in the case of success, returns differ across the agents. Given the permutation $\pi = (n_1, \ldots, n_N)$, agents who are positioned towards the beginning of the permutation are offered a higher net return (per unit of capital invested) than those positioned later in the permutation. The reason is that agents with a higher rank i condition on a larger set of other agents investing with the firm; thus, given the strategic complementarities, their participation constraints are less costly to satisfy. Clearly, in light of this result, a key question is how an optimal permutation π^* ranks the agents given the heterogeneity in the size of their investments. We turn to this question in the next subsection.

A useful property of the returns in Proposition 1 is that they maximally relax the firm's budget constraint. Specifically, since $k_i^* = 0$ for each $i \in S$, these returns minimize not only the firm's total costs but also its costs under success, $\sum_{i=1}^{N} r_i x_{n_i}$, for some permutation π. It follows that an optimal return schedule guaranteeing investments $(x_n)_{n \in S}$ exists if and only if a schedule with returns $(r_i^*, k_i^*) = (\theta/F(X_i), 0)$ for some permutation π satisfies (BC) given $(x_n)_{n \in S}$. As formalized in the next corollary, the latter requires that the firm's surplus A from success be large enough.

Corollary 1. *An optimal return schedule guaranteeing investments $(x_n)_{n \in S}$ exists if and only if there exists a permutation $\pi = (n_1, \ldots, n_N)$ such that*

$$\theta \sum_{i=1}^{N} \frac{x_{n_i}}{F(X_i)} \leq A.$$

We end our discussion of Proposition 1 with a remark. As noted above, the firm's scheme induces a supermodular game among the agents, namely one characterized by strategic complementarities. As a result, our requirement of unique implementation in Nash equilibria also yields unique implementation in rationalizable strategies.

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[25]These returns are strictly optimal for $i \in \{1, \ldots, N-1\}$ and weakly optimal for $i = N$.

Remark 1. *Take an optimal return schedule guaranteeing investments $(x_n)_{n \in S}$. Then these investments constitute the unique Nash equilibrium as well as the unique rationalizable outcome.*

3.2 Optimal permutation

We now turn to the question of how an optimal permutation ranks the agents. Assume hereafter that the condition in Corollary 1 holds, so an optimal return schedule guaranteeing investments $(x_n)_{n \in S}$ exists. By Proposition 1, it is optimal for the firm to specify some permutation $\pi = (n_1, \ldots, n_N)$ of the set of agents and returns $(r_i^*, k_i^*) = (\theta / F(X_i), 0)$ for each agent $n_i \in S$. Substituting in the firm's expected payoff yields

$$V((x_n)_{n \in S}) = \left(A - \theta \sum_{i=1}^{N} \frac{x_{n_i}}{F(X_i)} \right) F(X_N) \tag{4}$$

for some permutation $\pi = (n_1, \ldots, n_N)$. It follows from (4) and Corollary 1 that a permutation π is optimal if and only if it minimizes the firm's costs under success, given by

$$\theta \sum_{i=1}^{N} \frac{x_{n_i}}{F(X_i)}. \tag{5}$$

The next proposition shows that (5) is minimized by ranking the agents in decreasing order of the size of their investments, provided that a condition on the investment threshold distribution holds. This condition is that $1/F(x)$ be convex (over the relevant range), and it is satisfied by most commonly used distributions, as we explain subsequently (see Remark 2).

Proposition 2. *Suppose $1/F(x)$ is convex for $x \in [0, X]$, $X > 0$. Then for any investments $(x_n)_{n \in S}$ with $X_N \leq X$, an optimal permutation is $\pi^* = (n_1^*, \ldots, n_N^*)$ such that*

$$x_{n_1^*} \geq \ldots \geq x_{n_N^*} \tag{6}$$

Consequently, larger investors receive higher net returns than smaller investors.

The logic for the optimal permutation is as follows. Given a permutation $\pi = (n_1, \ldots, n_N)$, Proposition 1 shows that an optimal return schedule compensates each agent $n_i \in S$ on the *marginal* unit of capital invested in the project. Specifically,

for each unit invested by agent n_i, the firm pays the agent a return under success $r_i^* = \theta/F(X_i)$. As discussed in Section 3.1, the agent's return thus decreases with X_i; moreover, if $1/F(\cdot)$ is convex, $\theta/F(X_i)$ decreases at a decreasing rate with X_i. This means that the firm benefits from moving down the return curve as quickly as possible: the faster capital is accumulated along the sequence $(x_{n_1}, \ldots, x_{n_N})$, the lower is the sum of returns $x_{n_1}\theta/F(X_1) + x_{n_2}\theta/F(X_2) + \ldots + x_{n_N}\theta/F(X_N)$ that the firm has to pay under success. It follows that it is optimal to rank the agents in decreasing size order, from the largest investor to the smallest one.

Intuitively, to guarantee investment, the firm has to compensate the agents for the strategic risk that they face in addition to the fundamental risk. The risk premium for agent n_i is proportional to $1/F(X_i)$, which depends on the agent's rank i and her investment x_{n_i}. For any given rank, a large investor demands a lower risk premium than a small one because her large investment secures itself. That is, given X_{i-1} fixed, a larger investment x_{n_i} allows to reduce the risk premium $1/F(X_{i-1}+x_{n_i})$ that the firm has to pay on each unit of capital invested by n_i. Now the magnitude of this reduction depends on X_{i-1}: if $1/F(\cdot)$ is convex, the risk premium decreases most sharply with n_i's investment when the aggregate investment X_{i-1} of preceding agents is small, and thus when n_i's rank i is low. As a consequence, placing large investors early in the permutation minimizes the total risk premia that the firm has to pay when $1/F(\cdot)$ is convex.[26]

Figure 1 illustrates the result using the example described in the Introduction. We take F uniform over $[0, 30]$ and $\theta = 10\%$. The figure depicts the return curve $\theta/F(X_i)$, showing that the return that the firm pays under success declines at a decreasing rate with each additional unit of capital invested in the project. For $N = 2$ agents with investments $x_1 = 10$ and $x_2 = 20$, Proposition 2 implies that the optimal permutation is $\pi^* = (2, 1)$. That is, the firm sets $n_1^* = 2$ and $n_2^* = 1$ as agent 2's investment is larger than agent 1's. As shown in the left panel of Figure 1, the optimal returns under success are $r_1^* = 15\%$ for agent $n_1^* = 2$ and $r_2^* = 10\%$ for agent $n_2^* = 1$, yielding a cost for the firm of $20(15\%) + 10(10\%) = 4$. If the firm instead ranks the agents according to $\pi = (1, 2)$, as in the right panel of Figure 1, then the returns are $r_1 = 30\%$ for agent $n_1 = 1$ and $r_2 = 10\%$ for agent $n_2 = 2$, yielding a higher cost of $10(30\%) + 20(10\%) = 5$.

