Tau functions associated with linear systems

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Abstract. Let $(-A, B, C)$ be a linear system in continuous time $t > 0$ with input and output space $\mathbb{C}$ and state space H. The function $\phi_{(x)}(t) = Ce^{-(t+2x)A}B$ determines a Hankel integral operator $\Gamma_{\phi_{(x)}}$ on $L^2((0,\infty);\mathbb{C})$; if $\Gamma_{\phi_{(x)}}$ is trace class, then the Fredholm determinant $\tau(x) = \det(I + \Gamma_{\phi_{(x)}})$ defines the tau function of $(-A, B, C)$. Such tau functions arise in Tracy and Widom's theory of matrix models, where they describe the fundamental probability distributions of random matrix theory. Dyson considered such tau functions in the inverse spectral problem for Schrödinger's equation $-f'' + uf = \lambda f$, and derived the formula for the potential $u(x) = -2\frac{d^2}{dx^2}\log\tau(x)$ in the self-adjoint scattering case *Commun. Math. Phys.* **47** (1976), 171–183. This paper introduces a operator function R_x that satisfies Lyapunov's equation $\frac{dR_x}{dx} = -AR_x - R_xA$ and $\tau(x) = \det(I + R_x)$, without assumptions of self-adjointness. When $-A$ is sectorial, and B, C are Hilbert–Schmidt, there exists a non-commutative differential ring $\mathcal{A}$ of operators in H and a differential ring homomorphism $\lfloor\ \rfloor : \mathcal{A} \to \mathbb{C}[u, u', \dots]$ such that $u = -4\lfloor A\rfloor$, which extends the multiplication rules for Hankel operators considered by Pöppe, and McKean *Cent. Eur. J. Math.* **9** (2011), 205–243.

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1. Introduction

This paper is concerned with Fredholm determinants which arise in the theory of linear systems and their application to inverse spectral problem for Schrödinger's equation. For $\phi \in L^2((0,\infty);\mathbb{R})$, the Hankel integral operator corresponding to ϕ is Γ_ϕ where

$$\Gamma_\phi f(x) = \int_0^\infty \phi(x+y)f(y)\,dy \qquad (f \in L^2((0,\infty);\mathbb{C}). \tag{1.1}$$

Using the Laguerre system of orthogonal functions as in [31], one can express Γ_ϕ as a matrix $[\gamma_{j+k}]_{j,k=1}^\infty$ on ℓ^2, which has the characteristic shape of a Hankel matrix, and one can establish criteria for the operator to be bounded on $L^2((0,\infty);\mathbb{C})$. Megretskii, Peller and Treil [26] determined the possible spectrum and spectral multiplicity function that can arise from a bounded and self-adjoint Hankel operator. Thus they characterized the class of bounded self-adjoint Hankel operators up to unitary equivalence. Their method involved introducing suitable linear systems on a state space H, and this motivated the approach of our paper.

Following earlier works by Faddeev and others in the Russian literature, Dyson [8] considered the inverse spectral problem for Schrödinger's equation $-f''+uf=\lambda f$, for $u\in C^2(\mathbb{R};\mathbb{R})$ that decays rapidly as $x\to\pm\infty$. From the asymptotic solutions, he introduced a scattering function ϕ, considered the translations $\phi_{(x)}(y)=\phi(y+2x)$, and established connections with eigenvalue distributions in random matrix theory which are described in [38]. He showed that the potential can be recovered from the scattering data by means of the formula

$$u(x)=-2\frac{d^2}{dx^2}\log\det(I+\Gamma_{\phi_{(x)}}), \tag{1.2}$$

These results were developed further by Ercolani, McKean [10] and others [13], [39], [40] to describe the inverse spectral problem for self-adjoint Schrödinger operators on $\mathbb{R}$. Grudsky and Rybkin [17] describes the inverse scattering theory of the KdV equation in terms of Hankel and Toeplitz operators. The latter paper uses Sarason's algebra $H^\infty+C$ on the unit disc to describe compact Hankel operators. In the current paper, we use Hankel operators within the setting of linear systems in continuous time.

Remarkably, some of the methods of inverse scattering theory do not really need self-adjointness. However, a significant obstacle in this approach is that Hankel operators do not have a natural product structure, so it is unclear as to how one can fully exploit the multiplicative properties of determinants. This paper seeks to address this issue, by realizing Hankel operators from linear systems, and then introducing algebras of operators on state space that reflect the properties of Hankel operators and their Fredholm determinants. As in [26], the Lyapunov differential equation is fundamental to the development of the theory.

Definition 1.1. (i) *(Lyapunov equation).* Let H be a complex Hilbert space, known as the state space, and $\mathcal{L}(H)$ the algebra of bounded linear operators on H with the usual operator norm. Let $(e^{-tA})_{t\geq 0}$ be a strongly continuous (C_0) semigroup of bounded linear operators on H such that $\|e^{-tA}\|_{\mathcal{L}(H)}\leq M$ for all $t\geq 0$ and some $M<\infty$. Let $\mathcal{D}(A)$ be the domain of the generator $-A$ so that $\mathcal{D}(A)$ is itself a Hilbert space for the graph norm $\|\xi\|^2_{\mathcal{D}(A)}=\|\xi\|^2_H+\|A\xi\|^2_H$, and let $A^\dagger$ be the adjoint of A. Let $R:(0,\infty)\to\mathcal{L}(H)$ be a differentiable function. The Lyapunov equation is

$$-\frac{dR_z}{dz}=AR_z+R_zA \qquad (z>0), \tag{1.3}$$

where the right-hand side is to be interpreted as a bounded bilinear form on $\mathcal{D}(A) \times \mathcal{D}(A^\dagger)$. (This is a modified form of the version in [31] p. 502.)

(ii) *(Operator ideals).* Let $\mathcal{L}^2(H)$ be the space of Hilbert–Schmidt operators on H, and $\mathcal{L}^1(H)$ be the space of trace class operators on H, so $\mathcal{L}^1(H) = \{T : T = VW; V, W \in \mathcal{L}^2(H)\}$ and let det be the Fredholm determinant defined on $\{I + T : T \in \mathcal{L}^1(H)\}$; see [25].

Definition 1.2. (i) *(Linear system).* Let H_0 be a complex separable Hilbert space which serves as the input and output spaces; let $B : H_0 \to H$ and $C : H \to H_0$ be bounded linear operators. The continuous-time linear system $(-A, B, C)$ is

$$\begin{aligned} \frac{dX}{dt} &= -AX + BU \\ Y &= CX. \end{aligned} \tag{1.4}$$

(ii) *(Scattering function).* The scattering function is $\phi(x) = Ce^{-xA}B$, which is a bounded and weakly continuous function $\phi : (0, \infty) \to \mathcal{L}(H_0)$. The terminology is justified by [10] p.493. In control theory, the transfer function is the Laplace transform of ϕ; see [31] p. 467.

(iii) *(Hankel operator).* Suppose that $\phi \in L^2((0, \infty); \mathcal{L}(H_0))$. Then the corresponding Hankel operator is Γ_ϕ on $L^2((0, \infty); H_0)$, where $\Gamma_\phi f(x) = \int_0^\infty \phi(x + y) f(y)\, dy$; see [31], [29] for boundedness criteria.

Definition 1.3. *(Admissible linear system).* Let $(-A, B, C)$ be a linear system as above; suppose furthermore that the observability operator $\Theta_0 : L^2((0, \infty); H_0) \to H$ is bounded, where

$$\Theta_0 f = \int_0^\infty e^{-sA^\dagger} C^\dagger f(s)\, ds; \tag{1.5}$$

suppose that the controllability operator $\Xi_0 : L^2((0, \infty); H_0) \to H$ is also bounded, where

$$\Xi_0 f = \int_0^\infty e^{-sA} B f(s)\, ds. \tag{1.6}$$

(i) Then $(-A, B, C)$ is an admissible linear system. See [31] [page 469].

(ii) Suppose furthermore that Θ_0 and Ξ_0 belong to the ideal $\mathcal{L}^2$ of Hilbert–Schmidt operators. Then we say that $(-A, B, C)$ is $(2, 2)$-admissible.

The scattering map associates to any $(2, 2)$ admissible linear system $(-A, B, C)$ the corresponding scattering function $\phi(x) = Ce^{-xA}B$. The inverse scattering problem involves recovering data about u from ϕ, as in (1.2). In section 2 of this paper, we analyze the existence and uniqueness problem for the Lyapunov equation, and show that for any $(2, 2)$ admissible linear system, the operator, as in [5], [1],

$$R_x = \int_x^\infty e^{-tA} B C e^{-tA}\, dt \tag{1.7}$$

is trace class and gives the unique solution to (1.3) with the initial condition

$$\Big(\frac{dR_x}{dx}\Big)_{x=0} = -AR_0 - R_0A = -BC. \tag{1.8}$$

Also, $R_x \in \mathcal{L}^1(H)$ and the Fredholm determinant satisfies

$$\det(I + \lambda R_x) = \det(I + \lambda \Gamma_{\phi_{(x)}}) \qquad (x > 0, \lambda \in \mathbb{C}). \tag{1.9}$$

Definition 1.4. *(Tau function).* Given an $(2, 2)$ admissible linear system $(-A, B, C)$, we define

$$\tau(x) = \det(I + R_x). \tag{1.10}$$

Using this general definition of τ, we can unify several results from the scattering theory of ordinary differential equations. Under circumstances discussed in [17] and [35], this becomes the well-known Hitota tau function of soliton theory. Such tau functions are also strongly analogous to the tau functions introduced by Jimbo, Miwa and Date [27] to describe the isomonodromy of rational differential equations and generalize classical results on theta functions. The connection between Fredholm determinants and rational differential equations is further described in [11] and [38]; see also [20].

The Gelfand–Levitan–Marchenko equation [12] provides the linkage between ϕ and u via R_x. Consider

$$T(x, y) + \Phi(x + y) + \mu \int_x^\infty T(x, z)\Phi(z + y)\, dz = 0 \qquad (0 < x < y) \tag{1.11}$$

where $T(x, y)$ and $\Phi(x + y)$ are $m \times m$ matrices with scalar entries. In the context of $(-A, B, C)$ we assume that $\Phi(x) = Ce^{-xA}B$ is known and aim to find $T(x, y)$. In section two, we use R_x to construct solutions to the associated Gelfand–Levitan equation (1.11), and introduce a potential

$$u(x) = -2\frac{d^2}{dx^2} \log\det(I + R_x). \tag{1.12}$$

In section three, we obtain a differential equation linking $\Phi(x)$ to $u(x)$. In examples of interest in scattering theory, one can calculate $\det(I + \lambda R_x)$ more easily than the Hankel determinant of $\Gamma_{\Phi_{(x)}}$ directly [10], since R_x has additional properties that originate from Lyapunov's equation. In section four, we introduce a differential algebra of operators on the state space, and a homomorphism to the differential algebra $\mathbb{C}[u, u', \dots]$ that is generated by the potential. In section five, we describe the connection between this algebra and the stationary KdV hierarchy. There is a fundamental connection between theta functions and equations of KdV and KP type; see [28].

2. τ functions in terms of Lyapunov's equation and the Gelfand–Levitan equation

The following section proves existence and uniqueness of solutions of the Lyapunov equation (1.3), in a style suggested by [31] p 503]. Peller discusses scattering functions that produce bounded self-adjoint Hankel operators Γ_ϕ, and their realization in terms of continuous time linear systems. He observes that in some cases one needs a bounded semigroup with unbounded generator $(-A)$. We prove the uniqueness results for bounded and strongly continuous semigroups, then specialize to holomorphic semigroups. The main application is to the Gelfand–Levitan equation (1.11), and associated determinants.

