

FREE MINIMAL ACTIONS OF COUNTABLE GROUPS WITH INVARIANT PROBABILITY MEASURES

GÁBOR ELEK

ABSTRACT. We prove that for any countable group Γ there exists a free minimal continuous action $\alpha : \Gamma \curvearrowright \mathcal{C}$ on the Cantor set admitting an invariant Borel probability measure.

Keywords. Free minimal actions, uniformly recurrent subgroups, invariant measures

1. INTRODUCTION

In [6] Hjorth and Molberg proved that for any countable group Γ there exists a free and continuous action of Γ on the Cantor set \mathcal{C} that admits an invariant Borel probability measure. Our main goal is to prove that one can also assume the minimality of the action.

Theorem 1. *For any countably infinite group Γ , there exists a free minimal continuous action $\alpha : \Gamma \curvearrowright \mathcal{C}$ on the Cantor set admitting an invariant Borel probability measure.*

In fact, we show that α can be chosen to be universal in the following sense.

Theorem 2. *There exists a free, minimal action $\alpha : \Gamma \curvearrowright \mathcal{C}$ that satisfies the following condition. Let $\beta : \Gamma \curvearrowright X$ be a free Borel action of the above group Γ on the standard Borel space X admitting an invariant Borel probability measure μ . Then there exists a Borel embedding $\Phi : X' \rightarrow \mathcal{C}$ commuting with the actions α and β , such that X' is an invariant Borel set of X and $\mu(X') = 1$.*

It is important to note that Weiss proved [11] the existence of a topologically free minimal Cantor action satisfying the condition of Theorem 2. Notice however, that topologically free minimal Cantor actions of the free group admitting an invariant measure can be very far from being essentially free. In fact, it is possible that the equivalence relation associated to such topologically free minimal action is hyperfinite [1]. Our approach to prove Theorem 2 is based on the philosophy behind Weiss' proof and the notion of properness.

First we prove a result on Borel universality.

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Proposition 1.1. *For any countably infinite group Γ and for any free Borel action $\beta : \Gamma \curvearrowright X$ on the standard Borel space, there exists an injective equivariant Borel map $\Psi'_\beta : X \rightarrow \text{Free}(\mathcal{C}^\Gamma)$ (where $\text{Free}(\mathcal{C}^\Gamma)$ is the free part of the Bernoulli \mathcal{C} -shift space), such that the closure of the set $\Psi'_\beta(X)$ is still in $\text{Free}(\mathcal{C}^\Gamma)$.*

As a consequence, we will obtain the following theorem.

Theorem 3. *For any countably infinite group Γ , there exists a free continuous action $\zeta : \Gamma \curvearrowright \mathcal{C}$ such that for any free Borel action $\beta : \Gamma \curvearrowright X$ on the standard Borel space we have an injective Borel map $\Psi_\beta : X \rightarrow \mathcal{C}$ satisfying $\Psi_\beta \circ \beta = \zeta \circ \Psi_\beta$.*

Note that Seward and Tucker-Drob [10] (see also [2]) proved the following result: For any free Borel action $\alpha : \Gamma \curvearrowright X$ there exists a (not necessarily injective) equivariant Borel map $\Psi : X \rightarrow \text{Free}(\{0, 1\}^\Gamma)$ such that the closure of the set $\Psi(X)$ is still in $\text{Free}(\{0, 1\}^\Gamma)$.

In 1952 Oxtoby [9] proved that there exists a free, minimal Z -action on the Cantor set that is not uniquely ergodic, that is, it admits more than one ergodic invariant Borel probability measures. We will prove the following corollary of Theorem 2.

Corollary 1.1. *Any countably infinite group Γ has a free, minimal action on the Cantor set that is not uniquely ergodic.*

We will also prove a version of Theorem 3 for uniformly recurrent subgroups (Theorem 4) answering a question of Glasner and Weiss in the “universal sense”. Finally, we prove Theorem 1. The main idea is to construct an explicit free minimal action for any countably infinite group using an inductive “learning” algorithm. Then we combine this construction with Theorem 3 to obtain our main result.

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2. THE PROOF OF THEOREM 3

Let Γ be a countable infinite group and $\{\sigma_i\}_{i=1}^\infty$ be a generating system of Γ . Also for $n \geq 1$, let Γ_n be the subgroup of Γ generated by the elements $\{\sigma_i\}_{i=1}^n$. We will also assume that if $\Gamma_n \neq \Gamma$, then $\sigma_{n+1} \notin \Gamma_n$. Hence, if $\Gamma_n \neq \Gamma$ then $\Gamma_{n+1} \neq \Gamma_n$. Consequently, if Γ_n is finite, then $|\Gamma_n| \geq 2^n$. Let $\alpha : \Gamma \curvearrowright X$ be a Borel action of Γ on the standard Borel space X . We define a sequence $\{\mathcal{G}_n\}_{n=1}^\infty$ of Borel graph structures on X in the following way. If $p, q \in X$, $p \neq q$, then let $(p, q) \in E(\mathcal{G}_n)$ if there exists $1 \leq i \leq n$ such that $\alpha(\sigma_i)(p) = q$ or $\alpha(\sigma_i)(q) = p$. A Borel \mathcal{C} -coloring of X is a Borel map $\varphi : X \rightarrow \mathcal{C}$, where $\mathcal{C} = \{0, 1\}^\mathbb{N}$ is the Cantor set. We say that φ is a **proper** \mathcal{C} -coloring with

respect to $\alpha : \Gamma \curvearrowright X$ if for any $r > 0$ there exists $S_r > 0$ such that for any $p, q \in X$ $(\varphi(p))_{S_r} \neq (\varphi(q))_{S_r}$, provided that $0 < d_{\mathcal{G}_r}(p, q) \leq r$, where

- $d_{\mathcal{G}_r}$ is the shortest path metric on the components of the Borel graph \mathcal{G}_r .
- For $\kappa \in \mathcal{C}$ and $s > 0$, $(\kappa)_s \in \{0, 1\}^{[s]}$, denotes the projection of κ onto its first s coordinates. Here $[s]$ denotes the set $\{1, 2, \dots, s\}$.

The Borel coloring φ is called separating if for any $p \neq q \in X$, $\varphi(p) \neq \varphi(q)$.

Lemma 2.1. *For any Borel action $\alpha : \Gamma \curvearrowright X$ there exists a separating, proper \mathcal{C} -coloring with respect to α .*

Proof. First, for any $r \geq 1$ we construct a new Borel graph \mathcal{H}_r of bounded vertex degree on X such that $(p, q) \in E(\mathcal{H}_r)$ if $0 < d_{G_r}(p, q) \leq r$. By the classical result of Kechris, Solecki and Todorcevic [7], there exists an integer $m_r > 0$ and a Borel coloring $\psi_r : X \rightarrow \{0, 1\}^{[m_r]}$ such that $\psi_r(p) \neq \psi_r(q)$, whenever p and q are adjacent vertices in the Borel graph \mathcal{H}_r . Then $\varphi_1(p) = \{\psi_1(p)\psi_2(p) \dots\} \in \{0, 1\}^{\mathbb{N}}$, defines a proper \mathcal{C} -coloring of X with respect to α . Now we use the usual trick to obtain a separating coloring. Let $\varphi_2 : X \rightarrow \mathcal{C}$ be a Borel embedding. Then if $\varphi_1(p) = \{a_1 a_2 a_3 \dots\}$ and $\varphi_2(p) = \{b_1 b_2 b_3 \dots\}$ let $\varphi(p) = \{a_1 b_1 a_2 b_2 \dots\}$. Clearly, φ is a separating, proper \mathcal{C} -coloring with respect to α . \square

Now we prove Proposition 1.1. Let $\alpha : \Gamma \curvearrowright X$ be a free Borel action and let $\varphi : X \rightarrow \mathcal{C}$ be a separating, proper \mathcal{C} -coloring. Consider the Bernoulli shift \mathcal{C}^Γ with the natural left action

$$L_\delta(\rho)(\gamma) = \rho(\gamma\delta).$$

The map $\Psi'_\alpha : X \rightarrow \mathcal{C}^\Gamma$ is defined as usual by

$$\Psi'_\alpha(x)(\gamma) = \varphi(\alpha(\gamma)(x)).$$

Clearly, Ψ'_α is Borel and Γ -equivariant and since φ is separating, Ψ'_α is injective as well. Also, $\Psi'_\alpha(X) \subset \text{Free}(\mathcal{C}^\Gamma)$, where $\text{Free}(\mathcal{C}^\Gamma)$ is the free part of the Bernoulli shift, that is, the set of elements $\rho \in \mathcal{C}^\Gamma$ such that if $L_\delta(\rho) = \rho$, then $\delta = e_\Gamma$. We need to show that if $\rho \in \mathcal{C}^\Gamma$ is in the closure of $\Psi'_\alpha(X)$, then $L_\gamma(\rho) \neq \rho$, whenever $\gamma \neq e_\Gamma$. It is enough to see that if $\gamma \neq e_\Gamma$, then $\rho(\gamma) \neq \rho(e_\Gamma)$. Let $\lim_{n \rightarrow \infty} \Psi'_\alpha(x_n) = \rho \in \mathcal{C}^\Gamma$. Also, let $r > 0$ such that

- $\gamma \in \Gamma_r$.
- $d_{G_r}(e_\Gamma, \gamma) \leq r$, where d_{G_r} is the shortest path metric on the left Cayley graph $\text{Cay}(\Gamma_r, \{\sigma_i\}_{i=1}^r)$.

Thus, for any $n \geq 1$,

$$d_{G_r}(\alpha(\gamma)(x_n), x_n) \leq r.$$

Hence, for any $n \geq 1$,

$$(\varphi(\alpha(\gamma)(x_n)))_{S_r} \neq (\varphi(x_n))_{S_r}.$$

Since $\rho(\gamma) = \lim_{n \rightarrow \infty} \varphi(\alpha(\gamma)(x_n))$ and $\rho(e_\Gamma) = \lim_{n \rightarrow \infty} \varphi(x_n)$, we have that $\rho(\gamma) \neq \rho(e_\Gamma)$. Hence our proposition follows. \square

Now we prove Theorem 3. Let $\iota : \Gamma \curvearrowright \text{Free}(\mathcal{C}^\Gamma)$ be the natural free action and $\Psi'_\iota : \text{Free}(\mathcal{C}^\Gamma) \rightarrow \text{Free}(\mathcal{C}^\Gamma)$ be the injective Γ -equivariant Borel map defined in Proposition 1.1. Let W be the closure of $\Psi'_\iota(\text{Free}(\mathcal{C}^\Gamma))$ in \mathcal{C}^Γ . Thus, $W \subset \text{Free}(\mathcal{C}^\Gamma)$ is a closed invariant subset. Then, W can be written as the disjoint union of W_1 and W_2 , where W_1 is a countable invariant subset and W_2 is a closed invariant subset homeomorphic to the Cantor set. Indeed, W is an uncountable compact set. Hence W can be written as the disjoint union of W_a and W_b , where W_a is homeomorphic to the Cantor set and W_b is countable. Clearly, the orbit closure W_2 of W_a has no isolated point and $W_1 = W \setminus W_2$ is a countable invariant set. Now, let $\alpha : \Gamma \curvearrowright X$ be a free Borel action and $\Psi'_\alpha : X \rightarrow \text{Free}(\mathcal{C}^\Gamma)$ be the injective Γ -equivariant Borel map defined in Proposition 1.1. Let $\tilde{\Psi}_\alpha = \Psi'_\iota \circ \Psi'_\alpha$. Clearly, $\tilde{\Psi}_\alpha : X \rightarrow W$ is an injective Γ -equivariant Borel map. Now we define the injective Γ -equivariant Borel map $\Psi_\alpha : X \rightarrow W_2$ by modifying $\tilde{\Psi}_\alpha$ on countably many orbits. Let $Y = (\tilde{\Psi}_\alpha)^{-1}(W_1)$. We can assume that the complement W_3 of $\tilde{\Psi}_\alpha(X)$ in W_2 is infinite, so we can define an injective Γ -equivariant map $\Psi' : Y \rightarrow W_3$. Therefore, we can define the injective Γ -equivariant Borel map $\Psi_\alpha : X \rightarrow W_2$ by

- $\Psi_\alpha(x) = \tilde{\Psi}_\alpha(x)$ if $x \in X \setminus Y$.
- $\Psi_\alpha(x) = \Psi'(x)$ if $x \in Y$.

