The rigidity of countable frameworks in normed spaces

Sean Dewar

Department of Mathematics and Statistics

Lancaster University

This dissertation is submitted for the degree of

Doctor of Philosophy

October 2019

To my family, friends, and all who helped me get to this point.

Declaration

Hereby I declare that the present thesis was prepared by me and none of its contents was obtained by means that are against the law.

I also declare that the present thesis is a part of a PhD Programme at Lancaster University. The thesis has never before been a subject of any procedure of obtaining an academic degree.

Sean Dewar

October 2019

Acknowledgements

I've got a lot of awesome people to thank, so buckle up.

I'd first like to thank my supervisor Derek Kitson for introducing me to the wonderful world of rigidity. You taught me to calmly take a step back and look at a problem from all the angles, and not just rush in head first like I am prone to. Your advice and teachings have been priceless.

I'd like to also thank Tony Nixon, Bernd Schulze, Stephen Power, John Hewetson, Lefty Kastis, Viktória Kaszanitzky and Hattie Serocold of the Lancaster geometric rigidity research group. I always felt included, and you really can't ask for more than that.

The Lancaster Mathematics department as a whole has been brilliant, and to list you all would add another hundred pages to an already rambling thesis, but I would like to especially thank Yemon Choi, David Pauksztello and Julia Tawn for always being up for a chat.

I'd like to thank my fellow PhD students at Lancaster for all the help they have given me over the years. The opportunity to speak at the Post Graduate Forum has been a great boon to my public speaking, though I would like to apologise for anyone who witnessed a couple of the dud talks I have given over the years. I would like to especially thank Konrad Królicki for being the other half of our double act, and Chris Menez for being my mathematical senpai through out all the seven years we attended Lancaster University together.

Big shout out to all my friends from school and college who I still bother to this day; Roberts, Dan, TC, Tosh, Holt, Scott and Chris. You are some of the weirdest people I have ever met and I wouldn't have it any other way.

Without sounding like I am bragging (even though I most certainly am), I have had too many good friends over all my years at Lancaster University to name you all. You have all helped me grow from the shy introverted fresher to what I am today (so I guess you can all collectively take the blame for that). I would like to give special thanks to all the good friends I met through LURACS, keep up the good pubbing!

At the risk of getting too emotional, I would like to give a special acknowledgement to Kirstie Ferriday. For the last few years you have always been there to calm me when there was some hiccup along the road, and I cherish you dearly for it. I feel like a better person when I am with you, and I hope you feel the same.

Lastly, I'd like to thank my family; Mum, Dad, my brother Ben, Nan, Fred, Grandma, Grandad (I wish you could have been here for this), and all the rest of you. You have all been a solid rock for me, even if what I do sounds like a load of nonsense to all of you. I hope to one day repay all the help I have received off you, though I know that what you have given me is priceless.

Abstract

We present a rigorous study of framework rigidity in finite dimensional normed spaces using a wide array of tools to attack these problems, including differential and discrete geometry, matroid theory, convex analysis and graph theory. We shall first focus on giving a good grounding of the area of rigidity theory from a more general view point to allow us to deal with a variety of normed spaces. By observing orbits of placements from the perspective of Lie group actions on smooth manifolds, we obtain upper bounds for the dimension of the space of trivial motions for a framework.

Utilising aspects of differential geometry, we prove an extension of Asimow and Roth's 1978/9 result establishing the equivalence of local, continuous and infinitesimal rigidity for regular bar-and-joint frameworks in a d-dimensional Euclidean space. Further, we establish the independence of all graphs with $d+1$ vertices d-dimensional normed space, and also prove they will be flexible if the normed space is non-Euclidean.

Next, we prove that a graph has an infinitesimally rigid placement in a non-Euclidean normed plane if and only if it contains a $(2,2)$-tight spanning subgraph. The method uses an inductive construction based on generalised Henneberg moves and the geometric properties of the normed plane. As a key step, rigid placements are constructed for the complete graph K_4 by considering smoothness and strict convexity properties of the unit ball.

Finally, we carry our previous results to countably infinite frameworks where this is possible, and otherwise identify when such results cannot be brought forward. We first establish matroidal methods for identifying rigidity and flexibility, and apply these methods to a large class of normed spaces. We characterise a necessary and

sufficient condition for countably infinite graphs to have sequentially infinitesimally rigid placements in a general normed plane, and further stengthen the result for a large class normed planes. Finally, we prove that infinitesimal rigidity for countably infinite generic frameworks implies a weaker (but possibly equivalent) form of continuous rigidity, and infinitesimal rigidity for countably infinite algebraically generic frameworks implies continuous rigidity.

Table of contents

Introduction

A friend once put forth the following as the fundamental principal in structural engineering: if it moves, you have failed. While this is a gross oversimplification that ignores how structures are actually constructed to move so as to avoid breaking, the spirit of the idea is essentially correct; if we wish to make suitable buildings we should consider structures that are *rigid*, not those that are *flexible*. From a mathematical point of view, it is natural to first assess simplified models and use what we gleam from them to make predictions of what is rigid and what is flexible.

For what follows, we inform the reader that all specific terminology introduced shall be properly defined later in the text. We instead will give an informal definitions so as to give the reader a more intuitive look at the background and history of the areas studied.

The most simple abstract structure one can observe from this point of view is a *bar-joint framework*; a set of unbending, infinitesimally thin bars connected by joints that allow full range of motions, even those that allow bars to intersect each other. This can mathematically be modelled as a pair (G, p), where G is a simple finite graph and p - the *placement of* G - is a map from the vertices of G to the Euclidean space the framework sits in; the edges of G represent the bars of the framework, while the vertices of G represent the joints. This can be reversed; if we have a graph G and a placement p of G in some Euclidean space we can define a framework (G, p) in the said Euclidean space. With our method of modelling of bar-joint frameworks, we may now define a framework to be *rigid* if every continuous motion of the vertices that preserves the edge lengths corresponds to a rigid motion of the framework, i.e. some combination rotational and translational motion; otherwise, we shall define (G, p) to be *flexible*.

While research had been undertaken into rigid structures going as far back as Cauchy [12], the study of the rigidity of bar-joint frameworks from a more combinatorial point of view began with J. C. Maxwell. In 1864 he presented what is now known as the *Maxwell counting rule*: for a bar-joint framework with j joints in a 3-dimensional Euclidean space to be rigid we need at least

$$b \geq 3j - 6,$$

where b is the number of bars [49]. As a hand-wavy explanation, the count exists due to three things:

(i) Without any bars, a framework has $3j$ degrees of freedom, since each joint has 3 degrees of freedom.

(ii) We cannot remove the 6 degrees of freedom an object in 3-dimensional Euclidean space enjoys by any amount of bars.

(iii) Each bar will, usually, remove a degree of freedom from the framework, as it adds a single constraint between two joints, thus to remove all but 6 degrees of freedom we will need at least $3j - 6$ bars.

We can now see that the "3" is due to our framework sitting in a 3-dimensional space, and the "6" is as any object in 3-dimensional Euclidean space will have 6 degrees of freedom. If a bar-joint framework has any more bars than are required to obtain rigidity then it is *over-constrained*, since the extra bars must, in some sense, be adding nothing extra to the rigidity of the structure. As we wish to avoid over-constraining

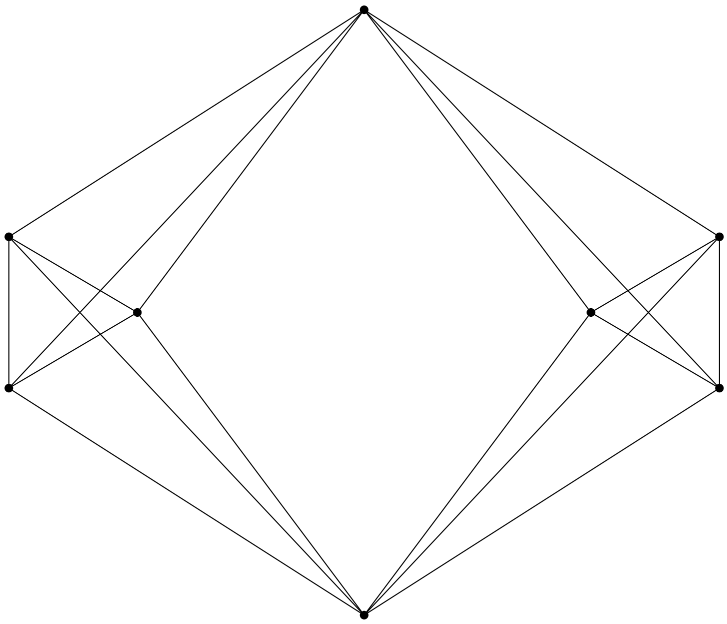

Fig. 1 The double banana framework

any part of the framework, we note as a corollary that for any part of the framework on $j' \geq 3$ joints with b' bars, we wish to have

$$b' \leq 3j' - 6.$$

Any rigid framework where Maxwell's counting rule also holds (i.e. is not over-constrained) is known as an *isostatic framework*. The Maxwell counting rule, however, is only a necessary condition, not a sufficient one; we direct the reader to Figure 1 for the *double-banana framework*, where Maxwell's counting rule holds but the framework is not rigid.

By noting that objects in the Euclidean plane have 3 degrees of freedom (2 translational + 1 rotational), Maxwell's counting rule needs to be changed to suit bar-joint frameworks (with j joints and b bars) in the Euclidean plane with the rule

$$b = 2j - 3,$$

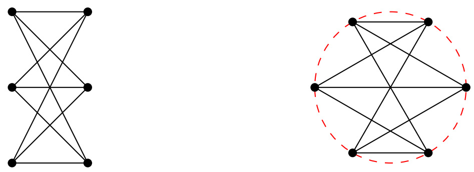

Fig. 2 A rigid placement of the bipartite graph $K_{3,3}$ (left) and flexible placement of same graph with vertices lying on a circle [10, Theorem 14] (right).

where for any part of the framework on $j' \geq 2$ joints with b' bars,

$$b' \leq 2j' - 3.$$

Although Maxwell's counting rule for the plane can also fail if our framework is placed in some special position (see Figure 2), unlike with 3-dimensional space, no counter-examples to Maxwell's counting rule could be found for "suitably generic" frameworks, raising the immediate question: is Maxwell's counting rule also sufficient for almost all bar-joint frameworks in the Euclidean plane? In her 1927 paper [58], H. Pollaczek-Geiringer proved that this was indeed true, but unfortunately her result was lost to the mists of time until recently. Her result showed that for "suitably generic" frameworks the placement of the framework's vertices could be forgotten, a landmark result.

H. Pollaczek-Geiringer's result was later rediscovered in 1970 by G. Laman [42] who - utilising an earlier result of L. Henneberg [27] - proved we can construct isostatic frameworks from a single edge by using *Henneberg moves*; the *(2-dimensional) 0-extension*, where we add a vertex and connect it to two distinct vertices, and the *(2-dimensional) 1-extension*, where we delete an edge and then add a vertex connected to the ends of the deleted edge and one other vertex (see Figure 3). The strength of such a characterisation lies in the fact that we can quickly check by eye whether a framework in the Euclidean plane is rigid.

Fig. 3 A 0-extension (left) and a 1-extension (right).

Fig. 4 Two continuously and locally rigid placements of the complete graph K_3 in the Euclidean plane. While the placement on the left is infinitesimally rigid, the placement on the right is infinitesimally flexible. The last bar of the right framework has been drawn curved so that the reader may view it.

There is, however, an issue with much of the commentary above, which is the vagueness of their definitions of rigidity. The type of rigidity they wish to prove - the definition we originally gave - is *continuous rigidity*, however there are many similar definitions that we could use for rigidity. Other possibilities we could have chosen include *infinitesimal rigidity* (whether the framework can be deformed under infinitesimal motions), or *local rigidity* (whether any other sufficiently close placement with equivalent edge lengths must also be congruent). For many of the above results, no distinction would be made between *finite rigidity* (continuous or local rigidity) and infinitesimal rigidity, since while the former was property that was desired, the latter only requires calculating the rank of a matrix. Although both properties are often equivalent, infinitesimal rigidity can often be seen to be a strictly stronger property, see Figure 4.

This ambiguity was finally solved by L. Asimow and B. Roth in 1978/79 [5] [6]. They proved that infinitesimal rigidity is a strictly stronger property than finite rigidity, and all types of rigidity are equivalent for almost all placements of a given framework; further, it also established that if a framework had a single infinitesimally rigid placement then almost all placements would be infinitesimally rigid. It follows from this and H. Pollaczek-Geiringer's result that we may define a graph to be *rigid* if it has a infinitesimally rigid placement and *flexible* otherwise.

In recent years the area of rigidity theory has expanded dramatically. For recent research we refer the reader to [30] [66] for *global rigidity* (whether a framework defined by certain edge lengths is unique up to isometry) and the closely linked *redundant rigidity* (whether a framework will remain rigid after the removal of any edge, often closely linked to global rigidity), [22] for *universal rigidity* (whether a framework in a given space defined by certain edge lengths is unique up to isometry in any higher dimensional space), [64] [31] for frameworks with symmetry, [33] [28] for infinitesimal rigidity concerning alternative types of frameworks and [54] for frameworks on surfaces.

The focus of this thesis shall be two specific areas.

The first shall be considering bar-joint frameworks in a *(finite dimensional real) normed space*, a (finite dimensional real) linear space X with a *norm*, function $\|\cdot\| : X \to \mathbb{R}$ such that the following holds:

(i) $\|x\| \geq 0$ for all $x \in X$ with equality if and only if $x = 0$,

(ii) $\|\lambda x\| = |\lambda|\|x\|$ for all $x \in X$ and $\lambda \in \mathbb{R}$,

(iii) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$.

Normed spaces (also referred to as *Minkowski spaces*) have been studied as far back as B. Riemann in 1868 [59] with the mention of the ℓ_4-norm, however as the name suggests, the study of them did not truly begin until H. Minkowski in 1894 [50]. An equivalent definition for a normed space is to first define a compact, centrally symmetric, convex set $B \subset X$ with non-empty interior and then define the map

$$\|\cdot\| : X \to \mathbb{R}, \ x \mapsto \inf \{\lambda > 0 : x \in \lambda B\}.$$

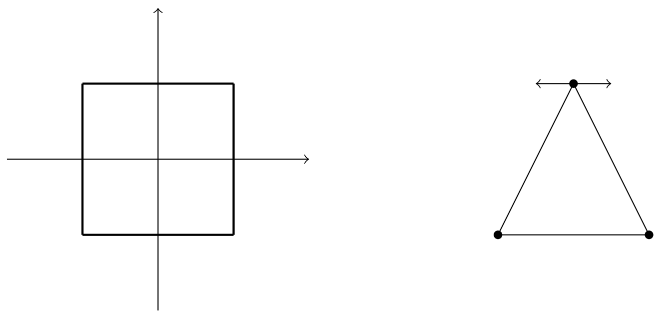

Fig. 5 (Left) Unit ball of a 2-dimensional normed space X; (right) a flexible framework in X.

By considering each bar to be a distance constraint between points, the latter definition of a normed space hints at how the geometry of the unit ball will effect whether a framework is rigid or not; see Figure 5. Recently research has been undertaken into framework rigidity in a variety of normed spaces, in particular spaces with ℓ_p norms ($p \in [1, \infty]$) [36], polyhedral norms [34] and matrix norms such as the Schatten p-norms [35]. We also recommend [48] [67] to the reader if they wish for more background and history on the topic of finite dimensional normed spaces.

The second area we shall focus on shall be considering countably infinite frameworks, i.e. frameworks with a countable set of vertices. Many of the results we have previously described no longer hold for infinite frameworks for a multitude of reasons; the Maxwell counting rule no longer applies as both b and j would be required to be infinite, and local rigidity now has a multitude of non-equivalent definitions. Even the property that infinitesimal rigidity implies finite rigidity fails, as can be seen by Figure 6. Due to all of these reasons, much work has been undertaken into this area, usually with the assumption that the framework has periodic symmetry [9] [56] [32]. A recent paper by D. Kitson and S. C. Power [38] instead deals with countably infinite frameworks with no such assumptions being made, and much of the inspiration of our work will stem from their research. Although study has been undertaken into frameworks with

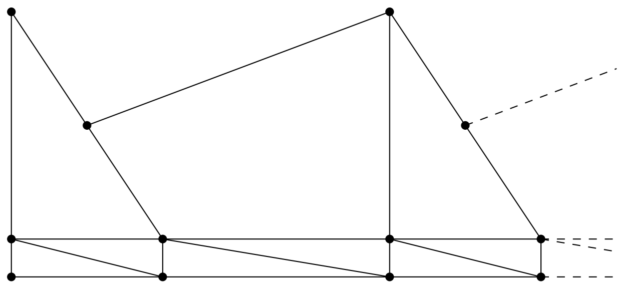

Fig. 6 An infinitesimally rigid framework in the Euclidean plane that is not continuously rigid, see [37, Example 6.4] for more details.

an uncountable set of vertices (for example [4]), we shall restrict ourselves just to countable frameworks.

The goals we wish to achieve in this thesis are the following:

(i) Developing precise definitions for infinitesimal, continuous and local rigidity for countable frameworks in normed spaces, and deciding when they are equivalent.

(ii) Identifying combinatorial methods to decide whether a given countable graph has an infinitesimally rigid placement in a given normed space, especially in the case of 2-dimensional normed spaces.

The former deals with the more geometric side of framework rigidity, and we refer the reader to the results Theorem 2.1.5, Theorem 4.4.1, Theorem 4.4.6 and Theorem 4.4.12 for some of our main results. The latter is more in keeping with the combinatorial nature of framework rigidity, and we refer the reader to Theorem 3.4.2, Theorem 4.1.20, Theorem 4.3.12 for our main results.

The thesis shall be set out as follows.

In Chapter 1, we shall set out all the required background material needed for rigidity theory. We shall generalise everything so as to apply in any normed space and any size framework unless stated otherwise. After introducing all of the required normed space geometry we shall give precise definitions of frameworks and placements, and present a rigorous study of the orbit and the trivial motion space of a set of points.

We will give an upper bound for the dimension of the space of trivial motions which will be achievable by most placements. We then define the varying types of rigidity for bar-joint frameworks, give some immediate necessary conditions for these to hold, and cover some of the more classical results for finite frameworks in Euclidean spaces.

In Chapter 2, we shall prove an extension of L. Asimow and B. Roth's 1978/9 result via differential geometry methods. We then establish that all graphs with $d+1$ vertices are independent in any d-dimensional normed space, and also flexible if the normed space is non-Euclidean. We finish by defining the graph substitution operation for all normed spaces, and detail for what normed spaces it preserves rigidity.

In Chapter 3, we shall extend H. Pollaczek-Geiringer's result to non-Euclidean normed planes. To prove this we employ a similar method to G. Laman, however, we require two additional graph operations: *vertex splitting* (see Section 3.3.3) and *vertex-to-K_4 extensions* (see Section 3.3.4). These graph operations were originally applied in the context of infinitesimal rigidity in [68] and [53] respectively. We first are required to prove that the complete graph on 4 vertices is rigid in all normed planes. To do this we shall split into three cases dependent on whether the normed plane X is *smooth* or *strictly convex* (see Section 1.1.3). The cases will be:

(i) X is not strictly convex,

(ii) X is strictly convex but not smooth,

(iii) X is both strictly convex and smooth.

For the first case we will construct an infinitesimally rigid placement of K_4 that takes advantage of the lack of strict convexity. In the second case we shall construct a sequence of placements p^n of K_4 and show that (K_4, p^n) will be infinitesimally rigid for large enough n. In the last case we shall use methods utilised in [16] to prove the existence of an infinitesimally rigid placement of K_4. We finish the chapter by giving some sufficient connectivity conditions for graph rigidity analogous to those given by Lovász & Yemini for the Euclidean plane in [46].

In Chapter 4, we will be extending the theory introduced in previous chapters to countably infinite frameworks and graphs. We first will outline the background for infinite frameworks and towers of frameworks, and also set up a matroidal structure for infinite frameworks that will be a vital tool in later sections. We then will discuss *generic spaces*, a class of normed space that is ideal for observing the combinatorial rigidity properties of countably infinite graphs. Afterwards, we shall extend H. Pollaczek-Geiringer's result to countably infinite graphs, and give a stronger classification for *generic placements* in generic spaces (see Section 4.2 for a full definition). Finally, we shall discuss the limitations of using infinitesimal rigidity to observe continuous rigidity, and what results we can gleam from it.

We shall finish with outlining possible further avenues of research in Chapter 5, which we hope will be of interest in future.

Chapter 1

Introduction to geometric rigidity theory in normed spaces

1.1 Normed space geometry

For a topological space X and set S, we shall denote S° to be the *interior* of S, $\overline{S}$ to be the *closure* of S and ∂S to be the *boundary* of S.

All normed spaces $(X, \|\cdot\|)$ shall be assumed to be over $\mathbb{R}$ and finite dimensional; it follows that all normed spaces will be complete and all norms on a given linear space will generate the same topology which we shall refer to as the *norm topology*. We shall denote a normed space by X when there is no ambiguity. For any normed space X we shall use the notation $B_r(x)$, $B_r[x]$ and $S_r[x]$ for the open ball, closed ball and the sphere with centre x and radius $r > 0$ respectively. If $\dim X = 2$ we shall refer to X as a *normed plane*.

Given normed spaces X, Y we shall denote by $L(X, Y)$ the normed space of all linear maps from X to Y with the operator norm $\|\cdot\|_{\text{op}}$ and $A(X, Y)$ to be space of all affine maps from X to Y with the norm topology. If $X = Y$ we shall abbreviate to $L(X)$ and $A(X)$. We denote by ι the identity map on X.

We define $X^* := L(X, \mathbb{R})$ to be the *dual space* of X, and refer to the operator norm as $\| \cdot \|$ when there is no ambiguity. We shall also define $B^*_r(f)$, $B^*_r[f]$ and $S^*_r[f]$ for the open ball, closed ball and the sphere of X^* with centre f and radius $r > 0$ respectively.

Given an affine map $h : X \to Y$ we can define its *linear component* to be the map $H : x \mapsto h(x) - h(0)$. If $X = Y$ it is immediate that h is invertible if and only if H is invertible. We denote $GL(X) \subset L(X)$ to be the invertible linear maps and $\mathrm{GA}(X) \subset A(X)$ to be the invertible affine maps. They are both groups and are referred to as the *general linear group (of X)* and the *affine group (of X)*.

For any $x_1, x_2 \in X$ we denote by

$$[x_1, x_2] := \{tx_1 + (1-t)x_2 : t \in [0,1]\} \qquad (x_1, x_2) := \{tx_1 + (1-t)x_2 : t \in (0,1)\}.$$

the *closed line segment (for x_1, x_2)* and *open line segment (for x_1, x_2)* respectively. For a set of points $S \subset X$ we define $\operatorname{conv} S$ to be the *convex hull of S*.

For a C^1-differentiable manifold M we shall denote by $T_x M$ the tangent space of M at $x \in M$. If $\dim T_x M = k$ for all $x \in M$ we shall define M to be k-dimensional, which we shall denote by $\dim M := k$. For a general reference on the theory of manifolds and the notation we shall be using, we refer the reader to Appendix A.1.

1.1.1 Euclidean and non-Euclidean normed spaces

We shall define a normed space X to be a *Euclidean space* if its norm is generated by an inner product $\langle \cdot, \cdot \rangle$ (i.e. for all $x \in X$, $\|x\| = \langle x, x \rangle^{\frac{1}{2}}$), otherwise X is a *non-Euclidean (normed) space.*

The following are some useful results for Euclidean spaces.

Theorem 1.1.1 (The Cauchy-Schwarz inequality)**.** Let X be a Euclidean normed space. Then for all $x, y \in X$,

$$|\langle x, y\rangle| \leq \|x\|\|y\|,$$

with equality if and only if x and y are linearly dependent.

Proof. Let $\alpha = 1$ if $\langle x, y\rangle \geq 0$ and $\alpha = -1$ otherwise. Define for any $t \in \mathbb{R}$, $f(t) := \|x - \alpha t y\|^2$. For all $t \in \mathbb{R}$, $f(t) \geq 0$ and

$$f(t) = \langle x - \alpha t y, x - \alpha t y\rangle = \|x\|^2 - 2t|\langle x, y\rangle| + t^2\|y\|^2.$$

If $\|y\| = 0$ then $|\langle x, y\rangle| = 0$, as otherwise $f(t) < 0$ for large enough t. If $\|y\| > 0$, then we note that if $t_0 = \frac{|\langle x,y\rangle|}{\|y\|^2}$ then

$$f(t_0) = \|x\|^2 - |\langle x, y\rangle|^2 \geq 0.$$

By rearranging we obtain the required inequality. □

Theorem 1.1.2 (The parallelogram equality)**.** [3, (1.1)] Let X be a normed space. Then X is Euclidean if and only if for all $x, y \in X$,

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

Proposition 1.1.3. Let X and Y be d-dimensional normed spaces and suppose X is Euclidean. Then Y is Euclidean if and only if there exists a linear isometry $T : Y \to X$.

Proof. Suppose Y is Euclidean. Choose an orthonormal basis $x_1, \ldots, x_d \in X$ and an orthonormal basis $y_1, \ldots, y_d \in Y$. If we define $T : Y \to X$ to be the linear map with $T(y_i) = x_i$ for all $i = 1, \ldots, d$ then T is an isometry.

Now suppose there exists a linear isometry $T : Y \to X$ and choose $y_1, y_2 \in Y$. By Theorem 1.1.2, the parallelogram equality holds in X. It now follows

$$\begin{aligned}\|y_1 + y_2\|^2 + \|y_1 - y_2\|^2 &= \|T(y_1) + T(y_2)\|^2 + \|T(y_1) - T(y_2)\|^2 \\ &= 2\|T(y_1)\|^2 + 2\|T(y_2)\|^2 \\ &= 2\|y_1\|^2 + 2\|y_2\|^2,\end{aligned}$$

thus the parallelogram equality holds. As this is true for all $y_1, y_2 \in Y$, by Theorem 1.1.2, Y is Euclidean. □

Remark 1.1.4. It follows from Proposition 1.1.3 that for each $d \in \mathbb{N}$, there exists a unique (up to isometry) d-dimensional Euclidean space. For $\mathbb{R}^d$ we define the *standard Euclidean norm* to be the norm generated by the *standard inner product*

$$\left\langle (x(i))_{i=1}^d, (y(i))_{i=1}^d \right\rangle := \sum_{i=1}^d x(i)y(i).$$

Example 1.1.5. Let $q \in [1, \infty)$ and define ℓ_q^d to be the linear space $\mathbb{R}^d$ with the ℓ_q*-norm*, i.e. the norm

$$\|(x(i))_{i=1}^d\|_q := \left(\sum_{i=1}^d |x(i)|^q \right)^{\frac{1}{q}}.$$

We also define ℓ_∞^d to be the linear space $\mathbb{R}^d$ with the ℓ_∞*-norm* (or *supremum norm*), i.e. the norm

$$\|(x(i))_{i=1}^d\|_\infty := \sup_{i=1,\ldots,d} |x(i)|.$$

If $d = 1$ then $\ell_q^1 = \ell_{q'}^1$ for all $q, q' \in [1, \infty]$. For all $d \in \mathbb{N}$, ℓ_2^d is the standard Euclidean space. For all $q \neq 2$ and $d > 1$, ℓ_q^d is non-Euclidean however, as the parallelogram inequality fails to hold for $x, y \in \ell_q^d$ with $x(1) = x(2) = y(1)$ and $x(i) = y(j) = 0$ for any $i > 2$, $j > 1$.

1.1.2 Differentiation in normed space

For normed spaces X, Y and $U \subset X$, $V \subset Y$ we define a map $f : U \to V$ to be *(Fréchet) differentiable* at $x_0 \in U^\circ$ (the interior of U) if there exists a linear map $df(x_0) : X \to Y$ such that

$$\frac{\|f(x_0 + h) - f(x_0) - df(x_0)h\|_Y}{\|h\|_X} \to 0$$

as $h \to 0$; we refer to $df(x_0)$ as the *(Fréchet) derivative of f at x_0*. If $U' \subset U^\circ$ is open and f is differentiable at all points in U' we say that f is *differentiable on U'*. If f is differentiable on U' and the map

$$df : U' \to L(X, Y), \ x \mapsto df(x)$$

is continuous then we say that f is C^1-differentiable on U' and define df to be the *C^1-derivative* of f; if $U' = U$ we just say that f is C^1-differentiable. For all $k \in \mathbb{N}$ we define inductively $d^k f := d(d^{k-1} f)$ where $d^1 f := df$ and $d^0 f := f$; by this we define f to be *C^k-differentiable* if $d^k f$ exists and is continuous. If f is C^k-differentiable for all $k \in \mathbb{N} \cup \{0\}$ we say f is *C^∞-differentiable* or *smooth*. If f is C^k-differentiable and bijective with C^k-differentiable inverse we say that f is a *C^k-diffeomorphism* or *smooth diffeomorphism* if $k = \infty$.

If we restrict ourselves to the case where $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$ then we may also define for certain points $x_0 \in U$ the *partial derivatives* $\partial f^i / \partial x^j (x_0)$ and *Jacobian matrix* $Jf(x_0)$ of f at x_0 in the standard way (see [47, Section 2.3, page 69] for more details). The following result shall help with computing specific derivatives.

Proposition 1.1.6. [47, Proposition 2.4.12] Let $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$ be differentiable on an open set U. Then the following holds:

(i) If f is differentiable at $x_0 \in U$, then each partial derivative of f at x_0 exists.

(ii) If f is differentiable at $x_0 \in U$, then $Jf(x_0)$ exists and is the matrix representation of $df(x_0)$.

(iii) f is C^1-differentiable on U if and only if the map

$$Jf : U \to M_{n\times m},\ x \mapsto Jf(x_0)$$

is well-defined and continuous, where $M_{n\times m}$ is the space of $n \times m$ real-valued matrices.

Some of the results referenced refer specifically to *Gâteaux differentiation*, however for Lipschitz maps between finite dimensional normed spaces Gâteaux differentiability is equivalent to differentiability (see [8, Proposition 4.3]).

The definitions for differentiability may be carried naturally to manifolds, and the definitions will be consistent if we consider open subsets of normed spaces to be manifolds. We refer the reader to [47, Section 3] for more details.

Remark 1.1.7. If we have a continuous path $\alpha : (a, b) \to X$ that is differentiable at $t \in (a, b)$ with differential $\alpha'(t)$ in the traditional sense i.e

$$\alpha'(t) := \lim_{h\to 0} \frac{\alpha(t+h) - \alpha(t)}{h},$$

then $\alpha'(t) = d\alpha(t)(1)$. Further, if M is a C^k*-submanifold* (see [47, Definition 3.2.1] for more detail) of some normed space X, then $\alpha_1 : (a, b) \to M$ is C^k-differentiable if and only if $\alpha_2 : (a, b) \to X$ is C^k-differentiable, where $\alpha_1(t) = \alpha_2(t)$ for all $t \in (a, b)$.

1.1.3 Support functionals

Let $x \in X$ and $f \in X^*$, then we say that f is *support functional* of x if $\|f\| = \|x\|$ and $f(x) = \|x\|^2$.

Proposition 1.1.8. Every point in a normed space has a support functional.

Proof. Choose $x_0 \in X$ with $\|x_0\| = 1$. Define the linear functional $\phi : \operatorname{span}\{x_0\} \to \mathbb{R}$ where $\phi(ax_0) = a$ for all $a \in \mathbb{R}$. We note that $\phi(x) \leq \|x\|$ for all $x \in \operatorname{span}\{x_0\}$, thus by the Hahn-Banach theorem there exists a linear map $f : X \to \mathbb{R}$ where $f(x) \leq \|x\|$ for all $x \in X$ and $f(x) = \phi(x)$ for all $x \in \operatorname{span}\{x_0\}$. It now follows f is a support functional of x_0.

Choose any $x_0 \in X$. If $x_0 = 0$ then the zero map is a support functional of x_0. Suppose $x_0 \neq 0$, then $x_0/\|x_0\|$ has support functional f and we note $\|x_0\|f$ is a support functional of x_0. □

As every point has at least one support functional we shall define for each $x \in X$ the set $\varphi[x]$ of support functionals of x.

We say that a non-zero point x is *smooth* if it has a unique support functional (i.e. $|\varphi[x]| = 1$) and define $\operatorname{smooth}(X) \subseteq X \setminus \{0\}$ to be the set of smooth points of X. If $\operatorname{smooth}(X) \cup \{0\} = X$ then we say that X is *smooth*. We define a norm to be *strictly convex* if $\|tx + (1-t)y\| < 1$ for all distinct $x, y \in S_1[0]$ and $t \in (0, 1)$.

The *dual map* of X is the map

$$\varphi : \operatorname{smooth}(X) \cup \{0\} \to X^*, \ x \mapsto \varphi(x),$$

where $\varphi(0) = 0$ and $\varphi(x)$ is the unique support functional of $x \in \operatorname{smooth}(X)$. It is immediate that φ is homogeneous since f is the support functional of x if and only if af is the support functional of ax for $a \neq 0$.

Proposition 1.1.9. Let X be a Euclidean normed space. Then the following holds:

(i) X is strictly convex.

(ii) X is smooth, and for each $x \in X$,

$$\varphi(x) : X \to \mathbb{R}, \; y \mapsto \langle x, y \rangle$$

is the unique support functional of x.

(iii) The dual map $\varphi : X \to X^*$ is a linear isometric isomorphism.

Proof. (i): For any distinct $x, y \in S_1[0]$ and $t \in (0,1)$,

$$\begin{aligned} \|tx + (1-t)y\|^2 &= t^2\|x\|^2 + 2t(1-t)\langle x, y \rangle + (1-t)^2\|y\|^2 \\ &< t^2 + 2t(1-t) + (1-t)^2 \\ &= 1, \end{aligned}$$

by the Cauchy-Schwarz inequality, as $x \neq y$.

(ii): Choose $x \in X$, then by the Cauchy-Schwarz inequality, $\varphi(x)$ supports x. Suppose that there also exists $f \in X^*$ that supports x. The map $z \mapsto \langle z, \cdot \rangle$ is an injective (thus surjective) linear map between X and X^*, thus there exists some $y \in X$ such that $f = \varphi(y)$. By the Cauchy-Schwarz inequality,

$$|\varphi(y)x| \leq \|x\|\|y\| = \|x\|^2$$

with equality if and only if $y = x$ or $y = -x$. As $\varphi(y)x = -\|x\|^2$ if $y = -x$, then $y = x$ and $f = \varphi(x)$. As this holds for all $x \in X \setminus \{0\}$, X is smooth.

(iii): As noted in (ii), φ is a linear isomorphism. As $\varphi(x)$ is the unique suport functional of $x \in \text{smooth}(X) \cup \{0\}$ then $\|\varphi(x)\| = \|x\|$ by definition. $\square$

The following result shows that if two normed spaces are isometrically isomorphic then they have equivalent support functions in some way.

Proposition 1.1.10. Let X and Y be normed spaces and $T : X \to Y$ a isometric isomorphism. Then the following holds:

(i) If $f \in X^*$ is a support functional of $x \in X$, then $f \circ T^{-1}$ is a support functional of $T(x)$.

(ii) x is a smooth point of X if and only if $T(x)$ is a smooth point of Y.

(iii) X is smooth if and only if Y is smooth.

(iv) X is strictly convex if and only if Y is strictly convex.

Proof. (i): As T is an isometric isomorphism, $\|T(x)\|_Y = \|x\|_X$, thus $f \circ T^{-1}(T(x)) = \|T(x)\|_Y^2$. Choose any $y \in Y$ with $\|y\|_Y = 1$, then as T is an isomorphism there exists $y' \in X$ with $T(y') = y$, and $\|y'\|_X = 1$ also. We now note that

$$|f \circ T^{-1}(y)| = |f(y')| \leq \|x\|_X = \|T(x)\|_Y,$$

thus $\|f \circ T^{-1}\|_Y = \|T(x)\|_Y$ as required.

(ii): Let $f, g \in X^*$ be distinct support functionals of $x \in X$. By (i), $f \circ T^{-1}$ and $g \circ T^{-1}$ are support functionals of $T(x)$. We note that f, g can not be linearly dependent; if $f = cg$ for some $c \neq 0, 1$ then $f(x) = c\|x\|_X \neq \|x\|_X$. As f, g are linearly independent we may choose any $y \in \ker f$ such that $y \notin \ker g$. It now follows that $T(y) \in \ker f \circ T^{-1}$ and $T(y) \notin \ker g \circ T^{-1}$, thus $f \circ T^{-1} \neq g \circ T^{-1}$. This implies that if x is a non-smooth point of X then $T(x)$ is a non-smooth point of Y. By symmetry we note the converse also holds as required.

(iii): This follows immediately from (ii).

(iv): Suppose X is strictly convex and choose any two points $x, y \in Y$ with $\|x\|_Y = \|y\|_Y = 1$. As T is an isomorphism there exist $x', y' \in X$ with $T(x') = x$, $T(y') = y$; we note that $\|x'\|_X = \|y'\|_X = 1$ also. It now follows that for any $t \in (0, 1)$,

$$\|tx + (1-t)y\|_Y = \|T(tx' + (1-t)y')\|_Y = \|tx' + (1-t)y'\|_X < 1$$

as X is strictly convex, thus Y is strictly convex as required. $\square$

For a d-dimensional normed space X we shall define $S \subset X$ to be *negligible* if for every $\epsilon > 0$ there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in S and $(r_n)_{n\in\mathbb{N}}$ in $(0, \infty)$ such that $\sum_{n\in\mathbb{N}} r_n^d < \epsilon$ and

$$S \subset \bigcup_{n\in\mathbb{N}} B_{r_n}(x_n).$$

We note that for two norms $\|\cdot\|, \|\cdot\|'$ of X, if S is a negligible subset of $(X, \|\cdot\|)$ it will also be a negligible subset of $(X, \|\cdot\|')$. The countable union of negligible sets is a negligible set, and the complement of a negligible set is a dense set (see Proposition B.2.7). If $S \subset \mathbb{R}^d$, then S is negligible if and only if it has Lebesgue measure zero (see Theorem B.2.6).

Proposition 1.1.11. For any normed space X the following properties hold:

(i) For $x_0 \neq 0$, $x_0 \in \operatorname{smooth}(X)$ if and only if $x \mapsto \|x\|$ is differentiable at x_0.

(ii) If $x \mapsto \|x\|$ is differentiable at x_0 then it has derivative $\frac{1}{\|x_0\|}\varphi(x_0)$.

(iii) The set $\operatorname{smooth}(X)$ is dense in X and $\operatorname{smooth}(X)^c$ is negligible.

(iv) The map φ is continuous.

(v) X is Euclidean if and only if X is smooth and the map φ is a linear map.

Proof. (i) & (ii): By [40, Lemma 1], $x \mapsto \|x\|$ is differentiable at x_0 if and only if $x_0 \in \operatorname{smooth}(X)$ with derivative $\frac{1}{\|x_0\|}\varphi(x_0)$.

(iii): The result follows from (i), [60, Theorem 25.5] and Theorem B.2.6 as $x \mapsto \|x\|$ is convex.

(iv): By [60, Theorem 25.5], the map $x \mapsto \frac{1}{\|x\|}\varphi(x)$ is continuous on $\operatorname{smooth}(X)$, thus φ is continuous on $\operatorname{smooth}(X)$ also. As $\|\varphi(x)\| = \|x\|$ it follows that φ is continuous at $0 \in X^*$ also as required.

(v): If X is Euclidean then by Proposition 1.1.9 (iii), X is smooth and φ is linear. Suppose φ is linear. If we define $\langle x, y\rangle := \frac{1}{2}(\varphi(x)y + \varphi(y)x)$ for each $x, y \in X$ then $\langle \cdot, \cdot \rangle$ is an inner product on X and $\|x\|^2 = \langle x, x\rangle$, thus X is Euclidean. □

The following are some examples of normed spaces and their corresponding dual maps.

Example 1.1.12 (Smooth and strictly convex)**.** For $q \in (1, \infty)$, the space ℓ_q^d is strictly convex, as the real-valued function $a \mapsto a^q$ is strictly convex and increasing. Let $\operatorname{sgn} : \mathbb{R} \to \{-1, 0, 1\}$ be the sign function, i.e.

$$\operatorname{sgn}(a) = \begin{cases} 1, \text{ if } a > 0 \\ 0, \text{ if } a = 0 \\ -1, \text{ if } a < 0. \end{cases}$$

For each $x := (x(i))_{i=1}^d \in \mathbb{R}^d$ and $r > 0$ define the vector $x^{(r)} \in \mathbb{R}^d$, where for each $i = 1, \dots, d$,

$$x^{(r)}(i) := \operatorname{sgn}(x(i))\, |x(i)|\,.$$

Take $x \in S_1[0]$, then we obtain the support functional

$$\varphi(x) : \mathbb{R}^d \to \mathbb{R},\ y \mapsto \frac{1}{\|x\|_q^{q-2}} \left\langle x^{(q-1)}, y \right\rangle .$$

As we can differentiate $\|\cdot\|_q$ at any point in $\mathbb{R}^d$, it follows from Proposition 1.1.11 (i) that ℓ_q^d is smooth.

Example 1.1.13 (Neither smooth nor strictly convex)**.** Fix $d \geq 2$ and choose a finite spanning set $F \subset (\mathbb{R}^d)^*$ such that $0 \notin F$ and if $f \in F$ then $-f \in F$. We may define a centrally symmetric (d-dimensional) polytope

$$\mathcal{P} := \{x \in \mathbb{R}^d : f(x) \leq 1 \text{ for all } f \in F\}.$$

Define X to be the linear space $\mathbb{R}^d$ with norm

$$\|x\|_{\mathcal{P}} := \max_{f \in F} |f(x)|.$$

By [34, Lemma 3], $\|x_0\|_{\mathcal{P}} f \in F$ is a support functional of x_0 if and only if $f(x_0) = \|x_0\|_{\mathcal{P}}$. We refer to these norm spaces as *polyhedral norm spaces.*

The set of points $x \in S_1[0]$ where $\|x\|_{\mathcal{P}} = f(x) = g(x)$ for distinct $f, g \in F$ is a non-empty negligible set as it is exactly the intersection of a finite set of hyperplanes, thus X is not smooth but does have an open set of smooth points. As $|F| < \infty$ there must exist two points $x, y \in S_1[0]$ that obtain their norm for the same linear functional $f \in F$, and so

$$\begin{aligned} t\|x\|_{\mathcal{P}} + (1-t)\|y\|_{\mathcal{P}} &= tf(x) + (1-t)f(y) \\ &= f(tx + (1-t)y) \\ &\leq \|tx + (1-t)y\|_{\mathcal{P}} \end{aligned}$$

$$\leq \quad t\|x\|_{\mathcal{P}} + (1-t)\|y\|_{\mathcal{P}},$$

thus X is also not strictly convex.

Example 1.1.14 (Smooth but not strictly convex)**.** Let X be the linear space $\mathbb{R}^2$ with norm

$$\|(x,y)\| = \begin{cases} |y|, & \text{if } |y| > |x| \\ \frac{x^2+y^2}{2|x|} & \text{if } |x| \geq |y|, \ x \neq 0 \\ 0 & \text{if } x = y = 0 \end{cases}$$

with unit sphere as described in Figure 1.1.

Let $z = (x,y) \in X$ and choose any $w \in X$. The norm is differentiable at all non-zero points (i.e. X is smooth), thus we have $\varphi(z)w = \|z\| \langle f(z), w \rangle$, where,

$$f(z) = \begin{cases} (0,1), & \text{if } |y| > |x| \\ \left(\frac{x^3 - xy^2}{2|x|^3}, \frac{y}{|x|}\right) & \text{if } |x| \geq |y|, \ x \neq 0 \\ (0,0), & \text{if } x = y = 0. \end{cases}$$

The norm is not strictly convex however, as the points $\{(t,1) : t \in (-1,1)\}$ all lie in $S_1[0]$.

Example 1.1.15 (Strictly convex but not smooth)**.** Let $\|\cdot\|_a$ be a strictly convex and smooth norm on $\mathbb{R}^d$ and $\|\cdot\|_b$ be a non-smooth norm on $\mathbb{R}^d$. We define X to be the linear space $\mathbb{R}^d$ with the norm $\|x\| := \|x\|_a + \|x\|_b$.

Choose any non-zero, non-smooth point $x \in (\mathbb{R}^d, \|\cdot\|_b)$. If x is a smooth point of X then by Proposition 1.1.11 (i), both $\|\cdot\|$ and $\|\cdot\|_a$ are differentiable at x. However, as $\|\cdot\|_b = \|\cdot\| - \|\cdot\|_a$, then $\|\cdot\|_b$ would be differentiable at x also, a contradiction. It

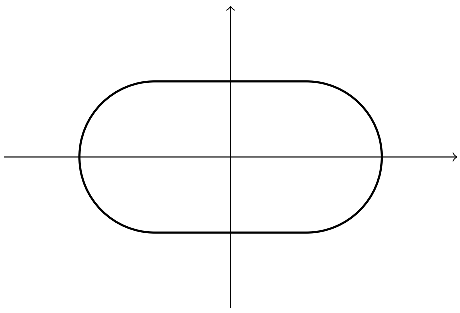

Fig. 1.1 The unit ball of the normed space described in Example 1.1.14.

follows that the smooth points of X are exactly the smooth points of $(\mathbb{R}^d, \|\cdot\|_b)$, thus X is not smooth.

Choose any two points $x, y \in S_1[0]$ and $t \in (0,1)$, then by the strict convexity of $\|\cdot\|_a$,

$$\begin{aligned} \|tx + (1-t)y\| &= \|tx + (1-t)y\|_a + \|tx + (1-t)y\|_b \\ &< (t\|x\|_a + (1-t)\|y\|_a) + (t\|x\|_b + (1-t)\|y\|_b) \\ &= t\|x\| + (1-t)\|y\|, \end{aligned}$$

thus X is also strictly convex.

The following are some useful properties regarding support functionals.

Proposition 1.1.16. Let X be a normed space and $x \in X \setminus \{0\}$. Then the following holds:

(i) $\varphi[x]$ is a compact and convex subset of $S^*_{\|x\|}[0]$.

(ii) If $\dim X = 2$ then $\varphi[x] = [f, g]$ for some $f, g \in X^*$, and $x \in \text{smooth}(X)$ if and only if $f = g$.

Proof. (i): For each $f \in \varphi[x]$ we have $\|f\| = \|x\|$ by definition, thus $\varphi[x] \subset S^*_{\|x\|}[0]$. Given $f, g \in \varphi[x]$ and $t \in [0,1]$ we note that $(tf + (1-t)g)(x) = \|x\|^2$ and

$$\|tf + (1-t)g\| \leq t\|f\| + (1-t)\|g\| = \|x\|,$$

thus $tf+(1-t)g \in \varphi[x]$ and $\varphi[x]$ is convex. Finally if $(f_n)_{n\in\mathbb{N}}$ is a convergent sequence of support functionals of x with limit f then $\|f\| = \|x\|$ and $f(x) = \lim_{n\to\infty} f_n(x) = \|x\|^2$, thus $f \in \varphi[x]$; since this implies $\varphi[x]$ is a closed subset of the compact set $S^*_{\|x\|}[0]$ then it too is compact.

(ii): If x is smooth then $\varphi[x] = \{\varphi(x)\} = [\varphi(x), \varphi(x)]$. Suppose x is not smooth, then by (i), $\varphi[x]$ is a compact convex subset of the 1-dimensional manifold $S^*_{\|x\|}[0]$, and hence is a line segment. □

Proposition 1.1.17. Let X be a normed space and $x \in S_1[0] \cap \operatorname{smooth}(X)$. Then the following holds:

(i) The set $\varphi(x)^{-1}[\{1\}] \cap S_1[0]$ is closed and convex.

(ii) If $\dim X = 2$, $\varphi(x)^{-1}[\{1\}] \cap S_1[0] = [x_1, x_2]$.

Proof. (i): Choose $y, z \in \varphi(x)^{-1}[\{1\}] \cap S_1[0]$, then $\varphi(x)(ty + (1-t)z) = 1$ for all $t \in [0,1]$. We further note that

$$1 = |\varphi(x)(ty + (1-t)z)| \leq \|ty + (1-t)z\| \leq 1,$$

thus $ty + (1-t)z \in S_1[0]$ also and $\varphi(x)^{-1}[\{1\}] \cap S_1[0]$ is convex. As $\varphi(x)$ is continuous then $\varphi(x)^{-1}[\{1\}] \cap S_1[0]$ is closed also.

(ii): If $\dim X = 2$ it follows that $\varphi(x)^{-1}[\{1\}] \cap S_1[0] = [x_1, x_2]$ as $S_1[0]$ is a 1-dimensional topological manifold homeomorphic to the circle. □

Example 1.1.18. Let V be a finite dimensional real vector space with dual V^*. Choose a non-empty set $F \subset V^*$ such that the following holds:

(i) If $f \in F$ then $-f \in F$.

(ii) For each $x \in V$, there exists $M > 0$ such that $f(x) \leq M$ for all $f \in F$.

(iii) $\operatorname{span} F = V^*$.

We may now define the normed space $X = (V, \|\cdot\|)$ where for all $x \in X$,

$$\|x\| := \sup_{f \in F} f(x).$$

We note that the following holds:

(i) As $\overline{F}$ is compact it follows that if $\|x\| = 1$, there exists $f \in \overline{F}$ such that $f(x) = 1$. It is immediate that f is a support functional of x.

(ii) If we define $\|\cdot\|'$ to be the norm generated by $\overline{F}$, $\|\cdot\|''$ to be the norm generated by $\operatorname{conv}(F)$ and $\|\cdot\|'''$ to be the norm generated by $\partial \operatorname{conv}(F)$, then

$$\|\cdot\| = \|\cdot\|' = \|\cdot\|'' = \|\cdot\|'''$$

(iii) By [60, Theorem 14.5], $B_1^*[0] = \operatorname{conv}(\overline{F})$; it follows that $S_1^*[0] = \partial \operatorname{conv}(F)$.

(iv) For any $x \in S_1[0]$, define $F_x \subset \overline{F}$ to be the set

$$F_x := \{f \in \overline{F} : f(x) = 1\},$$

then $\varphi[x] = \operatorname{conv}(F_x)$. It follows that $x \in \operatorname{smooth}(X)$ if and only if $|F_x| = 1$.

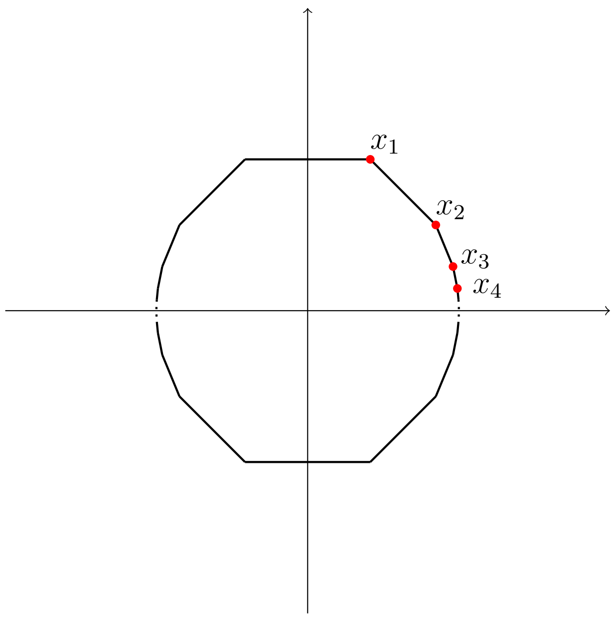

Fig. 1.2 The unit ball of the normed space described in Example 1.1.19 with the first four elements of the sequence $(x_n)_{n\in\mathbb{N}}$ of non-smooth points.

Example 1.1.19. Although most normed spaces that are commonly studied have an open set of smooths points, not all normed spaces do so. We shall now construct a normed plane with a smooth point that does not lie in smooth$(X)^\circ$.

For each $i, j \in \{-1, 1\}$ and $\theta \in [0, \frac{\pi}{2}]$, define the linear functional $s^*(i, j, \theta) \in (\mathbb{R}^2)^*$, where for any $x \in \mathbb{R}^2$,

$$s^*(i, j, \theta)(x) := \left\langle \Big((-1)^i \cos(\theta), (-1)^j \sin(\theta)\Big), x \right\rangle.$$

Define the sequence $(\theta_n)_{n\in\mathbb{N}}$ where $\theta_n := \frac{\pi}{2^n}$, and further define

$$F := \{s^*(i, j, \theta_n) : n \in \mathbb{N}, \ i, j \in \{-1, 1\}\}.$$

We note that

$$\overline{F} = F \cup \{s^*(-1, 1, 0), s^*(1, 1, 0)\},$$

with $s^*(i,-1,0) = s^*(i,1,0)$ for $i \in \{-1,1\}$.

Following Example 1.1.18, we define the norm $\|\cdot\|$ for $\mathbb{R}^2$ with F. For any $i,j \in \{-1,1\}$ and $n \in \mathbb{N}$,

$$s^*(i,j,\theta_n)((1,0)) = (-1)^i \cos\left(\frac{\pi}{2^n}\right) < 1$$

and

$$s^*(1,1,0)((1,0)) = 1,$$

thus $(1,0)$ is smooth (and similarly $(-1,0)$ is smooth).

For each $n \in \mathbb{N}$ define

$$x_n := \frac{2\sin\left(\frac{\theta_{n+1}}{2}\right)}{\sin(\theta_{n+1})}\left(\cos\left(\frac{3\theta_{n+1}}{2}\right), \sin\left(\frac{3\theta_{n+1}}{2}\right)\right),$$

and the function $f_n : [0, \frac{\pi}{2}] \to \mathbb{R}$ with

$$f_n(\phi) := s^*(1,1,\phi)(x_n) = \frac{2\sin\left(\frac{\theta_{n+1}}{2}\right)\cos\left(\frac{3\theta_{n+1}}{2} - \phi\right)}{\sin(\theta_{n+1})}.$$

We now note that

$$f_n(\theta_n) = f_n(\theta_{n+1}) = 1,$$

thus if $\|x_n\| = 1$ then $s^*(1,1,\theta_n)$ and $s^*(1,1,\theta_{n+1})$ support x_n. To see that $\|x_n\| = 1$ we first note by differentiating that f is strictly increasing on $[0, \frac{3\theta_{n+1}}{2}]$ and strictly decreasing on $[\frac{3\theta_{n+1}}{2}, \frac{\pi}{2}]$, thus for all $m \in \mathbb{N}$

$$f_n(\theta_m) \leq f_n(\theta_n) = 1.$$

As

$$s^*(i,j,\theta_m)(x_n) \leq f_n(\theta_m) \leq 1,$$

$\|x_n\| = 1$. The sequence $(x_n)_{n\in\mathbb{N}}$ is a sequence of non-smooth points converging to a smooth point $(1,0)$, thus smooth(X) is not open.

The following gives a useful characterisation for strictly convex normed spaces.

Proposition 1.1.20. Let X be a normed space, then X is strictly convex if and only if no support functional supports more than one point in X.

Proof. Suppose there exists $f \in X^*$ that supports two distinct points $x, y \in S_1[0]$. For all $t \in (0,1)$ we have

$$1 = tf(x) + (1-t)f(y) = f\left(tx + (1-t)y\right) \leq \|tx + (1-t)y\|,$$

thus X is not strictly convex. Now suppose there exists $f' \in X^*$ that supports two distinct points $x', y' \in X$, then by definition $\|x'\| = \|y'\| = \|f'\|$. Let $c = \frac{1}{\|f'\|}$ and define $f := cf'$, $x := cx'$ and $y := cy'$. It follows that f supports $x, y \in S_1[0]$, thus X^* is not strictly convex.

Now suppose $[x,y] \subset S_1[0]$ for distinct points $x, y \in X$. Choose $t \in (0,1)$ and let f be a support functional of $tx + (1-t)y$. As $\|f\| = 1$ then $f(x), f(y) \leq 1$. Suppose $f(x) < 1$, then

$$1 = f\left(tx + (1-t)y\right) = tf(x) + (1-t)f(y) < 1,$$

thus $f(x) = 1$. By symmetry, $f(y) = 1$, thus f supports x and y. $\square$

Remark 1.1.21. It follows from Proposition 1.1.20 that if the dual map of X is injective then X is strictly convex, and the converse holds if X is smooth.

