A Category Theoretic Approach to Extensions of Banach Algebras

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Declaration

I hereby declare that except where specific reference is made to the work of others, the contents of this thesis are original and have not been submitted in whole or in part for consideration for any other degree or qualification in this, or any other university. This thesis is my own work and contains nothing which is the outcome of work done in collaboration with others, except as specified in the text and Acknowledgements.

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Abstract

This thesis aims to generalise Busby's framework for studying extensions of C^*-algebras, to the Banach-algebraic setting, without requiring admissibility assumptions on extensions.

In the case where the canonical embedding ι_J of a faithful Banach algebra J into its multiplier algebra M_J has closed range, we classify all extensions of an arbitrary Banach algebra B by J. This is done by presenting two categories, one of extensions and another of Busby maps, and proving that these categories are equivalent.

We then consider cases where the canonical embedding of J need not have closed range, and provide some partial results in such cases under the extra assumption of a bounded linear lift for a given Busby map. These results are then applied to several examples, where we also compute explicit multiplier norms for M_J when J is a maximal ideal in $C^k([-1,1])$.

To go further, we study the quotient $M_J/\iota_J(J)$ not as a seminormed space but as an object in a suitable derived category. To lay the necessary foundations, the derived category construction of Grothendieck and Verdier is applied to the category of Banach spaces and bounded linear maps. Using this framework, we introduce a class of "Q-Busby maps" from an arbitrary Banach algebra B into $M_J/\iota_J(J)$, and obtain a restricted version of Busby's original correspondence, applicable whenever J is a faithful Banach algebra.

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Chapter 1

Introduction

1.1 Introduction

Let us motivate the problem of classifying extensions by starting with the version arising in group theory. The following passages can be found in Isaac's textbook [9, p.66].

> "*Given groups N and H, a group G is said to be an extension of H by N if there exists $N_0 \lhd G$ such that $N_0 \cong N$ and $G/N_0 \cong H$.*"
>
> "*... If G is an extension of H by N, then the normal subgroup N_0 of G such that $N_0 \cong N$ and $G/N_0 \cong H$ is not, in general, uniquely determined.*"

A consequence of G not being uniquely determined is that non-isomorphic extensions of H by N are possible.

Example 1.1.1. Fix $n \in \mathbb{N}$. Take D_{2n} and C_{2n}, the dihedral and cyclic groups of order $2n$ respectively. Both D_{2n} and C_{2n} contain normal subgroups isomorphic to $\mathbb{Z}_n$, let $N_1 \lhd D_{2n}$ and $N_2 \lhd C_{2n}$ be these normal subgroups. Since $D_{2n}/N_1 \cong C_{2n}/N_2 \cong \mathbb{Z}_2$, we have that both D_{2n} and C_{2n} are extensions of $\mathbb{Z}_2$ by $\mathbb{Z}_n$. However, we know for a fact that $D_{2n} \ncong C_{2n}$.

The definition of a group extension is given more formally below. Note that this has different notation from IV.3 of [11] but is equivalent.

Definition 1.1.2 (Group extension)**.** Let H and N be groups. An extension of H by N is a triple (ι, G, q) where G is a group, $\iota : N \to G$ is an injective group homomorphism, $q : G \to H$ is a surjective group homomorphism, and $\ker(q) = \mathrm{im}(\iota)$.

One can formulate similar definitions in other categories. For instance, in the category of C^*-algebras, there is an obvious analogue of the previous definition, with groups replaced by C^*-algebras and group homomorphisms by $*$-homomorphisms. Extensive work has been carried out on Toeplitz operators by Coburn and Douglas (see [5] for example) and they have particular relevance in index theory, with additional work from Schaeffer and Singer (see [6] and [13, Section 3.5]). An important example of a C^*-algebra extension is that of Toeplitz extension. We provide a brief description of the extension, the reader can find further details in [13, Section 3.5].

Example 1.1.3. Here, we take the Hardy space H^2 [13, p.96] and let J be $K(H^2)$, defined below. The Toeplitz algebra $\mathcal{T}$ is the algebra generated by the unilateral shift on H^2 (see [13, p. 102]). $\mathcal{T}$ contains $K(H^2)$, the set of compact operators on H^2, as a closed ideal, and the quotient of $\mathcal{T}$ by $K(H^2)$ is isomorphic to $C(\mathbb{T})$, the continuous functions on the circle. This extension is important in Fredholm theory as it shows that a Toeplitz operator can be assigned to a continuous function on the circle, modulo a compact operator. After that, the index of the Toeplitz operator can be evaluated as the negative of the winding number of the relevant function in $C(\mathbb{T})$.

There is a complete and satisfactory theory for C^*-algebra extensions, which is due to Busby in [2]. Given C^*-algebras B and J, the extensions of B by J are classified up to a natural notion of equivalence, by the various $*$-homomorphisms from B into the *corona algebra* of J. The corona algebra C_J is defined to be the quotient $M_J/\iota_J(J)$

where M_J is the *multiplier algebra* or *double centraliser algebra* of J and $\iota_J : J \to M_J$ is a canonical embedding.

A crucial ingredient in Busby's approach is the fact that J, being a C^*-algebra, has a bounded approximate identity. This ensures that the map ι_J is not only injective but also has closed range, so that C_J is a C^*-algebra and $*$-homomorphisms $B \to C_J$ are automatically continuous.

The primary goal of this thesis is to investigate how far Busby's approach can be extended to the setting of Banach algebras and continuous homomorphisms (precise definitions will be given in section 2.1). We note that in the intermediate setting of *non-self-adjoint operator algebras* and *completely bounded homomorphisms*, there has been previous work in this direction in the PhD thesis of Royce [14]. However, in Royce's framework, she only considers extensions of B by J where J has a contractive approximate identity. This generalises Busby's results for C^*-algebras, but cannot be applied to many natural examples of Banach algebras.

The multiplier algebra M_J can be defined for any Banach algebra J, not just for C^*-algebras, and so we can consider the corona algebra C_J even though it need not be a Banach algebra itself. In this thesis, we shall see that a modified version of Busby's correspondence still holds for extensions of Banach algebras, provided that we only consider extensions of B by J where J is "faithful". This corresponds to $\iota_J : J \to M_J$ being injective but also allows cases where ι_J does not have closed range, as in the following example.

Example 1.1.4. Consider $J = \ell^1(\mathbb{N})$ with pointwise product. The multiplier algebra of J is $M_J = \ell^\infty(\mathbb{N})$ and $\iota_J : \ell^1(\mathbb{N}) \to \ell^\infty(\mathbb{N})$ is the usual inclusion map. Since ι_J does not have closed range, C_J is not a Banach algebra.

1.2 Overview of the thesis

In Chapter 2, we present M_J, the multiplier algebra of a Banach algebra J, together with C_J, the corona algebra of J. We then present two categories, one of Banach algebra extensions by J and another of Busby maps with target C_J. These two categories are then proved to be equivalent in the sense of Definition A.1.15.

We then provide some partial, motivating results for the case where ι_J does not have closed range, culminating in the case where the base of the extension B has finite dimension. In Chapter 3 we apply these results, looking at several examples, calculating some explicit multiplier norms and algebras.

It remains that we would like a more abstract theory for examples where J is faithful but the canonical embedding $\iota_J : J \to M_J$ does not have closed range. This leads us to look at objects of the form A/I where A is a Banach algebra, I is an ideal in A but I is not necessarily closed in A. This leads us to take a derived category approach in the category of Banach spaces and bounded linear maps.

In Chapter 4 we construct a category which is capable of describing and working with these quotient Banach spaces. The approach we take is to construct the derived category in Section 4.3. The construction is due to Grothendieck and Verdier but we use Kashiwara and Schapira as a reference (see [10, chap. 7]). An important choice which we actively make is to follow through the construction in the category of Banach spaces, in order to gain an in-depth understanding of the topological properties of the objects in the category. This is not something easily gained from checking axioms and applying major theorems.

At the same time however, it is important to understand the derived category approach and not tackle this purely as a functional analysis problem.

The approach taken in this thesis for constructing a category in which we can study Banach algebra quotients is to carry out the derived category construction for Banach spaces. Objects in this category are monics in the category of Banach spaces, and morphisms in this category are equivalence classes of spans (see Definition 4.3.2). We then consider a wider class of "Busby maps" which are generated by this process.

Work at the end of Chapter 2 shows us that we can still run Busby's machinery when there is a bounded linear map from the base of the extension to the multiplier algebra. In Chapter 5, we provide a generalised version of the motivating case in Section 2.4, using the new derived category results.

We provide a list of the main results in this thesis for the ease of the reader:

- Theorem 2.3.9 - When a Banach algebra J is faithful and ι_J has closed range, the categories $\mathbf{Ext}(J)$ and $\mathbf{Bus}(J)$ are equivalent. This is an analogue for such J, of Busby's C^*-algebra classification result.
- Theorem 2.4.3 - When ι_J does not necessarily have closed range, but we have a bounded linear lift of $\varphi : B \to C_J$ with target M_J, we can equip a specific pullback with a Banach algebra norm in order to construction an extension.
- Theorem 3.4.2 - We can pin down the multiplier algebra of a maximal ideal of $C^k([-1, 1])$, the continuously k-th differentiable functions on $[-1, 1]$.
- Theorem 5.1.10 - A generalised version of Theorem 2.4.3 which works for Q-Busby maps. Here, Q-Busby maps are algebra homomorphisms with target C_J which are also morphisms in the derived category Q($\mathbf{Ban}$).
- Theorem 5.2.3 - A classification theorem for extensions where we use Q-Busby maps.

1.3 Comparisons with previous work in the literature

Outside of this thesis, there are many texts which investigate some of the concepts we cover. These range from books such as [11], which includes an account of extensions and short exact sequences in a categorical setting, to Busby's original papers which develop the understanding of double centralisers of C^*-algebras and their extensions. However, we argue that there are significant differences between some of these texts and the work laid out here.

In this thesis, faithfulness in the sense of Definition 2.1.9 is almost always assumed to be present. However, it differs from the definition taken by Dales in Definition 1.4.5 [7, p. 52]. Dales first defines left and right faithfulness, and then says an algebra is faithful if it is both left and right faithful. This is stronger than our definition of faithful, which in their setting would be left or right faithful. In our setting, it makes more sense to use the weaker definition, since this is precisely the condition which makes ι_J injective, as seen in Lemma 2.1.10.

When we consider the multiplier algebra of a Banach algebra J, as in Definition 2.1.4, we only consider left and right multipliers which are bounded. In the literature, Dales first considers multipliers in the algebraic sense in Definition 1.4.25 [7, p.59-60], before moving to settings where they are bounded. This is similarly true of Busby's approach in Corollary 2.4 [2, p.80], where multipliers are shown to be continuous using preceding results and properties of C^*-algebras. The aim in this thesis is to analyse extensions and Busby maps in the context of the category **Balg**, so it is enough to only consider bounded multipliers.

There has been extensive work on the case where J is nilpotent, meaning that there exists $n \in \mathbb{N}$ such that $J^n = \{0\}$. We will not attempt to give a full survey, since this thesis is concerned with the cases where the canonical map $\iota_J : J \to M_J$ (see Definition 2.1.5) is injective. See e.g. [1] for further details.

Another interesting and much-studied question is to decide when an extension of B by J (in the sense of Definition 2.1.3) is algebraically or topologically split. See [1] for further details. However, to keep the thesis focussed on the main classification problem, we shall not consider splitting questions, leaving them for possible future work.

There is a notion of equivalence, or more appropriately isomorphism, of extensions used in this thesis, for example in the proof of Theorem 2.3.9. This differs from the notions of equivalence found in certain texts. Busby considers two extensions to be equivalent if they have the same base B and ideal J, with an $*$-isomorphism between the C^*-algebras in the centre [2, p.88]. Our form of equivalence is weaker than Busby's and fulfils a different purpose to his notion of weak-equivalence [2, p.90].

The properties of categories considered, both by this thesis and in the general literature, are worth noting for comparison. The main results of [3] requires one to work with a faithful object in a sufficiently algebraic category (Definitions 1.5 and 1.6 [3]). Now **Balg** is a sufficiently algebraic category (see Example 2(d) in [3, p.48]) but for a Banach algebra J to be a "faithful object in **Balg**" in Busby's sense condition (a) of [3, Definition 1.5] requires $\iota_J : J \to M_J$ to have closed range.

In Section 3 of Chapter 4, [11, p.108], MacLane details a similar question of constructing extensions, albeit in a different setting. Given a group A and Π, he gives an account of the construction of all group extensions of Π by A. This amounts to using the homomorphisms from Π into $\mathrm{Aut}(A)$, similar to the way we use Busby maps. This thesis has the same aim, but is heavily restricted by the added topology in **Balg**. Busby solves the same problem for C^*-algebras, where certain topological problems can be circumvented. This is most apparent in the fact that the corona algebra of a C^*-algebra is always a C^*-algebra.

Later in his book, Section 6 of Chapter 12 [11, p.375], MacLane explores a category of short exact sequences, much like we do. He does this in an abelian category, which has a much richer structure than the additive setting of **Ban**, let alone **Balg** which fails to be additive.