As stated in Proposition 2, our characterization of an optimal permutation has

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[26]Conversely, if $1/F(\cdot)$ is concave, the firm would benefit from placing large investors late in the permutation. See Section 6.
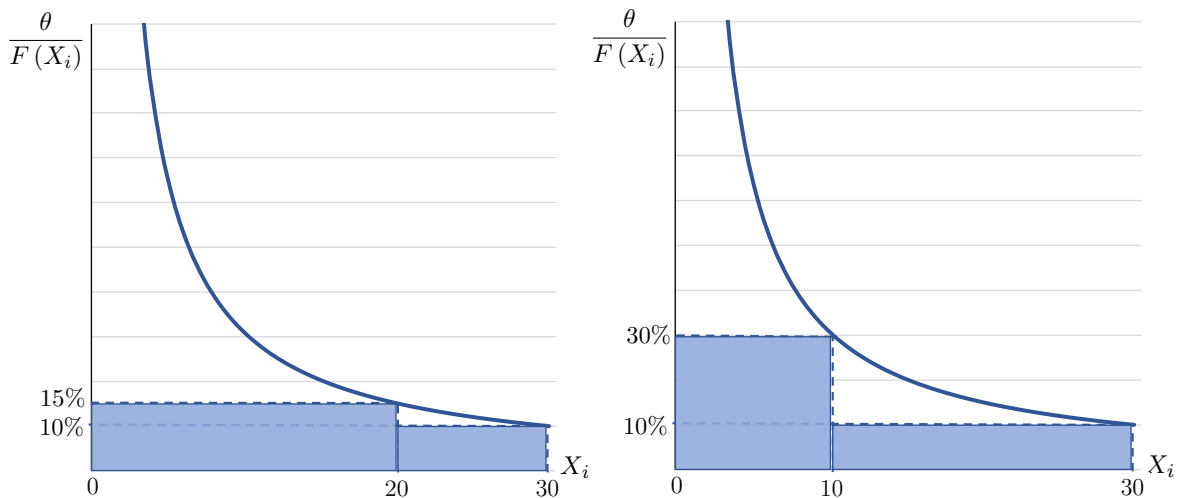
Figure 1: Return curve for F uniform over $[0, 30]$ and $\theta = 10\%$. Given $N = 2$ agents with investments $x_1 = 10$ and $x_2 = 20$, the left panel shows the returns paid by the firm under the optimal permutation $\pi^* = (2, 1)$. The right panel shows these returns under $\pi = (1, 2)$.

direct implications on investors' returns: given Proposition 1, it implies that larger investors receive higher net returns than smaller ones. The analysis therefore provides an explanation for the patterns of returns often observed in practice. As mentioned in the Introduction, our results are consistent with evidence from private equity. Tan (2016) and Clayton (2017), for example, point out an increasing tendency of private equity firms to give preferential treatment to limited partners based on size. The empirical findings in Dyck and Pomorski (2016) reveal that large investors receive higher net returns than small investors even when restricting attention to private equity investments without any preferential access.[27]

Proposition 2 suggests that the observed differences in returns across investors may arise as a firm's profit-maximizing solution to a coordination problem in investment. A natural question is whether other factors could instead explain the evidence. For example, a simple theory of transaction costs may justify firms' offering better terms to larger investors. However, the differential returns across investors are sizable, so these transaction costs would have to be too large to provide a justification. In fact, this discontinuity is a testable implication of our theory that distinguishes it from other explanations such as transaction costs: we find that small differences in agents' investments can yield large differences in the net returns that they receive.

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[27]From private conversations with industry experts, we find that similar patterns are observed in debtor-in-possession financing, often in the form of fee reductions for large investors.

By showing that larger investors get more per unit invested, Proposition 2 also has implications for dynamic capital markets. In particular, we will establish in Section 4.1 that the agents from whom the firm induces larger investments are precisely those who have larger endowments of capital to begin with. Therefore, as we discuss in that section, the differential treatment of investors described in Proposition 2 will imply that any differences in initial capital among the agents will be exacerbated by the firm's optimal scheme.

We close this section with two remarks. The result in Proposition 2 is shown to hold under a condition on the distribution of the investment threshold. First, it is worth noting that this condition is implied by log-concavity:

Remark 2. *If $F(x)$ is log-concave, then $1/F(x)$ is convex.*

Many familiar distributions have a log-concave cdf, including exponential, gamma, log-normal, Pareto, and uniform (see Bagnoli and Bergstrom, 2005).[28]

Second, one may wonder about the necessity of our condition on F. We can show that if $1/F(x)$ is non-convex for some $x \in [0, X]$, $X > 0$, then there exist investments $(x_n)_{n \in S}$ with $X_N \leq X$ such that a permutation that ranks the agents in decreasing size order is not optimal. Hence,

Remark 3. *Convexity of $1/F(x)$ over the relevant range is not only sufficient but also necessary for the statement in Proposition 2 to hold.*

Our emphasis is on the case in which $1/F(x)$ is convex because, as noted, most of the distributions that are frequently used satisfy this property. In fact, $1/F(x)$ cannot be globally concave (since $1/F(x) \to \infty$ as $x \to 0$), and thus an analysis under $1/F(x)$ concave must be conditioned on the range of capital $[\min\{x_n | n \in S\}, X_N]$ given $(x_n)_{n \in S}$. We discuss this possibility in Section 6.

4   Distribution of Capital

So far we have focused on step (i) of the firm's problem, taking the amounts of capital $(x_n)_{n \in S}$ that the firm raises as given. We now consider step (ii): given that an optimal return schedule guaranteeing investments $(x_n)_{n \in S}$ is characterized by Proposition 1 and

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[28]Log-concavity of the cdf is implied by, but weaker than, log-concavity of the probability density function.

Proposition 2, we study the optimal investments that the firm induces. Put differently, we ask: how does the distribution of capital among the agents impact the firm's payoff?

To address this question, we use the majorization partial ordering of vectors (Hardy, Littlewood and Pólya, 1934):

Definition 1. *For two N-vectors* $\mathbf{x} = (x_1, \ldots, x_N)$ *and* $\widehat{\mathbf{x}} = (\widehat{x}_1, \ldots, \widehat{x}_N)$, *vector* $\widehat{\mathbf{x}}$ *majorizes* $\mathbf{x}$ *if the components of* $\widehat{\mathbf{x}}$ *and* $\mathbf{x}$ *have the same total sum and, for all m, the sum of the m smallest components is weakly smaller in* $\widehat{\mathbf{x}}$ *than in* $\mathbf{x}$.

Majorization provides a formal definition of dispersion. Given a total investment X_N, the investments in $\widehat{\mathbf{x}} = (\widehat{x}_1, \ldots, \widehat{x}_N)$ are more unequal than those in $\mathbf{x} = (x_1, \ldots, x_N)$ if the vector $\widehat{\mathbf{x}}$ majorizes $\mathbf{x}$. This concept is analogous to that of mean-preserving spread for probability distributions.[29]

4.1   Optimal investments

The next proposition shows that for any given total investment X_N, the firm benefits from the individual investments $(x_n)_{n \in S}$ being more unequal:

Proposition 3. *Suppose* $1/F(x)$ *is convex for* $x \in [0, X]$, $X > 0$, *and consider investments* $(x_n)_{n \in S}$ *with* $X_N \leq X$. *Let the investments* $(\widehat{x}_n)_{n \in S}$ *majorize* $(x_n)_{n \in S}$. *The firm's expected payoff under* $(\widehat{x}_n)_{n \in S}$ *is higher than that under* $(x_n)_{n \in S}$.

For intuition, consider the example from the Introduction, with F uniform over $[0, 30]$ and $\theta = 10\%$. Suppose first that the firm raised capital from $N = 3$ agents with $x_1 = x_2 = x_3 = 10$. By our results in the previous section and as can be seen in Figure 1, the firm's optimal scheme would then entail costs equal to $10(30\% + 15\% + 10\%) = 5.5$. Now suppose that two of these investors were "merged" into a single larger investor, so the firm raises capital from $N = 2$ agents with $x_1 = 10$ and $x_2 = 20$ (which is equivalent to $N = 3$ agents with $x_1 = 10$, $x_2 = 20$, and $x_3 = 0$). The firm's costs under an optimal scheme would then be lower, equal to $20(15\%) + 10(10\%) = 4$. The reason is that merging the agents reduces the strategic uncertainty: while each separate agent faces uncertainty about the investment decision of the other agent, the merged agent

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[29]For any $x > 0$, let $H_{\mathbf{x}}(x)$ and $H_{\widehat{\mathbf{x}}}(x)$ denote the number of components of $\mathbf{x}$ and $\widehat{\mathbf{x}}$ respectively that do not exceed x. Then $H_{\widehat{\mathbf{x}}}(x)$ is a mean-preserving spread of $H_{\mathbf{x}}(x)$ if it is second-order stochastically dominated by $H_{\mathbf{x}}(x)$. By the results in Rothschild and Stiglitz (1970) and Machina and Pratt (1997), our analysis goes through without change with this definition of inequality.

knows that she will invest the whole capital amount in the firm's project. This allows the firm to guarantee the same total investment at a lower risk premium.

