RELATIVE EFFICIENCY OF CONTINUOUS AND DISCRETE METHODS OF DYNAMICAL CONTROL OF LASERS

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A direct comparison between continuous and discrete forms of control is investigated theoretically and numerically. Specifically we investigate energy-optimal control of switching of a periodically driven class B laser from stable to unstable pulsing regimes.

The dynamical control of nonlinear systems has been discussed intensively during recent years[1]. One aspect, how to control switching between the different regimes of multistable systems, is of special interest in view of possible applications. The main problem relates to the efficiency of switching between the stable and unstable states. In recent laser experiments [2,3], attention was concentrated on minimization of the *duration* of the transient processes. However, minimization of the *energy* of switching should also be considered. Control can be realized in either continuous or discrete time, with corresponding control schemes analyzed in terms of either continuous flows or maps.

In this presentation we consider theoretically and numerically a direct comparison between the continuous and discrete forms of control. The investigation is performed within the framework of theory developed in our earlier publications [4,5] and applied to the problem of energy-optimal migration between the stable and unstable states of a chaotic system. Specifically, we investigate the energy-optimal control of switching in a periodically-driven class-B laser between its stable and unstable pulsing regimes.

Our analysis is based on single-mode rate equations

$$\dot{u} = vu(y-1), \quad \dot{y} = Q - y - yu \ , \qquad (1)$$

where u and y are proportional to the density of radiation and carrier inversion respectively, v is the ratio of the photon damping rate in the cavity to the rate of carrier inversion relaxation, cavity loss is normalized to unity, and the pumping rate Q is driven, $Q= q + k\cos(\omega t)+ f(t)$, with k and ω being the amplitude and the frequency of external periodic modulation, $f(t)$ is the control force. For class-B lasers regimes of the spiking type are observed under deep modulation of the pumping rate. Their dynamics is determined by the two-dimensional Poincare map

$$c_{i+1} = G(c_i, \varphi_i), \quad \varphi_{i+1} = \varphi_i + \omega T(c_i, \varphi_i), \ \mathrm{mod}\ 2\pi, \qquad (2)$$

that we derived from Eqs.(1) following the asymptotic method[6]. Here the variables c_i, φ_i correspond to the inversion of population $y(t_i)$ and to the phase of modulation $\omega t_i \ mod\ 2\pi$ at the moments t_i of pulse onset, and the function $T(c_i, \varphi_i)$ gives the time interval between neighbouring pulses.

The fixed points of the map determine the spiking solutions of a period multiple to the period of driving, $T_n=nT_M$, $n=1,2,...$ They are born through the saddle-node bifurcation at the threshold level of modulation $k_{sn} = \sqrt{1+\omega^2}q(q-1)(nT_n)^2/12$, and the stable cycles undergo a period-doubling bifurcation if the modulation level exceeds $k_{pd} = \left(\sqrt{1+\omega^2}/\omega\right)(q-1)\left[1+2\pi\left(qnT_n/12\right)^2\right]$. In this way we determine analytically the regions where generalized multistability is realized as the coexistence of a number of cycles and approximate the location of saddles and stable cycles in the phase space.

We concentrate on the problem of controlled migration between the stable and saddle cycles related to the saddle-node bifurcation and which are of the same amplitude and period. Two forms of the control force *f(t)* are considered: one is continuous in time and the other is a sequence of discrete impulses $f(t)=\Sigma_i f_i\delta(t-t_i)$ at moments of system's crossing Poincare section, i.e. at the moment of laser spikes. The following energy-optimal control problem is considered: How can the system (1) with unconstrained control function *f(t)* be steered between coexisting states such that the "cost" functional

$$J_c = \inf_{f\in F} \tfrac{1}{2}\int_{t_0}^{t_1} f^2(t)dt \tag{3}$$

is minimized? Here t_1 is unspecified and F is the set of control functions. Following the Pontryagin theory of optimal control[7] we reduce the energy-minimal migration task to boundary problems for the Hamilton equation

$$\begin{cases} \dot{x} = z, \quad \dot{z} = q-1+k\cos(\Omega\tau) - e^x(1+\varepsilon z) - \varepsilon z + p_2, \\ \dot{p}_1 = p_2 e^x(1+\varepsilon z), \quad \dot{p}_2 = -p_1 + p_2\varepsilon(1+e^x). \end{cases} \tag{4}$$

where $\varepsilon = v^{-1/2}$, $\tau = \varepsilon^{-1}t$, $\Omega = \varepsilon\omega$, $z = \varepsilon^{-1}(y-1)$, $x = \ln u$, and it is assumed that the optimal control function *f(t)* at each instant takes those values $f(t)=p_2$ that maximize H_c over F. The corresponding area preserving map for the discrete control scheme takes the form

$$\begin{cases} c_{i+1} = q + (c_i - g - q - K\cos\psi_i)e^{-T} + K\cos(\omega T + \psi_i) + p_{i+1}^c, \\ \varphi_{i+1} = \varphi_i + \omega T, \quad \mathrm{mod}\, 2\pi, \end{cases}$$

$$\begin{pmatrix} p_{i+1}^c \\ p_{i+1}^\varphi \end{pmatrix} = \begin{pmatrix} \dfrac{\partial c_{i+1}}{\partial c_i} & \dfrac{\partial c_{i+1}}{\partial \varphi_i} \\ \dfrac{\partial \varphi_{i+1}}{\partial c_i} & \dfrac{\partial \varphi_{i+1}}{\partial \varphi_i} \end{pmatrix}^{-1} \begin{pmatrix} p_i^c \\ p_i^\varphi \end{pmatrix} \tag{5}$$