Proposition 2.1. *Let* $(e^{-tA})_{t\geq 0}$ *be a strongly continuous and weakly asymptotically stable semigroup on a complex Hilbert space* H*, so* $e^{-tA}f\to 0$ *weakly as* $t\to\infty$ *for all* $f\in H$*. Then*

(i) $S_t: R\mapsto e^{-tA}Re^{-tA}$ *for* $t\geq 0$ *defines a strongly continuous semigroup on* $\mathcal{L}^1(H)$*, which has generator* $(-L)$*, with dense domain of definition* $\mathcal{D}(L)$ *such that*

$$L(R)=AR+RA \qquad (R\in\mathcal{D}(L)). \tag{2.1}$$

(ii) The linear operator $L:\mathcal{D}(L)\to\mathcal{L}^1(H)$ *is injective, and for each* $R_0\in\mathcal{D}(L)$ *with* $L(R_0)=X$*, there exists a weakly convergent integral*

$$R_0=\int_0^\infty e^{-tA}Xe^{-tA}\,dt. \tag{2.2}$$

(iii) Suppose moreover that $\Vert e^{-t_0A}\Vert_{\mathcal{L}(H)}<1$ *for some* $t_0>0$*. Then* $L:\mathcal{D}(L)\to\mathcal{L}^1(H)$ *is surjective, the integral (2.2) converges absolutely in* $\mathcal{L}^1(H)$ *and* R_0 *gives the unique solution to* $AR_0+R_0A=X$*.*

Proof. (i) First observe that by the uniform boundedness theorem, there exists M such that $\Vert e^{-tA}\Vert_{\mathcal{L}(H)}\leq M$ for all $t\geq 0$, so $(e^{-tA})_{t\geq 0}$ is uniformly bounded. Also, the adjoint semigroup $(e^{-tA^\dagger})_{t\geq 0}$ is also strongly continuous and uniformly bounded, so A and $A^\dagger$ have dense domains $\mathcal{D}(A)$ and $\mathcal{D}(A^\dagger)$ in H.

Now $\mathcal{L}^1(H)=H\hat{\otimes}H$, the projective tensor product, so for all $X\in\mathcal{L}^1(H)$, there exists a nuclear decomposition $X=\sum_{j=1}^\infty B_jC_j$ where $B_j,C_j\in H$ satisfy $\Vert X\Vert_{\mathcal{L}^1(H)}=\sum_{j=1}^\infty\Vert B_j\Vert_H\Vert C_j\Vert_H$. Then

$$S_t(X)-X=\sum_{j=1}^\infty(e^{-tA}B_jC_je^{-tA}-B_jC_je^{-tA})+\sum_{j=1}^\infty(B_jC_je^{-tA}-B_jC_j)$$

where (e^{-tA}) is bounded, $\Vert e^{-tA}B_j-B_j\Vert_H\to 0$ and $\Vert e^{-tA^\dagger}C_j-C_j\Vert_H\to 0$ as $t\to 0+$; so $\Vert S_t(X)-X\Vert_{\mathcal{L}^1(H)}\to 0$ as $t\to 0+$; so $(S_t)_{t\geq 0}$ is strongly continuous on $\mathcal{L}^1(H)$. By the Hille–Yoshida theorem [15] p. 16, there exists a dense linear subspace $\mathcal{D}(L)$ of $\mathcal{L}^1(H)$ such that $S_t(R)$ is differentiable at $t=0+$ for all $R\in\mathcal{D}(L)$, and $(d/dt)_{t=0+}S_t(R)=-AR-RA$, so the generator is $(-L)$, where $L(R)=AR+RA$.

(ii) Certainly $\mathcal{D}(L)$ contains $\mathcal{D}(A^\dagger)\hat{\otimes}\mathcal{D}(A)$ in $\mathcal{L}^1(H)=H\hat{\otimes}H$. Choosing $f\in\mathcal{D}(A)$ and $g\in\mathcal{D}(A^\dagger)$, we find that

$$\begin{aligned}\frac{d}{dt}\big\langle e^{-tA}R_0e^{-tA}f,g\big\rangle&=-\big\langle e^{-tA}(AR_0+R_0A)e^{-tA}f,g\big\rangle\\&=-\big\langle e^{-tA}Xe^{-tA}f,g\big\rangle\end{aligned} \tag{2.3}$$

a continuous function of $t>0$; so integrating we obtain

$$\langle R_0 f, g\rangle - \langle e^{-sA}R_0 e^{-sA} f, g\rangle = \int_0^s \langle e^{-tA}Xe^{-tA}f, g\rangle\, dt. \tag{2.4}$$

We extend this identity to all $f, g \in H$ by joint continuity; then we let $s\to\infty$ and observe that $R_0 : H \to H$ is trace class and hence is completely continuous, hence R_0 maps the weakly null family $(e^{-sA}f)_{s\to\infty}$ to the norm convergent family $(R_0 e^{-sA}f)_{s\to\infty}$, so $\langle e^{-sA}R_0e^{-sA}f, g\rangle \to 0$ as $s\to\infty$; hence we have a weakly convergent improper integral

$$\langle R_0 f, g\rangle = \lim_{s\to\infty}\int_0^s \langle e^{-tA}Xe^{-tA}f, g\rangle\, dt \qquad (f, g\in H).$$

(iii) The function $t\mapsto e^{-tA}Xe^{-tA}$ takes values in the separable space $\mathcal{L}^1(H)$ and is weakly continuous, hence strongly measurable, by Pettis's theorem. By considering the spectral radius, Engel and Nagel [9] show that there exist $\delta>0$ and $M_\delta>0$ such that $\|e^{-tA}\|_{\mathcal{L}(H)} \leq M_\delta e^{-\delta t}$ for all $t\geq 0$; hence (2.2) converges as a Bochner–Lebesgue integral with

$$\begin{aligned}\|R_x\|_{\mathcal{L}^1(H)} &\leq \int_x^\infty M_\delta^2\|X\|_{\mathcal{L}^1(H)}e^{-2\delta t}\, dt\\ &\leq \frac{M_\delta^2}{2\delta}\|X\|_{\mathcal{L}^1(H)}e^{-2\delta x}.\end{aligned} \tag{2.5}$$

Furthermore, A is a closed linear operator and satisfies

$$\begin{aligned}A\int_x^s e^{-tA}Xe^{-tA}\, dt + \int_x^s e^{-tA}Xe^{-tA}\, dtA &= \int_x^s -\frac{d}{dt}\big(e^{-tA}Xe^{-tA}\big)\, dt\\ &= e^{-xA}Xe^{-xA} - e^{-sA}Xe^{-sA}\\ &\to e^{-xA}Xe^{-xA}\end{aligned} \tag{2.6}$$

as $s\to\infty$ where $\int_x^s e^{-tA}Xe^{-tA}dt \to R_x$; so $AR_x + R_xA = e^{-xA}Xe^{-xA}$ for all $x\geq 0$. We deduce that $x\mapsto R_x$ is a differentiable function from $(0,\infty)$ to $\mathcal{L}^1(H)$ and that the modified Lyapunov equation (1.3) holds.

□

The hypotheses (i) and (ii) are symmetrical under the adjoint $(A, R_0)\mapsto (A^\dagger, R_0^\dagger)$; however, the hypothesis (iii) is rather stringent, and in many applications one only needs existence of the integral (2.2).

Definition 2.2. *($(2,2)$ admissible linear systems).* (i) Let H be a complex Hilbert space and let $\Sigma = (-A, B, C)$ be a linear system with state space H. Suppose that the integral

$$W_c = \int_0^\infty e^{-tA}BB^\dagger e^{-tA^\dagger}\, dt \tag{2.7}$$

converges weakly and defines a bounded linear operator on H; then W_c is the controllability Gramian. Suppose further that the integral

$$W_o = \int_0^\infty e^{-tA^\dagger}C^\dagger Ce^{-tA}\, dt \tag{2.8}$$

converges weakly and defines a bounded linear operator on H; then W_o is the observability Gramian.

(ii) Then as in [5] p. 318 we define R_x to be the bounded linear operator on H determined by the weakly convergent integral

$$R_x = \int_x^\infty e^{-tA} BCe^{-tA}\, dt. \tag{2.9}$$

(iii) Then Σ satisfying (i) is said to be balanced if $W_c = W_o$ and $\ker(W_c) = 0$; see [31] p. 499.

(iv) Also, Σ satisfying (i) is said to be $(2,2)$ admissible if W_c and W_o are trace class, or equivalently Θ_0 and Ξ_0 are Hilbert-Schmidt; see [5].

(v) We introduce the scattering function $\phi(t) = Ce^{-tA}B$ and the shifted scattering function $\phi_{(x)}(t) = \phi(t+2x)$ for $x, t > 0$.

(vi) *(Sectorial operator).* For $0 < \theta \leq \pi$, we introduce the sector $S_\theta = \{z \in \mathbb{C} \setminus \{0\} : |\arg z| < \theta\}$. A closed and densely defined linear operator $-A$ is sectorial [9], [15] if there exists $\pi/2 < \theta < \pi$ such that S_θ is contained in the resolvent set of $-A$ and $|\lambda| \|(\lambda I + A)^{-1}\|_{\mathcal{L}(H)} \leq M$ for all $\lambda \in S_\theta$. Let $\mathcal{D}(A)$ be the domain of A and $\mathcal{D}(A^\infty) = \cap_{n=0}^\infty \mathcal{D}(A^n)$. See [15] p.37.

(vii) For $\pi/2 < \delta < \pi$, we introduce $X_\delta = \{\zeta \in S_\delta : -\zeta \in S_\delta\}$ which is an open set, symmetrical about $i\mathbb{R}$ and bounded by lines passing through 0.

Theorem 2.3. *Let $(-A, B, C)$ be a linear system such that $\|e^{-t_0 A}\|_{\mathcal{L}(H)} < 1$ for some $t_0 > 0$, and that B and C are Hilbert–Schmidt operators such that $\|B\|_{\mathcal{L}^2(H_0;H)} \|C\|_{\mathcal{L}^2(H;H_0)} \leq 1$. Suppose further that $-A$ is sectorial on S_θ for some $\pi/2 < \theta < \pi$.*

(i) Then $(-A, B, C)$ is $(2,2)$-admissible, so the trace class operators $(R_x)_{x>0}$ give the solution to Lyapunov's equation (1.3) for $x > 0$ that satisfies the initial condition (1.8), and the solution to (1.3) with (1.8) is unique.

(ii) The function $\tau(x) = \det(I + R_x)$ is differentiable for $x \in (0, \infty)$.

(iii) Then R_z extends to a holomorphic function that satisfies (1.3) on $S_{\theta-\pi/2}$, and $R_z \to 0$ as $z \to \infty$ in $S_{\theta-\varepsilon-\pi/2}$ for all $0 < \varepsilon < \theta - \pi/2$.

Proof. (i) Since $BC \in \mathcal{L}^1(H)$, the integrand of (2.9) takes values in $\mathcal{L}^1(H)$, and we can apply Proposition 2.1(iii) to $X = BC$.