Proposition 3 shows that this logic holds more generally. We find that any increase in the dispersion of investments, as formalized by the notion of majorization, increases the firm's payoff. To see why this is the case, take $(x_n)_{n \in S}$. Any capital amounts $(\widehat{x}_n)_{n \in S}$ that majorize $(x_n)_{n \in S}$ can be derived from $(x_n)_{n \in S}$ by performing a finite sequence of transfers from smaller to larger investors, increasing the gap between them (see Hardy et al., 1934). We show that each such transfer makes the firm better off. Fixing an optimal permutation $\pi = (n_1, \ldots, n_N)$ under $(x_n)_{n \in S}$, a transfer from a small to a large investor allows the firm to move down the optimal return curve $\theta / F(X_i)$ more quickly and thus reduce its costs. Intuitively, the transfer lowers the required risk premium by increasing the self-insurance of the large investor. This implies that the firm's payoff under the induced amounts $(\widehat{x}_n)_{n \in S}$ is higher than that under $(x_n)_{n \in S}$ given optimal returns and the original permutation π. Clearly, changing to a permutation that is optimal under $(\widehat{x}_n)_{n \in S}$ can only raise the firm's payoff further. It follows that this operation always benefits the firm.

In the limit, the operation in Proposition 3 would concentrate all the capital in one of the agents. In fact, if the firm raised capital from only one agent, this agent would face no strategic risk, and the firm would be able to raise the total investment X_N by offering a net return under success equal to $\theta / F(X_N)$. The firm's costs in this case would be minimized and equal to $X_N \theta$. The firm's costs are higher when raising capital from multiple agents because of the coordination problem governing the agents' interaction. The *price of coordination* is the additional cost above $X_N \theta$ that the firm pays when dealing with $N > 1$ agents, given by $\theta F(X_N) \sum_{i=1}^{N} x_{n_i^*} \left(\frac{1}{F(X_i)} - \frac{1}{F(X_N)} \right)$ for $X_i = \sum_{j=1}^{i} x_{n_j^*}$. Proposition 3 implies that the price of coordination is lower the more unequal the agents' investments. Figure 2 provides an illustration using the example discussed above.

Returning to the firm's problem, the result in Proposition 3 immediately tells us what are the optimal investments $(x_n^*)_{n \in S}$ that the firm induces from the agents given their endowments of capital $(\overline{x}_n)_{n \in S}$. For any given total investment X_N, we find that the firm raises as much capital as it can from the agents with the largest endowments. This solution yields the most unequal investments $(x_n)_{n \in S}$ that are feasible given the agents' endowments $(\overline{x}_n)_{n \in S}$, and so it is optimal by Proposition 3.
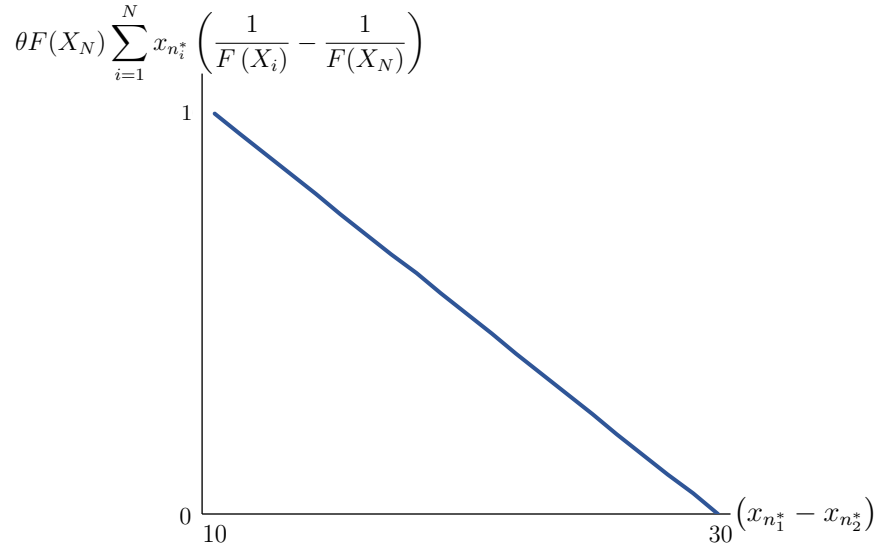
Figure 2: Price of coordination for F uniform over $[0, 30]$, $\theta = 10\%$, and $N = 2$ agents with total investment $X_2 = 30$, as we increase $\left(x_{n_1^*} - x_{n_2^*}\right)$ from 10 to 30.

Corollary 2. *Suppose $1/F(x)$ is convex for $x \in [0, X]$, $X > 0$, and agents' endowments satisfy $\sum_{n=1}^{N} \overline{x}_n \equiv \overline{X}_N \leq X$. Consider a permutation $\pi = (n_1, \dots, n_N)$ that ranks the agents in decreasing endowment order, i.e. with $i \leq i'$ if and only if $\overline{x}_{n_i} \geq \overline{x}_{n_{i'}}$. For any given total investment $X_N \leq \overline{X}_N$, an optimal scheme specifies investments $(x_{n_i}^*)_{i \in S}$ satisfying*

$$
x_{n_i}^* = \begin{cases} \overline{x}_{n_i} & \text{if } i < i^*, \\ x_{n_{i^*}}^* & \text{if } i = i^*, \\ 0 & \text{otherwise}, \end{cases}
$$

where $i^* \equiv \max\{i \in \{0, \dots, N+1\} : \sum_{i < i^*} \overline{x}_{n_i} \leq X_N\}$ and $x_{n_{i^*}}^* \equiv X_N - \sum_{i < i^*} \overline{x}_{n_i}$. Moreover, letting $X_i^* \equiv \sum_{j=1}^{i} x_{n_j}^*$ and noting that the investments $x_{n_i}^*$ are a function of X_N, the optimal total investment X_N^* then solves:

$$
\max_{X_N \in [0, \overline{X}_N]} \left(A - \theta \sum_{i=1}^{N} \frac{x_{n_i}^*}{F(X_i^*)} \right) F(X_N). \tag{7}
$$

Given the optimal individual investments as a function of X_N, the second part of Corollary 2 completes our characterization of the firm's optimal scheme by solving for the optimal total investment X_N^*. The program in (7) follows directly from our characterization in Proposition 1 and Proposition 2.

Corollary 2 has several important implications. First, since agents with larger endowments of capital make larger investments in the firm's project, our findings in Section 3 imply that agents with larger endowments receive higher net returns on their investments than those with smaller endowments. As such, our analysis highlights a mechanism through which capital becomes dispersed. We find that the firm's optimal scheme exacerbates differences in agents' initial capital. In fact, the results point to "winner-takes-all dynamics," whereby agents with large capital endowments become relatively larger over time, even when differences in initial endowments may be small. These effects resemble those that arise, albeit for different reasons, in tournament theory and models of superstars (Lazear and Rosen, 1981; Rosen, 1981).[30]

Second, we find that the firm's optimal scheme may imply differences among the agents not only in their net returns from investment but also in their access to investment opportunities. Specifically, if the total capital available $\overline{X}_N$ exceeds the amount X_N^* that the firm optimally raises, then the firm targets the largest investors and excludes smaller investors from the project.

Finally, our results have implications on the feasibility of investment. Since a more unequal distribution of capital among the agents increases the firm's payoff from any given total investment, such a distribution also reduces the minimum surplus A that is required from a project for investment to be profitable. As a consequence, we find that a larger range of investments can be undertaken when the population of investors is more heterogeneous.