1.4 Notational conventions

This section will contain some clarification and setup of notation used throughout this thesis.

Remark 1.4.1. Throughout this thesis, all vector spaces will be over the complex field.

Remark 1.4.2. In this thesis, when we use the phrase "linear map", it is a synonym for the phrase "linear operator".

Remark 1.4.3. In this thesis, the norm on a Banach algebra is always assumed to be submultiplicative.

Remark 1.4.4. In this thesis, algebra homomorphisms need not be unital, even when both the domain and codomain are unital (Banach) algebras. It is important that in this thesis, we allow the zero homomorphism so this convention is required.

Remark 1.4.5. This thesis will make use of concepts and results from category theory. We have included an appendix on the relevant theory, but we will also point out some notational conventions in this section.

For a category $\mathcal{C}$, the set of objects of the category will be denoted $\mathrm{obj}(\mathcal{C})$. For objects $A, B \in \mathrm{obj}(\mathcal{C})$, the set of morphisms from A to B will be denoted $\hom_{\mathcal{C}}(A, B)$. To simplify notation we will shorten $A \in \mathrm{obj}(\mathcal{C})$ to $A \in \mathcal{C}$. We will also simplify $\hom_{\mathcal{C}}(A, B)$ to $\hom(A, B)$ when there is no confusion as to what category we are working in (see Definition A.1.1).

Remark 1.4.6. In several places during this thesis we have the following situation: there is an injective function $\iota : X \to Y$, not necessarily surjective, and we have elements in $\iota(X)$ that we wish to "pull back" to elements of X. Consistent notation is required for the (unique) function $\iota(X) \to X$ which is defined in this way.

One possibility would be to use the notation ι^{-1}, it being understood from context that one would only consider expressions like $\iota^{-1}(y)$ when $y \in \iota(X)$. However, this could be misinterpreted as asserting the existence of a function $Y \to X$ that is left inverse to ι.

The most accurate notation would be something like

$$\left(\iota|^{\iota(X)}\right)^{-1} : \iota(X) \to X \tag{1.1}$$

where $\iota|^{\iota(X)} : X \to \iota(X)$ is the *corestriction* of ι to its image. However, this would make various formulas too cumbersome, and the extra precision would greatly reduce clarity. We therefore adopt the notation $\iota^{[-1]}$ for the function defined in (1.1); we will only use this notation in settings where ι is injective.

Chapter 2

Extensions of Banach algebras

2.1 The multiplier extension

Definition 2.1.1 (**Ban** and **Balg**)**.** We define the category **Ban** as follows: the objects of **Ban** are complex Banach spaces; and given $E, F \in \mathbf{Ban}$, define $\hom_{\mathbf{Ban}}(E, F)$ to be the set of all bounded linear maps from E to F.

We define the category **Balg** as follows: the objects of **Balg** are complex Banach algebras, not necessarily unital; and given $A, B \in \mathbf{Balg}$, define $\hom_{\mathbf{Balg}}(A, B)$ to be the set of all bounded algebra homomorphisms from A to B.

With this in mind, the theme of this thesis will be the study of extensions in **Balg** and the exploration of the category theory that will aid us in their classification. We begin with the definition of an extension in **Balg**.

Definition 2.1.2 (Short exact sequences in **Balg**)**.** By a short exact sequence in **Balg**, we mean the following data:

- objects J, A, B in **Balg**

- morphisms, $\iota \in \hom_{\mathbf{Balg}}(J, A)$ and $q \in \hom_{\mathbf{Balg}}(A, B)$

which satisfy the following properties: ι is injective, q is surjective and $\operatorname{Ran}\iota = \ker q$. In particular, $\iota(J)$ is a closed ideal in A.

Definition 2.1.3 (Extensions in **Balg**)**.** Given $B, J \in$ **Balg**, an extension of B by J is any triple (ι, A, q) where $A \in$ **Balg** and Diagram 2.1 below forms a short exact sequence in **Balg** in the sense of Definition 2.1.2.

$$0 \longrightarrow J \xrightarrow{\iota} A \xrightarrow{q} B \longrightarrow 0 \tag{2.1}$$

The aim of this chapter will be to explore and explain when it is currently possible to classify extensions. However, before we begin this process, it is evident from the work in [2] and [3] that it will be important to consider the multiplier and corona algebras of a Banach algebra J.

There is more than one way to describe the multiplier algebra of a given Banach algebra J. For the purposes of this thesis, it will be sufficient to consider it to be the algebra of double centralisers of J as Busby does in [2]. We provide the following definition for ease and in order to set up the notation which will be used throughout the thesis. However, the definitions are not new and can be found in standard sources such as [7, Section 1.4].

Definition 2.1.4 (Multiplier algebra - Double centraliser approach)**.** Let $J \in$ **Balg**. A multiplier of J is a pair of bounded linear maps from J into J, such that for all $x, y \in J$, we have that $L(xy) = L(x)y$, $R(xy) = xR(y)$ and $xL(y) = R(x)y$. We will denote the set of all multipliers of J by M_J.

There is an algebra structure on M_J with scalar multiplication and addition defined coordinate-wise. Multiplication, however, is defined in the following way for all $(L_1, R_1), (L_2, R_2) \in M_J$

$$(L_1, R_1)(L_2, R_2) = (L_1 L_2, R_2 R_1).$$

Further to this, the multiplier algebra is a Banach algebra when endowed with the norm $\|(L,R)\| = \max\{\|L\|, \|R\|\}$.

Definition 2.1.5 (The canonical map $\iota_J : J \to M_J$)**.** We can define a continuous homomorphism $\iota_J : J \to M_J, j \mapsto (L_j, R_j)$ where $L_j(x) = jx$ and $R_j(x) = xj$ for all $x \in J$. Moreover, $\iota_J(J)$ is always an ideal of M_J, we will prove this in Lemma 2.1.6.

Lemma 2.1.6. *Let J be a Banach algebra and let $\iota_J : J \to M_J$ be as in Definition 2.1.5. Then $\iota_J(J)$ is an ideal in M_J.*

Proof. We know $\iota_J(J)$ is a subalgebra of M_J since it is the image of an algebra homomorphism by definition. To see it is an ideal, let $j, j' \in J$, let $(L,R) \in M_J$ and consider $(L_jL, RR_j)(j')$. We see that $L_jL(j') = jL(j') = R(j)j'$, and we notice that $R(j)j' = L_{R(j)}(j')$.

Similarly $RR_j(j') = R(j'j) = j'R(j)$, which can be expressed as $R_{R(j)}(j')$. Since this is true for all $j' \in J$, we see that $(L_jL, RR_j) = (L_{R(j)}, R_{R(j)}) \in \iota_J(J)$.

A similar argument will yield that $(LL_j, R_jR) = (L_{L(j)}, R_{L(j)}) \in \iota_J(J)$. We have that $\iota(J)$ is an ideal in M_J. □

Definition 2.1.7 (Corona algebra)**.** Let $J \in$ **Balg** and consider its multiplier algebra M_J. By Lemma 2.1.6, $\iota_J(J)$ sits inside M_J as an ideal. We define the corona algebra to be the quotient $M_J/\iota_J(J)$, which we will refer to as C_J for ease. In order to have consistent notation throughout this thesis, we will always pair this with the quotient map $q_J : M_J \to C_J$.

Remark 2.1.8. In particular, C_J will be a Banach algebra if and only if ι_J has closed range, giving us that $\ker q_J$ is closed and hence that q_J is continuous.

Although this definition could be restricted purely to the case where ι_J has closed range, the case where ι_J does not have closed range will become central to our study in later chapters.

Definition 2.1.9 (Faithful)**.** Let J be an algebra. We say that J is faithful if for all $a \in J$ with $a \neq 0$, a does not annihilate J. That is, there exists some $b \in J$, such that either $ab \neq 0$ or $ba \neq 0$.

Lemma 2.1.10. *Let J be a Banach algebra with multiplier algebra M_J. Let $\iota_J : J \to M_J$ be as defined in Definition 2.1.5. Then J is faithful if and only if ι_J is injective.*

Lemma 2.1.10 is stated without proof in [7, p.60]. We provide a proof for the ease of the reader.

Proof. We begin by assuming that J is faithful. Take $j \in J$ and assume $\iota_J(j) = (0, 0)$. That is, we know that for all $x \in J$, that $xj = jx = 0$. But since J is faithful, we therefore have that $j = 0$, making ι_J injective.

Now assume that ι_J is injective and let $j \in J$ such that for all $x \in J$, both $xj = 0$ and $jx = 0$. It is clear that $\iota_J(j) = (0, 0)$ so by the injectivity of ι_J, we have that $j = 0$ as required. Therefore J is faithful. □

Corollary 2.1.11. *Let J be a Banach algebra with multiplier algebra M_J and let $\iota_J : J \to M_J$ be as defined in Definition 2.1.5. If J is faithful and ι_J has closed range, then (ι_J, M_J, q_J) is an extension of C_J by J.*

Proof. Since q_J is a quotient map, it is surjective and by construction $\ker q_J = \operatorname{Ran} \iota_J$. Further to this, $\ker q_J$ is closed by assumption so q_J must be continuous. Now since J is faithful, Lemma 2.1.10 gives us that ι_J is injective. We therefore have that (ι_J, M_J, q_J) is an extension of C_J by J. □

Definition 2.1.12 (The multiplier extension)**.** Let J be a faithful Banach algebra for which the canonical map $\iota_J : J \to M_J$, (as defined in Definition 2.1.5) has closed range. The extension (ι_J, M_J, q_J) described in Corollary 2.1.11 will be called the multiplier extension.

Definition 2.1.13 (Approximate and bounded approximate identity)**.** Let A be a Banach algebra. An approximate identity for A is a net $(e_\lambda)_{\lambda\in\Lambda}$ such that for all $\varepsilon > 0$ and $x \in A$, there exists $\lambda_0 \in \Lambda$ such that for all $\lambda \geq \lambda_0$, $\|x - e_\lambda x\| < \varepsilon$ and $\|x - xe_\lambda\| < \varepsilon$.

We say (e_λ) is a bounded approximate identity if $\sup_\lambda \|e_\lambda\| < \infty$.

Lemma 2.1.14. *Let J be a Banach algebra with a bounded approximate identity $(e_\lambda)_{\lambda\in\Lambda}$, then $\iota_J(J)$ is closed in M_J and J is faithful.*

Proof. Let $x \in J$, let $K = \sup_\lambda \|e_\lambda\|$ and consider $L_x(e_\lambda)$. We have that

$$\|xe_\lambda\|_J \leq \|L_x\|\|e_\lambda\|_J \leq K\|L_x\|.$$

Taking limits, we have that $\|x\|_J \leq K\|L_x\|$ and similarly that $\|x\|_J \leq K\|R_x\|$. We therefore have that ι_J is bounded below and hence is injective with closed range. □

Remark 2.1.15. We note here that all C^*-algebras have bounded approximate identities (see Theorem 3.1.1 [13, p.78]). Hence we have that when J is a C^*-algebra, J is faithful and ι_J has closed range.

2.2 A category of extensions and a category of Busby maps

In this section, we will set up two categories related to the study of extensions in **Balg**. The first category makes sense for any faithful J, but for the second, we will need to also assume that ι_J has closed range.

As mentioned in Section 1.3, these categories closely resemble known constructions in the literature, for groups or for modules or for C^*-algebras. We have chosen to give a self-contained presentation for the reader's benefit, since the extra details that we include will be useful for the later work in Chapter 5.

Remark 2.2.1 (Important convention)**.** Since we would always require that ι_J be an injective map, it will now be assumed that whenever we refer to an extension of a Banach algebra by a fixed J that J is faithful.

Our first category will be one whose objects are extensions. Before introducing it, we will first define what a morphism of two extensions by a fixed Banach algebra J is. Both objects and morphisms are similar in setup the the category of extensions described in [11, p.375], with an obvious exception being that we are fixing J.

Definition 2.2.2 (Extension morphism)**.** Let $B_1, B_2, J \in \mathbf{Balg}$ and for $i \in \{1, 2\}$ consider the extensions (ι_i, A_i, q_i) of B_i by J. A morphism from the first extension to the second will be a pair (θ, φ) where $\theta \in \hom_{\mathbf{Balg}}(A_1, A_2)$ and $\varphi \in \hom_{\mathbf{Balg}}(B_1, B_2)$, such that Diagram 2.2 commutes.

$$\begin{array}{ccccc} J & \xrightarrow{\iota_1} & A_1 & \xrightarrow{q_1} & B_1 \\ & \searrow^{\iota_2} & \downarrow{\scriptstyle\theta} & & \downarrow{\scriptstyle\varphi} \\ & & A_2 & \xrightarrow{q_2} & B_2 \end{array} \tag{2.2}$$

With this in mind, we will show that the set of all extensions of Banach algebras by J, varying the base, forms a category with these morphisms.