We define for $S_1[0]$ the *(inner) Löwner ellipsoid* S of $S_1[0]$, the unique convex body of maximal volume bounded by $S_1[0]$ which has a Minkowski functional $\|\cdot\|_S : X \to \mathbb{R}_{\geq 0}$ that can be induced by an inner product. It is immediate that $\|x\|_S \geq \|x\|$ for all $x \in X$ and the Euclidean space $(X, \|\cdot\|_S)$ has unit sphere S. For more information on Löwner ellipsoids see [67, Chapter 3.3].

Lemma 1.1.22. Let X be a d-dimensional normed space. Then there exists smooth points $y_1, \ldots, y_d \in S_1[0]$ so that $\varphi(y_1), \ldots, \varphi(y_d)$ are linearly independent.

Proof. By [3, Lemma 6.1] there exists $y_1, \ldots, y_d \in S_1[0]$ that lie on the Löwner ellipsoid S of $S_1[0]$. Suppose f_i is a support functional for y_i with respect to $\|\cdot\|$ and choose any $x \in S$. As $S \subset B_1[0]$ (the unit ball of $(X, \|\cdot\|)$) then $|f_i(x)| \leq 1$, thus f is a support functional for y_i with respect to $\|\cdot\|_S$ also. As $(X, \|\cdot\|_S)$ is Euclidean then it follows that $y_1, \ldots, y_d$ are smooth and $\varphi(y_1), \ldots, \varphi(y_d)$ are linearly independent. $\square$

Proposition 1.1.23. Suppose $\dim X = d \geq 2$. For all

$$x_1, \ldots, x_n \in S_1[0] \cap \operatorname{smooth}(X)$$

with $n < d$ there exists $y \in S_1[0] \cap \operatorname{smooth}(X)$ such that

$$y \notin \operatorname{span}\{x_1, \ldots, x_n\} \qquad \varphi(y) \notin \operatorname{span}\{\varphi(x_1), \ldots, \varphi(x_n)\}.$$

Proof. By Lemma 1.1.22, there exists smooth points $y_1, \ldots, y_d \in S_1[0]$ so that the support functionals $\varphi(y_1), \ldots, \varphi(y_d)$ are linearly independent. Define

$$Z := \operatorname{span}\{\varphi(x_1), \ldots, \varphi(x_n)\},$$

then $\dim Z \leq n < d$. If $\varphi(y_1), \ldots, \varphi(y_d) \in Z$ then $\dim Z = d$, thus there exists $y_i \notin Z$. We now choose $y := y_i$. $\square$

1.1.4 Lie groups and Lie group actions

A group Γ is a *Lie group* if it is a finite dimensional smooth manifold and the maps

$$G \times G \to G,\ (g,h) \mapsto gh \qquad G \to G,\ g \mapsto g^{-1}$$

are smooth. We define a subgroup $\Gamma' \leq \Gamma$ to be a *regular Lie subgroup* of Γ if Γ' is a submanifold of Γ and a Lie group under the inherited group operations.

Example 1.1.24. Let M_n denote the space of $n \times n$-matrices and $GL_n \subset M_n$ denote the subset of invertible matrices, then GL_n is a smooth submanifold of M_n; see Example A.1.7 for more detail. The formulae used to define multiplication and inverse operations are rational functions of the matrix components, thus GL_n is a Lie group. As M_n is isomorphic to $L(X)$ (as normed algebras) if $\dim X = n$, it follows that $GL(X)$ is also a Lie group; it further follows that $\mathrm{GA}(X)$ is also a Lie group.

The following is an important theorem that we shall often apply.

Theorem 1.1.25. (Closed subgroup theorem)[1, Proposition 4.1.12] Let Γ' be a closed subgroup of the Lie group Γ. Then Γ' is a regular Lie subgroup of Γ.

Let Γ be a Lie group and M a (finite dimensional) smooth manifold. If there exists a smooth group action

$$\phi : \Gamma \times M \to M,\ (g,x) \mapsto g.x$$

we say that ϕ is a *Lie group action of* Γ *on* M. We define the following for all $x \in M$:

(i) the *stabiliser* of x, $\mathrm{Stab}_x := \{g \in \Gamma : g.x = x\}$,

(ii) the *orbit* of x, $\mathcal{O}_x := \{g.x : g \in \Gamma\}$,

(iii) $\phi_x : \Gamma \to \mathcal{O}_x,\ g \mapsto g.x$.

We note immediately that Stab_x is a closed subgroup of Γ, thus by Theorem 1.1.25, Stab_x is a smooth submanifold of Γ.

We say that Γ acts *properly* on M if the map

$$\theta : \Gamma \times M \to M \times M, \ (g, x) \mapsto (\phi(g, x), x)$$

is proper i.e. the preimage of any compact set is compact. If H is a closed subgroup of Γ then by [47, Theorem 5.1.16], Γ/H (the set of left cosets gH, $g \in \Gamma$) has a unique manifold structure such that the quotient map $\pi : \Gamma \to \Gamma/H$ is a smooth surjective submersion i.e. $d\pi(g)$ is surjective for all $g \in \Gamma$.

Lemma 1.1.26. [1, Corollary 4.1.22] Let ϕ be a Lie group action of Γ on M. Suppose Γ acts properly on M, then $\mathcal{O}_x$ is a closed smooth submanifold of M that is diffeomorphic to Γ/Stab_x under the map $\tilde{\phi}_x : g\,\mathrm{Stab}_x \mapsto g.x$.

1.1.5 The group of isometries of a normed space

We shall define $\mathrm{Isom}(X, \|\cdot\|)$ to be the *group of isometries* of $(X, \|\cdot\|)$ and $\mathrm{Isom}^{\mathrm{Lin}}(X, \|\cdot\|)$ to be the *group of linear isometries* of $(X, \|\cdot\|)$ with the group actions being composition; we shall denote these as $\mathrm{Isom}(X)$ and $\mathrm{Isom}^{\mathrm{Lin}}(X)$ if there is no ambiguity.

Lemma 1.1.27. Let $g \in \mathrm{Isom}(X)$, then g is bijective.

Proof. As g is an isometry it is continuous and injective. By applying translations we may assume $g(0) = 0$, thus g maps $S_r[0]$ to $S_r[0]$ for all $r > 0$. Assume there exists $x \in S_r[0]$ that does not lie in the compact set $g(S_r[0])$; it follows that there exists $\epsilon > 0$ such that $\|x - y\| > \epsilon$ for all $y \in S_r[0]$. We define the sequence $(x_n)_{n \in \mathbb{N}}$ in $S_r[0]$ with $x_1 = x$ and $x_{n+1} = g(x_n)$. We note that for all $n, m \in \mathbb{N}$ with $n < m$,

$$\|x_n - x_m\| = \|g^{n-1}(x) - g^{n-1}(x_{m-n+1})\| = \|x - x_{m-n+1}\| > \epsilon,$$

contradicting that $S_r[0]$ is compact, thus $g(S_r[0]) = S_r[0]$. As this holds for all $r > 0$ and $g(0) = 0$, g is surjective. $\square$

Using the above lemma we may state a useful rewording of a famous theorem in normed space geometry.

Theorem 1.1.28 (Mazur-Ulam's theorem)**.** [67, Theorem 3.1.2] Let X be a normed space, then $\mathrm{Isom}(X)$ is a subset of the set $A(X)$ of affine transformations of X.

It now follows we may gift $\mathrm{Isom}(X)$ the topology inherited from $A(X)$, and we note that with this topology, $\mathrm{Isom}(X)$ is a closed subgroup of $A(X)$. By Theorem 1.1.25, $\mathrm{Isom}(X)$ is a regular Lie subgroup of $A(X)$, while $\mathrm{Isom}^{\mathrm{Lin}}(X)$ is a compact regular Lie subgroup of $GL(X)$ as it is closed and bounded in $L(X)$; further, $\mathrm{Isom}^{\mathrm{Lin}}(X)$ is a regular Lie subgroup of $\mathrm{Isom}(X)$.

Lemma 1.1.29. Let X be a d-dimensional normed space, then the following holds:

(i) There exists a unique Euclidean space $(X, \|\cdot\|_2)$ such that $\mathrm{Isom}(X, \|\cdot\|)$ is a regular Lie subgroup of $\mathrm{Isom}(X, \|\cdot\|_2)$ and $\mathrm{Isom}^{\mathrm{Lin}}(X, \|\cdot\|)$ is a regular Lie subgroup of $\mathrm{Isom}^{\mathrm{Lin}}(X, \|\cdot\|_2)$.

(ii) If X is Euclidean then:

  (a) $\dim \mathrm{Isom}(X) = \frac{d(d+1)}{2}$,

  (b) $\dim \mathrm{Isom}^{\mathrm{Lin}}(X) = \frac{d(d-1)}{2}$.

(iii) If X is non-Euclidean then:

  (a) $d \leq \dim \mathrm{Isom}(X) \leq \frac{d(d-1)}{2} + 1$,

  (b) $0 \leq \dim \mathrm{Isom}^{\mathrm{Lin}}(X) \leq \frac{(d-1)(d-2)}{2} + 1$.

Proof. (i): This follows from [67, Corollary 3.3.4] and Theorem 1.1.25.

(ii): See [25, Section 2.5.5] and [25, Section 2.5.9].

(iii): [51, Lemma 4].

□

Lemma 1.1.30. For any normed space X, $\dim \operatorname{Isom}(X) = \dim \operatorname{Isom}^{\mathrm{Lin}}(X) + \dim X$.

Proof. Choose any $g \in T_\iota \operatorname{Isom}(X)$. Denote $T(X)$ to be the set of constant maps $X \to X$. We note that there is a unique pair $g_0 \in T(X)$ and $g \in L(X)$ such that $g = g_0 + g_1$; we now need to show that $g_1 \in T_\iota \operatorname{Isom}^{\mathrm{Lin}}(X)$.

By definition there exists a continuous path $\alpha : (-1,1) \to \operatorname{Isom}(X)$ that is differentiable at $t = 0$ such that $\alpha_0 = \iota$ and $\alpha'_0 = g$. We note that $g_0 : x \mapsto g(0)$ is an affine map and $g_1 : x \mapsto g(x) - g(0)$ is a linear map. For each $t \in (-1,1)$ define the isometry $\beta_t : x \mapsto x + \alpha_t(0)$, then $\beta : (-1,1) \to \operatorname{Isom}(X)$ is a continuous map that is differentiable at $t = 0$ with $\beta'_0 = g_0$. Now define the continuous path $\gamma : (-1,1) \to \operatorname{Isom}^{\mathrm{Lin}}(X)$ where $\gamma_t := \beta_t^{-1} \circ \alpha_t$, and note that γ is differentiable at $t = 0$. By definition we have that $\alpha_t = \beta_t \circ \gamma_t$ for all $t \in (-1,1)$. As $\alpha_0 = \beta_0 = \gamma_0 = \iota$ then we note that

$$g = \alpha'_0 = (\beta_t \circ \gamma_t)'|_{t=0} = \beta'_0 \circ \gamma_0 + \beta_0 \circ \gamma'_0 = \beta'_0 + \gamma'_0.$$

As g_0, g_1 are unique, $\beta'_0 = g_0$ and $\gamma'_0 = g_1$, thus $g_1 \in T_\iota \operatorname{Isom}^{\mathrm{Lin}}(X)$.

As $T(X) \cap T_\iota \operatorname{Isom}^{\mathrm{Lin}}(X) = \{0\}$ and g was chosen arbitrarily it follows that $T_\iota \operatorname{Isom}(X) = T_\iota \operatorname{Isom}^{\mathrm{Lin}}(X) \oplus T(X)$. As $\dim T(X) = \dim X$, the result holds. □

Proposition 1.1.31. Let X and Y be normed spaces and $T : X \to Y$ be a isometric isomorphism. Then

$$\operatorname{Isom}(Y) = \{T \circ g \circ T^{-1} : g \in \operatorname{Isom}(X)\}.$$

Further, $\dim \operatorname{Isom}(X) = \dim \operatorname{Isom}(Y)$ and $\dim \operatorname{Isom}^{\mathrm{Lin}}(X) = \dim \operatorname{Isom}^{\mathrm{Lin}}(Y)$.

Proof. Define the linear isomorphism

$$F : A(X) \to A(Y), \ g \mapsto T \circ g \circ T^{-1}.$$

Choose any isometry $g \in \mathrm{Isom}(X)$, then it is clear that $F(g) = T \circ g \circ T^{-1}$ is an isometry of Y. Similarly, if $h \in \mathrm{Isom}(Y)$, then $F^{-1}(h) = h \circ T$ is an isometry of X. Choose any $h \in \mathrm{Isom}(Y)$ and define the isometry $g := T^{-1} \circ h \circ T$ of X. We now note that $F(g) = h$, thus $F(\mathrm{Isom}(X)) = \mathrm{Isom}(Y)$.

As F is a linear isomorphism then it is a diffeomorphism. It now follows that if we define $\tilde{F}$ to be the restriction of F to $\mathrm{Isom}(X)$ and $\mathrm{Isom}(Y)$ then $\tilde{F}$ is a diffeomorphism also, thus $\dim \mathrm{Isom}(X) = \dim \mathrm{Isom}(Y)$. By Lemma 1.1.30, $\dim \mathrm{Isom}^{\mathrm{Lin}}(X) = \dim \mathrm{Isom}^{\mathrm{Lin}}(Y)$ also. □

The following are useful properties of Euclidean spaces.

Lemma 1.1.32. Let X be a Euclidean space, then $T \in L(X)$ is an isometry if and only if for all $x, y \in X$,

$$\langle T(x), T(y) \rangle = \langle x, y \rangle .$$

Proof. It is immediate that if $\langle T(x), T(y) \rangle = \langle x, y \rangle$ for all $x, y \in X$ then $\|T(x)\| = \|x\|$ for all $x \in X$.

Suppose $\|T(x)\| = \|x\|$ for all $x \in X$, then for all $x, y \in X$,

$$\begin{aligned} \langle T(x), T(y) \rangle &= \frac{1}{2}\|T(x) - T(y)\|^2 - \frac{1}{2}\|T(x)\|^2 - \frac{1}{2}\|T(y)\|^2 \\ &= \frac{1}{2}\|x - y\|^2 - \frac{1}{2}\|x\| - \frac{1}{2}\|y\| \\ &= \langle x, y \rangle . \end{aligned}$$

□

Proposition 1.1.33. Let X be a Euclidean space and $S \subset X$. If $f : S \to X$ is an isometry then f can be extended to an isometry $\tilde{f} : X \to X$.

Proof. By applying suitable translations we may suppose $0 \in S$ and $f(0) = 0$. We first note that for all $x, y \in S$,

$$\begin{aligned}
\langle f(x), f(y) \rangle &= \frac{1}{2}\|f(x) - f(y)\|^2 - \frac{1}{2}\|f(x) - f(0)\|^2 - \frac{1}{2}\|f(y) - f(0)\|^2 \\
&= \frac{1}{2}\|x - y\|^2 - \frac{1}{2}\|x - 0\|^2 - \frac{1}{2}\|y - 0\|^2 \\
&= \langle x, y \rangle .
\end{aligned}$$

Let $B := \{x_1, \ldots, x_n\} \subset S$ be a basis of $\operatorname{span} S$. As the inner product is preserved by f we note that

$$\left\| \sum_{i=1}^{n} a_i f(x_i) \right\|^2 = \left\| \sum_{i=1}^{n} a_i x_i \right\|^2 ,$$

thus the map $\sum_{i=1}^{n} a_i x_i \mapsto \sum_{i=1}^{n} a_i f(x_i)$ is injective. As the map is also surjective, $\dim \operatorname{span} f(B) = \dim \operatorname{span} B$.

Suppose $\operatorname{span} f(B) \neq \operatorname{span} B$. By Proposition 1.1.3, there exists an isometry $g : \operatorname{span} f(B) \to \operatorname{span} B$. The map $g \circ f$ is an isometry with

$$\operatorname{span} B = \operatorname{span} g \circ f(B),$$

and $g \circ f$ can be extended to an isometry of X if and only if the same holds for f. It follows that we may without loss of generality assume $\operatorname{span} f(B) = \operatorname{span} B$. If we find an extension of f to a linear isometry $h \in \operatorname{Isom}^{\mathrm{Lin}}(\operatorname{span} S)$, we may trivially extend h to $\tilde{f} \in \operatorname{Isom}^{\mathrm{Lin}}(X)$ by letting $\tilde{f}(x) = x$ for all $x \perp \operatorname{span} B$, thus we may assume $\operatorname{span} B = X$ (i.e. $n = d$). As $B \subset S$, it follows that by our new assumption, $\operatorname{span} S = X$ and $\operatorname{span} f(S) = X$ also.

Define $\tilde{f} : X \to X$ to be the linear map where $\tilde{f}(x_i) = f(x_i)$ for $1 \leq i \leq d$ and

$$\tilde{f}\left(\sum_{i=1}^{d} a_i x_i\right) = \sum_{i=1}^{d} a_i \tilde{f}(x_i).$$

As

$$\left\langle \tilde{f}(x_i), \tilde{f}(x_j) \right\rangle = \langle x_i, x_j \rangle$$

for each $i, j = 1, \ldots, d$ then by Lemma 1.1.32, $\tilde{f}$ is a linear isometry. Choose any $x \in S$, then for all $i = 1, \ldots, d$,

$$\begin{aligned} 0 &= \langle x, x_i \rangle - \langle x, x_i \rangle \\ &= \left\langle \tilde{f}(x), \tilde{f}(x_i) \right\rangle - \langle f(x), f(x_i) \rangle \\ &= \left\langle \tilde{f}(x), \tilde{f}(x_i) \right\rangle - \left\langle f(x), \tilde{f}(x_i) \right\rangle \\ &= \left\langle \tilde{f}(x) - f(x), \tilde{f}(x_i) \right\rangle. \end{aligned}$$

As $\tilde{f}(x_1), \ldots, \tilde{f}(x_d)$ are linearly independent then $\tilde{f}(x) = f(x)$ as required. □

Proposition 1.1.33 does not hold for any non-Euclidean normed space however. In fact, many of the difficulties involved from the study of rigidity in general normed spaces stem from exactly this issue.

Proposition 1.1.34. *Let X be a non-Euclidean normed space, then there exists a finite set $S \subset X$ and isometry $f : S \to X$ such that f cannot be extended to an isometry of X.*

Proof. As X is non-Euclidean then by [3, (2.8)], there exists $x, y \in S_1[0]$ such that there exists no linear isometry that maps x to y. Take $S := \{0, x\}$ and $f : S \to X$ with $f(0) = 0$ and $f(x) = y$. If f could be extended to an isometry of X then there would exist a linear isometry that can map x to y, contradicting our choice of x, y. □

Let X be a Euclidean space. Then for all $T \in L(X)$ we may define the *adjoint of* T, the unique linear map $T^* \in L(X)$ where for all $x, y \in X$,

$$\langle x, T(y) \rangle = \langle T^*(x), y \rangle .$$

The following is a useful characterisation of the group of isometries of any Euclidean space.

Proposition 1.1.35. Let X be the Euclidean space and $T \in L(X)$. Then the following hold:

(i) $T \in \operatorname{Isom}^{\mathrm{Lin}}(X)$ if and only if $T^*T = \iota$.

(ii) $T \in T_\iota \operatorname{Isom}^{\mathrm{Lin}}(X)$ if and only if $T^* = -T$.

For the following proof we shall need to define the *exponential map* $\exp : L(X) \to L(X)$, where for all $A \in L(X)$,

$$\exp(A) := \sum_{n=0}^{\infty} \frac{1}{n!} A^n.$$

The exponential function has the following properties:

(i) $\exp(0) = \iota$ [24, Proposition 2.3 (1)].

(ii) $\exp(A^*) = \exp(A)^*$ [24, Proposition 2.3 (2)].

(iii) If $AB = BA$, $\exp(A)\exp(B) = \exp(A + B)$ [24, Proposition 2.3 (5)].

(iv) $\frac{d}{dt}\exp(tA) = \exp(tA)A$ [24, Proposition 2.4].

Proof. (i): By Lemma 1.1.32, for all $x, y \in X$,

$$\langle T^*T(x), y \rangle = \langle T(x), T(y) \rangle = \langle x, y \rangle \quad \Leftrightarrow \quad \langle (T^*T - \iota)(x), y \rangle = 0,$$

thus if $T^*T = \iota$ then $T \in \operatorname{Isom}^{\text{Lin}}(X)$. Suppose $(T^*T - \iota) \neq 0$, then there exists $x \in X$ such that $(T^*T - \iota)(x) \neq 0$. If we choose $y = (T^*T - \iota)(x)$ then

$$\langle (T^*T - \iota)(x), y \rangle = \|(T^*T - \iota)(x)\|^2 \neq 0,$$

thus $T \notin \operatorname{Isom}^{\text{Lin}}(X)$.

(ii): Suppose $T \in T_\iota \operatorname{Isom}^{\text{Lin}}(X)$, then there exists a continuous path $\gamma : (-1, 1) \to \operatorname{Isom}^{\text{Lin}}(X)$ that is differentiable at $t = 0$ with $\gamma(0) = \iota$ and $\gamma'(0) = T$. Define the continuous path $\gamma^* : (-1, 1) \to \operatorname{Isom}^{\text{Lin}}(X)$ with $\gamma^*(t) := \gamma(t)^*$, then γ^* is also differentiable at $t = 0$ with $\gamma^*(0) = \iota$ and $(\gamma^*)'(0) = T^*$. It now follows

$$0 = (\gamma^*(t)\gamma(t))'|_{t=0} = \gamma^*(0)\gamma'(0) + (\gamma^*)'(0)\gamma(0) = T + T^*,$$

thus $T^* = -T$ as required.

Now suppose $T^* = -T$. Define for each $t \in (-1, 1)$, $\gamma(t) := \exp(tT)$, then γ is a differentiable path in $L(X)$ with $\gamma(0) = \iota$ and $\gamma'(0) = T$. As $(tT^*)(tT) = (tT)(tT^*)$,

$$\gamma(t)^*\gamma(t) = \exp(tT)^* \exp(tT) = \exp(tT^* + tT) = \exp(0) = \iota.$$

for all $t \in (-1, 1)$, thus $T \in T_\iota \operatorname{Isom}^{\text{Lin}}(X)$. □

1.2 Frameworks and placements

1.2.1 Notation and graph sparsity

We shall assume that all graphs are simple i.e. no loops or parallel edges, and have a countable vertex set; we will allow them to have a countably infinite vertex set unless we explicitly state otherwise. We will denote $V(G)$ and $E(G)$ to be the vertex and edge sets of G respectively. If H is a subgraph of G we will represent this by $H \subseteq G$. For a given vertex $v \in V(G)$ we define $N_G(v) := \{w \in V(G) : vw \in E(G)$ and $d_G(v) := |N_G(v)|$.

For a set S we shall denote by K_S the complete graph on the set S. If $S \subset V(G)$ for some graph G then we define the *subgraph induced by* S to be the graph $G[S] := (S, E(S))$ with $E(S) := \{vw \in E(G) : v, w \in S\}$. Given a subset $T \subset E(G)$ for some graph G we define $V(T) := \{v \in V(G) : vw \in T$ for some $w \in V(G)\}$.

For a set S we shall denote the *power set of* S (the set of all subsets of S) by $\mathcal{P}(S)$.

If S is an infinite set and T is a finite subset of S we shall denote this by $T \subset\subset S$. Further, if G is a countably infinite graph and H is a finite subgraph we denote this by $H \subset\subset G$. We define a subset $T \subset S$ to be a *cofinite subset* if $S \setminus T$ is finite, and we say $H \subset G$ is a *cofinite subgraph* if both $E(G) \setminus E(H)$ and $V(G) \setminus V(H)$ are finite.

We shall note that the following definition can be applied to both finite and infinite graphs. We refer the reader to [43] for more details on graph sparsity.

Definition 1.2.1. Let $k, l \in \mathbb{N}$. We say a graph G is (k, l)*-sparse* if for every finite subgraph $H \subset G$ we have $|E(H)| \leq \max\{k|V(H)| - l, 0\}$. We say a finite (k, l)-sparse graph G is (k, l)*-tight* if $|E(G)| = k|V(G)| - l$.

1.2.2 Definitions for frameworks and placements

Let X be a normed space. For any set S we say $p \in X^S$ is a *placement of* S *in* X; we will denote this (p, S) if we need to clarify what set p is the placement of. For a graph G we say p is a *placement of* G *in* X if p is a placement of $V(G)$. We define a *(bar-joint) framework* to be a pair (G, p) where G is a graph and p is a placement of G in X. For all X and S we will gift X^S the product topology from X; if $|S| < \infty$ we define the norm

$$\|\cdot\|_S : (x_v)_{v\in S} \mapsto \max_{v\in S} \|x_v\|$$

on X^S. For $x \in X^S$ and $T \subset S$ we define $x|_T := (x_v)_{v\in T} \in X^T$.

A placement p is *spanning* in X if the set $\{p_v : v \in S\}$ affinely spans X. A placement p is in *general position* if for any choice of distinct vertices $v_0, v_1, \ldots, v_n \in S$ ($n \leq \dim X$) the set $\{p_{v_i} : i = 0, 1, \ldots, n\}$ is affinely independent. It is immediate that if p is in general position and $|S| \geq \dim X + 1$ then p is spanning. We denote the set of placements of S in general position by $\mathcal{G}(S) \subseteq X^S$; likewise for any graph G we let $\mathcal{G}(G) := \mathcal{G}(V(G))$. If S is finite, $\mathcal{G}(S)$ is an algebraic variety, thus by Proposition B.3.7, $\mathcal{G}(S)$ is an open dense subset of X^S and $X^S \setminus \mathcal{G}(S)$ is negligible.

For placements (q, T), (p, S) we say (q, T) is a *subplacement* of (p, S) (or $(q, T) \subseteq (p, S)$) if $T \subseteq S$ and $p_v = q_v$ for all $v \in T$. For frameworks (H, q) and (G, p) we say (H, q) is a *subframework* of (G, p) (or $(H, q) \subseteq (G, p)$) if $H \subseteq G$ and $p_v = q_v$ for all $v \in V(H)$. If H is also a spanning subgraph we say that (H, q) is a *spanning subframework* of (G, p).

1.2.3 The rigidity map and the rigidity matrix

We say that an edge $vw \in E(G)$ of a framework (G, p) is *well-positioned* if $p_v - p_w \in \mathrm{smooth}(X)$; if this holds we define the *edge support functional* $\varphi_{v,w} := \varphi\left(\frac{p_v - p_w}{\|p_v - p_w\|}\right)$. Unless stated otherwise, the edge support functional for an edge vw of a framework (G, p^δ) for some superscript δ will be defined to be $\varphi^\delta_{v,w}$. If all edges of (G, p) are well-positioned we say that (G, p) is *well-positioned* and p is a *well-positioned placement* of G. We shall denote the subset of well-positioned placements of G in X by the set $\mathcal{W}(G)$.

Example 1.2.2. If $X = \ell_q^d$, $q \in (1, \infty)$, $\mathcal{W}(G)$ is exactly the set of placements p of G with no zero length edge, i.e. for all $vw \in E(G)$, $\|p_v - p_w\| \neq 0$. We note that is also covers the Euclidean case.

Corollary 1.2.3. Let X and Y be normed spaces and $T : X \to Y$ be a isometric isomorphism. Let (G, p) be a framework in X and (G, q) be a framework in Y with $q_v = T(p_v)$ for all $v \in V(G)$. Then (G, p) is well-positioned in X if and only if (G, q) is well-positioned in Y.

Proof. As $q_v - q_w = T(p_v - p_w)$, this follows from Proposition 1.1.10 (ii). □

Lemma 1.2.4. Let G be finite, then $\mathcal{W}(G)$ is dense subset of $X^{V(G)}$ and $\mathcal{W}(G)^c$ is negligible.

Proof. By Proposition 1.1.11 (iii) the set smooth(X) is dense and its complement negligible, thus the result holds for all graphs with a single edge. Suppose the result holds for all graphs with $n - 1$ edges and let G be any graph with n edges. Choose $vw \in E(G)$, and define G_1, G_2 to be the spanning subgraphs of G where $E(G_1) := E(G) \setminus \{vw\}$ and $E(G_2) := \{vw\}$. By assumption, $\mathcal{W}(G_1)^c$ and $\mathcal{W}(G_2)^c$ are negligible. As $\mathcal{W}(G)^c = \mathcal{W}(G_1)^c \cup \mathcal{W}(G_2)^c$ then $\mathcal{W}(G)^c$ is negligible also; this further implies $\mathcal{W}(G)$ is also dense. The result now follows by induction. □

We can extend this result to placements where we fix some subset of points.

Lemma 1.2.5. Let G be a finite graph, $\emptyset \neq V \subsetneq V(G)$, X a normed space and $p \in X^V$ chosen such that $p_v - p_w \in$ smooth(X) for all $vw \in E(V)$. Then the set

$$\mathcal{W}(G)_V := \{q \in X^{V(G) \setminus V} : q \oplus p \in \mathcal{W}(G)\}$$

is dense in $X^{V(G) \setminus V}$ and $(\mathcal{W}(G)_V)^c$ is negligible.

Proof. If G has one edge the result can be seen to immediately follow from Lemma 1.1.11 (iii). Suppose this holds for all graphs with $n-1$ edges and let G be a graph with n edges. If there exists no edge connecting V and $V(G) \setminus V$ then $\mathcal{W}(G)_V = \mathcal{W}(G[V(G) \setminus V])$ and so the result follows from Lemma 1.2.4. Suppose there exists $vw \in E(G)$ such that $v \in V$ and $w \in V(G) \setminus V$. Define G_1, G_2 to be the spanning subgraphs of G where $E(G_1) := E(G) \setminus \{vw\}$ and $E(G_2) := \{vw\}$. By assumption $(\mathcal{W}(G_1)_V)^c$ and $(\mathcal{W}(G_2)_V)^c$ are negligible. As $(\mathcal{W}(G)_V)^c = (\mathcal{W}(G_1)_V)^c \cup (\mathcal{W}(G_2)_V)^c$ then $(\mathcal{W}(G)_V)^c$ is negligible also. As the complement of a negligible set is dense the result follows by induction. $\square$

We define the *rigidity map* of G (in X) to be the map

$$f_G : X^{V(G)} \to \mathbb{R}^{E(G)}, \ x = (x_v)_{v \in V(G)} \mapsto (\|x_v - x_w\|)_{vw \in E(G)}$$

and for well-positioned placements p we also define the *rigidity operator* of G at p in X to be the linear map

$$df_G(p) : X^{V(G)} \to \mathbb{R}^{E(G)}, \ x = (x_v)_{v \in V(G)} \mapsto (\varphi_{v,w}(x_v - x_w))_{vw \in E(G)}.$$

For any framework we define the *configuration space* of (G, p) in X to be the set $f_G^{-1}[f_G(p)]$.

Remark 1.2.6. For a finite or infinite graph G, the rigidity map f_G is continuous with respect to the product topologies for $X^{V(G)}$ and $\mathbb{R}^{E(G)}$. Likewise, if $p \in \mathcal{W}(G)$ then the rigidity operator $df_G(p)$ is continuous.

Proposition 1.2.7. [40, Proposition 6] Let X be a normed space and G a finite graph. Then f_G is differentiable at p if and only if p is a well-positioned placement of G;

further, if this holds then the rigidity operator at p is the derivative of the rigidity map at p.

Lemma 1.2.8. Let X be a normed space and G a finite graph. Then the map

$$df_G : \mathcal{W}(G) \to L(X^{V(G)}, \mathbb{R}^{E(G)}), \ x \mapsto df_G(x)$$

is continuous.

Proof. This follows from Proposition 1.1.11 (iv). □

For any well-positioned finite framework we can define the *rigidity matrix of* (G,p) *in* X to be the $|E(G)| \times |V(G)|$ matrix $R(G,p)$ with entries in the dual space X^* given by

$$a_{e,v} := \begin{cases} \varphi_{v,w}, & \text{if } e = vw \in E(G) \\ 0, & \text{otherwise} \end{cases}$$

for all $(e,v) \in E(G) \times V(G)$.

Remark 1.2.9. Let (G,p) be a well-positioned framework in the standard Euclidean space $\mathbb{R}^d$. The usual definition for the rigidity matrix R of (G,p) would be the $|E(G)| \times d|V(G)|$ real valued matrix with entries $a_{e,(v,i)}$ for each $e \in E(G)$, $v \in V(G)$ and $i \in \{1, \ldots, d\}$, where

$$a_{e,(v,i)} := \begin{cases} \text{the } i\text{-th coordinate of } p_v - p_w, & \text{if } e = vw \in E(G) \\ 0, & \text{otherwise} \end{cases}.$$

The matrix R can easily be changed into the rigidity matrix $R(G,p)$, and so in many ways most results are not affected. Our version of a rigidity matrix is defined differently so as to fit better with *pseudo-rigidity matrices* (see Section 1.3.3).

For any $|E(G)| \times |V(G)|$ matrix A with entries in the dual space X^* we may regard A as the linear transform from $X^{V(G)}$ to $\mathbb{R}^{E(G)}$ given by

$$u \mapsto A(u) := \left(\sum_{w' \in V(G)} a_{(vw,w')}(u_{w'}) \right)_{vw \in E(G)} .$$

By this definition we see that A has row independence if and only if A is surjective when considered as a linear transform. With this definition we note that $R(G,p)$ is a matrix representation of $df_G(p)$; we shall often use the notation $R(G,p)$ if we wish to observe properties involving the structure of the matrix and $df_G(p)$ if we wish to observe properties of the linear map. See Appendix B.1 for more details for matrices with vector entries.

1.2.4 Orbits of placements

For any set S, element $x \in X^S$ and affine map $g \in A(X)$ we define $g.x := (g(x_v))_{v \in S}$. With this notation we define for any S the map

$$\phi : \mathrm{Isom}(X) \times X^S \to X^S, \ (g,x) \mapsto g.x.$$

If $|S| < \infty$ then this is a Lie group action of $\mathrm{Isom}(X)$ on X^S; we shall always refer to this group action if we mention $\mathrm{Isom}(X)$ acting on X^S.

Lemma 1.2.10. For any normed space X and finite set S, the group of isometries $\mathrm{Isom}(X)$ acts properly on X^S.

Proof. Let $((g_n.p^n, p^n))_{n \in \mathbb{N}}$ be a convergent sequence in the image of

$$\theta : \mathrm{Isom}(X) \times X^S \to X^S \times X^S$$

with limit (q,p). By Mazur-Ulam's theorem (Theorem 1.1.28), for each $n \in \mathbb{N}$ there exists $G_n \in \mathrm{Isom}^{\mathrm{Lin}}(X)$ and $x_n \in X$ such that $g_n = T_{x_n} \circ G_n$, where T_{x_n} is the translation map $y \mapsto y + x_n$. As $((g_n.p^n, p^n))_{n\in\mathbb{N}}$ converges then $(g_n.p^n)_{n\in\mathbb{N}}$ and $(p^n)_{n\in\mathbb{N}}$ are bounded in X^S, thus $(x_n)_{n\in\mathbb{N}}$ is bounded as

$$\|x_n\| = \|(x_n)_{v\in S}\|_S \leq \|G_n.p^n + (x_n)_{v\in S}\|_S + \|G_n.p^n\|_S = \|g_n.p^n\|_S + \|p^n\|_S.$$

It follows by the Bolzano-Weierstrass theorem that there exists a convergent subsequence $(x_{n_k})_{k\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ with limit $x \in X$. Since $\mathrm{Isom}^{\mathrm{Lin}}(X)$ is compact, there exists a convergent subsequence $(G_{n_{k_l}})_{l\in\mathbb{N}}$ of $(G_{n_k})_{k\in\mathbb{N}}$ with limit $G \in \mathrm{Isom}^{\mathrm{Lin}}(X)$. This implies $(g_{n_{k_l}})_{l\in\mathbb{N}}$ converges to $g := T_x \circ G$, thus $((g_n, p^n))_{n\in\mathbb{N}}$ has a convergent subsequence.

Choose a compact set $C \subset X^S \times X^S$ and any sequence $((g_n, p^n))_{n\in\mathbb{N}}$ in the closed set $\theta^{-1}(C)$. As C is compact then $((g_n.p^n, p^n))_{n\in\mathbb{N}}$ has a convergent subsequence $((g_{n_k}.p^{n_k}, p^{n_k}))_{k\in\mathbb{N}}$. As shown previously, it follows that $((g_{n_k}, p^{n_k}))_{n\in\mathbb{N}}$ has a convergent subsequence, thus $((g_n, p^n))_{n\in\mathbb{N}}$ has a convergent subsequence. Since $\theta^{-1}(C)$ is closed then the convergent subsequence of $((g_n, p^n))_{n\in\mathbb{N}}$ converges in $\phi^{-1}(C)$. It now follows that $\phi^{-1}(C)$ is compact, thus as C was chosen arbitrarily, θ is proper as required. $\square$

Corollary 1.2.11. Let (p,S) be a placement in a normed space X, then Stab_p is a compact subgroup of $\mathrm{Isom}(X)$. Further, if $p_v = 0$ for some $v \in S$ then $\mathrm{Stab}_p \leq \mathrm{Isom}^{\mathrm{Lin}}(X)$.

Proof. It is immediate that Stab_p is a subgroup of $\mathrm{Isom}(X)$.

Suppose p is a finite placement, then by Lemma 1.2.10, ϕ is a proper map. As $\mathrm{Stab}_p := \{g : (g,p) \in \phi^{-1}[(p,p)]\}$ then Stab_p is compact. If p is not finite then we note that for any finite subplacement q, Stab_p is a closed subgroup of the compact group Stab_q, thus Stab_p is compact for any placement p. We finally note that if $p_v = 0$ then each element of Stab_p is linear. $\square$

Lemma 1.2.12. Let p be a placement of a finite set in a normed space X, then $\mathcal{O}_p$ is a closed smooth submanifold of $X^{V(G)}$ and the map

$$\tilde{\phi}_p : \operatorname{Isom}(X)/\operatorname{Stab}_p \to \mathcal{O}_p,\ g\operatorname{Stab}_p \mapsto g.p$$

is a smooth diffeomorphism.

Proof. By Lemma 1.2.10 and Lemma 1.1.26 it follows that $\mathcal{O}_p$ is a closed smooth submanifold of $X^{V(G)}$ diffeomorphic to $\operatorname{Isom}(X)/\operatorname{Stab}_p$ under the diffeomorphism $\tilde{\phi}_p$. □

For a placement (p, S) in a normed space X define the finite dimensional linear space $A_p := \{h.p : h \in A(X)\}$ with the topology inherited from X^S. We note that the topology on A_p is equivalent to the norm topology, thus A_p is a smooth manifold. We also note that if $\alpha : (-\delta, \delta) \to A_p$ is a differentiable path then $\alpha'(t) = (\alpha_v'(t))_{v\in S}$.

Lemma 1.2.13. Let $(q, T) \subset (p, S)$ be placements in X where the affine span of $\{p_v : v \in S\}$ is equal to the affine span of $\{q_v : v \in T\}$. If $|T| < \infty$ then $\mathcal{O}_p$ is a smooth manifold that is diffeomorphic to $\mathcal{O}_q$ and the restriction map

$$\rho : \mathcal{O}_p \to \mathcal{O}_q,\ (x_v)_{v\in S} \mapsto (x_v)_{v\in T}$$

is a smooth diffeomorphism.

Proof. We note that the linear map $\tilde{\rho} : A_p \to A_q$ where $\tilde{\rho}(x) := x|_T$ is a continuous linear isomorphism. This implies the map $\tilde{\rho}^{-1}|_{\mathcal{O}_q}$ is a smooth embedding into the smooth manifold A_p with image $\mathcal{O}_p$. By Lemma 1.2.12, $\mathcal{O}_q$ is a smooth manifold, thus $\mathcal{O}_q$ is diffeomorphic to $\mathcal{O}_p$ and $\rho := \tilde{\rho}|_{\mathcal{O}_p}^{\mathcal{O}_q}$ is a smooth diffeomorphism. □

Let (p, S) be a placement in a normed space X. We will define a *continuous path in X^S through p* to be a family $\alpha := (\alpha_v)_{v\in S}$ of continuous paths $\alpha_v : (-\delta, \delta) \to X$ (for some fixed $\delta > 0$) where $\alpha_v(0) = p_v$ for all $v \in S$. If $\alpha(t) := (\alpha_v(t))_{v \in S} \in \mathcal{O}_p$ for all $t \in (-\delta, \delta)$ then α is a *trivial finite motion*.

Let $u \in X^S$. If there exists a trivial finite motion α of p that is differentiable at $t = 0$ and $u_v = \alpha'_v(0)$ for all $v \in S$ then we say that u is a *trivial (infinitesimal) motion of* p. For any placement p we shall denote $\mathcal{T}(p)$ to be the set all trivial infinitesimal motions of p.

Theorem 1.2.14. Let p be a placement in a normed space X, then $\mathcal{O}_p$ is a smooth manifold with tangent space $\mathcal{T}(p)$ at p and

$$\tilde{\phi}_p : \operatorname{Isom}(X)/\operatorname{Stab}_p \to \mathcal{O}_p, \; g \operatorname{Stab}_p \mapsto g.p$$

is a smooth diffeomorphism.

Proof. Choose a finite subplacement (q, T) of (p, S) so that the p and q affinely span the same space, then $\operatorname{Stab}_p = \operatorname{Stab}_q$. By Lemma 1.2.12, $\mathcal{O}_q$ is a smooth manifold diffeomorphic to $\operatorname{Isom}(X)/\operatorname{Stab}_q$ under the smooth diffeomorphism $\tilde{\phi}_q$. By Lemma 1.2.13, $\mathcal{O}_p$ is a smooth manifold diffeomorphic to $\operatorname{Isom}(X)/\operatorname{Stab}_p$ and the restriction map $\rho : \mathcal{O}_p \to \mathcal{O}_q$ is a smooth diffeomorphism. As $\tilde{\phi}_p = \rho^{-1} \circ \tilde{\phi}_q$ then it is also a smooth diffeomorphism. It follows from its definition that $\mathcal{T}(p)$ is the tangent space of $\mathcal{O}_p$ at p. □

Remark 1.2.15. As the set of all trivial infinitesimal motions of a placement is a tangent space to a manifold, it follows that it must therefore be linear.

Corollary 1.2.16. Let p be a placement in a normed space X. Then the map ϕ_p is a smooth submersion and $d\phi_p(\iota) : T_\iota \operatorname{Isom}(X) \to \mathcal{T}(p)$ is surjective with $d\phi_p(\iota)g = g.p$ for all $g \in T_\iota \operatorname{Isom}(X)$. Further, $\ker d\phi_p(\iota) = T_\iota \operatorname{Stab}_p$.

Proof. By Theorem 1.2.14, $\tilde{\phi}_p$ is a smooth diffeomorphism. We note that $\phi_p = \tilde{\phi}_p \circ \pi$ where $\pi : \operatorname{Isom}(X) \to \operatorname{Isom}(X)/\operatorname{Stab}_p$ is the natural quotient map. By Theorem 1.1.25,

π is a smooth submersion, thus ϕ_p is a smooth submersion also and $d\phi_p(\iota)$ is surjective. As $\tilde{\phi}_p$ is a smooth diffeomorphism then $\ker d\phi_p(\iota) = \ker \pi$ as required. $\square$

Corollary 1.2.17. Let (p, S) and (q, T) be placements in a normed space X where the affine span of $\{p_v : v \in S\}$ is equal to the affine span of $\{q_v : v \in T\}$, then the following hold:

(i) The orbits $\mathcal{O}_p$ and $\mathcal{O}_q$ are diffeomorphic.

(ii) $\dim \mathcal{T}(p) = \dim \mathcal{T}(q)$.

(iii) If $(q, T) \subseteq (p, S)$ then the restriction map

$$\rho : \mathcal{O}_p \to \mathcal{O}_q, \ (x_v)_{v \in S} \mapsto (x_v)_{v \in T}$$

is a smooth diffeomorphism.

Proof. (iii): Choose a finite subplacement $(r, U) \subseteq (q, T) \subseteq (p, S)$ so that the affine span of $\{r_v : v \in U\}$ is equal to the affine span of $\{q_v : v \in T\}$. Define the restriction maps $\rho_T : \mathcal{O}_q \to \mathcal{O}_r$ and $\rho_S : \mathcal{O}_p \to \mathcal{O}_r$, then by Lemma 1.2.13, ρ_T, ρ_S are smooth diffeomorphisms. As $\rho = \rho_T^{-1} \circ \rho_S$ then it is also a smooth diffeomorphism.

(i): If $(q, T) \subseteq (p, S)$ this follows from (iii). Suppose (q, T) is not a subplacement of (p, S). Define $(t, S \sqcup T)$ to be the placement where $S \sqcup T$ is the disjoint union of S and T, $t|_S = p$ and $t|_T = q$. We note all three placements span the same affine subspace of X. As $(p, S) \subset (t, S \sqcup T)$ and $(q, T) \subseteq (t, S \sqcup T)$ then by (iii) we have $\mathcal{O}_p \cong \mathcal{O}_t \cong \mathcal{O}_q$ as required.

(ii): By Theorem 1.2.14, $\mathcal{O}_p$ has tangent space $\mathcal{T}(p)$ at p and $\mathcal{O}_q$ has tangent space $\mathcal{T}(q)$ at q. By (i), $\mathcal{O}_p \cong \mathcal{O}_q$ and so $\dim \mathcal{T}(p) = \dim \mathcal{T}(q)$. $\square$

1.2.5 Full placements and isometrically full placements

Proposition 1.2.18. Let p be a placement in a normed space X. Then the following holds:

(i) $d\phi_p(\iota)$ is injective if and only if ϕ_p is a smooth local diffeomorphism i.e. $d\phi_p(g)$ is bijective for all $g \in \operatorname{Isom}(X)$.

(ii) ϕ_p is injective if and only if ϕ_p is a smooth diffeomorphism.

(iii) If either (i) or (ii) hold then $\dim \mathcal{T}(p) = \dim \operatorname{Isom}(X)$.

Proof. (i): If ϕ_p is a local diffeomorphism then $d\phi_p(\iota)$ is bijective.

Suppose $d\phi_p(\iota)$ is injective. Choose any $g \in \operatorname{Isom}(X)$, then we note that $\ker d\phi_p(g) = g \ker d\phi_p(\iota)$. It now follows that $d\phi_p(g)$ is injective for all $g \in \operatorname{Isom}(X)$. By Corollary 1.2.16 it follows that $d\phi_p(g)$ is bijective for all $g \in \operatorname{Isom}(X)$, thus ϕ_p is a local diffeomorphism.

(ii): Suppose ϕ_p is injective. Then Stab_p is trivial, thus $\operatorname{Isom}(X)/\operatorname{Stab}_p = \operatorname{Isom}(X)$ and $\tilde{\phi}_p = \phi_p$. By Theorem 1.2.14, ϕ_p is a smooth diffeomorphism. Conversely, suppose ϕ_p is a smooth diffeomorphism. As $\phi_p = \tilde{\phi}_p \circ \pi$, the quotient map π is a diffeomorphism. This implies π is a group isomorphism, thus Stab_p is trivial and ϕ_p is injective.

(iii): If either (i) or (ii) hold then $d\phi_p(\iota)$ is bijective and $\dim \mathcal{T}(p) = \operatorname{Isom}(X)$. $\square$

Definition 1.2.19. We define a placement p to be *full* if ϕ_p is a local diffeomorphism and *isometrically full* if ϕ_p is a diffeomorphism.

It is immediate that any isometrically full placement is full. By Proposition 1.2.18 (i) our notion of full agrees with that given in [35]. The set of full placements of a set S will be denoted by $\operatorname{Full}(S)$ and likewise the set of full placements of a graph G will be denoted by $\operatorname{Full}(G)$.

Corollary 1.2.20. Let p be a placement in a normed space X. Then the following hold:

(i) p is full if and only if Stab_p is a finite set of points.

(ii) p is isometrically full if and only if $\operatorname{Stab}_p = \{\iota\}$.

Proof. (i): By Corollary 1.2.11, Stab_p is compact. By Corollary 1.2.16, $\ker d\phi_p(\iota) = T_\iota \operatorname{Stab}_p$. By Proposition 1.2.18 (i), it follows that p is full if and only if $T_\iota \operatorname{Stab}_p = \{0\}$, i.e. Stab_p is a 0-dimensional manifold. As all compact 0-dimensional manifolds are finite then the result follows.

(i): This follows immediately from Proposition 1.2.18 (ii). □

Corollary 1.2.21. Let X and Y be normed spaces and $T : X \to Y$ be a isometric isomorphism. Let p be a placement in X and q be a placement in Y with $q_v = T(p_v)$ for all $v \in V(G)$. Then the space of trivial infinitesimal motions of p is linearly isomorphic to the space of trivial infinitesimal motions of q. Further, p is full (respectively, isometrically full) in X if and only if q is full (respectively, isometrically full) in Y.

Proof. Define the linear isomorphism $F : A(X) \to A(Y)$ with $F(g) := T \circ g \circ T^{-1}$. It follows from Proposition 1.1.31 that

$$T_\iota \operatorname{Isom}(Y) = \{F(g) : g \in T_\iota \operatorname{Isom}(X)\},$$

thus $T_\iota \operatorname{Isom}(X)$ and $T_\iota \operatorname{Isom}(Y)$ are isomorphic. Now define $\mathcal{T}_X(p)$ to be the space of trivial infinitesimal motions of p in X, $\mathcal{T}_Y(q)$ to be the space of trivial infinitesimal motions of q in Y, and $\tilde{F} : \mathcal{T}_X(p) \to \mathcal{T}_Y(q)$ to be the linear map where $\tilde{F}(g.p) := F(g).q$. Suppose $\tilde{F}(g.p) = 0$, then

$$0 = \tilde{F}(g.p) = F(g).q = (T \circ g \circ T^{-1}(T(p_v)))_{v \in V(G)} = (T \circ g(p_v))_{v \in V(G)}.$$

As T is an isomorphism then $\tilde{F}$ is a bijection and $\mathcal{T}_X(p) \cong \mathcal{T}_Y(q)$.

By Corollary 1.2.20 and Proposition 1.1.31, it follows that p is full (respectively, isometrically full) in X if and only if q is full (respectively, isometrically full) in Y. $\square$

Corollary 1.2.22. Let p, q be placements in a normed space X. Suppose q is (isometrically) full and q is a subplacement of a placement p, then p is (isometrically) full.

Proof. As $\mathrm{Stab}_p \leq \mathrm{Stab}_q$ this follows from Corollary 1.2.20. $\square$

Corollary 1.2.23. Let p be a placement in a normed space X. Suppose p is (isometrically) full and $q \in \mathcal{O}_p$, then q is (isometrically) full.

Proof. Let $q := h.p$ for some $h \in \mathrm{Isom}(X)$, then $\mathrm{Stab}_q = h \,\mathrm{Stab}_p\, h^{-1}$, thus $|\,\mathrm{Stab}_q\,| = |\,\mathrm{Stab}_p\,|$. The result now follows Corollary 1.2.20. $\square$

Corollary 1.2.24. Let p be a placement in a normed space X where the affine span of $\{p_v : v \in V\}$ is a hyperplane of X, then p is full.

Proof. By Lemma 1.1.29 (i), there exists a Euclidean norm $\|\cdot\|_2$ so that $\mathrm{Isom}(X, \|\cdot\|)$ is a subgroup of $\mathrm{Isom}(X, \|\cdot\|_2)$, thus

$$\mathrm{Stab}_p \leq \mathrm{Stab}^2_p := \{h \in \mathrm{Isom}(X, \|\cdot\|_2) : h.p = p\}.$$

As the affine span of $\{p_v : v \in V\}$ is a hyperplane of X then $\mathrm{Stab}^2_p = \{\iota, h\}$, where h is the unique reflection of X about the hyperplane formed by the affine span of $\{p_v : v \in V\}$. It now follows that $|\,\mathrm{Stab}_p\,| \leq 2$, thus by Corollary 1.2.20 (i), p is full. $\square$

Corollary 1.2.25. Let p be a spanning placement in a normed space X, then p is isometrically full.

Proof. Suppose $g.p = p$ for some $g \in \mathrm{Isom}(X)$ and choose $v_0, \ldots, v_d \in S$ so that $p_{v_0}, \ldots, p_{v_d}$ is an affine basis of X. By our choice of isometry, $g(p_{v_i}) = p_{v_i}$ for all

$i = 0, \dots, d$. By Mazur-Ulam's theorem [67], g is affine, thus as $p_{v_0}, \dots, p_{v_d}$ is an affine basis of X this map must be unique. As $\iota.p = p$ then $g = \iota$ and ϕ_p is injective. The result now follows by Proposition 1.2.18 (ii). □

Example 1.2.26. Choose $q \in [1, \infty]$ with $q \neq 2$. The linear isometries of ℓ_q^2 are generated by the $\pi/2$ anticlockwise rotation around the origin and the reflection in the line $\{(t, 0) : t \in \mathbb{R}\}$. Let $S = \{v_1, v_2\}$ and (p^1, S) and (p^2, S) be the non-spanning placements in X where $p^1_{v_1} = p^2_{v_1} = 0$, $p^1_{v_2} = (1, 0)$ and $p^2_{v_2} = (1, 2)$. Both placements are full in X, however p^2 is isometrically full while p^1 is not. This example shows that while all spanning placements are isometrically full and all isometrically full placements are full the reverse is not necessarily true.

Proposition 1.2.27. Let $d + 1 \leq |S| < \infty$ and X a d-dimensional normed space. Then $\mathrm{Full}(S)$ is an open dense subset of X^S and $\mathrm{Full}(S)^c$ is negligible.

Proof. Since $|S| \geq d + 1$ then all placements in general position are spanning. By Corollary 1.2.25 we have $\mathcal{G}(S) \subset \mathrm{Full}(S)$, thus as $\mathcal{G}(S)$ is dense in X^S then $\mathrm{Full}(S)$ is dense in X^S. Since $\mathcal{G}(S)^c$ is an algebraic set then it is negligible, thus it follows $\mathrm{Full}(S)^c$ also is negligible.

Define the affine map

$$F : X^S \to L(T_\iota \operatorname{Isom}(X), X^S), \; p \mapsto d\phi_p(\iota).$$

The set of injective maps of $L(T_\iota \operatorname{Isom}(X), X^S)$ is open. We note $\mathrm{Full}(S)$ is the preimage of the set of injective maps of $L(T_\iota \operatorname{Isom}(X), X^S)$ under F by Proposition 1.2.18 (ii) and so $\mathrm{Full}(S)$ is open. □

Corollary 1.2.28. Let X be a normed space. Then all isometrically full placements in X are spanning if and only if X is Euclidean.

Proof. Suppose all isometrically full placements in X are spanning, then it follows by Proposition 1.2.18 (ii) that for all linear hyperplanes Y of X there exists a linear map $T_Y \neq \iota$ that is invariant on Y. By [3, (4.7)], X is Euclidean.

Conversely, suppose X is Euclidean. If p is a non-spanning placement in X then p lies in some affine hyperplane H. We note that if h is the reflection in H then $h.p = p$, thus by Proposition 1.2.18 (ii) p is not isometrically full. □

We may now give an upper and lower bound for the dimension of $\mathcal{T}(p)$.

Theorem 1.2.29. For any placement p in a d-dimensional normed space X,

$$d \leq \dim \mathcal{T}(p) \leq \dim \operatorname{Isom}(X)$$

with $\dim \mathcal{T}(p) = \dim \operatorname{Isom}(X)$ if and only if p is full.

Proof. By Corollary 1.2.16, $d\phi_p(\iota) : T_\iota \operatorname{Isom}(X) \to \mathcal{T}(p)$ is surjective, thus we have $\dim \mathcal{T}(p) \leq \dim \operatorname{Isom}(X)$. Let $x_1, \ldots, x_d \in X$ be a basis. Define for each $i \in \{1, \ldots, d\}$ the trivial finite motion $\alpha(i)$ of p where for each $v \in S$ we have

$$\alpha(i)_v : (-1, 1) \to X, \ t \mapsto p_v + tx_i.$$

We note that $(\alpha(i)'_v(0))_{v \in S} = (x_i)_{v \in S} \in \mathcal{T}(p)$ for each $i \in \{1, \ldots, d\}$, thus $\dim \mathcal{T}(p) \geq d$.