Since W_2 is homeomorphic to \mathcal{C} , our theorem follows. \square

3. UNIFORMLY RECURRENT SUBGROUPS

Let Γ be a countable group and let $\text{Sub}(\Gamma)$ be the space of subgroups of Γ [5]. Then $\text{Sub}(\Gamma)$ is a compact, metrizable space and conjugations define a continuous action $c : \Gamma \curvearrowright \text{Sub}(\Gamma)$. Now let $\beta : \Gamma \curvearrowright M$ be a continuous minimal nonfree action of a countable group on a compact metric space. Then we have a natural equivariant map $\text{Stab}_\beta : M \rightarrow \text{Sub}(\Gamma)$ from our space M to the compact space of subgroups of Γ , mapping each point $x \in M$ to its stabilizer subgroup. Glasner and Weiss (Proposition 1.2 [5]) proved that the set M_0 of the continuity points of the map Stab_β is a dense, invariant G_δ subset of M and the closure of $\text{Stab}_\beta(M_0)$ in $\text{Sub}(\Gamma)$ is a minimal closed invariant subset of $\text{Sub}(\Gamma)$, that is, a **uniformly recurrent subgroup**. They asked if for any uniformly recurrent subgroup $Z \subset \text{Sub}(\Gamma)$ there exists a minimal continuous action $\beta : \Gamma \curvearrowright M$ such that $M_0 = M$ and $\text{Stab}_\beta(M) = Z$. This question has been answered in [3] and [8]. Now we show that one can answer the question of Glasner and Weiss in the universal sense.

Theorem 4. *Let Γ be a countable group and $Z \subset \text{Sub}(\Gamma)$ be a uniformly recurrent subgroup. Then there exists a continuous action $\zeta_Z : \Gamma \curvearrowright \mathcal{C}$ such that*

- $\text{Stab}_{\zeta_Z} : \mathcal{C} \rightarrow \text{Sub}(\Gamma)$ is continuous everywhere and $\text{Stab}_{\zeta_Z}(\mathcal{C}) = Z$.
- For any Borel action $\alpha : \Gamma \curvearrowright X$ such that for any $x \in X$ the group $\text{Stab}_\alpha(x)$ is in Z (we call these actions (Γ, Z) -actions), there exists an injective Borel map $\Psi_\alpha : X \rightarrow \mathcal{C}$ such that $\Psi_\alpha \circ \alpha = \zeta_Z \circ \Psi_\alpha$.

It was proved in Section 5 [3] that there exist countable groups Γ and uniformly recurrent subgroups $Z \subset \text{Sub}(\Gamma)$ such that no Borel (Γ, Z) -action admits an invariant Borel probability measure. However, we have the following nonfree analogue of the aforementioned result of Hjorth and Molberg.

Corollary 3.1. *Let Γ be a countable group and let $Z \subset \text{Sub}(\Gamma)$ be a uniformly recurrent subgroup. If there exists a Borel (Γ, Z) -action $\alpha : \Gamma \curvearrowright X$ that admits an invariant Borel probability measure, then there exists a continuous (Γ, Z) -action $\beta : \Gamma \curvearrowright \mathcal{C}$ on the Cantor set admitting an invariant Borel probability measure such that*

- The map $\text{Stab}_\beta : \mathcal{C} \rightarrow \text{Sub}(\Gamma)$ is continuous everywhere and
- $\text{Stab}_\beta(\mathcal{C}) = Z$.

Proof. (of Theorem 4) Let $Z \subset \text{Sub}(\Gamma)$ be a uniformly recurrent subgroup. We define the Bernoulli shift space \mathcal{C}^Z of Z in the following way. Let

$$\mathcal{C}^Z = \cup_{H \in Z} \mathcal{F}(H),$$

where $\mathcal{F}(H)$ is the set of maps $\rho : \Gamma/H \rightarrow \mathcal{C}$ from the left coset space of H to the Cantor set. The action of Γ on \mathcal{C}^Z is defined as follows.

- If $\rho \in \mathcal{F}(H)$ then $L_\delta(\rho) \in \mathcal{F}(\delta H \delta^{-1})$ and
- $L_\delta(\rho)(\gamma \delta H \delta^{-1}) = \rho(\gamma \delta H)$.

Lemma 3.1. *$L : \Gamma \rightarrow \text{Homeo}(\mathcal{C}^Z)$ is a homomorphism.*

Proof. We need to show that if $\rho : \Gamma/H \rightarrow \mathcal{C}$ and $\delta_1, \delta_2 \in \Gamma$, then

$$L_{\delta_1}(L_{\delta_2}(\rho)) = L_{\delta_1 \delta_2}(\rho).$$

Observe that

$$L_{\delta_1}(L_{\delta_2}(\rho)) \in \mathcal{F}(\delta_1 \delta_2 H \delta_2^{-1} \delta_1^{-1}) \text{ and } L_{\delta_1 \delta_2}(\rho) \in \mathcal{F}(\delta_1 \delta_2 H \delta_2^{-1} \delta_1^{-1}).$$

Now

$$\begin{aligned} & L_{\delta_1}(L_{\delta_2}(\rho))(\gamma \delta_1 \delta_2 H \delta_2^{-1} \delta_1^{-1}) = \\ & = L_{\delta_2}(\rho)(\gamma \delta_1 \delta_2 H \delta_2^{-1}) = \rho(\gamma \delta_1 \delta_2 H) = L_{\delta_1 \delta_2}(\rho)(\gamma \delta_1 \delta_2 H \delta_2^{-1} \delta_1^{-1}), \end{aligned}$$

hence our lemma follows. \square

We can equip \mathcal{C}^Z with a compact metric structure d such that (\mathcal{C}^Z, d) is homeomorphic to the Cantor set and the Γ -action above is continuous. Let $\rho_1 : \Gamma/H_1 \rightarrow \mathcal{C}$, $\rho_2 : \Gamma/H_2 \rightarrow \mathcal{C}$ be elements of \mathcal{C}^Z . We say that ρ_1 and ρ_2 are n -equivalent, $\rho_1 \equiv_n \rho_2$ if

- For any $\gamma \in \Gamma$, $d_{G_n}(e_\Gamma, \gamma) \leq n$, $\gamma \in H_1$ if and only if $\gamma \in H_2$.

- For any $\gamma \in \Gamma$, $d_{G_n}(e_\Gamma, \gamma) \leq n$,

$$(\rho_1(\gamma H_1))_n = (\rho_2(\gamma H_2))_n.$$

Then we define $d(\rho_1, \rho_2) := \frac{1}{2^n}$ whenever $\rho_1 \equiv_n \rho_2$ and $\rho_1 \not\equiv_{n+1} \rho_2$. Let $\{H_n\}_{n=1}^\infty, H \in \text{Sub}(\Gamma)$, $\rho \in \mathcal{F}(H)$ and for any $n \geq 1$ let $\rho_n \in \mathcal{F}(H_n)$. Observe that $\{\rho_n\}_{n=1}^\infty \rightarrow \rho$ in the d -metric if and only if

- $H_n \rightarrow H$ in the compact space $\text{Sub}(\Gamma)$ and
- for any $\gamma \in \Gamma$, $\rho_n(\gamma H_n) \rightarrow \rho(\gamma H)$.

We can define $\text{Free}(\mathcal{C}^Z)$ in the usual way. We have that $\rho : \Gamma/H \rightarrow \mathcal{C} \in \text{Free}(\mathcal{C}^Z)$ if $L_\delta(\rho) \neq \rho$ for any $\delta \notin H$. Clearly, if for any $\delta \notin H$, $\rho(H) \neq \rho(\delta H)$, then $\rho \in \text{Free}(\mathcal{C}^Z)$. Also, $\text{Stab}_L(\rho) = H$. Indeed, if $\delta \in H$, then for any $\gamma \in \Gamma$ we have that

$$L_\delta(\rho)(\gamma H) = \rho(\gamma \delta H) = \rho(\gamma H).$$

On the other hand, if $\delta \notin H$, but $\delta H \delta^{-1} = H$, then

$$L_\delta(\rho)(H) = \rho(\delta H) \neq \rho(H).$$

Now let $\alpha : \Gamma \curvearrowright X$ be a Borel (Γ, Z) -action and $\varphi : X \rightarrow \mathcal{C}$ be a separating, proper \mathcal{C} -coloring with respect to α . We define $\Psi_\alpha^\varphi : X \rightarrow \mathcal{C}^Z$ as follows.

- $\Psi_\alpha^\varphi(x) \in \mathcal{F}(H)$, where $H = \text{Stab}_\alpha(x)$.
- $\Psi_\alpha^\varphi(x)(\gamma H) = \varphi(\alpha(\gamma)(x))$.

Clearly, $\Psi_\alpha^\varphi : X \rightarrow \mathcal{C}^Z$ is an injective Borel map and $\Psi_\alpha^\varphi(X) \subset \text{Free}(\mathcal{C}^Z)$.

Lemma 3.2. *The map $\Psi_\alpha^\varphi : X \rightarrow \text{Free}(\mathcal{C}^Z)$ is Γ -equivariant.*

Proof. Let $\delta \in \Gamma$. Then

$$L_\delta(\Psi_\alpha^\varphi(x))(\gamma \delta H \delta^{-1}) = \Psi_\alpha^\varphi(x)(\gamma \delta H) = \varphi(\alpha(\gamma \delta)(x)).$$

On the other hand,

$$\Psi_\alpha^\varphi(\alpha(\delta)(x))(\gamma \delta H \delta^{-1}) = \varphi(\alpha(\gamma) \alpha(\delta)(x)) = \varphi(\alpha(\gamma \delta)(x)). \quad \square$$

Now we prove the nonfree analogue of Proposition 1.1.