4.2 Distribution of returns

Our last main result concerns the relationship between the distribution of agents' investments and the distribution of their net investment returns. Consider an optimal return schedule guaranteeing investments $(x_n)_{n \in S}$. By Proposition 1 and Proposition 2, the schedule specifies a permutation $\pi^* = (n_1^*, \ldots, n_N^*)$ ranking the agents in decreasing size order and yields each agent n_i an expected net return $F(X_N)r_i^*$. Since $r_i^* \geq r_{i'}^*$ for $i \leq i'$, the *range of net returns* is equal to the difference between the largest and smallest investors' net returns, $F(X_N)(r_1^* - r_N^*)$. We find that if the distribution of

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[30]In a dynamic setting, further considerations may come into play, as the firm could potentially offer returns as a function of an agent's history of investments. See Rey and Tirole (2007) for an insightful related study in the context of cooperatives. A dynamic analysis is beyond the scope of this paper.

investments becomes more unequal, the range of net returns declines:

Proposition 4. *Suppose $1/F(x)$ is convex for $x \in [0, X]$, $X > 0$, and consider investments $(x_n)_{n \in S}$ with $X_N \leq X$. Let the investments $(\widehat{x}_n)_{n \in S}$ majorize $(x_n)_{n \in S}$. The range of net returns offered by the firm under $(\widehat{x}_n)_{n \in S}$ is smaller than that under $(x_n)_{n \in S}$.*

Recall that any investments $(\widehat{x}_n)_{n \in S}$ that majorize $(x_n)_{n \in S}$ can be obtained from the latter by performing a finite sequence of transfers from smaller to larger investors. To prove the proposition, we show that any such transfer keeps the smallest investor's net return unchanged (and equal to θ) while reducing the largest investor's net return (strictly if the transfer increases this investor's capital). These effects apply regardless of whether the identities of the smallest and largest investors change, and they imply that the range of net returns becomes smaller. In this sense, we find that the firm's scheme is less discriminatory when the agents are more heterogeneous.

The example from the Introduction offers an illustration. For F uniform over $[0, 30]$ and $\theta = 10\%$, compare two agents who invest the capital amounts $(x_1, x_2) = (10, 20)$ against two agents investing $(\widehat{x}_1, \widehat{x}_2) = (6, 24)$. Under an optimal return schedule, agent 1 and agent 2 receive expected net returns of 10% and 15% respectively in the former case, whereas in the latter case these expected net returns are 10% and 12.5%. The range of net returns is therefore smaller under the more unequal distribution of capital: $(12.5 - 10)\% < (15 - 10)\%$.

Of course, the range of agents' initial capital amounts is larger under the more unequal distribution. In the example, this range is $20 - 10$ under (x_1, x_2) and $24 - 6$ under $(\widehat{x}_1, \widehat{x}_2)$. The net effect of heterogeneity on the range of final capital is thus unclear, as final capital holdings depend on both the agents' investments and their net returns. In the example, the range of final capital is larger under $(\widehat{x}_1, \widehat{x}_2)$: $24(1 + 12.5\%) - 6(1 + 10\%) > 20(1 + 15\%) - 10(1 + 10\%)$. More generally, either direction is possible depending on parameters. That is, perhaps surprisingly, we find that by reducing the range of net returns, a more unequal distribution of initial capital can lead to a more equal distribution of final capital.[31]

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[31]For an example, take $F(x) = x^5$ for $x \in [0, 1]$, $\theta = 10\%$, and capital amounts $(x_1, x_2) = (\frac{1}{3}, \frac{2}{3})$ and $(\widehat{x}_1, \widehat{x}_2) = (\frac{1}{4}, \frac{3}{4})$. The range of final capital is 0.81 under (x_1, x_2) and 0.79 under $(\widehat{x}_1, \widehat{x}_2)$.

5 Firm's Initial Capital

Our model has considered a firm which owns no initial capital, so any payments it offers to the agents must be self-financed by its project. In this section, we study how the firm's problem changes when the firm has some capital of its own. We show that our main qualitative results continue to hold, with larger investors receiving higher net returns than smaller ones. What is new is that the firm now uses its funds to insure part of the investment, and we are able to provide a characterization of the level of insurance offered to different investors depending on their size.

Suppose the firm has initial capital $W > 0$ and wishes to raise an additional amount X_N from the set S of N agents. Consider a scheme specifying investments $(x_n)_{n \in S}$ and returns $(r_n, k_n)_{n \in S}$, where without loss we take $x_n > 0$ for each $n \in S$. The firm's budget constraint now requires that, for all profiles of agents' choices $\mathcal{Y} = (y_1, \ldots, y_N)$, the scheme satisfy

$$\sum_{n=1}^{N} r_n y_n x_n \leq W + A \quad \text{and} \quad \sum_{n=1}^{N} k_n y_n x_n \leq W. \tag{BC_W}$$

If $W \geq \theta X_N$, the problem is trivial: the firm can offer net returns $(r_n, k_n) = (\theta, \theta)$ to each agent $n \in S$ and fund its project at the safe rate. As all the agents are given full insurance, there is no coordination problem among them. In what follows, we thus assume that the firm's capital is limited, satisfying $W < \theta X_N$.

The firm's problem is the same as that in (P) but with the budget constraint given by (BC_W) above (and with the total investment in the project now including the firm's capital W in addition to the capital X_N raised from the agents). To solve this problem, observe first that Lemma 1 continues to hold in this setting. Given investments $(x_n)_{n \in S}$, the firm's return schedule must thus specify a permutation $\pi = (n_1, \ldots, n_N)$ such that, for each $i \in S$, agent n_i is willing to invest when agents $(n_1, \ldots, n_{i-1})$ invest, no matter the rest. The key difference relative to the analysis of Section 3.1 is that the firm can now pay positive returns under failure, and hence, in principle, offer returns satisfying $k_i > r_i$ to some agent $n_i \in S$. Such an agent's expected payoff from investing with the firm would be decreasing in the other agents' investments. That is, unlike when $W = 0$, inducing strategic substitutabiliy is now feasible.

Nevertheless, we are able to show that an optimal return schedule for the firm induces strategic complementarities among all the agents. Suppose by contradiction

that $k_i > r_i$ for some agent $n_i \in S$ in any optimal schedule. Such an agent's participation requires $k_i > \theta$, and so by (BC$_W$) and $W < \theta X_N$, there must exist $j \neq i$ with $k_j < \theta < r_j$. Furthermore, by analogous logic as in Section 3.1, agent n_i must be indifferent over investing with the firm when all other agents invest, whereas agent n_j must be indifferent conditioning on only agents $(n_1, \ldots, n_{j-1})$ investing. This means that agent n_i conditions on weakly more other agents investing, and hence on a weakly higher probability of success, than agent n_j. We thus consider a perturbation that reduces k_i and increases r_i while at the same time increasing k_j and reducing r_j. We show that this perturbation either contradicts the optimality of the original schedule or allows us to construct another optimal schedule which satisfies $r_i \geq k_i$ for all $i \in S$.

Using the strategic complementaries, we obtain the following characterization:

Proposition 5. *Consider the firm's problem with initial capital $W > 0$. Suppose $1/F(x)$ is convex for $x \in [0, X]$, $X > 0$, and there exists an optimal return schedule guaranteeing investments $(x_n)_{n \in S}$ with $W + X_N \leq X$. Then an optimal such schedule specifies a permutation $\pi^* = (n_1^*, \ldots, n_N^*)$ satisfying*

$$x_{n_1^*} \geq \ldots \geq x_{n_N^*}$$

and returns $(r_i^*, k_i^*)_{i \in S}$ satisfying

$$k_i^* = \frac{\min\{\theta x_{n_i^*}, W_i\}}{x_{n_i^*}} \quad and \quad r_i^* = \frac{\theta - k_i^*(1 - F(W + X_i))}{F(W + X_i)},$$

where $X_i = \sum_{j=1}^{i} x_{n_j^*}$, $W_N \equiv W$, and $W_i \equiv \max\{W - \sum_{j=i+1}^{N} k_j^* x_{n_j^*}, 0\}$ for $i \in \{1, \ldots, N-1\}$.