If p is full then by Proposition 1.2.18 (i), $\dim \mathcal{T}(p) = \dim \operatorname{Isom}(X)$. If $\dim \mathcal{T}(p) = \dim \operatorname{Isom}(X)$ then by Corollary 1.2.16, $d\phi_p(\iota)$ is bijective; it then follows by Proposition 1.2.18 (i) that p is full. □

A *rigid motion* of X is a family $\gamma := (\gamma_x)_{x \in X}$ of continuous maps

$$\gamma_x : (-\delta, \delta) \to X, \ x \in X$$

(for some fixed $\delta > 0$) where $\gamma_x(0) = x$ and $\|\gamma_x(t) - \gamma_y(t)\| = \|x - y\|$ for all $x, y \in X$ and $t \in (-\delta, \delta)$. We note immediately that for each $t \in (-\delta, \delta)$, the map $x \mapsto \gamma_x(t)$ is an isometry. The following shows that our definition of a trivial finite motion agrees with the definition given in [36] if a framework is isometrically full.

Proposition 1.2.30. Let p be an isometrically full placement in a normed space X. If α is a continuous path through p in X^S then the following are equivalent:

(i) α is a trivial finite motion.

(ii) There exists a unique continuous path $h : (-\delta, \delta) \to \mathrm{Isom}(X)$ such that $h_t(p_v) = \alpha_v(t)$ for all $t \in (-\delta, \delta)$ and $v \in S$.

(iii) There exists a unique rigid motion γ such that $\gamma_{p_v} = \alpha_v$ for all $v \in S$.

Proof. (i) $\Rightarrow$ (ii): As α is a continuous path in $\mathcal{O}_p$ and ϕ_p is a smooth diffeomorphism we define the unique continuous path $h := \phi_p^{-1} \circ \alpha$.

(ii) $\Rightarrow$ (iii): Define γ to be the unique family of maps γ where $\gamma_x(t) = h_t(x)$ for all $x \in X$ and $t \in (-\delta, \delta)$, then γ is a rigid motion as required.

(iii) $\Rightarrow$ (i): We note that γ restricted to the set $\{p_v : v \in S\}$ is a trivial finite motion. $\square$

1.3 Rigidity and independence

1.3.1 Local, continuous and infinitesimal rigidity

We define a *finite flex* of a framework (G, p) to be a continuous path α in $X^{V(G)}$ through a placement p where $\|\alpha_v(t) - \alpha_w(t)\| = \|p_v - p_w\|$ for all $vw \in E(G)$ and $t \in (-\delta, \delta)$. If α is a trivial finite motion of a placement p of G we say α is a *trivial finite flex* of (G, p); we note that α will automatically be a finite flex of (G, p) as isometries preserve distance.

We define $u \in X^{V(G)}$ to be a *trivial (infinitesimal) flex of* (G,p) if u is a trivial motion of p. If (G,p) is well-positioned we say that $u \in X^{V(G)}$ is an *(infinitesimal) flex of* (G,p) if $df_G(p)u = 0$. The following proposition shows a link between finite and infinitesimal flexes for frameworks.

Lemma 1.3.1. Let (G,p) be a well-positioned framework in X and α a finite flex of (G,p) that is differentiable at 0, then $(\alpha'_v(0))_{v\in V(G)}$ is an infinitesimal flex of (G,p).

Proof. This follows from the proof of [36, Lemma 2.1.(ii)]. $\square$

Since all trivial flexes of (G,p) are trivial motions of p we shall also denote $\mathcal{T}(p)$ to be the set all trivial infinitesimal flexes (G,p). If (G,p) is well-positioned we define $\mathcal{F}(G,p)$ to be the space of all infinitesimal flexes of (G,p). The latter is clearly a linear space as it is exactly the kernel of the rigidity operator. By Proposition 1.3.1 it follows $\mathcal{T}(p) \subseteq \mathcal{F}(G,p)$.

Definition 1.3.2. Let (G,p) be a framework in a normed space X. Then we define the following:

(i) (G,p) is *continuously rigid (in X)* if the only finite flexes of (G,p) are trivial, and (G,p) is *continuously flexible* if it is not continuously rigid.

(ii) Suppose (G,p) is finite. (G,p) is *locally rigid (in X)* if there exists a neighbourhood $U \subseteq X^{V(G)}$ of p such that $f_G^{-1}[f_G(p)] \cap U = \mathcal{O}_p \cap U$, and (G,p) is *locally flexible* if it is not locally rigid.

(iii) Suppose (G,p) is well-positioned. (G,p) is *infinitesimally rigid (in X)* if every flex is trivial, and (G,p) is *infinitesimally flexible (in X)* if it is not infinitesimally rigid.

We classify both continuous rigidity and local rigidity as *finite rigidity*.

1.3.2 Regular placements and independence for finite graphs

For a finite graph G we say that a well-positioned framework (G,p) in a normed space X is *regular* if for all $q \in \mathcal{W}(G)$ we have $\operatorname{rank} df_G(p) \geq \operatorname{rank} df_G(q)$. We shall denote the subset of $\mathcal{W}(G)$ of regular placements of G in X by $\mathcal{R}(G)$.

Lemma 1.3.3. Let G be a finite graph and X a normed space. Then the set of regular placements of G is a non-empty open subset of the set of well-positioned placements of G.

For this lemma we shall need to use the fact that the rank function on the space of linear maps between finite dimensional normed spaces X, Y is *lower semi-continuous* i.e. for all $c \geq 0$ the set $\{T \in L(X,Y) : \operatorname{rank} T \geq c\}$ is open.

Proof. Let $n := \sup\{\operatorname{rank} df_G(p) : p \in \mathcal{W}(G)\}$. The rank function $T \mapsto \operatorname{rank} T$ is lower semi-continuous and by Lemma 1.2.8, df_G is continuous, thus the map

$$f : \mathcal{W}(G) \to \mathbb{N} \cup \{0\}, \ p \mapsto \operatorname{rank} df_G(p)$$

is lower semi-continuous. As $\mathcal{R}(G) = f^{-1}[[n, \infty)]$ then $\mathcal{R}(G)$ is open. □

Lemma 1.3.4. Let G be a finite graph and X a normed space. Then the set of regular placements of G in general position is a non-empty open subset of the set of well-positioned placements of G.

Proof. By Lemma 1.2.4, $\mathcal{W}(G)^c$ is negligible. As $\mathcal{G}(G)^c$ is an algebraic set then it is a closed negligible set, thus $\mathcal{G}(G) \cap \mathcal{W}(G)$ is dense in $X^{V(G)}$ and $\mathcal{G}(G) \cap \mathcal{W}(G)$ is an open dense subset of $\mathcal{W}(G)$. By Lemma 1.3.3, $\mathcal{R}(G)$ is open in $\mathcal{W}(G)$ and so the result follows. □

Remark 1.3.5. If X is Euclidean then for any finite graph G the set $\mathcal{R}(G)$ is an open dense subset of $\mathcal{W}(G)$ (see [5, Section 3] for more details), however this is not always the case e.g. if X is a polyhedral normed space [34, Lemma 16].

Definition 1.3.6. Let (G,p) be a (possibly infinite) well-positioned framework in a normed space X. We define (G,p) to be *independent* if $df_G(p)$ is surjective and define (G,p) to be *dependent* otherwise. If (G,p) is infinitesimally rigid and independent we shall say that it is *isostatic*.

We shall use the convention that any framework with no edges (regardless of placement) is independent and that (K_1,p) is isostatic for any choice of placement p. It is immediate that if a finite framework is independent then its placement is regular, however the reverse does not necessarily hold.

We have a few equivalent definitions for independence. We first define for any well-positioned finite framework (G,p) an element $(a_{vw})_{vw\in E(G)} \in \mathbb{R}^{E(G)}$ to be a *stress* of (G,p) if it satisfies the *stress condition* at each vertex $v \in V(G)$, i.e.

$$\sum_{w\in N_G(v)} a_{vw}\varphi_{v,w} = 0.$$

Remark 1.3.7. Let (G,p) be a well-positioned framework in the standard Euclidean space $\mathbb{R}^d$ and $a \in \mathbb{R}^{E(G)}$. The usual definition for the stress condition is given as

$$\sum_{w\in N_G(v)} a_{vw}(p_v - p_w) = 0$$

for all $v \in V(G)$. If a satisfied the above version of the stress condition then a would not necessarily satisfy our version of the stress condition, since $\varphi_{v,w} = \varphi(\frac{p_v-p_w}{\|p_v-p_w\|})$, however $(a_{vw}\|p_v - p_w\|)_{vw\in E(G)}$ would, and so in many ways most results are not effected. Our version of a stress is defined differently so as to fit better with *pseudo-stresses* (see Section 1.3.3).

Proposition 1.3.8. Let (G, p) be a finite well-positioned framework in a normed space X. Then the following are equivalent:

(i) (G, p) is independent.

(ii) $R(G, p)$ has independent rows.

(iii) $|E(G)| = \operatorname{rank} df_G(p)$.

(iv) The only stress of (G, p) is the zero stress i.e. $a_{vw} = 0$ for all $vw \in E(G)$.

Proof. (i) $\Leftrightarrow$ (ii): If we consider $R(G, p)$ as a linear transform then it is surjective if and only if it has row independence. As $R(G, p) = df_G(p)$ when considered as a linear transform the result follows.

(i) $\Leftrightarrow$ (iii): This follows immediately as $\operatorname{im} df_G(p) \subseteq \mathbb{R}^{E(G)}$.

(ii) $\Leftrightarrow$ (iv): A non-zero stress is equivalent to a linear dependence on the edges of $R(G, p)$. $\square$

Remark 1.3.9. Let (G, p) be a well-positioned framework, then we may define a subset $E \subset E(G)$ to be independent if the subframework generated on the edge set E is an independent framework. Since framework independence is a property determined by matrix row independence then the power set of $E(G)$ with the independent sets as defined will be a *matroid*, see Appendix A.2.1 for more details. Because of this, it follows that every infinitesimally rigid framework has a spanning isostatic subframework; we remove any independent edges to form a maximally independent subframework (H, p) and note that $\operatorname{rank} df_G(p) = \operatorname{rank} df_H(p)$.

We note that rigidity and independence is all equivalent for isometrically isomorphic normed spaces.

Corollary 1.3.10. Let X and Y be normed spaces and $T : X \to Y$ be a isometric isomorphism. Let (G, p) be a framework in X and (G, q) be a framework in Y with $q_v = T(p_v)$ for all $v \in V(G)$. Then the following holds:

(i) If (G,p) is finite and well-positioned, (G,p) is regular in X if and only if (G,q) is regular in Y.

(ii) If (G,p) is well-positioned, (G,p) is independent in X if and only if (G,q) is independent in Y.

(iii) If (G,p) is well-positioned, (G,p) is infinitesimally rigid in X if and only if (G,q) is infinitesimally rigid in Y.

(iv) If (G,p) is finite, (G,p) is locally rigid in X if and only if (G,q) is locally rigid in Y.

(v) (G,p) is continuously rigid in X if and only if (G,q) is continuously rigid in Y.

Proof. We first note that by Corollary 1.2.3, if (G,p) is well-positioned in X then (G,q) is well-positioned in Y. Define the linear isomorphism $F : X^{V(G)} \to Y^{V(G)}$ with $F((x_v)_{v\in V(G)}) = (T(x_v))_{v\in V(G)}$, f_G^X to be the rigidity map of G in X and f_G^Y to be the rigidity map of G in Y. We note immediately that $f_G^Y = f_G^X \circ F^{-1}$ and (if (G,p) is well-positioned) $df_G^Y(q) = df_G^X(p) \circ F^{-1}$. It follows that, if (G,p) is well-positioned, $df_G^X(p)$ is surjective if and only if $df_G^Y(q)$ is surjective, and if (G,p) is also finite, $\operatorname{rank} df_G^X(p) = \operatorname{rank} df_G^Y(q)$, thus (i) and (ii) hold.

(iii): Define $\mathcal{T}_X(p)$ to be the space of trivial infinitesimal flexes of p in X and $\mathcal{T}_Y(q)$ to be the space of trivial infinitesimal flexes of q in Y. We note that $\dim\ker df_G^X(p) = \dim\ker df_G^Y(q)$ and, by Corollary 1.2.21, $\dim \mathcal{T}_X(p) = \dim \mathcal{T}_Y(q)$, thus the result holds.

(iv): Suppose that (G,p) is locally rigid in X, then there exists a neighbourhood $U \subseteq X^{V(G)}$ of p such that if $p' \in U$ and $f_G^X(p') = f_G^X(p')$ then there exists $g \in \operatorname{Isom}(X)$ such that $p' = g.p$. Define $U' := F(U)$ and suppose $q' \in U'$ with $f_G^Y(q') = f_G^Y(q)$. As $F^{-1}(q') \in U$ and

$$f_G^X(F^{-1}(q')) = f_G^Y(q') = f_G^Y(q) = f_G(p),$$

then there exists $g \in \mathrm{Isom}(X)$ such that

$$g.F^{-1}(q') = g \circ T^{-1}.q' = p.$$

By Proposition 1.1.31, $T \circ g \circ T^{-1}$ is an isometry of Y. As $T \circ g \circ T^{-1}.q' = q$, then (G, q) is locally rigid in Y. By a similar method we can show the converse holds also.

(v): This follows using a similar method to (iv). □

1.3.3 Pseudo-rigidity matrices

Often frameworks which are not well-positioned can be used to obtain information about well-positioned frameworks. We can apply the following method to test for independence, mainly applied in sections 3.2.2 and 3.3.

Suppose (G, p) is a not well-positioned framework in a normed space X, then there exists a non-empty subset $F \subset E(G)$ of non-well-positioned edges. For each $vw \in F$ we will choose some $f \in X^*$ and define $\varphi_{v,w} := f$. We define $\varphi_{v,w}$ to be the *pseudo-support functional of vw for p*. Using the support functionals of the edges in $E(G) \setminus F$ and the chosen pseudo-support functionals of the edges in F we define $\phi := \{\varphi_{v,w} : vw \in E(G)\}$ to be the set of support functionals and pseudo-support functionals for our framework and $R(G, p)^\phi$ to be the $|E(G)| \times |V(G)|$ *pseudo-rigidity matrix* generated by our set ϕ in the same manner as the rigidity matrix. We shall also use the notation $(G, p)^\phi$ to indicate that we are considering (G, p) with the pseudo-rigidity matrix $R(G, p)^\phi$.

We define $(G, p)^\phi$ to be *independent* if $R(G, p)^\phi$ has row independence and *dependent* otherwise. We define a vector $a := (a_{vw})_{vw \in E(G)} \in \mathbb{R}^{E(G)}$ to be a *pseudo-stress of* $(G, p)^\phi$ if it satisfies the *pseudo-stress condition* i.e. for all $v \in V(G)$, $\sum_{w \in N_G(v)} a_{vw}\varphi_{v,w} = 0$. Following from Proposition 1.3.8 we can see that $(G, p)^\phi$ is independent if and only if the only pseudo-stress is $(0)_{vw \in E(G)}$.

Suppose we have a sequence $(p^n)_{n\in\mathbb{N}}$ of well-positioned placements of G such that $p^n \to p$ as $n \to \infty$ and the sequences $(\varphi^n_{v,w})_{n\in\mathbb{N}}$ in X^* converge for all $vw \in E(G)$, where $\varphi^n_{v,w}$ is the support functional of vw in (G, p^n). If $vw \in E(G) \setminus F$ then by Proposition 1.1.11 (iv), $\varphi^n_{v,w} \to \varphi_{v,w}$ as $n \to \infty$. We say that $(G, p)^\phi$ is the *framework limit* of (G, p^n) (or $(G, p^n) \to (G, p)^\phi$ as $n \to \infty$) if $\varphi^n_{v,w} \to \varphi_{v,w}$ for all $vw \in E(G)$.

Proposition 1.3.11. Suppose $(G, p)^\phi$ is the framework limit of the sequence of well-positioned frameworks $((G, p^n))_{n\in\mathbb{N}}$ in a normed space X. If $R(G, p)^\phi$ has row independence then there exists $N \in \mathbb{N}$ such that (G, p^n) is independent for all $n \geq N$.

Proof. First note that if we consider $|E(G)| \times |V(G)|$ matrices with entries in X^* to be elements of $L(X^{V(G)}, \mathbb{R}^{E(G)})$ as described in Appendix B.1, then they will have row independence if and only if they are surjective. As $(G, p^n) \to (G, p)^\phi$ as $n \to \infty$ then $R(G, p^n) \to R(G, p)^\phi$ entrywise as $n \to \infty$. Since the set of surjective maps of $L(X^{V(G)}, \mathbb{R}^{E(G)})$ is an open subset and $R(G, p)^\phi$ is surjective then by Lemma 1.2.8 the result follows. □

1.3.4 Necessary conditions for rigidity of frameworks and graphs

The following gives us some necessary and sufficient conditions for infinitesimal rigidity.

Theorem 1.3.12. [40, Theorem 10] Let (G, p) be a finite well-positioned framework in a normed space X. Then the following hold:

(i) If (G, p) is independent then $|E(G)| = (\dim X)|V(G)| - \dim \mathcal{F}(G, p)$.

(ii) If (G, p) is infinitesimally rigid then $|E(G)| \geq (\dim X)|V(G)| - \dim \mathcal{T}(p)$.

The following gives an equivalence for isostaticity.

Proposition 1.3.13. Let (G, p) be a finite well-positioned framework in X. If any two of the following properties hold then so does the third (and (G, p) is isostatic):

(i) $|E(G)| = (\dim X)|V(G)| - \dim \mathcal{T}(p)$

(ii) (G, p) is infinitesimally rigid

(iii) (G, p) is independent.

Proof. Apply the Rank-Nullity theorem to the rigidity operator of G at p. The result follows the same method as [23, Lemma 2.6.1.c]. □

Lemma 1.3.14. Let (G, p) be a finite (possibly not spanning) framework in a d-dimensional normed space X with $|V(G)| \geq d + 1$. Suppose $q \in \mathcal{R}(G)$ is full, then the following hold:

(i) If (G, p) is independent then (G, p) is regular and (G, q) is independent.

(ii) If (G, p) is infinitesimally rigid then (G, p) is regular, p is full and (G, q) is infinitesimally rigid.

Proof. (i): As (G, p) is independent then $df_G(p)$ is surjective. As surjective linear maps have maximal possible rank then (G, p) is regular. Since q is regular it follows that (G, q) is independent.

(ii): As (G, q) is regular then by the Rank-Nullity theorem we have

$$d|V(G)| - \dim \mathcal{T}(p) = \operatorname{rank} df_G(p) \leq \operatorname{rank} df_G(q) \leq d|V(G)| - \dim \mathcal{T}(q),$$

thus by Theorem 1.2.29, $\dim \mathcal{T}(q) \leq \dim \mathcal{T}(p) \leq \dim \operatorname{Isom}(X)$. As q is full then by Theorem 1.2.29, $\dim \mathcal{T}(q) = \dim \operatorname{Isom}(X)$. It follows that $\dim \mathcal{T}(p) = \dim \operatorname{Isom}(X)$ and thus p is full. From the above inequality it now also follows that (G, q) is infinitesimally rigid. □

By using Lemma 1.3.14 we can extend Theorem 1.3.12 to obtain the following.

Corollary 1.3.15. Let (G, p) be a finite independent framework in a normed space X with $|V(G)| \geq \dim X + 1$. Then for all $H \subset G$ with $|V(H)| \geq \dim X + 1$,

$$|E(H)| \leq (\dim X)|V(H)| - \dim \mathrm{Isom}(X).$$

If (G, p) is also isostatic then

$$|E(G)| = (\dim X)|V(G)| - \dim \mathrm{Isom}(X).$$

Proof. As $\mathcal{G}(G)$ is an open dense subset of $X^{V(G)}$ and $\mathcal{G}(G)^c$ is negligible, by Lemma 1.2.4 and Lemma 1.3.3, the set $\mathcal{R}(G) \cap \mathcal{G}(G)$ is non-empty. Choose p' to be a regular placement of G in general position. Since (G, p') is regular it follows that it is also independent.

Define $q := p'|_{V(H)}$, then (H, q) is in general position. As $(H, q) \subseteq (G, p')$ then it follows from Remark 1.3.9 that (H, q) is independent; furthermore as H has at least $d + 1$ vertices then q is spanning. By Theorem 1.3.12,

$$|E(H)| = (\dim X)|V(H)| - \dim \mathcal{F}(H, q).$$

By Corollary 1.2.25 and Theorem 1.2.29, $\dim \mathcal{T}(q) = \dim \mathrm{Isom}(X)$. As $\mathcal{T}(q) \subset \mathcal{F}(H, q)$ we obtain the required inequality.

Suppose (G, p) is also isostatic. We note that the required equality holds due to Proposition 1.3.13, Lemma 1.3.14 (ii) and Theorem 1.2.29. □

Definition 1.3.16. Let X be a normed space and G a finite graph. We define the following:

(i) G is *rigid (in X)* if it has a infinitesimally rigid placement i.e. there exists a well-positioned placement p of G where (G,p) is infinitesimally rigid. If no such placement exists then G is *flexible (in X)*.

(ii) G is *independent (in X)* if it has a independent placement i.e. there exists a well-positioned placement p of G where (G,p) is independent. If no such placement exists then G is *dependent (in X)*.

(iii) G is *isostatic (in X)* if it is both rigid and independent.

If a given graph G is isostatic in a normed space X then by definition it will have independent placements and infinitesimally rigid placements, however from the definition it could possibly be that G has no placement that is both independent and infinitesimally rigid, i.e. isostatic. Fortunately, the following result tells us that this cannot happen.

Proposition 1.3.17. A finite graph G is isostatic if and only if it has an isostatic placement.

Proof. By later results in Chapter 2 (Theorem 2.2.8 and Proposition 2.2.7) we may suppose $|V(G)| \geq d+1$. If G has an isostatic placement then it is both rigid and independent as required. Suppose G is isostatic with infinitesimally rigid placement p. By Theorem 1.3.12,

$$|E(G)| \geq (\dim X)|V(G)| - \dim \mathcal{T}(p).$$

By Lemma 1.3.14 (ii), p is full, thus $\dim \mathcal{T}(p) = \dim \operatorname{Isom}(X)$. As G has an independent placement then it follows by Corollary 1.3.15 that

$$|E(G)| \leq (\dim X)|V(G)| - \dim \operatorname{Isom}(X),$$

thus

$$|E(G)| = (\dim X)|V(G)| - \dim \mathcal{T}(p).$$

By Proposition 1.3.13, (G, p) is isostatic as required. $\square$

The following shows that graph rigidity and graph independence is invariant under isometric isomorphisms.

Corollary 1.3.18. Let X and Y be normed spaces, $T : X \to Y$ be a isometric isomorphism and G a finite graph. Then the following holds:

(i) G is rigid in X if and only if G is rigid in Y.

(ii) G is independent in X if and only if G is independent in Y.

Proof. This follows immediately from Corollary 1.3.10. $\square$

1.3.5 Rigidity in the Euclidean spaces

Many results can be simplified for Euclidean spaces. We outline some that will be useful in this section. The first gives an equivalence of the different forms of rigidity for Euclidean spaces.

Theorem 1.3.19. [5][6] Let (G, p) be a regular finite framework in a Euclidean space X, then the following are equivalent:

(i) (G, p) is infinitesimally rigid in X,

(ii) (G, p) is locally rigid in X,

(iii) (G, p) is continuously rigid in X.

In her 1927 paper, H. Pollaczek-Geiringer proved the following.

Theorem 1.3.20. [58] For any graph G with $|V(G)| \geq 2$, G is isostatic in the Euclidean plane if and only if G is $(2,3)$-tight.

This result was later rediscovered by G. Laman who utilised the Henneberg moves to obtain the result [42]. His proof follows from three key results.

Proposition 1.3.21. [42, Theorem 5.6] Let G be a finite simple graph with $|V(G)| \geq 2$. If G is isostatic in the Euclidean plane then G is $(2,3)$-tight.

Proposition 1.3.22. [42, Theorem 6.4, Theorem 6.5] Henneberg moves preserve the $(2,3)$-tightness and $(2,3)$-sparsity of graphs. Further, any $(2,3)$-tight graph on 2 or more vertices can be constructed from K_2 by a finite sequence of Henneberg moves.

Proposition 1.3.23. [42, Proposition 5.3, Proposition 5.4] If G is isostatic in the Euclidean plane and G' is the graph formed from G by a Henneberg move then G' is also isostatic in the Euclidean plane.

We can now see that combining Proposition 1.3.21, Proposition 1.3.22 and Proposition 1.3.23 we obtain Theorem 1.3.20. Theorem 1.3.20 is the converse of Corollary 1.3.15 for dimension 2, however this cannot be extended to any dimension higher than 2. The counter-example to this for $\mathbb{R}^3$ with the Euclidean norm is the graph G of the double-banana framework (see Figure 1). For every $H \subset G$ with $|V(H)| \geq 3$ we have $|E(H)| \leq 3|V(H)| - 6$, however G is clearly flexible. It is, however, an open problem for all other 3-dimensional normed spaces whether the converse of Corollary 1.3.15 holds.

In the Euclidean spaces it is easy to determine rigidity and flexibility for non-spanning frameworks and/or complete frameworks, as the results below show.

Proposition 1.3.24. Let (G,p) be a finite framework in a d-dimensional Euclidean space X so that the following holds:

(i) The dimension of the affine span of p is n, with $0 \leq n < d$.

(ii) $|V(G)| > n+1$.

Then (G,p) is infinitesimally flexible.

Proof. By translation we may assume that $p_{v_0} = 0$ for some $v_0 \in V(G)$. Let $x_1, \ldots, x_d$ be an orthonormal basis and define $X_k := \operatorname{span}\{x_1, \ldots, x_k\}$ for each $1 \leq k \leq d$; we may assume we chose the basis such that (G,p) lies in X_n. As each space X_k is Euclidean, by Lemma 1.1.29 (ii), $\dim \operatorname{Isom}(X_k) = \frac{k(k+1)}{2}$. For $k \geq n$, we will denote by $\mathcal{T}_k(p)$ and $\mathcal{F}_k(G,p)$ the space of trivial flexes of p in X_k and the space of flexes of p in X_k respectively.

As (G,p) spans X_n then by Corollary 1.2.25, Corollary 1.2.24 and Theorem 1.2.29,

$$\dim \mathcal{T}_n(p) = \frac{n(n+1)}{2}, \qquad \dim \mathcal{T}_{n+1}(p) = \frac{(n+1)(n+2)}{2}.$$

Define for each $v \in V(G)$ the flex $u^v \in (X_n)^{V(G)}$ with $u^v_w = 0$ if $w \neq v$ and $u^v_v = x_{n+1}$. We now note that

$$\begin{aligned} \dim \mathcal{F}_{n+1}(G,p) &\geq \dim \mathcal{T}_n(p) + \dim\{u^v : v \in V(G)\} \\ &> \frac{n(n+1)}{2} + (n+1) \\ &= \frac{(n+1)(n+2)}{2} \\ &= \dim \mathcal{T}_{n+1}(p), \end{aligned}$$

thus (G,p) is infinitesimally flexible in X_{n+1}.

If $n+1 = d$ then $X_{n+1} = X$ and we are done. Suppose $n+1 < d$ and (G,p) is infinitesimally rigid in X. As (G,p) is infinitesimally flexible in X_{n+1} there exists a non-trivial flex u of (G,p) in X_{n+1}; by subtracting a suitable trivial flex we may assume $u_{v_0} = 0$. As (G,p) is infinitesimally rigid in X then u is a trivial flex of (G,p). By Corollary 1.2.16, there exists $T \in T_\iota \operatorname{Isom}^{\mathrm{Lin}}(X)$ such that $T.p = u$. By Proposition

1.1.35 (ii), $T^* = -T$. As $T(p_v) = u_v \in X_{n+1}$ for all $v \in V(G)$ and T is linear, then $T|_{X^n}$ maps into X^{n+1}. Define the linear map $R \in L(X^{n+1})$ such that $R(x) = T(x)$ for all $x \in X^n$ and $R(x_{n+1}) = 0$. It follows $R^* = -R$, thus by Proposition 1.1.35 (ii), $R \in T_\iota \operatorname{Isom}^{\mathrm{Lin}}(X^{n+1})$. By Corollary 1.2.16, $R.p = u \in \mathcal{T}_k(p)$, contradicting that u is non-trivial. $\square$

Proposition 1.3.25. Let (K,p) be a finite framework in a Euclidean space X where K is a complete graph. Then (K,p) is continuously and locally rigid. If (K,p) is also spanning then (K,p) is also infinitesimally rigid.

Proof. By Proposition 1.1.33, $f_K^{-1}[f_K(p)] = \mathcal{O}_p$, thus (K,p) is continuously and locally rigid.

Suppose (K,p) is spanning, then by [23, Corollary 2.3.1], $\mathcal{F}(K,p) = \frac{d(d+1)}{2}$. By Corollary 1.2.25, p is full and by Theorem 1.2.29 and Lemma 1.1.29 (ii), $\dim \mathcal{T}(p) = \frac{d(d+1)}{2}$. As $\mathcal{F}(K,p) = \mathcal{T}(p)$ then (K,p) is infinitesimally rigid. $\square$

Although complete frameworks are always continuously, locally and (in most cases) infinitesimally rigid, in non-Euclidean spaces this does not automatically hold. Take for instances any well-positioned placement p of K_3 in a non-Euclidean normed plane X. We first note that p either affinely spans X or a hyperplane of X, thus by Corollary 1.2.24, p is full. By Theorem 1.2.29 and Lemma 1.1.29 (iii), $\dim \mathcal{T}(p) = 2$, but

$$\dim \mathcal{F}(K_3, p) = 2|V(G)| - \operatorname{rank} df_{K_3}(p) \geq 6 - |E(G)| = 3,$$

thus (K_3, p) will be infinitesimally flexible.

Chapter 2

Framework rigidity in general normed spaces

In this chapter we shall generalise the regularity condition required for Theorem 1.3.19 to a new property - whether a framework is constant. Using this new property, we shall prove an analogue of Theorem 1.3.19 for this new class of framework; see Theorem 2.1.5. Following this we shall establish, among other properties, the flexibility and independence of small frameworks in general non-Euclidean normed spaces. We shall finish the chapter by defining the graph substitution operation for all normed spaces, and detail what properties a normed space requires for the operation to preserve graph rigidity.

2.1 Equivalence of local, continuous and infinitesimal rigidity

2.1.1 Properties of constant and regular placements

For a finite graph G we say that a well-positioned framework (G,p) in a normed space X is *constant* if there is an open neighbourhood $U \subset X^{V(G)}$ of p such that $U \subset \mathcal{W}(G)$ and $\operatorname{rank} df_G(q) = \operatorname{rank} df_G(p)$ for all $q \in U$. We shall define $\mathcal{C}(G)$ to be the subset of $\mathcal{W}(G)$ of constant placements of G.

For Euclidean spaces $\mathcal{R}(G) = \mathcal{C}(G)$ as $\mathcal{R}(G)$ is an open dense subset of $X^{V(G)}$ (see [5, Section 3] for more details). For a large class of normed spaces we can give an extension of this result.

Proposition 2.1.1. Let X be a normed space with an open set of smooth points and G be a finite graph. Then the following holds:

(i) $\mathcal{W}(G)$ is an open dense subset of $X^{V(G)}$.

(ii) $\mathcal{C}(G)$ is an open dense subset of $\mathcal{W}(G)$.

(iii) $\mathcal{R}(G)$ is an open subset of $\mathcal{C}(G)$.

Proof. (i): By Lemma 1.2.4, $\mathcal{W}(G)$ is a dense subset of $X^{V(G)}$. As $\operatorname{smooth}(X)$ is an open subset of X it follows that if $|E(G)| = 1$ then $\mathcal{W}(G)$ is open. Suppose for all finite graphs G with at most n edges the set $\mathcal{W}(G)$ is an open dense subset of $X^{V(G)}$. Let H be any graph with $n+1$ edges and choose an edge $e \in E(H)$. Define

$$H_1 = (V(H), \{e\}), \qquad H_2 = (V(H), E(H) \setminus \{e\}),$$

then $\mathcal{W}(H) = \mathcal{W}(H_1) \cap \mathcal{W}(H_2)$. As $|E(H_1)|, |E(H_2)| \leq n$ then $\mathcal{W}(H_1)$, $\mathcal{W}(H_2)$ are open dense subsets of $X^{V(H)}$ by assumption, thus $\mathcal{W}(H)$ is an open dense subset also.

By induction it now follows that $\mathcal{W}(G)$ is an open dense subset of $X^{V(G)}$ for any finite graph G.

(ii): For any open set $U \subseteq \mathcal{W}(G)$ and $n \in \mathbb{N}$ define $U_n \subseteq U$ to be the subset of placements $p \in U$ where $\operatorname{rank} df_G(p) \geq n$ for all $q \in U$. The rank function $T \mapsto \operatorname{rank} T$ is lower semi-continuous, thus by Lemma 1.2.8, the map

$$f : U \to \mathbb{N} \cup \{0\}, \ p \mapsto \operatorname{rank} df_G(p)$$

is lower semi-continuous. As $U_n = f^{-1}[[n, \infty)]$, U_n is open in $\mathcal{W}(G)$ for all $n \in \mathbb{N}$.

It follows from the definition that $\mathcal{C}(G)$ is open in $X^{V(G)}$, thus $\mathcal{C}(G)$ is open in $\mathcal{W}(G)$. Choose any $p \in \mathcal{W}(G)$, then by (i) we may choose an open neighbourhood $U \subset \mathcal{W}(G)$ of p. The rank function will be bounded by $\dim(\mathbb{R}^{E(G)}) = |E(G)|$, thus there exists $p' \in U$ where $\operatorname{rank} df_G(p') := n \geq \operatorname{rank} df_G(q)$ for all $q \in U$. Since U_n is open in $\mathcal{W}(G)$ and $U_m = \emptyset$ for all $m > n$ then the set of placements $q \in U$ with $\operatorname{rank} df_G(q) = n$ is the open set U_n, thus p' is a constant placement. Since this holds for all open neighbourhoods of p and p was chosen arbitrarily then $\mathcal{C}(G)$ is dense in $\mathcal{W}(G)$.

(iii): By Lemma 1.3.3, $\mathcal{R}(G)$ is an open subset of $\mathcal{W}(G)$. By (i), $\mathcal{W}(G)$ is open in $X^{V(G)}$. It now follows $\mathcal{R}(G)$ is open in $X^{V(G)}$, thus every regular placement is constant. □

2.1.2 Equivalence of types of rigidity for constant frameworks

The following theorem is a simplification of the original result for finite dimensional manifolds.

Theorem 2.1.2 (The Constant Rank Theorem)**.** [47, Theorem 2.5.15] Let X and Y be normed spaces with $\dim X = m$ and $\dim Y = n$, $U \subset X$ be an open set and $f : U \to Y$

be C^r-differentiable on U. Suppose f has *constant rank* k at $u \in U$ i.e. $\operatorname{rank} df(x) = k$ for all points x in a neighbourhood of u. Then there exists the following:

(i) Open neighbourhoods $U_2 \subset U$ and $V_1 \subset Y$ of u_0 and $f(u_0)$ respectively, with $f(U_2) \subset f(V_1)$.

(ii) Open sets $U_1 \subset \mathbb{R}^k \times \mathbb{R}^{m-k}$ and $V_2 \subset \mathbb{R}^k \times \mathbb{R}^{n-k}$.

(iii) C^r-diffeomorphisms $\psi : U_1 \to U_2$ and $\varphi : V_1 \to V_2$ such that $\varphi \circ f(u) = (0, 0)$, $\psi(0, 0) = u$ and $\varphi \circ f \circ \psi(x_1, x_2) = (x_1, 0)$ for all $(x_1, x_2) \in U_1$.

We can make the immediate corollary.

Corollary 2.1.3. Let X and Y be normed spaces, $U \subset X$ be an open set, $p \in U$ and $f : X \to Y$ be C^r-differentiable on U. Suppose f has constant rank at every point in an open neighbourhood of $f^{-1}[f(p)] \cap U$, then the following hold:

(i) $f^{-1}[f(p)] \cap U$ is a C^r-submanifold of X.

(ii) The tangent space of $f^{-1}[f(p)] \cap U$ at $u \in f^{-1}[f(p)] \cap U$ is $\ker df(u)$.

Proof. (i): Choose any $u \in f^{-1}[f(p)] \cap U$ and note that $f^{-1}[f(u)] \cap U = f^{-1}[f(p)] \cap U$. Let $m := \dim X$, $n := \dim Y$ and $k := \operatorname{rank} df(u)$, then $\dim \ker df(u) = m - k$. By applying Theorem 2.1.2 at the point u, there exists U_1, U_2, V_1, V_2, ψ and φ as described. Choose any $x \in f^{-1}[f(p)] \cap U_2$, then there exists a unique point $(x_1, x_2) \in U_1$ such that $\psi(x_1, x_2) = x$. As $x \in f^{-1}[f(p)] \cap U_2$ then

$$(x_1, 0) = \varphi \circ f \circ \psi(x_1, x_2) = \varphi \circ f(u) = (0, 0),$$

thus $x_1 = 0$. It now follows that

$$\psi^{-1}[f^{-1}[f(p)] \cap U_2] = U_2 \cap (\{0\} \times \mathbb{R}^{m-k}).$$

We now note that $\psi^{-1} : U_2 \to U_1$ is a chart of the C^r-manifold X with the submanifold property (see [47, Definition 3.2.1]). As this holds for all $u \in f^{-1}[f(p)] \cap U$ then $f^{-1}[f(p)] \cap U$ is a $(m-k)$-dimensional C^r-submanifold of X.

(ii): Choose any $u \in f^{-1}[f(p)] \cap U$. By definition,

$$T_u f^{-1}[f(p)] \cap U = \{\alpha'(0) : \alpha \text{ is } C^1\text{-differentiable in } f^{-1}[f(p)] \cap U,\ \alpha(0) = u\}.$$

Choose any C^1-differentiable path $\alpha : (-\delta, \delta) \to f^{-1}[f(p)] \cap U$ with $\alpha(0) = u$. As $f \circ \alpha(t) = f(p)$ for all $t \in (-\delta, \delta)$ then

$$df(u)(\alpha'(0)) = d(f \circ \alpha)(0) = \lim_{t \to 0} \frac{1}{t}(f \circ \alpha(t) - f \circ \alpha(0)) = 0,$$

thus $T_u f^{-1}[f(p)] \cap U \subset \ker df(u)$. As

$$\dim T_u f^{-1}[f(p)] \cap U = \dim f^{-1}[f(p)] \cap U = m - k = \dim \ker df(u)$$

the result now follows. □

The following key lemma shows why the notion of constant frameworks is required for linking the various types of rigidity.

Lemma 2.1.4. Let (G, p) a constant finite framework in a normed space X, then there exists an open neighbourhood $U \subset X^{V(G)}$ of p such that $f_G^{-1}[f_G(p)] \cap U$ is a C^1-manifold with tangent space $\mathcal{F}(G, p)$ at p and $\mathcal{O}_p \cap U$ is a C^1-submanifold of $f_G^{-1}[f_G(p)] \cap U$.

Proof. Since (G, p) is constant then p is an interior point of $\mathcal{W}(G)$. By Proposition 1.2.7 and Lemma 1.2.8, f_G is C^1-differentiable with constant rank on an open neighbourhood of p in $X^{V(G)}$. By Corollary 2.1.3, $f_G^{-1}[f_G(p)] \cap U$ is a C^1-manifold with tangent space $\ker df_G(p) = \mathcal{F}(G, p)$ at p.

By Lemma 1.2.12, $\mathcal{O}_p$ is a smooth submanifold of $X^{V(G)}$. As

$$\mathcal{O}_p \cap U \subseteq f_G^{-1}[f_G(p)] \cap U \subseteq X^{V(G)}$$

and both $\mathcal{O}_p \cap U$ and $f_G^{-1}[f_G(p)] \cap U$ are C^1-submanifolds of $X^{V(G)}$ then the inclusion map $\mathcal{O}_p \cap U \hookrightarrow f_G^{-1}[f_G(p)] \cap U$ is a C^1-embedding, thus $\mathcal{O}_p \cap U$ is a C^1-submanifold of $f_G^{-1}[f_G(p)] \cap U$. □

We are now ready to prove the following.

Theorem 2.1.5. Let (G, p) be a constant finite framework in a normed space X, then the following are equivalent:

(i) (G, p) is infinitesimally rigid in X,

(ii) (G, p) is locally rigid in X,

(iii) (G, p) is continuously rigid in X.

Proof. By Lemma 2.1.4, $\mathcal{O}_p \cap U$ is a C^1-submanifold of $f_G^{-1}[f_G(p)] \cap U$ for some open neighbourhood U of p. As manifolds are locally path-connected we may assume we chose U small enough such that $f_G^{-1}[f_G(p)] \cap U$ and $\mathcal{O}_p \cap U$ are path-connected.

(Infinitesimal rigidity $\Leftrightarrow$ Local rigidity): Since $\mathcal{O}_p \cap U$ is a C^1-submanifold of $f_G^{-1}[f_G(p)] \cap U$ we have

$$\begin{aligned}
& f_G^{-1}[f_G(p)] \cap U' = \mathcal{O}_p \cap U' \text{ for some open neighbourhood } U' \subseteq U \text{ of } p \\
\Leftrightarrow \quad & T_p(f_G^{-1}[f_G(p)] \cap U) = T_p(\mathcal{O}_p \cap U) \\
\Leftrightarrow \quad & \mathcal{F}(G, p) = \mathcal{T}(p);
\end{aligned}$$

this is equivalent to saying (G, p) is infinitesimally rigid if and only if (G, p) is locally rigid.

(Continuous rigidity $\Rightarrow$ Local rigidity): Suppose (G,p) is continuously rigid. Choose $q \in f_G^{-1}[f_G(p)] \cap U$, then there exists a continuous path from p to q in $f_G^{-1}[f_G(p)] \cap U$. This implies that we may define a finite flex α of (G,p) such that $\alpha(t_0) = q$ for some $t_0 \in (-\delta, \delta)$. Since (G,p) is continuously rigid then α is trivial and thus a continuous path in $\mathcal{O}_p$. It now follows $q \in \mathcal{O}_p \cap U$ as required.

(Local rigidity $\Rightarrow$ Continuous rigidity): For $r > 0$ and $s \in X^{V(G)}$, we shall define $B_r(s)$ to be the open ball of the normed space $X^{V(G)}$; we refer the reader to Section 1.2.2 for more details on this space.

Suppose (G,p) is locally rigid, then there exists $\epsilon > 0$ such that $B_\epsilon(p) \subset U$ and $f_G^{-1}[f_G(p)] \cap B_\epsilon(p) = \mathcal{O}_p \cap B_\epsilon(p)$. First note that both $f_G^{-1}[f_G(p)]$ and $\mathcal{O}_p$ are invariant under $\mathrm{Isom}(X)$. Choose any $q \in \mathcal{O}_p$, then there exists $g \in \mathrm{Isom}(X)$ such that $g.p = q$. We now note that

$$\mathcal{O}_p \cap B_\epsilon(q) = g.\left(\mathcal{O}_p \cap B_\epsilon(p)\right) = g.\left(f_G^{-1}[f_G(p)] \cap B_\epsilon(p)\right) = f_G^{-1}[f_G(p)] \cap B_\epsilon(q).$$

As this holds for all $q \in \mathcal{O}_p$ then $\mathcal{O}_p$ is open in $f_G^{-1}[f_G(p)]$. By Lemma 1.2.12, $\mathcal{O}_p$ is closed in $X^{V(G)}$, thus $\mathcal{O}_p$ is clopen in $f_G^{-1}[f_G(p)]$.

Define $f_G^{-1}[f_G(p)]^\Gamma$ to be the path-connected component of $f_G^{-1}[f_G(p)]$ that contains p with the subspace topology. As path-connected spaces are connected, the only clopen set in $f_G^{-1}[f_G(p)]^\Gamma$ is itself. Define $\mathcal{O}_p^\Gamma := \mathcal{O}_p \cap f_G^{-1}[f_G(p)]^\Gamma$, then $\mathcal{O}_p^\Gamma$ is clopen since $\mathcal{O}_p$ is clopen. This implies that $\mathcal{O}_p^\Gamma = f_G^{-1}[f_G(p)]^\Gamma$ and so any finite flex α lies in $\mathcal{O}_p$. $\square$

Remark 2.1.6. Suppose G is any finite graph and $\mathrm{smooth}(X)$ is an open subset of X (an example would be any ℓ_q^d space). By Proposition 2.1.1, every regular placement will be constant, thus by Theorem 2.1.5, if (G,p) is infinitesimally rigid then it will be continuously and locally rigid also.

2.2 Small frameworks and non-spanning placements

2.2.1 Orbits of non-spanning placements

For a normed space $(X, \|\cdot\|)$ we shall define $(X, \|\cdot\|_2)$ to be the unique Euclidean space for $(X, \|\cdot\|)$ as defined in Lemma 1.1.29; if we refer to just X we shall be referring to the general normed space $(X, \|\cdot\|)$. For any placement p in X we shall define $\mathcal{T}_2(p)$ to be the space of trivial motions of p in $(X, \|\cdot\|_2)$.

Lemma 2.2.1. Let p be a placement in a normed space X. Then $\mathcal{T}(p)$ is a linear subspace of $\mathcal{T}_2(p)$.

Proof. By Lemma 1.1.29, $\operatorname{Isom}(X, \|\cdot\|) \leq \operatorname{Isom}(X, \|\cdot\|_2)$. It now follows that $\mathcal{T}(p) \subseteq \mathcal{T}_2(p)$. □

For Euclidean spaces we have the following equality for the dimension of the space of trivial motions for non-spanning placements.

Lemma 2.2.2. [23, Lemma 2.3.3] Let (p, S) be a placement in a d-dimensional normed space X and let n be the dimension of the affine span of $\{p_v : v \in S\}$. Then

$$\dim \mathcal{T}_2(p) = \frac{(n+1)(2d-n)}{2}.$$

We now wish to obtain an upper and lower bound for the dimension of the space of trivial motions for non-spanning placements. To do this we shall first find an upper-bound for when $|S| = 2$ in non-Euclidean normed spaces and then use an inductive argument.

Lemma 2.2.3. Let $x_0 \in X \setminus \{0\}$ and $\dim X = d$. Then the set

$$\mathcal{O}(x_0) := \{T(x_0) : T \in \operatorname{Isom}^{\mathrm{Lin}}(X)\}$$

is a closed smooth submanifold of X; further $\dim \mathcal{O}(x_0) = d-1$ if and only if X is Euclidean.

Proof. Since $\mathrm{Isom}^{\mathrm{Lin}}(X)$ is compact then $\mathrm{Isom}^{\mathrm{Lin}}(X)$ gives rise to a proper Lie group action on X by $x \mapsto T(x)$ for all $T \in \mathrm{Isom}^{\mathrm{Lin}}(X)$, $x \in X$. As $\mathcal{O}(x_0)$ is the orbit of x_0 (with respect to $\mathrm{Isom}^{\mathrm{Lin}}(X)$) then by Lemma 1.1.26, $\mathcal{O}(x_0)$ is a closed smooth submanifold of X.

First suppose X is Euclidean. By [67, Corollary 3.3.3], $\mathrm{Isom}^{\mathrm{Lin}}(X)$ acts transitively on $S_{\|x_0\|}[0]$, thus $\mathcal{O}(x_0) = S_{\|x_0\|}[0]$. As the unit sphere of a Euclidean space is the d-sphere and $S_{\|x_0\|}[0] = \|x_0\|S_1[0]$ we have $\dim \mathcal{O}(x_0) = d-1$.

Now suppose $\dim \mathcal{O}(x_0) = d-1$. If $d=1$ then all normed spaces are Euclidean so assume $d>1$. The set $S_{\|x_0\|}[0]$ is a closed connected topological submanifold of X with dimension $d-1$ as it is homeomorphic to the d-sphere. Since $\mathcal{O}(x_0) \subset S_{\|x_0\|}[0]$ then $\mathcal{O}(x_0)$ is a closed subset of $S_{\|x_0\|}[0]$. As $\dim \mathcal{O}(x_0) = \dim S_{\|x_0\|}[0]$ it follows from Brouwer's theorem for invariance of domain [44, Theorem 1.18] that the inclusion map $\mathcal{O}(x_0) \hookrightarrow S_{\|x_0\|}[0]$ is an open map, thus $\mathcal{O}(x_0)$ is an open subset of $S_{\|x_0\|}[0]$. As the only clopen non-empty subset of a connected set is itself then $\mathcal{O}(x_0) = S_{\|x_0\|}[0]$. This implies $\mathrm{Isom}^{\mathrm{Lin}}(X)$ acts transitively on $S_{\|x_0\|}[0]$, thus by [67, Corollary 3.3.5], X is Euclidean. $\square$

Lemma 2.2.4. Let $S := \{v_1, v_2\}$ and p be a placement of S in a d-dimensional normed space X in general position. Then $\dim \mathcal{T}(p) \leq 2d-1$ with equality if and only if X is Euclidean.

Proof. By Lemma 2.2.1 and Lemma 2.2.2 it follows $\dim \mathcal{T}(p) \leq 2d-1$ with equality if X is Euclidean.

Now suppose $\dim \mathcal{T}(p) = 2d-1$. Without loss of generality we may assume $p_{v_1} = 0$. If $d=1$ then all normed spaces are Euclidean so assume $d>1$. As p is in general position then $p_{v_2} \neq 0$. By Lemma 2.2.3, $\mathcal{O}(p_{v_2})$ is a closed smooth submanifold of

X. We observe that the tangent space of $\mathcal{O}(p_{v_2})$ at p_{v_2} is $\{u \in X : (0,u) \in \mathcal{T}(p)\}$. Since $\dim \mathcal{T}(p) = d + (d-1)$ and the trivial motions generated by translations form a d-dimensional subspace then it follows $\mathcal{O}(p_{v_2})$ has dimension $d-1$. By Lemma 2.2.3, the space X is Euclidean as required. □

Lemma 2.2.5. Let X be a d-dimensional normed space and $2 \leq k \leq d$. Suppose $(p,S) \subset (p',S')$ are placements in X such that $\{p_v : v \in S\}$ has an affine span of dimension $k-1$ and $\{p'_v : v \in S'\}$ has an affine span of dimension k. Then

$$\dim \mathcal{T}(p') - \dim \mathcal{T}(p) \leq d - k.$$

Proof. Choose $T := \{v_0, \ldots, v_{k-1}\} \subset S$ and $T' := T \cup \{v_k\}$ such that $v_k \in S' \setminus S$ and $p'_{v_0}, \ldots, p'_{v_k}$ have affine span with dimension k. By translation we may assume $p_{v_0} = 0$. Define the linear restriction map

$$P : X^{T'} \to X^T, \ (x_{v_i})_{i=0}^{k} \mapsto (x_{v_i})_{i=0}^{k-1}$$

and the placements $(q,T) \subset (p,S)$ and $(q',T') \subset (p',S')$. We note

$$P(\mathcal{T}_2(q')) \subseteq \mathcal{T}_2(q), \qquad P(\mathcal{T}(q')) \subseteq \mathcal{T}(q).$$

Choose any $u \in \mathcal{T}(q)$, then by Corollary 1.2.16, there exists $h \in T_\iota \operatorname{Isom}(X)$ such that $u = h.q$. By Corollary 1.2.16, $h.q' \in \mathcal{T}(q')$, thus as $P(h.q') = h.q$ we have $P(\mathcal{T}(q')) = \mathcal{T}(q)$. By a similar argument we can see that $P(\mathcal{T}_2(q')) = \mathcal{T}_2(q)$ also.

By the Rank-Nullity theorem applied to $P|_{\mathcal{T}_2(q')}$ and Lemma 2.2.2

$$\dim \ker P|_{\mathcal{T}_2(q')} = \dim \mathcal{T}_2(q') - \dim \mathcal{T}_2(q) = \frac{(k+1)(2d-k)}{2} - \frac{k(2d-k+1)}{2} = d - k.$$

By Lemma 2.2.1 and the Rank-Nullity theorem applied to $P|_{\mathcal{T}(q')}$

$$\dim \mathcal{T}(q') - \dim \mathcal{T}(q) = \dim \ker P|_{\mathcal{T}(q')} \leq \dim \ker P|_{\mathcal{T}_2(q')} = d - k.$$

By Corollary 1.2.17 (ii), $\mathcal{T}(q) \cong \mathcal{T}(p)$ and $\mathcal{T}(q') \cong \mathcal{T}(p')$ and so the result follows. □

Proposition 2.2.6. Let (p, S) be a placement in a d-dimensional normed space X where $\{p_v : v \in S\}$ has an affine span of dimension $1 \leq n \leq d$. Then

$$\dim \mathcal{T}(p) \leq \frac{(n+1)(2d-n)}{2}$$

with equality if and only if X is Euclidean.

Proof. If X is Euclidean then the result follows by Lemma 2.2.2.

Suppose X is non-Euclidean. If $n = 1$ then the result follows by Lemma 2.2.4 and Corollary 1.2.17 (ii). Let $n > 1$ and suppose the theorem holds for all $m \leq n - 1$. Choose a subplacement $q \subset p$ with an affine span of dimension $n-1$, then by assumption $\dim \mathcal{T}(q) < \frac{n(2d-n+1)}{2}$. By Lemma 2.2.5 it follows that

$$\dim \mathcal{T}(p) \leq \dim \mathcal{T}(q) + d - n < \frac{n(2d-n+1)}{2} + d - n = \frac{(n+1)(2d-n)}{2}.$$

□

2.2.2 Infinitesimal flexibility and independence of small frameworks and non-full frameworks

We define a framework (G, p) in a d-dimensional normed space to be *small* if $|V(G)| \leq d + 1$. The following is a well known result for Euclidean spaces.

Proposition 2.2.7. Let (G,p) be a small well-positioned framework in a d-dimensional Euclidean space X. Then (G,p) is isostatic if and only if G is a complete graph and p is in general position.

Proof. ($\Rightarrow$): Let (G,p) be isostatic. By Proposition 1.3.13 and Lemma 2.2.2, $|E(G)| = \frac{|V(G)|(|V(G)|-1)}{2}$, thus G is a complete graph. By Proposition 1.3.24, p is in general position.

($\Leftarrow$): This follows from [23, Theorem 2.4.1.d]. □

Using Theorem 1.2.29 we can now state our own result for small frameworks for non-Euclidean spaces.

Theorem 2.2.8. Let (G,p) be a small well-positioned framework with in a d-dimensional non-Euclidean normed space X. Then (G,p) is infinitesimally flexible or $|V(G)| = 1$.

Proof. If $|V(G)| = 1$ then (G,p) is infinitesimally rigid by definition. Suppose $|V(G)| \geq 2$ and the affine span of $\{p_v : v \in V(G)\}$ has dimension n. We note that

$$n \leq |V(G)| - 1 \leq d.$$

Define the map $f : \mathbb{R} \to \mathbb{R}$ where

$$f(x) = \frac{(x+1)(2d-x)}{2}.$$

We note that f is increasing on the interval $[0, d-1]$ and $f(d-1) = f(d)$, thus it follows that $f(|V(G)|-1) \geq f(n)$. We note

$$\begin{aligned} |E(G)| &\leq \frac{|V(G)|(|V(G)|-1)}{2} \\ &= d|V(G)| - f(|V(G)|-1) \\ &\leq d|V(G)| - f(n) \end{aligned}$$

$$< \; d|V(G)| - \dim \mathcal{T}(p) \text{ by Lemma 2.2.6}$$

thus by Theorem 1.3.12, (G,p) is infinitesimally flexible. $\square$

We may also show that sufficiently small graphs will also be independent. We note that this differs from showing that small frameworks are always flexible and the two properties in this special case are unrelated.

Let (G,p) be well-positioned in a normed space X. We say (G,p) has the *graded independence property* if we can order the vertices $v_1,\ldots,v_n$ so that for each $1 < j \leq n$, the set

$$\varphi(G)_n := \{\varphi_{v_j,v_i} : 1 \leq i < j, \; v_iv_j \in E(G)\}$$

is linearly independent; we shall define v_n to be the *highest vertex* of (G,p). It is immediate that any framework with the graded independence property must be small. We also note that if (G,p) has the graded independence property with highest vertex v_n and (G',p') is the framework formed from (G,p) by deleting the vertex v_n then (G',p') also has the graded independence property.

Lemma 2.2.9. Let (G,p) be a well-positioned framework in a normed space X. If (G,p) has the graded independence property then (G,p) is independent.