(ii) The Fredholm determinant $R \mapsto \det(I + R)$ is a continuous function on $\mathcal{L}^1(H)$. Also the integral $R_x = \int_x^\infty e^{-tA} BCe^{-tA}\, dt$ belongs to $\mathcal{D}(L)$ and gives a differentiable function of $x > 0$ with values in $\mathcal{L}^1(H)$.

(iii) By classical results of Hille [15] p.34, $(e^{-zA})_{z \in S_{\theta-\pi/2}}$ defines an analytic semigroup on $S_{\theta-\pi/2}$, bounded on S_ν for all $0 < \nu < \theta - \pi/2$, so we can define $R_z = e^{-zA} R_0 e^{-zA}$ and obtain an analytic solution to Lyapunov's equation. For all $0 < \varepsilon < \theta - \pi/2$, there exists M'_ε such that $\|e^{-zA}\|_{\mathcal{L}(H)} \leq M'_\varepsilon$ for all $z \in S_\delta$ where $\delta = \theta - \varepsilon - \pi/2$. Now for $z \in S_{\delta/2}$, we write $z = x/2 + (x/2 + iy)$ with $x/2 + iy \in S_\delta$ and use the bound $\|e^{-zA}\|_{\mathcal{L}(H)} \leq \|e^{-xA/2}\|_{\mathcal{L}(H)} \|e^{-(x/2+iy)A}\|_{\mathcal{L}(H)}$ to obtain $\|e^{-zA}\|_{\mathcal{L}(H)} \leq M'^2_\varepsilon \|e^{-t_0 A}\|_{\mathcal{L}(H)}^{x/(4t_0)}$, so $\|e^{-zA}\|_{\mathcal{L}(H)} \to 0$ exponentially fast as $z \to \infty$ in the sector $S_{\delta/2}$. Hence R_z

is holomorphic and bounded on $S_{(\theta-\varepsilon-\pi/2)}$ and by (2.9), $R_z \to 0$ as $z \to \infty$ in $S_{(\theta-\varepsilon-\pi/2)/2}$.

□

Example. (i) Let $\Delta = -d^2/dx^2$ be the usual Laplace operator which is essentially self-adjoint and non-negative on $C_c^\infty(\mathbb{R};\mathbb{C})$ in $L^2(\mathbb{R};\mathbb{C})$. We introduce $A = \sqrt{I+\Delta}$ which is given by the Fourier multiplier $\mathcal{F}Af(\xi) = \sqrt{1+\xi^2}\mathcal{F}f(\xi)$. Then (e^{-zA}) and (e^{-zA^2}) give bounded holomorphic semigroups on H, as in Theorem 2.3, on the right half-plane $\{z \in \mathbb{C} : \Re z \geq 0\}$, which is the closure of $S_{\pi/2}$. On the imaginary axis, we have unitary groups (e^{itA}) and (e^{-itA^2}). By classical results from wave equations, we can write $e^{itA} + e^{-itA} = 2\cos(tA)$ where $u(x,t) = \cos(tA)f(x)$ for $f \in C_c^\infty(\mathbb{R};\mathbb{C})$ is given by

$$u(x,t) = \frac{1}{2}\big(f(x+t)+f(x-t)\big) + \frac{t}{2}\int_{x-t}^{x+t} f(y)\frac{J_0'(\sqrt{t^2-(x-s)^2})}{\sqrt{t^2-(x-s)^2}}ds, \quad (2.10)$$

where J_0 is Bessel's function of the first kind of order zero, and u satisfies

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} &= u(x,t), \\ u(x,0) &= f(x); \\ \frac{\partial u}{\partial t}(x,0) &= 0. \end{aligned} \quad (2.11)$$

See [15] p. 121. Note that $(\exp(t(iA)^{2j-1}))$ gives a unitary group on H for $j = 0,1,2,\dots$. This can be used to deform the linear system in the sense of Proposition 2.5(iii). Unitary deformation groups for tau functions are considered in [27]

(ii) In section 4 of [5], we introduced linear systems to describe

Schrödinger's equation when the potential is smooth and localized. In [17], the authors obtain detailed results about the corresponding Hankel operator.

Definition 2.4. (i) *(Block Hankel operators).* Say that $\Gamma \in \mathcal{L}(H)$ is block Hankel if there exists $1 \leq m < \infty$ such that Γ is unitarily equivalent to the block matrix $[A_{j+k-2}]_{j,k=1}^\infty$ on $\ell^2(\mathbb{C}^m)$ where $A_j \in \mathbb{C}^{m\times m}$ for $j = 0,1,\dots$.

(ii) Let $(-A,B,C)$ be a $(2,2)$ admissible linear system with input and output space H_0, where the dimension of H_0 over $\mathbb{C}$ is $m < \infty$. Then m is the number of outputs of the system, and systems with finite $m > 1$ are known as MIMO for multiple input, multiple output, and give rise to block Hankel operators with $\Phi(x) = Ce^{-xA}B$; see [59].

(iii) The Gelfand–Levitan integral equation for $(-A,B,C)$ as in (ii) is (1.11), where $T(x,y)$ and $\Phi(x+y)$ are $m \times m$ matrices with scalar entries, and $\mu \in \mathbb{C}$. We proceed to obtain a solution.

Proposition 2.5. *(i) In the notation of Theorem* 2.3*, there exists* $x_0 > 0$ *such that*

$$T_\mu(x,y) = -Ce^{-xA}(I+\mu R_x)^{-1}e^{-yA}B \quad (2.12)$$

*satisfies the integral equation (*1.11*) for* $x_0 < x < y$ *and* $|\mu| < 1$.

(ii) The determinant satisfies $\det(I + \mu R_x) = \det(I + \mu\Gamma_{\Phi_{(x)}})$ *and*

$$\mu \mathit{trace}\, T_\mu(x,x) = \frac{d}{dx}\log\det(I+\mu R_x). \tag{2.13}$$

(iii) Suppose that $t \mapsto U(t)$ *is a continuous function* $[0,1] \to \mathcal{L}(H)$ *such that* $U(t)A = AU(t)$ *and* $\|U(t)\|_{\mathcal{L}(H)} \leq 1$*. Then there is a family of* $(2,2)$ *admissible linear systems*

$$\Sigma(t) = (-A, U(t)B, CU(t)) \qquad (t \in [0,1]);$$

the corresponding tau function $\tau(x,t)$ *is continuous for* $(x,t) \in (0,\infty)\times[0,1]$.

Proof. (i) We choose x_0 so large that $e^{\delta x_0} \geq M_\delta/2\delta$, then by (2.7), we have $|\mu|\|R_x\|_{\mathcal{L}(H)} < 1$ for $x > x_0$, so $I + \mu R_x$ is invertible. Substituting $T_\mu(x,y)$ into the integral equation (1.11), we obtain

$$\begin{aligned} Ce^{-(x+y)A}B &- Ce^{-xA}(I+\mu R_x)^{-1}e^{-yA}B \\ &- \mu Ce^{-xA}(I+\mu R_x)^{-1}\int_x^\infty e^{-zA}BCe^{-zA}\,dz e^{-yA}B \\ &= Ce^{-(x+y)A}B - Ce^{-xA}(I+\mu R_x)^{-1}e^{-yA}B \\ &\quad - \mu Ce^{-xA}(I+\mu R_x)^{-1}R_x e^{-yA}B \\ &= 0. \end{aligned} \tag{2.14}$$

(ii) As in (1.5), the operator $\Theta_x : L^2(0,\infty) \to H$ is Hilbert–Schmidt; likewise $\Xi_x : L^2(0,\infty) \to H$ is Hilbert–Schmidt; so $(-A,B,C)$ is $(2,2)$-admissible. Hence $\Gamma_{\Phi_{(x)}} = \Theta_x^\dagger \Xi_x$ and $R_x = \Xi_x\Theta_x^\dagger$ are trace class, $(I+\mu R_x)$ is a holomorphic function of x on some sector S_δ as in Theorem 2.3 and

$$\det(I+\mu R_x) = \det(I+\mu\Xi_x\Theta_x^\dagger) = \det(I+\mu\Theta_x^\dagger\Xi_x) = \det(I+\mu\Gamma_{\Phi_{(x)}}).$$

By the Riesz functional calculus, $(I+\mu R_x)^{-1}$ is meromorphic for x in some S_δ. Correcting a typographic error in [5] p. 324, we rearrange terms and calculate the derivative

$$\begin{aligned} \mu T_\mu(x,x) &= -\mu \mathrm{trace}\left(Ce^{-xA}(I+\mu R_x)^{-1}e^{-xA}B\right) \\ &= -\mu\mathrm{trace}\Big((I+\mu R_x)^{-1}e^{-xA}BCe^{-xA}\Big) \\ &= \mu\mathrm{trace}\left((I+\mu R_x)^{-1}\frac{dR_x}{dx}\right) \\ &= \frac{d}{dx}\mathrm{trace}\log(I+\mu R_x). \end{aligned} \tag{2.15}$$

This identity is proved for $|\mu| < 1$ and extends by analytic continuation to the maximal domain of $T_\mu(x,x)$.

(iii) Since A commutes with $U(t)$, the domain $\mathcal{D}(A)$ is invariant under $U(t)$, and the multiplications $B \mapsto U(t)B$ and $C \mapsto CU(t)$ preserve the hypotheses of Theorem 2.3, so $(-A, U(t)B, CU(t))$ is $(2,2)$ admissible. By commutativity, we have $\tau(x,t) = \det(I + U(t)R_xU(t))$, which depends continuously on (x,t).

□

3. The Baker–Akhiezer function of an admissible linear system

In this section, we consider the Darboux addition rule for potentials and analyze the transformation $(-A, B, C) \mapsto (-A, B, -C)$ and the effect on the ratios and derivatives of τ functions. This generalizes section 3.4 of [10], and allows us to introduce a version of the Baker-Akhiezer function for a family of linear systems with properties that are similar to the classical case, as presented in [3] and [22].

Definition 3.1. *(Baker–Akhiezer function).* (i) Let $(-A, B, C)$ be as in Theorem 2.3, and let

$$\Sigma_\zeta = (-A, (\zeta I + A)(\zeta I - A)^{-1}B, C) \qquad (\zeta \in \mathbb{C} \cup \{\infty\} \setminus \mathrm{Spec}(A)) \tag{3.1}$$

so that Σ_ζ defines a $(2,2)$ admissible linear systems for ζ in an open subset of $\mathbb{C} \cup \{\infty\}$ which includes $\{\zeta \in \mathbb{C} : -\zeta \in S_\theta\}$ for some $\pi/2 < \theta < \pi$. We identify Σ_∞ with $(-A, B, C)$, and Σ_0 with $(-A, B, -C)$.

(ii) Let τ_ζ be the tau function of Σ_ζ, and let the Baker–Akhiezer function for the family of linear systems be

$$\psi_\zeta(x) = \frac{\tau_\zeta(x)}{\tau_\infty(x)} \exp\big(\zeta x\big). \tag{3.2}$$

(iii) Let $\tau^*_\zeta(x) = \overline{\tau_{\bar\zeta}(\bar x)}$ as in Schwarz's reflection principle, and let

$$\Sigma^*_\zeta = (-A^\dagger, C^\dagger, B^\dagger(\zeta I + A^\dagger)(\zeta I - A^\dagger)^{-1}) \qquad (\zeta \in \mathbb{C} \cup \{\infty\} \setminus \mathrm{Spec}(A^\dagger)) \tag{3.3}$$

so $\Sigma_\zeta \mapsto \Sigma^*_\zeta$ is an involution, and Σ^*_ζ has tau function τ^*.