Proposition 3.1. *For any countably infinite group Γ and for any free Borel (Γ, Z) -action $\alpha : \Gamma \curvearrowright X$, there exists an injective equivariant Borel map $\Psi'_\alpha : X \rightarrow \text{Free}(\mathcal{C}^Z)$, such that the closure of the set $\Psi'_\alpha(X)$ is still in $\text{Free}(\mathcal{C}^Z)$.*

Proof. Let φ and Ψ_α^φ be as above. Let $\{x_n\}_{n=1}^\infty \subset X$ such that

$$\lim_{n \rightarrow \infty} \Psi_\alpha^\varphi(x_n) = \rho \in \mathcal{F}(H)$$

and $\delta \notin H$. We need to show that $\rho \in \text{Free}(\mathcal{C}^Z)$. Observe that $\{\text{Stab}_\alpha(x_n)\}_{n=1}^\infty \rightarrow H$ in $\text{Sub}(\Gamma)$. Hence, there exists $N > 0$ such that $\delta \notin H_n$ if $n \geq N$. By properness, there exists $m > 0$ such that for all $n \geq N$

$$(\varphi(\alpha(\delta)(x_n)))_m \neq (\varphi(x_n))_m.$$

Since, $\lim_{n \rightarrow \infty} \varphi(\alpha(\delta)(x_n)) = \rho(\delta H)$ and $\lim_{n \rightarrow \infty} \varphi(x_n) = \rho(H)$ we have that $\rho(H) \neq \rho(\delta H)$. Hence, we have that $\rho \in \text{Free}(\mathcal{C}^Z)$. \square

Now Theorem 4 follows from Proposition 3.1 exactly the same way as Theorem 1 follows from Proposition 1.1. Note that if $Y \subset \text{Free}(\mathcal{C}^Z)$ is a closed, invariant subset, then the continuity of Stab_L on Y follows from the definition. \square

4. MINIMAL CANTOR LABELINGS I.

The goal of the next three sections is to present an explicit construction of free minimal actions for countably infinite groups. Before getting further let us recall some notions from graph theory. Let G be a graph, $S \subset V(G)$ be a subset and $r, s > 0$ be integers. Then S is an **s -net** if for any $x \in V(G)$ there exists $y \in S$ such that $d_G(x, y) \leq s$. Also, S is called an **r -sparse set** if for any $x \neq y \in S$, $d_G(x, y) > r$.

Definition 4.1. $T \subset V(G)$ is an S -maximal r -sparse set if T is maximal among sets that satisfy the following two properties.

- $T \subseteq S$
- T is r -sparse.

Note that an S -maximal r -sparse set is not necessarily a maximal r -sparse set. However, if T is an S -maximal r -sparse set, then T is also an S' -maximal r -sparse set, provided that $T \subseteq S' \subseteq S$.

Lemma 4.1. *If T is an S -maximal r -sparse set, where S is an s -net, then T is an $s + r$ -net.*

Proof. Let $x \in V(G)$. Then, we have $y \in S$ such that $d_G(x, y) \leq s$. By maximality, if $y \notin T$ then there exists $z \in T$ such that $d_G(y, z) \leq r$. Hence, $d_G(x, T) \leq r + s$. \square

Now let $\Gamma, \{\sigma_i\}_{i=1}^\infty, \{\Gamma_r\}_{r=1}^\infty$ be as above and for $r \geq 1$ let the graph G_r be defined in the following way.

- $V(G_r) = \Gamma$.
- $(p, q) \in E(G_r)$ if for some $1 \leq i \leq r$, $\sigma_i p = q$ or $\sigma_i q = p$.

We construct inductively sequence of positive integers

$$s_1 < r_1 < s_2 < r_2 < \dots ,$$

$$f(1) < f(2) < f(3) < \dots$$

and finite sets $\{F_m\}_{m=1}^\infty$ with distinguished elements $q_m \in F_m$ satisfying certain rules. In the following subsection we describe the rules.

4.1. The Rules. We choose the numbers $\{s_i\}_{i=1}^\infty, \{r_i\}_{i=1}^\infty, \{f(i)\}_{i=1}^\infty$ and $\{|F_m|\}_{m=1}^\infty$ in the following order.

$$s_1, f(1), r_1, |F_1|, s_2, f(2), r_2, |F_2|, s_3, \dots$$

Before stating the rules we need a lemma.

Lemma 4.2. *For any $T \geq 1$, there exists $n_T > 0$ such that if $n_T \leq n$, then there is an element $\gamma \in \Gamma_n$ so that $d_{G_n}(\gamma, e_\Gamma) \geq T$.*

Proof. Observe that for all $m \geq 1$, $|B_T(G_m, e_\Gamma)| \leq (3m)^T$. On the other hand, by our definition $|\Gamma_m| \geq 2^m$, hence our lemma follows. \square

Let $s_1 = 10$ and then define $f(1)$ in such a way that

$$10|B_{50}(G_{f(1)}, e_\Gamma)| < |\Gamma_{f(1)}|.$$

Note that if Γ_1 is infinite we can define $f(1)$ being equal to 1. Then, pick r_1 so that

$$|B_{\frac{1}{10}r_1}(G_{f(1)}, e_\Gamma)| \geq 10|B_{50}(G_{f(1)}, e_\Gamma)|.$$

Finally, choose $|F_1|$ in such a way that $|B_{r_1}(G_{f(1)}, e_\Gamma)| < |F_1|$.

Rule 1. Suppose that $s_1, f(1), r_1, |F_1|, s_2, f(2), \dots, |F_m|$ have already been chosen. Let $\kappa_m = |F_m|^{|\Gamma_{f(m)}, e_\Gamma|}$ be the number of labelings of the ball $B_{s_m}(\Gamma_{f(m)}, e_\Gamma)$ by the set F_m . Choose the number s_{m+1} so large (see Lemma 4.3 below) so that for any $n \geq n_{1000r_m\kappa_m}$ (see Lemma 4.2) and for any $2r_m$ -net $T \subset \Gamma_n$ one can pick a subset L_T satisfying the following three conditions.

- $|L_T| = \kappa_m$.
- For any $\delta \in L_T$,

$$\frac{1}{3}s_{m+1} \leq d_{G_n}(e_\Gamma, \delta) \leq \frac{2}{3}s_{m+1}.$$

- For any $\gamma \neq \delta \in L_T$,

$$d_{G_n}(\gamma, \delta) > 20r_m.$$

Lemma 4.3. *The number s_{m+1} can be chosen as it is required above.*

Proof. Let $s_{m+1} = 1000r_m\kappa_m$. Let $c_1 < c_2 < \dots < c_{\kappa_m}$ be integers such that for any $1 \leq i \leq \kappa_m$ we have $\frac{1}{3}s_{m+1} + 10r_m \leq c_i \leq \frac{2}{3}s_{m+1} - 10r_m$ and $|c_{i+1} - c_i| > 100r_m$. Now using Lemma 4.2, we pick elements $\{\gamma_i\}_{i=1}^{\kappa_m} \subset \Gamma$ such that $d_{G_n}(e_\Gamma, \gamma_i) = c_i$. For $1 \leq i \leq \kappa_m$, let $\delta_i \in T$ such that $d_{G_n}(\gamma_i, \delta_i) \leq 2r_m$. Then $L_T := \{\delta_i\}_{i=1}^{\kappa_m}$ satisfies the three conditions in Rule 1. \square

Rule 2. Suppose that the numbers

$s_1, f(1), r_1, |F_1|, s_2, f(2), r_2, \dots, s_{m+1}$ have already been chosen.

Choose $f(m+1)$ in such a way that $f(m+1) > f(m)$, $f(m+1) > n_{1000r_m\kappa_m}$ and

$$10^{m+1}|B_{5s_{m+1}}(G_{f(m+1)}, e_\Gamma)| < |\Gamma_{f(m+1)}|.$$

Then pick the number r_{m+1} so that both conditions below are satisfied.

- $r_{m+1} \geq 1000(\sum_{j=1}^{m+1} 5r_{j-1} + s_j)$
- $|B_{\frac{1}{10}r_{m+1}}(G_{f(m+1)}, e_\Gamma)| \geq 10^{m+1}|B_{5s_{m+1}}(G_{f(m+1)}, e_\Gamma)|$.

Rule 3. Suppose that the numbers $s_1, f(1), r_1, |F_1|, \dots, r_{m+1}$ have already been chosen. Then pick the number $|F_{m+1}|$ in such a way that

$$|B_{r_{m+1}}(G_{f(m+1)}, e_\Gamma)| < |F_{m+1}|.$$

In particular, one has a function $\varphi : \Gamma \rightarrow F_m$ such that if $0 < d_{G_{f(m)}}(x, y) \leq r_m$ then $\varphi(x) \neq \varphi(y)$.

Note that if the group Γ is torsion-free then for any $n \geq 1$, we can choose $f(n) = n$. On the other hand, if Γ is locally finite, then all the subgroups Γ_n are finite and we will need to choose all the values $f(n)$ as above.

4.2. Clean labelings. We assume that Γ is equipped with a labeling

$$\Theta = \Theta_{\mathcal{C}} \times \prod_{m=1}^{\infty} \Theta_m : \Gamma \rightarrow \mathcal{C} \times \prod_{m=1}^{\infty} F_m$$

satisfying the following conditions.

- (1) $(\Theta_1)^{-1}(q_1) \subset \Gamma$ is a maximal r_1 -sparse subset in the graph G_1 .
- (2) For any $m > 1$, $(\Theta_m)^{-1}(q_m) \subset \Gamma$ is a $(\Theta_{m-1})^{-1}(q_{m-1})$ -maximal r_m -sparse subset in the graph $G_{f(m)}$.
- (3) For any $m \geq 1$, $\Theta_m(x) \neq \Theta_m(y)$ whenever $0 < d_{G_{f(m)}}(x, y) \leq r_m$.
- (4) For any $m \geq 1$,

$$d_{G_{f(m)}}(e_\Gamma, (\Theta_m)^{-1}(q_m)) > 10s_m.$$

We call such a labeling a **clean labeling** of Γ . It is easy to see that such clean labelings exist using Rule 3. In Section 9 we will call a labeling **almost clean labeling** if it satisfies the first three conditions above. From now on we use \mathcal{F} as a shorthand for $\mathcal{C} \times \prod_{m=1}^{\infty} F_m$. Let $\Theta' = \Theta'_{\mathcal{C}} \times \prod_{m=1}^{\infty} \Theta'_m : \Gamma \rightarrow \mathcal{F}$ be a clean labeling and $j \geq 1$ be an integer. Let $\mathcal{F}_j = \{0, 1\}^{\{1, 2, \dots, j\}} \times \prod_{m=1}^j F_j$, let $z \in \Gamma$ and $t > 1$ be an integer. Then $B_t^{\Theta', j}(G_{f(j)}, z)$ is a \mathcal{F}_j -labeled $G_{f(j)}$ -ball around z of radius t , where the label of $y \in B_t(G_{f(j)}, z)$ is given as

$$\Theta'_{[j]}(y) = (\Theta'_{\mathcal{C}}(y))_j \times \prod_{m=1}^j \Theta'_m(y).$$

4.3. Patchings. Patching is an elementary construction that turns one clean labeling into another clean labeling using a third clean labeling. First, we define “regular n -patchings”. Suppose that two clean labelings $\Theta^a = \Theta_{\mathcal{C}}^a \times \prod_{m=1}^{\infty} \Theta_m^a : \Gamma \rightarrow \mathcal{F}$ and $\Theta^d = \Theta_{\mathcal{C}}^d \times \prod_{m=1}^{\infty} \Theta_m^d : \Gamma \rightarrow \mathcal{F}$ have already been given. Let $x \in \Gamma$ such that $\Theta_n^d(x) = q_n$. Consider the \mathcal{F}_n -labeled $G_{f(n)}$ -ball $B = B_{s_n+r_{n-1}}^{\Theta^d, n}(G_{f(n)}, x)$. The ball is the **patch** we wish to insert into Γ in order to turn Θ^a into the new clean labeling Θ^b . In the course of the paper we will refer to the change of labeling described in the next proposition as “patching the \mathcal{F}_n -labeled $G_{f(n)}$ -ball $B_{s_n+r_{n-1}}^{\Theta^d, n}(G_{f(n)}, x)$ into Θ^a around the vertex y .”