Proof. Let $|V(G)| = n$ and suppose the result holds for all frameworks with less than n vertices. Let a be a stress of (G,p). By observing the flex condition at v_n we note that as the set $\varphi(G)_n$ is linearly independent then $a_{v_iv_n} = 0$ for all $1 \leq i < n$. Define (G',p') to be subframework of (G,p) formed by deleting the highest vertex v_n, then $a|_{E(G')}$ is a stress of (G',p'). As $|V(G')| = n-1$ then by assumption, $a|_{E(G')} = 0$. It now follows $a = 0$ and (G,p) is independent as required. $\square$

Lemma 2.2.10. Let X be a d-dimensional normed space. Then for each $1 \leq n \leq d+1$, there exists a placement of K_n with the graded independence property.

Proof. We note that the graded independence property holds for well-positioned placements of K_1 and K_2. Suppose the result holds for all graphs on $n-1$ vertices or less for some $2 \leq n \leq d+1$. We shall now show the result holds for K_n.

Label the vertices of K_n as $v_1, \ldots, v_n$ and let K_{n-1} be the complete graph on $v_1, \ldots, v_{n-1}$. By assumption we have a placement q of K_{n-1} with the graded independence property; we shall further assume that $q_{v_{n-1}} = 0$. Define the set of points

$$A := \bigcap_{i=1}^{n-1} \{x \in X : x - q_{v_i} \in \operatorname{smooth}(X)\},$$

then it follows from Proposition 1.1.11 (i) that A is a dense subset of X.

By Proposition 1.1.23, there exists $y \in \operatorname{smooth}(X)$ so that

$$\varphi_{v_1,v_{n-1}}, \ldots, \varphi_{v_{n-2},v_{n-1}}, \varphi(y)$$

are linearly independent. Define the map

$$\tilde{\varphi} : \operatorname{smooth}(X) \to X^*, \ x \mapsto \frac{1}{\|x\|}\varphi(x)$$

and the sequence $(x_k)_{k\in\mathbb{N}}$ with $x_k := \frac{1}{k}y$; we note that $\tilde{\varphi}(x_k) = \tilde{\varphi}(y)$ for all $k \in \mathbb{N}$. By Proposition 1.1.11 (iv), $\tilde{\varphi}$ is continuous. As A is dense we may choose for each $k \in \mathbb{N}$ some element $y_k \in A$ sufficiently close to x_k so that

$$\|\tilde{\varphi}(y_k) - \tilde{\varphi}(y)\| = \|\tilde{\varphi}(y_k) - \tilde{\varphi}(x_k)\| < \frac{1}{k}.$$

Define the map $J : A \to L(X, \mathbb{R}^{n-1})$, where for all $x, x' \in X$, $J(x')$ is the map

$$J(x')x := \left(\tilde{\varphi}(x' - q_{v_1})x, \ \ldots \ , \ \tilde{\varphi}(x' - q_{v_{n-2}})x, \ \tilde{\varphi}(x')x\right).$$

We note J is continuous as $\tilde{\varphi}$ is continuous. Define $S \subset L(X, \mathbb{R}^{n-1})$ to be the subset of surjective maps, then S is an open non-empty subset. Let $T \in L(X, \mathbb{R}^{n-1})$ be the map where for all $x \in X$,

$$T(x) := (\varphi_{v_{n-1},v_1} x, \ \ldots \ , \varphi_{v_{n-1},v_{n-2}} x, \ \tilde{\varphi}(y)x),$$

then $T \in S$ by our choice of y. We now note that $J(y_k) \to T$ pointwise as $k \to \infty$, thus as S is open there exists some $N \in \mathbb{N}$ where $J(y_N) \in S$.

Define p to be the placement of K_n with $p_{v_i} = q_{v_i}$ for $1 \leq i \leq n-1$ and $p_{v_n} = y_N$. We now note that as $J(y_N)$ is surjective then the set of edge support functionals with end point v_n are linearly independent, thus (K_n, p) has the graded independence property as required. □

Theorem 2.2.11. Let G be a graph with $|V(G)| \leq d+1$ and X be a d-dimensional normed space. Then G is independent in X.

Proof. We may suppose $G = K_n$ for some $1 \leq n \leq d+1$. By Lemma 2.2.10, there exists a well-positioned placement p of K_n with the graded independence property. By Lemma 2.2.9, (K_n, p) is independent as required. □

Corollary 2.2.12. Let G be a graph with $|V(G)| \leq d+1$ and X be a d-dimensional normed space. Then G is independent and flexible in X.

Proof. This follows from Theorem 2.2.8 and Theorem 2.2.11. □

We may also give a weaker analogue to the infinitesimal flexibility of frameworks in Euclidean spaces that are not isometrically full (see Proposition 1.3.24 and Corollary 1.2.25).

Proposition 2.2.13. Let G be a finite graph with $|V(G)| \geq d+1$ and X be a d-dimensional normed space. If G is rigid then any regular and constant placement of G is full.

Proof. Suppose there exists a regular and constant placement p of G that is not full. As G is rigid we may choose some infinitesimally rigid placement p' of G. As (G,p) is regular then by Lemma 1.3.14 (ii) and Theorem 1.2.29,

$$\dim \mathcal{F}(G,p) = \dim \mathcal{F}(G,p') = \dim \mathrm{Isom}(X).$$

By Proposition 1.2.7 and Lemma 1.2.8, f_G is C^1-differentiable with constant rank on an open neighbourhood of p in $X^{V(G)}$. By the Constant Rank Theorem (Theorem 2.1.2), there exists the following:

(i) Open neighbourhoods $U_2 \subset \mathcal{R}(G)$ and $V_1 \subset \mathbb{R}^{E(G)}$ of p and $f_G(p)$ respectively, with $f_G(U_2) \subset f_G(V_1)$.

(ii) Open sets $U_1 \subset \mathbb{R}^{d|V(G)|-k} \times \mathbb{R}^k$ and $V_2 \subset \mathbb{R}^{|E(G)|-k} \times \mathbb{R}^k$, where $k := \dim \mathcal{F}(G,p)$.

(iii) C^1-diffeomorphisms $\psi : U_1 \to U_2$ and $\phi : V_1 \to V_2$ such that $\phi \circ f_G(p) = (0,0)$, $\psi(0,0) = p$ and

$$\tilde{f_G} := \phi \circ f_G \circ \psi : U_1 \to V_2, \ (x_1, x_2) \mapsto (x_1, 0).$$

We immediately note that for all $(x_1, x_2) \in U_1$ and $(x_1', x_2') \in \mathbb{R}^{d|V(G)|-k} \times \mathbb{R}^k$,

$$d\tilde{f_G}(x_1, x_2)(x_1', x_2') = (x_1', 0).$$

For any $q \in U_2$ and $u \in X^{V(G)}$ define $(q_1, q_2) := \psi^{-1}(q)$ and $(u_1, u_2) = d\psi(q_1, q_2)^{-1}(u)$. By the Chain Rule [47, Theorem 2.4.3],

$$d\tilde{f_G}(q_1, q_2) = d(\phi \circ f_G)(q) \circ d\psi(q_1, q_2)$$

As ϕ, ψ are C^1-diffeomorphisms then for $u \in X^{V(G)}$,

$$\begin{aligned} df_G(p)(u) = 0 &\Leftrightarrow d(\phi \circ f_G)(q)(u) = 0 \\ &\Leftrightarrow d\tilde{f_G}(q_1, q_2) \circ d\psi(q_1, q_2)^{-1}(u) = (u_1, 0) = 0, \end{aligned}$$

thus for all $q \in U_2$,

$$d\psi(q_1, q_2)\left(\{0\} \times \mathbb{R}^k\right) = \mathcal{F}(G, q).$$

As the set of spanning placements of G is an open dense subset of $X^{V(G)}$, choose a sequence $(p^n)_{n\in\mathbb{N}}$ of spanning placements of G such that $p^n \in U_2$ for all $n \in \mathbb{N}$ and $p^n \to p$ as $n \to \infty$; it follows from Corollary 1.2.25 and Theorem 1.2.29, (G, p^n) is infinitesimally rigid for all $n \in \mathbb{N}$. As (G, p) is regular but not full it follows from Theorem 1.2.29 that there exists $u \in \mathcal{F}(G, p) \setminus \mathcal{T}(p)$. For each $n \in \mathbb{N}$, define

$$u^n := d\psi(p_1^n, p_2^n)(0, u_2),$$

then $u^n \in \mathcal{F}(G, p^n)$. Since ψ is a C^1-diffeomorphism, $d\psi$ is continuous, thus $u^n \to u$ as $n \to \infty$.

Since (G, p^n) is infinitesimally rigid and full, there exists a unique element $g_n \in T_\iota \operatorname{Isom}(X)$ such that $g_n.p^n = u^n$. We note that the Lie group action

$$\theta : T_\iota(\operatorname{Isom}(X)) \times X^{V(G)} \to X^{V(G)} \times X^{V(G)}, \ (g, q) \to (g.q, q)$$

is proper as it is linear. As $(\theta(g_n, p^n))_{n\in\mathbb{N}}$ converges it follows that there exists a convergent subsequence $((g_{n_k}, p^{n_k}))_{k\in\mathbb{N}}$ with limit $(g, p) \in T_\iota(\operatorname{Isom}(X)) \times X^{V(G)}$. We now see that

$$u = \lim_{k\to\infty} u^{n_k} = \lim_{k\to\infty} g_{n_k}.p^{n_k} = g.p,$$

thus $u \in \mathcal{T}(p)$, contradicting our choice of u. □

Remark 2.2.14. We note that Proposition 2.1.1 need not hold if G is a finite flexible graph. Take for instance the graph

$$G := (\{v_1, v_2, v_3, v_4\}, \{v_1 v_2, v_2, v_3, v_3 v_4\}).$$

If we define the placement p of G in the standard Euclidean space $\mathbb{R}^3$ with

$$p_{v_1} = (0,0,0), \qquad p_{v_2} = (1,0,0), \qquad p_{v_3} = (2,0,0), \qquad p_{v_4} = (3,0,0),$$

then (G,p) is regular, constant but not full.

2.3 Composition and substitution of rigid graphs and frameworks

2.3.1 Composition of rigid frameworks

For a normed space X define

$$\min \operatorname{Full}(X) := \min\{|S| : S \neq \emptyset, \text{ there exists a full placement of } S \text{ in } X\}.$$

It follows from Corollary 1.2.24 that $1 \leq \min \operatorname{Full}(X) \leq \dim X$.

Example 2.3.1. If X is Euclidean then $\min \operatorname{Full}(X) = \dim X$, thus the top bound is exact. We shall see later (Proposition 2.3.5) that the lower bound is also exact.

Theorem 2.3.2. Let X be a normed space and G, H be finite graphs. Suppose the following holds:

(i) $|V(G) \cap V(H)| \geq \min \operatorname{Full}(X)$

(ii) G has infinitesimally rigid placement p.

(iii) H has infinitesimally rigid placement q.

(iv) $p_v = q_v$ for all $v \in V(G) \cap V(H)$.

(v) $p \cap q := p|_{V(G) \cap V(H)}$ is full.

Then $(G \cup H, r)$ is infinitesimally rigid, where $r_v = p_v$ for all $v \in V(G)$ and $r_v = q_v$ for all $v \in V(H)$.

Proof. Let u be a flex of $(G \cup H, r)$. We note that $u|_{V(G)}$ is a trivial infinitesimal flex of (G, p) and $u|_{V(H)}$ is a trivial infinitesimal flex of (H, q), thus there exists $g_1, g_2 \in T_\iota \operatorname{Isom}(X)$ such that $g_1.p = u|_{V(G)}$ and $g_2.q = u|_{V(H)}$. Since $p \cap q$ is a full placement then there exists a unique map $g \in T_\iota \operatorname{Isom}(X)$ such that $g.(p \cap q) = u|_{V(G) \cap V(H)}$. As $g_1.(p \cap q) = u|_{V(G) \cap V(H)}$ and $g_2.(p \cap q) = u|_{V(G) \cap V(H)}$ then $g = g_1 = g_2$, thus $g.p = u$. By Corollary 1.2.16, u is a trivial infinitesimal flex of $(G \cup H, r)$ as required. $\square$

Corollary 2.3.3. Suppose G is a graph with N vertices that is rigid in a normed space X, then K_n is rigid for all $n \geq N$.

Proof. Suppose $N < d + 1$. By Theorem 2.2.8, X is Euclidean, and by Proposition 2.2.7, K_{d+1} is rigid. It now follows that if the statement holds for $N \geq d + 1$ the statement holds, thus we shall assume $N \geq d + 1$.

Suppose K_n is rigid where $n \geq N$. We now define G to be a complete graph on the vertices $v_1, \ldots, v_{n-1}, a$ and H to be the complete graph on $v_1, \ldots, v_{n-1}, b$. As K_n is rigid, by Lemma 1.3.4 there exists a infinitesimally rigid placement p of G in general position. Let q be a placement of H where $q_{v_i} = p_{v_i}$ for all $i = 1, \ldots, n-1$ and $q_b = p_a$, then (H, q) is infinitesimally rigid also. By Theorem 2.3.2, $(G \cup H, r)$ is infinitesimally rigid, where $r_v = p_v$ for all $v \in V(G)$ and $r_v = q_v$ for all $v \in V(H)$, thus $G \cup H$ is rigid. As $G \cup H$ is the complete graph on $n + 1$ vertices minus an edge then K_{n+1} is rigid. By an inductive argument the result now follows. $\square$

2.3.2 Composition and substitution of rigid graphs in normed spaces with a finite number of linear isometries

Lemma 2.3.4. For a normed space X, $\min \mathrm{Full}(X) = 1$ if and only if there exist only finitely many linear isometries of X.

Proof. Let p be a placement of the single element set $\{v\}$ in X, then by Corollary 1.2.23 we may assume $p_v = 0$.

Suppose $|\operatorname{Isom}^{\mathrm{Lin}}(X)| < \infty$. By Corollary 1.2.11, $|\operatorname{Stab}_p| \leq |\operatorname{Isom}^{\mathrm{Lin}}(X)| < \infty$, thus by Corollary 1.2.20 (i), p is full and $\min \mathrm{Full}(X) = 1$ as required.

Now suppose $\min \mathrm{Full}(X) = 1$. As $\mathcal{O}_p = X$ then by Corollary 1.2.23, p is full. By Proposition 1.2.18 (i), $\dim \operatorname{Isom}(X) = \dim X$, thus by Lemma 1.1.30, $\dim \operatorname{Isom}^{\mathrm{Lin}}(X) = 0$. As $\operatorname{Isom}^{\mathrm{Lin}}(X)$ is compact and 0-dimensional then it is finite as required. □

Theorem 2.3.5. For any normed space X, the following are equivalent:

(i) $\dim \operatorname{Isom}(X) = \dim X$.

(ii) $|\operatorname{Isom}^{\mathrm{Lin}}(X)| < \infty$.

(iii) $\min \mathrm{Full}(X) = 1$.

(iv) All placements are full.

(v) For any set $S \neq \emptyset$ and any $p \in X^S$, $\mathcal{T}(p) = \dim X$.

Proof. (i) $\Leftrightarrow$ (ii): By Lemma 1.1.30, $\dim \operatorname{Isom}^{\mathrm{Lin}}(X) = \dim \operatorname{Isom}(X) - \dim X$. As $\operatorname{Isom}^{\mathrm{Lin}}(X)$ is a compact manifold, $\operatorname{Isom}^{\mathrm{Lin}}(X)$ is finite if and only if $\dim \operatorname{Isom}^{\mathrm{Lin}}(X) = 0$, thus the equivalence holds.

(ii) $\Leftrightarrow$ (iii): Lemma 2.3.4.

(iii) $\Rightarrow$ (iv): Let p be a placement in X, choose $v \in S$ and define $q := p|_v$. As $\min \mathrm{Full}(X) = 1$ then there exists a full placement q' of $\{v\}$, thus by Corollary 1.2.23, q is also full. As $q \subset p$ then by Corollary 1.2.22, p is full.

(iv) $\Rightarrow$ (iii): This is immediate.

(iv) $\Rightarrow$ (v): Choose any placement p, then p is full. Let S be any set with $|S| = 1$, then the placement (q, S) is also full, thus $\dim \mathcal{T}(p) = \dim \mathcal{T}(q) = \dim X$.

(v) $\Rightarrow$ (i): Let p be a spanning placement, then by Corollary 1.2.25, p is full. By Theorem 1.2.29, $\dim \operatorname{Isom}(X) = \mathcal{T}(p) = \dim X$. $\square$

Let H and G be finite graphs with $|V(G) \cap V(H)| = 1$ and $v_0 \in V(G) \cap V(H)$. We form a *H-substitution of G at v_0* from G by replacing v_0 with a copy of H and every edge $vv_0 \in E(G)$ with a new edge vw for some $w \in V(H)$.

Lemma 2.3.6. Let X be a normed space and $x \in S_1[0]$. Define $(x_n)_{n\in\mathbb{N}}$ to be a sequence of smooth points such that

$$\left\| \frac{1}{n}x - x_n \right\| < \frac{1}{n^2},$$

then $\frac{x_n}{\|x_n\|} \to x$ as $n \to \infty$.

Proof. We first note that

$$\left| \frac{1}{n} - \|x_n\| \right| \leq \left\| \frac{1}{n}x - x_n \right\| < \frac{1}{n^2},$$

thus

$$\frac{n-1}{n^2} < \|x_n\| < \frac{n+1}{n^2}.$$

We now note

$$\begin{aligned} \left\| \frac{x_n}{\|x_n\|} - x \right\| &= \frac{\|x_n - \|x_n\|x\|}{\|x_n\|} \\ &\leq \frac{\left\|x_n - \frac{1}{n}x\right\| + \left\|\frac{1}{n}x - \|x_n\|x\right\|}{\|x_n\|} \end{aligned}$$

$$
\begin{aligned}
&< \frac{\frac{1}{n^2} + \left|\frac{1}{n} - \|x_n\|\right|}{\|x_n\|} \\
&< \frac{\frac{2}{n^2}}{\|x_n\|} \\
&< \frac{\frac{2}{n^2}}{\frac{n-1}{n^2}} \\
&= \frac{2}{n-1}.
\end{aligned}
$$

□

Lemma 2.3.7. Let G, H be finite independent graphs, G' be a H-substitution of G at v_0 and p an independent placement of G in a normed space X. Then there exists an independent placement p' of G' such that $p'|_{V(G)} = p$.

Proof. We define q' to be the not well-positioned placement of G' that agrees with p on $V(G)$ and has $q'_v = p_{v_0}$ for all $v \in V(H)$. Let $x := (x_v)_{v \in V(H)}$ be an independent placement of H in general position with $x_{v_0} = 0$ (Lemma 1.3.4) and define for all $vw \in E(H)$ the pseudo-support functionals

$$\varphi'_{v,w} := \varphi\left(\frac{x_v - x_w}{\|x_v - x_w\|}\right);$$

by this we may define ϕ and $(G', q')^{\phi}$.

Let $a := (a_{vw})_{vw \in E(G')}$ be a pseudo-stress of $(G', q')^{\phi}$. Define $b := (b_{vw})_{vw \in E(G)}$ with $b_{vw} = a_{vw}$ for all $vw \in E(G) \cap E(G')$ and $b_{v_0 w} = a_{vw}$ for all $vw \in E(G')$ with $v \in V(H)$, $w \notin V(H)$. It is immediate that the stress condition of b holds at any vertex $v \neq v_0$. The stress condition of b at v_0 holds as

$$\sum_{w \in N_G(v_0)} b_{v_0 w} \varphi_{v_0,w} = \sum_{v \in V(H)} \sum_{w \in N_G(v)} a_{vw} \varphi'_{v,w} = \sum_{v \in V(H)} 0 = 0,$$

since for each $vw \in E(H)$ the stress vectors $a_{vw}\varphi'_{v,w}$ cancel each other out. It now follows b is a stress of an independent framework (G, p), thus $a_{vw} = 0$ for all $vw \notin E(H)$.

Since (H,x) is independent it follows that $a_{vw} = 0$ for all $vw \in E(H)$, thus $a = 0$ and $(G', q')^\phi$ is independent.

Define q^n to be the placement of G' where q^n agrees with q' on $V(G') \setminus V(H)$ and $q^n_v = q'_{v_0} + \frac{1}{n} x_v$ for all $v \in V(H)$. Define $V := (V(G') \setminus V(H)) \cup \{v_0\}$. By Lemma 1.2.5, we may choose $p^n \in \mathcal{W}(G)$ such that $\|p^n - q^n\|_{V(G)} < \frac{1}{n^2}$ and $p^n_v = q^n_v$ for all $v \in V$. By our choice of p^n we have that $\varphi^n_{v,w} = \varphi'_{v,w}$ for $vw \in E(G')$ with $v, w \in V$. By Lemma 2.3.6 and Proposition 1.1.11 (iv), $\varphi^n_{v,w} \to \varphi'_{v,w}$ as $n \to \infty$ for the remaining edges. This implies $(G', p^n) \to (G', q')^\phi$ as $n \to \infty$ and so by Proposition 1.3.11, there exists an independent placement $p' := p^n$ of G' for sufficiently large $n \in \mathbb{N}$. □

Theorem 2.3.8. Let X be a normed space, G, H be finite graphs and G' be a H-substitution of G at v_0. Then the following holds:

(i) If G and H are independent in X then G' is independent in X.

(ii) If G and H are rigid in X and $|\operatorname{Isom}^{\mathrm{Lin}}(X)| < \infty$ then G' is rigid in X.

Proof. (i): This follows from Lemma 2.3.7.

(ii): Suppose G, H are rigid in X and $\operatorname{Isom}^{\mathrm{Lin}}(X)$ is finite. We may suppose G and H are also independent in X by deleting edges. Let p, x, p' be regular placements of G, H, G' respectively, then by Theorem 2.3.5, all three are full. By Corollary 1.3.15, $|E(G)| = \dim X(|V(G)| - 1)$ and $|E(H)| = \dim X(|V(H)| - 1)$. We now note that

$$|E(G')| = |E(G)| + |E(H)| = \dim X(|V(G)| + |V(H)| - 1).$$

As (G', p') is independent then by Proposition 1.3.13, (G', p') is isostatic. □

Corollary 2.3.9. Let X be a normed space and G, H be finite graphs with $|V(G) \cap V(H)| = 1$. Then the following holds:

(i) If G and H are independent in X then $G \cup H$ is also independent in X.

(ii) If G and H are rigid in X and $|\operatorname{Isom}^{\mathrm{Lin}}(X)| < \infty$, then $G \cup H$ is rigid in X.

Proof. Let v_0 be the single vertex in $V(G) \cap V(H)$, then it is clear that $G \cup H$ is a H-substitution at v_0, thus by Theorem 2.3.8 the result holds. $\square$

Chapter 3

Graph rigidity in the normed plane

In this chapter we shall extend Theorem 1.3.20 to non-Euclidean normed planes; see Theorem 3.4.2. The proof is inductive, so requires a base of induction and an induction step.

For our base of induction, we shall prove in Section 3.2 that the graph K_4 is indeed rigid in all normed planes, as unlike for Euclidean spaces, this is far from obvious for general normed planes. The proof will be split into three cases:

(i) when X is not strictly convex (Section 3.2.1),

(ii) when X is strictly convex but not smooth (Section 3.2.3),

(iii) when X is both strictly convex and smooth (Section 3.2.2).

We shall next prove our induction step in Section 3.3 by proving that any graph that is formed from an isostatic graph by one of our graph operations will also be isostatic. The operations are actually proven to hold with stronger conditions than are required, as we shall require this later in Section 4.3. When combined with Proposition 3.4.1, we shall be able to prove the required result.

We shall finish the chapter by giving some sufficient connectivity conditions for graph rigidity analogous to those given by Lovász & Yemini for the Euclidean plane in [46], and by Jordán in [29].

3.1 Frameworks in normed planes

Many of our previous results may be simplified for the special case or normed planes. We shall outline the main changes here.

3.1.1 Isometries of a normed plane and full placements

For 2-dimensional normed spaces we can immediately categorize $\operatorname{Isom}(X)$ into one of two possibilities.

Proposition 3.1.1. Let X be a normed plane, then the following holds:

(i) If X is Euclidean then there are infinitely many linear isometries of X and the tangent space $T_\iota \operatorname{Isom}(X) = \operatorname{span}\{T_1, T_2, T_0\}$ where T_1, T_2 are linearly independent translations and T_0 is an invertible linear map.

(ii) If X is non-Euclidean then there are a finite number of linear isometries of X and the tangent space $T_\iota \operatorname{Isom}(X) = \operatorname{span}\{T_1, T_2\}$ where T_1, T_2 are linearly independent translations.

Proof. (i): Let x_1, x_2 be an orthonormal basis of X. By Proposition 1.1.35, $T \in T_\iota \operatorname{Isom}^{\mathrm{Lin}}(X)$ if and only if $T^* = -T$. We note that this is equivalent to T being a scalar multiple of the linear isometry

$$T : X \to X,\ ax_1 + bx_2 \mapsto ax_2 - bx_1.$$

(ii): As remarked in [67, pg. 83] there are only finitely many linear isometries $\iota := L_0, L_1, \ldots, L_n$ of X and so by Mazur-Ulam's theorem (Theorem 1.1.28) we have

$$\operatorname{Isom}(X) = \{T_x \circ L_i : x \in X,\ i = 0, \ldots, n\}$$

where $T_x(y) = x + y$ for all $y \in X$. We can now see that the tangent space at ι is exactly the space of constant maps from X to itself and the result follows. □

Corollary 3.1.2. Let (p, S) be a placement in normed plane X. Then p is full if and only if either:

(i) X is non-Euclidean,

(ii) X is Euclidean and the affine span of p is a line or X.

Proof. Suppose X is non-Euclidean. By Proposition 3.1.1 (ii), $|\operatorname{Isom}^{\mathrm{Lin}}(X)| < \infty$, thus by Theorem 2.3.5, all placements are full.

Suppose X is Euclidean. By Corollary 1.2.23, we may assume $p_w = 0$ for some $w \in S$. If the affine span of p is a line or X then by Corollary 1.2.24, p is full. If the affine span of p has dimension 0 then $p_v = 0$ for all $v \in S$. It follows from Proposition 3.1.1 (i) that $|\operatorname{Stab}_p| = \infty$, thus by Corollary 1.2.20, p is not full. □

3.1.2 Necessary conditions for graph rigidity in the normed plane

We shall now improve upon some previously given necessary conditions for rigidity for the specific case of normed planes.

Theorem 3.1.3. Let X be a normed plane. Define $k = 3$ if X is Euclidean and $k = 2$ if X is non-Euclidean. For any finite graph G with at least two vertices the following holds:

(i) If $|V(G)| = 2$ or 3 then G is rigid if and only if X is Euclidean and $G = K_2$ or K_3.

(ii) If G is independent then G is $(2, k)$-sparse.

(iii) If G is isostatic then G is $(2, k)$-tight.

(iv) If G is rigid then G contains a $(2, k)$-tight spanning subgraph.

Proof. (i): This follows from Theorem 2.2.8.

(ii): Choose $H \subset G$. We have three possibilities:

(i) If $|V(H)| \geq 3$ then by Corollary 1.3.15, $|E(H)| \leq 2|V(H)| - k$.

(ii) If $|V(H)| = 2$ then either $|E(H)| = 0$ or $|E(H)| = 1 \leq 2|V(H)| - k$.

(iii) If $|V(H)|$ then $|E(H)| = 0$.

As this holds for any $H \subset G$ then G is $(2, k)$-sparse.

(iii): As (i) holds we may assume $|V(G)| \geq 4$. Let p be a isostatic placement of G. By Corollary 1.3.15, $|E(G)| = 2|V(G)| - k$, thus G is $(2, k)$-tight.

(iv): As (i) holds we may assume $|V(G)| \geq 4$. Let p be an infinitesimally rigid placement of G. By Remark 1.3.9, there exists a spanning isostatic subframework (H, p) of (G, p). As (iii) holds it follows H is $(2, k)$-tight. □

Corollary 3.1.4. Let X be a normed plane and $k := \mathrm{Isom}(X)$. For any graph G with at least 3 vertices, if two of the following hold so does the third (and G is isostatic):

(i) $|E(G)| = 2|V(G)| - k$

(ii) G is independent

(iii) G is rigid.

Proof. By Lemma 1.3.4 we may choose a regular placement p of G in general position. By Corollary 1.2.25 and Theorem 1.2.29, $\dim \mathcal{T}(p) = \dim \operatorname{Isom}(X)$. We now apply Proposition 1.3.13. $\square$

3.2 Rigidity of K_4 in all normed planes

In this section we shall prove the following.

Theorem 3.2.1. *K_4 is rigid in all normed planes.*

This shall follow from Lemma 3.2.7, Lemma 3.2.11 and Lemma 3.2.23. We shall consider three separate cases; not strictly convex normed planes (Section 3.2.1), strictly convex but not smooth normed planes (Section 3.2.2), and strictly convex and smooth normed planes (Section 3.2.3).

3.2.1 The rigidity of K_4 in not strictly convex normed planes

We remember from Proposition 1.1.17 (ii) that if X is a normed plane and $x \in \operatorname{smooth}(X)$ then for some (possibly not distinct) $x_1, x_2 \in S_1[0]$,

$$\varphi(x)^{-1}[\{\|x\|\}] \cap S_1[0] = [x_1, x_2].$$

Lemma 3.2.2. *Let X be a normed space and $[x_1, x_2] \subset S_1[0]$. If*

$$x, y \in [x_1, x_2] \cap \operatorname{smooth}(X)$$

then $\varphi(x) = \varphi(y)$.

Proof. If $x_1 = x_2$ this is immediate so assume $x_1 \neq x_2$. Choose $x := t_0 x_1 + (1-t_0)x_2$ for $t_0 \in (0,1)$ and define the convex and differentiable map $f : [0,1] \to \mathbb{R}$ where

$$f(t) := \varphi(x)(tx_1 + (1-t)x_2) = t\varphi(x)x_1 + (1-t)\varphi(x)x_2.$$

We note $f(t_0) = 1$ and $f'(t) = \varphi(x)x_1 - \varphi(x)x_2$, thus if f is not constant then there exists $t \in [0,1]$ where $f(t) > 1$; however we note

$$|f(t)| \leq t|\varphi(x)x_1| + (1-t)|\varphi(x)x_2| \leq 1,$$

a contradiction. As f is constant then $f(t) = f(t_0) = 1$ for all $t \in [0,1]$, thus $\varphi(x)$ is a support functional for all $y \in [x_1, x_2]$ and the result follows. □

Lemma 3.2.3. Let X be a normed space and $x, y \in S_1[0] \cap \text{smooth}(X)$ with

$$\varphi(x)^{-1}[\{1\}] \cap S_1[0] = [x_1, x_2]$$

for distinct x_1, x_2. Define $a, b \in \mathbb{R}$ such that $y = ax_1 + bx_2$, then one of the following holds:

(i) $a, b \geq 0$ or $a, b \leq 0$ and $\varphi(x), \varphi(y)$ are linearly dependent.

(ii) $a < 0 < b$ or $b < 0 < a$ and $\varphi(x), \varphi(y)$ are linearly independent.

Proof. (i): If $a = 0$ then $y = x_2$ or $-x_2$ and $\varphi(x), \varphi(y)$ are linearly dependent; similarly if $b = 0$ then $\varphi(x), \varphi(y)$ are linearly dependent. We first note that $\varphi(x)y = a + b$. If $a, b > 0$ then

$$a + b = \varphi(x)y \leq \|y\| = \|ax_1 + bx_2\| \leq a + b,$$

thus $\varphi(x)$ is a support functional of y. If $a, b < 0$ then similarly we have $\varphi(y) = -\varphi(-y) = -\varphi(x)$; in either case $\varphi(x), \varphi(y)$ are linearly dependent.

(ii): Let $a < 0 < b$ and $\varphi(x), \varphi(y)$ be linearly dependent. As $\varphi(y) = -\varphi(-y)$ we may assume $\varphi(y) = \varphi(x)$, thus $\varphi(x)y = 1$. By assumption this implies $y \in [x_1, x_2]$; it follows that there exists $t \in [0, 1]$ such that

$$y = tx_1 + (1-t)x_2,$$

thus $a, b \geq 0$ contradicting our assumption. We see a similar contradiction if $b < 0 < a$ and $\varphi(x), \varphi(y)$ be linearly dependent, thus the result holds. □

Lemma 3.2.4. Let X be a normed plane that is not strictly convex, then there exists $x, y \in S_1[0] \cap \mathrm{smooth}(X)$ such that the following holds:

(i) $\varphi(x)^{-1}[\{1\}] \cap S_1[0] = [x_1, x_2]$ with $x_1 \neq x_2$.

(ii) $\varphi(x), \varphi(y)$ are linearly independent.

(iii) $y = ax_1 - bx_2$ for $a, b > 0$.

(iv) $-ax_1 + 2bx_2 \in \mathrm{smooth}(X)$.

Proof. As X is not strictly convex, we may choose $x \in S_1[0] \cap \mathrm{smooth}(X)$ such that (i) hold. We note that we need only now find an element $y \in \mathrm{smooth}(X)$ that satisfies (ii), (iii) and (iv) since we may always multiply y by a suitable scalar so that it lies in $S_1[0]$.

Define the linear isomorphism $T \in L(X)$ where $T(x_1) = -x_1$ and $T(x_2) = 2x_2$ and $D := T^{-1}(\mathrm{smooth}(X))$. By Proposition 1.1.11 (i), $\mathrm{smooth}(X)^c$ is negligible. As T^{-1} is linear then $D^c = T^{-1}(\mathrm{smooth}(X)^c)$ must also be negligible, thus $D \cap \mathrm{smooth}(X)$ is a dense subset in X. It follows that we may choose $y \in D \cap \mathrm{smooth}(X)$ such that $y = ax_1 - bx_2$ for $a, b > 0$.

We first note that y satifies (iii) and (iv). By Lemma 3.2.3, (ii) holds also as required. $\square$

We define for any $x_1, x_2 \in X$ the following sets:

(i) The *open cone*,

$$\mathrm{cone}^+(x_1, x_2) := \{ax_1 + bx_2 : a, b > 0\} = \{rx : x \in (x_1, x_2),\ r > 0\}.$$

(ii) The *closed cone*,

$$\mathrm{cone}^+[x_1, x_2] := \{ax_1 + bx_2 : a, b \geq 0\} = \{rx : x \in [x_1, x_2],\ r \geq 0\}.$$

(iii) The *two-sided open cone*,

$$\mathrm{cone}(x_1, x_2) := \mathrm{cone}^+(x_1, x_2) \cup \mathrm{cone}^+(-x_1, -x_2).$$

(iv) The *two-sided closed cone*,

$$\mathrm{cone}[x_1, x_2] := \mathrm{cone}^+[x_1, x_2] \cup \mathrm{cone}^+[-x_1, -x_2].$$

If x_1, x_2 are linearly independent then the (two-sided) open cone is open and the (two-sided) closed cone is cone.

Lemma 3.2.5. Let $x_1, x_2 \in S_1[0]$ be linearly independent in a normed plane X and $f \in X^*$ be a support functional of both x_1 and x_2. Then the following holds:

(i) If $y \in \mathrm{cone}^+[x_1, x_2]$ then $\|y\| f$ is a support functional for y.

(ii) If $y \in \mathrm{cone}^+(x_1, x_2)$ then y is smooth.

Proof. (i): Let $y \in \text{cone}^+[x_1, x_2]$. By scaling we may assume $\|y\| = 1$, thus $y = tx_1 + (1-t)x_2$ for some $t \in [0,1]$. We now note that

$$f(y) = tf(x_1) + (1-t)f(x_2) = 1$$

and thus f is a support functional for y.

(ii): Suppose $y \in \text{cone}^+(x_1, x_2)$ is not smooth. By scaling we may assume $\|y\| = 1$, thus $y = tx_1 + (1-t)x_2$ for some $t \in (0,1)$. As y is not smooth then y has support functional $g \in X^*$ with $f \neq g$. If g isn't a support functional for either x_1 or x_2 then

$$g(y) = tg(x_1) + (1-t)g(x_2) < 1,$$

thus g must be a support functional for both x_1, x_2. It follows by (i) that f, g are support functionals for all $x \in \text{cone}^+(x_1, x_2)$, thus $\text{cone}(x_1, x_2) \subseteq \text{smooth}(X)^c$. As $\text{cone}(x_1, x_2)$ is a non-empty open set this contradicts Proposition 1.1.11 (iii). □

Lemma 3.2.6. Let L be a line (i.e. a hyperplane) in a normed plane X that does not contain 0, then the set $\text{smooth}(X) \cap L$ is dense in L.

Proof. Suppose otherwise, then there exists distinct $x_1, x_2 \in L$ and $r > 0$ such that (x_1, x_2) lies in $L \setminus \text{smooth}(X)$. We note that x_1, x_2 must be linearly independent as $0 \notin L$, thus $\text{cone}^+(x_1, x_2)$ is a non-empty open subset of X. Since φ is homogeneous it follows that $\text{cone}^+(x_1, x_2) \subseteq \text{smooth}(X)^c$ which contradicts Proposition 1.1.11 (iii). □

We are now ready for our key lemma of the section.

Lemma 3.2.7. Let X be a normed plane that is not strictly convex, then K_4 is rigid in X.

Proof. Choose x, y as described in Lemma 3.2.4 and let $V(K_4) = \{v_1, v_2, v_3, v_4\}$. Define for $r > 0$ the placement p^r of K_4 where:

(i) $p^r_{v_1} = 0$,

(ii) $p^r_{v_2} = ax_1 - ry = (1-r)\,ax_1 + rbx_2$,

(iii) $p^r_{v_3} = bx_1 + ry = rax_1 + (1-r)\,bx_2$,

(iv) $p^r_{v_4} = (1-2r)\,y = (1-2r)\,ax_1 - (1-2r)\,bx_2$.

We note for all $0 < r < \frac{1}{3}$ the following holds:

(i) $p^r_{v_2} - p^r_{v_1}$, $p^r_{v_3} - p^r_{v_1}$, $p^r_{v_2} - p^r_{v_4} \in \mathrm{cone}^+(x_1, x_2)$.

(ii) $p^r_{v_2} - p^r_{v_3}$ and $p^r_{v_4} - p^r_{v_1}$ are positive scalar multiples of y.

(iii) $p^r_{v_4} - p^r_{v_3} = (1-3r)\,ax_1 - (2-3r)\,bx_2 \notin \mathrm{cone}[x_1, x_2]$.

Define the line

$$L := \{ax_1 - 2bx_2 + 3r(-ax_1 + bx_2) : r \in \mathbb{R}\}.$$

As $0 \notin L$, by Lemma 3.2.6 it follows we may choose $r \in \left(0, \frac{1}{3}\right)$ such that $p^r_{v_4} - p^r_{v_3}$ is smooth. Fix r so that this holds and define $\varphi^r_{v,w}$ to be the support functional of vw in (K_4, p^r). We now note the following holds:

(i) $\varphi^r_{v_2,v_1}$, $\varphi^r_{v_3,v_1}$, $\varphi^r_{v_2,v_4} = \varphi(x)$ (Lemma 3.2.5).

(ii) $\varphi^r_{v_2,v_3}$, $\varphi^r_{v_4,v_1} = \varphi(y)$.

(iii) $\varphi^r_{v_4,v_3} = f$ for some $f \in S^*_1[0]$ where $f, \varphi(x)$ are linearly independent (Lemma 3.2.3 (ii)).

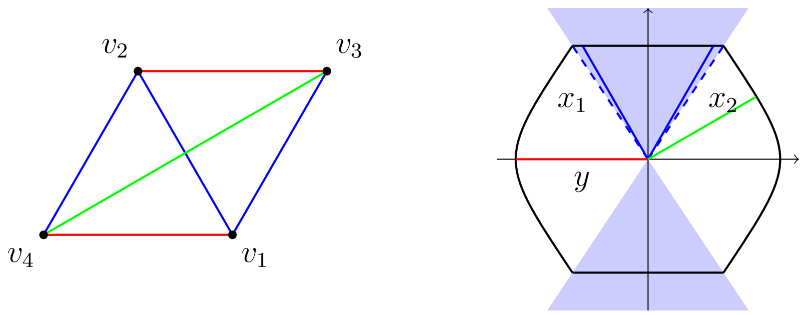

Fig. 3.1 A diagram to illustrate Lemma 3.2.7 applied to a not strictly convex normed plane X. (Left): The constructed infinitesimally rigid framework (K_4, p^r). (Right): The unit ball of X. The edge directions from our placement have been added as their corresponding colour lines, x_1, x_2 have been added as blue dashed lines and cone$[x_1, x_2]$ is shown as the blue area indicated.

We now obtain the following rigidity matrix for $R(K_4, p^r)$:

$$\begin{array}{c c}
 & \begin{array}{cccc} \quad v_1 \quad & \quad v_2 \quad & \quad v_3 \quad & \quad v_4 \quad \end{array} \\
\begin{array}{c} v_1v_2 \\ v_1v_3 \\ v_1v_4 \\ v_2v_3 \\ v_2v_4 \\ v_3v_4 \end{array} &
\begin{bmatrix}
-\varphi(x) & \varphi(x) & 0 & 0 \\
-\varphi(x) & 0 & \varphi(x) & 0 \\
-\varphi(y) & 0 & 0 & \varphi(y) \\
0 & \varphi(y) & -\varphi(y) & 0 \\
0 & \varphi(x) & 0 & -\varphi(x) \\
0 & 0 & -f & f
\end{bmatrix}
\end{array}$$

As $\varphi(x), \varphi(y)$ are linearly independent and $f, \varphi(x)$ are linearly independent then it follows that $R(K_4, p^r)$ has independent rows, thus (K_4, p^r) is independent. Since K_4 is independent in X, by Corollary 3.1.4, K_4 is isostatic in X as required. $\square$

3.2.2 The rigidity of K_4 in strictly convex but not smooth normed planes

The following technical lemmas will be of use later.

Lemma 3.2.8. Let K_4 be the complete graph on the vertex set $\{v_1, v_2, v_3, v_4\}$. Suppose we have a placement p of K_4 in a normed plane X where all edges but v_1v_4 are well-positioned. Further suppose the following:

(i) $\varphi_{v_1,v_2} = \varphi_{v_3,v_4} = \varphi(x)$.

(ii) $\varphi_{v_1,v_3} = \varphi_{v_2,v_4} = \varphi(y)$.

(iii) $\varphi_{v_2,v_3} = \varphi(\omega)$, where $\varphi(\omega) = a\varphi(x) + b\varphi(y)$ for some $a, b \in \mathbb{R}$.

(iv) $\varphi(x), \varphi(y), \varphi(\omega)$ are pairwise independent support functions.

(v) ϕ is the set of support functionals of (K_4, p) with the pseudo-support functional φ_{v_1,v_4}.

If φ_{v_1,v_4} and $a\varphi(x) - b\varphi(y)$ are linearly independent then $R(K_4, p)^\phi$ has row independence.

Proof. We see that with the given parameters $R(K_4, p)^\phi$ is of the form

$$
\begin{array}{cc}
 & \begin{array}{cccc} \quad v_1 \quad & \quad v_2 \quad & \quad v_3 \quad & \quad v_4 \quad \end{array} \\
\begin{array}{c} v_1v_2 \\ v_1v_3 \\ v_1v_4 \\ v_2v_3 \\ v_2v_4 \\ v_3v_4 \end{array} &
\begin{bmatrix}
\varphi(x) & -\varphi(x) & 0 & 0 \\
\varphi(y) & 0 & -\varphi(y) & 0 \\
\varphi_{v_1,v_4} & 0 & 0 & -\varphi_{v_1,v_4} \\
0 & \varphi(\omega) & -\varphi(\omega) & 0 \\
0 & \varphi(y) & 0 & -\varphi(y) \\
0 & 0 & \varphi(x) & -\varphi(x)
\end{bmatrix}
\end{array}
$$

Suppose $(c_{vw})_{vw \in E(G)}$ is a pseudo-stress of $(K_4, p)^\phi$. By the second column

$$-c_{v_1v_2}\varphi(x) + c_{v_2v_3}\varphi(\omega) + c_{v_2v_4}\varphi(y) = (c_{v_2v_3}a - c_{v_1v_2})\varphi(x) + (c_{v_2v_3}b + c_{v_2v_4})\varphi(y) = 0,$$

thus as $\varphi(x), \varphi(y)$ are linearly independent, $c_{v_1v_2} = c_{v_2v_3}a$ and $c_{v_2v_4} = -c_{v_2v_3}b$. By the third column

$$-c_{v_1v_3}\varphi(y) - c_{v_2v_3}\varphi(\omega) + c_{v_3v_4}\varphi(x) = -(c_{v_2v_3}a - c_{v_3v_4})\varphi(x) - (c_{v_2v_3}b + c_{v_1v_3})\varphi(y) = 0,$$

thus as $\varphi(x), \varphi(y)$ are linearly independent, $c_{v_3v_4} = c_{v_2v_3}a$ and $c_{v_1v_3} = -c_{v_2v_3}b$. By the first column combined with our previous results we see that

$$c_{v_1v_2}\varphi(x) + c_{v_1v_3}\varphi(y) + c_{v_1v_4}\varphi_{v_1,v_4} = c_{v_2v_3}(a\varphi(x) - b\varphi(y)) + c_{v_1v_4}\varphi_{v_1,v_4} = 0.$$

As φ_{v_1,v_4} is linearly independent of $a\varphi(x) - b\varphi(y)$ then $c_{v_2v_3} = c_{v_1v_4} = 0$. This implies $c = 0$ and thus $R(K_4, p)^\phi$ has row independence. □

Lemma 3.2.9. Let X be a normed space and $z \in X$. Then there exists $x, y \in \operatorname{smooth}(X)$ so that $x + y = z$ and $x - y \in \operatorname{smooth}(X)$. If $z \notin \operatorname{smooth}(X) \cup \{0\}$, then x, y are linearly independent.

Proof. If $z = 0$, choose any $x \in \operatorname{smooth}(X)$ and define $y := -x$. Similarly, if $z \in \operatorname{smooth}(X)$, let $x := 2z$ and $y := -z$.

Now suppose $z \notin \operatorname{smooth}(X) \cup \{0\}$. By Proposition 1.1.11 (iii), the sets

$$z + \operatorname{smooth}(X), \qquad z - \operatorname{smooth}(X)$$

have negligible complements, thus the complement of

$$A := (\operatorname{smooth}(X) - z) \cap (\operatorname{smooth}(X) + z)$$

is negligible. It follows that $A \neq \emptyset$ and so we may choose $w \in A$. If we define $x := \frac{1}{2}(z+w)$ and $y := \frac{1}{2}(z-w)$ then x, y and $x-y$ are smooth, and $z = x+y$. If x, y are linearly dependent then z is smooth, a contradiction, thus x, y are linearly independent. $\square$

Lemma 3.2.10. Let X be a strictly convex normed plane, $z \neq 0$ be non-smooth with $\|z\| = 1$ and $\varphi[z] = [f, g]$, and define

$$X^+ := (f-g)^{-1}(0,\infty), \qquad X^- := (f-g)^{-1}(-\infty, 0).$$

If $(z_n)_{n\in\mathbb{N}}$ is a sequence of smooth points that converges to z with $\|z_n\| = 1$, then the following properties hold:

(i) $(\varphi(z_n))_{n\in\mathbb{N}}$ has a convergent subsequence.

(ii) If $\varphi(z_n) \to h$ as $n \to \infty$ then $h = f$ or g.

(iii) If $\varphi(z_n) \to h$ as $n \to \infty$ and $\varphi(z_n) \in X^+$ for large enough n then $h = f$.

(iv) If $\varphi(z_n) \to h$ as $n \to \infty$ and $\varphi(z_n) \in X^-$ for large enough n then $h = g$.

Proof. (i): This holds as $S_1^*[0]$ is compact.

(ii): Choose any $\epsilon > 0$, then we may choose $N \in \mathbb{N}$ such that for all $n \geq N$

$$\|h - \varphi(z_n)\| < \frac{\epsilon}{2} \quad \text{ and } \quad \|z - z_n\| < \frac{\epsilon}{2}.$$

We now note,

$$\begin{aligned} |1 - h(z)| &= |\varphi(z_n)(z_n) - h(z)| \\ &\leq |\varphi(z_n)(z_n) - \varphi(z_n)(z)| + |\varphi(z_n)(z) - h(z)| \\ &\leq \|z_n - z\| + \|\varphi(z_n) - h\| \end{aligned}$$

$$< \epsilon.$$

As this holds for all $\epsilon > 0$ then $h(z) = 1$. As $\|h\| = 1$ then h supports z, thus $h \in [f, g]$.

If h lies in the interior of $[f, g]$ then for large enough $n \in \mathbb{N}$ we would have $\varphi(z_n)$ in the interior of $[f, g]$ (with respect to $S_1^*[0]$), thus $\varphi(z_n)$ is a support functional of z. If $z \neq z_n$ then we note that $[z, z_n] \in S_1[0]$ as for any $t \in [0, 1]$

$$1 = \varphi(z_n)(tz + (1-t)z_n) \leq \|tz + (1-t)z_n\| \leq 1,$$

however this contradicts the strict convexity of X. If $z = z_n$ then as z_n is smooth z is also smooth, however this contradicts the assumption that z is non-smooth. As the only non-interior points are f, g the result follows.

(iii): Suppose for contradiction that $\varphi(z_n) \to g$ as $n \to \infty$. As $f \neq g$ then f, g must be linearly independent (as otherwise $0 \in [f, g] \subset S_1^*[0]$), thus for each $n \in \mathbb{N}$ there exists $a_n, b_n \in \mathbb{R}$ such that $\varphi(z_n) = a_n f + b_n g$. Since $\varphi(z_n) \to g$ then for large enough n we have that $b_n > 0$.

Suppose $a_n, b_n \geq 0$ for large enough n, then

$$\begin{aligned}
\|\varphi(z_n)\| &= \|a_n f + b_n g\| \\
&\leq a_n + b_n \\
&= a_n f(z) + b_n g(z) \\
&= \varphi(z_n)(z) \\
&\leq \|\varphi(z_n)\|,
\end{aligned}$$

thus $\varphi(z_n)$ is a support functional of z. As noted previously, this either contradicts that X is strictly convex or that z_n is smooth and z is non-smooth.

Suppose instead that for large enough n we have $a_n < 0 < b_n$. We now note that

$$\varphi(z_n)(z_n) = a_n f(z_n) + b_n g(z_n)$$

$$\begin{aligned}
&= a_n(f-g)(z_n) + (a_n+b_n)g(z_n) \\
&< (a_n+b_n)g(z_n) \quad \text{as } z_n \in X^+ \text{ and } a_n < 0 \\
&\leq a_n + b_n \\
&= \|b_n g\| - \|-a_n f\| \text{ as } \|f\| = \|g\| = 1 \\
&\leq \|a_n f + b_n g\| \\
&= \|\varphi(z_n)\|
\end{aligned}$$

which implies $\varphi(z_n)(z_n) < 1$ contradicting that $\varphi(z_n)$ is the support functional of z_n and $\|z_n\| = 1$. It follows that $\varphi(z_n) \nrightarrow g$, thus $\varphi(z_n) \to f$ by (ii).

(iv) now follows by the same method given above. □

We are now ready for our key lemma.

Lemma 3.2.11. Let X be a strictly convex normed plane with non-zero non-smooth points, then K_4 is rigid in X.

Proof. We consider K_4 to be the complete graph on the vertex set $\{v_1, v_2, v_3, v_4\}$. Let z be a non-zero non-smooth point of X with $\|z\| = 1$. By Lemma 3.2.9, we can choose smooth linearly independent $x, y \in X$ such that $z = x + y$ and $w := x - y$ is smooth.

Define the placements p, q^k of K_4 for $k \in \mathbb{Z} \setminus \{0\}$ where

$$p_{v_1} = 0, \quad p_{v_2} = x, \quad p_{v_3} = y, \quad p_{v_4} = x + y = z,$$

and:

$$q^k_{v_1} = 0, \quad q^k_{v_2} = x + \frac{1}{k}x, \quad q^k_{v_3} = y, \quad q^k_{v_4} = x + y + \frac{1}{k}x = z + \frac{1}{k}x.$$

By Lemma 1.2.5 there exists for each $k \in \mathbb{Z} \setminus \{0\}$ a well-positioned placement p^k such that $\|p^k - q^k\|_{V(K_4)} < \frac{1}{k^2}$ and $p^k_{v_1} = 0$.

By Proposition 1.1.11 (iv), the support functionals $\varphi^k_{v,w}$ for p^k satisfy the following:

(i)

$$\lim_{k\to\infty} \varphi^k_{v_2,v_1} = \lim_{k\to-\infty} \varphi^k_{v_2,v_1} = \lim_{k\to\infty} \varphi^k_{v_4,v_3} = \lim_{k\to-\infty} \varphi^k_{v_4,v_3} = \frac{1}{\|x\|}\varphi(x),$$

(ii)

$$\lim_{k\to\infty} \varphi^k_{v_3,v_1} = \lim_{k\to-\infty} \varphi^k_{v_3,v_1} = \lim_{k\to\infty} \varphi^k_{v_4,v_2} = \lim_{k\to-\infty} \varphi^k_{v_4,v_2} = \frac{1}{\|y\|}\varphi(y),$$

(iii)

$$\lim_{k\to\infty} \varphi^k_{v_2,v_3} = \lim_{k\to-\infty} \varphi^k_{v_2,v_3} = \frac{1}{\|w\|}\varphi(w).$$

By Proposition 1.1.16 (ii), $\varphi[z] = [f, g]$ for some $f \neq g$. We now further define

$$X^+ := (f-g)^{-1}(0,\infty), \qquad X^- := (f-g)^{-1}(-\infty,0).$$

We note that $(f-g)x \neq 0$ (as otherwise x, z are linearly independent), thus without loss of generality we may assume $x \in X^+$. For each $k \in \mathbb{Z} \setminus \{0\}$ define $d_k := p^k_{v_4} - q^k_{v_4}$, then $\|d_k\| < \frac{1}{k^2}$. As

$$(f-g)\left(p^k_{v_4} - p^k_{v_1}\right) = (f-g)\left(z + \frac{1}{k}x + d_k\right) = \frac{1}{k}(f-g)(x) + (f-g)(d_k)$$

and $\|f-g\| \leq 2$ it follows that

$$\left|(f-g)\left(p^k_{v_4} - p^k_{v_1}\right) - \frac{1}{k}(f-g)(x)\right| = (f-g)(d_k) \leq \frac{2}{k^2}.$$

We may rewrite this as

$$\frac{1}{k}\left((f-g)(x)-\frac{2}{k}\right) \leq (f-g)\left(p^k_{v_4}-p^k_{v_1}\right) \leq \frac{1}{k}\left((f-g)(x)+\frac{2}{k}\right),$$

thus there exists $N \in \mathbb{N}$ such that if $k \geq N$ then $p^k_{v_4} - p^k_{v_1} \in X^+$ and if $k \leq -N$ then $p^k_{v_4} - p^k_{v_1} \in X^-$. By Lemma 3.2.10, there exists a strictly increasing sequence $(n_i)_{i\in\mathbb{N}}$ in $\mathbb{N}$ such that

$$\lim_{i\to\infty} \varphi^{n_i}_{v_4,v_1} = f \qquad \lim_{i\to\infty} \varphi^{-n_i}_{v_4,v_1} = g.$$

Define ϕ_f to be the support functionals of (K_4, p) with pseudo-support functional $\varphi_{v_4,v_1} = f$ and likewise define ϕ_g to be the support functionals of (K_4, p) with pseudo-support functional $\varphi_{v_4,v_1} = g$. We note that $R(K_4, p^{n_i}) \to R(K_4, p)^{\phi_f}$ and $R(K_4, p^{-n_i}) \to R(K_4, p)^{\phi_g}$ as $i \to \infty$.