An optimal scheme for the firm includes full-insurance contracts, with returns under success and failure equal to the safe rate θ. That is, we find that the firm uses its initial capital W to fully insure some of the capital X_N that it raises from the agents. Since W is limited, only an amount of capital W/θ can be insured. Once W is depleted, the firm faces the same problem that we solved in the previous sections, and hence it guarantees investment using a schedule analogous to that characterized in Proposition 1 and Proposition 2.

Proposition 5 shows that the smallest investors are the ones who receive insurance. The intuition is simple. The firm's cost of fully insuring the portion of capital W/θ is

equal to W and thus independent of how this capital is distributed among the agents. In contrast, the firm's cost of raising the additional capital $X_N - W/\theta$ does depend on its distribution: by Proposition 3, this cost is minimized when $X_N - W/\theta$ is raised from the largest investors. Consequently, it follows that it is optimal for the firm to raise the fully insured portion W/θ from the smallest investors.

The characterization in Proposition 5 shows that our results in Section 3 and Section 4 are robust to the firm owning initial capital.[32] In addition, this characterization offers predictions on the levels of risk afforded to investors of different size. Interestingly, empirical studies find that large investors hold riskier portfolios than small investors, and some of the explanations discussed in the literature include capital market imperfections and investors' risk aversion declining with wealth (see Carroll, 2000).[33] We contribute to this discussion from a different perspective, that of optimal design. Proposition 5 indeed predicts a high-risk, high-return investment for large investors and a low-risk, low-return investment for small investors. Here, however, the distinction arises as an optimal solution to the firm's problem of raising capital in the presence of strategic risk.

6   Discussion

Below we discuss some extensions and applications of our model and results.

Social planner and policy implications.   We have solved the problem of a firm that seeks to maximize its profits while guaranteeing a unique outcome. We point out here that our results also have implications for a social planner who internalizes agents' welfare.

Consider the problem of a planner who maximizes the probability of project success, subject to budget and unique implementation constraints as those in program (P). Because the budget constraint requires limiting the cost of raising capital, the solution to this problem coincides with that of the firm when the budget constraint is tight enough, namely when the surplus A from project success is sufficiently small. Specifically, the planner may have to give higher net returns to larger investors compared to smaller

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[32]Note that the smallest investors who receive full insurance also receive a lower net return on their investment compared to other investors, since $\theta \leq F(W + X_N)r_i^* + (1 - F(W + X_N))k_i^*$ for any $i \in S$.

[33]Capital market imperfections may cause entrepreneurs to finance their activities with their own capital and to earn a high return on their investments.

investors in order to be able to finance the investment in the project. Furthermore, the planner may benefit from a more unequal distribution of capital among the agents, as such a distribution allows to reduce costs and make the investment viable.

Of course, things change if we take different social welfare functions, and in particular if we include a concern for inequality. Our analysis shows that the need to address strategic risk gives rise to an important tradeoff between equality and efficiency.

Policy responses aimed at limiting inequality should support small investors, who we find will be at a disadvantage relative to larger investors. A direct measure would be to prevent the use of differential returns based on investor size, either by regulation or by design. For example, in the context of venture capital, all participants of the same investment round receive the same price, as the firm has only one valuation at any given point in time. Our model and results show that constraining differential pricing can protect small investors from getting worse terms than others, possibly at the expense of a lower revenue for the firm or greater strategic uncertainty.

Intermediaries that bundle the capital of small private investors into a single larger investment could also help limit inequality, provided that they do not extract any additional surplus by charging high fees. Additionally, regulators can generate instruments to facilitate coordination. For example, it may be possible to promote platforms where small investors can make commitments to invest that are legally binding but contingent on a minimum total investment. Such instruments would reduce the strategic risk which, we have shown, drives inequality.

Threshold distribution. Our analysis has focused on situations in which the distribution F of the investment threshold satisfies the condition of $1/F(x)$ being convex. As noted in Section 3.2, $1/F(x)$ cannot be globally concave, and it is indeed globally convex for most commonly used distribution functions. Yet, it is worth considering how our results would change if the condition on F is not met.

Given capital amounts $(x_n)_{n \in S}$, suppose $1/F(x)$ is concave over the whole relevant range, namely for $x \in [\min\{x_n | n \in S\}, X_N]$. Then our results in Proposition 2 would be reversed: given optimal returns $(r_i^*, k_i^*)_{i \in S}$ as characterized in Proposition 1, we would find that an optimal permutation $\pi^* = (n_1^*, \ldots, n_N^*)$ ranks the agents in increasing as opposed to decreasing size order. The intuition is the same as in Proposition 2 but also reversed: the firm benefits from placing large investors in the permutation according to when the risk premium drops most sharply with investment, and if $1/F(x)$

is concave, this occurs at the end of the permutation, when $X_i = \sum_{j=1}^{i} x_{n_j}$ is largest. The implication is that larger investors would now receive lower net returns than smaller investors, as opposed to the case in which $1/F(x)$ is convex. The contrasting results that we obtain in the two cases offer predictions that could be empirically tested.

Regarding the analysis in Section 4, we maintained the assumption of $1/F(x)$ convex throughout that section for consistency with our results in Section 3. However, our results on the distribution of capital in Section 4 are more general. In fact, if $1/F(x)$ is concave over the whole relevant range, one can follow the same proof strategy used for Proposition 3 to verify that the result sill applies, namely that the firm benefits from distributions of capital which are more unequal.

General equilibrium. We have taken a standard mechanism design approach by considering a single firm that makes take-it-or-leave-it offers to the agents. We assumed that the agents have the same outside option and differ only in their wealth, which is the focus of our study. From a theoretical perspective, this monopolistic setting permits a clean analysis where differential returns are not driven by considerations other than wealth. From an applied perspective, while there are multiple entrepreneurs with whom an investor may choose to contract in practice, these markets are far from competitive. Entrepreneurs do not sell "identical goods," and shifting from one enterprise to another is costly. Indeed, investors spend substantial time and effort on understanding the nature of the enterprise they may invest in (see, e.g., Fried and Hisrich, 1994), which creates a holdup problem and grants entrepreneurs a certain degree of monopoly power. This does not mean that investors' only outside option is a risk-free asset, but we believe modeling the outside market as one in which investors are price-takers approximates reality better than a model in which they can costlessly switch entrepreneurs.

That said, it is still of interest to explore how our analysis translates to a competitive market setting. With that goal, we study in the Online Appendix a simple extension of our model in which two firms compete for the capital of two heterogeneous investors. In this oligopolistic environment, the investors' outside options are not necessarily the same, as they are endogenously determined by the firms' equilibrium offers which can condition on investor size. Analogous to our analysis of the one-firm problem, we study equilibria where the firms' offers yield a unique outcome in the interaction between the investors (given an assumption on behavior under indifference) and where, upon a firm's deviation, the investors play the equilibrium that is worst for the firm. We show that

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under weak conditions, our main qualitative results continue to hold: in equilibrium, larger investors receive higher expected returns than smaller investors, on a per-dollar basis.

Proportional surplus. Our model has assumed that project success yields a fixed surplus $A > 0$, and only the probability of success varies with the amount of capital invested in the project. More generally, the surplus from project success may also be a function of the investment. Consider a simple case in which success yields a net surplus Rx if capital x is invested in the project, for some $R > \theta$. Given a scheme specifying investments $(x_n)_{n \in S}$ and returns $(r_n, k_n)_{n \in S}$, the firm's budget constraint then requires that, for all profiles $\mathcal{Y} = (y_1, \ldots, y_N)$,

$$\sum_{n=1}^{N} r_n y_n x_n \leq \sum_{n=1}^{N} R y_n x_n \quad \text{and} \quad \sum_{n=1}^{N} k_n y_n x_n \leq 0. \tag{BC$_R$}$$

Relative to the original budget constraint (BC), this constraint places further restrictions on the firm's scheme. In fact, note that given R, (BC$_R$) implies (BC) under a fixed surplus $A_R \equiv R X_N$, as both constraints require that the sum of payments under success do not exceed this amount. But (BC$_R$) adds restrictions, by requiring that the payment to any agent under success be no larger than the surplus generated by the project when only such an agent has invested. That is, the firm's budget constraint now requires $\max_{n \in S} r_n \leq R$.