Proposition 4.1. *Let $y \in \Gamma$ such that $\Theta_n^a(y) = q_n$. Then, there exists a clean labeling $\Theta^b = \Theta_c^b \times \prod_{m=1}^{\infty} \Theta_m^b : \Gamma \rightarrow \mathcal{F}$ such that*

- For any $z \in \Gamma, (\Theta_c^b)_{n+} = (\Theta_c^a)_{n+}$, where for $c \in \{0, 1\}^{\mathbb{N}}$, $(c)_{n+}$ denotes the projection of c onto $\{0, 1\}^{\mathbb{N} \setminus \{1, 2, \dots, n\}}$.
- For any $z \in \Gamma$ such that $d_{G_{f(n)}}(y, z) \geq s_n + 4r_{n-1}$, we have $\Theta_c^b(z) = \Theta_c^a(z)$ (in particular, $\Theta_c^b(z) = \Theta_c^a(z)$, if z is not in the $\Gamma_{f(n)}$ -orbit of y).
- For any $z \in \Gamma$ such that $d_{G_{f(n)}}(y, z) \leq s_n + r_{n-1}$, we have $(\Theta_c^b(z))_n = (\Theta_c^d(z)y^{-1}x)_n$.
- For $m \geq n + 1$, we have $\Theta_m^b = \Theta_m^a$.
- For $m \leq n$ and $z \in \Gamma$ such that $d_{G_{f(n)}}(y, z) \geq s_n + 4r_{n-1}$, we have $\Theta_m^b(z) = \Theta_m^a(z)$ (again, $\Theta_m^b(z) = \Theta_m^a(z)$, if z is not in the $\Gamma_{f(n)}$ -orbit of y).
- For $m \leq n$ and $z \in \Gamma$ such that $d_{G_{f(n)}}(y, z) \leq s_n + r_{n-1}$, we have $\Theta_m^b(z) = \Theta_m^d(z)y^{-1}x$.

Proof. We need to define only $\Theta_m^b : \Gamma \rightarrow F_m$ for each $m \leq n - 1$. In the first round we will define $(\Theta_m^b)^{-1}(q_m)$ for each m , in the second (much easier) round we will define the remaining part of the maps Θ_m^b . For $1 \leq i \leq n - 1$, let

$$\kappa_i \subset B_{s_n+2r_{n-1}}(G_{f(n)}, y) \cup (\Gamma \setminus B_{s_n+3r_{n-1}}(G_{f(n)}, y))$$

be defined in the following way.

- If $d_{G_{f(n)}}(z, y) > s_n + 3r_{n-1}$ then let $z \in \kappa_i$ if and only if $\Theta_i^a(z) = q_i$.
- If $d_{G_{f(n)}}(z, y) \leq s_n + 2r_{n-1}$, then let $z \in \kappa_i$ if and only if $\Theta_i^d(z)y^{-1}x = q_i$.

We also define κ_0 as the empty set. One can observe that $\kappa_{n-1} \subset \kappa_{n-2} \subset \dots \subset \kappa_1$ and for any $1 \leq i \leq n - 1$, the set κ_i is r_i -sparse in $G_{f(i)}$. Let $\kappa_1 \subset \lambda_1$ be an arbitrary maximal r_1 -sparse set. Now, let $\kappa_2 \subset \lambda_2$ be a λ_1 -maximal r_2 -sparse set. Inductively, we construct the sets

$$\lambda_{n-1} \subset \lambda_{n-2} \subset \dots \subset \lambda_1$$

so that for any $1 \leq i \leq n - 1$ we have $\kappa_i \subset \lambda_i$ and λ_i is a λ_{i-1} -maximal r_i -sparse set.

Lemma 4.4. *Let $1 \leq i \leq n - 1$ and $z_i \in B_{s_n+r_{n-1}}(G_{f(n)}, y)$. Also, let $z_i \in \lambda_i$. Then $z_i \in \kappa_i$. Similarly, if $z_i \in \Gamma \setminus B_{s_n+4r_{n-1}}(G_{f(n)}, y)$ and $z_i \in \lambda_i$, then $z_i \in \kappa_i$.*

Proof. Suppose that $z_1 \in B_{s_n+r_{n-1}}(G_{f(n)}, y)$ such that $z_1 \in \lambda_1 \setminus \kappa_1$. Since $(\Theta_1^d)^{-1}(q_1)$ is a maximal r_1 -sparse set, there exists $w_1 \in (\Theta_1^d)^{-1}(q_1)$ such that $d_{G_{f(1)}}(z_1 y^{-1}x, w_1) \leq r_1$. That is, $d_{G_{f(1)}}(z_1, w_1 x^{-1}y) \leq r_1$ and $d_{G_{f(1)}}(y, w_1 x^{-1}y) \leq s_n + 2r_{n-1}$. Thus, $w_1 x^{-1}y \in \kappa_1 \subset \lambda_1$ leading to a contradiction. Suppose that our statement holds for all $1 \leq k \leq i$. Let $z_{i+1} \in B_{s_n+r_{n-1}}$ and $z_{i+1} \in \lambda_{i+1} \setminus \kappa_{i+1}$. By the induction argument, $z_{i+1} \in \kappa_i$. Since $(\Theta_{i+1}^d)^{-1}(q_{i+1})$ is a $(\Theta_i^d)^{-1}(q_i)$ -maximal r_{i+1} -sparse set, there exists $w_{i+1} \in$

$(\Theta_{i+1}^d)^{-1}(q_{i+1})$ such that $d_{G_{i+1}}(z_{i+1}y^{-1}x, w_{i+1}) \leq r_{i+1}$. That is, $d_{G_{i+1}}(z_{i+1}, w_{i+1}x^{-1}y) \leq r_{i+1}$ and $d_{G_{i+1}}(y, w_{i+1}x^{-1}y) \leq s_n + 2r_{n-1}$. Therefore, $w_{i+1}x^{-1}y \in \lambda_{i+1}$ in contradiction with the fact that λ_{i+1} is r_{i+1} -sparse. The proof of the second part can be done similarly. \square

Now, for $1 \leq i \leq n-1$ let

- $\Theta_i^b(z) = q_i$ if $z \in \lambda_i$.
- $\Theta_i^b(z) = \Theta_i^d(zy^{-1}x)$ if $z \in B_{s_n+r_{n-1}}(G_{f(n)}, y)$.
- $\Theta_i^b(z) = \Theta_i^a(z)$ if $z \in (\Gamma \setminus B_{s_n+4r_{n-1}}(G_{f(n)}, y))$.

Also, let $\Theta_n^b(z) = \Theta_n^d(zy^{-1}x)$ if $z \in B_{s_n+r_{n-1}}(G_{f(n)}, y)$ and $\Theta_n^b(z) = \Theta_n^a(z)$ if $z \in (\Gamma \setminus B_{s_n+4r_{n-1}}(G_{f(n)}, y))$. Then for $1 \leq i \leq n$, let us extend Θ_i^b onto the set

$$\{z \mid s_n + r_{n-1} < d_{G_{f(n)}}(y, z) \leq s_n + 4r_{n-1}\} \setminus (\Theta_i^b)^{-1}(q_i)$$

to obtain a clean labeling. Note that $1_\Gamma \notin B_{s_n+5r_{n-1}}(G_{f(n)}, y)$, hence by Rule 2 such extensions Θ_i^b clearly exist. \square

We need another kind of patching in our construction that we call ‘‘supersize n -patchings’’. Let $x \in \Gamma$ such that $\Theta_n^d(x) = q_n$. Consider the \mathcal{F}_n -labeled $G_{f(n)}$ -ball $B = B_{3r_n}^{\Theta_n^d}(G_{f(n)}, x)$. The following proposition is about how to insert B into Θ^a . Notice that in the case of regular patchings the ball B we use as a patch contained only one element of $(\Theta_n^d)^{-1}(q_n)$. In the case of supersize patchings the ball B contains many elements of $(\Theta_n^d)^{-1}(q_n)$. Nevertheless, B will not contain any element of the set $(\Theta_{n+1}^d)^{-1}(q_{n+1})$. The next proposition is the ‘‘supersized’’ version of Proposition 4.1.

Proposition 4.2. *Let $y \in \Gamma$ such that $\Theta_n^a(y) = q_n$ and $d_G(y, (\Theta_{n+1}^a)^{-1}(q_{n+1})) > 20r_n$. Then, there exists a clean labeling $\Theta^b = \Theta_C^b \times \prod_{m=1}^\infty \Theta_m^b : \Gamma \rightarrow \mathcal{F}$ such that*

- For any $z \in \Gamma, (\Theta_C^b)_{n+} = (\Theta_C^a)_{n+}$.
- For any $z \in \Gamma$ such that $d_G(y, z) \geq 7r_n$, we have $\Theta_C^b(z) = \Theta_C^a(z)$.
- For any $z \in \Gamma$ such that $d_G(y, z) \leq 3r_n$, we have $(\Theta_C^b(z))_n = (\Theta_C^d(zy^{-1}x))_n$.
- For $m \geq n+1$, let $\Theta_m^b = \Theta_m^a$.
- For $m \leq n$ and $z \in \Gamma$ such that $d_{G_{f(n)}}(y, z) \geq 7r_n$, we have $\Theta_m^b(z) = \Theta_m^a(z)$.
- For $m \leq n$ and $z \in \Gamma$ such that $d_{G_{f(n)}}(y, z) \leq 3r_n$, we have $\Theta_m^b(z) = \Theta_m^d(zy^{-1}x)$.

Proof. First we construct sets $\lambda_n \subset \lambda_{n-1} \subset \dots \subset \lambda_1$, where $\kappa_i \subset \lambda_i$ for any $1 \leq i \leq n$, λ_1 is a maximal r_1 -sparse set and if $i > 1$, then λ_i is a λ_{i-1} -maximal r_i -sparse set. Then we can proceed in the same way as in the proof of Proposition 4.1. \square

5. MINIMAL CANTOR LABELINGS II. (CODEBALLS)

Let $\Theta' = \Theta'_C \times \prod_{m=1}^{\infty} \Theta'_m : \Gamma \rightarrow \mathcal{F}$ and $\Theta'' = \Theta''_C \times \prod_{m=1}^{\infty} \Theta''_m : \Gamma \rightarrow \mathcal{F}$ be clean labelings. Let $\rho > 0$ be an integer. We say that the \mathcal{F}_j -labeled $G_{f(j)}$ -ball $B_t^{\Theta',j}(G_{f(j)}, z)$ is (ρ, j) -**syndetic** in Θ'' if for any $g \in \Gamma$ there exists $h \in \Gamma$ such that

- $d_{G_{f(j)}}(g, h) \leq \rho$.
- $B_t^{\Theta',j}(G_{f(j)}, z)$ is isomorphic (as \mathcal{F}_j -labeled $G_{f(j)}$ -balls) to $B_t^{\Theta'',j}(G_{f(j)}, h)$,
 $B_t^{\Theta',j}(G_{f(j)}, z) \cong B_t^{\Theta'',j}(G_{f(j)}, h)$, that is for any $y \in B_t(G_{f(j)}, z)$ we have that

$$\Theta'_{[j]}(y) = \Theta''_{[j]}(yz^{-1}h).$$

Definition 5.1. Let Θ' and Θ'' be labelings as above. We say that the \mathcal{F}_j -labeled $G_{f(j)}$ -ball $B_l^{\Theta'',j}(G_{f(j)}, x)$ **fully contains** an isomorphic copy of the $G_{f(j)}$ -ball $B_{l'}^{\Theta',j}(G_{f(j)}, x')$ if there exists $w \in \Gamma$, $d_{G_{f(j)}}(x, w) + l' < l$ such that $B_{l'}^{\Theta',j}(G_{f(j)}, x')$ is isomorphic to $B_{l'}^{\Theta'',j}(G_{f(j)}, w)$ as \mathcal{F}_j -labeled $G_{f(j)}$ -balls.