There exists unique $a, b \in \mathbb{R}$ such that $\varphi(w) = a\varphi(x) + b\varphi(y)$. By Lemma 3.2.8, $R(K_4, p)^{\phi_f}$ has row independence if f is linearly independent of $a\varphi(x) - b\varphi(y)$ and $R(K_4, p)^{\phi_g}$ has row independence if g is linearly independent of $a\varphi(x) - b\varphi(y)$. Both f, g cannot be linearly dependent to $a\varphi(x) - b\varphi(y)$ as f, g are linearly independent, thus either $R(K_4, p)^{\phi_f}$ or $R(K_4, p)^{\phi_g}$ has row independence. By Lemma 1.3.11 this implies that for large enough i we have either (K_4, p^{n_i}) or (K_4, p^{-n_i}) are independent and thus there exists an independent placement of K_4. It now follows by Proposition 1.3.13 that K_4 is rigid also. □

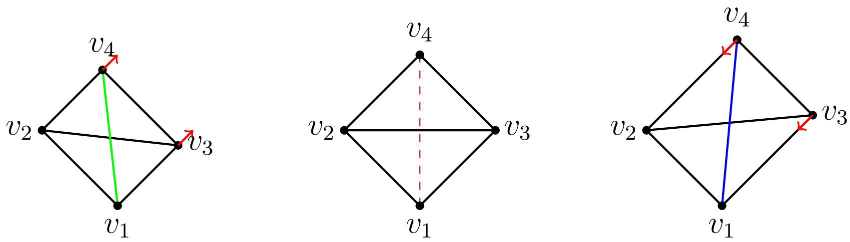

Fig. 3.2 From left to right: (K_4, p^{-n_i}), (K_4, p) and (K_4, p^{n_i}) for $i \in \mathbb{N}$. The red dashed edge indicates the edge v_1v_4 of (K_4, p) is not well-positioned. We note that the support functional of the green edge will approximate g while the support functional of the blue edge will approximate f.

3.2.3 The rigidity of K_4 in strictly convex and smooth normed planes

For this section we shall define $\{v_1, v_2, v_3, v_4\}$ to be the vertex set of K_4 and $e := v_1v_4$. The graph $K_4 - e$ will be the subgraph of K_4 formed by removing the edge e. Given a normed plane X we shall fix a basis $b_1, b_2 \in S_1[0]$.

Definition 3.2.12. Let (G, p) be a framework in a normed space X. We say (G, p) is in *3-cycle general position* if every subframework $(H, q) \subset (G, p)$ with $H \cong K_3$ is in general position.

Lemma 3.2.13. Let X be a strictly convex normed space and $x, y \in X$ linearly independent and smooth. Then $\varphi(x), \varphi(y)$ are linearly independent.

Proof. Suppose $\varphi(x), \varphi(y)$ are linearly dependent, then $\varphi(x) = c\varphi(y)$ for some $c \in \mathbb{R}$. As φ is homogenous it follows $\varphi(x) = \varphi(cy)$, thus by Proposition 1.1.20, $x = cy$ as required. $\square$

Lemma 3.2.14. Let $(K_4 - e, p)$ be in 3-cycle general position in a strictly convex normed plane X. Then the following holds:

(i) For all $q \in f^{-1}_{K_4-e}[f_{K_4}(p)]$, $(K_4 - e, q)$ is in 3-cycle general position.

(ii) If $(K_4 - e, p)$ is well-positioned, then $(K_4 - e, p)$ is independent.

Proof. (i): Suppose $(K_4 - e, q)$ is not in 3-cycle general position, then without loss of generality we may assume $q_{v_1}, q_{v_2}, q_{v_3}$ lie on a line. By possibly reordering vertices we note that we have

$$\|q_{v_1} - q_{v_2}\| + \|q_{v_2} - q_{v_3}\| = \|q_{v_1} - q_{v_3}\|.$$

Define $a_{12} = \|p_{v_1} - p_{v_2}\|$, $a_{23} = \|p_{v_2} - p_{v_3}\|$, $x_{12} = (p_{v_1} - p_{v_2})/a_{12}$ and $x_{23} = (p_{v_2} - p_{v_3})/a_{23}$. As p is in general position we note $a_{12}, a_{23} > 0$ and x_{12}, x_{23} are linearly independent. As $f_{K_4}(q) = f_{K_4}(p)$, then

$$\|a_{12}x_{12} + a_{23}x_{23}\| = \|a_{12}x_{12}\| + \|a_{23}x_{23}\|.$$

We note that

$$\frac{a_{23}}{a_{12} + a_{23}} = 1 - \frac{a_{12}}{a_{12} + a_{23}}.$$

If we let $t := \frac{a_{12}}{a_{12}+a_{23}}$ then $t \in (0, 1)$ and

$$\begin{aligned} \|tx_{12} + (1-t)x_{23}\| &= \frac{\|a_{12}x_{12} + a_{23}x_{23}\|}{a_{12} + a_{23}} \\ &= \frac{\|a_{12}x_{12}\| + \|a_{23}x_{23}\|}{a_{12} + a_{23}} \\ &= t\|x_{12}\| + (1-t)\|x_{23}\| \\ &= 1, \end{aligned}$$

which contradicts the strict convexity of X.

(ii): Suppose $a \in \mathbb{R}^{E(K_4)\setminus\{e\}}$ is a stress of $(K_4 - e, p)$. By observing the stress condition at v_1 we note

$$a_{v_1v_2}\varphi_{v_1,v_2} + a_{v_1v_3}\varphi_{v_1,v_3} = 0.$$

As $(K_4 - e, p)$ is in 3-cycle general position then by Lemma 3.2.13, $a_{v_1v_2} = a_{v_1v_3} = 0$. Similarly, if we observe the stress condition at v_4 we see that $a_{v_2v_4} = a_{v_3v_4} = 0$. We now see that the stress condition at v_2 is

$$a_{v_1v_2}\varphi_{v_2,v_1} + a_{v_2v_3}\varphi_{v_2,v_3} + a_{v_2v_4}\varphi_{v_2,v_4} = a_{v_2v_3}\varphi_{v_2,v_3} = 0,$$

thus $a = 0$ and $(K_4 - e, p)$ is independent. □

Define for any graph G and vertex $v \in V(G)$ the map

$$f_{G,v} : X^{V(G)} \to \mathbb{R}^{E(G)} \times X, \ p \mapsto (f_G(p), p_v).$$

It is immediate that $f_{G,v}$ is differentiable at p if and only if p is well-positioned. We note that the kernel of $df_{G,v}(p)$ is exactly the space of infinitesimal flexes u of (G,p) where $u_v = 0$.

Lemma 3.2.15. Let X be a strictly convex and smooth normed plane and suppose $(K_4 - e, p)$ is in 3-cycle general position with $p_{v_1} = 0$, then $V(p) := f^{-1}_{K_4-e,v_1}[f_{K_4-e,v_1}(p)]$ is a 1-dimensional compact Hausdorff C^1-manifold.

Proof. As $K_4 - e$ is connected, $V(p)$ is bounded. As f_{K_4-e,v_1} is continuous then $V(p)$ is closed, thus $V(p)$ is compact; further, as $X^{V(K_4)}$ is Hausdorff so too is $V(p)$.

Choose any $q \in V(p)$, then by Lemma 3.2.14 (i), $(K_4 - e, q)$ is in 3-cycle general position. By Lemma 3.2.14 (ii), $(K_4 - e, q)$ is independent, thus for all $q \in V(p)$ we have that $df_{K_4,v_1}(q)$ is surjective i.e. p is a regular point of f_{K_4-e,v_1}. It now follows from

[47, Theorem 3.5.2(ii)] that $V(p)$ is a C^1-manifold with dimension $\dim\ker df_{K_4-e,v_1}(p)$. Since $K_4 - e$ is independent then $\dim\ker df_{K_4-e,v_1}(p) = 1$ as required. □

We denote by $\mathbb{T}$ the circle group i.e. the set $\{e^{i\phi} : \phi \in (-\pi, \pi]\}$ with topology and group operation inherited from $\mathbb{C} \setminus \{0\}$. Let X be a normed plane with basis b_1, b_2. We note there exists a surjective continuous map $\theta : X \setminus \{0\} \to \mathbb{T}$ given by

$$x = \lambda b_1 + \mu b_2 \mapsto \frac{\lambda + \mu i}{\sqrt{\lambda^2 + \mu^2}}.$$

If we restrict θ to $S_1[0]$ then it is a homeomorphism. Let $x, y \in X \setminus \{0\}$ be linearly independent, then $\theta(x)\theta(y)^{-1} = e^{i\phi} \neq \pm 1$; if $\phi \in (0, \pi)$ then we say $x\theta y$, while if $\phi \in (-\pi, 0)$ then we say $y\theta x$.

Choose any two linearly independent points x, y in a normed plane X and define $L(x, y)$ to be the unique line through x and y. By abuse of notation we also denote by $L(x, y)$ the unique linear functional $L(x, y) : X \to \mathbb{R}$ where $L(x, y)x = L(x, y)y = 1$. We say that $z, z' \in X$ are *on opposite sides of the line* $L(x, y)$ if and only if $L(x, y)z < 1 < L(x, y)z'$ or vice versa.

Lemma 3.2.16. Let X, p and $V(p)$ be as defined in Lemma 3.2.15. Define the maps $f, g : V(p) \to \{-1, 1\}$ where

$$f(q) = \begin{cases} 1, & \text{if } q_{v_2}\theta q_{v_3} \\ -1, & \text{if } q_{v_3}\theta q_{v_2} \end{cases}$$

and

$$g(q) = \begin{cases} 1, & \text{if } L(q_{v_2}, q_{v_3})(q_{v_4}) > 1 \\ -1, & \text{if } L(q_{v_2}, q_{v_3})(q_{v_4}) < 1, \end{cases}$$

then f, g are well-defined and continuous.

Proof. We note that f is not well-defined at q if and only if q_{v_2}, q_{v_3} are linearly dependent. By Lemma 3.2.14 (i), as $(K_4 - e, q)$ is in 3-cycle general position and $q_{v_1} = 0$ then q_{v_2}, q_{v_3} are linearly independent, thus f is well-defined at all $q \in V(p)$.

The map g is not well-defined at q if and only if either q_{v_2}, q_{v_3} are linearly dependent or q_{v_4} lies on $L(q_{v_2}, q_{v_3})$, thus g is not well-defined at q if and only if q is not in 3-cycle general position. By Lemma 3.2.14 (i), $(K_4 - e, q)$ is in 3-cycle general position for all $q \in V(p)$, thus g is well-defined.

As f and g are locally constant they are continuous. $\square$

Lemma 3.2.17. [48, Proposition 31] Let X be a strictly convex normed plane and $a, b, c \in X \setminus \{0\}$ be distinct with $\|b\| = \|c\|$. If either,

(i) $a\theta b$, $b\theta c$ and $a\theta c$,

(ii) $c\theta b$, $b\theta a$ and $c\theta a$,

then $\|a - b\| < \|a - c\|$.

Lemma 3.2.18. Let X be a strictly convex normed plane, $x, y \in X$ be distinct and $r_x, r_y > 0$. If $S_{r_x}[x] \cap S_{r_y}[y] \neq \emptyset$ then one of the following holds:

(i) $S_{r_x}[x] \cap S_{r_y}[y] = \{z\}$ and x, y, z are colinear.

(ii) $S_{r_x}[x] \cap S_{r_y}[y] = \{z_1, z_2\}$ for $z_1 \neq z_2$. Further, if $x = 0$ then $z_1 \theta y$ and $y \theta z_2$ or vice versa, and if x, y are linearly independent then z_1, z_2 are on opposite sides of the line $L(x, y)$.

Proof. Let $\theta : S_1[0] \to \mathbb{T}$ be as previously described. Define the continuous map

$$\phi : [-\pi, \pi] \to S_{r_x}[x], \ \phi(t) := r_x \theta^{-1}(e^{i(t+t_0)}) + x,$$

where $r_x\theta^{-1}(e^{it_0})$ the unique point between x, y on $S_{r_x}[x]$. Now define the map

$$h : [-\pi, \pi] \to \mathbb{R}, \ h(t) := \|\phi(t) - y\|.$$

We note that $\phi(-\pi) = \phi(\pi)$ and $h(-\pi) = h(\pi)$. It follows from Lemma 3.2.17 that h is strictly increasing on $[0, \pi]$ and strictly decreasing on $[-\pi, 0]$.

If $\phi(0) \in S_{r_x}[x] \cap S_{r_y}[y]$ then for all $t \neq 0$,

$$\|\phi(t) - y\| = h(t) > h(0) = r_y,$$

thus $S_{r_x}[x] \cap S_{r_y}[y] = \{z\}$ with $z := \phi(0)$; similarly if $\phi(\pi) \in S_{r_x}[x] \cap S_{r_y}[y]$ then $S_{r_x}[x] \cap S_{r_y}[y] = \{z\}$ with $z := \phi(\pi)$ and so (i) holds.

Suppose $\phi(0), \phi(\pi) \notin S_{r_x}[x] \cap S_{r_y}[y]$. We note that as $S_{r_x}[x] \cap S_{r_y}[y] \neq \emptyset$ then there exists $t_1 \in (-\pi, \pi) \setminus \{0\}$ so that $h(t_1) = r_y$. First suppose $t_1 \in (-\pi, 0)$, then for all $t \in (t_1, 0)$ and $t' \in (-\pi, t_1)$ we have $h(t) < h(t_1) < h(t')$, thus there are no other intersection points in $(-\pi, 0)$. As $h|_{[0,\pi]}$ is strictly increasing and

$$h(0) < h(t_1) = r_y < h(-\pi) = h(\pi)$$

then by the Intermediate Value Theorem there exists a unique value $t_2 \in (0, \pi)$ so that $h(t_2) = r_y$, thus $S_{r_x}[x] \cap S_{r_y}[y] = \{\phi(t_1), \phi(t_2)\}$ with $-\pi < t_1 < 0 < t_2 < \pi$. Similarly if $t_1 \in (0, \pi)$ then $S_{r_x}[x] \cap S_{r_y}[y] = \{\phi(t_1), \phi(t_2)\}$ with $-\pi < t_2 < 0 < t_1 < \pi$.

If $x = 0$ then it is immediate that $\phi(t_1)\theta\phi(0)$ and $\phi(0)\theta\phi(t_2)$. As $\phi(0)$ is a positive scalar multiple of y then $\phi(t_1)\theta y$ and $y\theta\phi(t_2)$. Now suppose x, y are linearly independent, then we now note that $\phi(t_1)$ and $\phi(t_2)$ lie on opposite sides of the line through x, y as $e^{i(t_1+t_0)}$ and $e^{i(t_2+t_0)}$ lie on opposite sides of the line through e^{it_0} and e^{-it_0}. $\square$

Lemma 3.2.19. Let X, p and $V(p)$ be as defined in Lemma 3.2.15. Let $q^1, q^2 \in V(p)$ with $f(q^1) = f(q^2)$, $g(q^1) = g(q^2)$ and $q^1_{v_2} = q^2_{v_2}$, then $q^1 = q^2$.

Proof. By Lemma 3.2.14 (i), q^1, q^2 are in 3-cycle general position. As $q^1_{v_1}, q^1_{v_2}, q^1_{v_3}$ are not colinear then by Lemma 3.2.18 there exists exactly one other point $z \in X$ such that $\|z - q^1_{v_1}\| = \|q^1_{v_3} - q^1_{v_1}\|$ and $\|z - q^1_{v_2}\| = \|q^1_{v_3} - q^1_{v_2}\|$. We note that as $q^1_{v_1} = q^2_{v_1} = 0$ and $q^1_{v_2} = q^2_{v_2}$ then $q^2_{v_3} = q^1_{v_3}$ or $q^2_{v_3} = z$. By Lemma 3.2.18 (ii), either $z\theta q^1_{v_2}$ and $q^1_{v_3}\theta z$ or vice versa. If $q^2_{v_3} = z$ then $f(q^2) = -f(q^1)$, thus $q^2_{v_3} = q^1_{v_3}$.

Similarly, as $q^1_{v_2}, q^1_{v_3}, q^1_{v_4}$ are not colinear then by Lemma 3.2.18 there exists exactly one other point $z' \in X$ such that $\|z' - q^1_{v_2}\| = \|q^1_{v_4} - q^1_{v_2}\|$ and $\|z' - q^1_{v_3}\| = \|q^1_{v_4} - q^1_{v_3}\|$. By Lemma 3.2.18 (ii), $z', q^1_{v_4}$ are on the opposite sides of $L(q^1_{v_2}, q^1_{v_3})$. If $q^2_{v_4} = z'$ then $g(q^2) = -g(q^1)$, thus $q^2_{v_4} = q^1_{v_4}$. $\square$

Lemma 3.2.20. Let X, p and $V(p)$ be as defined in Lemma 3.2.15. The path-connected components of $V(p)$ are exactly $f^{-1}[1] \cap g^{-1}[1]$, $f^{-1}[1] \cap g^{-1}[-1]$, $f^{-1}[-1] \cap g^{-1}[1]$ and $f^{-1}[-1] \cap g^{-1}[-1]$. Further, each $f^{-1}[i] \cap g^{-1}[j]$ component is a path-connected compact Hausdorff 1-dimensional C^1-manifold.

Proof. By multiple applications of Lemma 3.2.18 it follows that each $f^{-1}[i] \cap g^{-1}[j]$ is non-empty.

Choose $i, j \in \{1, -1\}$. Suppose there exists disjoint path-connected components of $A, B \subset f^{-1}[i] \cap g^{-1}[j]$, then by Lemma 3.2.15, A, B are both path-connected compact Hausdorff 1-dimensional C^1-manifolds. As every path-connected compact Hausdorff 1-dimensional manifold is homeomorphic to a circle (see [45, Theorem 5.27]) we may define the homeomorphisms $\alpha : \mathbb{T} \to A$ and $\beta : \mathbb{T} \to B$. We will define α_{v_i}, β_{v_i} to be the v_i components of α and β respectively.

Suppose there exists $z_1, z_2 \in \mathbb{T}$ such that $\alpha_{v_2}(z_1) = \alpha_{v_2}(z_2)$, then by Lemma 3.2.19, $\alpha(z_1) = \alpha(z_2)$, thus the map $\alpha_{v_2} : \mathbb{T} \to S_{\|p_{v_2}\|}[0]$ is injective; similarly, the map $\beta_{v_2} : \mathbb{T} \to S_{\|p_{v_2}\|}[0]$ is also injective. As $\mathbb{T}$ is compact then by the Brouwer's theorem for invariance of domain [44, Theorem 1.18] it follows $\alpha_{v_2}, \beta_{v_2}$ are homeomorphisms, thus we may choose $z, z' \in \mathbb{T}$ so that $\alpha_{v_2}(z) = \beta_{v_2}(z')$. By Lemma 3.2.19 it follows $\alpha(z) = \beta(z')$ and A, B are not disjoint path-connected components. $\square$

Lemma 3.2.21. Let X, p and $V(p)$ be as defined in Lemma 3.2.15 and $V_0(p)$ be the path-connected component of $V(p)$ that contains p. Suppose $p_{v_4} = p_{v_2} + p_{v_3}$, then for all $q \in V_0(p)$ we have $q_{v_4} = q_{v_2} + q_{v_3}$.

Proof. Choose $q \in V_0(p)$ then by Lemma 3.2.20, $f(q) = f(p)$ and $g(q) = g(p)$. Define q' to be the placement of $K_4 - e$ where $q'_{v_i} = q_{v_i}$ for $i = 1, 2, 3$ and $q'_{v_4} = q'_{v_2} + q'_{v_3}$. We immediately note $q' \in V(p)$ and $f(q') = f(p)$. Suppose $q' \neq q$, then by Lemma 3.2.19 we must have $-g(q') = g(q) = g(p)$; however

$$L(p_{v_2}, p_{v_3})(p_{v_4}) = L(p_{v_2}, p_{v_3})(p_{v_2} + p_{v_3}) = 2 > 1$$

and

$$L(q'_{v_2}, q'_{v_3})(q'_{v_4}) = L(q'_{v_2}, q'_{v_3})(q'_{v_2} + q'_{v_3}) = 2 > 1,$$

and so $g(q') = 1 = g(p)$, a contradiction, thus $q' = q$ and the result follows. $\square$

We will finally need the following result which will help us separate when we are dealing with Euclidean and non-Euclidean normed planes.

Theorem 3.2.22. Let X be a normed plane and

$$D(\epsilon) := \{\|a + b\| : \|a - b\| = \epsilon, \ \|a\| = \|b\| = 1\}$$

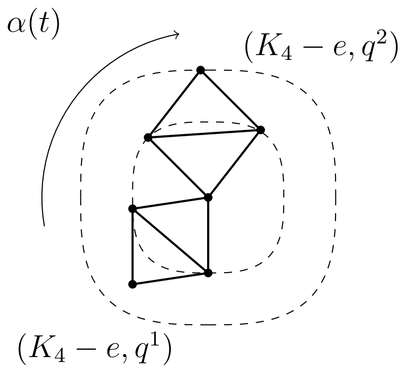

Fig. 3.3 The frameworks $(K_4 - e, q^1)$ and $(K_4 - e, q^2)$ in some strictly convex and smooth normed plane X, as described in Lemma 3.2.23. The inner dotted shape represents the unit sphere of X and the outer dotted shape represents the sphere of X with radius $\|q^2_{v_4}\|$. As the framework follows the differentiable path $\alpha(t)$ the distance $\|\alpha_{v_1}(t) - \alpha_{v_4}(t)\|$ is non-constant; when the derivative of $t \mapsto \|\alpha_{v_1}(t) - \alpha_{v_4}(t)\|$ is non-zero at point s we add the edge v_1v_4 and note $(K_4, \alpha(s))$ will be infinitesimally rigid.

If X is a non-Euclidean normed plane then for all $0 < \epsilon < 2$ where $\epsilon \neq 2\cos(k\pi/2n)$ $(n, k \in \mathbb{N},\ 1 \leq k \leq n)$,

$$\inf D(\epsilon) < \sup D(\epsilon).$$

Proof. This follows by [2, p. 323] and noting that if Property Q_ϵ does not hold then $D(\epsilon)$ is not a single point. □

We are now ready for our key lemma.

Lemma 3.2.23. Let X be a normed plane that is strictly convex and smooth, then K_4 is rigid in X.

Proof. If X is Euclidean this follows from Theorem 1.3.20 so suppose X is non-Euclidean.

Choose any $0 < \epsilon < 2$ so that $\epsilon \neq 2\cos(k\pi/2n)$ for all $n, k \in \mathbb{N}$ with $1 \leq k \leq n$. By the continuity of the norm we may choose a placement p of K_4 so that:

(i) $p_{v_1} = 0$,

(ii) $\|p_{v_2}\| = \|p_{v_3}\| = 1$,

(iii) $p_{v_2} \theta p_{v_3}$,

(iv) $\|p_{v_2} - p_{v_3}\| = \epsilon$,

(v) $p_{v_4} = p_{v_2} + p_{v_3}$,

We note $(K_4 - e, p)$ is in 3-cycle general position, $f(p) = 1$ as $p_{v_2} \theta p_{v_3}$, and $g(p) = 1$ as

$$L(p_{v_2}, p_{v_3})(p_{v_4}) = L(p_{v_2}, p_{v_3})(p_{v_2} + p_{v_3}) = 2 > 1.$$

By Lemma 3.2.15 and Lemma 3.2.20, $V_0(p) = f^{-1}[1] \cap g^{-1}[1]$ is a path-connected compact Hausdorff 1-dimensional C^1-manifold. We note that for every pair a, b in $S_1[0]$ with $\|a - b\| = \epsilon$ there exists $q \in V_0(p)$ so that $q_{v_2} = a$ and $q_{v_3} = b$ or vice versa, thus there exists $q^1, q^2 \in V_0(p)$ so that

$$\|q^1_{v_4}\| = \inf\{\|a + b\| : \|a - b\| = \epsilon,\ \|a\| = \|b\| = 1\}$$
$$\|q^2_{v_4}\| = \sup\{\|a + b\| : \|a - b\| = \epsilon,\ \|a\| = \|b\| = 1\};$$

further, by Theorem 3.2.22 we have that $\|q^2_{v_4}\| - \|q^1_{v_4}\| > 0$.

As $V_0(p)$ is a path connected C^1-manifold that is C^1-diffeomorphic to $\mathbb{T}$ we may define a C^1-differentiable path $\alpha : [0, 1] \to V_0(p)$ where $\alpha(0) = q^1$, $\alpha(1) = q^2$ and $\alpha'(t) \neq 0$ for all $t \in [0, 1]$. By Lemma 3.2.21, $\alpha_{v_4}(t) = \alpha_{v_2}(t) + \alpha_{v_3}(t)$ for all $t \in [0, 1]$; further, as $\alpha_{v_2}(t), \alpha_{v_3}(t)$ are linearly independent, $\alpha_{v_4}(t) \neq 0$ for all $t \in [0, 1]$.

As X is smooth, $(K_4, \alpha(t))$ is well-positioned for all $t \in [0,1]$. By Proposition 1.1.11 (i) and Proposition 1.1.11 (ii), for all $1 \leq i < j \leq 4$, $(i,j) \neq (1,4)$ and $t \in [0,1]$,

$$0 = \frac{d}{dt}\|\alpha_{v_i}(t) - \alpha_{v_j}(t)\| = \varphi\left(\frac{\alpha_{v_i}(t) - \alpha_{v_j}(t)}{\|\alpha_{v_i}(t) - \alpha_{v_j}(t)\|}\right)(\alpha'_{v_i}(t) - \alpha'_{v_j}(t)),$$

thus $\alpha'(t)$ is a non-trivial flex of $(K_4 - e, \alpha(t))$ with $\alpha'_{v_1}(t) = 0$. By Lemma 3.2.14 (ii), $(K_4 - e, \alpha(t))$ is independent and so it follows from Theorem 1.3.12 that $\alpha'(t)$ is the unique (up to scalar multiplication) non-trivial flex of $(K_4 - e, \alpha(t))$ with $\alpha'_{v_1}(t) = 0$.

By the Mean Value Theorem it follows that there exists $s \in [0,1]$ so that

$$\varphi\left(\frac{\alpha_{v_4}(s)}{\|\alpha_{v_4}(s)\|}\right)(\alpha'_{v_4}(s)) = \frac{d}{dt}\|\alpha_{v_4}(t)\||_{t=s} = \|q^2_{v_4}\| - \|q^1_{v_4}\| > 0,$$

thus $\alpha'(s)$ is not a flex of $(K_4, \alpha(s))$. As $\mathcal{F}(K_4, \alpha(s)) \subsetneq \mathcal{F}(K_4 - e, \alpha(s))$ then $(K_4, \alpha(s))$ is infinitesimally rigid as required. □

3.3 Graph operations for the normed plane

In this section we shall define a set of graph operations and prove that they preserve isostaticity in non-Euclidean normed planes. The Henneberg moves and the *vertex split* (see Section 3.3.3) have also been shown to preserve isostaticity in the Euclidean normed plane and can even be generalised to higher dimensions [23] [68], however the *vertex-to-K_4 extension* (see Section 3.3.4) is strictly a non-Euclidean normed plane graph operation as it will not preserve $(2,3)$-sparsity.

3.3.1 0-extensions

We will first prove a more general result concerning frameworks.

Lemma 3.3.1. Let (G,p) be a finite framework in general position in a normed plane X and G' a 0-extension of G from the vertices $v_1, v_2 \in V(G)$ by a vertex v_0. Suppose p' is a placement of G' such that $(G,p) \subset (G',p')$, p' is in general position and $\varphi'_{v_0,v_1}, \varphi'_{v_0,v_2}$ are linearly independent. Then (G,p) is independent if and only if (G',p') is independent.

Proof. Choose any stress $a = (a_{vw})_{vw \in E(G')}$ of (G',p'), then by observing the stress condition at v_0 we note that

$$0 = a_{v_0 v_1}\varphi'_{v_0,v_1} + a_{v_0 v_2}\varphi'_{v_0,v_2}.$$

As $\varphi'_{v_0,v_1}, \varphi'_{v_0,v_2}$ are linearly independent then $a_{v_0 v_i} = 0$ for $i = 1,2$ and $a|_{E(G)}$ is a stress of (G,p). It now follows that there exists a non-zero stress of (G',p') if and only if there exists a non-zero stress of (G,p). By Proposition 1.3.8, (G',p') is independent if and only if (G,p) is independent. □

Lemma 3.3.2. Let (G,p) be a finite independent framework in general position in a normed plane X and G' a 0-extension of G from the vertices $v_1, v_2 \in V(G)$ by a vertex v_0. Then there exists a p' a placement of G' in general position such that $p'|_{V(G)} = p$ and (G',p') is independent.

Proof. By Proposition 1.1.23 we may choose linearly independent $y_1, y_2 \in \text{smooth}(X)$ such that $\|y_1\| = \|y_2\| = 1$ and $\varphi(y_1), \varphi(y_2) \in X^*$ are linearly independent; we note that if y_1, y_2 and $p_{v_1} - p_{v_2}$ are not pairwise linearly independent then by Proposition 1.1.11 (iv) and Proposition 1.1.11 (iii), we may perturb y_1 and y_2 so that they are smooth, $\varphi(y_1), \varphi(y_2)$ are linearly independent and y_1, y_2, $p_{v_1} - p_{v_2}$ are pairwise linearly independent.

Define for $i = 1, 2$ the lines

$$L_i := \{p_{v_i} + ty_i : t \in \mathbb{R}\},$$

then since $p_{v_1} \neq p_{v_2}$ (as p is in general position) and y_1, y_2, $p_{v_1} - p_{v_2}$ are pairwise linearly independent then there exists a unique point $z \in L_1 \cap L_2$ and $z \neq p_{v_i}$ for $i = 1, 2$. Define p' to be the placement of G' that agrees with p on $V(G)$ with $p'_{v_0} = z$. By Lemma 3.3.2, (G', p') is independent as required.

We finally note that if p' is not in general position, by Lemma 1.2.5 and Lemma 1.3.4, we may perturb the vertex v_0 such that p' is in general position and independent. $\square$

We may now use Lemma 3.3.2 to prove the following.

Lemma 3.3.3. 0-extensions preserve independence, dependence and isostaticity in any normed plane.

Proof. Let G be a finite graph in a normed plane X. Since we can only apply 0-extensions to graphs with at least two vertices we may assume that $|V(G)| \geq 2$ and define $v_1, v_2 \in V(G)$ to be the vertices where we are applying the 0-extension. Let G' be the 0-extension of G at v_1, v_2 with added vertex v_0.

Suppose G is dependent. Let p' be a well-positioned placement of G' and $p := p'|_{V(G)}$ be a well-positioned placement of G. As (G, p) is dependent and $(G, p) \subset (G', p')$ then (G', p') is dependent, thus G' is dependent.

Now suppose G is independent. By Lemma 1.3.4, we may choose an independent placement $p \in \mathcal{R}(G) \cap \mathcal{G}(G)$. By Lemma 3.3.2, there exists an independent placement p' of G', thus G' is independent.

As G was chosen arbitrarily then it follows that 0-extensions preserve independence and dependence. By Proposition 1.3.22 and Proposition 3.4.1, $(2, k)$-tightness is preserved by 0-extensions (for $k = 2, 3$), thus it follows from Corollary 3.1.4 that isostaticity is also preserved. $\square$

3.3.2 1-extensions

Lemma 3.3.4. Let (G, p) be a finite independent framework in general position in a normed plane X and G' a 1-extension of G formed by deleting the edge $v_1v_2 \in E(G)$ and adding a vertex v_0 connected to the end points and some other distinct vertex $v_3 \in V(G)$. Then there exists a placement p' of G' in general position such that $p'|_{V(G)} = p$ and (G', p') is independent.

Proof. By Proposition 1.1.23 there exists $y \in \text{smooth}(X)$, $\|y\| = 1$, such that $y, p_{v_1} - p_{v_2}$ are linearly independent and $\varphi(y), \varphi_{v_1,v_2}$ are linearly independent. We note that as $y, p_{v_1} - p_{v_2}$ are linearly independent and p_{v_1}, p_{v_2}, p_{v_3} are not colinear (since (G, p) is in general position) then the line through p_{v_1}, p_{v_2} and the line through p_{v_3} in the direction y must intersect uniquely at some point $z \neq p_{v_3}$. By Proposition 1.1.11 (iii) and Proposition 1.1.11 (iv), if $z = p_{v_i}$ for some $i = 1, 2$ we may perturb y to some sufficiently close $y' \in \text{smooth}(X)$ such that the pairs $y', p_{v_1} - p_{v_2}$ and $\varphi(y'), \varphi_{v_1,v_2}$ are linearly independent and our new intersection point z' is not equal to p_{v_i} for $i = 1, 2$; we will now assume y is chosen so that this holds.

Define q' to be the placement of G' where $q'_v = p_v$ for all $v \in V(G)$ and $q'_{v_0} = z$. We recall that $\varphi'_{v,w}$ is the support functional $vw \in E(G')$ in (G', q'); it is immediate that if $vw \in E(G) \setminus \{v_1v_2\}$ then $\varphi'_{v,w} = \varphi_{v,w}$. We note that φ'_{v_1,v_0}, φ'_{v_0,v_2}, φ'_{v_1,v_2} are all pairwise linearly dependent, thus there exists $f \in S_1^*[0]$ and $\sigma_{v_i,v_j} \in \{-1, 1\}$ such that $\varphi'_{v_i,v_j} = \sigma_{v_i,v_j} f$ for distinct $i, j \in \{0, 1, 2\}$, with $\sigma_{v_j,v_i} = -\sigma_{v_i,v_j}$. We further note that, due to our choice placement, at least one of φ'_{v_1,v_0}, φ'_{v_0,v_2} must be equal to φ'_{v_1,v_2}; we may assume by our ordering of v_1, v_2 and choice of f that $\sigma_{v_1,v_0} = \sigma_{v_1,v_2} = 1$. We may also assume $\varphi'_{v_0,v_3} = \varphi(y)$, as if $\varphi'_{v_0,v_3} = -\varphi(y)$ we may assume we originally chose y to be $-y$. Lastly, note that $\varphi(y)$ is linearly independent of f by our choice of z.

Choose any stress $a := (a_{vw})_{vw \in E(G')}$ of (G', q'). If we observe a at v_0 we note

$$a_{v_0v_1}\varphi'_{v_0,v_1} + a_{v_0v_2}\varphi'_{v_0,v_2} + a_{v_0v_3}\varphi'_{v_0,v_3} = (\sigma_{v_0,v_2}a_{v_0v_2} - a_{v_0v_1})f + a_{v_0v_3}\varphi(y) = 0,$$

thus since $f, \varphi(y)$ are linearly independent, $a_{v_0v_3} = 0$ and $\sigma_{v_0,v_2}a_{v_0v_1} = a_{v_0v_2}$. Define $b := (b_{vw})_{vw \in E(G)}$ where $b_{vw} = a_{vw}$ for all $vw \in E(G) \setminus \{v_1v_2\}$ and $b_{v_1v_2} = a_{v_0v_1} = \sigma_{v_0,v_2}a_{v_0v_2}$. For each $v \in V(G) \setminus \{v_1, v_2\}$ it is immediate that

$$\sum_{w \in N_G(v)} b_{vw}\varphi_{v,w} = \sum_{w \in N_{G'}(v)} a_{vw}\varphi'_{v,w} = 0;$$

we note that this will also hold for v_3 as $a_{v_0v_3} = 0$. If we observe whether the stress condition of b holds at v_1 we note

$$\sum_{w \in N_G(v_1)} b_{vw}\varphi_{v,w} = b_{v_1v_2}f + \sum_{\substack{w \in N_G(v_1) \\ w \neq v_2}} b_{vw}\varphi_{v,w} = a_{v_0v_1}\varphi'_{v_1,v_0} + \sum_{\substack{w \in N_{G'}(v_1) \\ w \neq v_0}} a_{vw}\varphi'_{v,w} = 0,$$

while if we observe whether the stress condition of b holds at v_2 we note

$$\sum_{w \in N_G(v_2)} b_{vw}\varphi_{v,w} = -b_{v_1v_2}f + \sum_{\substack{w \in N_G(v_2) \\ w \neq v_1}} b_{vw}\varphi_{v,w} = a_{v_0v_2}\varphi'_{v_2,v_0} + \sum_{\substack{w \in N_{G'}(v_2) \\ w \neq v_0}} a_{vw}\varphi'_{v,w} = 0,$$

thus b is a stress of (G, p). Since (G, p) is independent then $b = 0$ which in turn implies $a = 0$. As a was chosen arbitrarily then (G', q') is independent.

By applying Lemma 1.2.5 with $V := V(G)$ to (G', q') and Lemma 1.3.3 it follows that we may choose a well-positioned placement p' of G' in general position with $p'|_V = q'|_V = p$ sufficiently close to q' so that (G', p') is independent as required. $\square$

Fig. 3.4 A vertex split (left) and a vertex-to-K_4 extension (right).

Lemma 3.3.5. 1-extensions preserve independence and isostaticity in any normed plane.

Proof. Let G be independent, with 1-extension G'. By Lemma 1.3.4 (applied to G) and Lemma 3.3.4, it follows that G' is independent. By Proposition 1.3.22 and Proposition 3.4.1, $(2,k)$-tightness is preserved by 1-extensions (for $k = 2,3$), thus it follows from Corollary 3.1.4 that isostaticity is also preserved. $\square$

3.3.3 Vertex splitting

A *vertex split* is given by the following process applied to any graph G (see Figure 3.4):

1. Choose an edge $v_0w_0 \in E(G)$,
2. Add a new vertex w_1 to $V(G)$ and edges v_0w_1, w_0w_1 to $E(G)$,
3. For every edge $vw_0 \in E(G)$ we may either leave it or replace it with vw_1.

Lemma 3.3.6. Let (G,p) be a finite independent framework in general position in a normed plane X and G' formed by a vertex split applied to G at the edge v_0w_0 with added vertex w_1. Then there exists a placement p' of G' in general position such that $p'|_{V(G)} = p$ and (G',p') is independent.

Proof. We shall define q' to be the not well-positioned placement of G' with $q'_{w_1} = q'_{w_0} = p_{w_0}$ and $q'_v = p_v$ for all $v \in V(G') \setminus \{w_1\}$. By Proposition 1.1.23, we may choose smooth $x \in S_1[0]$ such that $\|x\| = 1$, the pair $x, p_{v_0} - p_{w_0}$ are linearly independent, and the pair $\varphi(x)$, φ_{v_0,w_0} are linearly independent. We shall define the pseudo-support functional $\varphi'_{w_0,w_1} := \varphi(x)$ and thus define $(G',q')^\phi$ with $\phi := \{\varphi'_{v,w} : vw \in E(G')\}$.

Let $a := (a_{vw})_{vw\in E(G')}$ be a pseudo-stress of $(G',q')^\phi$. Define $b := (b_{vw})_{vw\in E(G)}$ with $b_{v_0w_0} = a_{v_0w_0} + a_{v_0w_1}$, $b_{vw_1} = a_{vw_0}$ if $v \neq v_0$ and $b_{vw} = a_{vw}$ for all other edges of G. We shall now show b is a stress of (G,p). We first note that for any $v \in V(G) \setminus \{v_0, w_0\}$ the stress condition of b at v holds as the pseudo-stress of a holds at v, and the stress condition of b at v_0 holds as

$$b_{v_0w_0}\varphi_{v_0,w_0} = a_{v_0w_0}\varphi'_{v_0,w_0} + a_{v_0w_1}\varphi'_{v_0,w_1};$$

further, if we observe the stress condition of b at w_0 we note

$$\sum_{v\in N_G(w_0)} b_{w_0v}\varphi_{w_0,v} = \sum_{v\in N_{G'}(w_0)} a_{w_0v}\varphi'_{w_0,v} + \sum_{v\in N_{G'}(w_1)} a_{w_1v}\varphi'_{w_1,v} = 0 + 0 = 0,$$

thus b is a stress of (G,p). As (G,p) is independent then $b = 0$, thus $a_{vw} = 0$ for all edges $vw \neq w_0w_1, v_0w_0, v_0w_1$ of G', and $a_{v_0w_0} + a_{v_0w_1} = 0$. We note by observing the pseudo-stress condition of a at w_0,

$$0 = \sum_{v\in N_{G'}(w_0)} a_{w_0v}\varphi'_{w_0,v} = a_{w_0w_1}\varphi'_{w_0,w_1} + a_{v_0w_0}\varphi'_{w_0,v_0} = a_{w_0w_1}\varphi(x) + a_{v_0w_0}\varphi_{w_0,v_0},$$

thus $a_{v_0w_0} = a_{w_0w_1} = 0$; similarly, by observing the pseudo-stress condition of a at w_1 we note $a_{v_0w_1} = 0$. It now follows $a = 0$, thus $R(G',q')^\phi$ has row independence.

Define $q^n \in X^{V(G')}$ to be the placement of G' that agrees with q' on $V(G)$ with $q^n_{w_1} = q'_{w_0} - \frac{1}{n}x$. By Lemma 1.2.5 (with $V = V(G)$) we may choose $p^n \in \mathcal{W}(G') \cap \mathcal{G}(G')$ such that $p^n|_{V(G)} = q^n|_{V(G)} = p$ and $\|p^n - q^n\|_{V(G')} < \frac{1}{n^2}$. By Proposition 1.1.11 (iv), $\varphi^n_{v,w_1} \to \varphi'_{v,w_1}$ as $n \to \infty$ for all $vw_1 \in E(G')$ with $v \neq w_0$. By Lemma 2.3.6 and Proposition 1.1.11 (iv), $\varphi^n_{w_0,w_1} \to \varphi'_{w_0,w_1}$ as $n \to \infty$ also. This implies $(G',p^n) \to$

$(G', q')^\phi$ as $n \to \infty$ and so by Proposition 1.3.11, there exists an independent placement $p' := p^n$ of G' in general position for sufficiently large $n \in \mathbb{N}$. $\square$

Lemma 3.3.7. Vertex splitting preserves independence and isostaticity in any non-Euclidean normed plane.

Proof. Let G be independent and G' formed from G by a vertex split. By Lemma 1.3.4 (applied to G) and Lemma 3.3.6, it follows that G' is independent. By Proposition 3.4.1, $(2,2)$-tightness is preserved by vertex splitting, thus it follows from Corollary 3.1.4 that isostaticity is also preserved. $\square$

3.3.4 Vertex-to-K_4 extensions

The *vertex-to-K_4 extension* is given by the following process applied to any graph G (see Figure 3.4):

1. Choose a vertex $v_0 \in V(G)$,
2. Add the vertices v_1, v_2, v_3 to $V(G)$ and edges $v_i v_j$ to $E(G)$, $0 \leq i < j \leq 3$,
3. Replace any edge $v_0 w \in E(G)$ with $v_i w$ for some $i = 0, 1, 2, 3$.

Lemma 3.3.8. Let (G, p) be an independent framework in a normed plane X and G' a vertex-to-K_4 extension. Then there exists an independent placement p' of G' in X so that $p'|_{V(G)} = p$.

Proof. By Theorem 3.2.1 and Corollary 3.1.4, K_4 is isostatic in any non-Euclidean normed plane. We now note a vertex-to-K_4 substitution is a K_4-substitution (see Section 2.3.2), thus by Lemma 2.3.7, the result follows. $\square$

Lemma 3.3.9. Vertex-to-K_4 moves preserve independence and isostaticity in any non-Euclidean normed plane.

Proof. By Theorem 3.2.1 and Corollary 3.1.4, K_4 is isostatic in any non-Euclidean normed plane. We now note a vertex-to-K_4 substitution is a K_4-substitution (see Section 2.3.2), thus by Proposition 3.1.1 (ii) and Theorem 2.3.8, the result follows. □

3.4 Graph sparsity and connectivity conditions for rigidity

3.4.1 A characterisation of rigid graphs in normed planes

The following result provides an analogue for Proposition 1.3.22.

Proposition 3.4.1. [53, Theorem 1.5] Henneberg moves, vertex splitting and vertex-to-K_4 extensions preserve $(2,2)$-tightness and $(2,2)$-sparsity. Further, if G is $(2,2)$-tight then it may constructed from K_1 by a finite sequence of Henneberg moves, vertex splitting and vertex-to-K_4 extensions.

We are now ready to prove our main theorem of the chapter.

Theorem 3.4.2. Let X be a non-Euclidean normed plane. Then a graph G is isostatic in X if and only if G is $(2,2)$-tight.

Proof. Suppose $|V(G)| \leq 2$, then G is either K_1, K_2 or $K_1 \sqcup K_1$ (the graph on 2 vertices with no edges). We note all three are $(2,2)$-sparse but only K_1 is $(2,2)$-tight. By definition, K_1 is rigid and by Theorem 3.1.3 (i), both K_2 and $K_1 \sqcup K_1$ are infinitesimally flexible as required.

Let G be isostatic with $|V(G)| \geq 3$, then by Theorem 3.1.3 (iii), G is $(2,2)$-tight.

Now let G be $(2,2)$-tight with $|V(G)| \geq 3$, then by Proposition 3.4.1 it can be obtained from K_4 by a finite sequence of 0-extensions, 1-extensions, vertex splitting and vertex-to-K_4 extensions. By Theorem 3.2.1 and Corollary 3.1.4 K_4 is isostatic and so by Lemma 3.3.3, Lemma 3.3.5, Lemma 3.3.7 and Lemma 3.3.9, G is isostatic. □

We now have an immediate corollary.

Corollary 3.4.3. A graph is rigid in all normed planes if and only if it contains a proper $(2,3)$-tight spanning subgraph.

Proof. Let G contain a proper $(2,3)$-tight spanning subgraph H. As H is proper there exists $e \in E(G) \setminus E(H)$. It follows that $H \cup \{e\}$ is a $(2,2)$-tight spanning subgraph of G. By Theorem 1.3.20 and Theorem 3.4.2, G is rigid in all normed planes.

Suppose G is rigid in all normed planes, then by Theorem 1.3.20, G contains a $(2,3)$-tight spanning subgraph H and $|E(G)| \geq 2|V(G)| - 2$. As $|E(H)| < |E(G)|$, H is proper. □

We note that there exist $(2,2)$-tight graphs which are not rigid in the Euclidean plane, e.g. consider two copies of K_4 joined at a single vertex (see Figure 3.5).

3.4.2 Analogues of Lovász & Yemini's theorem for non-Euclidean normed planes

We say that a connected graph is *k-connected* if G remains connected after the removal of any $k-1$ vertices and *k-edge-connected* if G remains connected after the removal of any $k-1$ edges. This section shall deal with how we may obtain sufficient conditions for rigidity from the connectivity of the graph. The first result is the famous connectivity result given by Lovász & Yemini in [46].

Theorem 3.4.4. Any 6-connected graph is rigid in the Euclidean plane.

The following is a corollary of a famous result of Nash-Williams [52, Theorem 1].

Corollary 3.4.5. The following properties hold:

(i) G is (k,k)-tight if and only if G contains k edge-disjoint spanning trees $T_1, \ldots, T_k$ where $E(G) = \bigcup_{i=1}^{k} E(T_i)$

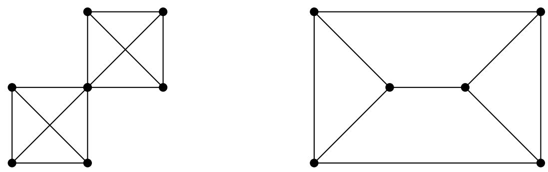

Fig. 3.5 (Left): A $(2,2)$-tight graph that is not rigid in the Euclidean plane. (Right): A 3-connected (and hence 3-edge-connected) graph that does not contain a $(2,2)$-tight spanning subgraph.

(ii) If G is k-edge-connected then G contains k edge-disjoint spanning trees.

Using Corollary 3.4.5 we may obtain an analogous result.

Theorem 3.4.6. Any 4-edge-connected graph is rigid in all non-Euclidean normed planes.

Proof. By Corollary 3.4.5 if G is 4-edge-connected then it will contain two edge-disjoint spanning trees, thus by Corollary 3.4.5, G must have a $(2,2)$-tight spanning subgraph H. By Theorem 3.4.2 we have that G is rigid in any non-Euclidean normed plane as required. □

Since k-connectivity implies k-edge-connectivity then we can see that a 4-connected graph will also be rigid in all non-Euclidean normed planes. We note that this is the best possible result as we can find graphs that are 3-edge-connected but do not contain a $(2,2)$-tight spanning subgraph (see Figure 3.5).

Corollary 3.4.7. Any 6-connected graph is rigid in all normed planes.

Proof. As G is 6-connected then by Theorem 3.4.4, G is rigid in the Euclidean normed plane. As 6-connected implies 6-edge-connected then G is 4-edge-connected, thus by Theorem 3.4.6, G is rigid in any non-Euclidean normed plane. □

This following result is generalisation of Lovász & Yemini's theorem given by Tibor Jordán on the number of rigid spanning subgraphs contained in a graph.

Theorem 3.4.8. [29, Theorem 3.1] Any $6k$-connected graph contains k edge-disjoint $(2,3)$-tight spanning subgraphs.

Yet again we may obtain an analogous result.

Theorem 3.4.9. Any $4k$-edge-connected graph contains k edge-disjoint $(2,2)$-tight spanning subgraphs.

Proof. By Corollary 3.4.5 if G is $4k$-edge-connected then it will contain $2k$ edge-disjoint spanning trees, thus by Corollary 3.4.5, G has k $(2,2)$-tight spanning subgraphs. □

Combining this we have the final generalisation.

Corollary 3.4.10. Any $6k$-connected graph contains k edge-disjoint spanning subgraphs $H_1, \ldots, H_k$ that are rigid in any normed plane.

Proof. Since $6k$-connected implies $6k$-edge-connected then by Theorem 3.4.8 there exists k edge-disjoint $(2,3)$-tight spanning subgraphs $A_1, \ldots, A_k$ and by Theorem 3.4.8 k edge-disjoint $(2,2)$-tight spanning subgraphs $B_1, \ldots, B_k$. We shall define $A := \bigcup_{i=1}^k A_i$ and $B := \bigcup_{i=1}^k B_i$, then $|E(B)| - |E(A)| = k$ and so we may choose $e_1, \ldots, e_k \in E(B) \setminus E(A)$. For any $i = 1, \ldots, k$ we note that $H_i := A_i \cup \{e_i\}$ will be a $(2,2)$-tight spanning subgraph that contains a $(2,3)$-tight spanning subgraph A_i, thus by Corollary 3.4.3, H_i is rigid in all normed planes. We now note $E(H_i) \cap E(H_j) = \emptyset$ for $i \neq j$ as required. □

Remark 3.4.11. Corollary 3.4.10 only gives that for any normed plane X a graph G will contain k edge-disjoint spanning subgraphs $H_1, \ldots, H_k$ with infinitesimally rigid placements (H_1, p^1), ..., (H_k, p^k) in X. In general this does not guarantee the existence of a single placement p of G such that $(H_1, p), \ldots, (H_k, p)$ are infinitesimally rigid in X. However if $\mathcal{R}(H)$ is dense in $\mathcal{W}(H)$ for any subgraph $H \subset G$ then such a placement does exist. An example where this occurs would be any graph in any smooth ℓ_p space (see [38, Lemma 2.7]). In contrast, if X has a polyhedral unit ball then this property does not hold in general (see [34, Lemma 16]).

Chapter 4

Rigidity for countable frameworks

In this chapter we will be extending the theory introduced in previous chapters to countably infinite frameworks and graphs. We shall define a matroidal structure for infinite frameworks that will be a vital tool when dealing with *generic placements* and *generic spaces*; see Section 4.2 for more details. We shall then extend Theorem 1.3.20 and Theorem 3.4.2 to countably infinite graphs (see Theorem 4.3.12), with a stronger classification for generic placements in generic spaces (see Theorem 4.3.14). We shall finish the chapter by investigating how we may utilise infinitesimal rigidity to detect continuous rigidity, especially in the case of *algebraically generic* frameworks; see Section 4.4.

4.1 Preliminaries on countably infinite frameworks

4.1.1 Well-positioned and completely well-positioned frameworks

We define a placement (p, S) in a normed space X to be *completely well-positioned* if the framework (K_S, p) is well-positioned in X. We define a framework (G, p) in a normed space X to be *completely well-positioned* if $(p, V(G))$ is a completely well-positioned placement.

Remark 4.1.1. For much of the material of this chapter we shall require that a framework is completely well-positioned. This is as we shall often be observing the infinitesimal flex spaces of all possible frameworks with a given placement.

For a placement (p, S) and normed space X, we usually consider X^S so that the rigidity operator of a given framework is continuous. For the following, however, we shall consider instead the *box topology*, the topology of X^S generated by all sets of the form $\prod_{s\in S} U_s$, where each U_s is an open set in X. We note that if a set is dense in X^S with respect to the box topology then it is automatically dense in X^S with respect to the product topology.

Proposition 4.1.2. Let X be a normed space, $S := \{s_1, s_2, \ldots\}$ a countable set and $S_n := \{s_1, \ldots, s_n\}$ for all $n \in \mathbb{N}$. Suppose $(P_n)_{n\in\mathbb{N}}$ are a sequence of sets with the following properties:

(i) $P_n \subset X^{S_n}$ and $X^{S_n} \setminus P_n$ is negligible for each $n \in \mathbb{N}$.

(ii) For all $n \leq m$, if $(x_s)_{s\in S_m} \in P_m$ then $(x_s)_{s\in S_n} \in P_n$.

If $P \subset X^S$ is the set of all points $(x_s)_{s\in S} \in X^S$ such that $(x_s)_{s\in S_n} \in P_n$ for each $n \in \mathbb{N}$, then P is dense in X^S with respect to the box topology.

Proof. Choose $p \in X^S$ and a sequence $(r_n)_{n\in\mathbb{N}}$ of strictly positive real numbers. Define $U := \prod_{n\in\mathbb{N}} B_{r_n}(p_{s_n})$, the sets $U_n := \prod_{i=1}^{n} B_{r_n}(p_{s_n})$, and the surjective maps

$$\rho_{n,m} : U_m \to U_n, \ (x_{s_i})_{i=1}^m \mapsto (x_{s_i})_{i=1}^n$$

for any $n, m \in \mathbb{N}$ with $n \leq m$. We note it is sufficient to find $q \in P \cap U$ for the result to hold.

Choose any $n \in \mathbb{N}$, then $U_n \cap P_n \neq \emptyset$. Choose any $C \subseteq U_n \cap P_n$ such that C^c is negligible in U_n and define for each $m \geq n+1$,

$$A_m(C) := \rho_{n,m}^{-1}(C) \cap P_m.$$

As $A_m(C)^c = \rho_{n,m}^{-1}(C^c)$ then by Proposition B.2.10 (i), $A_m(C)^c$ is negligible in U_m. We note

$$A_{n+1}(C) \supset \rho_{n+1,n+2}(A_{n+2}(C)) \supset \rho_{n+1,n+3}(A_{n+3}(C)) \supset \dots$$

By Proposition B.2.10 (ii), $\rho_{n+1,m}(A_m(C))^c$ is negligible in U_m. It now follows that if we define

$$B_{n+1}(C) := \bigcap_{m \geq n+1} \rho_{n+1,m}(A_m(C)) \subset U_{n+1} \cap P_{n+1}$$

then $B_{n+1}(C)^c$ is negligible in U_{n+1}. The set $B_{n+1}(C)$ is now the set of all elements of U_m that are an extension of a point from C which in turn can for each $m \geq n+1$ be extended to a point in $U_m \cap P_m$.

Define $C_1 := U_1 \cap P_1$ and $C_{n+1} := B_{n+1}(C_n)$ for all $n \in \mathbb{N}$. As shown prior, each C_n is non-empty, thus we may choose $q^1 \in C_1$. Due to how the sequence $(C_n)_{n\in\mathbb{N}}$ was constructed, we now may choose a sequence $(q^n)_{n\in\mathbb{N}}$ where $q^n \in C_n$ and $\rho_{n,m}(q^m) = q^n$ for all $n \leq m$. We now define $q \in X^S$ where $q_{s_n} := q^n_{s_n}$ for all $n \in \mathbb{N}$. □

Corollary 4.1.3. Let G be a countably infinite graph and X a normed space. Then $\mathcal{W}(G)$ is dense in $X^{V(G)}$ with respect to the box topology.

Proof. This follows from Lemma 1.2.4 and Proposition 4.1.2. □

Corollary 4.1.4. Let V be a countable set, then the set of completely well-positioned placements is dense in X^V with respect to the box topology.

Proof. We note that the set of completely well-positioned placements of V is exactly $\mathcal{W}(K_V)$, thus we apply Corollary 4.1.3 to K_V. $\square$

4.1.2 Towers of frameworks

We define a *tower (of frameworks)* to be a sequence of finite frameworks $((G^n, p^n))_{n\in\mathbb{N}}$ in a normed space X where $(G^n, p^n) \subset (G^{n+1}, p^{n+1})$ for all $n \in \mathbb{N}$. Given a framework (G, p) in X we define a tower $((G^n, p^n))_{n\in\mathbb{N}}$ to be a *tower of* (G, p) if $(G^n, p^n) \subset (G, p)$ for all $n \in \mathbb{N}$. A tower $((G^n, p^n))_{n\in\mathbb{N}}$ of (G, p) is *(completely) well-positioned* if each framework in the sequence is (completely) well-positioned, *vertex-complete* if $\bigcup_{n\in\mathbb{N}} V(G^n) = V(G)$, *edge-complete* if $\bigcup_{n\in\mathbb{N}} E(G^n) = E(G)$ and *complete* if it is both vertex-complete and edge-complete. We note that an edge-complete tower of a framework with no isolated vertices is a complete tower.