Despite this difference, we can show that our analysis continues to apply to this setting. Specifically, given R, consider the firm's problem in (P) when project success yields a fixed surplus equal to A_R. As just explained, this is a relaxed problem relative to the firm's proportional surplus problem that is subject to (BC$_R$). Hence, it follows that if the solution to (P) described in Proposition 1 and Proposition 2 satisfies (BC$_R$)—namely, if this solution specifies $r_{n_1^*}^* \leq R$—then it is also a solution to the firm's proportional surplus problem. Moreover, note that among all return schedules guaranteeing investments $(x_n)_{n \in S}$ subject to (BC), the solution to (P) minimizes the highest return that the firm has to pay to any agent $n \in S$ under success. Therefore, if the solution described in Proposition 1 and Proposition 2 specifies $r_{n_1^*}^* > R$, no schedule can guarantee investments $(x_n)_{n \in S}$ while satisfying (BC$_R$).

Applications. We have formulated our problem in the context of a firm that raises capital to fund a project. There are various examples that may fit this description. As mentioned, our results resonate with evidence from private equity investments. The project in our model could also concern the building of a property to which agents contribute with purchase commitments, or fund-raising for a charity as in Andreoni (1998). We next discuss some further applications that relate to other literatures.

Exclusive contracts: A number of influential papers study how an incumbent firm may coordinate buyers on signing exclusive dealing contracts (see Rasmusen, Ramseyer and Wiley, 1991; Innes and Sexton, 1994; Segal and Whinston, 2000). Our analysis can be applied to this question. Consider an incumbent monopolist offering exclusive dealing contracts to buyers of different size, namely who differ in the number of units that they demand.[34] A potential entrant enters the market only if the total demand that has contracted with the monopolist is below a stochastic threshold, and the monopolist offers prices contingent on entry to guarantee a given total demand. Our results suggest that under certain conditions on the threshold distribution, the monopolist will offer lower unit prices to larger buyers compared to smaller ones. Moreover, the more heterogeneous the buyer population, the higher the monopolist's incentive to offer exclusive dealing contracts to fight market entry.

Joint task: Consider a team incentive problem similar to that in Winter (2004) but allowing for heterogeneity. A principal contracts with multiple agents who can contribute towards a joint task. Agents differ in their ability, with more skilled agents being able to make larger contributions than less skilled ones. Suppose that the probability of completing the joint task is increasing in the sum of agents' contributions, and the principal offers rewards contingent on task completion in order to guarantee a level of participation. Applying our results to this setting suggests that optimal rewards will be convex: the principal compensates agents with high ability more than proportionally relative to those with lower ability.

Bank runs: A sizable literature studies bank runs and how to prevent them. Consider a simple setting in which N agents have their funds deposited in a bank and can withdraw them at any time. Suppose there is a random threshold such that if the total withdrawal exceeds it, a bank run occurs and the bank collapses. To exclude a run,

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[34]Note that due to compatibility and cost considerations, these demands are often indivisible.

the bank can offer depositors collateral (to be paid in the case of a run) or a higher interest rate on deposits (absent a run). A conjecture that can be derived from our analysis is that large depositors will be treated more favorably than small ones even on a per-dollar basis, whether it is collateral or an increased interest rate that is used to prevent the run.

A   Proofs

This Appendix provides proofs for the results in Section 3 and Section 4. Supplementary proofs and results can be found in the Online Appendix. We abbreviate Nash equilibrium by NE.

A.1   Proof of Lemma 1

As defined in Section 3.1, denote the aggregate capital of the first i agents in a permutation $\pi = (n_1, \ldots, n_N)$ by $X_i \equiv \sum_{j=1}^{i} x_{n_j}$.

($\Longrightarrow$)   We begin by proving that (U) implies a permutation as described in the lemma. Suppose (U) holds. Note that there must exist an agent n_1 who is willing to invest with the firm when no other agent does. If this was not true, there would be a NE in which no agent invests, contradicting (U). Hence, we have:

$$r_1 F(X_1) + k_1 (1 - F(X_1)) \geq \theta. \tag{8}$$

Additionally, agent n_1 must be willing to invest with the firm when all other agents do. Otherwise, there would not be a NE in which all agents invest with the firm, contradicting (U). Hence, we also have:

$$r_1 F(X_N) + k_1 (1 - F(X_N)) \geq \theta. \tag{9}$$

For any set of agents $S_I \subseteq S$, let $X(S_I) \equiv \sum_{i \in S_I} x_{n_i}$ be the aggregate capital of the agents in S_I. Since $F(X_1) \leq F(X_1 + X(S_I)) \leq F(X_N)$ for $S_I \subseteq \{2, \ldots, N\}$, equations (8) and (9) imply

$$r_1 F(X_1 + X(S_I)) + k_1 (1 - F(X_1 + X(S_I))) \geq \theta$$

for all $S_I \subseteq \{2, \ldots, N\}$. Therefore, agent n_1 is willing to invest with the firm no matter what the other agents do.

We now proceed by induction: for any $i \in \{2, \ldots, N-1\}$, suppose that there is an agent n_i who is willing to invest with the firm if agents $(n_1, \ldots, n_{i-1})$ invest, regardless of what the other agents do. Then there must be an agent n_{i+1} who is willing to invest with the firm if agents $(n_1, \ldots, n_i)$ invest and the other agents do not. Otherwise, there would be a NE in which agents $(n_1, \ldots, n_i)$ invest with the firm and agents $(n_{i+1}, \ldots, n_N)$ do not, contradicting (U). Thus, we have

$$r_{i+1} F(X_{i+1}) + k_{i+1} (1 - F(X_{i+1})) \geq \theta. \tag{10}$$

Moreover, by (U), agent n_{i+1} must also be willing to invest with the firm when all other agents do:

$$r_{i+1} F(X_N) + k_{i+1} (1 - F(X_N)) \geq \theta. \tag{11}$$

Since $F(X_{i+1}) \leq F(X_{i+1} + X(S_I)) \leq F(X_N)$ for $S_I \subseteq \{i+2, \ldots, N\}$, equations (10) and (11) imply

$$r_{i+1} F(X_{i+1} + X(S_I)) + k_{i+1} (1 - F(X_{i+1} + X(S_I))) \geq \theta$$

for all $S_I \subseteq \{i+2, \ldots, N\}$. Therefore, agent n_{i+1} is willing to invest with the firm if agents $(n_1, \ldots, n_i)$ invest with the firm, regardless of what the other agents do.

($\Longleftarrow$) We next prove that a permutation as described in the lemma implies (U). First, note that since each agent $n_i \in S$ is willing to invest if $(n_1, \ldots, n_{i-1})$ invest no matter what the rest does, it must be that each agent n_i is willing to invest when all other agents invest. Hence, there exists a NE in which all agents invest.

Next, to show uniqueness, suppose towards a contradiction that there exists a NE in which some agents do not invest with the firm. Recall that all such agents must strictly prefer not to invest. Call the set of non-investing agents S_{NI}. We claim that S_{NI} must be empty. Clearly, n_1 cannot be in S_{NI}, as n_1 is willing to invest with the firm no matter what the other agents do. So n_1 must be in the set of agents who invest, call it S_I. Now proceed by induction: for any $i \in \{2, \ldots, N-1\}$, suppose agents $(n_1, \ldots, n_i)$ are in S_I. Then by the permutation stated in the lemma, agent n_{i+1} is willing to invest with the firm, and thus she cannot be in S_{NI} either. The claim follows.