The following result introduces a family of solutions of Schrödinger equation corresponding to the Σ_ζ with an addition rule in the style of Darboux.

Proposition 3.2. *Let $(-A, B, C)$ be as in Theorem* 2.3.

(i) Then for $-\zeta \in S_\theta$, the linear system Σ_ζ is also $(2,2)$ admissible, and the Baker–Akhiezer function satisfies

$$-\frac{d^2}{dx^2}\psi_\zeta(x) + u_\infty(x)\psi_\zeta(x) = -\zeta^2\psi_\zeta(x). \tag{3.4}$$

(ii) There exist $h_j \in C^\infty((0,\infty); \mathbb{C})$ such that there is an asymptotic expansion

$$\psi_\zeta(x) \asymp e^{\zeta x}\Big(1 + \frac{h_1(x)}{\zeta} + \frac{h_2(x)}{\zeta^2} + \dots\Big) \tag{3.5}$$

as $\zeta \to \pm i\infty$, and the expansion is uniform for x in compact subsets of $(0,\infty)$.

Proof. (i) For all $\zeta \in \mathbb{C} \setminus \mathrm{Spec}(A)$, there exists $x_0(\zeta)$ such that $\|(\zeta I + A)(\zeta I - A)^{-1}R_x\|_{\mathcal{L}^1(H)} < 1$ for all $x > x_0(\zeta)$, so that $\tau_\zeta(x)$ is continuously differentiable and non-zero as a function of $x \in (x_0(\zeta), \infty)$. In particular, suppose that $\Re\zeta < 0$, then $-\zeta \in S_\theta$ so $\zeta I - A$ is invertible. Using the R function for Σ_ζ, we write

$$\begin{aligned}\frac{\tau_\zeta(x)}{\tau_\infty(x)} &= \frac{\det\big(I + (\zeta I + A)(\zeta I - A)^{-1} R_x\big)}{\det\big(I + R_x\big)}\\ &= \frac{\det\big(I + (\zeta I - A)^{-1}((\zeta I - A)R_x + AR_x + R_x A)\big)}{\det\big(I + R_x\big)}\\ &= \frac{\det\big(I + R_x + (\zeta I - A)^{-1}(AR_x + R_x A)\big)}{\det\big(I + R_x\big)} \qquad (3.6)\end{aligned}$$

so that when $AR_x + R_x A$ has rank one, the perturbing term $(\zeta I - A)^{-1}(AR_x + R_x A)$ has rank one; continuing we find

$$\begin{aligned}\frac{\tau_\zeta(x)}{\tau_\infty(x)} &= \det\big(I + (\zeta I - A)^{-1} e^{-xA} B C e^{-xA} (I + R_x)^{-1}\big)\\ &= \det\big(I + C e^{-xA}(I + R_x)^{-1}(\zeta I - A)^{-1} e^{-xA} B\big)\\ &= 1 + C e^{-xA}(I + R_x)^{-1}(\zeta I - A)^{-1} e^{-xA} B, \qquad (3.7)\end{aligned}$$

since $B : \mathbb{C} \to H$ and $C : H \to \mathbb{C}$ have rank one. Hence

$$\begin{aligned}\psi_\zeta(x) &= \frac{\tau_\zeta(x)}{\tau_\infty(x)} \exp\big(\zeta x\big)\\ &= \exp\big(\zeta x\big) + C e^{-xA}(I + R_x)^{-1}(\zeta I - A)^{-1} e^{-xA} B \exp\big(\zeta\big)\\ &= \exp\big(\zeta x\big) - \int_x^\infty C e^{-xA}(I + R_x)^{-1} e^{-yA} B \exp\big(\zeta y\big)\, dy\\ &= \exp\big(\zeta x\big) + \int_x^\infty T(x, y) \exp\big(\zeta y\big)\, dy. \qquad (3.8)\end{aligned}$$

Here T satisfies the Gelfand–Levitan equation, and by integrating by parts, we see that

$$\frac{\partial^2 T}{\partial x^2} - \frac{\partial^2 T}{\partial y^2} = u(x) T(x, y) \qquad (3.9)$$

where $u(x) = -2\frac{d^2}{dx^2} \log \tau(x)$. Then by integrating by parts, we see that ψ_ζ satisfies Schrödinger's equation.

The solutions of the differential equation depend analytically on ζ at those points where the potential depends analytically on ζ; note that $\zeta \mapsto \tau_\zeta(x)$ is holomorphic and non zero for $\|R_x\| < 1$ and $-\zeta \in S_\theta$. Then we continue the solutions analytically to all $-\zeta$ in the sector S_θ, on which $\psi_\zeta(x)$ is holomorphic as a function of ζ for $x > 0$.

(ii) Observe that $X_\theta = S_\theta \cap (-S_\theta)$ contains $i\mathbb{R} \setminus \{0\}$. For $\zeta \in S_\theta \cap (-S_\theta)$, by (i) there exist solutions $\psi_\zeta(x)$ and $\psi_{-\zeta}(x)$ to (3.4). In particular, ψ_{ik} and

$\psi_{-ik}(x)$ are solutions for $k > 0$. We integrate by parts repeatedly

$$e^{-xA}(\zeta I - A)^{-1} = e^{-xA}\int_0^\infty e^{\zeta s}e^{-sA}\,ds$$
$$= \frac{e^{-xA}}{\zeta} + \frac{Ae^{-xA}}{\zeta^2} + \dots + \frac{A^{k-1}e^{-xA}}{\zeta^k}$$
$$+ \int_0^\infty \frac{A^k e^{-xA}}{\zeta^k} e^{\zeta s}e^{-sA}\,ds, \tag{3.10}$$

where the integral converges by the hypothesis of Theorem 2.3. Also, (e^{-zA}) is an analytic semigroup in the sector $S_{\theta-\pi/2}$, so $\mathcal{D}(A^j)$ is a dense linear subspace of H for all $j = 1, 2, \dots$ and $A^j e^{-xA} \in \mathcal{L}(H)$ and by Cauchy's estimates there exists $C > 0$ such that $\|A^j e^{-xA}\|_{\mathcal{L}(H)} \leq Cj!/x^j$ for all $x > 0$. So we can generate an asymptotic expansion of (3.6) with terms

$$h_j(x) = Ce^{-xA}(I + R_x)^{-1}A^{j-1}e^{-xA}B$$

which are bounded on compact subsets of $(0, \infty)$.

□

Definition 3.3. *(Darboux transforms).* Let $(-A, B, C)$ be an $(2, 2)$ admissible linear system with tau function $\tau_\infty(x; \mu) = \det(I + \mu R_x)$. Define the Darboux transform of $(-A, B, C)$ to be $(-A, B, -C)$ with tau function transform $\tau_0(x; \mu) = \det(I - \mu R_x)$. Let

$$v = \frac{1}{\mu}\frac{d}{dx}\log\frac{\tau_\infty}{\tau_0}, \quad w = \frac{1}{\mu}\frac{d}{dx}\log\bigl(\tau_0\tau_\infty\bigr),$$
$$u_\infty = -\frac{2}{\mu^2}\frac{d^2}{dx^2}\log\tau_\infty, \quad u_0 = -\frac{2}{\mu^2}\frac{d^2}{dx^2}\log\tau_0. \tag{3.11}$$

In the following result, we show how products and quotients of τ functions can be linked by the Gelfand–Levitan equation for 2×2 matrices, and satisfy the identities usually associated with Darboux transforms in the theory of integrable systems. See [23]

Theorem 3.4. *Let $(-A, B, C)$ be a $(2, 2)$-admissible linear system with input and output spaces $\mathbb{C}$, and let $\phi(x) = Ce^{-xA}B$.*

(i) Then there exists $\delta > 0$ such that for all $\mu \in \mathbb{C}$ such that $|\mu| < \delta$, the Gelfand-Levitan equation (1.11) with

$$T(x, y) = \begin{bmatrix} W(x, y) & V(x, y) \\ V(x, y) & W(x, y) \end{bmatrix}, \tag{3.12}$$

$$\Phi(x + y) = \begin{bmatrix} 0 & \phi(x + y) \\ \phi(x + y) & 0 \end{bmatrix} \tag{3.13}$$

has a solution such that

$$W(x, x) = \frac{1}{2\mu}\frac{d}{dx}\log\bigl(\tau_\infty(x; \mu)\tau_0(x; \mu)\bigr), \tag{3.14}$$

$$V(x, x) = \frac{1}{2\mu}\frac{d}{dx}\log\frac{\tau_\infty(x; \mu)}{\tau_0(x; \mu)} \tag{3.15}$$

and

$$\frac{1}{2\mu}\frac{d}{dx}W(x,x) = -V(x,x)^2; \tag{3.16}$$

(ii) also Toda's equation holds in the form

$$\tau_0''\tau_\infty - 2\tau_0'\tau_\infty' + \tau_0\tau_\infty'' = 0. \tag{3.17}$$

Proof. (i) Let

$$T_\infty(x,y) = -Ce^{-xA}(I+\mu R_x)^{-1}e^{-yA}B,$$
$$T_0(x,y) = Ce^{-xA}(I-\mu R_x)^{-1}e^{-yA}B$$

and

$$\Phi(x) = \begin{bmatrix} 0 & \phi(x) \\ \phi(x) & 0 \end{bmatrix}.$$

Now let

$$T(x,y) = \frac{1}{2}\begin{bmatrix} T_\infty + T_0 & T_\infty - T_0 \\ T_\infty - T_0 & T_\infty + T_0 \end{bmatrix}$$

so that

$$T(x,y) = -\begin{bmatrix} C & 0 \\ 0 & C \end{bmatrix}\begin{bmatrix} e^{-xA} & 0 \\ 0 & e^{-xA} \end{bmatrix}\begin{bmatrix} I & \mu R_x \\ \mu R_x & I \end{bmatrix}^{-1}\begin{bmatrix} e^{-yA} & 0 \\ 0 & e^{-yA} \end{bmatrix}\begin{bmatrix} 0 & B \\ B & 0 \end{bmatrix}$$

hence T satisfies the Gelfand–Levitan equation (1.11).

ii) As in Proposition 2.5,

$$T_\infty(x,x) = \frac{1}{\mu}\frac{d}{dx}\log\tau_\infty(x),$$
$$T_0(x,x) = \frac{1}{\mu}\frac{d}{dx}\log\tau_0(x);$$

hence (3.17) is equivalent to the condition

$$\frac{d}{dx}T_0(x,x) + \mu\big(T_0(x,x) - T_\infty(x,x)\big)^2 + \frac{d}{dx}T_\infty(x,x) = 0, \tag{3.18}$$

which we now verify. The left-hand side of (3.18) equals

$$\begin{aligned} &Ce^{-xA}\big(-A(I-\mu R_x)^{-1} - (I-\mu R_x)^{-1}\mu(AR_x + R_xA)(I-\mu R_x)^{-1} \\ &- (I-\mu R_x)^{-1}A\big)e^{-xA}B + Ce^{-xA}\big((I-\mu R_x)^{-1} + (I+\mu R_x)^{-1}\big)e^{-xA}\mu BCe^{-xA} \\ &\big((I-\mu R_x)^{-1} + (I+\mu R_x)^{-1}\big)e^{-xA} + Ce^{-xA}\big(A(I+\mu R_x)^{-1} \\ &- (I+\mu R_x)^{-1}\mu(AR_x + R_xA)(I+\mu R_x)^{-1} + (I+\mu R_x)^{-1}A\big)e^{-xA}B \end{aligned} \tag{3.19}$$