Given a well-positioned tower $((G^n, p^n))_{n\in\mathbb{N}}$ we define the projection maps

$$\rho_{j,k} : X^{V(G^k)} \to X^{V(G^j)},\ (x_v)_{v\in V(G^k)} \mapsto (x_v)_{v\in V(G^j)}$$

for all $1 \leq j \leq k$; if $((G^n, p^n))_{n\in\mathbb{N}}$ is a tower of (G, p) we also define the projection maps

$$\rho_k : X^{V(G)} \to X^{V(G^k)},\ (x_v)_{v\in V(G)} \mapsto (x_v)_{v\in V(G^k)}$$

for all $k \in \mathbb{N}$.

Let $(I, \leq)$ be a partially ordered set where any two elements have an upper bound. An *inverse system (of vector spaces and linear maps)* is a triple $((V_i)_{i\in I}, (f_{i,j})_{i\leq j}, (I, \leq))$ where:

(i) For all $i \in I$, V_i is a vector space

(ii) For all $i, j \in I$ with $i \leq j$, $f_{i,j} : V_j \to V_i$ is a linear map.

(iii) For all $i, j, k \in I$ with $i \leq j \leq k$, $f_{i,i}$ is the identity map and $f_{i,j} \circ f_{j,k} = f_{i,k}$

If clear we shall denote this by $(V_i, f_{i,j})$. For an inverse system $(V_i, f_{i,j})$ we define the *inverse limit* to be the linear space

$$\lim_{\leftarrow} V_i := \left\{ (v_i)_{i \in I} \in \prod_{i \in I} V_i : f_{i,j}(v_j) = v_i \text{ for all } i \leq j \right\}.$$

We note that $\rho_{j,k}(\mathcal{F}(G^k, p^k)) \subset \mathcal{F}(G^j, p^j)$, thus we have the inverse system

$$(\mathcal{F}(G^k, p^k), \rho_{j,k})$$

and inverse limit

$$\lim_{\leftarrow} \mathcal{F}(G^k, p^k) := \{(x^k)_{k \in \mathbb{N}} \in \prod_{k \in \mathbb{N}} \mathcal{F}(G^k, p^k) : \rho_{j,k}(x^k) = x^j \text{ for all } 1 \leq j \leq k\}.$$

A tower $((G^n, p^n))_{n \in \mathbb{N}}$ has the *flex-cancellation property* if for each $j \in \mathbb{N}$ there exists $k > j$ such that $\rho_{j,k}(\mathcal{F}(G^k, p^k)) \subset \mathcal{T}(p^j)$. We define a non-trivial flex $u \in \mathcal{F}(G^1, p^1)$ to be an *enduring flex* of $((G^n, p^n))_{n \in \mathbb{N}}$ if for each $n \in \mathbb{N}$ there exists $u' \in \mathcal{F}(G^n, p^n)$ such that $u'|_{V(G^1)} = u$. We say a tower is *relatively infinitesimally rigid* if $\rho_{k,k+1}(\mathcal{F}(G^{k+1}, p^{k+1})) \subset \mathcal{T}(p^k)$.

The following are useful results with regard to towers of frameworks.

Lemma 4.1.5. (Lemma 3.6, [38]) Let $((G^n, p^n))_{n \in \mathbb{N}}$ be a tower in a normed space X with an enduring flex $u \in \mathcal{F}(G^1, p^1)$, then there exists a sequence $(u^n)_{n \in \mathbb{N}}$ such that $u^n \in \mathcal{F}(G^n, p^n)$, $u^{n+1}|_{V(G^n)} = u^n$ and $u^1 = u$.

Proposition 4.1.6. If $((G^n, p^n))_{n \in \mathbb{N}}$ is a complete tower of (G, p) then $\mathcal{F}(G, p)$ is isomorphic (as a vector space) to $\lim_{\leftarrow} \mathcal{F}(G^k, p^k)$.

Proof. Let $u \in \mathcal{F}(G,p)$, then $(\rho_k(u))_{k\in\mathbb{N}} \in \varprojlim \mathcal{F}(G^k,p^k)$, and so we may define the linear map $f : \mathcal{F}(G,p) \to \varprojlim \mathcal{F}(G^k,p^k)$, $f(u) := (\rho_k(u))_{k\in\mathbb{N}}$. If $f(u) = 0$ then $u_v = 0$ for all $v \in \bigcup_{n\in\mathbb{N}} V(G^n) = V(G)$, thus f is injective. Choose $(u^k)_{k\in\mathbb{N}} \in \varprojlim \mathcal{F}(G^k,p^k)$, then as $((G^n,p^n))_{n\in\mathbb{N}}$ is vertex-complete there exists $u \in X^{V(G)}$ so that $\rho_k(u) = u^k$ for all $k \in \mathbb{N}$. As $((G^n,p^n))_{n\in\mathbb{N}}$ is edge-complete it follows that $u \in \mathcal{F}(G,p)$, thus f is also surjective as required. $\square$

Theorem 4.1.7. [38, Proposition 3.10, Theorem 3.14] Let (G,p) be a countably infinite framework in a normed space X, then the following are equivalent:

(i) (G,p) is infinitesimally rigid.

(ii) Every complete tower of (G,p) has the flex cancellation property.

(iii) (G,p) contains a vertex-complete tower with the flex cancellation property.

(iv) (G,p) contains a vertex-complete relatively infinitesimally rigid tower.

We define a tower $((G^n,p^n))_{n\in\mathbb{N}}$ in a normed space X to be *sequentially infinitesimally rigid* if (G^n,p^n) is infinitesimally rigid for all $n \in \mathbb{N}$ and *sequentially isostatic* if (G^n,p^n) is isostatic for all $n \in \mathbb{N}$. If a framework contains a complete sequentially infinitesimally rigid tower we shall also call the framework *sequentially infinitesimally rigid*; likewise if a framework contains a complete sequentially isostatic tower we shall also call the framework *sequentially isostatic*.

Corollary 4.1.8. [38, Corollary 3.16] If a tower $((G^n,p^n))_{n\in\mathbb{N}}$ is sequentially infinitesimally rigid in a normed space X then it is relatively infinitesimally rigid. Similarly, if a framework (G,p) is sequentially infinitesimally rigid in a normed space X then it is infinitesimally rigid.

It should be noted that not every relatively infinitesimally rigid tower is sequentially infinitesimally rigid, see Figure 6 for such a framework.

4.1.3 Independence for countably infinite frameworks

We define a tower to be *independent/isostatic* if every element of the sequence is an independent/isostatic framework. For the next result we remember from Definition 1.3.6 that a well-positioned framework is independent if its rigidity operator is surjective.

Lemma 4.1.9. Let $((G^n, p^n))_{n \in \mathbb{N}}$ be a complete tower of a well-positioned framework (G, p) in a normed space X. Then (G, p) is independent if and only if $((G^n, p^n))_{n \in \mathbb{N}}$ is independent.

Proof. Suppose (G, p) is independent. Choose $n \in \mathbb{N}$ and $a^n \in \mathbb{R}^{E(G^n)}$. As $df_G(p)$ is surjective then there exists $x \in X^{V(G)}$ so that $df_G(p)x = a$, where $a_{vw} = a^n_{vw}$ for all $vw \in E(G^n)$ and $a_{vw} = 0$ otherwise. We now note that $df_{G^n}(p^n)(x|_{V(G^n)}) = a^n$ as required.

Suppose (G^n, p^n) is independent for all $n \in \mathbb{N}$. Choose $a \in \mathbb{R}^{E(G)}$ and define $a^n := a|_{V(G^n)}$. Define for each $n \in \mathbb{N}$ the affine space

$$A_n := df_{G^n}(p^n)^{-1}[a^n] \subset X^{V(G^n)},$$

then $\rho_{n,m}(A_m) \subset A_n$ for all $m \geq n$. We now define for each $n \in \mathbb{N}$,

$$B_n := \bigcap_{m \geq n} \rho_{n,m}(A_m) \subset X^{V(G^n)}.$$

As $(\rho_{n,m}(A_m))_{m=n}^{\infty}$ is a nested sequence of finite dimensional affine spaces, there exists $f(n) \in \mathbb{N}$ such that $\rho_{n,m}(A_m) = B_n$ for all $m \geq f(n)$. By assumption we may choose each $f(n)$ such that the map $f : \mathbb{N} \to \mathbb{N}$ is increasing. We now note that

$$\rho_{n,m}(B_m) = \rho_{n,m} \circ \rho_{m,f(m)}(A_{f(m)}) = \rho_{n,f(m)}(A_{f(m)}) = B_n$$

for all $n \leq m$.

Choose $x^1 \in B_1$, then there exists a sequence $(x^1)_{n\in\mathbb{N}}$ such that $x^n \in B_n$ and $\rho_{n,m}(x^m) = x^n$; further, for each $n \in \mathbb{N}$, $df_{G^n}(p^n)x^n = a^n$. As $\rho_{n,m}(x^m) = x^n$ for all $1 \leq n \leq m$ and $((G^n, p^n))_{n\in\mathbb{N}}$ is complete we may define $x \in X^{V(G)}$ with $x|_{V(G^n)} = x^n$ for all $n \in \mathbb{N}$. We now note that $df_G(p)(x) = a$ as required. $\square$

Proposition 4.1.10. A framework is independent if and only if all of its finite subframeworks are independent.

Proof. For finite frameworks the result follows from Remark 1.3.9. Let (G, p) be a well-positioned countably infinite framework. Suppose all of the finite subframeworks of (G, p) are independent, then we can construct an independent complete tower of (G, p), thus by Lemma 4.1.9, (G, p) is independent. Now suppose (G, p) is independent and choose any $(H, q) \subset\subset (G, p)$. Let $((G^n, p^n))_{n\in\mathbb{N}}$ be a complete tower of (G, p), then by Lemma 4.1.9, $((G^n, p^n))_{n\in\mathbb{N}}$ is independent. As $((G^n, p^n))_{n\in\mathbb{N}}$ is complete there exists $k \in \mathbb{N}$ such that $(H, q) \subset (G^k, p^k)$, thus (H, q) is independent as required. $\square$

We can extend the concept of stresses to infinite frameworks as such; for a well-positioned framework (G, p) in a normed space X we define $a \in \mathbb{R}^{E(G)}$ to a *finitely supported stress* if the following holds:

(i) The *support* of a, the set $\{vw \in E(G) : a_{vw} \neq 0\}$, is finite.

(ii) a satisfies the *stress condition* i.e. for each $v \in V(G)$, $\sum_{w\in N_G(v)} a_{vw}\varphi_{v,w} = 0$.

Corollary 4.1.11. A well-positioned framework (G, p) is independent if and only if the only finitely supported stress of (G, p) is the zero stress i.e. $(0)_{vw\in E(G)}$.

Proof. Suppose (G, p) is independent, then by Proposition 4.1.10, all subframeworks of (G, p) are independent. Let $a \in \mathbb{R}^{E(G)}$ be a finitely supported stress of (G, p) with support $E \subset E(G)$. If $E \neq \emptyset$ then we note that $a|_E$ is a non-zero stress of the finite framework $((V(E), E), p)$, thus by Proposition 1.3.8, $((V(E), E), p)$ is dependent, contradicting our assumption that (G,p) is independent. As $E=\emptyset$ then a is the zero stress as required.

Suppose (G,p) is dependent, then by Proposition 4.1.10, there exists dependent $(H,q) \subset\subset (G,p)$. By Proposition 1.3.8, there exists a non-zero stress $a \in \mathbb{R}^{E(H)}$ of (H,q). Define $b \in \mathbb{R}^{E(G)}$ as the non-zero element where $b_{vw} = a_{vw}$ for all $vw \in E(H)$ and $b_{vw} = 0$ otherwise. We now note b satisfies the stress condition and is finitely supported as required. $\square$

The following result illustrates how we may in some ways replace the idea of a regular framework (which is no longer well-defined for infinite frameworks) with independent frameworks, reflecting that independence implies regularity for finite frameworks.

Proposition 4.1.12. Let (G,p) be sequentially isostatic in a normed space X and q be an independent full placement of G. Then (G,q) is sequentially isostatic also.

Proof. Let $((G^n,p^n))_{n\in\mathbb{N}}$ be the complete isostatic tower of (G,p), then this also defines a complete tower $((G^n,q^n))_{n\in\mathbb{N}}$ of (G,q). We may assume that we chose our tower so that each q^n has the same affine span as q, thus by Corollary 1.2.17 (ii) and Theorem 1.2.29, q^n is full for all $n \in \mathbb{N}$. By Lemma 4.1.9, $((G^n,q^n))_{n\in\mathbb{N}}$ is independent, thus as each q^n is full then $((G^n,q^n))_{n\in\mathbb{N}}$ is isostatic also as required. $\square$

4.1.4 The closure operator

In this section we shall define some tools that will be vital in proving later results in Section 4.2.

Let p be a completely well-positioned placement of a countable set V in a normed space X. We define for each $e = vw \in E(K_V)$ the linear function

$$e_p : X^{V(G)} \to \mathbb{R},\ (x_v)_{v\in V(G)} \mapsto \varphi_{v,w}(x_v - x_w).$$

It is immediate from definition that $df_G(p)(u) = 0$ if and only if $e_p(u) = 0$ for all $e \in E(G)$.

We define the map $\langle \cdot \rangle_p : \mathcal{P}(E(K_V)) \to \mathcal{P}(E(K_V))$, where for any set $E \subset E(K_V)$,

$$\langle E \rangle_p := \{e \in E(K_V) : e_p \in \operatorname{span}\{f_p : f \in E\}\},$$

to be the *closure operator (with respect to (p, V))*.

Lemma 4.1.13. Let p be a completely well-positioned placement of V in a normed space X. Then the following holds for all $E \subseteq F \subseteq E(K_V)$:

(i) **CL1**: $E \subseteq \langle E \rangle_p$.

(ii) **CL2**: $\left\langle \langle E \rangle_p \right\rangle_p = \langle E \rangle_p$.

(iii) **CL3**: $\langle E \rangle_p \subseteq \langle F \rangle_p$.

(iv) **CL4**: For all $e, f \in E(K_V) \setminus \langle E \rangle_p$, if $e \in \langle E \cup \{f\} \rangle_p$ then $f \in \langle E \cup \{e\} \rangle_p$.

(v) **CL5**: $\langle E \rangle_p = \bigcup_{F \subset\subset E} \langle F \rangle_p$.

Proof. (i): If $e_p \in \{f_p : f \in E\}$ then $e_p \in \operatorname{span}\{f_p : f \in E\}$.

(ii): This follows as $\operatorname{span}\operatorname{span}\{f_p : f \in E\} = \operatorname{span}\{f_p : f \in E\}$.

(iii): This follows as $\operatorname{span}\{f_p : f \in E\} \subseteq \operatorname{span}\{f_p : f \in F\}$.

(iv): As $e \in \langle E \cup \{f\} \rangle_p$ then

$$e_p = \sum_{i=1}^{n} a_i (e_i)_p + b f_p$$

for some $a_1, \ldots, a_n, b \in \mathbb{R}$, $e_1, \ldots, e_n \in E$. If $b = 0$ then $e \in \langle E \rangle_p$, thus $b \neq 0$. We now note

$$f_p = (e_p - \sum_{i=1}^{n} a_i (e_i)_p)/b,$$

thus $f \in \langle E \cup \{e\} \rangle_p$.

(v): By **CL3** it follows $\bigcup_{F \subset\subset E} \langle F \rangle_p \subseteq \langle E \rangle_p$. Suppose $e \in \langle E \rangle_p$, then $e_p = \sum_{i=1}^n a_i (e_i)_p$ for some $a_1, \ldots, a_n \in \mathbb{R}$, $e_1, \ldots, e_n \in E$. If we define $F := \{e_1, \ldots, e_n\}$ then $e \in F$ and $F \subset\subset E$, thus $\langle E \rangle_p \subseteq \bigcup_{F \subset\subset E} \langle F \rangle_p$. □

Given a completely well-positioned placement (p, V) in a normed space X we say a set $E \subset E(K_V)$ is *independent (with respect to* (p, V)*)* if for all $e \in E$, $e \notin \langle E \setminus \{e\} \rangle_p$; otherwise E is *dependent (with respect to* (p, V)*)*. We denote by $\mathcal{I}_p$ the set of all subsets of $E(K_V)$ that are independent (with respect to (p, V)). If the context is clear we shall just refer to these properties as *independence* and *dependence*. By Lemma 4.1.13 we see that $(E(K_V), \mathcal{I}_p)$ is a *finitary matroid*; see Appendix A.2.2 for more details.

Proposition 4.1.14. Let (p, V) be a completely well-positioned placement in a normed space X, $E \subset E(K_V)$ and $G = (V, E)$. Then the following are equivalent:

(i) (G, p) is independent.

(ii) $\{f_p : f \in E\}$ is an independent set of linear functions.

(iii) E is an independent set in $(E(K_V), \mathcal{I}_p)$.

Proof. (i) $\Leftrightarrow$ (ii): From the definition of linear independence, $\{f_p : f \in E\}$ is a dependent set of linear functions if and only if the only there exists a non-zero finitely supported stress of (G, p). The result now follows from Corollary 4.1.11.

(ii) $\Rightarrow$ (iii): Suppose $\{f_p : f \in E\}$ is independent and choose $e \in E$, then $e_p \notin \operatorname{span}\{f_p : f \in E \setminus \{e\}\}$. Thus $e \notin \langle E \rangle_p$; as this holds for any $e \in E$ then E is independent.

(iii) $\Rightarrow$ (ii): Suppose E is independent and choose any $e_p \in \{f_p : f \in E\}$, then $e_p \notin \operatorname{span}\{f_p : f \in E \setminus \{e\}\}$. Thus e_p is linearly independent of $\{f_p : f \in E \setminus \{e\}\}$; as this holds for any $e_p \in \{f_p : f \in E\}$ then $\{f_p : f \in E\}$ is independent. □

Let (G, p) be a framework in a normed space X. By choosing $V = V(G)$ we define $\langle G\rangle_p := (V(G), \langle E(G)\rangle_p)$.

Proposition 4.1.15. Let (G, p) be completely well-positioned in a normed space X with complete tower $((G^n, p^n))_{n\in\mathbb{N}}$, then

$$\langle G\rangle_p = \bigcup_{n\in\mathbb{N}} \langle G^n\rangle_p = \bigcup_{H\subset\subset G} \langle H\rangle_p .$$

Proof. Since $(G^n)_{n\in\mathbb{N}}$ is a complete tower of G then

$$V(\langle G\rangle_p) = V\left(\bigcup_{n\in\mathbb{N}} \langle G^n\rangle_p\right) = V\left(\bigcup_{H\subset\subset G} \langle H\rangle_p\right).$$

By **CL5** it follows $\langle G\rangle_p = \bigcup_{H\subset\subset G} \langle H\rangle_p$. By **CL3** it follows $\bigcup_{n\in\mathbb{N}} \langle G^n\rangle_p \subseteq \langle G\rangle_p$. Choose $e \in \langle E(G)\rangle_p$, then by **CL5**, $e \in \langle F\rangle_p$ for some $F \subset\subset E(G)$. As $(G^n)_{n\in\mathbb{N}}$ is a complete tower of G then there exists $n \in \mathbb{N}$ so that $F \subset E(G^n)$. By **CL1**, $e \in \bigcup_{n\in\mathbb{N}} \langle E(G^n)\rangle_p$ as required. $\square$

Lemma 4.1.16. For any completely well-positioned framework (G, p) in a normed space X,

$$\mathcal{F}(\langle G\rangle_p , p) = \mathcal{F}(G, p).$$

Proof. As (G, p) is a spanning subframework of $(\langle G\rangle_p , p)$ then $\mathcal{F}(\langle G\rangle_p , p) \subseteq \mathcal{F}(G, p)$. Choose any $u \in \mathcal{F}(G, p)$ and any edge $e \in E(\langle G\rangle_p)$, then by **CL5** there exists $F \subset\subset E(G)$ where $e \in \langle F\rangle_p$; it follows that $e_p = \sum_{f\in F} a_f f_p$, where $a_f \in \mathbb{R}$. As $u \in \mathcal{F}(G, p)$ then $f_p(u) = 0$ for all $f \in F$, thus $e_p(u) = 0$. As this holds for all $e \in \langle E(G)\rangle_p$ then $u \in \mathcal{F}(\langle G\rangle_p , p)$. $\square$

Lemma 4.1.17. Let (G, p) be a finite completely well-positioned framework in a normed space X, then the following are equivalent:

(i) $e \in E(\langle G \rangle_p)$.

(ii) $\mathcal{F}(G,p) = \mathcal{F}(G \cup \{e\}, p)$.

(iii) $\operatorname{rank} df_G(p) = \operatorname{rank} df_{G\cup\{e\}}(p)$.

Proof. (i) $\Rightarrow$ (ii): Suppose $e \in E(\langle G \rangle_p)$, then $\langle G \cup \{e\} \rangle_p = \langle G \rangle_p$. By Proposition 4.1.16 we have that

$$\mathcal{F}(G,p) = \mathcal{F}(\langle G \rangle_p, p) = \mathcal{F}(\langle G \cup \{e\} \rangle_p, p) = \mathcal{F}(G \cup \{e\}, p).$$

(ii) $\Rightarrow$ (i): Suppose $e \notin E(\langle G \rangle_p)$. Define $\mathcal{E} := \operatorname{span}\{f_p : f \in E(G)\}$, then $e_p \notin \mathcal{E}$. Since G is finite then $\mathcal{E}$ is a subspace of the dual space of $X^{V(G)}$. By the Hahn-Banach theorem there exists $u \in X^{V(G)}$ such that $f(u) = 0$ for all $f \in \mathcal{E}$ and $e_p(u) = 1$, thus $\mathcal{F}(G,p) \neq \mathcal{F}(G \cup \{e\}, p)$.

(ii) $\Leftrightarrow$ (iii): This follows from the Rank-Nullity theorem applied to $df_G(p)$. $\square$

Lemma 4.1.18. Let (G,p) be a completely well-positioned framework in a normed space X, then the following are equivalent:

(i) $e \in E(\langle G \rangle_p)$.

(ii) $\mathcal{F}(G,p) = \mathcal{F}(G \cup \{e\}, p)$.

Proof. If G is finite this follows from Lemma 4.1.17 so we shall assume G is countably infinite.

Suppose $e \in \langle E(G) \rangle_p$, then $\langle G \cup \{e\} \rangle_p = \langle G \rangle_p$, thus by Proposition 4.1.16 we have that

$$\mathcal{F}(G,p) = \mathcal{F}(\langle G \rangle_p, p) = \mathcal{F}(\langle G \cup \{e\} \rangle_p, p) = \mathcal{F}(G \cup \{e\}, p).$$

Conversely suppose $e = vw \notin \langle E(G) \rangle_p$. Let $((G^n, p^n))_{n\in\mathbb{N}}$ be a complete tower of (G, p) with $v, w \in V(G^1)$, then by Proposition 4.1.15, $e \notin \langle E(G^n) \rangle_p$ for all $n \in \mathbb{N}$. By Lemma 4.1.17, for each $n \in \mathbb{N}$ there exists $x^n \in \mathcal{F}(G^n, p^n)$ such that $\varphi_{v,w}(x^n_v - x^n_w) = 1$. For all $n \in \mathbb{N}$ define U_n to be the finite dimensional affine subspace of $\mathcal{F}(G^n, p^n)$ such that $x \in U_n$ if and only if $\varphi_{v,w}(x_v - x_w) = 1$, then $U_n \neq \emptyset$. It is clear that $\rho_{n,m}(U_m) \subseteq U_n$ for all $n \leq m$ and so we have the inclusion

$$U_1 \supseteq \rho_{1,2}(U_2) \supseteq \rho_{1,3}(U_3) \supseteq \ldots.$$

As they are all finite dimensional affine spaces there exists some $N \in \mathbb{N}$ such that $\rho_{1,n}(U_n) = \rho_{1,N}(U_N)$ for all $n \geq N$. Choose $u^1 \in \rho_{1,N}(U_N) \subset \mathcal{F}_q(G^1, p^1)$, then u^1 is an enduring flex of $((G^n, p^n))_{n\in\mathbb{N}}$. By Lemma 4.1.5, there exists $u \in \mathcal{F}(G, p)$ such that $u|_{V(G^1)} = u^1$. As

$$\varphi_{v,w}(u_v - u_w) = \varphi_{v,w}(u^n_v - u^n_w) = 1 \neq 0$$

for all $n \in \mathbb{N}$, then $u \in \mathcal{F}(G, p) \setminus \mathcal{F}(G \cup \{e\}, p)$. It now follows $\mathcal{F}(G, p) \neq \mathcal{F}(G \cup \{e\}, p)$ as required. $\square$

We may now state the following key lemma.

Lemma 4.1.19. Let (p, V) be a completely well-positioned placement in a normed space X. Suppose G, H are graphs with $V(G) = V(H) = V$, then the following are equivalent:

(i) $\mathcal{F}(G, p) = \mathcal{F}(H, p)$.

(ii) $\langle G \rangle_p = \langle H \rangle_p$.

Proof. Suppose that $\langle G\rangle_p \neq \langle H\rangle_p$ then without loss of generality there exists $e \in \langle E(G)\rangle_p \setminus \langle E(H)\rangle_p$. By Lemma 4.1.18, there exists $u \in \mathcal{F}(H,p)$ such that $e_p(u) \neq 0$. As $e \in E(G)$ then $u \notin \mathcal{F}(G,p)$, thus $\mathcal{F}(G,p) \neq \mathcal{F}(H,p)$ as required.

Now suppose $\langle G\rangle_p = \langle H\rangle_p$, then by Lemma 4.1.16,

$$\mathcal{F}(G,p) = \mathcal{F}(\langle G\rangle_p, p) = \mathcal{F}(\langle H\rangle_p, p) = \mathcal{F}(H,p)$$

as required. $\square$

Theorem 4.1.20. Let (G,p) be a completely well-positioned framework in a normed space X. Then the following are equivalent:

(i) (G,p) is infinitesimally rigid in X.

(ii) $\langle G\rangle_p = K_{V(G)}$ and $(K_{V(G)},p)$ is infinitesimally rigid in X.

Proof. Suppose (G,p) is infinitesimally rigid, then as (G,p) is a spanning subframework of $(K_{V(G)},p)$ we have that

$$\mathcal{T}(p) \subseteq \mathcal{F}(K_{V(G)},p) \subseteq \mathcal{F}(G,p) = \mathcal{T}(p)$$

and thus $(K_{V(G)},p)$ is infinitesimally rigid. Since $\mathcal{F}(G,p) = \mathcal{F}(K_{V(G)},p)$ then by Lemma 4.1.19 $\langle G\rangle_p = K_{V(G)}$ as required.

Now suppose $\langle G\rangle_p = K_{V(G)}$ and $(K_{V(G)},p)$ is infinitesimally rigid. Then by Lemma 4.1.19

$$\mathcal{F}(G,p) = \mathcal{F}(\langle G\rangle_p, p) = \mathcal{F}(K_{V(G)},p) = \mathcal{T}(p)$$

and thus (G,p) is infinitesimally rigid. $\square$

4.2 Countably infinite frameworks in generic spaces

4.2.1 Generic placements, spaces and properties

We define a completely well-positioned placement (p, V) in a normed space X to be *generic* if every finite subframework of (K_V, p) is regular; we likewise define a framework to be *generic* if it has a generic placement, and define a tower to be *generic* if every framework in its sequence is generic.

Proposition 4.2.1. Let (G, p) be independent and (G, q) be generic in a normed space X. Then (G, q) is independent also.

Proof. As (G, p) is independent then by Proposition 4.1.10, every finite subframework of (G, p) is independent. It now follows that as (G, q) is generic the same holds, thus by Proposition 4.1.10, (G, q) is also independent. $\square$

Proposition 4.2.2. Let (G, p) be sequentially infinitesimally rigid and (G, q) be generic and full in a normed space X. Then (G, q) is sequentially infinitesimally rigid also.

Proof. Let $((G^n, p^n))_{n \in \mathbb{N}}$ be a complete sequentially infinitesimally rigid tower of (G, p). As (G, q) is generic then for each $n \in \mathbb{N}$, (G^n, q^n) is regular, where $q^n := q|_{V(G^n)}$. Without loss of generality we may assume that our original tower was chosen so that the affine span of q^n is the same as the affine span of q for all $n \in \mathbb{N}$, thus by Corollary 1.2.17 (ii) and Theorem 1.2.29, each (G^n, q^n) is full. It now follows that each (G^n, q^n) is infinitesimally rigid, thus (G, q) is sequentially infinitesimally rigid. $\square$

Motivated by Proposition 4.2.1, for a given normed space X we shall define a countably infinite graph G to be *independent (in X)* if it has a independent placement and *dependent (in X)* otherwise, extending the definitions from finite graphs. With some further motivation from Proposition 4.1.12 and Proposition 4.2.2, we shall also define a countably infinite graph G to be *sequentially rigid (in X)* if it has a sequentially infinitesimally rigid placement, *sequentially isostatic (in X)* if it has a sequentially isostatic placement and *sequentially flexible (in X)* otherwise. We shall not do the same for the graph terms of rigidity, isostaticity and flexiblity due to reasons that will become clear during this section.

We define a normed space X to be *quasi-generic* if for all finite graphs G the set $\mathcal{R}(G)$ is an open dense subset of $X^{V(G)}$. If X is a quasi-generic space and there exists a finite graph on two or more vertices which is rigid in X then X is a *generic* normed space.

If X is a quasi-generic space then we note a few things immediately:

(i) For any finite set V, the set of generic placements of V is an open dense subset of X^V.

(ii) By Proposition 2.1.1, the set $\mathcal{W}(G)$ is open and $\mathcal{R}(G) = \mathcal{C}(G)$ for any finite graph G.

(iii) By Proposition 2.2.13, any finite rigid graph will have only full generic placements.

We conjecture that the following holds.

Conjecture 4.2.3. Every quasi-generic normed space is a generic normed space.

It follows from Theorem 1.3.20 and Theorem 3.4.2 that the conjecture holds in dimension 2. Conjecture 4.2.3 is still open in higher dimensions, however.

Example 4.2.4. For $q \in (1, \infty)$ the normed space ℓ_q^d (see Example 1.1.5) is a quasi-generic space by [38, Lemma 2.7]. By Proposition 1.3.25, ℓ_2^d - the standard Euclidean space - is generic. If $q \neq 2$ then by a result in an upcoming paper [20], K_{2d} is isostatic in ℓ_q^d, thus ℓ_q^d is a generic space. These spaces are, in many ways, the motivating cases for the study of rigidity in generic spaces.

Example 4.2.5. Let X be a polyhedral normed space (see Example 1.1.13), then by [34, Lemma 17 (ii)], X is not a quasi-generic space.

Example 4.2.6. Let X be a d-dimensional normed space for $d \geq 2$ with a dual map φ that is constant on some open set U of $S_1[0]$. We may assume we chose U such that for each $x, y \in U$, $\frac{x+y}{\|x+y\|} \in U$. By Lemma 2.2.11, the complete graph K_3 on $\{v_1, v_2, v_3\}$ has an independent placement p in X and $\operatorname{rank} df_{K_3}(p) = 3$. Define the open set of placements

$$O := \{q \in \mathcal{W}(G) : q_{v_2} = q_{v_1} + x,\ q_{v_3} = q_{v_2} + y,\ x, y \in U\},$$

then for each $q \in O$, $\operatorname{rank} df_{K_3}(q) = 2$. It follows that $O \subset \mathcal{R}(K_3)^c$, thus X is not a quasi-generic space. It follows that all generic normed planes must be strictly convex.

We define a norm $\|\cdot\| : \mathbb{R}^d \to \mathbb{R}$ to be *analytic* if it is an *analytic function* (see Appendix B.3) when restricted to $\mathbb{R}^d \setminus \{0\}$. By [63, Theorem (b)] and [67, Theorem 2.2.13], any norm $\|\cdot\|$ of $\mathbb{R}^d$ can be uniformly approximated by a sequence of analytic norms $(\|\cdot\|_n)_{n\in\mathbb{N}}$ such that

$$\operatorname{Isom}(\mathbb{R}^d, \|\cdot\|) = \operatorname{Isom}(\mathbb{R}^d, \|\cdot\|_n).$$

We shall now prove that all analytic normed spaces are quasi-generic, thus giving a large class of such spaces. The following uses methods similar to those given in [38, Lemma 2.7].

Proposition 4.2.7. Let $X := (\mathbb{R}^d, \|\cdot\|)$ be a normed space where $\|\cdot\|$ is analytic. Then X is a quasi-generic space.

Proof. If $d = 1$ the X is isometrically isomorphic to the Euclidean space $(\mathbb{R}, |\cdot|)$, thus X is quasi-generic. Suppose $d > 1$, then $\mathbb{R}^d \setminus \{0\}$ is connected. Choose a finite graph

G and note that

$$\mathcal{W}(G) := \left\{ x \in X^{V(G)} : x_v \neq x_w \text{ for all } vw \in E(G) \right\},$$

and $\mathcal{W}(G)$ is a connected open subset of $X^{V(G)}$.

Define $\{e_i : i = 1, \ldots, d\}$ to be the standard basis of $\mathbb{R}^d$. For any $p \in \mathcal{W}(G)$, define the $|E(G)| \times d|V(G)|$ real matrix $\tilde{R}(G,p)$ with entries $a_{e,(v,i)}$, where

$$a_{e,(v,i)} := \begin{cases} \varphi_{v,w}(e_i), & \text{if } e = vw \in E(G) \\ 0, & \text{otherwise.} \end{cases}$$

We note that $\operatorname{rank} \tilde{R}(G,p) = \operatorname{rank} df_G(p)$. Define

$$n := \sup_{p \in \mathcal{W}(G)} \tilde{R}(G,p)$$

and for each $p \in \mathcal{W}(G)$ define the finite set of $n \times n$ submatrices of $\tilde{R}(G,p)$,

$$\Phi(p) := \{\Phi_i(p) : i \in I\}.$$

We note that each $\det \Phi_i$ will be an analytic function on $\mathcal{W}(G)$, and for some $j \in I$, $\det \Phi_j$ is non-zero (as otherwise we will have a drop in rank). As each $\det \Phi_i$ is an analytic function with connected domain, then the set

$$V(\det \Phi_i) := \{p \in \mathcal{W}(G) : \det \Phi_i(p) = 0\}$$

is negligible if and only if $\det \Phi_i$ is non-zero (see Theorem B.3.6). We now note that

$$\mathcal{R}(G) = \mathcal{W}(G) \setminus \left(\bigcup_{i \in I} V(\det \Phi_i) \right),$$

thus as $\det \Phi_j$ is non-zero, $\mathcal{R}(G)^c$ is a negligible closed set. It now follows $\mathcal{R}(G)$ is an open dense set as required. $\square$

Following from Proposition 4.2.7, we make the immediate following conjecture.

Conjecture 4.2.8. Every analytic normed space is a generic space.

Example 4.2.9. While Example 4.2.6 shows that strict convexity can be seen to be a necessary condition for a normed plane to be generic it is not sufficient. We shall now construct a strictly convex and smooth normed space that is not generic (or even quasi-generic). We define the smooth norm $\|\cdot\|$ for $\mathbb{R}^2$ where for any $(x,y) \in \mathbb{R}^2$

$$\|(x,y)\| = \begin{cases} (|x|^2 + |y|^2)^{1/2} \text{ if } |x| \geq |y| \\ \\ (|x|^4 + |y|^4)^{1/4} \text{ if } |y| \geq |x|. \end{cases}$$

For any $(x,y) \in \mathbb{R}^2$ this norm has dual map

$$\varphi((x,y)) = \begin{cases} ((x,y), \ \cdot \ ) \text{ if } |x| \geq |y| \\ \\ \frac{1}{(\|(x,y)\|_4)^2}((x^3,y^3), \ \cdot \ ) \text{ if } |y| \geq |x| \end{cases}$$

where $((a,b), \ \cdot \ )$ is the linear functional that maps (c,d) to $(a,b).(c,d) = ac + bd$ and $\|\cdot\|_4$ is the standard ℓ_4 norm. Since $(\mathbb{R}^2, \|\cdot\|)$ is non-Euclidean then by Lemma 1.1.29, $\dim \mathrm{Isom}(X, \|\cdot\|) = 2$.

We define $C_2(K_4)$ to be the set of placements $p := (p_i(x), p_i(y))_{i=1}^4$ of K_4 in $\mathbb{R}^2$ where $|p_i(x) - p_j(x)| > |p_i(y) - p_j(y)|$ for all $1 \leq i < j \leq 4$ and $C_4(K_4)$ to be the set of placements of K_4 in $\mathbb{R}^2$ where $|p_i(y) - p_j(y)| > |p_i(x) - p_j(x)|$ for all $1 \leq i < j \leq 4$. We note that $C_2(K_4)$ and $C_4(K_4)$ are non-empty open subsets of $\mathcal{W}(K_4)$.

As ℓ_4^2 is a generic space then the set of isostatic placements of K_4 in $(\mathbb{R}^2, \|\cdot\|_4)$ is an open dense subset U of $(\mathbb{R}^2)^{V(K_4)}$. It now follows that if $p \in C_4(K_4) \cap U$ then

(K_4, p) is independent in $(\mathbb{R}^2, \|\cdot\|)$ also, thus all regular placements of K_4 in $(\mathbb{R}^2, \|\cdot\|)$ are exactly the independent placements.

If we now however choose any placement $p \in C_2(K_4)$ then we note that $R(G, p)$ will be exactly the rigidity matrix of (K_4, p) in the Euclidean space $(\mathbb{R}^2, \|\cdot\|_2)$, thus for any $p \in C_2(K_4)$, (K_4, p) is independent in $(\mathbb{R}^2, \|\cdot\|_2)$ if and only if (K_4, p) is independent in $(\mathbb{R}^2, \|\cdot\|)$. As K_4 is $(2, 2)$-tight then K_4 is dependent in $(\mathbb{R}^2, \|\cdot\|_2)$, thus for all $p \in C_2(K_4)$, (K_4, p) is dependent. As the regular placements of K_4 in $(\mathbb{R}^2, \|\cdot\|)$ are exactly the independent placements then $C_2(K_4) \subset \mathcal{R}(K_4)^c$. As $\mathcal{R}(K_4)^c$ contains an open set, thus $\mathcal{R}(K_4)$ is not dense in $\mathcal{W}(K_4)$.

A notable property of quasi-generic spaces is that we may always approximate non-generic placements by generic placements.

Corollary 4.2.10. Let (p, V) be a countable placement in a quasi-generic space X. Then the set of generic placements is a dense subset of X^V in the box topology.

Proof. As the set of generic placements of a finite set S are dense in X^S, this follows from Proposition 4.1.2. □

We define a property of frameworks (i.e. independence, rigidity, ect.) to be a *generic property (for finite graphs)* if, given any finite graph G and any generic space X, the property holds for all generic placements of G in X or no generic placements of G in X. If for any finite or countably infinite graph G the property holds for all generic placements of G or no generic placements of G then we define it to be a *generic property for infinite graphs*. It follows from the definition that a generic property for infinite graphs is a generic property for finite graphs.

As we can observe generic properties of graphs, generic spaces are the normed spaces where the most "combinatorial" rigidity results hold. This is as we can solve many rigidity related problems by observing only the graph of the framework for a large class of placements.

Proposition 4.2.11. The following are generic properties:

(i) Independence and dependence.

(ii) Infinitesimal rigidity and flexibility.

(iii) Continuous rigidity and flexibility.

(iv) Local rigidity and flexibility.

Proof. We shall fix our finite graph G and let p, q be generic placements of G. We note that both (G, p) and (G, q) are regular, and by Proposition 2.1.6), both are also constant.

(i): Suppose (G, p) is independent, then as (G, q) is regular then (G, q) is also independent. By symmetry it follows that if (G, q) is independent then (G, p) is also independent, thus the required result holds.

(ii): Suppose (G, p) is infinitesimally rigid, then by Proposition 2.2.13, both (G, p) and (G, q) are full. As (G, q) is regular then

$$\dim \mathcal{F}(G, q) = \dim \mathcal{F}(G, p) = \dim \mathcal{T}(p) = \dim \mathcal{T}(q),$$

thus (G, q) is infinitesimally flexible. By symmetry it follows that if (G, q) is infinitesimally rigid then (G, p) is also infinitesimally rigid, thus the required result holds.

(iii): This follows from (ii) and Theorem 2.1.5.

(iv): This follows from (ii) and Theorem 2.1.5.

□

Corollary 4.2.12. Independence and dependence are generic properties for infinite graphs.

Proof. This follows from Proposition 4.2.1. □

4.2.2 Generic rigidity for infinite graphs

Lemma 4.2.13. Let G be a finite graph and p, q be completely well-positioned placements of G in a normed space X. If p is generic and q is regular, then $\langle G \rangle_p \subseteq \langle G \rangle_q$, with equality if q is also generic.

Proof. As both placements are regular, $\operatorname{rank} df_G(p) = \operatorname{rank} df_G(q)$. If $e \in \langle G \rangle_p$ then by Lemma 4.1.17, $\operatorname{rank} df_G(p) = \operatorname{rank} df_{G\cup\{e\}}(p)$. It now follows

$$\operatorname{rank} df_G(q) \leq \operatorname{rank} df_{G\cup\{e\}}(q) \leq \operatorname{rank} df_{G\cup\{e\}}(p) = \operatorname{rank} df_G(p) = \operatorname{rank} df_G(q),$$

thus by Lemma 4.1.17, $e \in \langle G \rangle_q$ and $\langle G \rangle_p \subseteq \langle G \rangle_q$. $\square$

Lemma 4.2.14. Let p, q be independent, completely well-positioned placements of a countable graph G in a normed space X. If p is generic, then $\langle G \rangle_p \subseteq \langle G \rangle_q$, with equality if q is also generic.

Proof. We note that all subframeworks of (G, p) and (G, q) are regular as they are independent, thus by Lemma 4.2.13 and **CL5**,

$$\langle G \rangle_p = \bigcup_{H \subset\subset G} \langle H \rangle_p \subset \bigcup_{H \subset\subset G} \langle H \rangle_q = \langle G \rangle_q$$

as required. $\square$

Theorem 4.2.15. Let p, q be generic placements of a countable graph G in a normed space X, then $\langle G \rangle_p = \langle G \rangle_q$.

Proof. By Lemma 4.2.13, $\langle H \rangle_p = \langle H \rangle_q$ for all $H \subset\subset G$. The result now follows from **CL5**. $\square$

Let V be countable, then for any set $E \subset E(K_V)$ and generic placements p, p' of V in a generic space X we have that $\langle E \rangle_p = \langle E \rangle_{p'}$. If we fix the generic space X and choose any generic placement p of V in X we may define the following:

(i) The *generic closure operator (for X)*; the map $\langle \cdot \rangle : \mathcal{P}(E(K_V)) \to \mathcal{P}(E(K_V))$ where $\langle E \rangle := \langle E \rangle_p$.

(ii) *Independent edge sets of X*; an edge set $E \subseteq E(K_V)$ which is independent with respect to p. We further define $\mathcal{I}(X) := \mathcal{I}_p$.

(iii) The *closure of a graph (in X)*; for a graph G we define $\langle G \rangle := \langle G \rangle_p$.

We immediately note that the pair $(E(K_V), \mathcal{I}(X))$ will be a finitary matroid.

Lemma 4.2.16. Let (p, V) be a generic countably infinite placement in a generic space X, then the following holds:

(i) (p, V) is full.

(ii) (K_V, p) is sequentially infinitesimally rigid.

Proof. (i): Choose any rigid finite graph $G \subset K_V$ in X, then $(G, p|_{V(G)})$ is regular and constant. By Proposition 2.2.13, $p|_{V(G)}$ is full, thus by Corollary 1.2.22, p is full.

(ii): As V is countable we may label $V = \{v_1, v_2, \ldots\}$ and define $V_n := \{v_1, \ldots, v_n\}$, then $((K_{V_n}, p^n))_{n \in \mathbb{N}}$ is a complete tower of (K_V, p). As X is generic then for some $N \in \mathbb{N}$, K_{V_N} is rigid. By Corollary 2.3.3 and Proposition 2.2.13, (K_{V_n}, p^n) is infinitesimally rigid for all $n \geq N$, thus (K_V, p) is sequentially infinitesimally rigid. □

Theorem 4.2.17. The following are generic properties of infinite graphs:

(i) Independence and dependence.

(ii) Infinitesimal rigidity and flexibility.

(iii) Sequential infinitesimal rigidity and flexibility.

Proof. Fix a countably infinite graph G and let p, q be generic placements of G in a generic space X. By Lemma 4.2.16 (i), p, q are full.

(i): Lemma 4.2.12.

(ii): Suppose (G, p) is infinitesimally rigid. As (G, p) is infinitesimally rigid then by Theorem 4.1.20, $\langle G \rangle_p = K_{V(G)}$ and $(K_{V(G)}, p)$ is infinitesimally rigid. As X is generic and q is a generic placement then by Corollary 4.2.16, $(K_{V(G)}, q)$ is infinitesimally rigid. By Theorem 4.2.15, $\langle G \rangle_q = \langle G \rangle_p = K_{V(G)}$, thus by Theorem 4.1.20, (G, q) is infinitesimally rigid also. It follows from symmetry that if (G, q) is infinitesimally rigid then (G, p) is infinitesimally rigid also, thus both infinitesimal rigidity and flexibility are generic properties for infinite graphs.

(iii): Suppose that (G, p) is sequentially infinitesimally rigid. As q is generic and full then by Proposition 4.2.2, (G, q) is sequentially infinitesimally rigid. It follows from symmetry that if (G, q) is sequentially infinitesimally rigid then (G, p) is sequentially infinitesimally rigid also, thus both sequentially infinitesimal rigidity and flexibility are generic properties for infinite graphs. $\square$

Motivated Theorem 4.2.17, for a generic space X we shall define a graph G to be *generically rigid (in X)* if there exists some generic placement p such that G is (G, p) is infinitesimally rigid, *generically isostatic (in X)* if there exists some generic placement p such that G is (G, p) is isostatic, and *generically flexible* otherwise. For finite graphs we note that as any generic placement is regular, a graph is rigid/isostatic if and only if it is generically rigid/isostatic.

Corollary 4.2.18. Let G be generically rigid in a generic space X and p a independent placement of G in X where $(K_{V(G)}, p)$ is infinitesimally rigid. Then G is generically isostatic and (G, p) is isostatic.

Proof. Let q be a generic placement of G. By Theorem 4.2.17, (G, q) is isostatic, thus G is generically isostatic. By Lemma 4.2.14,

$$K_{V(G)} = \langle G \rangle \subset \langle G \rangle_p \subset K_{V(G)},$$

thus the result holds by Theorem 4.1.20. □

Corollary 4.2.19. Let G be a countable graph and p a independent full placement of G in a generic space X. Then G is sequentially isostatic if and only if (G, p) is sequentially isostatic.

Proof. Suppose G is sequentially isostatic. Let q be a generic placement of G, then by Theorem 4.2.17, (G, q) is sequentially isostatic with complete sequentially isostatic tower $((G^n, q^n))_{n\in\mathbb{N}}$. Let $((G^n, p^n))_{n\in\mathbb{N}}$ be the corresponding complete tower of (G, p), then $((G^n, p^n))_{n\in\mathbb{N}}$ is independent. By Proposition 2.2.13, $((G^n, p^n))_{n\in\mathbb{N}}$ is isostatic as required. □

It follows from Propositon 4.2.2 that if G has a sequentially infinitesimally rigid placement then every generic placement of G is sequentially infinitesimally rigid. However, there exist countably infinite graphs with infinitesimally rigid placements that are not generically rigid. An example is the framework described in Figure 6; it is an isostatic framework with a graph that is not generically isostatic.

4.3 Combinatorial rigidity of countable graphs

4.3.1 Rigidity and independence in normed planes

We now wish to obtain some combinatorial rigidity results for graphs in normed planes for countably infinite graphs, similar to Theorem 1.3.20 and Theorem 3.4.2. To do so we shall need to alter our definition for towers of frameworks so as to remove the requirement of specific placements.

A *tower (of graphs)* is a sequence $(G^n)_{n\in\mathbb{N}}$ of finite graphs where $G^n \subset G^{n+1}$ for all $n \in \mathbb{N}$. Given a graph G we define a tower $(G^n)_{n\in\mathbb{N}}$ to be a *tower of* G if $G^n \subset G$ for all $n \in \mathbb{N}$. A tower $(G^n)_{n\in\mathbb{N}}$ of G is *vertex-complete* if $\cup_{n\in\mathbb{N}} V(G^n) = V(G)$, *edge-complete* if $\cup_{n\in\mathbb{N}} E(G^n) = E(G)$ and *complete* if it is both vertex-complete and edge-complete. For $k, l \in \mathbb{N}$, we define a tower to be (k,l)-*sparse* if each G^n is (k,l)-sparse, and (k,l)-*tight* if each G^n is (k,l)-tight; we note that a countably infinite graph will be (k,l)-sparse if and only if it contains a (k,l)-sparse complete tower.

We can immediately give a combinatorial result regarding graph sparsity.

Lemma 4.3.1. Let G be a countable graph, X a normed plane and $k \in \{2,3\}$, where $k = 3$ if X is Euclidean and $k = 2$ if X is non-Euclidean. If G is independent then G is $(2,k)$-sparse. If X is generic then the reverse also holds.

Proof. Let p be an independent placement of G and choose any $H \subset\subset G$. By Proposition 4.1.10, $(H,q) \subset\subset (G,p)$ is independent, thus by applying either Theorem 1.3.20 or Theorem 3.4.2 (depending on if X is Euclidean or non-Euclidean) we see that H is $(2,k)$-sparse as required.

Suppose X is generic and G is $(2,k)$-sparse and let p be a generic placement of G. Choose any $H \subset\subset G$, then by either Theorem 1.3.20 or Theorem 3.4.2 (depending on if X is Euclidean or non-Euclidean), $(H, p|_{V(H)})$ is independent. As this holds for any finite subframework of (G,p) then be Proposition 4.1.10, (G,p) is independent as required. □

For Theorem 4.3.12 we will need the following lemmas.

Lemma 4.3.2. Let $H \subsetneq G$ be finite $(2,k)$-tight graphs for some fixed $k \in \{2,3\}$. Then there exists a vertex $v_0 \in V(G) \setminus V(H)$ such that $d_G(v_0) = 2$ or 3.

Proof. Fix $k \in \{2,3\}$. We first note that as $H \neq G$ and both are $(2,k)$-tight then $V(G) \setminus V(H) \neq \emptyset$.

Suppose that for all $v \in V(G) \setminus V(H)$, $d_G(v) \geq 4$. Define

$$\partial H := \{vw \in E(G) : v \in V(H), w \notin V(H)\}.$$

As G is $(2,k)$-tight it is connected, thus $|\partial H| > 0$. By the Hand Shaking Lemma,

$$\begin{aligned} |E(G)| - |E(H)| &= \frac{1}{2}\sum_{v\in V(G)} d_G(v) - \frac{1}{2}\sum_{v\in V(H)} d_H(v) \\ &= \frac{1}{2}\sum_{v\in V(G)\setminus V(H)} d_G(v) + \frac{1}{2}|\partial H| \\ &\geq \frac{1}{2}4|V(G)\setminus V(H)| + \frac{1}{2}|\partial H| \\ &> 2|V(G)| - 2|V(H)|. \end{aligned}$$

However, as H and G are $(2,k)$-tight then $|E(G)| - |E(H)| = 2|V(G)| - 2|V(H)|$, a contradiction. □

Remark 4.3.3. The above method will work for $(2,1)$-tight graphs also, however it will fail for $(2,0)$-tight graphs. This is as we can no longer guarantee that G will be connected, and thus we can have $\partial H = \emptyset$. It is not just the method that fails however; we note Lemma 4.3.2 fails in the $k = 0$ case for $H = K_5$ and $G = K_5 \sqcup K_5$, the disjoint union of two K_5 graphs.

Lemma 4.3.4. Let G, H be finite $(2,2)$-tight graphs where $G \cup H$ is $(2,2)$-sparse and $V(G) \cap V(H) \neq \emptyset$. Then $G \cup H$ and $G \cap H$ are $(2,2)$-tight.

Proof. As $G \cap H \subset G \cup H$, $G \cap H$ is $(2,2)$-sparse. We now note

$$\begin{aligned} |E(G\cap H)| &= |E(G)| + |E(H)| - |E(G\cup H)| \\ &\geq 2(|V(G)| + |V(H)| - |V(G\cup H)|) - 2 \\ &= 2|V(G\cap H)| - 2, \end{aligned}$$

thus $G \cap H$ is $(2,2)$-tight. It now follows

$$\begin{aligned} |E(G\cup H)| &= |E(G)| + |E(H)| - |E(G\cap H)| \\ &= 2(|V(G)| + |V(H)| - |V(G\cap H)|) - 2 \\ &= 2|V(G\cup H)| - 2, \end{aligned}$$

as required. $\square$

Lemma 4.3.5. Let G be a finite $(2,k)$-tight graph for some fixed $k \in \{2,3\}$, $v_1 \in V(G)$ and $N_G(v_1) = \{v_2, v_3, v_4\}$ for distinct vertices $v_2, v_3, v_4 \in V(G)$. Suppose that v_1 does not lie in a K_4 subgraph of G, then $(G - v_1) + v_i v_j$ is $(2,k)$-tight for some $2 \leq i < j \leq 4$.

Proof. If $k = 3$ this follows by [30, Lemma 2.1.4], while if $k = 2$ this follows by [53, Lemma 3.1]. $\square$

Lemma 4.3.6. [53, Lemma 3.3] Let G be a finite $(2,2)$-tight graph with complete subgraph $K_V \subset G$ for some set $V := \{a,b,c,d\} \subset V(G)$. Suppose that there is no vertex $v \in V(G) \setminus V$ connected to more than one vertex of V. Then for each $j \in \{a,b,c,d\}$ the graph G' is $(2,2)$-tight, where $V(G') := (V(G) \setminus V) \cup \{j\}$ and $E(G')$ is the set:

$$\{vw \in E(G) : v, w \in V(G')\} \cup \{jw : w \notin V, iw \in E(G) \text{ for some } i \in V \setminus \{j\}\}.$$

Let G be a finite $(2,2)$-tight graph. We define a sequence T of graphs

$$T_4 \subset T_5 \subset \ldots \subset T_n$$

to be a $(2,2)$-*simplex sequence* of G (or a *simplex sequence*) if

(i) $V(T_n) = \{v_1, \ldots, v_n\}$,

(ii) $|V(T_i)| = i$ and $V(T_i) = \{v_1, \ldots, v_i\}$ for $4 \leq i \leq n$,

(iii) $T_4 \cong K_4$,

(iv) $E(T_{i+1}) := E(T_i) \cup \{v_{i+1}v_i, v_{i+1}v_{i-1}\}$.

We define a simplex sequence $T = (T_i)_{i=4}^n$ to be *maximal* if there is no vertex in $V(G) \setminus V(T_n)$ that is attached to both v_{n-1} and v_n. Simplex sequences are based on *triangle sequences* (see [53, Definition 3.5]).

Lemma 4.3.7. Let G be a finite $(2,2)$-tight with a simplex sequence $T = (T_i)_{i=4}^n$. Then the following holds:

(i) T_i is $(2,2)$-tight for all $i = 4, \ldots, n$.

(ii) Either T is maximal or there exists $T_{n+1}, \ldots, T_{n+m}$ such that $(T_i)_{i=4}^{n+m}$ is a maximal simplex sequence.

(iii) If $H \subseteq T_n$ is $(2,2)$-tight, then either H is a single vertex or $H = T_k$ for some $4 \leq k \leq n$.

(iv) If $H \subseteq G$ is $(2,2)$-tight and $|V(H) \cap V(T_n)| > 1$, then for some $4 \leq k \leq n$, $H \cap T_n = T_k$.

(v) If T is maximal, there exists no vertex $v \in V(G) \setminus \{v_{n-2}\}$ that shares an edge with both v_{n-1} and v_n.

Proof. (i): As T_{i+1} is formed from T_i by performing a 0-extension and K_4 is $(2,2)$-tight, this follows from Proposition 3.4.1.