A.2 Proof of Proposition 1

We begin by proving the first part of the proposition. By Lemma 1, any optimal return schedule specifies some permutation $\pi = (n_1, \ldots, n_N)$ and returns $(r_i, k_i)_{i \in S}$ which satisfy, for each $i \in S$ and each $j \in \{i, \ldots, N\}$,

$$r_i F(X_j) + k_i (1 - F(X_j)) \geq \theta. \tag{12}$$

As argued in the text, the firm's budget constraint (BC) requires $k_i \leq 0$ for each $i \in S$. Given this and $\theta > 0$, equation (12) then requires $r_i > 0$ for each $i \in S$. It follows that $r_i > 0 \geq k_i$ for each $i \in S$, and thus the left-hand side of (12) is increasing in $F(X_j)$. Since $F(X_j)$ is increasing in j, it follows that (12) holds for each $i \in S$ and each $j \in \{i, \ldots, N\}$ if and only if, for each $i \in S$,

$$r_i F(X_i) + k_i (1 - F(X_i)) \geq \theta. \tag{13}$$

We show that optimality requires (13) to hold with equality for each $i \in S$. Suppose by contradiction that there is an optimal return schedule under which (13) holds as a strict inequality for some $i' \in S$. Then consider a perturbation in which we reduce $r_{i'}$ by $\varepsilon > 0$ arbitrarily small while keeping all other returns unchanged. Since (13) was a strict inequality for i', this constraint continues to be satisfied for all $i \in S$. It is also clear that the budget constraint (BC) is relaxed by the perturbation. Moreover, note that the firm's expected payoff is

$$\left(A - \sum_{i=1}^{N} r_i x_{n_i} \right) F(X_N) - \sum_{i=1}^{N} k_i x_{n_i} (1 - F(X_N)), \tag{14}$$

which is decreasing in r_i for any $i \in S$. Therefore, we obtain that the perturbation increases the firm's expected payoff while preserving the agents' incentives to participate and the firm's budget constraint, and thus the original return schedule cannot be optimal.

We next prove the second part of the proposition. By the claims above, any optimal return schedule specifies some permutation $\pi = (n_1, \ldots, n_N)$ and returns $(r_i, k_i)_{i \in S}$ satisfying

$$r_i F(X_i) + k_i (1 - F(X_i)) = \theta \tag{15}$$

for each $i \in S$. We show that it is optimal to set $k_i = 0$ for each $i \in S$, which combined with (15) implies $r_i = \theta / F(X_i)$ for each $i \in S$. Suppose by contradiction that this is not the case, i.e. any optimal schedule has $k_{i'} < 0$ for some $i' \in S$. (Recall that by the firm's budget constraint, $k_i \leq 0$ for all $i \in S$.) Then consider the following perturbation: for any such i', we increase $k_{i'}$ by $\varepsilon > 0$ arbitrarily small and reduce $r_{i'}$ by $\varepsilon \eta_{i'}$, where

$$\eta_{i'} \equiv \frac{1 - F(X_{i'})}{F(X_{i'})}.$$

Since we had $k_{i'} < 0$, the perturbed schedule continues to satisfy the firm's budget constraint (BC). Moreover, by construction, the left-hand side of (15) is unchanged by the perturbation, so the agents' incentives to participate are preserved. Finally, note that the perturbation changes the firm's expected payoff in (14) by

$$\varepsilon \frac{(F(X_N) - F(X_{i'}))}{F(X_{i'})},$$

which is positive (and strictly positive if $i' \in \{1, \ldots, N-1\}$). Therefore, the perturbation increases the firm's expected payoff while preserving the agent's incentives to participate and the budget constraint. Since we can perform this perturbation whenever $k_i < 0$ for some $i \in S$, this contradicts the assumption that an optimal schedule with $k_i = 0$ for each $i \in S$ does not exist.

Finally, we prove that if an optimal return schedule given investments $(x_n)_{n \in S}$ exists, there exists an optimal schedule specifying some permutation $\pi = (n_1, \ldots, n_N)$ and returns $(r_i^*, k_i^*) = (\theta / F(X_i), 0)$ for each $i \in S$. As shown above, any optimal schedule specifies some permutation $\pi = (n_1, \ldots, n_N)$ and returns $(r_i, k_i)_{i \in S}$ such that (15) holds for each $i \in S$. It is clear that for each agent n_i, the return r_i that satisfies this binding participation constraint is decreasing in k_i. Thus, given a permutation π, setting k_i as high as possible for each $i \in S$, subject to (BC), minimizes the firm's costs under success, $\sum_{i=1}^{N} r_i x_{n_i}$. It follows that setting $k_i = 0$ for each $i \in S$ maximally relaxes the firm's budget constraint. As we have shown that setting $(r_i^*, k_i^*) = (\theta / F(X_i), 0)$ for some permutation π is optimal subject to the budget constraint, this proves the claim.

A.3   Proof of Proposition 2

Assume that $1/F(x)$ is convex for all $x \in [0, X_N]$. We proceed in two steps.

Step 1. Define

$$\Psi\left(a, b, c\right) \equiv \frac{1}{c}\left(\frac{1}{F\left(a+b\right)} - \frac{1}{F\left(a+b+c\right)}\right) - \frac{1}{b}\left(\frac{1}{F\left(a+c\right)} - \frac{1}{F\left(a+b+c\right)}\right). \quad (16)$$

We show that for any $a \geq 0$ and $b > c > 0$ satisfying $a + b + c \leq X_N$,

$$\Psi\left(a, b, c\right) \leq 0. \quad (17)$$

To prove this claim, observe that

$$\Psi\left(a, b, c\right) = \frac{1}{c}\int_b^{b+c}\frac{F'\left(a+z\right)}{\left(F\left(a+z\right)\right)^2}dz - \frac{1}{b}\int_c^{b+c}\frac{F'\left(a+z\right)}{\left(F\left(a+z\right)\right)^2}dz.$$

Define $\psi\left(\widetilde{z}\right) = \frac{c\widetilde{z}+b^2-c^2}{b}$. Note that ψ is linear with $\psi\left(c\right) = b$, $\psi\left(b+c\right) = b+c$, and $\psi'\left(\widetilde{z}\right) = \frac{c}{b}$. Hence, a change of variables yields:

$$\Psi\left(a, b, c\right) = \frac{1}{b}\int_c^{b+c}\left(\frac{F'\left(a+\psi(\widetilde{z})\right)}{\left(F\left(a+\psi(\widetilde{z})\right)\right)^2} - \frac{F'\left(a+\widetilde{z}\right)}{\left(F\left(a+\widetilde{z}\right)\right)^2}\right)d\widetilde{z}. \quad (18)$$

Note that given $b > c$, $\psi\left(\widetilde{z}\right) \geq \widetilde{z}$ for all $\widetilde{z}$ in the integration region. Given $a \geq 0$ and $a + b + c \leq X_N$, the assumption that $1/F(x)$ is convex for all $x \in [0, X_N]$ then implies that the integrand in (18) is (weakly) negative. The claim follows.