All of the terms begin with Ce^{-xA} and end with $e^{-xA}B$, and we can replace $e^{-xA}\mu BCe^{-xA}$ by $\mu(AR_x + R_xA)$ to obtain

$$\begin{aligned} (3.19) = Ce^{-xA}\Big(&-2(I-\mu R_x)^{-1}A(I-\mu R_x)^{-1} \\ &+ 4(I-\mu^2R_x^2)^{-1}\mu(AR_x + R_xA)(I-\mu^2R_x^2)^{-1} \\ &+ 2(I+\mu R_x)^{-1}A(I+\mu R_x)^{-1}\Big)e^{-xA}B \\ = 0.& \end{aligned} \tag{3.20}$$

This proves (3.18), and one can easily check that (3.17) is equivalent to

$$u_0(x) = \frac{1}{\mu}\frac{dv}{dx} + v(x)^2, \quad v(x)^2 = -\frac{1}{\mu}\frac{dw}{dx}.$$

The entries of T satisfy the pair of coupled integral equations

$$\begin{aligned} 0 &= W(x,y) + \mu\int_x^\infty V(x,s)\phi(s+y)\,ds \\ 0 &= V(x,y) + \phi(x+y) + \mu\int_x^\infty W(x,s)\phi(s+y)\,ds; \end{aligned} \tag{3.21}$$

so W satisfies

$$\begin{aligned} 0 = -W(x,z) &+ \mu\int_x^\infty \phi(x+y)\phi(y+z)\,dy \\ &+ \mu^2\int_x^\infty W(x,s)\int_x^\infty \phi(s+y)\phi(y+z)\,dyds, \end{aligned} \tag{3.22}$$

which explains how $\mu^2\Gamma_\phi^2$ enters into several determinant formulas [38].

□

Definition 3.5. (i) *(Darboux Addition).* For $-\zeta \in S_\theta \cup \{0\}$ we define the Darboux addition rule on $(2,2)$ admissible linear systems by

$$M_\zeta : (-A, B, C) \mapsto (-A, (\zeta I + A)(\zeta I - A)^{-1}B, C)$$

and correspondingly on potentials by

$$u_\infty \mapsto u_\zeta = u_\infty - 2(\log\psi_\zeta)''. \tag{3.23}$$

(ii) Let $\mathrm{Wr}(\varphi,\psi)$ be the Wronskian of $\psi, \varphi \in C^1((0,\infty);\mathbb{C})$.

Corollary 3.6. *The set $\{M_\zeta,\ (\zeta \in X_\theta), M_0, M_\infty = I\}$ generates a group such that $M_0^2 = I$, $M_\zeta M_{-\zeta} = I$ and $M_\zeta M_\eta$ corresponds to adding*

$$-2\frac{d^2}{dx^2}\log Wr(\psi_\zeta, \psi_\eta) \tag{3.24}$$

to the potential.

Proof. The definition is consistent with [10] p. 484]. In particular, $\psi_0(x) = \tau_0(x)/\tau_\infty(x)$, and $u_0(x) = u_\infty(x) - 2\frac{d^2}{dx^2}\log\psi_0(x)$, which is consistent with (3.18).

For $\zeta_1 \neq \zeta_2$, let $\Psi(x) = \mathrm{Wr}(\psi_{\zeta_1}, \psi_{\zeta_2})/\psi_{\zeta_2}$, and observe that

$$\Psi'' = \big(\zeta_2^2 + u_\infty - 2(\log\psi_{\zeta_1})''\big)\Psi.$$

This gives the basic composition rule for $M_{\zeta_2}M_{\zeta_1}$. The other statements follow from Proposition 3.2 and Theorem 3.4. See [25]

□

4. The state ring associated with an admissible linear system

Gelfand and Dikii [11] considered the algebra $\mathcal{A}_u = \mathbb{C}[u, u', u'', \dots]$ of complex polynomials in a smooth potential u and its derivatives. They showed that if u satisfies the stationary higher order KdV equations (5.1), then $\mathcal{A}_u$ is a Noetherian ring [2] and the associated Schrödinger equation is integrable by quadratures; see [7]. In this section, we introduce an analogue $\mathcal{A}_\Sigma$ for an admissible linear system.

We develop a calculus for R_x which is the counterpart of Pöppe's functional calculus for Hankel operators from [32], [25], [33]. As we see in other papers , our theory of state rings has wider scope for generalization.

Definition 4.1. (1) *(Differential rings).* Let $\mathcal{R}$ be a ring with ideal $\mathcal{J}$, and let $\partial : \mathcal{R} \to \mathcal{R}$ be a derivation. Then $\mathcal{R}_\mathcal{J} = \{r \in \mathcal{R} : \partial(r) \in \mathcal{J}\}$ gives a subring of $\mathcal{R}$, the ring of constants relative to $\mathcal{J}$. When $\mathcal{R}$ is an algebra over $\mathbb{C}$ and $\mathcal{J} = (0)$, we call $\mathcal{R}_0$ the constants; see [34].

(2) *(State ring of a linear system).* Let $(-A, B, C)$ be a linear system such that $A \in \mathcal{L}(H)$. Suppose that:

(i) $\mathcal{S}$ is a differential subring of $C^\infty((0,\infty); \mathcal{L}(H))$;

(ii) I, A and BC are constant elements of $\mathcal{S}$;

(iii) e^{-xA}, R_x and $F_x = (I + R_x)^{-1}$ belong to $\mathcal{S}$.

Then $\mathcal{S}$ is a state ring for $(-A, B, C)$.

Lemma 4.2. *Suppose that $(-A, B, C)$ is a linear system with $A \in \mathcal{L}(H)$ and that R_x gives a solution of Lyapunov's equation (1.3) such that $I + R_x$ is invertible for $x > 0$ with inverse F_x. Then the free associative algebra $\mathcal{S}$ generated by $I, R_0, A, F_0, e^{-xA}, R_x$ and F_x is a state ring for $(-A, B, C)$ on $(0, \infty)$. For all $t > 0$, there exists a ring homomorphism $S_t : \mathcal{S} \to \mathcal{S}$ given by $S_t : G(x) \mapsto G(x + t)$ such that S_t commutes with d/dx*

Proof. We can regard $\mathcal{S}$ as a subring of $C_b((0,\infty), \mathcal{L}(H))$, so the multiplication is well defined. Then we note that $BC = AR_0 + R_0A$ belongs to $\mathcal{S}$, as required. We also note that $(d/dx)e^{-xA} = -Ae^{-xA}$ and that Lyapunov's equation (1.3) gives

$$\frac{d}{dx}(I + R_x)^{-1} = (I + R_x)^{-1}(AR_x + R_xA)(I + R_x)^{-1}, \tag{4.1}$$

which implies

$$\frac{dF_x}{dx} = AF_x + F_xA - 2F_xAF_x, \tag{4.2}$$

with the initial condition

$$AF_0 + F_0A - 2F_0AF_0 = F_0BCF_0.$$

Hence $\mathcal{S}$ is a differential ring.

We can map $I \mapsto I$, $e^{-xA} \mapsto e^{-(x+t)A}$, $R_0 \mapsto e^{-tA}R_0e^{-tA}$, $R_x \mapsto e^{-tA}R_xe^{-tA}$ and $F_x \mapsto (I + e^{-tA}R_xe^{-tA})^{-1}$, and thus produce a ring homomorphism $G(x) \mapsto G(x + t)$ which satisfies $(d/dx)S_tG(x) = G'(x + t) = S_t(d/dx)G(x)$.

□

Definition 4.3. *(Products and brackets).* (i) Given a state ring $\mathcal{S}$ for $(-A, B, C)$, let $\mathcal{B}$ be any differential ring of functions from $(0, \infty) \to \mathcal{L}(H_0)$. Let

$$\mathcal{A}_\Sigma = \mathrm{span}_{\mathbb{C}}\{A^{n_1}, A^{n_1} F_x A^{n_2} \dots F_x A^{n_r} : n_j \in \mathbb{N}\}. \tag{4.3}$$

(ii) On $\mathcal{S}$ we introduce the associative product $*$ by

$$P * Q = P(AF + FA - 2FAF)Q \qquad (P, Q \in \mathcal{S}), \tag{4.4}$$

which is distributive over the standard addition, and the derivation $\partial : \mathcal{S} \to \mathcal{S}$ by

$$\partial P = A(I - 2F)P + \frac{dP}{dx} + P(I - 2F)A \qquad (P \in \mathcal{S}). \tag{4.5}$$

(iii) Let $\lfloor \cdot \rfloor : \mathcal{S} \to \mathcal{B}$ be the linear map

$$\lfloor Y \rfloor = Ce^{-xA} F_x Y F_x e^{-xA} B \qquad (Y \in \mathcal{S}), \tag{4.6}$$

so that $x \mapsto \lfloor Y \rfloor$ is a differentiable function $(x_0, \infty) \to \mathcal{L}(H_0)$.

For $x_0 \geq 0$ and $0 < \phi < \pi$, let $S_\delta^{x_0}$ be the translated sector $S_\delta^{x_0} = \{z = x_0 + w : w \in \mathbb{C} \setminus \{0\}; |\arg w| < \delta\}$ and let $H^\infty(S_\delta^{x_0})$ the the bounded holomorphic complex functions on $S_\delta^{x_0}$. Then let $H_\infty^\infty = \cup_{x_0 > 0} H^\infty(S_\delta^{x_0})$ be the algebra of complex functions which are bounded on some translated sector $S_\delta^{x_0}$, with the usual pointwise multiplication.

Theorem 4.4. *Let $(-A, B, C)$ be a $(2, 2)$-admissible linear system with $H_0 = \mathbb{C}$ as in Theorem 2.3, so (e^{-zA}) for $z \in S_\phi^0$ is a bounded holomorphic semigroup on H. Let $\Theta_0 = \{P \in \mathcal{A}_\Sigma : \lfloor P \rfloor = 0\}$.*

(i) Then $(\mathcal{A}_\Sigma, *, \partial)$ is a differential ring with bracket $\lfloor \cdot \rfloor$;

(ii) there is a homomorphism of differential rings $\lfloor \cdot \rfloor : (\mathcal{A}_\Sigma, *, \partial) \to (H_\infty^\infty, \cdot, d/dz)$;

(iii) Θ_0 is a differential ideal in $(\mathcal{A}_\Sigma, *, \partial)$ such that $\mathcal{A}_\Sigma/\Theta_0$ is a commutative differential ring, and an integral domain.

Proof. (i) We can multiply elements in $\mathcal{S}$ by concatenating words and taking linear combinations. Since all words in $\mathcal{A}_\Sigma$ begin and end with A, we obtain words of the required form, hence $\mathcal{A}_\Sigma$ is a subring of $\mathcal{S}$. To differentiate a word in $\mathcal{A}_\Sigma$ we add words in which we successively replace each F_x by $AF_x + F_x A - 2F_x AF_x$, giving a linear combination of words of the required form. The basic observation is that $dF/dx = AF + FA - 2FAF$, so one can check that

$$\partial(P * Q) = (\partial P) * Q + P * (\partial Q); \tag{4.7}$$

hence $(\mathcal{S}, *, \partial)$ is a differential ring with differential subring $(\mathcal{A}_\Sigma, *, \partial)$.