Proposition 2.2.3. *Fix a faithful Banach algebra J. There is a category $\boldsymbol{Ext}(J)$, whose objects are extensions in **Balg** of the form*

$$0 \longrightarrow J \xrightarrow{\iota} A \xrightarrow{q} B \longrightarrow 0 \tag{2.3}$$

as in Definition 2.1.3, and whose morphisms are the extension morphisms from Definition 2.2.2.

Proof. We will check the axioms for a category as defined in Definition A.1.1.

Firstly, let (ι_i, A_i, q_i) be extensions of B_i by J for $i \in \{1,2,3\}$, let (θ, φ) be a morphism from (ι_1, A_1, q_1) to (ι_2, A_2, q_2) and let (θ', φ') be a morphism from (ι_2, A_2, q_2) to (ι_3, A_3, q_3). We need to check that $(\theta'\theta, \varphi'\varphi)$ is a morphism from (ι_1, A_1, q_1) to (ι_3, A_3, q_3). We have that both $\theta'\theta$ and $\varphi'\varphi$ are continuous homomorphisms as they are compositions of continuous homomorphisms so all that remains is to check that the Diagram 2.4 commutes.

$$\begin{array}{ccccc} J & \xrightarrow{\iota_1} & A_1 & \xrightarrow{q_1} & B_1 \\ & \searrow^{\iota_3} & \downarrow{\scriptstyle \theta'\theta} & & \downarrow{\scriptstyle \varphi'\varphi} \\ & & A_3 & \xrightarrow{q_3} & B_3 \end{array} \tag{2.4}$$

This is true as $\theta'\theta\iota_1 = \theta'\iota_2 = \iota_3$ and $\varphi'\varphi q_1 = \varphi' q_2 \theta = q_3\theta'\theta$.

Secondly we note for any given extension (ι, A, q) of $B \in \mathbf{Balg}$ by J, there exists an identity morphism, this being $(\mathrm{id}_A, \mathrm{id}_B)$.

Lastly we need to check associativity of our morphism composition. This is true since the composition of continuous homomorphisms is associative. □

Now that $\mathbf{Ext}(J)$ has been defined, we can explore a useful property which highlights the importance of the multiplier extension. For the next lemma, we remind the reader of the notational convention from remark 1.4.6.

Lemma 2.2.4. *Let $J, B \in \boldsymbol{Balg}$ with J faithful, and let (ι, A, q) be an extension of B by J in the sense of Definition 2.1.3. For each $a \in A$ define $L_a^A, R_a^A : J \to J$ by*

$$L_a^A(j) = \iota^{[-1]}(a\iota(j)), \quad R_a^A(j) = \iota^{[-1]}(\iota(j)a).$$

Then:

1. $(L_a^A, R_a^A) \in M_J$

2. *The function* $\theta_A : A \to M_J$ *defined by* $\theta_A(a) = (L_a^A, R_a^A)$ *belongs to* $\hom_{\boldsymbol{Balg}}(A, M_J)$

3. $\theta_A \iota = \iota_J$

4. *There exists a unique algebra homomorphism* $\varphi_A : B \to C_J$ *such that* $\varphi_A q = q_J \theta_A$*.*

5. *If, moreover,* $\iota_J : J \to J$ *has closed range, then* $C_J \in$ ***Balg*** *and* $\varphi_A \in \hom_{\boldsymbol{Balg}}(B, C_J)$*.*

Proof. 1. First we need to check that (L_a^A, R_a^A) is a bounded multiplier of J. It is easy to see that both L_a^A and R_a^A are linear maps from J into itself since ι is linear. We will only show that L_a^A is continuous, as the proof for R_a^A is almost identical. To this end, let $(j_n) \in J$ with limit $j \in J$, since ι and multiplication in A is continuous we have that $a\iota(j_n)$ has limit $a\iota(j)$. Now since ι has closed range and is injective, $\iota^{[-1]} : \iota(J) \to J$ is a bounded linear map (by the Open Mapping Theorem) and hence L_a^A is bounded linear as required. Similarly, R_a^A is also bounded linear.

Next we show that the multiplier axioms from Definition 2.1.4 hold. We compute for $j_1, j_2 \in J$, that $L_a^A(j_1 j_2) = \iota^{[-1]}(a\iota(j_1 j_2)) = \iota^{[-1]}(a\iota(j_1)\iota(j_2)) = \iota^{[-1]}(a\iota(j_1))j_2$ as required. The right-hand property holds in the same way for R_a^A and $j_1 L_a^A(j_2) = j_1\iota^{[-1]}(a\iota(j_2)) = \iota^{[-1]}(\iota(j_1)a\iota(j_2)) = \iota^{[-1]}(\iota(j_1)a)j_2 = R_a^A(j_1)j_2$ as required for (L_a^A, R_a^A) to be in M_J.

2. Now we can check whether θ_A is a continuous homomorphism. The fact that θ_A is a homomorphism follows easily from the algebra structure on A together with the linearity of ι. For continuity, take $\|L_a^A(j)\| = \|\iota^{[-1]}(a\iota(j))\|$ and note that ι is bounded below since ι is injective with closed range. Therefore $\|\iota^{[-1]}(a\iota(j))\| \leq (1/c)\|\iota\|\|j\|\|a\|$ where c is the bounded below constant. Now it is clear that $\|\theta_A(a)\| = \|(L_a^A, R_a^A)\| = \max\{\|L_a^A\|, \|R_a^A\|\}$ which is less than $(1/c)\|\iota\|\|a\|$ giving us that θ_A is bounded.

3. Next we can check that $\theta_A \iota = \iota_J$. We have that for any $j \in J$, that $\theta_A \iota(j) = (L_{\iota(j)}, R_{\iota(j)})$. We check that for $j' \in J$, that $L^A_{\iota(j)}(j') = \iota^{[-1]}(\iota(j)\iota(j')) = jj' = L^A_j(j')$ where L_j is the left-hand component of $\iota_J(j)$. This similarly holds for the right hand components so we have that $\theta_A \iota = \iota_J$ as required.

4. Since $\ker q = \iota(J)$ and $\theta_A \iota(J) = \iota_J(J) = \ker q_J$, it is clear that $\ker q \subseteq \ker q_J \theta_A$. Let $b \in B$, and as q is surjective, there exists $a \in A$ such that $q(a) = b$. Define $\varphi_A(b) = q_J \theta_A(a)$, this is well defined since $\ker q \subseteq \ker q_J \theta_A$ and can easily be checked to be an algebra homomorphism. Clearly, $\varphi_A q = q_J \theta_A$ by construction, and uniqueness of φ_A follows from the surjectivity of q. We illustrate this with Diagram 2.5

$$\begin{array}{ccccc} J & \xrightarrow{\iota} & A & \xrightarrow{q} & B \\ & \searrow^{\iota_J} & \downarrow{\theta_A} & & \vdots{\varphi_A} \\ & & M_J & \xrightarrow{q_J} & C_J \end{array} \tag{2.5}$$

5. If ι_J has closed range, then $\ker q_J = \operatorname{Ran} \iota_J$ is closed and hence C_J is a Banach algebra. To see that φ_A is now continuous, let $b \in B$ and note that in the construction of φ_A, using Corollary A.2.5, we can choose $a \in A$ with $\|a\|_A < K\|b\|_B$ for some $K > 0$. We therefore have that $\|\varphi_A(b)\|_{C_J} \leq K\|q_J\|\|\theta_A\|\|b\|_B$, and hence that φ_A is continuous as required.

□

The second category we wish to define will be the category of Busby maps which have target C_J. It is important to note that our results now begin to hinge on the corona algebra being a Banach algebra, as C_J did in Lemma 2.2.4. The situation where ι_J does not have closed range will become the focus of later chapters.

Definition 2.2.5 (Busby maps and Busby map morphisms)**.** Let $B, J \in$ **Balg** where J is faithful and ι_J has closed range. We define the Busby maps to be the continuous homomorphisms from B into C_J.

Let $\varphi_i \in \hom_{\mathbf{Balg}}(B_i, C_J)$ for $i \in \{1, 2\}$. A Busby map morphism from φ_1 to φ_2 will be a continuous homomorphism $u : B_1 \to B_2$ such that Diagram 2.6 commutes.

$$\begin{array}{ccc} B_1 & & \\ \Big\downarrow{\scriptstyle u} & \searrow^{\varphi_1} & \\ & & C_J \\ & \nearrow_{\varphi_2} & \\ B_2 & & \end{array} \tag{2.6}$$

We will now provide a reference to show the Busby maps form a category.

Proposition 2.2.6. *Fix a faithful Banach algebra J such that ι_J has closed range. The collection of all Busby maps for J, together with the morphisms defined in Definition 2.2.5, forms a category.*

Proof. This is a slice category in **Balg** over the object C_J. This can be found in [12, p.45-46]. We will refer to this category as $\mathbf{Bus}(J)$. □

2.3 Functors and categorical equivalence

This section aims to define a functor Bus from $\mathbf{Ext}(J)$ to $\mathbf{Bus}(J)$, and another functor Pull in the reverse direction. These two functors will show that $\mathbf{Bus}(J)$ and $\mathbf{Ext}(J)$ are equivalent as categories. This equivalence is more precise than the statements in [2] and [3], which do not consider the set of Busby maps as a category. Throughout this section, we assume ι_J has closed range and that J is faithful.

Definition 2.3.1. Given an extension in **Balg**

$$0 \longrightarrow J \xrightarrow{\ \iota\ } A \xrightarrow{\ q\ } B \longrightarrow 0$$

such that $\iota_J : J \to M_J$ is injective with closed range, defined $\mathrm{Bus}(((\iota, A, q))$ to be the map $\varphi_A \in \hom_{\mathbf{Balg}}(B, C_J)$ that is given by Lemma 2.2.4 (5).

Now that we have a way of taking extensions to Busby maps, we need a way of taking morphisms between extensions to morphisms between the corresponding Busby maps. To this end, let $(\iota_i, A_i, q_i) \in \mathbf{Ext}(J)$ be extensions of B_i for $i \in \{1,2\}$, let $\xi_i = \mathrm{Bus}((\iota_i, A_i, q_i))$ and suppose there exists $(\theta, \varphi) \in \hom_{\mathbf{Ext}(J)}((\iota_1, A_1, q_1), (\iota_2, A_2, q_2))$. Set $\mathrm{Bus}((\theta,\varphi)) = \varphi$.

Proposition 2.3.2. *Fix a faithful Banach algebra J such that ι_J has closed range.* Bus *is a functor from $\boldsymbol{Ext}(J)$ to $\boldsymbol{Bus}(J)$.*

Proof. Let $(\iota_1, A_1, q_1) \in \mathbf{Ext}(J)$ be an extension of B_1 by J and suppose that $\xi_1 = \mathrm{Bus}((\iota_1, A_1, q_1))$. By construction $\mathrm{Bus}((\mathrm{id}_{A_1}, \mathrm{id}_{B_1})) = \mathrm{id}_{B_1}$ and id_{B_1} is the identity morphism for ξ_1 in $\mathbf{Bus}(J)$.

Now suppose also that $(\iota_2, A_2, q_2) \in \mathbf{Ext}(J)$ is an extension of B_2 by J, that $\xi_2 = \mathrm{Bus}((\iota_2, A_2, q_2))$, and that $(\theta, \varphi) \in \hom_{\mathbf{Ext}(J)}((\iota_1, A_1, q_1), (\iota_2, A_2, q_2))$. We wish to check that $\mathrm{Bus}((\theta,\varphi)) = \varphi \in \hom_{\mathbf{Bus}(J)}(\xi_1, \xi_2)$, which amounts to checking whether $\xi_1 = \xi_2\varphi$. Since q_1 is surjective, we can check whether $\xi_2\varphi q_1 = \xi_1 q_1$. Let $\theta_i : A_i \to M_J$ for $i \in \{1,2\}$ be the continuous homomorphisms associated with the extensions (ι_i, A_i, q_i) as constructed in Lemma 2.2.4. Now $\xi_2\varphi q_1 = \xi_2 q_2 \theta$ and $\xi_2 q_2\theta = q_J\theta_2\theta$. Now $q_J\theta_2\theta = q_J\theta_1$ if $\mathrm{Ran}(\theta_2\theta - \theta_1) \in \iota_J(J)$, so let $a \in A_1$ and consider $\theta_2\theta(a) - \theta_1(a)$. This is the multiplier $(L^{A_2}_{\theta(a)} - L^{A_1}_a, R^{A_2}_{\theta(a)} - R^{A_1}_a) \in M_J$. Next, letting $j \in J$ and taking the left-hand multiplier, $L^{A_2}_{\theta(a)}(j) - L^{A_1}_a(j) = {\iota_2}^{[-1]}(\theta(a)\iota_2(j)) - {\iota_1}^{[-1]}(a\iota_1(j))$, which in turn is equal to ${\iota_2}^{[-1]}(\theta(a)\theta\iota_1(j)) - {\iota_1}^{[-1]}(a\iota_1(j))$. Applying the homomorphism property of θ we get that ${\iota_2}^{[-1]}(\theta(a)\theta\iota_1(j)) - {\iota_1}^{[-1]}(a\iota_1(j)) = {\iota_2}^{[-1]}(\theta(a\iota_1(j)) - {\iota_1}^{[-1]}(a\iota_1(j)))$. From the fact that $\theta\iota_1 = \iota_2$, we have that ${\iota_2}^{[-1]}(\theta(a\iota_1(j)) - {\iota_1}^{[-1]}(a\iota_1(j))) = 0$. A similar property holds for the right-hand side, giving us that $q_J\theta_2\theta = q_J\theta_1 = \xi_1 q_1$. Therefore $\xi_2\varphi = \xi_1$ as required.