Step 2. By Step 1, (17) holds for any $a \geq 0$ and $b > c > 0$ satisfying $a+b+c \leq X_N$. Using (16), this inequality can be rewritten as

$$\frac{b}{F\left(a+b\right)} + \frac{c}{F\left(a+b+c\right)} \leq \frac{c}{F\left(a+c\right)} + \frac{b}{F\left(a+b+c\right)}. \quad (19)$$

We now show that there is an optimal permutation $\pi^* = \left(n_1^*, \ldots, n_N^*\right)$ satisfying

$$x_{n_1^*} \geq \ldots \geq x_{n_N^*}. \quad (20)$$

Suppose that some permutation $\pi = \left(n_1, \ldots, n_N\right)$ is optimal. If π satisfies (20), we are done. Suppose instead that (20) is not satisfied. Take the lowest index $j < N$ for which $x_{n_j} < x_{n_{j+1}}$. We perform a perturbation in which we swap agents n_j and n_{j+1}. Note that this swap has no effect on X_i for any $i < j$ or $i > j + 1$. Hence, the

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perturbation only affects the terms j and $j+1$ of the sum in the firm's costs in (5). Under the original permutation, these terms sum to:

$$\frac{x_{n_j}}{F\left(X_{j-1}+x_{n_j}\right)} + \frac{x_{n_{j+1}}}{F\left(X_{j-1}+x_{n_j}+x_{n_{j+1}}\right)}. \tag{21}$$

The perturbation changes the sum of these terms to:

$$\frac{x_{n_{j+1}}}{F\left(X_{j-1}+x_{n_{j+1}}\right)} + \frac{x_{n_j}}{F\left(X_{j-1}+x_{n_j}+x_{n_{j+1}}\right)} \tag{22}$$

Letting $a = X_{j-1}$, $b = x_{n_{j+1}}$, and $c = x_{n_j}$, it follows from (19) that the sum in (22) is no larger than the sum in (21). Therefore, the perturbation (weakly) reduces the firm's costs and thus increases the firm's expected payoff. Note that we can proceed by performing this perturbation for the next pair of agents with (higher) indices $(i, i+1)$ such that $x_{n_i} < x_{n_{i+1}}$, repeating until the permutation satisfies (20). Since each perturbation increases the firm's expected payoff and the original permutation was optimal, we obtain that a permutation satisfying (20) is optimal.

A.4 Proof of Proposition 3

The capital amounts $(\widehat{x}_n)_{n\in S}$ can be obtained from the original amounts $(x_n)_{n\in S}$ by performing a finite sequence of transfers from smaller to larger investors (Hardy et al., 1934). Thus, it suffices to show that each such transfer makes the firm better off. Without loss of generality, consider the first such transfer. Let the permutation $\pi = (n_1, \ldots, n_N)$ be optimal under $(x_n)_{n\in S}$. Take any two agents n_j and n_ℓ where $j < \ell$ and, thus, $x_{n_j} \geq x_{n_\ell}$. For any $\Delta \in (0, x_{n_\ell}]$, let $(\widehat{x}_n)_{n\in S}$ be the result of transferring Δ units of capital from agent n_ℓ to agent n_j. We will show that the firm's minimized costs under $(\widehat{x}_n)_{n\in S}$ are lower than its minimized costs under $(x_n)_{n\in S}$ when keeping the permutation π unchanged. Since the transfer does not change the probability of project success (as it does not affect the total amount of capital invested in the project), it will follow that the firm's expected payoff under $(\widehat{x}_n)_{n\in S}$ is higher than that under $(x_n)_{n\in S}$ when keeping the permutation π unchanged. Clearly, changing to a permutation that is optimal under $(\widehat{x}_n)_{n\in S}$ can only increase the firm's payoff from these investments further, so this is sufficient to prove the claim.

To show that the transfer from agent n_ℓ to agent n_j reduces the firm's costs when

keeping the permutation unchanged, note first that the costs from returns paid to agents n_i with ranks $i < j$ or $i > \ell$ are unaffected. The change in the firm's minimized costs in (5), divided by the constant $\theta > 0$, is thus equal to

$$\frac{\Delta}{F(X_j + \Delta)} - \frac{\Delta}{F(X_\ell)} - \sum_{i=j}^{\ell-1} \left[\frac{x_{n_i}}{F(X_i)} - \frac{x_{n_i}}{F(X_i + \Delta)} \right]. \tag{23}$$

Replacing $F(X_j + \Delta)$ by $F(X_j)$ and x_{n_i} by $x_{n_{i+1}}$, (23) is no larger than

$$\frac{\Delta}{F(X_j)} - \frac{\Delta}{F(X_\ell)} - \sum_{i=j}^{\ell-1} x_{n_{i+1}} \left[\frac{1}{F(X_i)} - \frac{1}{F(X_i + \Delta)} \right].$$

This expression can be rewritten as

$$\Delta \sum_{i=j}^{\ell-1} x_{n_{i+1}} \Lambda_i,$$

where

$$\begin{aligned}
\Lambda_i &= \frac{1}{x_{n_{i+1}}} \left[\frac{1}{F(X_i)} - \frac{1}{F(X_i + x_{n_{i+1}})} \right] - \frac{1}{\Delta} \left[\frac{1}{F(X_i)} - \frac{1}{F(X_i + \Delta)} \right] \\
&= \frac{1}{x_{n_{i+1}}} \int_{X_i}^{X_i + x_{n_{i+1}}} \frac{F'(z)}{(F(z))^2} dz - \frac{1}{\Delta} \int_{X_i}^{X_i + \Delta} \frac{F'(z)}{(F(z))^2} dz.
\end{aligned}$$

Define $\psi(\widetilde{z}) = \frac{x_{n_{i+1}} \widetilde{z} - X_i (x_{n_{i+1}} - \Delta)}{\Delta}$. Note that ψ is linear with $\psi(X_i) = X_i$, $\psi(X_i + \Delta) = X_i + x_{n_{i+1}}$, and $\psi'(\widetilde{z}) = \frac{x_{n_{i+1}}}{\Delta}$. Hence, a change of variables yields:

$$\Lambda_i = \frac{1}{\Delta} \int_{X_i}^{X_i + \Delta} \left(\frac{F'(\psi(\widetilde{z}))}{(F(\psi(\widetilde{z})))^2} - \frac{F'(\widetilde{z})}{(F(\widetilde{z}))^2} \right) d\widetilde{z}. \tag{24}$$

Since $\Delta \leq x_{n_\ell}$ and $x_{n_\ell} \leq x_{n_{i+1}}$ for all $j \leq i \leq \ell - 1$, we have $\Delta \leq x_{n_{i+1}}$ for all $j \leq i \leq \ell - 1$. Thus, one can verify that $\psi(\widetilde{z}) \geq \widetilde{z}$ for all $\widetilde{z}$ in the integration region and $j \leq i \leq \ell - 1$. Given $X_i \geq 0$ and $X_i + x_{n_{i+1}} \leq X_N$ for all $j \leq i \leq \ell - 1$, the assumption that $1/F(x)$ is convex for all $x \in [0, X_N]$ then implies that the integrand in (24) is (weakly) negative. It follows that $\Lambda_i \leq 0$ for all $j \leq i \leq \ell - 1$, so the change in costs in (23) is no larger than a (weakly) negative number. The claim follows.

A.5 Proof of Proposition 4

Let $\pi = (n_1, \ldots, n_N)$ and $\widehat{\pi} = (\widehat{n}_1, \ldots, \widehat{n}_N)$ be optimal permutations under $(x_n)_{n \in S}$ and $(\widehat{x}_n)_{n \in S}$ respectively, where we consider only agents with strictly positive investments. The smallest investor's expected net return is the same under $(x_n)_{n \in S}$ and $(\widehat{x}_n)_{n \in S}$, as it is equal to $F(X_N) \frac{\theta}{F(X_N)} = \theta$ regardless of how X_N is distributed among the agents. The largest investor's expected net return is equal to $F(X_N) \frac{\theta}{F(x_{n_1})}$ under $(x_n)_{n \in S}$ and $F(X_N) \frac{\theta}{F(\widehat{x}_{\widehat{n}_1})}$ under $(\widehat{x}_n)_{n \in S}$. Recall that the capital amounts $(\widehat{x}_n)_{n \in S}$ can be obtained from $(x_n)_{n \in S}$ by performing a finite sequence of transfers from smaller to larger investors (Hardy et al., 1934). It follows that $\widehat{x}_{\widehat{n}_1} \geq x_{n_1}$, and since F is increasing, the largest investor's net return is weakly lower under $(\widehat{x}_n)_{n \in S}$ compared to $(x_n)_{n \in S}$. The claim follows.

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