(ii) Now we verify that there is a homomorphism of differential rings $(\mathcal{A}_\Sigma, *, \partial) \to (\mathcal{B}, \cdot, d/dx)$ given by $P \mapsto \lfloor P \rfloor$. From the definition of R_x, we have $AR_x + R_x A = e^{-xA} BCe^{-xA}$, and hence

$$F_x e^{-xA} BCe^{-xA} F_x = AF_x + F_x A - 2F_x AF_x,$$

which implies

$$\begin{aligned}\lfloor P\rfloor\lfloor Q\rfloor &= Ce^{-xA}F_xPF_xe^{-xA}BCe^{-xA}F_xQF_xe^{-xA}B\\ &= Ce^{-xA}F_xP(AF_x+F_xA-2F_xAF_x)QF_xe^{-xA}B\\ &= \lfloor P(AF_x+F_xA-2F_xAF_x)Q\rfloor\\ &= \lfloor P*Q\rfloor. \end{aligned}\tag{4.8}$$

Moreover, the first and last terms in $\lfloor P\rfloor$ have derivatives

$$\frac{d}{dx}Ce^{-xA}F_x = Ce^{-xA}F_xA(I-2F_x), \qquad \frac{d}{dx}F_xe^{-xA}B = (I-2F_x)AF_xe^{-xA}B,$$

so the bracket operation satisfies

$$\frac{d}{dx}\lfloor P\rfloor = \Big\lfloor A(I-2F_x)P + \frac{dP}{dx} + P(I-2F_x)A\Big\rfloor = \lfloor \partial P\rfloor. \tag{4.9}$$

In this case A is possibly unbounded as an operator, so we use the holomorphic semigroup to ensure that products (4.5) and brackets (4.7) are well defined. We observe that $\mathcal{A}_\Sigma$ has a grading $\mathcal{A}_\Sigma=\oplus_{n=1}^\infty A_n$, where A_n is the span of the elements that have total degree n when viewed as products of A and F. For $X_n\in A_n$ and $Y_m\in A_m$, we have $X_n*Y_m\in A_{n+m+2}\oplus A_{n+m+3}$ and $\partial X_n\in A_{n+1}\oplus A_{n+2}$.

Also we have $A^ke^{-zA}\in\mathcal{L}(H)$ for all $z\in S^0_\phi$ and $\|A^ke^{-zA}\|_{\mathcal{L}(H)}\to 0$ as $z\to\infty$ in S^0_ϕ; hence $R_zA^k\to 0$ and $A^kR_z\to 0$ in $\mathcal{L}(H)$ as $z\to\infty$ in S^0_ϕ. Hence there exists an increasing positive sequence $(x_k)_{k=0}^\infty$ such that $A^kF_z-A^k\in\mathcal{L}(H)$ for all $z\in S^{x_k}_\phi$ and $A^kF_z-A^k\to 0$ in $\mathcal{L}(H)$ as $z\to\infty$ in $S^{x_k}_\phi$. Let $X_n\in A_n$ and consider a typical summand $AF_zA^kF_z\dots A$ in X_n; we replace each factor like A^kF_z by the sum of $A^k(F_z-I)$ and A^k where $k\leq n$; then we observe that there in an initial factor Ce^{-zA} and a final factor $e^{-zA}B$ in $\lfloor X_n\rfloor$; hence $\lfloor X_n\rfloor$ determines an element of $H^\infty(S^{x_n}_\phi)$.

We can identify H^∞_∞ with the algebraic direct limit $H^\infty_\infty=\lim_{n\to\infty}H^\infty(S^{x_0+n}_\phi)$. By the principle of isolated zeros, the multiplication on H^∞_∞ is consistently defined, and H^∞_∞ is an integral domain. Now each $f\in H^\infty_\infty$ gives $f\in H^\infty(S^{x_0}_\phi)$ so $f'\in H^\infty(S^{x_0+1}_\phi)$ by Cauchy's estimates, so $f'\in H^\infty_\infty$. From (i) we deduce that $\lfloor\cdot\rfloor:\oplus_{n=1}^\infty\mathcal{A}_n\to\cup_{n=1}^\infty H^\infty(S^{x_n}_\phi)$ is well-defined and the bracket is multiplicative with respect to $*$, and behaves naturally with respect to differentiation.

(iii) We check that $\lfloor\cdot\rfloor$ is commutative on $(\mathcal{A}_\Sigma,*,\partial)$, by computing

$$\begin{aligned}\lfloor P*Q\rfloor &= \mathrm{trace}\big(Ce^{-xA}FPFe^{-xA}BCe^{-xA}FQFe^{-xA}B\big)\\ &= \mathrm{trace}\big(Ce^{-xA}FQFe^{-xA}BCe^{-xA}FPFe^{-xA}B\big)\\ &= \lfloor Q*P\rfloor.\end{aligned}\tag{4.10}$$

Hence Θ_0 contains all the commutators $P*Q-Q*P$, and Θ_0 is the kernel of the homomorphism $\lfloor\cdot\rfloor$, hence is an ideal for $*$. Also, we observe that for all $Q\in\Theta_0$, we have $\partial Q\in\Theta_0$ since $\lfloor\partial Q\rfloor=(d/dx)\lfloor Q\rfloor=0$. Hence Θ_0 is a differential ideal which contains the commutator subspace of $(\mathcal{A}_\Sigma,*)$, so

$\mathcal{A}_\Sigma/\Theta_0$ is a commutative algebra. Also, ∂ determines a unique derivation $\bar\partial$ on $\mathcal{A}_\Sigma/\Theta_0$ by $\bar\partial Q = \partial Q + \Theta_0$ for all $Q \in \mathcal{A}_\Sigma$; hence $\mathcal{A}_\Sigma/\Theta_0$ is a differential algebra. We can identify $\mathcal{A}_\Sigma/\Theta_0$ with a subalgebra of H^∞_∞, which is an integral domain. □

Remark 4.5. Pöppe [32] introduced a linear functional $\lceil\,.\,\rceil$ on Fredholm kernels $K(x,y)$ on $L^2(0,\infty)$ by $\lceil K\rceil = K(0,0)$. In particular, let K, G, H, L be integral operators on $L^2(0,\infty)$ that have smooth kernels of compact support, let $\Gamma = \Gamma_{\phi_{(x)}}$ have kernel $\phi(s+t+2x)$, let $\Gamma' = \frac{d}{dx}\Gamma$ and $G = \Gamma_{\psi_{(x)}}$ be another Hankel operator; then the trace satisfies

$$\lceil\Gamma\rceil = -\frac{d}{dx}\operatorname{trace}\Gamma, \tag{4.11}$$

$$\lceil\Gamma K G\rceil = -\frac{1}{2}\frac{d}{dx}\operatorname{trace}\Gamma K G, \tag{4.12}$$

$$\lceil (I+\Gamma)^{-1}\Gamma\rceil = -\operatorname{trace}\big((I+\Gamma)^{-1}\Gamma'\big), \tag{4.13}$$

$$\lceil K\Gamma\rceil\lceil GL\rceil = -\frac{1}{2}\lceil K(\Gamma' G + \Gamma G')L\rceil, \tag{4.14}$$

where (4.14) is known as the product formula. The easiest way to prove (4.11)-(4.14) is to observe that $\Gamma' G + \Gamma G'$ is the integral operator with kernel $-2\phi_{(x)}(s)\psi_{(x)}(t)$, which has rank one. These ideas were subsequently revived by McKean [25], and are implicit in some results of [38]. Our formulas (4.7) and (4.9) incorporate a similar idea, and are the basis of the proof of Theorem 4.4. The results we obtain appear to be more general than those of Pöppe, and extend to periodic linear systems [6].

For the remainder of this section, we let A be a $n \times n$ complex matrix with eigenvalues λ_j $(j = 1, \dots, m)$ with geometric multiplicity n_j such that $\lambda_j + \lambda_k \neq 0$ for all $j, k \in \{1, \dots, m\}$; let $\mathbb{K} = \mathbb{C}(e^{-\lambda_1 t}, \dots, e^{-\lambda_m t}, t)$. Also, let $B \in \mathbb{C}^{n\times 1}$ and $C \in \mathbb{C}^{1\times n}$. The formula (4.18) resembles the expressions used to obtain soliton solutions of KdV, as in [19] (14.12.11) and [16]. In [17, (6.25)], there is a discussion of how the scattering data evolve under the time evolution associated with the KdV flow.

Proposition 4.6. *(i) There exists a solution R_t to Lyapunov's equation (1.3) with $R_0 = BC$, such that the entries of R_t belong to $\mathbb{K}$, and $\tau(t) \in \mathbb{K}$;*

(ii) $\phi \in \mathbb{K}$ satisfies a linear differential equation with constant coefficients.

(iii) Suppose further that all the eigenvalues of A are simple. Then there exists an invertible matrix S such that $S^{-1}B = (b_j)_{j=1}^n \in \mathbb{C}^{n\times 1}$ and

$CS = (c_j)_{j=1}^n \in \mathbb{C}^{1\times n}$ *and the tau function is given by*

$$\begin{aligned}\tau(t) =& 1 + \sum_{j=1}^n \frac{b_j c_j e^{-2\lambda_j t}}{2\lambda_j}\\ &+ \sum_{(j,k),(m,p):j\neq m;k\neq p} (-1)^{j+k+m+p}\frac{b_j b_m c_k c_p e^{-(\lambda_j+\lambda_k+\lambda_m+\lambda_p)t}}{(\lambda_j+\lambda_m)(\lambda_k+\lambda_p)} + \dots\\ &+ \prod_{j=1}^n \frac{b_j c_j}{2\lambda_j}\prod_{1\leq j<k\leq n}\frac{(\lambda_j-\lambda_k)^2}{(\lambda_j+\lambda_k)^2} e^{-2\sum_{j=1}^n \lambda_j t}. \end{aligned} \tag{4.15}$$

Proof. (i) By the hypothesis, we can introduce a chain of circles $\mathcal{C}$ that go once round each λ_j in the positive sense and have all the points $-\lambda_k$ in their exterior. Then by [4], the matrix

$$R_0 = \frac{-1}{2\pi i}\int_{\mathcal{C}} (A+\lambda I)^{-1} BC (A-\lambda I)^{-1} d\lambda$$

gives a solution to Sylvester's equation in the form $-AR_0 - R_0 A = -BC$. To see this, one considers $(A+\lambda I)R_0 + R_0(A-\lambda I)$ and then uses the calculus of residues. By the Riesz functional calculus, we also have

$$e^{-tA} = \frac{1}{2\pi i}\int_{\mathcal{C}} \big(\lambda I - A\big)^{-1} e^{-t\lambda} d\lambda;$$

hence by Cauchy's residue theorem, there exist complex polynomials p_j and q_j, and integers $m_j \geq 0$ such that

$$e^{-tA} = \sum_{j=1}^m q_j(t) e^{-t\lambda_j} p_j(A), \tag{4.16}$$

where $q_j(t)$ is constant if the corresponding eigenvalue is simple. We let $R_t = e^{-tA} R_0 e^{-tA}$, which gives a solution to Lyapunov's equation with initial condition $-BC$. From (4.16), we see that all the entries of R_t belong to $\mathbb{K}$. By the Laplace expansion of the determinant, we see that all entries of $\tau(t) = \det(I + R_t)$ also belong to $\mathbb{K}$.

(ii) We have $\phi(t) = Ce^{-tA}B \in \mathbb{K}$ by (4.16). Also, we introduce the characteristic polynomial of $(-A)$ by $\det(\lambda I + A) = \sum_{j=0}^n a_j \lambda^j$. Then by the Cayley–Hamilton theorem, $\sum_{j=0}^n a_j \phi^{(j)}(t) = 0$.