Lastly, suppose also that $(\iota_3, A_3, q_3) \in \mathbf{Ext}(J)$ is an extension of B_3, let $\mathrm{Bus}((\iota_3, A_3, q_3)) = \xi_3$ and let $(\theta', \varphi') \in \hom_{\mathbf{Ext}(J)}((\iota_2, A_2, q_2), (\iota_3, A_3, q_3))$. We have that $\mathrm{Bus}((\theta'\theta, \varphi'\varphi)) = \varphi'\varphi = \mathrm{Bus}((\theta', \varphi'))\,\mathrm{Bus}((\theta, \varphi))$. Therefore Bus is a functor. □

We now aim to construct a functor from $\mathbf{Bus}(J)$ to $\mathbf{Ext}(J)$.

Definition 2.3.3 $((\iota_\xi, P_\xi, \pi_B))$**.** Let $\xi \in \hom_{\mathbf{Balg}}(B, C_J)$, hence we have that $\xi \in \mathbf{Bus}(J)$. We shall take the pullback in **Balg** of ξ and q_J and aim to construct an extension. The pullback, which we will call P_ξ, will be the following set.

$$P_\xi = \{(m, b) \in M_J \times B : q_J(m) = \xi(b)\}.$$

This is an algebra with the usual scalar multiplication and coordinate-wise multiplication. Moreover, P_ξ is a Banach algebra when equipped with the max norm.

Note that there exists an injective, continuous homomorphism $\iota_\xi : J \to P_\xi$ given by $j \mapsto (\iota_J(j), 0)$. Where π_B is the surjective continuous homomorphism given by $(m, b) \mapsto b$, we have that $(\iota_\xi, P_\xi, \pi_B) \in \mathbf{Ext}(J)$.

Set $\mathrm{Pull}(\xi) = (\iota_\xi, P_\xi, \pi_B)$.

Next suppose we have two Busby maps $\xi_i \in \hom_{\mathbf{Balg}}(B_i, C_J)$ for $i = \{1, 2\}$ and $u \in \hom_{\mathbf{Bus}(J)}(\xi_1, \xi_2)$. We aim to show there exists a continuous homomorphism $\theta : P_{\xi_1} \to P_{\xi_2}$ such that when paired with u, forms a morphism in $\mathbf{Ext}(J)$ from $\mathrm{Pull}(\xi_1)$ to $\mathrm{Pull}(\xi_2)$.

Let π_{M_1} and π_{M_2} be the projections from P_{ξ_1} and P_{ξ_2} respectively to M_J, and similarly let π_{B_1} and π_{B_2} be the projections from P_{ξ_1} and P_{ξ_2} to B_1 and B_2 respectively. Since $\xi_1 = \xi_2 u$, we have that Diagram 2.7 commutes.

$$\begin{array}{ccccc} P_{\xi_1} & & \xrightarrow{\quad u\pi_{B_1} \quad} & & \\ & \searrow^{\pi_{M_1}} & P_{\xi_2} & \xrightarrow{\pi_{B_2}} & B_2 \\ & & \Big\downarrow{\scriptstyle \pi_{M_2}} & & \Big\downarrow{\scriptstyle \xi_2} \\ & & M_J & \xrightarrow[q_J]{} & C_J \end{array} \tag{2.7}$$

Therefore by the pullback property, there exists a unique continuous homomorphism $\theta \in \hom_{\mathbf{Balg}}(P_{\xi_1}, P_{\xi_2})$ such that $\pi_{B_2}\theta = u\pi_{B_1}$ and $\pi_{M_2}\theta = \pi_{M_1}$.

We claim that (θ, u) is a morphism from $\mathrm{Pull}(\xi_1)$ to $\mathrm{Pull}(\xi_2)$. Checking this amounts to verifying that Diagram 2.8 commutes.

$$\begin{array}{ccccc} J & \xrightarrow{\iota_{\xi_1}} & P_{\xi_1} & \xrightarrow{\pi_{B_1}} & B_1 \\ & \searrow^{\iota_{\xi_2}} & \Big\downarrow{\scriptstyle \theta} & & \Big\downarrow{\scriptstyle u} \\ & & P_{\xi_2} & \xrightarrow{\pi_{B_2}} & B_2 \end{array} \tag{2.8}$$

Firstly, let $j \in J$ and check

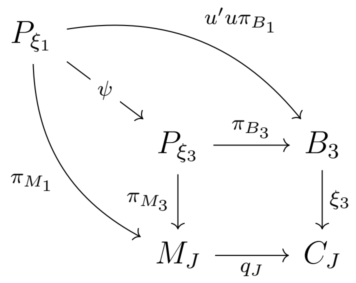
$$\theta\iota_{\xi_1}(j) = \theta((L_j, R_j), 0) = (\pi_{M_1}((L_j, R_j), 0), 0) = ((L_j, R_j), 0) = \iota_{\xi_2}(j).$$

The fact that $u\pi_{B_1} = \pi_{B_2}\theta$ is evident from the construction of θ. Now set $\mathrm{Pull}(u) = (\theta, u)$.

Proposition 2.3.4. *Fix a faithful Banach algebra J.* Pull *is a functor from **Bus**(J) to **Ext**(J).*

Proof. It is easy to verify that if $\xi \in \mathbf{Balg}(B, C_J)$, we have that $\mathrm{Pull}(\mathrm{id}_B) = (\mathrm{id}_{P_\xi}, \mathrm{id}_B)$, which is the identity morphism for $(\iota_\xi, P_\xi, \pi_B)$.

Now suppose that $\xi_i \in \mathbf{Balg}(B_i, C_J)$ for $i = \{1, 2, 3\}$ with $u \in \mathbf{Balg}(B_1, B_2)$ and $u' \in \mathbf{Balg}(B_2, B_3)$ as morphisms from ξ_1 to ξ_2 and ξ_2 to ξ_3 respectively. Take $\mathrm{Pull}(u) = (\theta, u)$ and $\mathrm{Pull}(u') = (\theta', u')$, we need to check that $\mathrm{Pull}(u'u) = \mathrm{Pull}(u')\,\mathrm{Pull}(u)$. We have that $\mathrm{Pull}(u'u) = (\psi, u)$ where ψ is the unique continuous homomorphism that makes Diagram 2.9 commute.



(2.9)

However, $\theta'\theta$ also makes this diagram commute, so we must have that $\psi = \theta'\theta$ because ψ was unique. Therefore $\mathrm{Pull}(u'u) = \mathrm{Pull}(u')\,\mathrm{Pull}(u)$ as required and we have that Pull is a functor. □

Remark 2.3.5. The constructions involved in Propositions 2.3.2 and 2.3.4 are known ones. However, the proof that they are functorial is new work.

We will show that Bus and Pull provide an equivalence between the categories $\mathbf{Ext}(J)$ and $\mathbf{Bus}(J)$. However, before we prove this claim, we require a technical lemma.

This is a special case of the short five lemma in the category of vector spaces, applied within **Balg**. We will provide a proof for the ease of the reader.

Lemma 2.3.6. *Let B and J be in* ***Balg*** *such that J is faithful. Let $(\iota_i, A_i, q_i) \in$* ***Ext***(J) *be extensions of B by J for $i \in \{1, 2\}$. If (θ, id_B) is a morphism from (ι_1, A_1, q_1) to (ι_2, A_2, q_2), then θ is an isomorphism in* ***Balg****.*

Proof. Let $x \in \iota_1(J) \cap \ker\theta$. We have that $x = \iota_1(j)$ for some unique $j \in J$ and that $\iota_2(j) = \theta(x) = 0$, but ι_2 was injective so $j = 0$ which implies $x = 0$. Therefore $\ker\theta \cap \iota_1(J) = \{0\}$.

Now let $x \in \ker\theta$. Then $x \in \ker q_2\theta = \ker q_1 = \iota_1(J)$. Therefore $\ker\theta \subseteq \iota_1(J)$ which we can combine with the fact that $\ker\theta \cap \iota_1(J) = \{0\}$ to give us that $\ker\theta = \{0\}$ and therefore that θ is injective.

Now we show that θ is surjective. To this end, let $a_2 \in A_2$. Since q_1 is surjective, there exists $a_1 \in A_1$ such that $q_2(a_2) = q_1(a_1) = q_2\theta(a_1)$. We therefore have that $q_2(a_2 - \theta(a_1)) = 0$ which implies that $a_2 - \theta(a_1) \in \iota_2(J)$. There then exists $j \in J$ such that $a_2 = \iota_2(j) + \theta(a_1) = \theta\iota_1(j) + \theta(a_1)$ so we have that $a_2 \in \operatorname{Ran}\theta$ as required and so θ is surjective.

From this, θ is bijective and by the Banach Isomorphism Theorem (Theorem A.2.4), θ has continuous inverse. In summary, we have that θ is an isomorphism of the extensions (ι_1, A_1, q_1) and (ι_2, A_2, q_2). □

Lemma 2.3.7. *Let $J, B \in$ **Balg** with J faithful, and suppose further that ι_J has closed range. Let $\xi \in \hom_{\boldsymbol{Balg}}(B, C_J)$ be a Busby map and let $\theta_\xi : P_\xi \to M_J$ be the function given by applying Lemma 2.2.4 to the extension $(\iota_\xi, P_\xi, \pi_B)$. Then $\theta_\xi = \pi_M$.*

Proof. Let $A = P_\xi$ for ease of notation and let $(m, b) \in A$. We have to show that $(L^A_{(m,b)}, R^A_{(m,b)}) = m$. Let $j \in J$. Then

$$\begin{aligned} L^A_{(m,b)}(j) &= {\iota_\xi}^{[-1]}((m,b)\iota_\xi(j)) \\ &= {\iota_\xi}^{[-1]}((m,b)(\iota_J(j),0)) \\ &= {\iota_\xi}^{[-1]}((m\iota_J(j),0)) \\ &= {\iota_J}^{[-1]}(m\iota_J(j)) \end{aligned}$$

and similarly $R^A_{(m,b)}(j) = {\iota_J}^{[-1]}(\iota_J(j)m)$. Now if $m = (L, R)$, then by the proof of Lemma 2.1.6

$$m\iota_J(j) = (L,R)(L_j, R_j) = (L_{L(j)}, R_{L(j)}) = \iota_J(L(j))$$

and similarly $\iota_J(j)m = \iota_J(R(j))$. Hence $L^A_{(m,b)}(j) = L(j)$ and $R^A_{(m,b)}(j) = R(j)$ as required. $\square$

Corollary 2.3.8. *Let $J \in$ **Balg** be faithful and suppose that ι_J has closed range. Then* Bus Pull *is the identity functor.*

Proof. We will first check Bus Pull is the identity on objects. If $\xi \in \mathrm{Bus}(J)$, then $\mathrm{Pull}(\xi) = (\iota_x i, P_\xi, \pi_B)$ and by Lemma 2.3.7, $\theta_\xi = \pi_M$. Now by the uniqueness part of Lemma 2.2.4 4), we see that $\mathrm{Bus\,Pull}(\xi) = \xi$ (see Diagram 2.5).

We now check that we have the identity on morphisms. If

$$\begin{array}{ccc} B_1 & & \\ & \searrow^{\varphi_1} & \\ \downarrow u & & C_J \\ & \nearrow_{\varphi_2} & \\ B_2 & & \end{array} \tag{2.10}$$

is a morphism in $\mathrm{Bus}(J)$. Then applying Pull gives us

$$\begin{array}{ccccc} J & \xrightarrow{\iota_{\xi_1}} & P_{\xi_1} & \xrightarrow{\pi_{B_1}} & B_1 \\ & \searrow^{\iota_{\xi_2}} & \downarrow \theta & & \downarrow u \\ & & P_{\xi_2} & \xrightarrow{\pi_{B_2}} & B_2 \end{array} \tag{2.11}$$

and $\mathbf{Bus}\,\mathrm{Pull}(u) = \mathbf{Bus}((\theta, u)) = u$ as required. Hence $\mathbf{Bus}\,\mathrm{Pull} = \mathrm{id}_{\mathrm{Bus}(J)}$.

$\square$

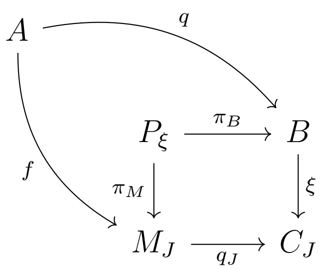
Theorem 2.3.9. *Fix a faithful Banach algebra J. The categories, **Ext**(J) and **Bus**(J) are equivalent in the sense of Definition A.1.15.*

Proof. By Corollary 2.3.8 $\mathrm{Bus\,Pull} = \mathrm{id}_{\mathbf{Bus}(J)}$. There is therefore no need to look for a natural isomorphism here since the functors are equal.