We will start with a clean labeling Θ and inductively construct a sequence of clean labelings $\{\Theta^i\}_{i=1}^{\infty}$ such that $\Theta^1 = \Theta$ and for any $z \in \Gamma$ there exists an integer $i_z > 0$ such that if $i \geq i_z$ then $\Theta^i(z) = \Theta^{i_z}(z)$. Hence, for each $z \in \Gamma$, $\lim_{i \rightarrow \infty} \Theta^i(z) = \Theta^\infty(z)$ exists. The clean labeling $\Theta^\infty : \Gamma \rightarrow \mathcal{F}$ will generate a free, minimal subshift in the Bernoulli shift space \mathcal{F}^Γ .

The construction of Θ^2 . Our input is the clean labeling $\Theta = \Theta^1$. Pick any $z_2 \in \Gamma$ such that $\Theta_2^1(z_2) = q_2$. First we modify the labeling Θ^1 on the $G_{f(2)}$ -ball $B_{s_2}(G_{f(2)}, z_2)$. Let $\mathcal{A}^1 = \{A_i^1\}_{i=1}^{\tau_1}$ be the set of all \mathcal{F}_1 -labeled $G_{f(1)}$ -balls of radius s_1 in Θ^1 up to isomorphism. Let $\{w_i^1\}_{i=1}^{\tau_1} \subset B_{s_2}(G_{f(2)}, z_2) \cap (\Theta_1^1)^{-1}(q_1)$ be a subset of vertices such that

- For any $1 \leq i \leq \tau_1$,

$$\frac{1}{3}s_2 \leq d_{G_{f(2)}}(z_2, w_i^1) \leq \frac{2}{3}s_2.$$

- For any $1 \leq i \neq j \leq \tau_1$,

$$d_{G_{f(2)}}(w_i^1, w_j^1) > 20r_1.$$

Then for each $1 \leq i \leq \tau_1$ pick $z_i^1 \in (\Theta_1^1)^{-1}(q_1)$ such that $B_{2r_1}^{\Theta^1,1}(G_{f(1)}, z_i^1)$ fully contains an isomorphic copy of the $G_{f(1)}$ -ball A_i^1 . Then using Proposition 4.2, for each $1 \leq i \leq \tau_1$ let us simultaneously patch the balls $B_{3r_1}^{\Theta^1,1}(G_{f(1)}, z_i^1)$ into Θ^1 around w_i^1 . Since $(\Theta_1^1)^{-1}(q_1)$ is a maximal r_1 -sparse set such a family of vertices $\{z_i^1\}_{i=1}^{\tau_1}$ indeed exists. Hence, we obtain a modified clean labeling $\hat{\Theta}^2 : \Gamma \rightarrow \mathcal{F}$. We call the \mathcal{F}_2 -labeled $G_{f(2)}$ -ball $B_2 = B_{s_2}^{\hat{\Theta}^2,2}(G_{f(2)}, z_2)$ the **2-codeball**. We finish the construction of Θ^2 by patching

the ball $B_{s_2+r_1}^{\hat{\Theta}^2,2}(G_{f(2)}, z_2)$ simultaneously into Θ^1 centered around all vertices $y \in (\Theta_2^1)^{-1}(q_2)$, using Proposition 4.1. Since $(\Theta_1^1)^{-1}(q_1)$ is an r_1 -net and $(\Theta_2^1)^{-1}(q_2)$ is a $(\Theta_1^1)^{-1}(q_1)$ -maximal r_2 -sparse net, by Lemma 4.1 each ball $A_j^1 \in \mathcal{A}^1$ is $(2r_2, 2)$ -syndetic in Θ^2 . Note that the 2-codeballs are the analogues of the 2-welcome words of [11]. We will see that for any $i \geq 1$ and $x \in (\Theta_2^i)^{-1}(q_2)$,

$$B_{s_2}^{\Theta^i,2}(G_{f(2)}, x) \cong B_2.$$

Therefore, all balls $A_j^1 \in \mathcal{A}^1$ remain $(2r_2, 2)$ -syndetic in the process. It is important to note that it is possible that the set of \mathcal{F}_1 -labeled $G_{f(1)}$ -balls of radius r_1 will increase when we turn the clean labeling Θ^1 into Θ^2 .

The construction of Θ^3 . Our input is now the clean labeling Θ^2 . Pick any $z_3 \in \Gamma$ such that $\Theta_3^2(z_3) = q_3$. We modify the labeling Θ^2 on the ball $B_{s_3}(G_{f(3)}, z_3)$. Let $\mathcal{A}^2 = \{A_i^2\}_{i=1}^{\tau_2}$ be the set of all \mathcal{F}_2 -labeled $G_{f(2)}$ -balls of radius s_2 in Θ^2 up to isomorphism. Let $\{w_i^2\}_{i=1}^{\tau_2} \subset B_{s_3}(G_{f(3)}, z_3) \cap (\Theta_2^2)^{-1}(q_2)$ be a subset of vertices such that:

- For any $1 \leq i \leq \tau_2$,

$$\frac{1}{3}s_3 \leq d_{G_{f(3)}}(z_3, w_i^2) \leq \frac{2}{3}s_3.$$

- For any $1 \leq i \neq j \leq \tau_2$,

$$d_{G_{f(3)}}(w_i^2, w_j^2) > 20r_2.$$

Then for each $1 \leq i \leq \tau_2$ pick $z_i^2 \in (\Theta_2^2)^{-1}(q_2)$ such that $B_{2r_2}^{\Theta^2,2}(G_{f(2)}, z_i^2)$ fully contains an isomorphic copy of the ball A_i^2 . Then using Proposition 4.2, for each $1 \leq i \leq \tau_2$ we simultaneously patch the balls $B_{3r_2}^{\Theta^2,2}(G_{f(2)}, z_i^2)$ into Θ^2 around w_i^2 . Hence, we obtain a modified clean labeling $\hat{\Theta}^3 : \Gamma \rightarrow \mathcal{F}$. So far, the construction of the 3-codeball was identical to the one of B_2 . Now, we need some further considerations. It is possible that after the patching we will have some $g \in (\hat{\Theta}^3)^{-1}(q_2)$ such that the ball $B_{s_2}^{\hat{\Theta}^3,2}(G_{f(2)}, g)$ is not isomorphic to the codeball B_2 . Using Proposition 4.1, we patch the ball $B_{s_2+r_1}^{\Theta^2,2}(G_{f(2)}, z_2)$ into $\hat{\Theta}^3$ around all such elements g as above to obtain the clean labeling $\hat{\Theta}_1^3$. We call the \mathcal{F}_3 -labeled $G_{f(3)}$ -ball $B_3 = B_{s_3}^{\hat{\Theta}_1^3,3}(G_{f(3)}, z_3)$ the **3-codeball**. Now we construct the clean labeling $\bar{\Theta}^3$ by patching the ball $B_{s_3+r_2}^{\hat{\Theta}_1^3,3}(G_{f(3)}, z_3)$ into Θ^2 around all vertices $y \in (\Theta_3^2)^{-1}(q_3)$. Again, due to the patchings, it is possible that for some $g \in (\bar{\Theta}^3)^{-1}(q_2)$, $B_{s_2}^{\bar{\Theta}^3,2}(G_{f(2)}, g)$ is not isomorphic to the 2-codeball B_2 . We obtain the clean labeling Θ^3 by patching the ball $B_{s_2+r_1}^{\Theta^2,2}(G_{f(2)}, z_2)$ into $\bar{\Theta}^3$ around the vertices g above. Then

- For all $z \in (\Theta_3^3)^{-1}(q_3)$ we have $B_{s_3}^{\Theta^3,3}(G_{f(3)}, z) \cong B_3$.
- For all $z \in (\Theta_2^3)^{-1}(q_2)$ we have $B_{s_2}^{\Theta^3,2}(G_{f(2)}, z) \cong B_2$.

6. MINIMAL CANTOR LABELINGS III. (THE INDUCTION)

Let us suppose that we have already constructed the clean labelings $\Theta^1, \Theta^2, \dots, \Theta^n$ satisfying the following properties.

- If $j \leq k \leq n$ and $(\Theta_j^k)(z) = q_j$, then the \mathcal{F}_j -labeled $G_{f(j)}$ -ball $B_{s_j}^{\Theta_j^k, j}(G_{f(j)}, z)$ is isomorphic to a given \mathcal{F}_j -labeled $G_{f(j)}$ -ball B_j , the **j-codeball**.
- For $1 < j \leq n$, let $\mathcal{A}^{j-1} = \{A_i^{j-1}\}_{i=1}^{\tau_{j-1}}$ be the set of all \mathcal{F}_{j-1} -labeled $G_{f(j-1)}$ -balls of radius s_{j-1} in Θ^{j-1} up to isomorphism. Then for any $1 \leq i \leq \tau_{j-1}$, the j -codeball B_j fully contains a copy of A_i^{j-1} .

Now we construct the clean labeling Θ^{n+1} in two rounds.

Round one: The $n+1$ -codeball. Pick $z_{n+1} \in \Gamma$ such that $\Theta_{n+1}^n(z_{n+1}) = q_{n+1}$. We modify the labeling Θ^n on the ball $B_{s_{n+1}}(G_{f(n+1)}, z_{n+1})$. Let $\mathcal{A}^n = \{A_i^n\}_{i=1}^{\tau_n}$ be the set of all \mathcal{F}_n -labeled $G_{f(n)}$ -balls of radius s_n in Θ^n up to isomorphism. Let $\{w_i^n\}_{i=1}^{\tau_n} \subset B_{s_{n+1}}(G_{f(n+1)}, z_{n+1}) \cap (\Theta^n)^{-1}(q_n)$ be a subset of vertices such that

- For any $1 \leq i \leq \tau_n$,

$$\frac{1}{3}s_{n+1} \leq d_{G_{f(n+1)}}(z_{n+1}, w_i^n) \leq \frac{2}{3}s_{n+1}.$$

- For any $1 \leq i \neq j \leq \tau_n$,

$$d_{G_{f(n+1)}}(w_i^n, w_j^n) > 20r_n.$$

By Rule 1, such system indeed exists. Now, for each $1 \leq i \leq \tau_n$ pick $z_i^n \in (\Theta^n)^{-1}(q_n)$ such that $B_{2r_n}^{\Theta^n, n}(G_{f(n)}, z_i^n)$ fully contains an isomorphic copy of the ball A_i^n . Finally, using Proposition 4.2, for each $1 \leq i \leq \tau_n$ we simultaneously patch the balls $B_{3r_n}^{\Theta^n, n}(G_{f(n)}, z_i^n)$ into Θ^n around w_i^n . Hence, we obtain a modified clean labeling $\hat{\Theta}^{n+1} : \Gamma \rightarrow \mathcal{F}$. Now we need a definition to describe our procedure. Let $\Theta' : \Gamma \rightarrow \mathcal{F}$ be a clean labeling and let $2 \leq i \leq n$. Let $\text{Bad}_i(\Theta')$ be the set of all $y \in \Gamma$ such that $\Theta'_i(y) = q_i$ and $B_{s_i}^{\Theta', i}(G_{f(i)}, y)$ is not isomorphic to the i -codeball B_i . The **i -repair** of Θ' , $\Theta'_{\langle i \rangle}$ is constructed in the following way. Using Proposition 4.1, we simultaneously patch the \mathcal{F}_i -labeled $G_{f(i)}$ -ball $B_{s_i+r_{i-1}}^{\Theta', i}(G_{f(i)}, z_i)$ into Θ' around all elements $y \in \text{Bad}_i(\Theta')$. Here, $z_i \in \Gamma$ is an element for which $B_{s_i}^{\Theta', i}(G_{f(i)}, z_i)$ is isomorphic to the i -codeball B_i . Clearly, $\text{Bad}_i(\Theta'_{\langle i \rangle})$ is empty. Note however, that if $\text{Bad}_j(\Theta')$ is empty for some $j < i$, it is possible that $\text{Bad}_j(\Theta'_{\langle i \rangle})$ is non-empty. Now, we construct inductively the clean labelings $\{\hat{\Theta}^{n+1, i}\}_{i=2}^{n+1}$. Let $\hat{\Theta}^{n+1, n+1} = \hat{\Theta}^{n+1}$ and for $2 \leq i \leq n$, let $\hat{\Theta}^{n+1, i} = \hat{\Theta}_{\langle i \rangle}^{n+1, i+1}$.