(ii): As G is finite, this follows immediately.

(iii): We note that if we remove any edge or vertex from T_i we either obtain a $(2,3)$-sparse graph or T_{i-1}. The result now follows immediately.

(iv): By Lemma 4.3.4, $H \cap T_n$ is $(2,2)$-tight. The result now follows from (iii).

(v): Suppose there exists $v \in V(G) \setminus \{v_{n-2}\}$ that shares an edge with both v_{n-1} and v_n. If $v \notin V(T_n)$ then T is not maximal, thus $v \in V(T_n) \setminus \{v_{n-2}\}$. Let $F = T_n \cup \{vv_{n-1}, vv_n\}$, then $T_n \subsetneq F \subset G$. By (i), T_n is a spanning $(2,2)$-tight subgraph of F, thus F is not $(2,2)$-sparse, contradicting the $(2,2)$-sparsity of G. $\square$

Lemma 4.3.8. [53, Lemma 3.4] Let G be a finite $(2,2)$-tight graph with complete subgraph $K_V \subset G$ for some set $V := \{w_0, w_1, w_2\} \subset V(G)$. Suppose that:

(i) There exists no vertex $v \in V(G) \setminus V$ connected to both w_1 and w_2.

(ii) There exists no $(2,2)$-tight subgraph $F \subset G$ where $w_1, w_2 \in V(F)$ but $w_0 \notin V(F)$.

Then the graph H' is $(2,2)$-tight, where

$$\begin{aligned} V(G') &:= V(G) \setminus \{w_2\}, \\ E(G') &:= \{vw \in E(G) : v, w \neq w_2\} \cup \{vw_1 : v \notin V,\ vw_2 \in E(G)\}. \end{aligned}$$

The following two lemmas give the same result to [37, Proposition 4.6 and Proposition 4.9], however we do not use a minimality argument on the set of all pairs of graphs (H, G) with $H \subset G$ and no *Henneberg construction* from H to G.

Lemma 4.3.9. Let $H \subsetneq G$ be $(2,k)$-tight graphs for some $k \in \{2,3\}$. Then there exists a $(2,k)$-tight graph $G' \supset H$ such that either:

(i) $k = 3$ and G is either a 0-extension or 1-extension of G'.

(ii) $k = 2$ and G is either a 0-extension, 1-extension, vertex split or vertex-to-K_4 extension of G'.

Proof. By Lemma 4.3.2, there exists a vertex $v_1 \in V(G) \setminus V(H)$ with $d_G(v) \in \{2,3\}$. We shall now discuss the various case and prove for each one we can construct the required graph G'.

First suppose $d_G(v_0) = 2$. Let $G' := G \setminus \{v_1\}$. Then $G' \supset H$, G' is $(2,k)$-tight and G is a 0-extension of G' as required.

Now suppose that $d_G(v_1) = 3$ with neighbours v_2, v_3, v_4 and define $V := \{v_1, v_2, v_3, v_4\}$.

Suppose $G[V] \neq K_V$. By Lemma 4.3.5, there exists $2 \leq i < j \leq 4$ such that $G' := (G \setminus \{v_1\}) \cup \{v_i v_j\}$ is $(2,k)$-tight. We note that $G' \supset H$ and G is a 1-extension of G' as required.

Now suppose $G[V] = K_V$. As K_V is $(2,2)$-tight we note that we must have that $k = 2$. As $v_1 \notin V(H)$ we have that $H \cap K_V$ is either the empty graph or a copy of K_1, K_2 or K_3. If $V(H) \cap V \neq \emptyset$ then by Lemma 4.3.4, $H \cap K_V \cong K_1$ as K_2, K_3 are not $(2,2)$-tight.

Suppose there is no vertex in $V(G) \setminus V$ that shares an edge with more than one of V. By Lemma 4.3.6, we may define a $(2,2)$-tight graph G' as described with j chosen to be the vertex of V that lies in $V(H)$; if no vertex of V lies in H we may choose j to be any of V. We note that G is a vertex-to-K_4 extension of G' and $G' \supset H$ as required.

Now suppose that there is a vertex v_5 in $V(G)\setminus V$ that shares an edge with more than one of V; by relabelling we may assume that v_5 shares an edge with v_3, v_4. We note that $(K_V, K_{V\cup\{v_5\}})$ is a simplex sequence, thus by Lemma 4.3.7 (ii), there exists a maximal simplex sequence $T := (T_i)_{i=4}^n$ with the ordered list of vertices $\{v_1, \ldots, v_n\}$ (note that $T_4 = K_V$ and $T_5 = K_{V\cup\{v_5\}}$). As $v_1 \notin H$, by Lemma 4.3.7 (iv), $|V(H) \cap T_n| \leq 1$.

By Lemma 4.3.7 (v), there is no vertex except v_{n-2} that shares an edge with both v_{n-1} and v_n. Define $w_0 := v_{n-2}$, and let $w_1 = v_{n-1}$, $w_2 = v_n$ if $v_{n-1} \in V(H)$ and $w_1 = v_n$, $w_2 = v_{n-1}$ otherwise. By Lemma 4.3.8 we have a $(2,2)$-tight graph G' as described. We now note that G is a vertex split of G' and $G' \supset H$ as required. $\square$

Lemma 4.3.10. Let $H \subset G$ be finite $(2,k)$-tight graphs for some fixed $k \in \{2,3\}$. Then there exists a sequence $H_1, \ldots, H_n$ of $(2,k)$-tight graphs such that:

(i) $H_1 = H$ and $H_n = G$,

(ii) $H \subset H_i$ for all $1 \leq i \leq n$,

(iii) H_{i+1} is a either a 0-extension or 1-extension of H_i for all $1 \leq i \leq n-1$ if $k = 3$, or,

(iv) H_{i+1} is a either a 0-extension, 1-extension, vertex split or vertex-to-K_4 extension of H_i for all $1 \leq i \leq n-1$ if $k = 2$.

Proof. This follows from applying Lemma 4.3.9 to each pair $H \subset H_i$ to obtain H_{i-1}. □

Lemma 4.3.11. Let $H \subset G$ be finite isostatic graphs in a normed plane X and let q be an independent placement of H in general position in X. Then there exists an independent placement p of G in general position so that $p|_{V(G)} = q$.

Proof. Fix $k = 3$ if X is Euclidean and $k = 2$ if X is non-Euclidean. By either Theorem 1.3.20 or Theorem 3.4.2 (depending on if X is Euclidean or non-Euclidean), H and G are $(2,k)$-tight. Let $H_1, \ldots, H_n$ be the sequence of $(2,k)$-tight graphs described in Lemma 4.3.10. By Lemma 3.3.2, Lemma 3.3.4, Lemma 3.3.6 and Lemma 3.3.8, there exists a sequence $q^1, \ldots, q^n$ of isostatic placements in general position of $H^1, \ldots, H^n$ respectively so that $q^1 := q$ and $q^i|_{V(H)} = q$ for each $1 \leq i \leq n$. We now define $p := q^n$. □

Theorem 4.3.12. Let G be a countable graph, X a normed plane and $k \in \{2,3\}$, with $k = 3$ if X is Euclidean and $k = 2$ if X is non-Euclidean. Then the following are equivalent:

1. G is sequentially rigid in X.

2. G contains a vertex-complete $(2,k)$-tight tower.

Proof. Suppose G has a sequentially infinitesimally rigid placement p in X, then there exists a sequentially infinitesimally rigid vertex-complete tower $((G^n,p^n))_{n\in\mathbb{N}}$ of (G,p). For each $n \in \mathbb{N}$ let (F^n,p^n) be a spanning isostatic subframework of (G^n,p^n). Let

$(H^1, p^1) := (F^1, p^1)$. Suppose we now have spanning isostatic subframeworks

$$(H^1, p^1), \ldots, (H^n, p^n)$$

for some $n \in \mathbb{N}$ so that $H^1 \subset \ldots \subset H^n$. As $(E(K_{V(G^{n+1})}), \mathcal{I}_{p^{n+1}})$ is a matroid and $|E(H^n)| < |E(Fn+1)|$ then there exists a graph H^{n+1} such that

$$H^n \subset H^{n+1}, \qquad |E(H^{n+1})| = |F^{n+1}|, \qquad (H^{n+1}, p^{n+1}) \text{ is isostatic.}$$

By induction we obtain a vertex-complete sequentially isostatic tower $((H^n, p^n))_{n\in\mathbb{N}}$ of G. By either Theorem 1.3.20 or Theorem 3.4.2 (depending on if X is Euclidean or non-Euclidean), $(H^n)_{n\in\mathbb{N}}$ is a vertex-complete $(2,k)$-tight tower of G.

Now suppose G contains a vertex-complete $(2,k)$-tight tower $(G^n)_{n\in\mathbb{N}}$; without loss of generality we may assume that $|V(G^1)| \geq 3$. By either Theorem 1.3.20 or Theorem 3.4.2 (depending on if X is Euclidean or non-Euclidean), each G^n is isostatic in X. Let p^1 be an isostatic placement of G^1 in general position (Lemma 1.2.5). By Lemma 4.3.11, there exists a sequence $(p^n)_{n\in\mathbb{N}}$ such that p^n is an isostatic placement G^n in general position and $p^m|_{V(G^n)} = p^n$ for all $n \leq m$. Then $((G^n, p^n))_{n\in\mathbb{N}}$ is an sequentially isostatic tower. By letting $p_v := p^n_v$ for $v \in V(G^n) \subset V(G)$ we have that $((G^n, p^n))_{n\in\mathbb{N}}$ is a complete tower of the well-positioned placement (G,p), thus (G,p) is sequentially isostatic as required. □

4.3.2 Rigidity and independence in generic normed planes

We shall now strengthen Theorem 4.3.12 for generic placements in generic normed planes. Here we can make much stronger statements due to the inherently "combinatorial nature" of generic spaces. We remind ourselves that both infinitesimal rigidity and sequential infinitesimal rigidity are generic properties for infinite graphs.

Let $H \subset G$ be finite graphs and X a generic space. We say H is *relatively rigid* in G if $K_{V(H)}$ is rigid and $\langle G \rangle = \left\langle G \cup K_{V(H)} \right\rangle$, and we say H has a *rigid container* in G if there exists a rigid graph H' with $H \subset H' \subset G$.

Lemma 4.3.13. Let $H \subset G$ be finite graphs and X be a generic normed plane, where $|V(H)| \geq 2$ if X is Euclidean and $|V(H)| \geq 4$ otherwise. Then H is relatively rigid in G if and only if H has a rigid container in G.

Proof. As X is a normed plane then by Theorem 1.3.20 and Theorem 3.4.2, a graph is isostatic if and only if it is $(2, k)$-tight, where $k = 3$ if X is Euclidean and $k = 2$ if X is non-Euclidean. We now note that the proof is identical to the proof of [38, Theorem 3.6]. □

The following is an extension of [38, Theorem 1.1] and [38, Theorem 4.1].

Theorem 4.3.14. Let G be a countable graph and X a generic normed plane. Let $k = 3$ if X is Euclidean and $k = 2$ is non-Euclidean, then the following are equivalent:

(i) G is generically rigid in X.

(ii) G is sequentially rigid in X.

(iii) G contains a vertex-complete $(2, k)$-tight tower.

Proof. By applying either Theorem 1.3.20 or Theorem 3.4.2 (depending on if X is Euclidean or non-Euclidean) and Corollary 4.1.8 to a generic placement of G we have that (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i).

(i) $\Rightarrow$ (ii): Let p be any generic placement of G, then (G, p) is infinitesimally rigid, and by Theorem 4.1.7, there exists a vertex-complete relatively infinitesimally rigid tower $((G^n, p^n))_{n\in\mathbb{N}}$. By Lemma 4.3.13, there exists a sequence $(H^n)_{n\in\mathbb{N}}$ so that

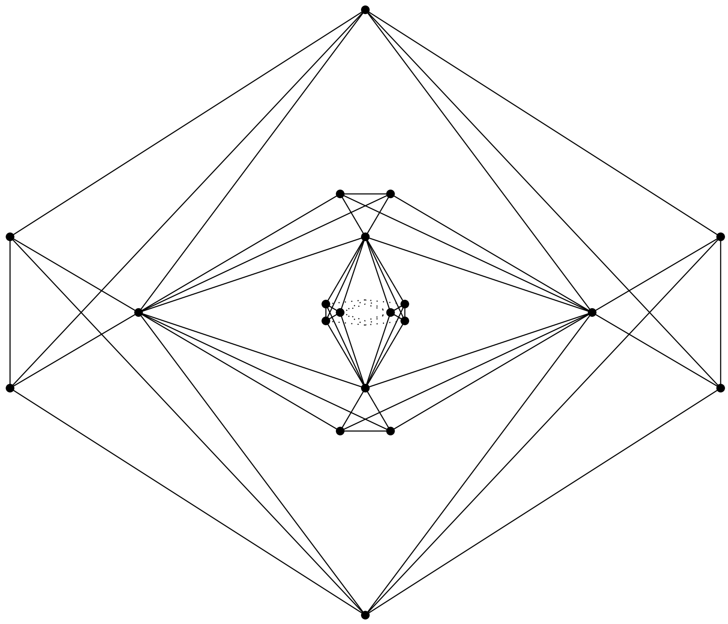

Fig. 4.1 The infinite double banana graph, based on [38, Figure 2]. Every finite subframework is flexible, however the graph is infinitesimally rigid for any generic placement.

$G^n \subset H^n \subset G^{n+1}$ and each H^n is rigid. It now follows $((H^n, p|_{V(H^n)}))_{n\in\mathbb{N}}$ is a vertex-complete sequentially infinitesimally rigid tower of (G, p) and so (G, p) is sequentially infinitesimally rigid as required.

(ii) $\Rightarrow$ (iii): This follows from Theorem 4.3.12. $\square$

Theorem 4.3.14 can only be applied to generic placements, see Figure 6 for an example of an isostatic framework with no $(2, 3)$-tight tower. It also fails to hold in Euclidean 3-space, as can be seen by the counter-example in Figure 4.1.

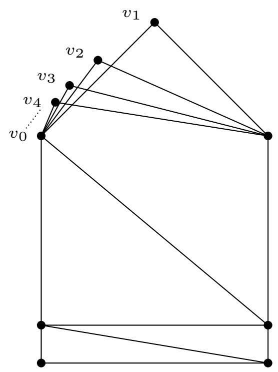

Fig. 4.2 An infinitesimally flexible independent framework (G, p) in the Euclidean plane that is continuously rigid. The sequence $\{p_{v_1}, p_{v_2}, \ldots\}$ converges to p_{v_0}. We may choose $p_{v_0}, p_{v_1}, p_{v_2}, \ldots$ so as to be an algebraically independent set, thus we can assume (G, p) is also generic.

4.4 Continuous rigidity for countable frameworks

4.4.1 Continuous rigidity for countable generic frameworks in generic spaces

We would naively hope that Theorem 2.1.5 would extend to infinite independent frameworks. Unfortunately, we can construct frameworks in normed planes with open sets of smooth points that are continuously rigid, independent - even generic if the normed plane is generic - but infinitesimally flexible, see Figure 4.2.

If continuous rigidity does not imply infinitesimal rigidity, then does infinitesimal rigidity imply continuous rigidity? We have more luck with this direction, especially for sequentially infinitesimally rigid frameworks, although there are still many open questions.

For the following we define a tower to be *constant* and/or *regular* if each framework in the sequence is constant and/or regular.

Theorem 4.4.1. Let (G, p) be a framework in a normed space X. Suppose (G, p) contains a vertex-complete sequentially infinitesimally rigid tower that is constant, then (G, p) is continuously rigid.

Proof. Suppose $\alpha := (\alpha_v)_{v \in V(G)}$ with $\alpha_v : (-\delta, \delta) \to X$ is a finite flex of (G, p) and let $((G^n, p^n))_{n \in \mathbb{N}}$ be a vertex-complete sequentially infinitesimally rigid tower of (G, p). We note that $\alpha|_{V(G^n)}$ must also be a finite flex of (G^n, p^n) for each $n \in \mathbb{N}$. By Theorem 2.1.5, $\alpha|_{V(G^n)}$ is trivial for each $n \in \mathbb{N}$, thus for all $t \in (-\epsilon, \epsilon)$ and all $n \in \mathbb{N}$, $\alpha(t)|_{V(G^n)} \in \mathcal{O}_{p^n}$, i.e. there exists for each $t \in (-\delta, \delta)$ and $n \in \mathbb{N}$ some isometry $h^n_t \in \operatorname{Isom}(X)$ so that $h^n_t.p^n = \alpha(t)|_{V(G^n)}$.

Define for each $t \in (-\epsilon, \epsilon)$ and $n \in \mathbb{N}$ the set H^n_t of such isometries g where $g.p^n = \alpha(t)|_{V(G^n)}$, then we note $H^n_t = h^n_t \operatorname{Stab}_p$. By Corollary 1.2.11, H^n_t is compact. We note that for $n \leq m$, $H^m_t \subseteq H^n_t$, and so as $H^n_t \neq \emptyset$ for all $n \in \mathbb{N}$, there exists $h_t \in \cap_{n \in \mathbb{N}} H^n_t$. We now note $h_t.p = \alpha(t)$, thus α is trivial as required. □

Corollary 4.4.2. Suppose (G, p) is sequentially infinitesimally rigid in a normed space X with an open set of smooth points, then (G, p) is continuously rigid.

Proof. By Proposition 2.1.1, every regular framework will be constant. Since (G, p) is sequentially infinitesimally rigid it contains a sequentially infinitesimally rigid tower. As this tower will be regular (and so constant), the result follows from Theorem 4.4.1. □

Corollary 4.4.3. Suppose (G, p) is infinitesimally rigid and generic in a generic normed plane X, then (G, p) is continuously rigid.

Proof. By Theorem 4.3.14, (G, p) is sequentially rigid. The result now follows from Corollary 4.4.2. □

For our following results we are required to weaken continuous rigidity somewhat.

Let (G,p) be a framework in a normed space X and α be a finite flex of (G,p). We define α to be *proper* if there exists $\epsilon > 0$ and $v, w \in V(G)$ so that

$$\|\alpha_v(t) - \alpha_w(t)\| \neq \|p_v - p_w\|$$

for all $t \in (-\epsilon, 0) \cup (0, \epsilon)$. We note that all proper finite flexes are non-trivial. If (G,p) has no proper finite flexes then (G,p) is *weakly continuously rigid*. It is immediate that continuous rigidity implies weak continuous rigidity, and for finite frameworks, weak continuous rigidity and continuous rigidity are equivalent. We have the following conjecture.

Conjecture 4.4.4. A framework is weakly continuously rigid if and only if it is continuously rigid, or equivalently, a framework has a proper finite flex if it has a non-trivial finite flex.

It is possible we would have to restrict to certain categories of frameworks (e.g. generic frameworks, periodic frameworks) for the above conjecture. So far the conjecture holds for all known examples.

Lemma 4.4.5. Suppose (G,p) is a finite generic framework in a normed space X with an open set of smooth points, then there exists an open neighbourhood U of p such that

$$f_G^{-1}[f_G(p)] \cap U = f_{\langle G \rangle}^{-1}[f_{\langle G \rangle}(p)] \cap U.$$

Proof. As (G,p) is generic then (G,p) and $(\langle G \rangle, p)$ are regular, thus by Proposition 2.1.1, both are constant. By Lemma 4.1.19, $\mathcal{F}(\langle G \rangle, p) = \mathcal{F}(G,p)$, thus it follows from Lemma 2.1.4 that there exists an open neighbourhoods U' of p such that both $f_G^{-1}[f_G(p)] \cap U'$ and $f_{\langle G \rangle}^{-1}[f_{\langle G \rangle}(p)] \cap U'$ are C^1-submanifolds of $X^{V(G))}$, and both have tangent space $\mathcal{F}(G,p)$ at p.

As

$$f_{\langle G\rangle}^{-1}[f_{\langle G\rangle}(p)] \cap U' \subseteq f_G^{-1}[f_G(p)] \cap U' \subseteq X^{V(G)}$$

and both are C^1-submanifolds of $X^{V(G)}$, the inclusion map

$$f_{\langle G\rangle}^{-1}[f_{\langle G\rangle}(p)] \cap U' \hookrightarrow f_G^{-1}[f_G(p)] \cap U'$$

is a C^1-embedding, thus $f_{\langle G\rangle}^{-1}[f_{\langle G\rangle}(p)] \cap U'$ is a C^1-submanifold of $f_G^{-1}[f_G(p)] \cap U'$. As both have the same tangent space at p, there exists an open neighbourhood U of p such that $f_G^{-1}[f_G(p)] \cap U = f_{\langle G\rangle}^{-1}[f_{\langle G\rangle}(p)] \cap U$ as required. $\square$

Theorem 4.4.6. Suppose (G, p) is infinitesimally rigid and generic in a normed space X with an open set of smooth points, then (G, p) is weakly continuously rigid.

Proof. Suppose there exists a proper flex $\alpha : (-\delta, \delta) \to X^{V(G)}$ of (G, p), then there $\epsilon > 0$ and $v, w \in V(G)$ such that

$$\|\alpha_v(t) - \alpha_w(t)\| \neq \|p_v - p_w\|$$

for all $t \in (-\epsilon, 0) \cup (0, \epsilon)$. Let $((G^n, p^n))_{n\in\mathbb{N}}$ be a complete relatively rigid tower of (G, p), then for each $n \in \mathbb{N}$,

$$\mathcal{F}(G^{n+1}, p^{n+1}) = \mathcal{F}(G^{n+1} \cup K_{V(G^n)}, p^{n+1}).$$

As (G, p) is generic it is also completely well-positioned, thus by Lemma 4.1.19

$$\left\langle G^{n+1} \right\rangle_p = \left\langle G^{n+1} \cup K_{V(G^n)} \right\rangle_p$$

for all $n \in \mathbb{N}$. As $((G^n, p^n))_{n\in\mathbb{N}}$ is a complete tower of (G, p), there exists $m \in \mathbb{N}$ such that $v, w \in V(G^m)$.

By Lemma 4.4.5, there exists an open neighbourhood U of p^{m+1} such that

$$f^{-1}_{G^{m+1}}[f_{G^{m+1}}(p^{m+1})] \cap U = f^{-1}_{\langle G^{m+1} \cup K_{V(G^m)} \rangle}\left[f_{\langle G^{m+1} \cup K_{V(G^m)} \rangle}(p^{m+1})\right] \cap U.$$

As $\alpha|_{V(G^{m+1})}$ is a finite flex of (G^{m+1}, p^{m+1}) it now follows that for some $\epsilon' > 0$,

$$\|\alpha_v(t) - \alpha_w(t)\| = \|p_v - p_w\|$$

for all $t \in (-\epsilon', \epsilon')$, a contradiction. $\square$

Remark 4.4.7. Theorem 4.4.6 requires that our framework is generic; for instance, figure 6 is shown in [38, Example 6.4] to be infinitesimally rigid but weakly continuously flexible.

4.4.2 Continuous rigidity for countable algebraically generic frameworks in Euclidean spaces

Let p be a finite placement in $\mathbb{R}^d$, and define $p_v(i)$ to be the i-th coordinate of p_v. We define p to be *algebraically generic* if the set $\{p_v(i) : v \in V,\ i = 1, \dots, d\}$ is algebraically independent over $\mathbb{Q}$. If p is a countably infinite placement we define p to be *algebraically generic* if every finite subplacement is algebraically independent. We likewise define a framework (G, p) to be *algebraically generic* if p is algebraically generic.

Lemma 4.4.8. Let $|V| < \infty$, then the set of algebraically generic placements in $\mathbb{R}^d$ is a dense subset of $(\mathbb{R}^d)^V$ with negligible complement.

Proof. Fix $k = |V|$. Let F be the set of all polynomials $f \in \mathbb{R}[X_1, \ldots, X_{dk}]$ with integer coefficients. We note that the set of algebraic placements is exactly

$$V(F) := \{x \in \mathbb{R}^{dk} : f(x) = 0 \text{ for all } f \in F\},$$

thus the result holds as $V(F)$ is an *algebraic set* (see Corollary B.3.7). $\square$

Corollary 4.4.9. Let V be a countable set. Then the set of algebraically generic placements of V are dense in $(\mathbb{R}^d)^V$ with respect to the box topology.

Proof. This follows Lemma 4.4.8 and Proposition 4.1.2 $\square$

Proposition 4.4.10. Let $X := \mathbb{R}^d$ have the standard Euclidean norm and (G, p) be an algebraically generic framework in X. Then (G, p) is generic. Further, if (G, p) is finite, then every placement $q \in f_G^{-1}[f_G(p)]$ is regular, and $f_G^{-1}[f_G(p)]$ is a C^1-manifold with dimension $\dim \mathcal{F}(G, p)$.

Proof. To see that (G, p) is generic we apply [15, Proposition 3.1] to any finite subframework. Suppose (G, p) is finite. By [15, Proposition 3.3], every point $q \in f_G^{-1}[f_G(p)]$ is regular. As X is Euclidean, every regular placement is well-positioned and constant, thus by Proposition 1.2.7 and Lemma 1.2.8, f_G is C^1-differentiable on $\mathcal{W}(G)$ and has constant rank at each point $q \in f_G^{-1}[f_G(p)]$. By Corollary 2.1.3, $f_G^{-1}[f_G(p)]$ is a C^1-manifold with dimension $\dim \mathcal{F}(G, p)$ as required. $\square$

For a framework (G, p) we shall denote by $f_G^{-1}[f_G(p)]^\Gamma$ the path-connected component of the configuration space of (G, p) that contains p.

Lemma 4.4.11. Let $X := \mathbb{R}^d$ have the standard Euclidean norm. Suppose (G, p) is a finite algebraically generic framework, then

$$f_G^{-1}[f_G(p)]^\Gamma = f_{\langle G \rangle}^{-1}[f_{\langle G \rangle}(p)]^\Gamma.$$

Proof. By Proposition 4.4.10 and Lemma 4.1.16, both $f_G^{-1}[f_G(p)]^\Gamma$ and $f_{\langle G\rangle}^{-1}[f_{\langle G\rangle}(p)]^\Gamma$ are smooth manifolds of equal dimension. We note immediately that $f_{\langle G\rangle}^{-1}[f_{\langle G\rangle}(p)]^\Gamma$ is a closed C^1-submanifold of $f_G^{-1}[f_G(p)]^\Gamma$. Since both have the same dimension we also have that $f_{\langle G\rangle}^{-1}[f_{\langle G\rangle}(p)]^\Gamma$ is an open smooth submanifold of $f_G^{-1}[f_G(p)]^\Gamma$. As $f_G^{-1}[f_G(p)]^\Gamma$ is connected then its only clopen non-empty subset is itself, hence the result holds. □

Theorem 4.4.12. Suppose (G, p) is infinitesimally rigid and algebraically generic in $X := \mathbb{R}^d$ with the standard Euclidean norm, then (G, p) is continuously rigid.

Proof. Since (G, p) is infinitesimally rigid, by Proposition 1.3.24, (G, p) is spanning. Let $\alpha : (-\delta, \delta) \to X^{V(G)}$ be a finite flex of (G, p), then as (G, p) is spanning in a Euclidean space, α is trivial if and only if

$$\|\alpha_v(t) - \alpha_w(t)\| = \|p_v - p_w\|$$

for all $v, w \in V(G)$ and $t \in (-\delta, \delta)$ by Proposition 1.1.33. By Theorem 4.1.7, there exists a vertex-complete relatively infinitesimally rigid tower $((G^n, p^n))_{n\in\mathbb{N}}$ of (G, p), where we may assume (G^1, p^1) is spanning. Then for each $n \in \mathbb{N}$,

$$\mathcal{F}(G^{n+1}, p^{n+1}) = \mathcal{F}(G^{n+1} \cup K_{V(G^n)}, p^{n+1}).$$

As (G, p) is algebraically generic it is also completely well-positioned, thus by Lemma 4.1.19

$$\left\langle G^{n+1} \right\rangle = \left\langle G^{n+1} \cup K_{V(G^n)} \right\rangle$$

for all $n \in \mathbb{N}$.

By Lemma 4.4.11, for each $n \in \mathbb{N}$,

$$f_{G^{n+1}}^{-1}[f_{G^{n+1}}(p^{n+1})]^\Gamma \quad = \quad f_{\langle G^{n+1}\rangle}^{-1}[f_{\langle G^{n+1}\rangle}(p^{n+1})]^\Gamma$$

$$
\begin{aligned}
&= f^{-1}_{\langle G^{n+1} \cup K_{V(G^n)} \rangle}[f_{\langle G^{n+1} \cup K_{V(G^n)} \rangle}(p^{n+1})]^{\Gamma} \\
&= f^{-1}_{G^{n+1} \cup K_{V(G^n)}}[f_{G^{n+1} \cup K_{V(G^n)}}(p^{n+1})]^{\Gamma}.
\end{aligned}
$$

Choose any $v, w \in V(G)$, then there exists $n \in \mathbb{N}$ such that $v, w \in V(G^n)$. We now note that $\alpha(t)|_{V(G^{n+1})} \in f^{-1}_{G^{n+1} \cup K_{V(G^n)}}[f_{G^{n+1} \cup K_{V(G^n)}}(p^{n+1})]^{\Gamma}$, thus

$$\|\alpha_v(t) - \alpha_w(t)\| = \|p_v - p_w\|$$

for all $t \in (-\delta, \delta)$ as required. $\square$

Chapter 5

Further research and open problems

The natural avenue of research would be to expand geometric rigidity theory in normed spaces to other classical topics such as redundant and global rigidity [26], universal rigidity [22], formation control [55] and periodic symmetry-forced rigidity [9]. We have below a list of some other areas of research and open problems that stem from our previous results.

5.1 Expanding Theorem 2.1.5 to a larger class of frameworks

While Theorem 2.1.5 requires that a framework is constant, Theorem 1.3.19 only requires that the framework is regular. For a large class of normed spaces, regular implies constant - i.e. those with an open set of smooth points, see Proposition 2.1.1 - however there do exist normed spaces without an open set of smooth points such as Example 1.1.19.

Some hope in how to remedy this can be seen in a paper by F. H. Clarke [14]. Given a Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}^n$ he notes that by Rademacher's theorem, f is differentiable on a set U where U^c is negligible. It follows we can define for each $x_0 \in \mathbb{R}^n$ the set $Df(x)$ to be the convex hull of all linear operators T where there exists a sequence of differentiable points $(x_n)_{n\in\mathbb{N}}$ such that $x_n \to x$ as $n \to \infty$ and $T = \lim_{n\to\infty} df(x_n)$, i.e.

$$Df(x) := \operatorname{conv}\left\{\lim_{n\to\infty} df(x_n) : x_n \in U,\ x_n \to x \text{ as } n \to \infty\right\}.$$

As noted in [14, Proposition 1], each $Df(x)$ is a compact convex set, and if the map $df : U \to \mathbb{R}^n$ is continuous then $Df(x) = \{df(x)\}$. Using this generalisation, Clarke forms a version of the Inverse Function Theorem for Lipschitz functions.

Theorem 5.1.1. [14, Theorem 1] Suppose $Df(x_0)$ has maximal rank i.e. for all $T \in Df(x_0)$, T is bijective. Then there exists open neighbourhoods U and V of x_0 and $f(x_0)$ respectively and Lipschitz function $g : V \to U$ such that $g(f(u)) = u$ and $f(g(v)) = v$ for all $u \in U$ and $v \in V$.

To be able to utilise Theorem 5.1.1 with the rigidity map we would need to naturally extend the definition of regular and constant frameworks to non-well-positioned frameworks. Some viable definitions would be as follows:

(i) A finite framework (G,p) is *regular* if $Df_G(p)$ has maximal rank i.e. for all $T \in Df_G(p)$ and $S \in Df_G(q)$ for some $q \in X^{V(G)}$, $\operatorname{rank} T \geq \operatorname{rank} S$

(ii) A finite framework (G,p) is *constant* if there exists an open neighbourhood U of p and $k \in \mathbb{N}$ such that for all $q \in U$ and $T \in Df_G(q)$, $\operatorname{rank} T = k$.

We would then also define a framework to be *infinitesimally rigid* if for all $T \in Df_G(p)$, $\ker T = \mathcal{T}(p)$ and *independent* if for all $T \in Df_G(p)$, T is surjective. Using these definitions we would make the following conjectures.

Conjecture 5.1.2. If (G, p) is regular then (G, p) is constant.

Conjecture 5.1.3. Let (G, p) be a constant finite framework in X, then the following are equivalent:

(i) (G, p) is infinitesimally rigid in X,

(ii) (G, p) is locally rigid in X,

(iii) (G, p) is continuously rigid in X.

It is possible that Conjecture 5.1.2 will fail but we can still apply Theorem 5.1.1 to obtain a similar result, in which case we conjecture the following.

Conjecture 5.1.4. Let (G, p) be a regular finite framework in X, then the following are equivalent:

(i) (G, p) is infinitesimally rigid in X,

(ii) (G, p) is locally rigid in X,

(iii) (G, p) is continuously rigid in X.

Much of the work required would be towards obtaining some version of the Constant Rank Theorem for Lipschitz functions. This would allow us to obtain a *Lipschitz manifold*, a topological manifold where we require each chart is a Lipschitz map; this is a weaker condition than C^1-manifolds but stronger than topological manifolds.

5.2 Infinitesimal rigidity in linear metric spaces

All of the theory we have used is for finite dimensional normed spaces, however there is no reason that we cannot extend the ideas presented to a larger class of spaces; namely *(finite dimensional) linear metric spaces*, spaces $(\mathbb{R}^n, d)$ where addition and scalar multiplication are continuous. Any such space will be a Hausdorff *topological vector space* and all Hausdorff topological vector spaces of the same dimension are TVS isomorphic (i.e. there exists a bijective linear homeomorphism between any two). This implies there will exist a bijective homeomorphism from $(\mathbb{R}^n, d)$ to $\mathbb{R}^n$ with the standard Euclidean topology.

Example 5.2.1. A motivating example would be $(\mathbb{R}^n, d_q)$, where $q \in (0,1)$ and

$$d_q(x,y) := \sum_{i=1}^{n} |x(i) - y(i)|^q$$

for all $x = (x(1), \ldots, x(n))$, $y = (y(1), \ldots, y(n)) \in \mathbb{R}^n$. The metric is differentiable at all values with no zero coordinate, and the set of isometries is identical to any n-dimensional non-Euclidean ℓ_q-normed space, so we may talk about infinitesimal rigidity with no issues.

There would be two main objectives:

(i) Extending Theorem 2.1.5 to linear metric spaces.

(ii) Obtaining a combinatorial theory similar to Theorem 1.3.20 and Theorem 3.4.2 for linear metric planes.

To work on either objective we would need to know what it means for a metric to "differentiable". Some possibilities are given in [7], though fortunately for linear metric spaces all are equivalent, bar a version of Gâteaux differentiation which they all imply.

We would also wish for the smooth points of our metric to have a negligible complement. This does not seem to be immediately true, however if we restrict to linear metric spaces that are *Lipschitz equivalent* to the Euclidean normed space of the same dimension (i.e. there exists a bijective Lipschitz map between them that has Lipschitz inverse) then this will be automatic.

Much of the differential geometry of linear metric spaces would need to be understood, especially the isometries of a given space. If $(\mathbb{R}^n, d)$ is *translation-invariant* ($d(x+z, y+z) = d(x,y)$ for all $x, y, z \in \mathbb{R}^n$) then we can show that $\mathrm{Isom}(\mathbb{R}^n, d)$ is a closed subgroup of the affine maps on $\mathbb{R}^n$ in the topology of pointwise convergence, see [13, Theorem 1] for more details. We cannot say the same if $(\mathbb{R}^n, d)$ is not translation-invariant, though it is possible that some similar result exists.

For Objective (i) we would need to show two things. Firstly, we would wish to show that given a "suitably nice" placement, the configuration space is some kind of manifold in a neighbourhood of the placement. After that, we would need to prove that the orbit of any placement is a submanifold of the configuration space, after which we can use basic differentiable geometry to prove our required result.

Regarding Objective (ii), we would wish to break this down into set cases. We would conjecture the following.

Conjecture 5.2.2. Let $(\mathbb{R}^2, d)$ be a linear metric plane. Then $\mathrm{Isom}(\mathbb{R}^2, d)$ is a smooth manifold of dimension $k \in \{0, 1, 2, 3\}$.

If Conjecture 5.2.2 was true, and we have a good idea of what infinitesimal rigidity is in a given linear metric plane, we would further conjecture the following.

Conjecture 5.2.3. Let $(\mathbb{R}^2, d)$ be a linear metric plane with $\dim \mathrm{Isom}(\mathbb{R}^2, d) = k$ for $k \in \{0, 1, 2, 3\}$. Then G is minimally rigid in $(\mathbb{R}^2, d)$ if and only if G is $(2, k)$-tight.

This highlights the more interesting properties at play for graphs in a linear metric plane, especially if they lack translation-invariance.

It is worth noting the work of Stacey and Mahoney [65], who generalised Euclidean rigidity by replacing normed spaces with isometry groups to smooth manifolds with a Lie group acting on them. Although this research would not be the same, both draw from the wish to generalise to more unusual geometries.

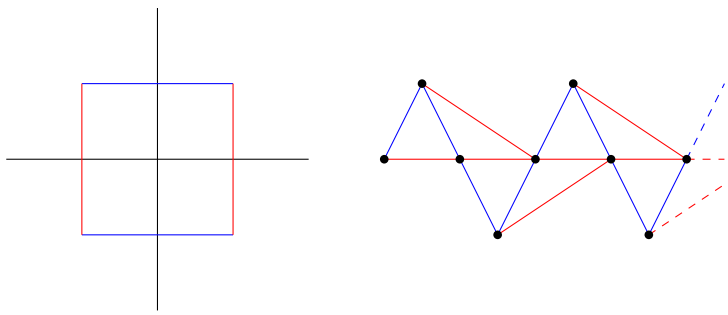

Fig. 5.1 (Left) Unit ball of ℓ_∞-normed plane; (right) a framework in ℓ_∞-normed plane that is infinitesimally rigid but contains no $(2,2)$-tight vertex-complete tower.

5.3 Open problems regarding countable frameworks

5.3.1 Edge partitions

We direct the reader to Figure 5.1 as an example of an infinitesimally rigid, but not sequentially infinitesimally rigid, framework in the ℓ_∞-normed plane. The red edges have edge support functional $(x,y) \mapsto x$ and the blue edges have edge support functional $(x,y) \mapsto y$, for any $(x,y) \in \mathbb{R}^2$ (see [34] for more details).

Figure 5.1 is an interesting example, as in it leads us to believe that rather than looking at density counts as a necessary condition to whether a graph has an infinitesimally rigid placement, we should instead consider tree partitions. We first define the following for Conjecture 5.3.1.

Let $k,l \in \mathbb{N}$. A graph G has a *lTk-partition* if there exists l edge-disjoint trees $T_1, \ldots, T_l \subset G$ such that every edge of G lies in exactly one of the trees and every vertex of G lies in exactly k of the trees. A lTk-partition is *proper* if for every finite set $U \subsetneq V$,

$$\sum_{i=1}^{l} |\{H \subseteq G[U] \cap T_i : H \text{ is a connected component of } G[U] \cap T_i\}| \geq l.$$

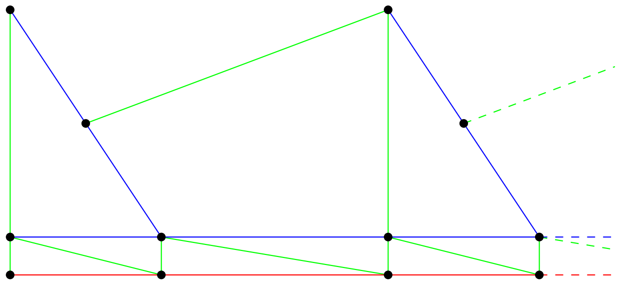

Fig. 5.2 A graph with a proper $3T2$-partition but contains no $(2, 3)$-tight tower. It was shown in [37] that the above does have an infinitesimally rigid placement in the Euclidean plane

The two types of partition we shall be interested in are $3T2$-partitions and $2T2$-partitions. We note that every $2T2$-partition is proper and a $2T2$-partition is equivalent to the graph containing exactly two edge-disjoint spanning trees. For a finite graph G the following holds:

(i) G has a proper $3T2$-partition if and only if G is $(2, 3)$-tight ([17, Theorem 1]).

(ii) G has a $2T2$-partition if and only if G is $(2, 2)$-tight (Corollary 3.4.5).

It is not so difficult to show that if a graph contains a $(2, k)$-tight tower then it has a proper $kT2$-partition for $k \in \{2, 3\}$, thus if G has a sequentially isostatic placement, by Theorem 4.3.12, G has a proper $3T2$-partition. The reverse, however, is not true; for a counter-example we note that Figure 5.1 has a $2T2$-partition, however it does not contain a $(2, 2)$-tight tower. Another example is given in Figure 5.2; while the graph has a proper $3T2$-partition it does not contain a $(2, 3)$-tight tower. Both examples, however, have an infinitesimally rigid placement in the ℓ_∞-normed plane and the Euclidean plane respectively. This leads to the following conjecture.

Conjecture 5.3.1. Suppose (G, p) is a countably infinite isostatic framework in normed plane X, then the following holds:

(i) If X is Euclidean then G has a proper $3T2$-partition.

(ii) If X is non-Euclidean then the edges of G can be partitioned into 2 edge-disjoint spanning trees.

If this holds to be true, we would also wish for the following, much stronger, conjecture to hold.

Conjecture 5.3.2. Let G be a countable graph and X a normed plane, then the following holds:

(i) If X is Euclidean, G has an isostatic placement in X if and only if G has a proper $3T2$-partition.

(ii) If X is non-Euclidean, G has an isostatic placement in X if and only if G can be partitioned into 2 edge-disjoint spanning trees.

It is possible that we would need to restrict ourselves to generic spaces, however the hope would be that this is not the case.

5.3.2 Sequences of graph operations

The following questions stem from research by Derek Kitson and Stephen Power [39].

We shall define a *graph operation sequence* to be a sequence of graphs $(G^n)_{n\in\mathbb{N}}$ so that G^{n+1} is obtained by applying a finite sequence of graph operation to G^n; just 0-extensions and 1-extensions if X is Euclidean, or 0-extensions, 1-extensions, vertex splitting and vertex-to-K_4 if X is non-Euclidean.

Let G have a sequentially isostatic placement in a normed plane X and let $k = \dim \operatorname{Isom}(X) \in \{2,3\}$. We note from the proof of Theorem 4.3.12 that there exists a graph operation sequence $(G^n)_{n\in\mathbb{N}}$ such that each graph is $(2,k)$-tight; we shall call this a $(2,k)$-*tight sequence.* The obvious question is, if given a $(2,k)$-tight sequence, do we have a good idea of its "limit" and what can we say about such a graph?

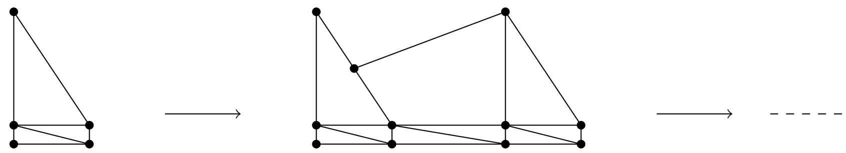

Fig. 5.3 A $(2,3)$-tight sequence with the graph given in Figure 6 as the limit.

Suppose we have graph operation sequence $(G^n)_{n\in\mathbb{N}}$. We can define the limit to be the graph G with

$$V(G) := \bigcup_{n\in\mathbb{N}} V(G^n), \qquad E(G) := \bigcup_{N\in\mathbb{N}} \bigcap_{n\geq N} E(G^n);$$

we may define $V(G)$ as such since $V(G^n) \subset V(G^{n+1})$ for all $n \in \mathbb{N}$.

For each edge $e \in E(G^n)$ we define an *edge sequence* to be a sequence of edges $(e_i)_{i\geq n}^{\infty}$ such that $e_i \in E(G^i)$, $e_n = e$ and e_{i+1} is either e_i or one of the edges that replaces e_i during some graph operation move between G^i and G^{i+1}. If every edge sequence converges (i.e. is eventually constant) then we define $(G^n)_{n\in\mathbb{N}}$ to be *edge stable*.

We will require edge stability to be a property if we wish to say anything about the limit; for instance if we take G^1 to be any $(2,k)$-tight graph and obtain G^{n+1} from G^n by applying 1-extensions to every edge of G^n, then $(G^n)_{n\in\mathbb{N}}$ is not edge stable and $G \cong (\mathbb{N}, \emptyset)$, a graph with only has infinitesimally flexible placements. It follows from Lemma 4.3.10 that any $(2,k)$-tight tower is an edge stable $(2,k)$-tight sequence, however not every edge stable $(2,k)$-tight sequence has a limit G that contains a $(2,k)$-tight tower, see Figure 5.3 for an example. We instead conjecture the following.

Conjecture 5.3.3. Let (G^n) be an edge stable graph operation sequence with limit G, X a normed plane and $k := \dim \operatorname{Isom}(X)$. Then G has an infinitesimally rigid placement in X if and only if $(G^n)_{n\in\mathbb{N}}$ is a $(2,k)$-tight sequence.

It is yet again possible that we would need to restrict ourselves to generic spaces, however the hope would be that this is not the case.

Appendix A

Background on manifolds and matroids

A.1 Manifolds

We shall outline some of the more crucial ideas involving manifolds here, however we refer the reader to [47] for more background on the topic.

A.1.1 Basic definitions

We are reminded for this section that only finite dimensional normed spaces are considered, however many of the definitions and results carry through with little amount of changes to infinite dimensional Banach spaces; see [47] for more detail.

Let S be any set. A *chart* of S is a pair (U, ϕ) such that $U \subset S$, $\phi : U \to X$ (for some normed space X) is injective and $\phi(U)$ is open in X; we define $\dim X$ to be the dimension of (U, ϕ), or $\dim(U, \phi)$ for short. Given $k \in \mathbb{N} \cup \{0, \infty\}$, a C^k-*atlas* of S is a set $\mathcal{A} = \{(U_i, \phi_i) : i \in I\}$ of charts of S such that $\bigcup_{i \in I} U_i = S$ and for any $i, j \in I$ with $U_i \cap U_j \neq \emptyset$ (i.e. the charts (U_1, ϕ_1) and (U_2, ϕ_2) *intersect non-trivially*), the bijective map

$$\phi_{j,i} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j), \ x \mapsto \phi_j \circ \phi_i^{-1}(x)$$

is a C^k-diffeomorphism. It follows from Brouwer's theorem for invariance of domain [44, Theorem 1.18] that if (U_1, ϕ_1) and (U_2, ϕ_2) intersect non-trivially then $\dim(U_1, \phi_1) = \dim(U_2, \phi_2)$.

For $k \in \mathbb{N} \cup \{0, \infty\}$, we define that two C^k-atlases $\mathcal{A}_1$ and $\mathcal{A}_2$ of S are *C^k-equivalent* if $\mathcal{A}_1 \cup \mathcal{A}_2$ is a C^k-atlas of S also. As this forms a equivalence relation on the set of all possible atlases of S, we define a *C^k-differential structure* of S to be any equivalence class $\mathcal{D}$ of C^k-equivalent C^k-atlases of S. It follows that given a C^k-atlas $\mathcal{A}$ of S we may generate a unique differential structure $\mathcal{D}_{\mathcal{A}}$ from $\mathcal{A}$.

We define a *C^k-manifold* to be a pair $M := (S, \mathcal{D})$ where $\mathcal{D}$ is a C^k-differential structure of the set S; if $k = 0$ we define M to be a *topological manifold* and if $k = \infty$ we define M to be a *smooth manifold.* Given a C^k-manifold $M = (S, \mathcal{D})$, we may define the *maximal atlas* of M to be the C^k-atlas

$$\mathcal{A}_{\mathcal{D}} := \{\mathcal{A} : \mathcal{A} \in \mathcal{D}\}.$$

We shall often by abuse of notation refer to S as M and a chart $(U, \phi) \in M$ if (U, ϕ) lies in the maximal atlas of M.

We may now define a topology on M by defining $U \subset M$ to be open if and only if $(U, \phi) \in \mathcal{A}_{\mathcal{D}}$. We note that for any path-connected connected component $C \subset M$, if (U_i, ϕ_i) for $i = 1, 2$ are charts with $U_i \subset C$, then $\dim(U_1, \phi_1) = \dim(U_2, \phi_2)$. If all charts have dimension d then we define M to be d-dimensional.

Proposition A.1.1. Let M be a connected C^k-manifold, then M is a path-connected C^k-manifold.

Proof. Choose $x \in M$ and let C be the path-connected component of M containing x. Choose any $y \in C$ and $y' \in M \setminus C$ and any charts (U, ϕ), $(U', \phi') \in M$ containing y and y' respectively. As $\phi(U)$ and $\phi'(U')$ are open we may without loss of generality assume that both are convex. For any $z \in U$ we may define the line $L : [0, 1] \to \phi(U)$ with $L(t) := t\phi(z) + (1 - t)\phi(y)$. As $\phi^{-1} \circ L$ is a continuous path from y to z, it follows that $z \in C$, thus $U \subset C$. By a similar method we note that $U' \subset M \setminus C$. Since y, y' were chosen arbitrarily it follows that C and $M \setminus C$ are open, thus C is a clopen set. As M is connected, $C = M$ as required. □

Remark A.1.2. We have at no point assumed that our manifolds will be Hausdorff or separable. Fortunately, all of the manifolds mentioned in this body of text lie inside some separable metric space, thus the Hausdorff property and separability may be assumed.

Example A.1.3. Define the equivalence relation $\sim$ on $\mathbb{R} \times \{0, 1\}$ with $(x, 0) \sim (x, 1)$ if $x \neq 0$. We now define the *line with two ends*, the topological manifold $S := \mathbb{R} \times \{0, 1\}/ \sim$ with the inherited topology. The line with two ends is not Hausdorff as any neighbourhood of $(0, 0)$ contains a neighbourhood of $(0, 1)$.

Example A.1.4. We define the *long line* to be the set $S := [0, 1) \times \mathbb{R}$. For any two points $z_1 := (x_1, y_1)$ and $z_2 := (x_2, y_2)$ in S, we define $z_1 < z_2$ if $y_1 < y_2$ or $y_1 = y_2$ and $x_1 < x_2$. This is a total ordering on S and so can be used to generate a topology for S. Under this topology, S is a topological manifold, however it is not separable.

A.1.2 Tangent spaces

Let M be a C^k-manifold for $k \in \mathbb{N} \cup \{\infty\}$ and let each $\gamma_i : (a_i, b_i) \to M$ for $i \in \{1, 2\}$ be a C^1-*differentiable path*, i.e. for any point $t \in (a, b)$ and any chart $(U, \phi) \in M$ that contains $\gamma_i(t)$, the map $\phi \circ \gamma_i$ is C^1-differentiable (as a map between normed spaces).

Suppose γ_1 and γ_2 *pass through* $x \in M$ i.e. $x := \gamma_1(t_1) = \gamma_2(t_2)$ for some $t_1 \in (a_1, b_1)$ and $t_2 \in (a_2, b_2)$. Then γ_1, γ_2 are *tangent at* x if there exists a chart $(U, \phi) \in M$ containing x such that $(\phi \circ \gamma)'(t_1) = (\phi \circ \gamma)'(t_2)$; if this holds for one chart then it will hold for all charts that contain x, see [47, Proposition 3.3.2].

As this is a equivalence relation on all the C^1-differentiable paths γ passing through a point $x \in M$, we define the equivalence classes $[\gamma]_x$ of all C^1-differentiable paths that pass through x that are tangent with γ at x. We now define the *tangent space* of M at x to be the set

$$T_x M := \{[\gamma]_x : \gamma \text{ passes through } x\}$$

and the *tangent bundle*

$$TM := \{[\gamma]_x : x \in X,\ \gamma \text{ passes through } x\}$$

Remark A.1.5. Let $(U, \phi) \in M$ be a C^k-chart at $x \in M$ with $\phi : U \to X$. Choose $y \in X$ and define the smooth path

$$c : (-\delta, \delta) \to \phi(U),\ t \mapsto \phi(x) + ty,$$

then $\phi \circ c$ is a C^1-differentiable curve that passes through x, and $(\phi \circ c)'(0) = y$. We can similarly prove that every element $[\gamma]_x$ can be associated to some vector $y \in X$ given a map ϕ [47, Lemma 3.3.4]. It follows that $T_m M$ is a linear space that is linearly isomorphic to X.

A.1.3 Maps between manifolds

Let M and N be C^k-manifolds. We define a map $f : M \to N$ to be C^k*-differentiable* if for all $x \in M$ and chart (V, ψ) of N with $f(x) \in N$ there exists a chart (U, ϕ) of M such that $x \in U$, $f(U) \subset V$ and $\psi \circ f \circ \phi^{-1}$ is C^k-differentiable on U; in fact, we need only show that for each $x \in M$ this holds for a single chart (V, ψ) with $f(x) \in V$ [47, Proposition 3.2.6]. If f is bijective with C^k-differentiable inverse then f is a C^k*-diffeomorphism*.

Remark A.1.6. If M and N are normed spaces we note that $f : M \to N$ is C^k-differentiable as a map between manifolds if and only if it is C^k-differentiable as a map between normed spaces.

Given a C^k-differentiable map $f : M \to N$ we may define for each $x \in M$ the map

$$df(x) : T_m M \to T_{f(m)} M, \; [\gamma]_m \mapsto [f \circ \gamma]_{f(m)}.$$

The map is well defined (see [47, Lemma 3.3.5]) and also linear (this follows from methods similar to Remark A.1.5); from this it follows that $df(x)(T_x M)$ is a linear subspace of $T_{f(x)} N$. We define the following properties for f:

(i) f is a *submersion* if $df(x)$ is surjective for all $x \in M$.

(ii) f is an *immersion* if $df(x)$ is injective for all $x \in M$.

(iii) f is a *local diffeomorphism* if it is both a submersion and an immersion.

If f is a local diffeomorphism then at any point $x \in M$ there exists neighbourhoods U and V of x and $f(x)$ respectively so that $f|_U^V$ is a C^k-diffeomorphism [47, Theorem 3.5.1]; further, if f is a bijective local diffeomorphism then f is a C^k-diffeomorphism.

A.1.4 Submanifolds

Given a C^k-manifold M, a set $N \subset M$ is a C^k*-submanifold* of M if for all $x \in N$, there exists a chart (U, ϕ) of M such that $x \in U$ and ϕ has the *submanifold property*, i.e. $\phi : U \to X \times Y$ for some normed spaces X, Y, and

$$\phi(U \cap N) = \phi(U) \cap (X \times \{0\}).$$

The submanifold N is itself a C^k-manifold with differential structure generated by the C^k-atlas

$$\{(U \cap N, \phi|_N) : U \cap N \neq \emptyset,\ (U, \phi) \in M \text{ and has the submanifold property}\};$$

further, the topology generated will be the relative topology of $N \subset M$ [47, Proposition 3.2.2]. We notice immediately that any open subset of a manifold will trivially be a submanifold.

Example A.1.7. Let M_n be the linear space of $n \times n$ matrices with real coefficients, and let GL_n be the subset of M_n of invertible matrices. As the determinant map $\det : M_n \to \mathbb{R}$ is continuous and $GL_n = M_n \setminus \det\{0\}$ then GL_n is an open subset of M_n, thus GL_n is a smooth submanifold of M_n.

Proposition A.1.8. Let M be a C^k-manifold for $k \in \mathbb{N} \cup \{\infty\}$ and N, N' a C^k-submanifolds of M. Then the following hold:

(i) The inclusion map $\iota : N \hookrightarrow M$ is a C^k-differentiable immersion and a closed map.

(ii) If $\dim N = \dim M$ then N is an open subset of M.

(iii) If $N \subset N'$ then N is a C^k-submanifold of N'.

Proof. (i): We must first prove ι is a C^k-differenitable map. Choose a point $x \in X$ and a chart (V, ψ) such that $\iota(x) = x \in V$. As N is a submanifold we may choose a chart $(U, \phi) \in M$ that contains x with the submanifold property such that $U \subset V$. It now follows that the map (restricted to $\phi(U \cap N)$) $\psi \circ \iota \circ \phi^{-1} = \psi \circ \phi^{-1}$, thus as M is a manifold, $\psi \circ \iota \circ \phi^{-1}$ (restricted to $\phi(U \cap N)$) is a C^k-differentiable map.