We now look for a natural isomorphism between Pull Bus and $\mathrm{id}_{\mathbf{Ext}(J)}$. Let $B \in$ **Balg**, let (ι, A, q) be an extension of B by J and let $\xi : B \to C_J$ be a Busby map.

We shall construct $\psi : A \to P_\xi$ such that (ξ, id_B) is an isomorphism in $\mathbf{Ext}(J)$ from (ι, A, q) to $(\iota_\xi, P_\xi, \pi_B)$.

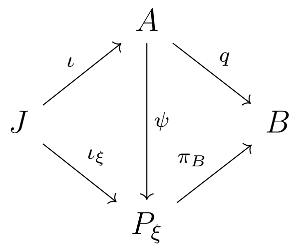
We have that Diagram 2.12 commutes.



(2.12)

By the pullback property, there exists a unique continuous homomorphism $\psi : A \to P_\xi$ such that $\pi_B\psi = q$ and $\pi_M\psi = f$.

From this, we can verify that Diagram 2.13 commutes.



(2.13)

The fact that $q = \psi\pi_B$ is evident from the construction of ψ. Let $j \in J$, and take $\psi\iota(j) = (f\iota(j), q\iota(j)) = (\iota_J(j), 0) = \iota_\xi(j)$. Therefore, $\psi\iota = \iota_\xi$ and the diagram commutes. This makes (ψ, id_B) into a morphism from (ι, A, q) to $(\iota_\xi, P_\xi, \pi_B)$ so by Lemma 2.3.6, (ψ, id_B) is an isomorphism of our extensions.

Let $B_1, B_2 \in \mathbf{Balg}$ and let (ι_i, A_i, q_i) be extensions of B_i by J for $i \in \{1, 2\}$. Together with a morphism (θ, φ) from (ι_1, A_1, q_1) to (ι_2, A_2, q_2). Explicitely, $\theta \in \hom_{\mathbf{Balg}}(A_1, A_2)$, $\varphi \in \hom_{\mathbf{Balg}}(B_1, B_2)$ and $\varphi q_1 = q_2\theta$.

Then $\operatorname{Pull}\operatorname{Bus}((\theta, \varphi))$ is given by the commuting square.

$$\begin{array}{ccc} P_{\xi_1} & \xrightarrow{\pi_{B_1}} & B_1 \\ \big\downarrow{\scriptstyle \theta'} & & \big\downarrow{\scriptstyle \varphi} \\ P_{\xi_2} & \xrightarrow{\pi_{B_2}} & B_2 \end{array} \tag{2.14}$$

where $\theta'((m, b)) = (m, \varphi(b))$. We also have isomorphims $\psi_i : A_i \to P_{\xi_i}$ in **Balg** for $i \in \{1, 2\}$ which make the analogues of Diagram 2.13 commute. To prove naturality, it suffices to show that $\theta'\psi_1 = \psi_2\theta$.

Let $a_1 \in A_1$, we compute that $\theta'\psi_1(a_1) = \theta'((\theta_1(a_1), q_1(a_1))) = (\theta_1(a_1), \varphi q_1(a_1)) = (\theta_1(a_1), q_2\theta(a_1))$. Now $(\theta_2\theta(a_1), q_2\theta(a_1)) = \psi_2\theta(a_1)$ as required. Therefore Pull Bus is naturally isomorphic to $\mathrm{id}_{\mathbf{Ext}(J)}$ and we have that $\mathbf{Ext}(J)$ and $\mathbf{Bus}(J)$ are equivalent as categories. □

In particular this equivalence means that extensions by J, up to isomorphism, are in one-to-one correspondence with Busby maps up to isomorphism. This reduces their study and classification to the study of continuous homomorphisms into the corona algebra of J. However, this feels unsatisfying as this correspondence is established subject to ι_J being injective with closed range. It is important to note that ι_J can be injective with non-closed range even in some relatively simple cases. We will cover some of these in Chapter 3.

2.4 Partial results when ι_J has non-closed range

Although we will still require J to be faithful in this section, we shall not assume that ι_J has closed range. This causes some parts of the general theory to fail, stemming from the possibility that the "multiplier extension" (see Definition 2.1.12) is not an extension of Banach algebras since the corona algebra is not necessarily a Banach algebra. Lemma 2.2.4 shows us that even when C_J is not a Banach algebra, an extension of Banach algebras still gives rise to an algebra homomorphism from the base of the extension to

C_J. In this section we begin the investigation of certain algebra homomorphisms from a Banach algebra B into C_J, which still give rise to extensions of B by J. A fuller investigation will be carried out in Chapter 5.

Proposition 2.4.1. *Let $B, J \in$ **Balg** such that J is faithful and let φ be an algebra homomorphism from B to the corona algebra, C_J. If there exists a bounded linear map $g : B \to M_J$ such that $\varphi = q_J g$, then P_φ is a Banach space with norm*

$$\|(m, b)\|_{P_\varphi} = \|m\|_{M_J} + \|b\|_B + \|{\iota_J}^{[-1]}(m - g(b))\|_J.$$

Proof. We will first show that $\| \cdot \|_{P_\varphi}$ is a norm for the underlying vector space of P_φ. Let $(m, b) \in P_\varphi$ and let $\lambda \in \mathbb{C}$. Then

$$\begin{aligned} \|\lambda(m, b)\|_{P_\varphi} &= \|\lambda m\|_{M_J} + \|\lambda b\|_B + \|{\iota_J}^{[-1]}(\lambda m - g(\lambda b))\|_J \\ &= \|\lambda m\|_{M_J} + \|\lambda b\|_B + \|{\iota_J}^{[-1]}(\lambda m - \lambda g(b))\|_J \\ &= \|\lambda m\|_{M_J} + \|\lambda b\|_B + \|\lambda {\iota_J}^{[-1]}(m - g(b))\|_J \\ &= |\lambda|\|m\|_{M_J} + |\lambda|\|b\|_B + |\lambda|\|{\iota_J}^{[-1]}(m - g(b))\|_J \\ &= |\lambda|\|(m, b)\|_{P_\varphi}. \end{aligned}$$

Now suppose $\|(m, b)\|_{P_\varphi} = 0$. Then $\|m\|_{M_J} + \|b\|_B + \|{\iota_J}^{[-1]}(m - g(b))\|_J = 0$, and so in particular $\|m\|_{M_J} = \|b\|_B = 0$. Therefore $m = 0_{M_J}$ and $b = 0_B$, so we have that $(m, b) = (0_M, 0_B)$.

Lastly let $(m_1, b_1), (m_2, b_2) \in P_\varphi$. We check

$$\begin{aligned}
\|(m_1 + m_2, b_1 + b_2)\|_{P_\varphi} =& \|m_1 + m_2\|_{M_J} + \|b_1 + b_2\|_B + \|{\iota_J}^{[-1]}(m_1 + m_2 - g(b_1 + b_2))\|_J \\
=& \|m_1 + m_2\|_{M_J} + \|b_1 + b_2\|_B \\
&+ \|{\iota_J}^{[-1]}(m_1 - g(b_1)) + {\iota_J}^{[-1]}(m_2 - g(b_2))\|_J \\
\leq& \|m_1\|_{M_J} + \|b_1\|_B + \|{\iota_J}^{[-1]}(m_1 - g(b_1))\|_J \\
+& \|m_2\|_{M_J} + \|b_2\|_B + \|{\iota_J}^{[-1]}(m_2 - g(b_2))\|_J \\
=& \|(m_1, b_1)\|_{P_\varphi} + \|(m_2, b_2)\|_{P_\varphi}.
\end{aligned}$$

So indeed, $\|\cdot\|_{P_\varphi}$ is a norm on the underlying vector space of P_φ. Next we check if P_φ is complete with respect to $\|\cdot\|_{P_\varphi}$.

Let $(m_n, b_n) \in P_\varphi$ be a Cauchy sequence. We have that (m_n), (b_n) and $({\iota_J}^{[-1]}(m_n - g(b_n)))$ are Cauchy sequences in M_J, B and J respectively. We therefore have that (m_n) and (b_n) converge to limits $m \in M_J$ and $b \in B$ respectively, and moreover, if $\lim_{n\to\infty} {\iota_J}^{[-1]}(m_n - g(b_n)) = y \in J$, we have that $\iota_J(y) = \lim_{n\to\infty} m_n - g(b_n) = m - g(b)$. So $\iota^{[-1]}(m - g(b)) \in J$ and hence our Cauchy sequence (m_n, b_n) converges to $(m, b) \in P_\varphi$. We therefore have that P_φ is a Banach space. □

Proposition 2.4.2. *Let B and J be Banach algebras such that J is faithful. Let $\varphi : B \to C_J$ and $g : B \to M_J$ be as in Proposition 2.4.1. There exists a constant K such that for all $(m_1, b_1), (m_2, b_2) \in P_\varphi$,*

$$\|(m_1 m_2, b_1 b_2)\|_{P_\varphi} \leq K \|(m_1, b_1)\|_{P_\varphi} \|(m_2, b_2)\|_{P_\varphi}.$$

Proof. We have that $\|(m_1 m_2, b_1 b_2)\|_{P_\varphi} = \|m_1 m_2\|_{M_J} + \|b_1 b_2\|_B + \|{\iota_J}^{[-1]}(m_1 m_2 - g(b_1 b_2))\|_J$. It is clear that since B and M_J are Banach algebras in their own right that $\|m_1 m_2\|_{M_J} \leq \|m_1\|_{M_J} \|m_2\|_{M_J}$ and $\|b_1 b_2\|_B \leq \|b_1\|_B \|b_2\|_B$, hence we need only focus on

$\|{\iota_J}^{[-1]}(m_1m_2 - g(b_1b_2))\|_J$. This can be rewritten as $\|{\iota_J}^{[-1]}((m_1 - g(b_1))(m_2 - g(b_2)) + m_1g(b_2) + g(b_1)m_2 - g(b_1)g(b_2) - g(b_1b_2))\|_J$. Since $m_i - g(b_i) \in \iota_J(J)$ for $i \in \{1, 2\}$, this is in turn less than or equal to

$$\|{\iota_J}^{[-1]}((m_1-g(b_1))(m_2-g(b_2)))\|_J+\|{\iota_J}^{[-1]}(m_1g(b_2)+g(b_1)m_2-g(b_1)g(b_2)-g(b_1b_2))\|_J.$$

Since ι_J is a homomorphism and the norm on J is submultiplicative, we have that

$$\|{\iota_J}^{[-1]}((m_1-g(b_1))(m_2-g(b_2)))\|_J \leq \|{\iota_J}^{[-1]}(m_1-g(b_1))\|_J\|{\iota_J}^{[-1]}(m_2-g(b_2))\|_J \tag{2.15}$$

Now $\|{\iota_J}^{[-1]}(m_1g(b_2) + g(b_1)m_2 - g(b_1)g(b_2) - g(b_1b_2))\|_J$ can be expressed as

$$\begin{aligned}
&\|{\iota_J}^{[-1]}((m_1 - g(b_1))g(b_2) + g(b_1)m_2 - g(b_1b_2))\|_J \\
&= \|{\iota_J}^{[-1]}((m_1 - g(b_1))g(b_2) + g(b_1)m_2 - g(b_1b_2) + g(b_1)g(b_2) - g(b_1)g(b_2))\|_J \\
&= \|{\iota_J}^{[-1]}((m_1 - g(b_1))g(b_2) + g(b_1)(m_2 - g(b_2)) - g(b_1b_2) + g(b_1)g(b_2))\|_J.
\end{aligned}$$

Note here that applying q_J to $g(b_1b_2) - g(b_1)g(b_2)$ gives $\varphi(b_1b_2) - \varphi(b_1)\varphi(b_2) = 0$. Since $(m_1 - g(b_1))g(b_2)$, $g(b_1)(m_2 - g(b_2))$, $g(b_1)g(b_2) - g(b_1b_2) \in \iota_J(J)$, we know this is less than or equal to

$$\|{\iota_J}^{[-1]}((m_1-g(b_1))g(b_2))\|_J+\|{\iota_J}^{[-1]}(g(b_1)(m_2-g(b_2)))\|_J+\|{\iota_J}^{[-1]}(g(b_1)g(b_2)-g(b_1b_2))\|_J.$$

Initially we focus on $\|{\iota_J}^{[-1]}((m_1-g(b_1))g(b_2))\|_J$. Note that since $m_1-g(b_1) \in \iota_J(J)$, we can express $m_1 - q(b_1)$ as (L_j, R_j) for some $j \in J$. Now let $(L, R) = g(b_2)$ and note that $(L_j, R_j)(L, R) = (L_jL, RR_j)$. We can compute that for $j' \in J$, $RR_j(j') = R(j'j) = j'R(j) = R_{R(j)}(j')$ and similarly $L_jL(j') = jL(j') = R(j)j' = L_{R(j)}(j')$. So