Definition 6.1. Let $2 \leq i \leq n$. Let $i > j_l > j_{l-1} > \dots > j_1 \geq 2$ be integers and for $1 \leq k \leq l$ let $g_k \in \Gamma$ such that if $k < l$ then the set

$$B_{s_{j_k}+5r_{j_k-1}}(G_{f(j_k)}, g_k) \cap B_{s_{j_{k+1}}+5r_{j_{k+1}-1}}(G_{f(j_{k+1})}, g_{k+1})$$

is non-empty. Then, the set $\mathcal{L} = \cup_{k=1}^l B_{s_{j_k}+5r_{j_k-1}}(G_{f(j_k)}, g_k)$ is called an i -chain of length l .

Lemma 6.1. *The $G_{f(i)}$ -diameter of an i -chain \mathcal{L} is less than $\frac{1}{10}r_{i-1}$.*

Proof. We have that

$$\text{diam}_{G_{f(i)}}(\mathcal{L}) \leq \sum_{k=1}^l \text{diam}_{G_{f(i)}}(B_{s_{j_k}+5r_{j_k-1}}(G_{f(j_k)}, g_k)),$$

hence the lemma follows from Rule 2. \square

Proposition 6.1. *For any $2 \leq i \leq n$, $\text{Bad}_i(\hat{\Theta}^{n+1,2})$ is empty.*

Proof. Let $\hat{\Theta}_i^{n+1,i}(w) = q_i$. It is enough to prove that $B_{s_i}^{\hat{\Theta}^{n+1,2},i}(G_{f(i)}, w) \cong B_i$. Suppose that the ball $B_{s_i}^{\hat{\Theta}^{n+1,2},i}(G_{f(i)}, w)$ is not isomorphic to the i -codeball B_i . Since $B_{s_i}^{\hat{\Theta}^{n+1,i},i}(G_{f(i)}, w) \cong B_i$, there exists $h \in \Gamma$ and $j < i$ such that

- $h \in \text{Bad}_{s_j}(\hat{\Theta}^{n+1,j+1})$
- $B_{s_j+5r_{j-1}}(G_{f(j)}, h) \cap B_{s_i}(G_{f(i)}, w) \neq \emptyset$.

Let l be the largest integer such that we have an i -chain \mathcal{L} of length l satisfying the following properties.

- $\mathcal{L} \cap B_{s_i}(G_{f(i)}, w) \neq \emptyset$.
- $g_k \in \text{Bad}_{s_{j_k}}(\hat{\Theta}^{n+1,j_k+1})$, where

$$\mathcal{L} = \cup_{k=1}^l B_{s_{j_k}+5r_{j_k-1}}(G_{f(j_k)}, g_k).$$

So, by our previous observation we have that $l \geq 1$.

Lemma 6.2. *The set*

$$B_{s_{j_l}+5r_{j_l-1}}(G_{f(j_l)}, g_l) \cap (\Gamma \setminus B_{s_i+\frac{1}{2}r_{i-1}}(G_{f(i)}, w))$$

is nonempty.

Proof. If $B_{s_{j_l}+5r_{j_l-1}}(G_{f(j_l)}, g_l)$ is contained in the ball $B_{s_i+\frac{1}{2}r_{i-1}}(G_{f(i)}, w)$, then $B_{s_{j_l}}^{\hat{\Theta}^{n+1,i},j_l}(G_{f(j_l)}, g_l)$ is isomorphic to the j_l -codeball B_{j_l} . Thus, there is some $j_l < k < i$ and $g \in \Gamma$ such that $g \in \text{Bad}_k(\hat{\Theta}^{n+1,k+1})$ and the ball $B_{s_k+5r_{k-1}}(G_{f(k)}, g)$ intersects the ball $B_{s_{j_l}+5r_{j_l-1}}(G_{f(j_l)}, g_l)$, in contradiction with the maximality of l . Hence the lemma follows. \square

By Lemma 6.1, $\text{diam}_{G_{f(i)}}(\mathcal{L}) \leq \frac{1}{10}r_{i-1}$. Therefore, it is not possible that both $\mathcal{L} \cap B_{s_i}(G_{f(i)}, w) \neq \emptyset$ and $\mathcal{L} \cap (\Gamma \setminus B_{s_i+\frac{1}{2}r_{i-1}}(G_{f(i)}, w)) \neq \emptyset$ hold. Therefore, our proposition follows. \square

We call the \mathcal{F}_{n+1} -labeled $G_{f(n+1)}$ -ball $B_{n+1} = B_{s_{n+1}}^{\hat{\Theta}^{n+1,2},n+1}(G_{f(n+1)}, z_{n+1})$ the $n+1$ -**codeball**. Now we construct the clean labeling $\bar{\Theta}^{n+1}$ by patching the ball $B_{s_{n+1}+r_n}^{\hat{\Theta}^{n+1,2},n+1}(G_{f(n+1)}, z_{n+1})$ into Θ^n around all the vertices $y \in (\Theta_{n+1}^n)^{-1}(q_{n+1})$.

Round two: The construction of Θ^{n+1} . Again, it is possible that for some $2 \leq i \leq n$ the set $\text{Bad}_i(\overline{\Theta}^{n+1})$ is non-empty. Now, we inductively construct the clean labelings $\{\overline{\Theta}^{n+1,i}\}_{i=2}^{n+1}$. Let $\overline{\Theta}^{n+1,n+1} = \overline{\Theta}^{n+1}$ and for $2 \leq i \leq n$, let $\overline{\Theta}^{n+1,i} = \overline{\Theta}_{\langle i \rangle}^{n+1,i+1}$. Copying the proof of Proposition 6.1, we can prove the following proposition.

Proposition 6.2. *For any $2 \leq i \leq n$, $\text{Bad}_i(\overline{\Theta}^{n+1,2})$ is empty.*

Let $\Theta^{n+1} = \overline{\Theta}^{n+1,2}$. Let us summarize what we have already proved about the clean labeling Θ^{n+1} .

Proposition 6.3.

- For all $1 < j \leq n+1$ and $z \in (\Theta_j^{\Theta^{n+1}})^{-1}(q_j)$ we have $B_{s_j}^{\Theta^{n+1},j}(G_{f(j)}, z) \cong B_j$.
- For any $1 < j \leq n+1$ and $1 \leq i \leq \tau_{j-1}$, the j -codeball B_j fully contains a copy of the ball A_i^{j-1} .

Finally, we have the following lemma.

Lemma 6.3. *Let $z \in \Gamma$ such that $\Theta^{n+1}(z) \neq \Theta^n(z)$. Then $d_{G_{f(n+1)}}(z, e_\Gamma) \geq 5s_{n+1}$.*

Proof. By our construction if $\Theta^{n+1}(z) \neq \Theta^n(z)$ then there exists some $y \in \Gamma$ such that

- $\Theta_{n+1}^n(y) = q_{n+1}$ and there is an l -chain \mathcal{L} for some $l \leq n$, intersecting $B_{2s_{n+1}}(G_{f(n+1)}, y)$ for which $z \in B_{2s_{n+1}}(G_{f(n+1)}, y) \cup \mathcal{L}$.

Therefore, $z \in B_{5s_{n+1}}(G_{f(n+1)}, y)$, hence the lemma follows from the definition of the clean labelings. \square

7. EXPLICIT CONSTRUCTION OF A FREE, MINIMAL ACTION

Let $\{\Theta^n\}_{n=1}^\infty$ be the clean labelings constructed in the previous sections. By Lemma 6.3, for any $y \in \Gamma$ there exists $n_y > 0$ such that if $n \geq n_y$, then $\Theta^n(y) = \Theta^{n_y}(y)$. Hence, $\lim_{n \rightarrow \infty} \Theta^n(y) = \Theta^\infty(y)$ is a well-defined clean labeling.

Lemma 7.1. *For any $i \geq 2$ and $z \in (\Theta_i^\infty)^{-1}(q_i)$, $B_{s_i}^{\Theta^\infty,i}(G_{f(i)}, z) \cong B_i$.*

Proof. Let $n \geq i$ such that $\Theta^\infty(y) = \Theta^n(y)$ for all $y \in B_{s_i}(G_{f(i)}, z)$. Then, $(\Theta_i^n)(z) = q_i$. By Proposition 6.3, $B_{s_i}^{\Theta^n,i}(G_{f(i)}, z) \cong B_i$. Hence our lemma follows. \square

Lemma 7.2. *Let $z \in \Gamma$, and $t, j \geq 1$ be integers. Then, there exists some $\rho > 0$ such that the \mathcal{F}_j -labeled ball $B_t^{\Theta^\infty,j}(G_{f(j)}, z)$ is (ρ, j) -syndetic in Θ^∞ .*

Proof. Let $q > j$ be an integer such that the \mathcal{F}_j -labeled ball $B_t^{\Theta^m, j}(G_{f(j)}, z)$ is isomorphic to $B_t^{\Theta^\infty, j}(G_{f(j)}, z)$, whenever $m \geq q$. Let $n > q$ be an integer such that $s_n > t$. Then the $n + 1$ -codeball B_{n+1} contains a copy of the \mathcal{F}_n -labeled $G_{f(n)}$ -ball $B_{s_n}^{\Theta^n, n}(G_{f(n)}, z)$. In particular, B_{n+1} contains a copy of the ball $B_t^{\Theta^n, j}(G_{f(j)}, z) \cong B_t^{\Theta^\infty, j}(G_{f(j)}, z)$. The set $(\Theta_1^\infty)^{-1}(q_1)$ is a maximal r_1 -sparse set, hence an r_1 -net. The set $(\Theta_2^\infty)^{-1}(q_2)$ is a $(\Theta_1^\infty)^{-1}(q_1)$ -maximal r_2 -sparse set, hence by Lemma 4.1 an $r_2 + r_1$ -net. Inductively, $(\Theta_{n+1}^\infty)^{-1}(q_{n+1})$ is a $\sum_{i=1}^{n+1} r_i$ -net, thus a $2r_{n+1}$ -net. So, for every $x \in \Gamma$ the ball $B_{3r_{n+1}}^{\Theta^\infty, n+1}(G_{f(n+1)}, x)$ contains a copy of $B_t^{\Theta^\infty, j}(G_{f(j)}, z)$. That is, the \mathcal{F}_j -labeled $G_{f(j)}$ -ball $B_t^{\Theta^\infty, j}(G_{f(j)}, z)$ is $(3r_{n+1}, j)$ -syndetic in Θ^∞ . \square

Theorem 5. *The orbit closure of Θ^∞ in the Bernoulli shift space \mathcal{F}^Γ is free and minimal.*

By Lemma 7.2, the orbit closure of Θ^∞ is minimal. Since Θ^∞ is a clean labeling, it is proper in the sense that for any $r > 0$ there exists some $s_r > 0$ such that if $0 < d_{G_r}(x, y) \leq r$, then

$$\Theta_{[s_r]}^\infty(x) \neq \Theta_{[s_r]}^\infty(y).$$

Therefore, by repeating the argument of the proof of Proposition 1.1, we immediately obtain that the action of Γ on the orbit closure of Θ^∞ is free. \square

8. THE DYNAMICAL VERSION OF CLEAN LABELINGS

In this section we prove some propositions and lemmas that we need for the proof of Theorem 2. For the whole section let the integers $\{s_i\}_{i=1}^\infty$, $\{r_i\}_{i=1}^\infty$ and the finite sets $\{F_m\}_{m=1}^\infty$ be as in the previous sections. Our first proposition is about the existence of the dynamical analogue of clean labelings.