(iii) There exists an invertible matrix S such that SAS^{-1} is the $n\times n$ diagonal matrix $D = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$, and we observe that

$$R_t = \Big[\frac{b_j c_k e^{-(\lambda_j+\lambda_k)t}}{\lambda_j+\lambda_k}\Big]_{j,k=1}^n \tag{4.17}$$

satisfies $\frac{d}{dt}R_t = -[b_j c_k e^{-(\lambda_j+\lambda_k)t}]_{j,k=1}^n$ and $-DR_t - R_t D = -[b_j c_k e^{-(\lambda_j+\lambda_k)t}]_{j,k=1}^n$; so R_t gives a solution of the Lyapunov equation with generator $-D$ and initial condition given by the rank-one matrix $-S^{-1}BCS = -[b_j c_k]_{j,k=1}^n$. Hence the tau function is given by $\tau(t) = \det(I + R_t)$ for this matrix, and there is an expansion

$$\det\Bigl[\delta_{jk} + \frac{b_j c_k e^{-(\lambda_j+\lambda_k)x}}{\lambda_j+\lambda_k}\Bigr]_{j,k=1}^{n} = \sum_{\sigma\subseteq\{1,\dots,n\}} \det\Bigl[\frac{b_j c_k e^{-\lambda_j x-\lambda_k x}}{\lambda_j+\lambda_k}\Bigr]_{j,k\in\sigma} \tag{4.18}$$

in which each subset σ of $\{1,\dots,n\}$ of order $\sharp\sigma$, contributes a minor indexed by $j,k\in\sigma$. From the Cauchy determinant formula, we obtain the identity

$$\det\Bigl[\frac{b_j c_k e^{-\lambda_j x-\lambda_k x}}{\lambda_j+\lambda_k}\Bigr]_{j,k\in\sigma} = \prod_{j\in\sigma}\frac{b_j c_j e^{-2\lambda_j x}}{2\lambda_j}\prod_{j,k\in\sigma:j\neq k}\frac{\lambda_j-\lambda_k}{\lambda_j+\lambda_k}. \tag{4.19}$$

□

5. Diagonal Green's function and stationary KdV hierarchy

In this section, we obtain properties of $\mathcal{A}_\Sigma$ in terms of the brackets of odd powers of A. Thus we obtain some sufficient conditions for some differential equations to be integrable. Throughout this section, we suppose that the hypotheses of Theorem 4.4 are in force, so that any finite set of elements of $\mathcal{A}_\Sigma$ are holomorphic functions on a some sector Ω containing (x_0,∞) for some $x_0\geq 0$. We do not generally require u to be real valued, although in Theorem 5.4(iv) we impose this further condition so that we can compare our results with the classical spectral theory for the Schrödinger equation on the real line.

Definition 5.1. *(Stationary KdV hierarchy).* (i) Let $f_0 = 1$ and $f_1 = (1/2)u$. Then the KdV recursion formula is

$$4\frac{d}{dx}f_{m+1}(x) = 4f_1(x)\frac{d}{dx}f_m(x) + 4\frac{d}{dx}\bigl(f_1(x)f_m(x)\bigr) - \frac{d^3}{dx^3}f_m(x). \tag{5.1}$$

(ii) If u satisfies $f_m = 0$ for all m greater than or equal to some m_0, then u satisfies the stationary *KdV* hierarchy and is said to be an algebro-geometric (finite gap) potential; see [10], [11], [13], [36], [30].

(iii) Suppose that $u(x)\to 0$ as $x\to\infty$, and likewise for all the partial derivatives $\partial^\ell u/\partial x^\ell$; suppose further that $f_j(x)\to 0$ as $x\to 0$ as $x\to\infty$ for all $j = 1,2,\dots$. Then we say that the f_j are homogeneous solutions of the KdV hierarchy, and we consider cases where the system of differential equations (5.1) has no arbitrary constants of integration.

Proposition 5.2. *Let $\mathcal{A}_\Sigma$ be as in Theorem 4.4. Then $f_m = (-1)^m 2\lfloor A^{2m-1}\rfloor$ for $m = 1,2,\dots$ satisfies the stationary KdV hierarchy (Novikov's equations), since*

$$4\frac{d}{dx}\lfloor A^{2m+3}\rfloor = \frac{d^3}{dx^3}\lfloor A^{2m+1}\rfloor + 8\Bigl(\frac{d}{dx}\lfloor A\rfloor\Bigr)\lfloor A^{2m+1}\rfloor + 16\lfloor A\rfloor\Bigl(\frac{d}{dx}\lfloor A^{2m+1}\rfloor\Bigr). \tag{5.2}$$

Proof. (i) We have the basic identities

$$\lfloor A(I-2F)A(I-2F)X\rfloor = \lfloor A^2X\rfloor - 2\lfloor A\rfloor\lfloor X\rfloor; \tag{5.3}$$

$$-2A(AF+FA-2FAF) = A(I-2F)A(I-2F) - A^2 \tag{5.4}$$

and their mirror images. Hence

$$\frac{d}{dx}\lfloor A^{2m+1}\rfloor = \lfloor A(I-2F)A^{2m+1} + A^{2m+1}(I-2F)A\rfloor, \tag{5.5}$$

so

$$\begin{aligned}\frac{d^2}{dx^2}\lfloor A^{2m+1}\rfloor = {} & \lfloor A(I-2F)A(I-2F)A^{2m+1} + 2A(I-2F)A^{2m+1}(I-2F)A \\ & + A^{2m+1}(I-2F)A(I-2F)A \\ & - 2A(AF+AF-2FAF)A^{2m+1} \\ & - 2A^{2m+1}(AF+FA-2FAF)A\rfloor \\ = {} & \lfloor A(I-2F)A(I-2F)A^{2m+1} + 2A(I-2F)A^{2m+1}(I-2F)A \\ & + A^{2m+1}(I-2F)A(I-2F)A \\ & + A(I-2F)A(I-2F)A^{2m+1} - A^{2m+3} \\ & + A^{2m+1}(I-2F)A(I-2F)A - A^{2m+3}\rfloor\end{aligned} \tag{5.6}$$

and by the basic identities (5.3) and (5.4)

$$\begin{aligned}\frac{d^2}{dx^2}\lfloor A^{2m+1}\rfloor = {} & 2\lfloor A(I-2F)A^{2m+1}(I-2F)A\rfloor - 2\lfloor A^{2m+3}\rfloor \\ & + 2\lfloor A(I-2F)A(I-2F)A^{2m+1}\rfloor \\ & + 2\lfloor A^{2m+1}(I-2F)A(I-2F)A\rfloor \\ = {} & 2\lfloor A(I-2F)A^{2m+1}(I-2F)A\rfloor + 2\lfloor A^{2m+3}\rfloor \\ & - 4\lfloor A^{2m+1}\rfloor\lfloor A\rfloor - 4\lfloor A\rfloor\lfloor A^{2m+1}\rfloor.\end{aligned} \tag{5.7}$$

Now we differentiate the first summand of the final term

$$\begin{aligned}&\frac{d}{dx}2\lfloor A(I-2F)A^{2m+1}(I-2F)A\rfloor \\ &\quad = 2\lfloor A(I-2F)A(I-2F)A^{2m+1}(I-2F)A\rfloor \\ &\quad\quad + 2\lfloor A(I-2F)A^{2m+1}(I-2F)A(I-2F)A\rfloor \\ &\quad\quad - 4\lfloor A(AF+FA-2FAF)A^{2m+1}(I-2F)A\rfloor \\ &\quad\quad - 4\lfloor A(I-2F)A^{2m+1}(AF+FA-2FAF)A\rfloor\end{aligned} \tag{5.8}$$

$$\begin{aligned}&= 2\lfloor A(I-2F)A(I-2F)A^{2m+1}(I-2F)A\rfloor\\&\quad + 2\lfloor A(I-2F)A^{2m+1}(I-2F)A(I-2F)A\rfloor\\&\quad + 2\lfloor A(I-2F)A(I-2F)A^{2m+1}(I-2F)A\rfloor\\&\quad - 2\lfloor A^{2m+3}(I-2F)A\rfloor\\&\quad + 2\lfloor A(I-2F)A^{2m+1}(I-2F)A(I-2F)A\rfloor\\&\quad - 2\lfloor A(I-2F)A^{2m+3}\rfloor \end{aligned}\tag{5.9}$$

thus we obtain

$$\begin{aligned}\frac{d^2}{dx^2}\lfloor A^{2m+1}\rfloor &= 4\lfloor A(I-2F)A(I-2F)A^{2m+1}(I-2F)A\rfloor\\&\quad + 4\lfloor A(I-2F)A^{2m+1}(I-2F)A(I-2F)A\rfloor\\&\quad - 2\lfloor A(I-2F)A^{2m+3} + A^{2m+3}(I-2F)A\rfloor\\&= -8\lfloor A\rfloor\lfloor A^{2m+1}(I-2F)A\rfloor + 4\lfloor A^{2m+3}(I-2F)A\rfloor\\&\quad - 8\lfloor A\rfloor\lfloor A(I-2F)A^{2m+1}\rfloor\\&\quad + 4\lfloor A(I-2F)A^{2m+3}\rfloor - 2\frac{d}{dx}\lfloor A^{2m+3}\rfloor\\&= -8\lfloor A\rfloor\lfloor A(I-2F)A^{2m+1} + A^{2m+1}(I-2F)A\rfloor\\&\quad + 4\lfloor A(I-2F)A^{2m+3} + A^{2m+3}(I-2F)A\rfloor - 2\frac{d}{dx}\lfloor A^{2m+3}\rfloor\\&= -8\lfloor A\rfloor\frac{d}{dx}\lfloor A^{2m+1}\rfloor + 2\frac{d}{dx}\lfloor A^{2m+3}\rfloor; \end{aligned}\tag{5.10}$$

hence

$$\frac{d^3}{dx^3}\lfloor A^{2m+1}\rfloor = -8\lfloor A\rfloor\frac{d}{dx}\lfloor A^{2m+1}\rfloor + 4\frac{d}{dx}\lfloor A^{2m+3}\rfloor - 8\frac{d}{dx}\Big(\lfloor A\rfloor\lfloor A^{2m+1}\rfloor\Big);\tag{5.11}$$

which gives the stated result (5.2).

□

Definition 5.3. *(Diagonal Green's function).* Let $(-A,B,C)$ be as in Theorem 2.3. Then the diagonal Green's function is $g_0(x;\zeta)/\sqrt{\zeta}$ where

$$g_0(x;\zeta) = (1/2) + \lfloor A(\zeta I - A^2)^{-1}\rfloor.\tag{5.12}$$

The notation $g_0(x;\zeta)$ is chosen to indicate a generating function and also the diagonal of a Green's function; now in Theorem 5.4(iv) we explain the latter connection. Let $\mathbb{C}_+ = \{\lambda\in\mathbb{C} : \Im\lambda > 0\}$.

Theorem 5.4. *Let* $(-A,B,C)$ *be as in Theorem* 2.3.