$(m_1 - g(b_1))g(b_2) = (L_{R(j)}, R_{R(j)})$ and hence

$$\|{\iota_J}^{[-1]}((m_1 - g(b_1))g(b_2))\|_J = \|R(j)\|_J \leq \|R\|\|j\|_J \leq \|g\|\|b_2\|_B\|{\iota_J}^{[-1]}(m_1 - g(b_1))\|_J \tag{2.16}$$

A similar argument gives us that

$$\|{\iota_J}^{[-1]}(g(b_1)(m_2 - g(b_2)))\|_J \leq \|g\|\|b_2\|_B\|\iota_J^{-1}(m_2 - g(b_2))\|_J. \tag{2.17}$$

Lastly we tackle $\|{\iota_J}^{[-1]}(g(b_1)g(b_2) - g(b_1b_2))\|_J$. Define the map $G : B \times B \to M_J, (b_1, b_2) \mapsto g(b_1b_2) - g(b_1)g(b_2)$. It is easily checked that ${\iota_J}^{[-1]}G$ is well defined and bilinear since ι_J is injective and $\operatorname{Ran} G \subseteq \operatorname{Ran} \iota_J$. We aim to show that ${\iota_J}^{[-1]}G$ is bounded in the sense that $\|\iota_J^{[-1]}G(b_1, b_2)\|_J \leq M\|b_1\|_B\|b_2\|_B$ for some fixed $M > 0$. If we show ${\iota_J}^{[-1]}G$ is continuous in each variable, then Banach-Steinhaus gives us the desired result. To this end, fix $a \in B$ and take a sequence $b_n \in B$ converging to $0 \in B$. Now assume ${\iota_J}^{[-1]}G(a, b_n)$ converges to $j \in J$. Since ι_J is continuous, $G(a, b_n)$ converges to $\iota_J(j)$, but since G is continuous in the second variable, $G(a, b_n)$ converges to 0 and hence $j = 0$. The Closed Graph Theorem now gives us that $\iota^{-1}G$ is continuous in the second variable. A similar argument gives us continuity in the first variable and Banach-Steinhaus gives us the existence of a $M > 0$ such that

$$\|{\iota_J}^{[-1]}(g(b_1)g(b_2) - g(b_1b_2))\|_J = \|{\iota_J}^{[-1]}G(b_1, b_2)\|_J \leq M\|b_1\|_B\|b_2\|_B. \tag{2.18}$$

We now see that for all $(m_1, b_1), (m_2, b_2) \in P_\varphi$, that

$$\|(m_1m_2, b_1b_2)\|_{P_\varphi} \leq (1 + \|g\| + M)\|(m_1, b_1)\|_{P_\varphi}\|(m_2, b_2)\|_{P_\varphi}.$$

Setting $K = 1 + \|g\| + M$ we have our result as required. □

Theorem 2.4.3. *Let B and J be Banach algebras such that J is faithful. Let $\varphi : B \to C_J$ be an algebra homomorphism. If there exists a bounded linear map $g : B \to M_J$ such that $\varphi = q_J g$, then there exists a norm which makes the pullback P_φ into a Banach algebra.*

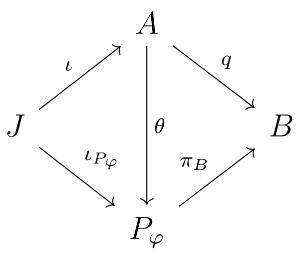
Proof. We know from Propositions 2.4.1 and 2.4.2 that there is a norm $\|\cdot\|_{P_\varphi}$ which makes P_φ into a Banach space and that this norm is submultiplicative up to a constant. We can now use Proposition 2.1.9 in [7, p. 156] to renorm P_φ in such a way that it is a Banach algebra. □

Remark 2.4.4. The fact that we can make the pullback into a Banach algebra with our choice of norm is not a method explored in the literature. In Chapter 5 we use Theorem 2.4.3 as a motivating case for a more general theory.

Before we move on to a case which motivates the rest of the thesis, we wish to show why this norming process is important. In particular, it will allow us to have a similar situation to the categorical equivalence described in Theorem 2.3.9.

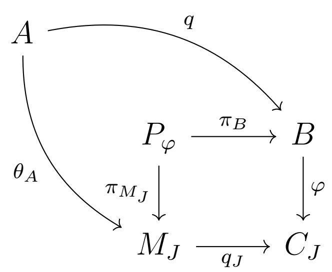
Remark 2.4.5. Note that even when ι_J does not have closed range, we can use the Bus functor to construct an algebra homorphism $\varphi : B \to C_J$ for any extension $(\iota, A, q) \in \mathbf{Ext}(J)$.

Theorem 2.4.6. *Let J and B be Banach algebras such that J is faithful. Let $(\iota, A, q) \in$ **Ext**(J) and let $\varphi = \mathrm{Bus}((\iota, A, q))$ in the sense of Remark 2.4.5. If there exists a bounded linear map $g : B \to M_J$ such that $h = q_J g$, then there exists a continuous isomorphism $\theta : A \to P_\varphi$ such that Diagram 2.19 commutes and hence (ι, A, q) is isomorphic to $(\iota_{P_\varphi}, P_\varphi, \pi_B)$.*



(2.19)

Proof. We have by the construction of φ that Diagram 2.19 commutes in the category of algebras with algebra homomorphisms.



(2.20)

Though C_J fails to be a Banach algebra in general, all the above objects are still algebras, and P_φ is still a pullback in the category of algebras, so there exists a homomorphism $\theta : A \to P_\varphi$ such that $q = \pi_B\theta$ and $\theta_A = \pi_{M_J}\theta$. With reference to Diagram 2.19, it is clear by construction that $q = \pi_B\theta$. We check the left-hand triangle, letting $j \in J$ and taking $\theta\iota(j) = (\theta_A\iota(j), q\iota(j)) = (\iota_J(j), 0) = \iota_{P_\varphi}(j)$. Now θ is a bijective linear homomorphism due to the short five lemma in the category of algebras and algebra homomorphisms, we need only check θ is continuous.

Our aim will be to use the Closed Graph Theorem, so we let $(a_n) \in A$ be a sequence with limit $a \in A$ such that $\theta(a_n)$ has limit $(m, b) \in P_\varphi$. However, since the norms of M_J and B are dominated by the norm of P_φ, we have that $\lim_{n\to\infty} \theta_A(a_n) = m$ in M_J and $\lim_{n\to\infty} q(a_n) = b$ in B. So $\theta(a) = (m, b)$ as required and hence θ is continuous. Hence we have that the extensions (ι, A, q) and $(\iota_{P_\varphi}, P_\varphi, \pi_B)$ are isomorphic.

$\square$

Theorem 2.4.6 shows that we can recover A, up to topological isomorphism, from the Busby map φ, and its lift g. This is useful for a specific class of extensions that always have a lift map g, allowing us to classify all extensions by J even when ι_J does not have closed range.

Corollary 2.4.7. *Let J be a faithful Banach algebra, let B be a finite-dimensional Banach algebra, and let (ι, A, q) be an extension of B by J. There exists an algebra homomorphism $\varphi : B \to C_J$, from which we can construct an extension $(\iota_{P_\varphi}, P_\varphi, \pi_B)$ of B by J such that (ι, A, q) is isomorphic to $(\iota_{P_\varphi}, P_\varphi, \pi_B)$.*

Proof. We can construct an algebra homomorphism from B into C_J by applying Bus in the sense of Remark 2.4.5. Let $\varphi = \mathrm{Bus}((\iota, A, q))$.

Since B is finite dimensional, say with $\dim(B) = n$, it has a basis $\{e_i \in B : 1 \leq i \leq n\}$. We can now use the fact that q_J is surjective to choose $m_i \in M_J$ such that $q_J(m_i) = \varphi(e_i)$. This prescription can be extended to a linear map $g : B \to M_J$ and since B is finite dimensional, g is bounded. Now we can apply Theorem 2.4.3 and construct an extension $(\iota_{P_\varphi}, P_\varphi, \pi_B)$ of B by J.

Now we can apply Theorem 2.4.6 to show that the pullback extension constructed from this homomorphism is isomorphic to (ι, A, q) as required. □

Notably, we see in the proof of Corollary 2.4.7 that when B is finite dimensional, we can construct an extension of B by J from any algebra homomorphism $\varphi : B \to C_J$.

In this case, although C_J is not assumed to be a Banach algebra, the finite-dimensionality of B allows us to construct a Banach algebra extension of B by J. This is motivation for exploring Banach algebra extensions even when the corona algebra is not a Banach algebra. As such, Chapter 4 will focus on constructing a category which can generalise this setup. However, before moving onto this, we will discuss some examples.

Chapter 3

Examples of multiplier algebras, Busby maps and extensions

Where Chapter 2 focussed on the abstract theory of extensions of Banach algebras, here we will look at some specific examples. These will highlight some of the concepts discussed in previous sections. We will highlight some cases where ι_J does not have closed range, look at some extensions where the base has finite dimension and show that we can pin down some Busby maps and extensions using our theory. In all of these cases, J will be a function algebra. In this setting, we will have an alternative definition of the multiplier algebra.

3.1 An equivalent description for multiplier algebras of function algebras

It is well known that there are multiple ways to describe multiplier algebras. In this thesis, we focus on the double centraliser approach taken by Busby in [2]. However, in the case where J is a function algebra with certain easily satisfied properties, M_J is also a function algebra. In this section, we will show that under certain conditions, the double centraliser approach is equivalent to M_J being a function algebra. We will then provide a result of Wang from [18] concerning these function algebras which we will use in the next section to analyse some specific examples.

Definition 3.1.1 (Notation for sets of functions)**.** In this thesis, we will use the notation Y^X to describe the functions from X to Y.

The next result is well known but we provide a proof.

Theorem 3.1.2. *Let $J \subseteq \mathbb{C}^X$ be an algebra of functions where X is a set. If for all $x \in X$, there exists $f \in J$ with $f(x) \neq 0$, then for all $(L, R) \in M_J$ there exists $h \in \mathbb{C}^X$ such that $L(g) = R(g) = hg$.*

Proof. Let $(L, R) \in M_J$, let $x \in X$ and let $f_1, f_2 \in J$ such that $f_1(x) \neq 0$ and $f_2(x) \neq 0$. Consider $L(f_1 f_2)$ evaluated at x. Since L is a left multiplier, we have that $L(f_1 f_2)(x) = L(f_1)(x) f_2(x)$. However, since J is commutative, we also have that this is equal to $L(f_2)(x) f_1(x)$ and hence that

$$\frac{L(f_1)(x)}{f_2(x)} = \frac{L(f_2)(x)}{f_1(x)}.$$

The quotient $L(f)(x)/f(x)$ is well defined since we can always choose an f which is not 0 at x and since the quotient is independent of our choice of f. Let $h(x) = L(f)(x)/f(x)$ for each $x \in X$, we now aim to show that $h(x) = R(f)(x)/f(x)$.

To see this, fix $x \in X$ and choose an $f \in J$ such that $f(x) \neq 0$. Now $L(ff)(x) = L(f)(x)f(x)$ and hence $L(f)(x)f(x) = f(x)L(f)(x) = R(f)(x)f(x)$. Dividing through by $f(x)$ gives us that $L(f)(x)/f(x) = R(f)(x)/f(x) = h(x)$ as required. □

Therefore, when J is an algebra of functions on a set X such that the conditions in Theorem 3.1.2 are satisfied, we can work with a function algebra definition of the multiplier algebra. This is useful, since it allows us to narrow down exactly what the multipliers are, as the Theorem 3.1.3 and the next section will illustrate.

Theorem 3.1.3 ([18] Theorem 3.1)**.** *Let X and J be as in Theorem 2.1.2, and suppose in addition that:*

- *X is a topological space and $J \subseteq C(X)$;*
- *J is a Banach algebra for some norm satisfying $|f(x)| \leq \|f\|$ for all $x \in X$.*

Then for all $(L, R) \in M_J$, the corresponding function $h \in \mathbb{C}^X$ is continuous and bounded, with $\|h\|_\infty \leq \|(L, R)\|$.

Remark 3.1.4. It is noted that Wang states this result for semisimple Banach algebras [18, Theorem 3.1] but upon inspection, his proof holds in the generality stated here in Theorem 3.1.3.

In particular, this is the first instance we see in this thesis of the multiplier algebra being narrowed down, in this case to $C_b(X)$.

In the rest of this chapter we will go through some examples of J where we can describe the multiplier algebra explicitly. In these cases, we will also look at extensions where B is $\mathbb{C}$. Since Busby maps are algebra homomorphisms, they must take $1 \in \mathbb{C}$ to an idempotent in C_J, whether C_J is a Banach algebra or not. Describing Busby maps is therefore equivalent to identifying idempotents in the corona. It is then possible to run these examples through Corollary 2.4.7 to construct extensions.

3.2 A C^*-algebra case

The first example will be a C^*-algebra, in this case take the C^*-algebra $C([-1, 1])$ and consider the maximal ideal $J = \{f \in C([-1, 1]) : f(0) = 0\}$. We will provide this as an example for ease and reference. Here we shall determine the multiplier algebra of J

and show that there are exactly four idempotents in C_J. This is not a new result, but we provide full details in order to motivate our approach in examples which are more in line with the theme of this thesis.