Proposition 8.1. *Let $\alpha : \Gamma \curvearrowright \mathcal{C}$ be a free continuous action as in the previous sections. Then there exists a continuous map*

$$\Sigma = \Sigma_{\mathcal{C}} \times \prod_{m=1}^{\infty} \Sigma_m : \mathcal{C} \rightarrow \mathcal{F}$$

*such that for any $x \in \mathcal{C}$, $\Sigma_x(\gamma) := \Sigma(\alpha(\gamma)(x))$ defines an almost clean labeling of the group Γ (see Section 4). We call Σ a **dynamical clean labeling**.*

Proof. Before starting the proof let us fix some definitions and notations. As in Section 2, we will consider the Borel graphs \mathcal{G}_r on the Cantor set \mathcal{C} associated to the free action α . If U is a clopen subset of \mathcal{C} , then we define a subset of our countable group Γ by

$$O_x^U := \{\gamma \in \Gamma \mid \alpha(\gamma)(x) \in U\}.$$

Let $t > 0$ be an integer. We call a clopen set $U \subset \mathcal{C}$ **t -separated** in the Borel graph \mathcal{G}_r if for any $x \neq y \in U$, $d_{\mathcal{G}_r}(x, y) > t$. Clearly, U is t -separated if and only if for any $x \in \mathcal{C}$ the set O_x^U is t -sparse in the graph G_r .

Lemma 8.1. *Let $U \subset \mathcal{C}$ be a clopen set and $t > 0$ be an integer. Then there exists a clopen set $V \subset \mathcal{C}$ such that for any $x \in \mathcal{C}$, the set O_x^V is an O_x^U -maximal t -sparse set in the graph G_r .*

Proof. By continuity and freeness, there exists some $s > 0$ such that if for some $x, y \in \mathcal{C}$ we have $0 < d_{G_r}(x, y) \leq t$, then $(x)_s \neq (y)_s$. Let a_1, a_2, \dots, a_{2^s} be an enumeration of the set $\{0, 1\}^{\{1, 2, \dots, s\}}$. For $1 \leq i \leq 2^s$, let

$$W_i := \{x \in \mathcal{C} \mid (x)_s = a_i\}.$$

Now let $V_1 := U \cap W_1$ and

$$Z_1 = \cup_{d_{G_r}(\gamma, e_\Gamma) \leq t} \alpha(\gamma) V_1.$$

Clearly, Z_1 is a clopen set. Let Z_1^c denote the complement of Z_1 . Now, let $V_2 := W_2 \cap U \cap Z_1^c$. Then for any $x \in \mathcal{C}$, $O_x^{V_1 \cup V_2}$ is a t -sparse set in G_r contained in the set O_x^U . Inductively, suppose that for $1 \leq i \leq n < 2^s$,

- $V_i \subset W_i \cap U$ is a clopen set,
- $Z_i = \cup_{d_{G_r}(\gamma, e_\Gamma) \leq t} \alpha(\gamma) (\cup_{j=1}^i V_j)$,
- $V_{i+1} = W_{i+1} \cap U \cap Z_i^c$.

Then for any $1 \leq i \leq n$, $O_x^{\cup_{j=1}^i V_j}$ is a t -sparse set in the graph G_r which is contained in the set O_x^U . We define

$$V_{n+1} := W_{n+1} \cap U \cap Z_n^c.$$

Then for any $x \in \mathcal{C}$, $O_x^{\cup_{j=1}^{n+1} V_j} \subset O_x^U$ is a t -sparse set in the graph G_r . We claim that for any $x \in \mathcal{C}$, $O_x^{\cup_{j=1}^{2^s} V_j}$ is an O_x^U -maximal t -sparse set in G_r . Clearly, $O_x^{\cup_{j=1}^{2^s} V_j} \subset O_x^U$ and $O_x^{\cup_{j=1}^{2^s} V_j}$ is a t -sparse set in the graph G_r . Suppose that $O_x^{\cup_{j=1}^{2^s} V_j}$ is not an O_x^U -maximal t -sparse set in the graph G_r . Then there exists $\delta \in \Gamma_r$ such that $\alpha(\delta)(x) \in U$ and

$$(1) \quad d_{G_r}(\delta, O_x^{\cup_{j=1}^{2^s} V_j}) > t.$$

Let $(\alpha(\delta)(x))_s = a_i$. Then by (1),

$$\alpha(\delta)(x) \in W_i \cap U \cap Z_{i-1}^c = V_i$$

in contradiction with our assumption. □

Lemma 8.2. *Let V be an r_m -separated clopen set in the Borel graph $\mathcal{G}_{f(m)}$. Then there exists a continuous function $\varphi : \mathcal{C} \rightarrow F_m$ such that*

- $V = \varphi^{-1}(q_m)$.
- For any $q \in F_m$, the clopen set $\varphi^{-1}(q)$ is r_m -separated in the Borel graph $\mathcal{G}_{f(m)}$.

Proof. Let $V_1 := V$. Using Lemma 8.1, we pick V_2 in such a way that for any $x \in \mathcal{C}$, $O_x^{V_2}$ is an $O_x^{V_1}$ -maximal r_m -sparse set in the graph $G_{f(2)}$. Inductively, suppose that the clopen sets V_1, V_2, \dots, V_i have already been picked. Then,

let $V_{i+1} \subset \mathcal{C}$ be a clopen set such that for any $x \in \mathcal{C}$, $O_x^{V_{i+1}}$ is an $O_x^{(\cup_{j=1}^i V_j)^c}$ -maximal r_m -sparse set in the graph $G_{f(m)}$. It is enough to show that $\cup_{i=1}^{|F_m|} V_i = \mathcal{C}$. Suppose that $x \in \mathcal{C} \setminus \cup_{i=1}^{|F_m|} V_i$. By Rule 3, there exists $1 \leq i \leq |F_m|$, such that for any γ for which $d_{G_{f(m)}}(\gamma, e_\Gamma) \leq r_m$ holds, we have that $\alpha(\gamma)(x) \notin V_i$. However, this is in contradiction with the definition of the set V_i . \square

Now we finish the proof of our proposition. First, using Lemmas 8.1 and 8.2 we construct a continuous map $\Sigma_1 : \mathcal{C} \rightarrow F_1$ such that for any $q \in F_1$ the set $(\Sigma_1)^{-1}(q)$ is r_1 -separated in the Borel graph $\mathcal{G}_{f(1)}$. Then, using the lemmas we construct $\Sigma_2 : \mathcal{C} \rightarrow F_2$ such that for any $q \in F_2$, $(\Sigma_2)^{-1}(q)$ is r_2 -separated in $\mathcal{G}_{f(2)}$ and for any $x \in \mathcal{C}$, $O_x^{(\Sigma_2)^{-1}(q_2)}$ is a $(\Sigma_1)^{-1}(q_1)$ -maximal r_2 -sparse set in the graph $G_{f(2)}$. Inductively, we construct Σ_m for all $m \geq 1$. \square

Remark: The inductive part in the proof of the previous proposition is the reason that in the definition of clean labelings we required $(\Theta_m)^{-1}(q_m)$ to be a $(\Theta_{m-1})^{-1}(q_{m-1})$ -maximal r_m -sparse subset instead of being a maximal r_m -sparse subset that is contained in $(\Theta_{m-1})^{-1}(q_{m-1})$.

The following two lemmas are straightforward consequences of the definitions.

Lemma 8.3. *Let*

$$\Sigma' = \Sigma'_\mathcal{C} \times \prod_{m=1}^{\infty} \Sigma'_m : \mathcal{C} \rightarrow \mathcal{F}$$

and

$$\Theta' = \Theta'_\mathcal{C} \times \prod_{m=1}^{\infty} \Theta'_m : \mathcal{C} \rightarrow \mathcal{F}$$

be continuous functions. Let $U \subset \mathcal{C}$ be a $2t$ -separated clopen set in the Borel graph \mathcal{G}_r . Also, let $B_t^{\Theta, i}(G_r, e_\Gamma)$ be a \mathcal{F}_i -labeled ball of radius t in the graph G_r . Let

$$\Sigma'' = \Sigma''_\mathcal{C} \times \prod_{m=1}^{\infty} \Sigma''_m : \mathcal{C} \rightarrow \mathcal{C} \times \prod_{m=1}^{\infty} F_m$$

be defined in the following way.

- If $d_{\mathcal{G}_r}(x, U) > t$, then $\Sigma''(x) = \Sigma'(x)$.
- If there exists $y \in U$ and $\gamma \in \Gamma$, $d_{G_r}(\gamma, e_\Gamma) \leq t$ such that $\alpha(\gamma)(y) = x$, then let $\Sigma''_{[i]}(x) = \Theta'_{[i]}(\gamma)$, $(\Sigma''_\mathcal{C}(x))_{i+} = (\Sigma'_\mathcal{C}(x))_{i+}$ and for $j > i$, let $\Sigma''_j(x) = \Sigma'_j(x)$.

Then Σ'' is a continuous map as well.

Lemma 8.4. *Let $V \subset \mathcal{C}$ be a t -separated clopen set. Let $s < t$ and suppose that for some $\varepsilon > 0$ we have*

$$|B_s(G_r, e_\Gamma)| \leq \varepsilon |B_t(G_r, e_\Gamma)|.$$

Let $U \subset \mathcal{C}$ be the set of elements x such that there exists $y \in V$ so that $d_{\mathcal{G}_r}(x, y) \leq s$. Then, for any Borel probability measure μ on \mathcal{C} that is invariant under the action α ,

$$\mu(U) < \varepsilon.$$

9. THE PROOF OF THEOREM 2

Let $\zeta : \Gamma \curvearrowright \mathcal{C}$ be a free continuous action. Let $\{\Sigma^i : \mathcal{C} \rightarrow \mathcal{F}\}_{i=1}^\infty$ be a sequence of dynamical clean labelings. We call $y \in \mathcal{C}$ stable (with respect to the system $\{\Sigma^i\}_{i=1}^\infty$) if for any element $x \in \mathcal{C}$ in the Γ -orbit of y , there exists $n_x > 0$ such that if $n \geq n_x$ then $\Sigma^n(x) = \Sigma^{n_x}(x)$. So, for the Γ -invariant subset of stable points y , $\lim_{n \rightarrow \infty} \Sigma^n(y) = \Sigma^\infty(y)$ exists.