As ι is an inclusion map it is automatically closed. We now note that for each $x \in N$, $d(\iota)(x)([\gamma]_x) = [\gamma]_x$, thus ι is an immersion also.

(ii): Suppose $\dim N = \dim M$. We note that ι is a submersion also as $\dim T_x N = \dim T_x M$ for all $x \in N$. Choose any point $x \in N$, then there exists a chart $(U, \phi) \in M$ that contains x with $\phi : U \to X \times Y$ such that

$$\phi(U \cap N) = \phi(U) \cap (X \times \{0\}).$$

Since $\dim N = \dim M$ then $\dim X \times \{0\} = \dim X \times Y$, thus $Y = \{0\}$. This implies (U, ϕ) is a chart of N, thus U is an open neighbourhood of x in N as required.

(iii): The inclusion map from N to N' can be seen to be an immersion, thus the result follows. $\square$

Remark A.1.9. Let X be a normed space. For each $x, y \in X$ we can define the smooth path

$$\gamma_y : (-1, 1) \to X, \ t \mapsto x + ty.$$

We can now define a linear isomorphism

$$I : X \to T_x X, \ y \mapsto [\gamma_y]_x.$$

The map I gives us a natural way to embed the tangent spaces of each point of X inside X as a subspace. Further, for any C^k-submanifold M of X with $x \in M$, by Proposition A.1.8 (i), ι is an immersion, thus we can embed the tangent space T_xM into X as a linear subspace under the unique injective linear map $I^{-1} \circ d\iota(x)$.

A.2 Matroids

A.2.1 Finite matroids

Definition A.2.1. A *matroid* is a pair $M = (S, \mathcal{I})$ where S is a finite set and $\mathcal{I} \subset \mathcal{P}(S)$ is a set where the following holds:

(i) **I1**: $\emptyset \in \mathcal{I}$.

(ii) **I2**: If $I_1 \subset I_2$ and $I_2 \in \mathcal{I}$ then $I_1 \in \mathcal{I}$.

(iii) **I3**: If $I_1, I_2 \in \mathcal{I}$ and $|I_1| < |I_2|$ then there exists $e \in I_2$ such that $I_1 \cup \{e\} \in \mathcal{I}$.

We define any set in $\mathcal{I}$ to be *independent* and any set not in $\mathcal{I}$ to be *dependent*.

We may also define the following combinatorial structures for a matroid $(S, \mathcal{I})$:

(i) Any maximally independent subset of S (with respect to set inclusion) is a *base*.

(ii) Any minimally dependent subset of S (with respect to set inclusion) is a *circuit*.

(iii) The *rank function* is the map $r : \mathcal{P}(S) \to \mathbb{N} \cup \{0\}$ where

$$r(A) := \max \{|I| : I \subset A,\ I \in \mathcal{I}\}.$$

(iv) The *closure operator* is the map $\langle\cdot\rangle : \mathcal{P}(S) \to \mathcal{P}(S)$ where

$$\langle A \rangle := \{x \in S : x \in A \text{ or } I \cup \{x\} \notin \mathcal{I} \text{ for some independent } I \subset A\}.$$

There are many equivalent definitions for a matroid, in the sense that the combinatorial structures defined will generate the same set of independent sets. Here we outline a few.

Definition A.2.2. A *matroid* is a pair $M = (S, \mathcal{B})$ where S is a finite set and $\mathcal{B} \subset \mathcal{P}(S)$ is a set where the following holds:

(i) **B1**: $\mathcal{B} \neq \emptyset$.

(ii) **B2**: If $B_1, B_2 \in \mathcal{B}$ and $e \in B_1 \setminus B_2$ then there exists $f \in B_2 \setminus B_1$ such that $(B_1 \setminus \{e\}) \cup \{f\} \in \mathcal{B}$.

We define any set in $\mathcal{B}$ to be a *base*, and define any subset of a base to be *independent* and any set that is not a subset of a base to be *dependent*.

Definition A.2.3. A *matroid* is a pair $M = (S, \mathcal{C})$ where S is a finite set and $\mathcal{C} \subset \mathcal{P}(S)$ is a set where the following holds:

(i) **C1**: $\emptyset \notin \mathcal{C}$.

(ii) **C2**: If $C_1, C_2 \in \mathcal{C}$ and $C_1 \subset C_2$ then $C_1 = C_2$.

(iii) **C3**: If $C_1, C_2 \in \mathcal{C}$, $C_1 \neq C_2$ and $e \in C_1 \cap C_2$ then there exists $C_3 \in \mathcal{C}$ such that $C_3 \subset (C_1 \cup C_2) \setminus \{e\}$.

We define any set in $\mathcal{C}$ to be a *circuit*, and define any set that does not contain a circuit to be *independent* and any set that contains a circuit to be *dependent*.

Definition A.2.4. A *matroid* is a pair $M = (S, r)$ where S is a finite set and $r : \mathcal{P}(S) \to \mathbb{N} \cup \{0\}$ is a function where the following holds:

(i) **R1**: For all $A \subseteq S$, $0 \leq r(A) \leq |A|$.

(ii) **R2**: If $A \subseteq B \subseteq S$ then $r(A) \leq r(B)$.

(iii) **R3**: For all $A, B \subseteq S$,

$$r(A \cup B) + r(A \cap B) \leq r(A) + r(B).$$

We define r to be the *rank function*, and define any $A \subseteq S$ to be *independent* if $r(A) = |A|$ and *dependent* otherwise.

Definition A.2.5. A *matroid* is a pair $M = (S, \langle \cdot \rangle)$ where S is a finite set and $r : \mathcal{P}(S) \to \mathcal{P}(S)$ is a map where the following holds:

(i) **CL1**: For all $A \subseteq S$, $A \subseteq \langle A \rangle$.

(ii) **CL2**: For all $A \subseteq S$, $\langle \langle A \rangle \rangle = \langle A \rangle$.

(iii) **CL3**: For all $A \subseteq B \subseteq S$, $\langle A \rangle \subseteq \langle B \rangle$.

(iv) **CL4**: For all $A \subseteq S$ and $e, f \in S \setminus \langle A \rangle$, if $e \in \langle A \cup \{f\} \rangle$ then $f \in \langle A \cup \{e\} \rangle$.

We define $\langle \cdot \rangle$ to be the *closure operator*, and define any $A \subseteq S$ to be *independent* if $e \notin \langle A \setminus \{e\} \rangle$ for all $e \in A$ and *dependent* otherwise.

Theorem A.2.6. Let S be a finite set. Then the following holds:

(i) [57, Theorem 1.2.3] Let $(S, \mathcal{B})$ be a matroid as defined in Definition A.2.2. If $\mathcal{I}$ is the set of independent sets then $(S, \mathcal{I})$ is a matroid as defined in Definition A.2.1 with $\mathcal{B}$ as its set of bases.

(ii) [57, Theorem 1.1.4] Let $(S, \mathcal{C})$ be a matroid as defined in Definition A.2.3. If $\mathcal{I}$ is the set of independent sets then $(S, \mathcal{I})$ is a matroid as defined in Definition A.2.1 with $\mathcal{C}$ as its set of circuits.

(iii) [57, Theorem 1.3.2] Let (S, r) be a matroid as defined in Definition A.2.4. If $\mathcal{I}$ is the set of independent sets then $(S, \mathcal{I})$ is a matroid as defined in Definition A.2.1 with r as its rank function.

(iv) [57, Theorem 1.4.4] Let $(S, \langle\cdot\rangle)$ be a matroid as defined in Definition A.2.5. If $\mathcal{I}$ is the set of independent sets then $(S, \mathcal{I})$ is a matroid as defined in Definition A.2.1 with $\langle\cdot\rangle$ as its closure operator.

Theorem A.2.7. Let $(S, \mathcal{I})$ be a matroid as defined in Definition A.2.1. Then the following holds:

(i) [57, Corollary 1.2.5] If $\mathcal{B}$ is the set of bases of $(S, \mathcal{I})$ then $(S, \mathcal{B})$ is a matroid as defined in Definition A.2.2. Further, the independent sets of $(S, \mathcal{B})$ are exactly the independent set of $(S, \mathcal{I})$.

(ii) [57, Corollary 1.1.5] If $\mathcal{C}$ is the set of circuits of $(S, \mathcal{I})$ then $(S, \mathcal{C})$ is a matroid as defined in Definition A.2.3. Further, the independent sets of $(S, \mathcal{C})$ are exactly the independent set of $(S, \mathcal{I})$.

(iii) [57, Corollary 1.3.4] If r is the rank function of $(S, \mathcal{I})$ then (S, r) is a matroid as defined in Definition A.2.4. Further, the independent sets of (S, r) are exactly the independent set of $(S, \mathcal{I})$.

(iv) [57, Corollary 1.4.6] If $\langle\cdot\rangle$ is the closure operator of $(S, \mathcal{I})$ then $(S, \langle\cdot\rangle)$ is a matroid as defined in Definition A.2.5. Further, the independent sets of $(S, \langle\cdot\rangle)$ are exactly the independent set of $(S, \mathcal{I})$.

A.2.2 Infinite matroids

There are many ways of extending matroids to infinite sets. The two most common are *finitary matroids* and *B-matroids*. We shall only describe finitary matroids, the stronger property out of the two, and we refer the reader to [11].

Definition A.2.8. A *finitary matroid* is a pair $M = (S, \mathcal{I})$ where S is a set and $\mathcal{I} \subset \mathcal{P}(S)$ is a set where the following holds:

(i) **I1**: $\emptyset \in \mathcal{I}$.

(ii) **I2**: $I \in \mathcal{I}$ if and only if for all $J \subset\subset I$, $J \in \mathcal{I}$.

(iii) **I3**: If $I_1, I_2 \in \mathcal{I}$ and $|I_1| < |I_2| < \infty$ then there exists $e \in I_2$ such that $I_1 \cup \{e\} \in \mathcal{I}$.

We define any set in $\mathcal{I}$ to be *independent* and any set not in $\mathcal{I}$ to be *dependent*.

We may also define the following combinatorial structures for a finitary matroid $(S, \mathcal{I})$:

(i) Any maximally independent subset of S (with respect to set inclusion) is a *base*.

(ii) Any minimally dependent subset of S (with respect to set inclusion) is a *circuit*.

(iii) The *closure operator* is the map $\langle \cdot \rangle : \mathcal{P}(S) \to \mathcal{P}(S)$ where

$$\langle A \rangle := \{x \in S : x \in A \text{ or } I \cup \{x\} \notin \mathcal{I} \text{ for some independent } I \subset A\}.$$

Similar to matroids, there are many equivalent definitions for a finitary matroid. Here we outline a few.

Definition A.2.9. A *finitary matroid* is a pair $M = (S, \mathcal{C})$ where S is a set and $\mathcal{C} \subset \mathcal{P}(S)$ is a set where the following holds:

(i) **C1**: $\emptyset \notin \mathcal{C}$.

(ii) **C2**: If $C_1, C_2 \in \mathcal{C}$ and $C_1 \subset C_2$ then $C_1 = C_2$.

(iii) **C3**: If $C_1, C_2 \in \mathcal{C}$, $C_1 \neq C_2$ and $e \in C_1 \cap C_2$ then there exists $C_3 \in \mathcal{C}$ such that $C_3 \subset (C_1 \cup C_2) \setminus \{e\}$.

(iv) **C4**: Every set in $\mathcal{C}$ is finite.

We define any set in $\mathcal{C}$ to be a *circuit*, and define any set that does not contain a circuit to be *independent* and any set that contains a circuit to be *dependent*.

Definition A.2.10. A *finitary matroid* is a pair $M = (S, \langle\cdot\rangle)$ where S is a set and $r : \mathcal{P}(S) \to \mathcal{P}(S)$ is a map where the following holds:

(i) **CL1**: For all $A \subseteq S$, $A \subseteq \langle A\rangle$.

(ii) **CL2**: For all $A \subseteq S$, $\langle\langle A\rangle\rangle = \langle A\rangle$.

(iii) **CL3**: For all $A \subseteq B \subseteq S$, $\langle A\rangle \subseteq \langle B\rangle$.

(iv) **CL4**: For all $A \subseteq S$ and $e, f \in S \setminus \langle A\rangle$, if $e \in \langle A \cup \{f\}\rangle$ then $f \in \langle A \cup \{e\}\rangle$.

(v) **CL5**: $\langle A\rangle = \bigcup_{B \subset\subset A} \langle B\rangle$.

We define $\langle\cdot\rangle$ to be the *closure operator*, and define any $A \subseteq S$ to be *independent* if $e \notin \langle A \setminus \{e\}\rangle$ for all $e \in A$ and *dependent* otherwise.

Theorem A.2.11. Let S be any set. Then the following holds:

(i) Let $(S, \mathcal{C})$ be a matroid as defined in Definition A.2.3. If $\mathcal{I}$ is the set of independent sets then $(S, \mathcal{I})$ is a matroid as defined in Definition A.2.1 with $\mathcal{C}$ as its set of circuits.

(ii) Let $(S, \langle\cdot\rangle)$ be a matroid as defined in Definition A.2.5. If $\mathcal{I}$ is the set of independent sets then $(S, \mathcal{I})$ is a matroid as defined in Definition A.2.1 with $\langle\cdot\rangle$ as its closure operator.

Proof. (i): We first check the independence axioms:

- **I1**: By **C1**, $\emptyset \notin \mathcal{C}$. As the only subset of $\emptyset$ is itself, $\emptyset \in \mathcal{I}$.

- **I2**: Suppose $I \in \mathcal{I}$, then I contains no circuits. It follows that for any subset $J \subset\subset I$, J can also not contain a circuit, thus J is also independent.

  Now suppose that for all $J \subset\subset I$, $J \in \mathcal{I}$, but I is not independent. Then I contains a circuit C. By **C4**, C is a finite subset of I, thus C is independent. As $C \subset C$ and C is a circuit, we obtain a contradiction as required.

- **I3**: The proof is identical to the finite matroid case, see [57, Theorem 1.1.4].

Let C be a circuit of the finitary matroid $(S, \mathcal{I})$, then C is a minimally dependent subset. Due to how we defined our independent sets, this implies any proper subset of C does not contain an element of $\mathcal{C}$ while C does, thus $C \in \mathcal{C}$ as required. Now choose a circuit in $(S, \mathcal{C})$, then C is dependent in $(S, \mathcal{I})$. By **C2**, all proper subsets of C are independent in $(S, \mathcal{I})$, thus C is a circuit of $(S, \mathcal{I})$.

(ii): We first check the independence axioms:

- **I1**: As $\emptyset$ contains no elements it is trivially an independent subset of $(S, \langle\cdot\rangle)$.

- **I2**: Suppose $I \in \mathcal{I}$ but there exists $J \subset\subset I$ such that J is not independent. Then there exists $e \in J$ such that $e \in \langle J \setminus \{e\}\rangle$. We now note by **CL3**,

$$e \in \langle J \setminus \{e\}\rangle \subset \langle I \setminus \{e\}\rangle ,$$

  a contradiction.

  Now suppose that for all $J \subset\subset I$, $J \in \mathcal{I}$, but $I \notin \mathcal{I}$. Then there exists $e \in \langle I \setminus \{e\}\rangle$. By **CL5**, there exists $J \setminus \{e\} \subset\subset I \setminus \{e\}$ such that $e \in \langle J \setminus \{e\}\rangle$, thus J is not independent. However $J \subset\subset I$, a contradiction.

- **I3**: The proof is identical to the finite matroid case, see [57, Theorem 1.4.4].

Let $\langle\cdot\rangle'$ to be the closure operator generated by the finitary matroid $(S,\mathcal{I})$. By Theorem A.2.6 (iv) $\langle A\rangle' = \langle A\rangle$ for all finite sets, thus by **CL5** applied to both closure operators, $\langle\cdot\rangle' = \langle\cdot\rangle$ as required. $\square$

Theorem A.2.12. Let $(S,\mathcal{I})$ be a matroid as defined in Definition A.2.1. Then the following holds:

(i) If $\mathcal{C}$ is the set of circuits of $(S,\mathcal{I})$ then $(S,\mathcal{C})$ is a matroid as defined in Definition A.2.3. Further, the independent sets of $(S,\mathcal{C})$ are exactly the independent set of $(S,\mathcal{I})$.

(ii) If $\langle\cdot\rangle$ is the closure operator of $(S,\mathcal{I})$ then $(S,\langle\cdot\rangle)$ is a matroid as defined in Definition A.2.5. Further, the independent sets of $(S,\langle\cdot\rangle)$ are exactly the independent set of $(S,\mathcal{I})$.

Proof. (i): We first check the circuit axioms (however, we shall prove **C4** before **C3**):

- **C1**: This follows from **I1**.
- **C2**: This follows as the circuits are exactly the minimally dependent subsets.
- **C4**: Suppose A is infinite and dependent, then by **I4**, there exists a finite dependent subset $A' \subset\subset A$. It now follows A is not a minimally dependent subset as required.
- **C3**: As **C4** holds then all circuits are finite. The result now follows from the same method employed in [57, Lemma 1.1.3].

Suppose I is an independent set in the finitary matroid $(S,\mathcal{C})$, then I contains no circuits $C \in \mathcal{C}$. If I is dependent in $(S,\mathcal{I})$ then I would contain a circuit, thus I is independent in $(S,\mathcal{I})$. Now suppose I is a dependent set in the finitary matroid $(S,\mathcal{C})$, then there exists $C \in \mathcal{C}$ such that $C \subset I$. By **C4**, C is finite, thus by **I2** I is dependent in $(S,\mathcal{I})$.

(ii): We note that **CL1-C4** hold for $\langle\cdot\rangle$ restricted to the finite subsets of S, thus if **C5** holds then **CL1-C4** hold also.

CL5: Choose $A \subset S$ and $e \in \langle A\rangle \setminus A$, then there exists an independent set $I \subset A$ where $I \cup \{e\} \notin \mathcal{I}$. By **I2**, there exists a dependent set $J \cup \{e\} \subset\subset I \cup \{e\}$ and $J \in \mathcal{I}$. By definition, $e \in \langle J\rangle$, thus $e \in \bigcup_{B \subset\subset A} \langle B\rangle$. Now choose $e \in B$ for some $B \subset\subset A$, then there exists an independent set $I \subset B$ where $I \cup \{e\} \notin \mathcal{I}$. As $I \subset A$ then $e \in \langle A\rangle$ as required.

Define $\mathcal{I}'$ to be the independent subsets of $(S, \langle\cdot\rangle)$. By Theorem A.2.7 (iv), $\mathcal{I}, \mathcal{I}'$ have the same finite sets. By applying **I2** to both we have that $\mathcal{I} = \mathcal{I}'$. □

While the following proposition is immediate for matroids, it requires slightly more thought for finitary matroids.

Proposition A.2.13. [11, Corollary 4.4] Let $(S, \mathcal{I})$ be a finitary matroid and $A \subset S$. Then A contains an independent subset I such that if $J \subset A$ is independent and $I \subset J$ then $I = J$.

Appendix B

Miscellaneous results

B.1 Matrices with vector values

For any normed space we shall denote $M_{n\times m}(X)$ to be the set of all $n \times m$ matrices

$$A = (a_{i,j})_{\substack{1\leq i\leq n\\ 1\leq j\leq m}}$$

where $a_{i,j} \in X$ for all i, j; for this section we shall refer to matrices with upper case letters and their entries to be the corresponding lower letter.

For $A \in M_{n\times m}(X^*)$ we may define the linear maps

$$T_A : X^m \to \mathbb{R}^n,\ (x_j)_{j=1}^m \mapsto \left(\sum_{j=1}^m a_{i,j}(x_j)\right)_{i=1}^n$$

and

$$T_A^* : \mathbb{R}^n \to (X^*)^m,\ (y_i)_{i=1}^n \mapsto \left(\sum_{i=1}^n y_i a_{i,j}\right)_{j=1}^m .$$

Given a normed space X and $m \in \mathbb{N}$ we shall define the map

$$\langle \cdot, \cdot \rangle : X^m \times (X^*)^m \to \mathbb{R},\ ((x_j)_{j=1}^m, (f_j)_{j=1}^m) \mapsto \left\langle (x_j)_{j=1}^m, (f_j)_{j=1}^m \right\rangle := \sum_{j=1}^m f_j(x_j).$$

Lemma B.1.1. Let X be a normed space, $A \in M_{n\times m}(X^*)$ and $\langle \cdot, \cdot \rangle$ be the standard inner product of $\mathbb{R}^n$. Then for all $x := (x_j)_{j=1}^m \in X^m$ and $y := (y_i)_{i=1}^n \in \mathbb{R}^n$,

$$\langle T_A(x), y \rangle = \langle x, T_A^*(y) \rangle .$$

Proof. We observe that

$$\langle T_A(x), y \rangle = \left\langle \left(\sum_{j=1}^m a_{i,j}(x_j) \right)_{i=1}^n, (y_i)_{i=1}^n \right\rangle = \sum_{i=1}^n \sum_{j=1}^m y_j a_{i,j}(x_j),$$

and

$$\langle x, T_A^*(y) \rangle = \left\langle (x_j)_{j=1}^m, \left(\sum_{i=1}^n y_i a_{i,j} \right)_{j=1}^m \right\rangle = \sum_{i=1}^n \sum_{j=1}^m y_j a_{i,j}(x_j).$$

□

We define a matrix $A \in M_{n\times m}(X)$ to have *row independence* if the set

$$\{(a_{i,j})_{1\le j\le m} \in X^m : i = 1, \dots, n\}$$

is linearly independent.

Lemma B.1.2. Let X be a normed space and $A \in M_{n\times m}(X^*)$. Then A has row independence if and only if T_A^* is injective.

Proof. We note that $y \in \ker T_A^*$ if and only if y is a linear dependence on the rows of A, thus A has row independence if and only if T_A^* is injective. □

Proposition B.1.3. Let X be a normed space and $A \in M_{n\times m}(X^*)$. Then A has row independence if and only if T_A is surjective.

Proof. Suppose A has row independence but $T_A(X^m) \neq \mathbb{R}^n$, then we may choose $z \in \mathbb{R}^n$ such that $\langle T_A(x), z\rangle = 0$ for all $x \in X^m$. By Lemma B.1.1, $\langle x, T_A^*(z)\rangle = 0$ for all $x \in X^m$, thus $T_A(z) = 0$. However, by Lemma B.1.2, T_A^* is injective, a contradiction.

Now suppose T_A is surjective but A does not have row independence. By Lemma B.1.2, T_A^* is not injective, thus there exists $z \in \mathbb{R}^n$ such that $T_A^*(z) = 0$. By Lemma B.1.1, it follows that $\langle T_A(x), z\rangle = 0$ for all $x \in X^m$. As T_A is surjective, there exists $w \in X^m$ such that $T_A(w) = z$. However, $\langle z, z\rangle \neq 0$, a contradiction. □

B.2 Negligible sets and complete Haar measures on normed spaces

Let S be any set and $\Sigma(S)$ a set of subsets of S, then $\Sigma(S)$ is a *σ-algebra* if the following holds:

(i) $S \in \Sigma(S)$.

(ii) If $A \in \Sigma(S)$ then $X \setminus A \in \Sigma(S)$.

(iii) If $(A_n)_{n\in\mathbb{N}}$ is a sequence in $\Sigma(S)$ then $\bigcup_{n\in\mathbb{N}} A_n \in \Sigma(S)$.

A *measure* is a map $m : \Sigma(S) \to [0, \infty]$ such that the following holds:

(i) $m(\emptyset) = 0$.

(ii) For a sequence $(A_n)_{n\in\mathbb{N}}$ in $\Sigma(S)$ with $A_i \cap A_j = \emptyset$ for all $i \neq j$,

$$m\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \sum_{n\in\mathbb{N}} m(A_n).$$

Let X be a normed space and define $B(X)$ to be the set of *Borel sets* of X, sets formed by the countable union and intersections of open and closed sets. By definition, $B(X)$ is a σ-algebra. A function $m : B(X) \to [0, \infty]$ is a *regular (Borel) measure* if the following holds:

(i) m is *inner regular*, i.e. for any $S \in B(X)$,

$$m(S) = \sup \{m(C) : C \subset S,\ C \text{ is a compact set } \}.$$

(ii) m is *outer regular*, i.e. for any $S \in B(X)$,

$$m(S) = \inf \{m(O) : S \subset O,\ O \text{ is an open set } \}.$$

A regular measure m is a *Haar measure* of X if it also has the following properties:

(i) (Finiteness on compact sets) If $C \subset X$ is compact then $m(C) < \infty$.

(ii) (Positivity on open sets) If $O \subset X$ is open then $m(O) > 0$.

(iii) (Translation invariance) For all $S \in B(X)$ and $x \in X$, $m(S + x) = m(S)$.

By [67, Theorem 1.4.2], there exists a Haar measure m on any normed space X; further, by [67, Theorem 1.4.3], if m' is another Haar measure on X, there exists $\lambda > 0$ such that $m' = \lambda m$. It follows that we may consider the unique (up to scalar multiplication) Haar measure m on any given normed space X.

Define for some Haar measure m the sets

$$\begin{aligned} B_0(X) &:= \{S \in B(X) : m(S) = 0\} \\ N(X) &:= \{N \subset X : N \subset S \text{ for some } S \in B_0(X)\}. \end{aligned}$$

As m is unique up to scalar multiplication, the sets $B_0(X)$ and $N(X)$ are independent of our choice of Haar measure. By letting $A \triangle B$ be the symmetric difference of two sets, we may extend $B(X)$ to a larger set

$$\overline{B}(X) := \{S \subset X : S \triangle S' \in N(X) \text{ for some } S' \in B(X)\}.$$

The set $\overline{B}(X)$ is the smallest σ-algebra that contains $B(X) \cup N(X)$.

We may now extend m to the *complete Haar measure* $m : \overline{B}(X) \to [0, \infty]$, the measure on $\overline{B}(X)$ where given $S \in \overline{B}(X)$ with $S \triangle S' \in N(X)$ for some $S' \in B(X)$, $m(S) = m(S')$. This definition is well-defined; if $S \triangle S'' \in N(X)$ also for some $S'' \in B(X)$, then

$$S' \triangle S'' \subset (S \triangle S') \cup (S \triangle S'') \in N(X),$$

and so it follows that $m(S') = m(S'')$. It is immediate that $S \in N(X)$ if and only if $m(S) = 0$. The complete Haar measure will have positivity and invariance, and for any $S \in \overline{B}(X)$,

$$\begin{aligned} m(S) &= \sup \{m(C) : C \subset S,\ C \text{ is a compact set }\} \\ &= \inf \{m(O) : S \subset O,\ O \text{ is an open set }\}. \end{aligned}$$

It follows immediately that a set $S \subset X$ will lie in $N(X)$ if and only if there exists a sequence of open sets $(A_n)_{n \in \mathbb{N}}$ such that $S \subset A_n$ and $m(A_n) \to 0$ as $n \to \infty$.

Example B.2.1. For $\mathbb{R}^d$, we define the *Lebesgue measure* to be the complete Haar measure d where $d([0,1]^d) = 1$.

For the following results (Lemma B.2.2, Corollary B.2.3 and Corollary B.2.4) it is sufficient to show the result holds for Haar measures.

Lemma B.2.2. Let m be a complete Haar measure on a normed space X and $g \in \mathrm{GA}(X)$. Then there exists $C > 0$ such that $m(g(S)) = Cm(S)$ for all $S \in \overline{B}(S)$.

Proof. We note that the measure $m_g : B(X) \to [0, \infty]$ with $m_g(S) = m(g(S))$ is also a Haar measure of X, thus there exists $C > 0$ such that $m_g = Cm$. $\square$

For a basis $e_1, \ldots, e_d$ of a normed space X we define the *unit box*

$$\mathrm{Box} := \mathrm{conv} \left\{ \sum_{i=1}^{d} \sigma_i e_i : \sigma_i \in \{0, 1\} \right\}.$$

Corollary B.2.3. Let m be a complete Haar measure on a d-dimensional normed space X. If $S \in \overline{B}(S)$ and $\lambda > 0$ then $m(\lambda S) = \lambda^d m(S)$.

Proof. Choose a basis $e_1, \ldots, e_d$ of X and define Box accordingly. As m has invariance,

$$\begin{aligned} m(\mathrm{Box}) &= m \left(\bigcup_{k_1=0}^{n-1} \ldots \bigcup_{k_d=0}^{n-1} \left(\frac{1}{n} \mathrm{Box} + \sum_{j=1}^{d} k_j e_j \right) \right) \\ &= \sum_{k_1=0}^{n-1} \ldots \sum_{k_d=0}^{n-1} m \left(\frac{1}{n} \mathrm{Box} \right) \\ &= n^d m \left(\frac{1}{n} \mathrm{Box} \right), \end{aligned}$$

thus $m(n^{-1} \mathrm{Box}) = n^{-d} m(\mathrm{Box})$ for all $n \in \mathbb{N}$. By a similar method we see that $m(n \,\mathrm{Box}) = nm(\mathrm{Box})$ for all $n \in \mathbb{N}$, thus $m(\lambda \,\mathrm{Box}) = \lambda^d m(\mathrm{Box})$ for all rational $\lambda > 0$. As m is outer regular, $m(\lambda \,\mathrm{Box}) = \lambda^d m(\mathrm{Box})$ for all $\lambda > 0$. The result now follows from Lemma B.2.2. $\square$

Corollary B.2.4. Let m be a complete Haar measure of a normed space X and $g \in \mathrm{Isom}(X)$. Then $m(g(S)) = m(S)$ for all $S \subset \overline{B}(X)$.

Proof. As m is translation invariant, we may assume g is linear. As $T(B_1[0]) = B_1[0]$ then $m(g(B_1[0])) = m(B_1[0])$. The result now follows from Lemma B.2.2. $\square$

Lemma B.2.5. Let X be a normed space and C a compact set that lies in a hyperplane Y of X, then $m(C) = 0$.

Proof. Let $d := \dim X$, $e_1, \ldots, e_d \in X$ be a basis and m be a complete Haar measure of X. Define the closed set

$$S := \left\{ \sum_{i=1}^{d-1} a_i e_i : 0 \leq a_i \leq 1 \text{ for all } i = 1, \ldots, d-1 \right\}.$$

By Lemma B.2.2, we may assume $C \subset S$. It now follows that if $m(S) = 0$ then the result holds.

Define for each $n \in \mathbb{N} \cup \{0\}$ the open sets

$$S_n := \left\{ \sum_{i=1}^{d} a_i e_i : |a_d| < \frac{1}{2^n}, \ |a_i| < 2 \text{ for all } i = 1, \ldots, d-1 \right\}.$$

Also define the bijective linear transform $T : X \to X$ with

$$T(\sum_{i=1}^{d} a_i e_i) = \sum_{i=1}^{d-1} a_i e_i + \frac{a_d}{2} e_d,$$

then $S_n = T^n(S_0)$. Since $\overline{S_0}$ is compact then $m(S_0) < \infty$. By Lemma B.2.2, there exists some $C > 0$ such that $m(S_n) = C^n m(S_0)$. As

$$\left(S_1 + \frac{1}{2}e_d\right) \cup \left(S_1 - \frac{1}{2}e_d\right) \subset S_0, \quad \left(S_1 + \frac{1}{2}e_d\right) \cap \left(S_1 - \frac{1}{2}e_d\right) = \emptyset$$

then $C < 1$, thus $m(S_n) \to 0$ as $n \to \infty$. As $S \subset S_n$ and m is outer regular, $m(S) = 0$ also. □

Theorem B.2.6. Let X be a normed space with complete Haar measure m. For any set $S \subset X$, the following are equivalent:

(i) S is negligible in X.

(ii) $m(S) = 0$.

Proof. (i) $\Rightarrow$ (ii): Suppose S is negligible and choose any $\epsilon > 0$. Let $c := m(B_1(0))$. As S is negligible, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in S and $(r_n)_{n\in\mathbb{N}}$ in $(0,\infty)$ such that such that $\sum_{n\in\mathbb{N}} r_n^d < \frac{\epsilon}{c}$ and

$$S \subset O := \bigcup_{n\in\mathbb{N}} B_{r_n}(x_n).$$

By Corollary B.2.3,

$$m(O) \leq \sum_{n\in\mathbb{N}} cr_k^d < \epsilon.$$

As m is outer regular it follows that $m(S) = 0$.

(ii) $\Rightarrow$ (i): Suppose $m(S) = 0$ and choose any $\epsilon > 0$. Choose a basis $e_1, \ldots, e_d$ of X and define Box accordingly. Fix $\delta > 0$ such that Box $\subset B_\delta(0)$. As $m(S) = 0$ and m is outer regular, there exists an open set O such that $S \subset O$ and $m(O) < \frac{\epsilon m(\text{Box})}{\delta^d}$. For each $n \in \mathbb{N}$ define the sets

$$\begin{aligned}
X_n &:= \left\{\sum_{i=1}^{d} \frac{a_i}{2^n} e_i : a_i \in \mathbb{Z} \text{ for all } i = 1, \ldots, d\right\}, \\
\mathcal{B}_n &:= \left\{\frac{1}{2^n}\operatorname{Box} + x : x \in O \cap X_n,\ \left(\frac{1}{2^n}\operatorname{Box} + x\right) \subset O\right\}, \\
\mathcal{B} &:= \left\{A \in \bigcup_{n\in\mathbb{N}} \mathcal{B}_n : A \text{ is not a proper subset of any } B \in \bigcup_{n\in\mathbb{N}} \mathcal{B}_n\right\}
\end{aligned}$$

We now note 3 things:

(i) $\mathcal{B}$ is a countable set.

(ii) For $A, B \in \mathcal{B}$, $A \cap B$ has measure zero (Lemma B.2.5).

(iii) $O = \bigcup_{A\in\mathcal{B}} A$.

This implies that $m(O) = \sum_{A\in\mathcal{B}} m(A)$.

As $\mathcal{B}$ is countable we may define a sequence $(r_n)_{n\in\mathbb{N}}$ in $(0,\infty)$ and a sequence $(x_n)_{n\in\mathbb{N}}$ in O such that

$$\mathcal{B} = \{r_n \operatorname{Box} + x_n : n \in \mathbb{N}\},$$

thus

$$O = \bigcup_{n\in\mathbb{N}} (r_n \operatorname{Box} + x_n) \subset \bigcup_{n\in\mathbb{N}} B_{\delta r_n}(x_n).$$

By Corollary B.2.3,

$$\sum_{n\in\mathbb{N}} r_n^d m(\operatorname{Box}) = \sum_{n\in\mathbb{N}} m\left(r_n \operatorname{Box} + x_n\right) = m(O) < \frac{\epsilon m(\operatorname{Box})}{\delta^d},$$

thus

$$\sum_{n\in\mathbb{N}} (\delta r_n)^d < \epsilon$$

as required. □

Proposition B.2.7. Let X be a normed space. Then the following hold:

(i) If $(S_n)_{n\in\mathbb{N}}$ is a sequence of negligible sets then $S := \bigcup_{n\in\mathbb{N}} S_n$ is negligible.

(ii) If S is negligible then $X \setminus S$ is dense in X.

Proof. Let m be a complete Haar measure of X.

(i): By Theorem B.2.6, $m(S_n) = 0$ for each $n \in \mathbb{N}$, thus $m(S) = 0$. By Theorem B.2.6, S is negligible.

(ii): Suppose $X \setminus S$ is not dense in X, then there exists $x \in X$ and $r > 0$ such that $B_r(x) \subset S$. By positivity we note that

$$m(S) \geq m(B_r(x)) > 0.$$

However, by Theorem B.2.6, $m(S) = 0$, a contradiction. □

Lemma B.2.8. Let X be a normed space. Then every proper affine subspace of X is negligible.

Proof. This follows by Lemma B.2.5 and Proposition B.2.7 (i). □

Proposition B.2.9. Let M be a n-dimensional C^k-submanifold of a d-dimensional normed space X, where $k \in \mathbb{N} \cup \{\infty\}$ and $n < d$. Then M is negligible.

Proof. Choose any point $x \in S$. As M is a submanifold of X, there exists an open neighbourhood U_x of x in X, an open set $U \subset \mathbb{R}^d$ and a diffeomorphism $\phi : U_x \to U$ such that the restriction

$$\phi' : U_x \cap M \to U' := U \cap (\mathbb{R}^n \times \{0\}^{d-n}), y \mapsto \phi(y)$$

is well-defined. By Lemma B.2.8, set $\mathbb{R} \times \{0\}^{n-1}$ is negligible in $\mathbb{R}^d$, thus by [47, Lemma 3.6.1],

$$\phi^{-1}(\mathbb{R}^n \times \{0\}^{d-n} \cap U') = U_x \cap M$$

is negligible.

As $\{U_x : x \in M\}$ is a cover of M and X is paracompact then there exists a countable set S such that $M = \bigcup_{x \in S} U_x \cap M$. By Proposition B.2.7 (i), M is negligible. □

Proposition B.2.10. Let $P : \mathbb{R}^d \to \mathbb{R}^k$ be the projection $P((x_n)_{n=1}^d) := (x_n)_{n=1}^k$. Then the following holds:

(i) If $S \subset \mathbb{R}^k$ is negligible then $P^{-1}[S]$ is negligible in $\mathbb{R}^d$.

(ii) If $S \subset \mathbb{R}^d$ is a subset such that S^c is negligible then $P(S)^c$ is negligible in $\mathbb{R}^k$.

Proof. Let m_d and m_k be the Lebesgue measure of $\mathbb{R}^d$ and $\mathbb{R}^k$ respectively. By Theorem B.2.6, we need only show the sets have measure zero.

(i): As P is continuous then $P^{-1}[S] \in \overline{B}(X)$. We note that if

$$S_n := S \times [n, n+1]^{d-k}$$

then $P^{-1}[S] = \bigcup_{n\in\mathbb{Z}} S_n$. We note that $m_d(S\times[n,n+1]^{d-k}) = m_k(S)$, thus $m_d(P^{-1}[S]) = 0$.

(ii): By Proposition B.2.7 (i) and possible translation, it follows we may assume that $S \subset [0,1]^d$. We may further assume that S has the property that if $(x,a) \in \mathbb{R}^k \times \mathbb{R}^{d-k}$ lies in S for some $a \in [0,1]$ then $(x,b) \in S$ for all $b \in [0,1]$ as this will not change the value of $m_k(P(S))$. We note immediately that $m_k(P(S)) = m_d(S) = 1$, thus $m_k(P(S)^c) = 0$. □

B.3 Zero sets of algebraic and analytic functions

For this section we shall assume $\mathbb{R}^d$ has the standard Euclidean norm.

Define $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. For any $k = (k_i)_{i=1}^d \in \mathbb{N}_0^d$ and $x = (x_i)_{i=1}^d \in \mathbb{R}$, we define

$$|k| := \sum_{i=1}^d k_i, \qquad k! := k_1! \dots k_d!, \qquad x^k := x_1^{k_1} \dots x_d^{k_d}.$$

Let $U \subset \mathbb{R}^d$ be an open set. A function $f : U \subset \mathbb{R}^d \to \mathbb{R}$ is *analytic* if for each point $z \in U$ there exists a set $(a_k)_{k\in\mathbb{N}_0^d}$ and $r > 0$ such that for all $x \in B_r(x)$,

$$f(x) := \sum_{k\in\mathbb{N}_0^d} a_k (x-z)^k,$$

where the summation is absolutely convergent.

For an open set $U \subset \mathbb{R}^d$ we define the following:

(i) The *zero set* f in U

$$V_U(f) := \{x \in U : f(x) = 0\}.$$

(ii) The sets

$$R_U(f) := \{x \in U : f(x) = 0,\ df(x) \neq 0\}, \qquad S_U(f) := V_U(f) \setminus R_U(f).$$

We will denote $V(f) := V_{\mathbb{R}^d}(f)$ (and similarly $R(f)$ and $S(f)$) for brevity. Given a set F of analytic functions with the same domain U we may define the *analytic set*

$$V_U(F) := \{x \in U : f(x) = 0 \text{ for all } f \in F\}.$$

Likewise, we denote $V(F) := V_{\mathbb{R}^d}(F)$ for brevity, and if each of F is an algebraic function we define $V(F)$ to be an *algebraic set*.

We define for each $k \in \mathbb{N}_0$ the *k-th partial derivative* at $x \in U$ by

$$\partial^k f := \frac{\partial^{j_1}}{\partial x_1^{j_1}} \cdots \frac{\partial^{j_d} f(x)}{\partial x_d^{j_d}}.$$

If the map f is smooth (as shall be shown in the following result) the order in which we apply the partial derivatives does not matter [61, Theorem 9.41].

Proposition B.3.1. [41, Proposition 1.6.3] Let f is analytic and for $z \in U$ suppose $f(x) = \sum_{k \in \mathbb{N}_0^d} a_k (x-z)^k$ for all $x \in B_r(z)$. Then for each $j \in \mathbb{N}_0^d$ (including $(0)_{k=1}^d$), the map $\partial^j f : U \to \mathbb{R}$ is well-defined, smooth and analytic, where for all $x \in B_r(z)$,

$$\partial^j f(x) = \sum_{k \in \mathbb{N}_0^d} \frac{(k+j)!}{k!} a_{k+j}(x-z)^k.$$

Lemma B.3.2. Let $U \subset \mathbb{R}^d$ be open set and let $f : U \to \mathbb{R}^d$ be the analytic function with $f(x) := \sum_{k \in \mathbb{N}_0^d} a_k(x-z)^k$ for some $z \in U$. Then f is uniformly zero on U if and only if $a_k = 0$ for all $k \in \mathbb{N}_0$.

Proof. If $a_k = 0$ for all $k \in \mathbb{N}_0$ then it is immediate that $f = 0$. Suppose f is uniformly zero on U but there exists $a_j \neq 0$ for some $j \in \mathbb{N}_0$. As f is uniformly zero on U, then for each $k \in \mathbb{N}_0$ and $x \in U$, $\partial^k f(x) = 0$. We note that $\partial^j f(z) = j! a_j \neq 0$, a contradiction. □

Lemma B.3.3. Let $U \subset \mathbb{R}^d$ a connected open set and $f : U \subset \mathbb{R}^d \to \mathbb{R}$ be a non-zero analytic function. Then

$$V_U(f) \subset \bigcup_{k \in \mathbb{N}_0^d} R_U(\partial^k f).$$

Proof. Suppose there exists $z \in V_U(f)$ with $z \notin R_U(\partial^k f)$ for all $k \in \mathbb{N}_0^d$. It follows from Proposition 1.1.6 that $\partial^k f(z) = 0$ for all $k \in \mathbb{N}_0^d$. As f is analytic there exists there exists a set $(a_k)_{k \in \mathbb{N}_0^d}$ and $r > 0$ such that for all $x \in B_r(x)$, $f(x) := \sum_{k \in \mathbb{N}_0^d} a_k(x-z)^k$. By Proposition B.3.1, $\partial^j f(z) = j! a_j \neq 0$, a contradiction. □

Lemma B.3.4. Let $U \subset \mathbb{R}^d$ a connected open set and $f : U \subset \mathbb{R} \to \mathbb{R}$ be a non-zero analytic function. Then $R_U(f)$ is negligible.

Proof. As f can be extended to a complex analytic function on some complex open set Ω, the result follows from [62, Theorem 10.18]. □

Lemma B.3.5. Let $U \subset \mathbb{R}^d$ a connected open set and $f : U \subset \mathbb{R}^d \to \mathbb{R}$ be a non-zero analytic function. Then $R_U(f)$ is negligible.

Proof. If $d = 1$ then by Lemma B.3.4, $V_U(f)$ has a countable set of zeroes, thus $R_U(f)$ is negligible. Suppose $d > 1$. We note that $R_U(f)$ is a non-empty open subset of $V_U(f)$ (as $S_U(f)$ is closed) and $\operatorname{rank} df(x) = 1$ for all $x \in R_U(f)$, thus by Corollary 2.1.3, $R_U(f)$ is a 1-dimensional submanifold of $\mathbb{R}^d$. By Proposition B.2.9, $R_U(f)$ is negligible. □

Theorem B.3.6. Let $U \subset \mathbb{R}^d$ a connected open set and $f : U \subset \mathbb{R}^d \to \mathbb{R}$ be a non-zero analytic function. Then $V_U(f)$ is negligible.

Proof. By Lemma B.3.3,

$$V_U(f) \subset \bigcup_{k \in \mathbb{N}_0^d} R_U(\partial^k f).$$

By Lemma B.3.5, each $R_U(\partial^k f)$ is negligible, thus $V_U(f)$ is negligible also. □

Corollary B.3.7. Let V be an algebraic set in $\mathbb{R}^d$ where $V \neq \mathbb{R}^d$, then V is negligible.

Proof. By definition, there exists a set F of algebraic functions such that $V(F) = V$. We now note that

$$V = \bigcap_{f \in F} V(f) \subset V(f),$$

thus by Theorem B.3.6, V is negligible. □

List of symbols

References

[1] R. Abraham, J. E. Marsden, *Foundations of mechanics, second edition*, Addison-Wesley, Reading Massachusetts, 1978.

[2] J. Alonso, C, Benítez, *Some characteristic and non-characteristic properties of inner product spaces*, Journal of approximation theory, Volume 55 Issue 3, pp 318-325, 1988.

[3] D. Amir, *Characterization of inner product spaces*, Birkhauser Verlag Basel, Switzerland, 1986.

[4] E. B. Ashton, *Exploring continuous tensegrities*, PhD thesis, University of Georgia, 2007.

[5] L. Asimow, B. Roth, *The rigidity of graphs*, Transactions of the American Mathematical Society, Volume 245, pp. 279-289, 1978.

[6] L. Asimow, B. Roth, *The rigidity of graphs II*, Journal of Mathematical Analysis and Applications, Volume 68 Issue 1, pp. 171-190, 1979.

[7] V. I. Averbukh, O. G. Smolyanov, *The theory of differentiation in linear topological spaces*, Russian Mathematical Surveys, Volume 22 no. 6, pp 201-258, 1967.

[8] Y. Benyamini, J. Lindenstrauss, *Geometric nonlinear functional analysis: Volume 1*, American Mathematics Society, 2000.

[9] C. S. Borcea, I. Streinu, *Periodic frameworks and flexibility*, Proceedings of the Royal Society A, Volume 466 Issue 2121, pp. 2633–2649, 2010.

[10] E. D. Bolker, B. Roth, *When is a bipartite graph a rigid framework?*, Pacific Journal of Mathematics, Volume 90 No. 1, pp. 27-44, 1980

[11] H. Bruhn, R. Diestel, M. Kriesell, R. Pendavingh, P. Wollan, *Axioms for infinite matroids*, Advances in Mathematics, Volume 239 pp. 14-46, 2013.

[12] A. L. Cauchy, *Recherche sur les polyèdres – premier mémoire*, Journal de l'Ecole Polytechnique, Volume 9, pp.66-86, 1813.

[13] Z. Charzyński, *Sur les transformations isométriques des espaces du type (F)*, Studia Mathematica, Volume 13, pp. 94-121, 1953.

[14] F. H. Clarke, *On the inverse function theorem*, Pacific Journal of Mathematics, Volume 64 no. 1, 1976.

[15] R. Connelly, *Generic global rigidity*, Discrete & Computational Geometry, Volume 33 Issue 4, pp. 549-563, Springer-Verlag, 2005.

[16] J. Cook, J. Lovett, F. Morgan, *Rotations in a normed plane*, The American Mathematical Monthly, Volume 114 Issue 7, pp. 628-632, 2007.

[17] H. Crapo, *On the generic rigidity of structures in the plane*, Tech. Rep. 1278, Institut de recherche d'informatique et d'automatique, 1988.

[18] S. Dewar, *Equivalence of continuous, local and infinitesimal rigidity in normed spaces*, arXiv.org preprint, https://arxiv.org/abs/1809.01871, 2018.

[19] S. Dewar, *Infinitesimal rigidity in normed planes*, arXiv.org preprint, https://arxiv.org/abs/1812.06022, 2018.

[20] S. Dewar, D. Kitson, A. Nixon, *Which graphs are rigid in ℓ^p-space?*, upcoming paper.

[21] H. Gluck, *Almost all simply connected closed surfaces are rigid*, Geometric topology: Lecture Notes in Mathematics, Volume 438, pp. 225-239, Springer, Berlin, 1975.

[22] S. J. Gortler *Characterizing the universal rigidity of generic frameworks*, Discrete & Computational Geometry, Volume 51 Issue 4, pp. 1017-1036, Springer US, 2014.

[23] J. Graver, B. Servatius, H. Servatius, *Combinatorial rigidity*, Graduate Studies in Mathematics Volume 2, American Mathematics Society, 1993.

[24] B. C. Hall, *An elementary introduction to groups and representations*, arXiv.org online lecture notes, https://arxiv.org/abs/math-ph/0005032, 2000.

[25] B. Hall, *Lie groups, Lie algebras, and representations: An elementary introduction*, Graduate Texts in Mathematics Volume 222, Springer Science & Business Media, 2003.

[26] B. Hendrickson, *Conditions for unique graph realizations*, SIAM Journal on Computing, Volume 21 Issue 1, pp. 65–84, 1992.

[27] L. Henneberg, *Die Graphische Statik der Starren Körper*, Encyklopädie der Mathematischen Wissenschaften mit Einschluss ihrer Anwendungen, pp. 345-434, 1908.

[28] B. Jackson, T. Jordán, *The generic rank of body–bar-and-hinge frameworks*, European Journal of Combinatorics, Volume 31 Issue 2, pp. 574-588 (2010).

[29] T. Jordán, *On the existence of k edge-disjoint 2-connected spanning subgraphs*, Journal of Combinatorial Theory Series B, Volume 95 Issue 2, pp. 257-262, 2005.

[30] T. Jordán, *Combinatorial rigidity: graphs and matroids in the theory of rigid frameworks*, Technical report TR-2014-12, Egerváry Research Group, Budapest, 2014.

[31] T. Jordán, V. E. Kaszanitzky, S. Tanigawa, *Gain-sparsity and symmetry-forced rigidity in the plane* Discrete & Computational Geometry, Volume 55 Issue 2, pp. 314-372, 2016.

[32] E. Kastis, S. C. Power, *Algebraic spectral synthesis and crystal rigidity*, Journal of Pure and Applied Algebra, Volume 223 Issue 11, 2019.

[33] N. Katoh, S. Tanigawa, *A proof of the molecular conjecture*, Discrete & Computational Geometry, Volume 45 Issue 4, pp. 647–700, 2011.

[34] D. Kitson, *Finite and infinitesimal rigidity with polyhedral norms*, Discrete & Computational Geometry, Volume 54 Issue 2, pp. 390-411, Springer US, 2015.

[35] D. Kitson, R. H. Levene, *Graph rigidity for unitarily invariant matrix norms*, arXiv.org preprint, https://arxiv.org/abs/1709.08967, 2017.

[36] D. Kitson, S. C. Power, *Infinitesimal rigidity for non-Euclidean bar-joint frameworks*, Bulletin of the London Mathematical Society, Volume 46 Issue 4, pp. 685-697, 2014.

[37] D. Kitson, S. C. Power, *The rigidity of infinite graphs*, arXiv.org preprint, https://arxiv.org/abs/1310.1860, 2013.

[38] D. Kitson, S. C. Power, *The rigidity of infinite graphs*, Discrete Computational Geometry, Volume 60 Issue 3, pp 531-537, Springer US, 2018.

[39] D. Kitson, S. C. Power, *The rigidity of infinite graphs II*, upcoming paper.

[40] D. Kitson, B. Schulze, *Maxwell-Laman counts for bar-joint frameworks in normed spaces*, Linear Algebra and its Applications, Volume 481, pp. 313-329, 2015.

[41] S. G. Krantz, H. R. Parks, *A primer of real analytic functions*, Birkhäuser Advanced Texts Basler Lehrbücher, Birkhäuser Basel, 1992.

[42] G. Laman, *On graphs and rigidity of plane skeletal structures*, Journal of Engineering Mathematics, Volume 4 Issue 4, pp. 331-340, 1970.

[43] A. Lee, I. Streinu, *Pebble game algorithms and sparse graphs*, Discrete Mathematics, Volume 308 Issue 8, pp. 1425-1437, 2008.

[44] J. Lee, *Manifolds and differential geometry*, Graduate studies in mathematics, Volume 107, American Mathematical Society, 2010.

[45] J. Lee, *Introduction to topological manifolds, second edition*, Graduate Texts in Mathematics, Springer-Verlag New York, 2011.

[46] L. Lovász, Y. Yemini, *On generic rigidity in the plane*, SIAM Journal of Algebraic Discrete Methods, Volume 3 Issue 1, pp. 91-98, 1982.

[47] J. E. Marsden, T. Raitu, R. Abraham, *Manifolds, tensor analysis, and applications, third edition*, Springer-Verlag New York, 2002.

[48] H. Martini, K. J. Swanepoel, Gunter Weiß, *The geometry of Minkowski spaces - A survey. Part 1*, Expositiones Mathematicae, Volume 19 Issue 2, pp. 97-142, Urban & Fischer Verlag, 2001.

[49] J. C. Maxwell, *On the calculation of the equilibrium and stiffness of frames*, The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science, Volume 27 Issue 182, pp. 294-299, 1864.

[50] H. Minkowski, *Sur les propriétés des nombres entiers qui sont dérivées de l'intuition de l'espace*, Nouvelles annales de mathématiques : journal des candidats aux écoles polytechnique et normale, Serie 3, Volume 15, pp. 393-403, 1896.

[51] D. Montgomery, H. Samelson, *Transformation groups of spheres*, Annals of Mathematics, Second Series, Vol. 44, No. 3, pp. 454-470, 1943.

[52] C. St.J. A. Nash-Williams, *Edge-disjoint spanning trees of finite graphs*, Journal of the London Mathematical Society, Volume 36 Issue 1 pp. 445-450, Oxford University Press, 1961.

[53] A. Nixon, J. C. Owen, *An inductive construction of* $(2,1)$*-tight graphs*, Contributions to Discrete Mathematics, Volume 9 Issue 2, pp. 1-16, 2014.

[54] A. Nixon, J.C. Owen, S. C. Power, *Rigidity of frameworks supported on surfaces*, SIAM Journal on Discrete Mathematics, Volume 26 Issue 4, pp. 1733-1757, 2012.

[55] K. Oh, M. Park, H. Ahn, *A survey of multi-agent formation control*, Automatica, Volume 53, pp. 424-440, 2015.

[56] J. C. Owen, S. C. Power, *Infinite bar-joint frameworks, crystals and operator theory*, New York Journal of Mathematics, Volume 17, pp. 445-490, 2011.

[57] J. G. Oxley, *Matroid theory*, Oxford University Press, 1992.

[58] H. Pollaczek-Geiringer, *Ueber die Gliederung ebener Fachwerke*, Zeitschrift für Angewandte Mathematik und Mechanik, Volume 7 Issue 1, pp. 58-72, 1927.

[59] B. Riemann, *Ueber die Hypothesen, welche der Geometrie zu Grunde liegen*, Abhandlungen der Königlichen Gesellschaft der Wissenschaften zu Göttingen, Volume 13, pp. 133-150, 1868.

[60] R. T. Rockafellar *Convex analysis*, Princeton Landmarks in Mathematics and Physics, Princeton University Press, 1970.

[61] W. Rudin, *Principals of mathematical analysis, third edition*, International Series in Pure & Applied Mathematics, McGraw-Hill Higher Education, 1976.

[62] W. Rudin, *Real and complex analysis, third edition*, McGraw-Hill, Inc., New York, NY, USA, 1987.

[63] R. Schneider, *Smooth approximation of convex bodies*, Rendiconti del Circolo Matematico di Palermo, Volume 33 Issue 3, pp. 436–440, 1984.

[64] B. Schulze, S. Tanigawa, *Infinitesimal rigidity of symmetric bar-joint frameworks*, SIAM Journal on Discrete Mathematics, Volume 29 Issue 3, pp. 1259-1286, 2015.

[65] G. Stacey, R. Mahoney, *The role of symmetry in rigidity analysis: A tool for network localization and formation control*, IEEE Transactions on Automatic Control, Volume 63 No. 5, pp. 1313-1328, May 2018.

[66] S. Tanigawa, *Sufficient conditions for the global rigidity of graphs*, Journal of Combinatorial Theory Series B, Volume 113, pp. 123-140, 2015.

[67] A. C. Thompson, *Minkowski geometry*, Encyclopedia of Mathematics and its Applications, Cambridge University Press, 1996.

[68] W. Whiteley, *Vertex splitting in isostatic frameworks*, Structural Topology, Volume 16, pp. 23-30, 1991.