Example 3.2.1. We see that $J = \ker \varepsilon_0$ where $\varepsilon_0(f) = f(0)$. This gives us that $C([-1,1])$ is an extension of $\mathbb{C}$ by J. Our aim will be to show there are exactly four Busby maps from $\mathbb{C}$ into C_J. This is equivalent to identifying idempotents in C_J as discussed at the end of Section 2.1.

We can use Theorem 3.1.3 to see that the multiplier algebra M_J is $C_\mathrm{b}(X)$ where $X = [-1,0) \cup (0,1]$. It is still unknown what C_J is, but we can show it has at least four idempotents. Two of these are clearly $q_J(1)$ and $q_J(0)$ where 1 and 0 are the usual constant functions. However, consider the following two functions

$$f_1(x) = \begin{cases} 1 & x < 0 \\ 0 & x > 0 \end{cases} \qquad f_2(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases}$$

It is easy to see that $f_i^2 - f_i \in J$ for $i \in \{1,2\}$, so $q_J(f_i)$ is an idempotent for $i \in \{1,2\}$. Moreover, $f_1 - f_2 \notin J$ so these idempotents are distinct and neither $1 - f_i$ nor $0 - f_i$ is in J, there are at least four idempotents in C_J.

However, we can go further and say there are exactly four idempotents in C_J. Suppose there is $h \in M_J$ such that $g = h^2 - h \in J$. Then for any $\varepsilon > 0$, there exists $\delta > 0$ such that $|g(x)| < \varepsilon^2$ whenever $|x| < \delta$. This means that on this interval, either $|h(x)| < \varepsilon$ or $|h(x) - 1| < \varepsilon$. Suppose that there existed $x_1, x_2 \in (-\delta, 0)$ with $x_1 < x_2$ such that $|h(x_1)| < \varepsilon$ and $|h(x_2) - 1| < \varepsilon$, the latter implying $|h(x_2)| > 1 - \varepsilon$. Now by the Intermediate Value Theorem, there exists an $x_0 \in (x_1, x_2)$ such that $|h(x_0)| = 1/2$. Choosing $0 < \varepsilon^2 < 1/4$ yields a contradiction, since we found an $x_0 \in (\delta, 0)$ such that $|g(x)| > 1/4$. Therefore we have that either $h(x) \to 0$ or $h(x) \to 1$ as $x \nearrow 0$. A similar argument gives us that $h(x) \to 0$ or $h(x) \to 1$ as $x \searrow 0$.

We can now split this into four cases. If $h(x) \to 1$ as $x \to 0$ then $q_J(h) = q_J(1)$ and if $h(x) \to 0$ as $x \to 0$, then $q_J(h) = q_J(0)$. Now if $h(x) \to 1$ as $x \nearrow 0$ and $h(x) \to 0$ as $x \searrow 0$, then $h - f_1 \in J$ so $q_J(h) = q_J(f_1)$. Similarly if $h(x) \to 0$ as $x \nearrow 0$ and $h(x) \to 1$ as $x \searrow 0$, then $q_J(h) = q_J(f_2)$.

Since J is a C^*-algebra, it has a bounded approximate identity and hence by Lemma 2.1.14, we know it is closed in M_J, so this leaves us with a very clear picture of the multiplier extension. We can now look at extensions when the base is $\mathbb{C}$. We will start by stating all the Busby maps from $\mathbb{C}$ into C_J and then stating all the extensions of $\mathbb{C}$ by J.

Example 3.2.2. We know that C_J has four idempotents from Example 3.2.1. The Busby maps from $\mathbb{C}$ into C_J are precisely those that send 1 to one of the four idempotents in C_J. This follows trivially from the fact that Busby maps are homomorphisms. The four Busby maps are therefore the trivial homomorphism φ_{00}, φ_{11}, φ_{10} and φ_{01}. These are given by $1 \mapsto [q_J(0)]$, $1 \mapsto [q_J(1)]$, $1 \mapsto [q_J(f_1)]$ and $1 \mapsto [q_J(f_2)]$ respectively.

It is easy to see that $\mathrm{Pull}(\varphi_{00}) = (\iota_{00}, J \oplus \mathbb{C}, q_{00})$ from the pullback construction. Similarly, $\mathrm{Pull}(\varphi_{11}) = (\iota_{11}, C([-1,1]), q_{11})$, the extension we started with. Lastly, $\mathrm{Pull}(\varphi_{01}) = (\iota_{01}, C_0([-1,0)) \oplus C([0,1]), q_{01})$ and $\mathrm{Pull}(\varphi_{10}) = (\iota_{10}, C([-1,0]) \oplus C_0((0,1]), q_{10})$. It is interesting to note that while $C_0([-1,0)) \oplus C([0,1])$ and $C([-1,0]) \oplus C_0((0,1])$ are isomorphic as Banach algebras, the extensions are not isomorphic in the category $\mathbf{Ext}(J)$.

In the next two examples, we shall see what happens when we change the base. Initially we will consider $\mathbb{C}^2$.

Example 3.2.3. As we had with Example 3.2.2, the Busby maps will be characterised by where they send the generators of $\mathbb{C}^2$, namely $e_1 = (1,0)$ and $e_2 = (0,1)$. Not only are both e_1 and e_2 idempotents, but their product is $(0,0)$. This prevents us from using combinations such as $e_1 \mapsto [1]$ and $e_2 \mapsto [1]$. Moreover, there is one, non-trivial automorphism of $\mathbb{C}^2$, namely

$$\theta : \mathbb{C}^2 \to \mathbb{C}^2, \quad (\lambda_1 e_1, \lambda_2 e_2) \mapsto (\lambda_2 e_1, \lambda_1 e_2). \tag{3.1}$$

We can use this to identify isomorphic Busby maps, and hence isomorphic extensions.

In total, there are five Busby maps up to isomorphism, together with their extensions. We will go through each in turn. First we have the trivial Busby map given by $\varphi_1(e_1) = \varphi_1(e_2) = [q_J(0)]$. This corresponds to $(\iota_1, J \oplus \mathbb{C}^2, q_1)$. We next have the map given by $\varphi_2(e_1) = [q_J(1)]$ and $\varphi_2(e_2) = [q_J(0)]$, which corresponds to the extension $(\iota_2, C([-1,1]) \oplus \mathbb{C}, q_2)$. The automorphism θ gives us that φ_2 is isomorphic to the Busby map given by $\varphi_2'(e_1) = [q_J(0)]$ and $\varphi_2'(e_2) = [q_J(1)]$ since $\varphi_2' = \varphi_2\theta$.

Now we have the map given by $\varphi_3(e_1) = [q_J(f_1)]$ and $\varphi_3(e_2) = [q_J(0)]$. Its extension is $(\iota_3, C([-1,0]) \oplus C_0((0,1]) \oplus \mathbb{C}, q_3)$. Then we have the map given by $\varphi_4(e_1) = [q_J(f_2)]$ and $\varphi_4(e_2) = [q_J(0)]$. Similarly, its corresponding extension is $(\iota_4, C_0([-1,0)) \oplus C([0,1]) \oplus \mathbb{C}, q_4)$. Although these extensions have isomorphic Banach algebras, they are not isomorphic as extensions since there is no automorphism of $\mathbb{C}^2$ which makes φ_3 and φ_4 isomorphic in $\mathbf{Bus}(J)$. Lastly, we have the Busby map given by $\varphi_5(e_1) = [q_J(f_1)]$ and $\varphi_5(e_2) = [q_J(f_2)]$.

Before moving on to objects which are not C^*-algebras, we wish to introduce a third base which we will refer back to.

Example 3.2.4. Consider

$$B = \left\{ \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} : a, b \in \mathbb{C} \right\}. \tag{3.2}$$

For all $x \in B$, $x = \lambda I_2 + \mu z$ where $\lambda, \mu \in \mathbb{C}$ and

$$z = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \tag{3.3}$$

It is clear that I_2 and z generate B. In the same vein as Examples 3.2.2 and 3.2.3, we must have that I_2 is mapped to an idempotent. However, z is nilpotent and must therefore be mapped to a nilpotent element of C_J. Since there is no $h \in C_b(X) \backslash J$ such that $h^2 \in J$, all Busby maps must send z to $[q_J(0)]$. We therefore have a situation similar to Example 3.2.2 where there are four Busby maps and hence four extensions.

3.3 Introducing differentiability

After looking at continuous functions on $[-1, 1]$, it is natural to consider differentiable functions. Consider $C^1([-1, 1]) = \{f \in C([-1, 1]) : f' \in C([-1, 1])\}$. This is a Banach algebra with norm $\|f\|_{C^1} = \|f\|_\infty + \|f'\|_\infty$. Once again, we shall take the maximal ideal $\ker \varepsilon_0$ and in this case provide some new results. It should be noted that the multiplier algebras of the Banach algebras in the remainder of this section are not explored in the literature. This is perhaps due to the fact that the Banach algebras are not closed in their multiplier algebras and therefore do not fit into the current machinery. Since this thesis is interested in the cases where J is not closed in M_J, we have explored them in detail.

For these new results, a version of Taylor's theorem will be required, specifically the Peano remainder version. We state and reference this here for ease.

Theorem 3.3.1 (Taylor's theorem with Peano remainder)**.** *[8, Sec. 151, p. 509] Let $n \in \mathbb{N}$, let $a \in \mathbb{R}$, and let f be a function which is n times differentiable on some* *neighbourhood of a. Then there exists a function g_n such that*

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n + g_n(x)(x-a)^n$$

where $\lim_{x \to a} g_n(x) = 0$.

Proposition 3.3.2. *Let $J_1 = \{f \in C^1([-1,1]) : f(0) = 0\}$ and let $X = [-1,0) \cup (0,1]$. The multiplier algebra of J_1 is $\{h \in C([-1,1]) : h \in C^1(X) \& \lim_{x \to 0} xh'(x) = 0\}$.*

Proof. We will show that these conditions are necessary, and then that those conditions are sufficient. Suppose $h \in M_{J_1}$. Observe that J_1 is contained in the set of continuous functions vanishing at 0, so we know from Theorem 3.1.3 that $h \in C_{\mathrm{b}}(X)$. Let $n \in \mathbb{N}$ and consider the following family of functions.

$$f_n(x) = \begin{cases} 1 & 1/n \leq x \leq 1 \\ \frac{1}{2}(-\cos(n\pi x) + 1) & -1/n \leq x \leq 1/n \\ 1 & -1 \leq x \leq -1/n \end{cases} \tag{3.4}$$

For all $n \in \mathbb{N}$, $f_n \in J_1$ and hence $hf_n \in J_1$. However, for all $x \in X$, there exists $n_x \in \mathbb{N}$ such that $f_{n_x}(x) = 1$ so we have that h is differentiable away from 0.

Now take $u(x) = x$, clearly $u \in J_1$. Since $uh \in J_1$, it must be differentiable at least once. Applying Theorem 3.3.1 with $a = 0$ we see that $xh(x) = (uh)(0) + x(uh)'(0) + xg_n(x)$ where $\lim_{x \to 0} g_n(x) = 0$. Dividing through by x gives us that $h(x) = (uh)'(0) + g_n(x)$. However, since $\lim_{x \to 0} g_n(x) = 0$, we have that $\lim_{x \to 0} h(x) = (uh)'(0)$ so we can extend h continuously from X to $[-1,1]$.

Now observe that for all $x \in X$, we can say that $(uh)'(x) = xh'(x) + h(x)$. Taking limits as $x \to 0$ we see that $\lim_{x \to 0} xh'(x) = 0$. Therefore, all our conditions are necessary.

Now suppose that $h \in C([-1,1])$, $h \in C^1(X)$ and that $\lim_{x \to 0} xh'(x) = 0$. Let $f \in J_1$ Since h is differentiable away from 0, it is clear that hf is also differentiable away from 0. It is also clear that $hf(0) = 0$ since $h \in C([-1,1])$, so we need only check that hf is differentiable at 0 and that $(hf)'(x) \to (hf)'(0)$ as $x \to 0$.

Now for any $x \in X$, $(hf)'(x) = h(x)f'(x)+h'(x)f(x)$. We have that $\lim_{x \nearrow 0} h(x)f'(x) = \lim_{x \searrow 0} h(x)f'(x)$ leaving us with $h'(x)f(x)$. Using the mean value theorem, we see that $|f(x)| \leq |x|\|f'\|_\infty$ and hence that $|h'(x)f(x)| \leq |h'(x)||x|\|f'\|_\infty$. Lastly, we see that $\lim_{x \nearrow 0} h'(x)f(x) \leq \lim_{x \nearrow 0} h'(x)|x|\|f'\|_\infty = 0$ and similarly $\lim_{x \searrow 0} h'(x)f(x) = 0$. Therefore $\lim_{x \nearrow 0}(hf)'(x) = \lim_{x \searrow 0}(hf)'(x) = h(0)f'(0)$ as required.