(i) Then $g_0(x;\zeta)$ *is bounded and continuously differentiable in* x *and has a unique asymptotic expansion depending on the bracketed odd powers of* A*,*

$$g_0(x;\zeta) \asymp \frac{1}{2} + \frac{\lfloor A\rfloor}{\zeta} + \frac{\lfloor A^3\rfloor}{\zeta^2} + \frac{\lfloor A^5\rfloor}{\zeta^3} + \dots \qquad (\zeta\to-\infty);\tag{5.13}$$

(ii) $g_0(x;\zeta)$ satisfies Drach's equation

$$\frac{d^3 g_0}{dx^3} = 4(u+\zeta)\frac{dg_0}{dx} + 2\frac{du}{dx}g_0 \qquad (x > x_0; -\zeta > \omega); \tag{5.14}$$

(iii) there exists $x_1 > 0$ such that

$$\psi_\pm(x,\zeta) = \sqrt{g_0(x,-\zeta)}\exp\Bigl(\mp\sqrt{-\zeta}\int_{x_1}^{x}\frac{dy}{2g_0(y;-\zeta)}\Bigr) \tag{5.15}$$

satisfies Schrödinger's equation

$$-\psi_\pm''(x;\zeta) + u(x)\psi_\pm(x,\zeta) = \zeta\psi_\pm(x;\zeta) \qquad (x > x_1, \zeta > \omega). \tag{5.16}$$

(iv) Suppose that u is a continuous real function that is bounded below, and that $\psi_\pm$ from (iii) satisfy $\psi_+(x;\zeta) \in L^2((0,\infty);\mathbb{C})$ and $\psi_-(x;\zeta) \in L^2((-\infty,0);\mathbb{C})$ for all $\zeta \in \mathbb{C}_+$. Then $L = -\frac{d^2}{dx^2} + u(x)$ defines an essentially self-adjoint operator in $L^2(\mathbb{R};\mathbb{C})$, and the Green's function $G(x,y;\zeta)$ which represents $(\zeta I - L)^{-1}$ has a diagonal that satisfies

$$G(x,x;\zeta) = \frac{g_0(x;-\zeta)}{\sqrt{-\zeta}}. \tag{5.17}$$

Proof. (i) Let $\pi - \theta < \arg\lambda < \theta$, so λ and $-\lambda$ both lie in S_θ, hence $\zeta = \lambda^2$ satisfies $2\pi - 2\theta < \arg\zeta < 2\theta$, so ζ lies close to $(-\infty, 0)$. Then $\zeta I - A^2$ is invertible and $|\zeta|\|(\zeta I - A^2)^{-1}\|_{\mathcal{L}(H)} \leq M$. The function

$$g_0(x;\zeta) = \frac{1}{2} + Ce^{-xA}(I+R_x)^{-1}A(\zeta I - A^2)^{-1}(I+R_x)^{-1}e^{-xA}B \qquad (x > 0)$$

is well defined by Theorem 2.3(iii).

To obtain the asymptotic expansion, we note that $e^{-xA}(I+R_x)^{-1}$ and $(I+R_x)e^{-xA}$ involve the factor e^{-xA}, where (e^{-zA}) is a holomorphic semigroup on $S_{\theta-\pi/2}$. Hence $A^{2j+1}e^{-xA} \in \mathcal{L}(H)$ and by Cauchy's estimates there exist $\delta, x_0, M_0 > 0$ such that $\|A^{2j+1}e^{-xA}\|_{\mathcal{L}(H)} \leq M_0(2j+1)!$ for all $x \geq x_0 > 0$, and $\|e^{-sA}\|_{\mathcal{L}(H)} \leq M_0 e^{-s\delta}$. As in Proposition 3.2, we have an asymptotic expansion of

$$\begin{aligned}
e^{-zA}\bigl((\lambda I - A)^{-1} &- (\lambda I + A)^{-1}\bigr) \\
&= -e^{-zA}\int_0^\infty e^{\lambda s}e^{-sA}\,ds - e^{-zA}\int_0^\infty e^{-\lambda s}e^{-sA}\,ds \\
&= e^{-zA}\Bigl(\frac{A}{\lambda^2} + \frac{A^3}{\lambda^4} + \dots + \frac{A^{2j-1}}{\lambda^{2j}}\Bigr) \\
&\quad + \frac{e^{-zA}}{\lambda^{2j+1}}\int_0^\infty A^{2j+1}e^{-sA}(e^{s\lambda} - e^{-\lambda s})\,ds,
\end{aligned} \tag{5.18}$$

in which all the summands are in $\mathcal{L}(H)$ due to the factor e^{-zA} for $z \in S_{\theta-\pi/2}$. Hence

$$Ce^{-xA}(I+R_x)^{-1}\int_0^\infty A^{2j+1}e^{-sA}(e^{s\lambda} - e^{-s\lambda})\,ds(I+R_x)^{-1}e^{-xA}B \to 0 \qquad (x > 0)$$

as $\lambda \to i\infty$, or equivalently $\zeta \to -\infty$, so

$$g_0(x,\zeta) = \frac{1}{2} + Ce^{-xA}(I+R_x)^{-1}\Big(\frac{A}{\zeta} + \frac{A^3}{\zeta^2} + \dots + \frac{A^{2j-1}}{\zeta^j}\Big)(I+R_x)^{-1}e^{-xA}B + O\Big(\frac{1}{\zeta^{j+1}}\Big). \tag{5.19}$$

This gives the asymptotic series; generally, the series is not convergent since the implied constants in the term $O(\zeta^{-(j+1)})$ involve $(2j+1)!$.

(ii) From Proposition 5.2 we have

$$4\frac{d}{dx}\sum_{m=0}^{\infty}\frac{\lfloor A^{2m+3}\rfloor}{\zeta^{m+1}} = \frac{d^3}{dx^3}\sum_{m=0}^{\infty}\frac{\lfloor A^{2m+1}\rfloor}{\zeta^{m+1}} + 8\Big(\frac{d}{dx}\lfloor A\rfloor\Big)\sum_{m=0}^{\infty}\frac{\lfloor A^{2m+1}\rfloor}{\zeta^{m+1}} + 16\lfloor A\rfloor\frac{d}{dx}\sum_{m=0}^{\infty}\frac{\lfloor A^{2m+1}\rfloor}{\zeta^{m+1}}; \tag{5.20}$$

the required result follows on rearranging.

Conversely, suppose that g_0 as defined in (5.12) has an asymptotic expansion with coefficients in $C^\infty((0,\infty);\mathbb{C})$ as $\zeta \to -\infty$ and that $g_0(x;\zeta)$ satisfies (5.14). Then the coefficients of ζ^{-j} satisfy a recurrence relation which is equivalent to the systems of differential equations (5.1).

The asymptotic expansion is unique in the following sense. Suppose momentarily that $t \mapsto \lfloor Ae^{-tA^2}\rfloor$ is bounded and repeatedly differentiable on $(0,\infty)$, with $M,\omega > 0$ such that $\|\lfloor Ae^{-tA^2}\rfloor\| \le Me^{\omega t}$ for $t > 0$, and that there is a Maclaurin expansion

$$\lfloor Ae^{-tA^2}\rfloor = \lfloor A\rfloor - \lfloor A^3\rfloor t + \frac{\lfloor A^5\rfloor t^2}{2!} - \dots + O(t^k)$$

on some neighbourhood of 0+. Then by Watson's Lemma [37] p. 188, the integral $\int_0^\infty \lfloor Ae^{-tA^2}\rfloor e^{t\zeta}\,dt$ has an asymptotic expansion as $\zeta \to -\infty$, where the coefficients give the formula (5.13).

(iii) Since $(e^{-tA})_{t>0}$ is a contraction semigroup on H, we have $\mathcal{D}(A^2) \subseteq \mathcal{D}(A)$ and $\|Af\|_H^2 \le 2\|A^2f\|_H\|f\|_H$ for all $f \in \mathcal{D}(A^2)$ by the Hardy-Littlewood-Landau inequality [15] p.65, so $\|\zeta f + A^2 f\|_H \ge \sqrt{\zeta}\|Af\|_H$ for $\zeta > 0$. We deduce that $A^2 - 2A + \zeta I$ is invertible for $\zeta > 9$ and generally for all $\zeta \in \mathbb{C}$ such that $\Re\zeta$ is sufficiently large. By Proposition 5.2 and the multiplicative property of the bracket, we have

$$\frac{1}{2g_0(x;-\zeta)} = 1 + \big\lfloor 2A(\zeta I + A^2 - 2A)^{-1}\big\rfloor,$$

and we observe that $g_0(x;-\zeta) \to 1/2$ as $x \to \infty$, so there exists $x_1 > 0$ such that $g_0(x,-\zeta) > 0$ for all $x > x_1$ and the differential equation integrates to

$$g_0\frac{d^2g_0}{dx^2} - \frac{1}{2}\Big(\frac{dg_0}{dx}\Big)^2 = 2(u-\zeta)g_0^2 + \frac{\zeta}{2}. \tag{5.21}$$

So one can define $\psi(x;\zeta)$ as in (5.15), and then one verifies the differential equation for $\psi(x;\zeta)$ by using (5.21).

(iv) By a theorem of Weyl [18] 10.1.4, L is of limit point type at $\pm\infty$, and there exist nontrivial solutions $\psi_\pm(x;\zeta)$ to $-\psi_\pm''(x;\zeta)+u(x)\psi_\pm(x;\zeta)=\zeta\psi_\pm(x;\zeta)$ such that $\psi_+(x;\zeta)\in L^2(0,\infty)$ and $\psi_-(x;\zeta)\in L^2(-\infty,0)$, and these are unique up to constant multiples. Also the inverse operator $(-\zeta I+L)^{-1}$ may be represented as an integral operator in $L^2(\mathbb{R};\mathbb{C})$ with kernel $G(x,y;\zeta)$, which has diagonal

$$G(x,x;\zeta)=\frac{\psi_+(x;\zeta)\psi_-(x;\zeta)}{\mathrm{Wr}(\psi_+(\,;\zeta),\psi_-(\,;\zeta))}\qquad(\Im\zeta>0).$$

Given $\psi_\mp$ as in (iii), we can compute $\psi_+(x;\zeta)\psi_-(x;\zeta)=g_0(x;-\zeta)$ and their Wronskian is $\mathrm{Wr}(\psi_+,\psi_-)=\sqrt{-\zeta}$, hence the result.

□

Remark 5.5. (i) The importance of the diagonal Green's function is emphasized in [14]. Gesztesy and Holden [13] obtain an asymptotic expansion of the diagonal $G(x,x;\zeta)$ which is consistent with Theorem 5.4(i). Under conditions discussed in (5.46), we have similar asymptotics as $-\zeta\to\infty$.

(ii) Drach observed that one can start with the differential equation (5.14), and produce the solutions (5.24); see [7]. He showed that Schrödinger's equation is integrable by quadratures, if and only if (5.14) can be integrated by quadratures for typical values of ζ, and Brezhnev translated his results into the modern theory of finite gap integration [7]. Having established integrability of Schrödinger's equation by quadratures, one can introduce the hyperelliptic spectral curve $\mathcal{E}$ with $g<\infty$ and proceed to express the solution in terms of the Baker–Akhiezer function. Hence one can integrate the equation and express the solution in terms of the Riemann's theta function on the Jacobian of $\mathcal{E}$, as in [10], [3], [13].

(iii) Kotani [21] has introduced the Baker-Akhiezer function and the τ function via the Weyl m-function for a suitable class of potentials that included multi-solitons and algebro-geometric potentials. There is a determinant formula for τ correspnding to (1.9) and (3.2), and the theory develops themes from [36].

(iv) The deformation theory for rational differential equations is discussed in [20].

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