The solutions of (4) and (5) are found using a method based on the analogy between the Hamiltonian theory of optimal control and the Hamiltonian theory of large fluctuations[5,6]. In this approach, the optimal force is identified with the optimal fluctuational force. It is determined by substituting for the control function *f(t)* a random force and collecting those realizations of the random force that switch the laser between its stable and unstable pulsating regimes. The ensemble-averaged realizations of the random force approximate closely the solution of the optimal control problem[5,6].

For the discrete control scheme, the optimization problem was addressed by solving a boundary value problem for (5). An example of the numerical results are shown in the Fig. 1 where the realization of the coordinate *x(t)* corresponds to the switching of minimal energy between stable limit cycle C_3 and the unstable limit cycle S_3. Using prehistory analysis, two escape paths were found for transition $C_2 \to S_2$ with practically equal probability (55% and 45%), hence, two control forces were determined and shown in the table (in the last two columns of the second line). It can be seen that the continuous force is significantly more efficient energetically. Experiments show that by increasing the duration τ of the impulses of discrete force, the energy required for control can be substantially reduced.

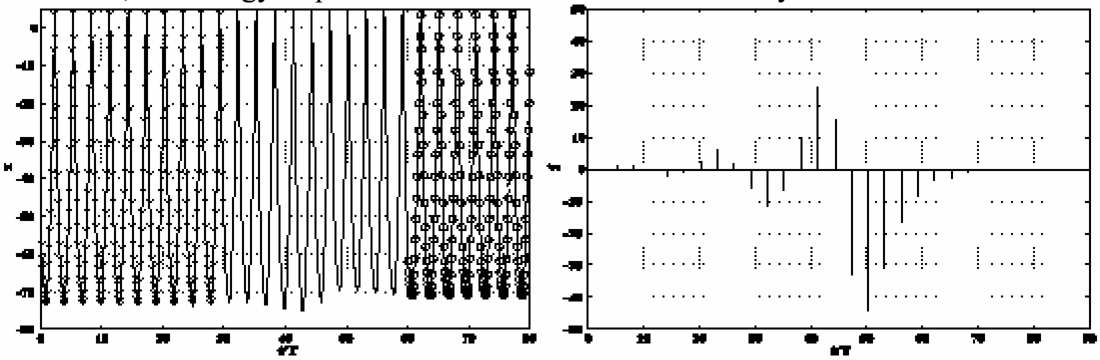

Fig. 1. Time realizations of coordinate $x(t)$ (a) and impulse control force $f(t)$ (b) are shown during migration from C_3 to S_3. Stable cycle C_3 is marked by "x", saddle cycle S_3 by "o". The control force is based on the map force determined by prehistory fluctuation approach.

Moreover it was found that there is an optimal value of the duration where energy takes a minimum. The energy values for optimized impulse duration are shown in Table 1. As can be seen, the energy can be decreased by several orders through optimization of the impulse duration, but it still exceeds the optimal energy of the continuous force.

	J_c	J^H_d	J^P_d	J_{dopt}
C_3 - S_3	0.00028	26.76	16.55	0.55
C_2 - S_2	0.0015	88.26	73.3/80.3	1.935/2.147

Table. 1. The energy of the control function: J_c corresponds to the continuous force, J^H_d is the discrete force determined by solving the boundary problem, J^P_d is the discrete force determined by prehistory approach, J_{dopt} is the discrete force with optimizations of impulse duration for the force determined by prehistory approach. The energy is shown for two transitions $C_3 \to S_3$ and $C_2 \to S_2$.

Summarizing, we have obtained the protocols of energy-minimal migration from stable to saddle cycles differing from each other by only their phase relative to the modulation signal. Within the framework of the method proposed, switching between stable cycles with different periods and amplitudes can be also considered for an amplitude coding scheme, as well as for a combination of phase and amplitude coding.

1. S. Boccaletti, S.C. Grebogi, Y.C. Lai, H. Mancini, D. Maza, *Phys. Rep.,* **329**, 103-197, (2000).
2. V. N. Chizhevsky. *J. Opt. B: Quantum Semiclass., Opt.* **2**, 711-717, (2000).
3. N.V. Chizhevsky, E.V. Grigorieva, S.A. Kaschenko, *Opt. Commun.,* 133, 189-195, (1997).
4. I.A. Khovanov, D.G. Luchinsky, R. Mannella, P.V.E. McClintock, *Phys. Rev. Lett.,* **85**, 2100-2559, (2000).
5. D.G. Luchinsky, S. Beri, R. Mannella, P.V.E. McClintock, I.A. Khovanov, *Int. J. Bif. and Chaos,* **12**, 583-604, (2002).
6. V.A. Gaisyonok, E.V. Grigorieva, S.A. Kaschenko, *Opt. Commun.,* **124**, 408-417, (1996).
7. L.S. Pontryagin, *The Mathematical Theory of Optimal Processes*, (Macmillan, 1964).