Proposition 9.1. *There exists a sequence of dynamical clean labelings $\{\Sigma^i\}_{i=1}^\infty$ such that*

- For any Borel probability measure μ that is invariant under the action ζ , the μ -measure of the stable points is 1.
- There exists a free and minimal Bernoulli subshift $M \subset \mathcal{F}^\Gamma$ such that for each stable point $y \in \mathcal{C}$,

$$\gamma \rightarrow \Sigma^\infty(\zeta(\gamma)(y))$$

defines an injective equivariant Borel map from the Borel set of stable points to M .

Proof. We proceed very similarly as we did in the construction of Θ^∞ . The following lemma is an immediate corollary of Lemma 8.3.

Lemma 9.1. *Let*

$$\Sigma^a : \Sigma_{\mathcal{C}}^a \times \prod_{m=1}^{\infty} \Sigma_m^a : \mathcal{C} \rightarrow \mathcal{F}$$

be a dynamical clean labeling and

$$\Theta^d : \Theta_{\mathcal{C}}^d \times \prod_{m=1}^{\infty} \Theta_m^d : \Gamma \rightarrow \mathcal{F}$$

be an almost clean labeling. Let $\Theta_n^d(x) = q_n$. Let $V \subset \mathcal{C}$ be an r_n -separated clopen subset. For all $y \in V$ we patch the ball $B_{s_n+r_{n-1}}^{\Theta^d, n}(G_{f(n)}, x)$ into Σ_y^a around y . Then we obtain a new dynamical clean labeling Σ^b .

The construction of Σ^2 . Let Σ^1 be a dynamical clean labeling such that for $x = (x_1, x_2, x_3, \dots) \in \mathcal{C}$, $\Sigma_{\mathcal{C}}^1(x) = (y_1, y_2, y_3, \dots)$, where for $n \geq 0$, $y_{2^n} = x_1$ and if $2^n < a < 2^{n+1}$, then $y_a = x_{a-2^n+1}$. Let $x \in \mathcal{C}$ and $\Theta^1 := \Sigma_x^1$. Let $\hat{\mathcal{A}}^1 = \{\hat{A}_i^1\}_{i=1}^{\hat{r}_1}$ be the set of all \mathcal{F}_1 -labeled $G_{f(1)}$ -balls of radius s_1 in the almost clean labelings $\{\Sigma_w^1\}_{w \in \mathcal{C}}$. Note that $\hat{\mathcal{A}}^1$ might contain balls that are not in Θ^1 . Let $z_2 \in (\Theta_2^1)^{-1}(q_2)$. Proceeding in the same way as in Section 5 (using $\hat{\mathcal{A}}^1$ instead of \mathcal{A}^1), we construct the modified labeling $\hat{\Theta}^2 : \Gamma \rightarrow \mathcal{F}$. Now,

our new 2-codeball will be $\hat{B}_2 = B_{s_2}^{\hat{\Theta}^2, 2}(G_{f(2)}, z_2)$. Then, we patch the ball $B_{s_2+r_1}^{\hat{\Theta}^2, 2}(G_{f(2)}, z_2)$ into Σ^1 around all vertices $y \in (\Sigma^1)^{-1}(q_2)$ to obtain (Lemma 9.1) the dynamical clean labeling Σ^2 .

The induction. Let us suppose that we have already constructed the dynamical clean labelings $\Sigma^1, \Sigma^2, \dots, \Sigma^n$ satisfying the following properties.

- If $j \leq k \leq n$, $x \in \mathcal{C}$, $\Theta^k = \Sigma_x^k$ and $(\Theta_j^k)(z) = q_j$, then the \mathcal{F}_j -labeled $G_{f(j)}$ -ball $B_{s_j}^{\Theta_j^k, j}(G_{f(j)}, z)$ is isomorphic to a given \mathcal{F}_j -labeled $G_{f(j)}$ -ball \hat{B}_j , our new j -codeball.
- For $1 < j \leq n$, let $\hat{\mathcal{A}}^{j-1} = \{\hat{A}_i^{j-1}\}_{i=1}^{\hat{\tau}_{j-1}}$ be the set of all \mathcal{F}_{j-1} -labeled $G_{f(j-1)}$ -balls of radius s_{j-1} in the almost clean labelings $\{\Sigma_w^{j-1}\}_{w \in \mathcal{C}}$ up to isomorphism. Then for any $1 \leq i \leq \hat{\tau}_{j-1}$, the j -codeball \hat{B}_j fully contains a copy of \hat{A}_i^{j-1} .

Again, we construct the dynamical clean labeling Σ^{n+1} in two rounds.

Round one: The new $n+1$ -codeball. Let $x \in \mathcal{C}$ and $\Theta^n = \Sigma_x^n$. Also, let $z_{n+1} \in \Gamma$ such that $\Theta_{n+1}^n(z_{n+1}) = q_{n+1}$. Let $\hat{\mathcal{A}}^n = \{\hat{A}_i^n\}_{i=1}^{\hat{\tau}_n}$ be the set of all \mathcal{F}_n -labeled $G_{f(n)}$ -balls of radius s_n in the almost clean labelings $\{\Sigma_w^n\}_{w \in \mathcal{C}}$ up to isomorphism. Now we construct the almost clean labeling $\hat{\Theta}^{n+1, 2} : \Gamma \rightarrow \mathcal{F}$ exactly the same way as in Section 6. The \mathcal{F}_{n+1} -labeled $G_{f(n+1)}$ -ball $\hat{B}_{n+1} = B_{s_{n+1}}^{\hat{\Theta}^{n+1, 2}, n+1}(G_{f(n+1)}, z_{n+1})$ will be the new $n+1$ -codeball. Finally, we construct the dynamical clean labeling $\bar{\Sigma}^{n+1}$ by patching the ball $B_{s_{n+1}+r_n}^{\hat{\Theta}^{n+1, 2}, n+1}(G_{f(n+1)}, z_{n+1})$ into Σ^n around all the vertices $y \in (\Sigma_{n+1}^n)^{-1}(q_{n+1})$.

Round two: The construction of Σ^{n+1} . For a dynamical clean labeling Σ' , the set $\text{Bad}_i(\Sigma')$ is clopen so by Lemma 8.3 we can construct the dynamical clean labelings $\bar{\Sigma}^{n+1, n+1}$, $\bar{\Sigma}^{n+1, i} = \bar{\Sigma}_{\langle i \rangle}^{n+1, i+1}$ in the same way as in Section 6. Let $\Sigma^{n+1} = \bar{\Sigma}^{n+1, 2}$ and let $Y \subset \mathcal{C}$ be the set of stable points with respect to the family of dynamical clean labelings $\{\Sigma^i\}_{i=1}^\infty$. Then $\Sigma^\infty(y) = \lim_{n \rightarrow \infty} \Sigma^n(y)$ is a well-defined Borel map on the invariant Borel set Y . By Theorem 5, for any $y \in Y$, the orbit closure of Θ_y^∞ in the Bernoulli shift space \mathcal{F}^Γ is free and minimal. By our construction of the codeballs, if $y_1, y_2 \in Y$, then for any $i \geq 1$, the set of \mathcal{F}_i -labeled $G_{f(i)}$ -balls in $\Theta_{y_1}^\infty$ and $\Theta_{y_2}^\infty$ coincide. Hence the orbit closures of $\Theta_{y_1}^\infty$ and $\Theta_{y_2}^\infty$ coincide as well. So, let M be the orbit closure of Θ_y^∞ in \mathcal{F}^Γ , where $y \in Y$. Notice that the map $\Phi : Y \rightarrow M$ defined by $\Phi(y)(\gamma) = \Sigma^\infty(\zeta(\gamma)(y))$ is injective. Indeed, if $y_1 \neq y_2 \in Y$, then $\Sigma_{\mathcal{C}}^1(y_1)$ and $\Sigma_{\mathcal{C}}^1(y_2)$ differ from each other in infinitely many digits. Hence, $\Sigma^\infty(y_1) \neq \Sigma^\infty(y_2)$. We finish the proof of our proposition by showing that for any Borel probability measure μ invariant under the action ζ , $\mu(Y) = 1$. Let $Q_n \subset \mathcal{C}$ be the set of elements w such that $\Sigma^{n+1}(w) \neq \Sigma^n(w)$. By the

Borel-Cantelli Lemma, it is enough to prove that

$$(2) \quad \sum_{n=1}^{\infty} \mu(Q_n) < \infty.$$

Let $z \in Q_n$ and let $V = (\Sigma_{n+1}^n)^{-1}(q_{n+1})$. As we observed in the proof of Lemma 6.3, there exists $y \in V$ and $\gamma \in B_{5s_{n+1}}(G_{f(n+1)}, e_\Gamma)$ such that $\zeta(\gamma)(y) = z$. Note that V is a $\frac{1}{10}r_{n+1}$ -separated clopen set and by Rule 2,

$$|B_{\frac{1}{10}r_{n+1}}(G_{f(n+1)}, e_\Gamma)| \geq 10^{n+1} |B_{5s_{n+1}}(G_{f(n+1)}, e_\Gamma)|.$$

Thus by Lemma 8.4 we have that $\mu(Q_n) < 10^{-(n+1)}$, therefore (2) holds. This finishes the proof of our proposition. \square

Now we can finish the proof of Theorem 2. Let $\zeta : \Gamma \curvearrowright \mathcal{C}$ be the universal action of Theorem 3. Let $\alpha : \Gamma \curvearrowright M$ be the Bernoulli subshift and $\{\Sigma^i\}_{i=1}^{\infty}$ be the sequence of dynamical clean labelings in Proposition 9.1. Let $\beta : \Gamma \curvearrowright X$ be a Borel action, let ν be a Borel probability measure on X invariant under β , and finally let $\Psi_\beta : X \rightarrow \mathcal{C}$ be the equivariant map in Theorem 3. Then $(\Psi_\beta)_*(\nu)$ is a Borel probability measure invariant under the action ζ . Let $Y \subset \mathcal{C}$ be the set of stable points with respect to the system $\{\Sigma^i\}_{i=1}^{\infty}$. Then, $\Phi(y)(\gamma) = \Sigma^\infty(\zeta(\gamma)(y))$ defines an injective, equivariant Borel map $\Phi : Y \rightarrow M$. Let $X' = (\Psi_\beta)^{-1}(Y)$. Then by Proposition 9.1, $\nu(X') = 1$ and $\Phi \circ \Psi_\beta : X' \rightarrow M$ is an injective, equivariant Borel map. Hence, our theorem follows. \square

Finally, we prove Corollary 1.1. Let $\beta : \Gamma \curvearrowright \mathcal{C}$ be the action in Theorem 2. First let us suppose that Γ is amenable. Let $\alpha : \Gamma \curvearrowright (X, \mu)$ be an ergodic action and $\Phi_\alpha(X') \rightarrow \mathcal{C}$ be an equivariant injective map. Then the Kolmogorov-Sinai entropies of α and $(\Phi_\alpha)_*(\mu)$ coincide. Also, $(\Phi_\alpha)_*(\mu)$ is ergodic. Since for any $0 < c \leq \infty$ we have an ergodic action with Kolmogorov-Sinai entropy c , our corollary follows for amenable groups. Now let us suppose that Γ is non-amenable. Again, we need to prove that there exist more than one non-isomorphic essentially free ergodic actions of Γ . However, by the famous theorem of Epstein every non-amenable group has uncountably many pairwise orbit-inequivalent actions [4] so our corollary follows for non-amenable groups as well.

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DEPARTMENT OF MATHEMATICS AND STATISTICS, FYLDE COLLEGE, LANCASTER UNIVERSITY, LANCASTER, LA1 4YF, UNITED KINGDOM

E-mail address: g.elek@lancaster.ac.uk