Now we compute the difference quotient for hf at $x = 0$. Observe that

$$\frac{(hf)(x) - (hf)(0)}{x} = \frac{h(x)f(x)}{x} = h(x)\frac{f(x) - f(0)}{x}.$$

This has limit $h(0)f'(0)$ as $x \to 0$ as required. The conditions are therefore sufficient.

□

The description of M_{J_1} in Proposition 3.3.2 is in line with that of M_J in Example 3.2.1. Here we allow for the derivative of a multiplier to be discontinuous at 0 which we can compare to the multiplier itself having this property in M_J. However, we will find that this is where the similarities end. We begin by proving a result concerning the number of idempotents in C_{J_1}.

Proposition 3.3.3. *Let J_1 be as in Proposition 3.3.2. There are exactly two idempotents in C_{J_1}.*

Proof. It is clear that $q_{J_1}(0)$ and $q_{J_1}(1)$ are distinct idempotents in C_{J_1} where 0 and 1 represent constant functions taking these values. We therefore need to show that only two idempotents exist to complete the proof.

To this end, let $h \in M_{J_1}$ such that $g = h^2 - h \in J_1$. We have that the $q_{J_1}(h)$ must be an idempotent in C_{J_1}. Moreover, since both g and h are continuous at 0, $h(0)$ is either 0 or 1. Now we differentiate to give us $g'(x) = 2h'(x)h(x) - h(x) = h'(x)(2h(x) - 1)$ for all $x \in X$. Taking limits as $x \to 0$ we see that $\lim_{x \to 0} h'(x)$ is $\pm g'(0)$ and that h is differentiable at 0.

Finally, we check both cases. If $h(0) = 1$, then $h - 1 \in J_1$ and therefore $q_{J_1}(h) = q_{J_1}(1)$. If $h(0) = 0$, then $h \in J_1$ and $q_{J_1}(h) = 0$ as required. □

It is interesting to note that it is the addition of differentiability that restricts our available Busby maps and hence our available extensions when the base is $\mathbb{C}$.

Example 3.3.4. Consider the case where the base is $\mathbb{C}$. In contrast with Example 3.2.2, we now only have two Busby maps from $\mathbb{C}$ into C_{J_1}. These are given by $\varphi_1(1) = [q_{J_1}(1)]$ and $\varphi_2(1) = [q_{J_1}(0)]$ and yield the extensions $(\iota_1, J_1 \oplus \mathbb{C}, q_1)$ and $(\iota_2, C^1([-1, 1]), q_2)$ respectively.

We now turn our attention to the case where the base is $\mathbb{C}^2$.

Example 3.3.5. Let $e_1 = (1, 0)$ and $e_2 = (0, 1)$ as in Example 3.2.3. We now have that there are only two idempotents to map these generators to. There are now only two Busby maps up to isomorphism. The first is given by $\varphi_1(e_1) = \varphi_1(e_2) = [q_{J_1}(0)]$, yielding the extension $(\iota_1, J_1 \oplus \mathbb{C}^2, q_1)$. The second is given by $\varphi_2(e_1) = [q_{J_1}(1)]$ and $\varphi_2(e_2) = [q_{J_1}(0)]$, yielding the extension $(\iota_2, C^1([-1, 1]) \oplus \mathbb{C}, q_2)$.

However, we will see that if the base is the B described in Equation 3.2, that our Busby maps are not as restricted as they have been so far.

Example 3.3.6. Consider the following function

$$f(x) = \begin{cases} i\sqrt{-x} & x < 0 \\ \sqrt{x} & x \geq 0 \end{cases} \tag{3.5}$$

Clearly, $f \notin J_1$. We have that $f \in M_{J_1}$ since it is continuous on $[-1,1]$ and $xf'(x) \to 0$ as $x \to 0$. However, $f^2 = u \in J_1$ where $u(x) = x$. Therefore $[q_{J_1}(f)]$ is nilpotent in C_{J_1}. We therefore have at least four Busby maps.

Recall that z is the matrix in Equation 3.3. The first is given by $\varphi_1(I_2) = \varphi_1(z) = [q_{J_1}(0)]$. The second is given by $\varphi_2(I_2) = [q_{J_1}(1)]$ and $\varphi_2(z) = [q_{J_1}(0)]$. The third is given by $\varphi_3(I_2) = [q_{J_1}(0)]$ and $\varphi_3(z) = [q_{J_1}(f)]$. The fourth is given by $\varphi_4(I_2) = [q_{J_1}(1)]$ and $\varphi_4(z) = [q_{J_1}(f)]$.

At this point, it is not known whether this is all the Busby maps since there may be other nilpotent elements in C_{J_1}.

However, J_1 is even further removed from the C^* case. In this case, J_1 is not closed in M_{J_1} despite retaining a similar feel to a well known C^*-algebra. Before showing this, we will show that the multiplier norm is equivalent to a norm which is easier to work with.

Proposition 3.3.7. *Let J_1 and M_{J_1} be as in Proposition 3.3.2 and let $u(x) = x$. The norms $\|h\|_{M_{J_1}} = \sup\{\|hf\|_{J_1} : \|f\|_{J_1} = 1\}$ and $\|h\|_{\mathcal{M}} = \|h\|_\infty + \|uh'\|_\infty$ are equivalent.*

Proof. Take $\|h\|_{M_{J_1}} = \sup\{\|hf\|_{J_1} : \|f\|_{J_1} = 1\}$. Since $\|hf\|_\infty \leq \|h\|_\infty \|f\|_\infty \leq \|h\|_\infty$, we can see that $\|h\|_{M_{J_1}} \leq \|h\|_\infty + \sup\{\|(hf)'\|_\infty : \|f\|_{J_1} = 1\}$. We then apply the same technique, showing $\|(hf)'\|_\infty = \|hf' + h'f\|_\infty \leq \|h\|_\infty + \|h'f\|_\infty$. The mean value theorem then gives us that $\|h'f\|_\infty \leq \|uh'\|_\infty \|u^{-1}f\|_\infty \leq \|uh'\|_\infty \|f'\|_\infty \leq \|uh'\|_\infty$. Therefore we have that $\|h\|_{M_{J_1}} \leq 2\|h\|_{\mathcal{M}}$.

For the second inequality, we spot that $\|h\|_\infty + \|uh'\|_\infty = 2\|h\|_\infty + \|uh'\|_\infty - \|h\|_\infty$. This is in turn less than or equal to $2\|h\|_\infty + \|uh' - h\|_\infty = 2\|h\|_\infty + \|(uh)'\|_\infty$. We have from Theorem 3.1.3 that $\|h\|_\infty \leq \|h\|_{M_{J_1}}$ and $\|(uh)'\|_\infty \leq \|uh\|_{J_1} \leq 2\|h\|_{M_{J_1}}$. Therefore $\|h\|_M \leq 4\|h\|_{M_{J_1}}$ and the norms are therefore equivalent.

□

Proposition 3.3.8. *If J_1 and M_{J_1} are as in Proposition 3.3.2, then J_1 is not closed in M_{J_1}.*

Proof. Showing that J_1 is not closed with respect to $\| \cdot \|_{M_{J_1}}$ is equivalent to showing that it is not closed with respect to $\| \cdot \|_{\mathcal{M}}$. Consider the sequence of functions

$$f_n(x) = \begin{cases} 0 & x < 0 \\ x^{\frac{n+1}{n}} & x > 0 \end{cases}$$

where $n \in \mathbb{N}$. It is easily checked that $f_n \in J_1$ for each $n \in \mathbb{N}$, however we claim that (f_n) has limit

$$f(x) = \begin{cases} 0 & x < 0 \\ x & x > 0 \end{cases}$$

which is not differentiable at 0. To this end take $\|f - f_n\|_{\mathcal{M}} = \|f - f_n\|_\infty + \|u(f - f_n)'\|_\infty$. As $n \to \infty$, we have that $\sup\limits_{x \geq 0} |x - x^{\frac{n+1}{n}}| \to 0$. Now $(x - x^{\frac{n+1}{n}})' = (1 - \frac{n+1}{n} x^{\frac{1}{n}})$ so $\sup\limits_{x \geq 0} |x(x - x^{\frac{n+1}{n}})'| = \sup\limits_{x \geq 0} |x - x^{\frac{n+1}{n}}|$ which we have shown tends to 0 as $n \to \infty$. We therefore have that J_1 is not closed in M_{J_1} as required. □

It is therefore evident that the addition of differentiability has removed us from the C^* case described in Example 3.2.1.

3.4 Higher levels of differentiability

A natural question to consider here is whether we can say something similar about higher differentiable functions. However, we must first prove a technical lemma.

Lemma 3.4.1. *Let $m \in \mathbb{N}$ and let $f \in C^{m-1}([-1,1])$. Define*

$$F(t) = \frac{1}{t}\int_0^t f(s)ds \tag{3.6}$$

for $t \neq 0$. If $t^{-m}f(t) \to 0$, then $t^{-(m-i)}F^{(i)}(t) \to 0$ for all $i \in \{0,\ldots,m\}$.

Proof. We seek to prove this by induction on i. For our basis of induction, let $i = 0$.

Since $x^{-m}f(x) \to 0$ as $x \to 0$, we have that for $\varepsilon > 0$, there exists $\delta > 0$ such that for all $t \in [-\delta, 0) \cup (0, \delta]$, that $|t^{-m}f(t)| < \varepsilon$. Now for all $t \in [-\delta, 0) \cup (0, \delta]$, it follows that

$$|t^{-m}F(t)| = \left|\frac{1}{t^{m+1}}\int_0^t f(s)ds\right| \leq \frac{1}{|t^{m+1}|}\int_0^t |f(s)|ds \leq \frac{1}{|t^{m+1}|}\int_0^t \varepsilon|s^m|ds \leq \frac{1}{m+1}\varepsilon.$$

Hence we have that $t^{-m}F(t) \to 0$.

Now assume this is true for $i = n$ and consider $t^{-(m-(n+1))}F^{(n+1)}$. Note that since $f \in C^{m-1}([-1,1])$, that $F \in C^m(X)$ and observe that $f = (uF)' = uF' + F$. Differentiating this n times gives us that $f^{(n)} = uF^{(n+1)} + (n+1)F^{(n)}$. Now we have that

$$t^{-(m-(n+1))}F^{(n+1)}(t) = t^{-(m-n)}tF^{(n+1)}(t) = t^{-(m-n)}f^{(n)}(t) - (n+1)t^{-(m-n)}F^{(n)}(t)$$

which has limit 0 as $t \to 0$. We therefore have that the statement is true by induction.

□

Theorem 3.4.2. *Let $h \in C_b([-1,1])$, let $X = [-1,0)\cup(0,1]$, let $J_k = \{f \in C^k([-1,1]) : f(0) = 0\}$, and let u be the function given by $u(x) = x$. The following are equivalent:*

1. *$h \in M_{J_k}$*

2. *$uh \in J_k$*

3. $h \in C^k(X)$*,* $h \in C^{k-1}([-1,1])$ *and* $uh^{(k)}(x) \to 0$ *as* $x \to 0$*.*

Proof. $(1 \Rightarrow 2)$ holds trivially for all k.

We aim to show $(2 \Rightarrow 3)$. First note that repeated use of the quotient rule on $h = uh/u$ gives us that $h \in C^k(X)$. Now take $g = (uh)'$ where $u(x) = x$ as usual. Since $uh \in J_k$, we must have that $g \in C^{(k-1)}([-1,1])$. Applying Theorem 3.3.1 to g its derivatives with $a = 0$ we have that

$$\begin{aligned}
&x^{-(k-1)}\left(g(x) - \left(g(0) + g'(0)x + \ldots + g^{(k-1)}\frac{x^{k-1}}{(n+1)!}\right)\right) && \to 0 \\
&\vdots && \vdots \\
&x^{-1}\left(g^{(k-2)} - \left(g^{(k-2)}(0) + g^{(k-1)}(0)x\right)\right) && \to 0 \\
&g^{(k-1)}(x) - g^{(k-1)}(0) && \to 0
\end{aligned}$$

as $x \to 0$. Let $r(x)$ be the Peano remainder for the first equation so that

$$g(x) = g(0) + g'(0)x + \ldots + g^{(k-1)}\frac{x^{k-1}}{(n+1)!} + r(x).$$

By definition, $x^{-(k-1)}r(x) \to 0$ as $x \to 0$. Moreover, differentiating r and substituting into the above identities gives us that $x^{-i}r^{(k-1-i)} \to 0$ as $x \to 0$ for $i \in \{0, \ldots, k-1\}$.

By construction, we have that

$$h(t) = \frac{1}{t}\int_0^t g(s)ds.$$

Substituting the Taylor expansion of g for g in this gives

$$h(t) = g(0) + g'(0)\frac{t}{2!} + \ldots + g^{(k-1)}(0)\frac{t^{k-1}}{k!} + R(t) \tag{3.7}$$

where

$$R(t) = \frac{1}{t}\int\limits_0^t r(s)ds.$$

Note that $R \in C^k(X)$ since $h \in C^k(X)$ and that

$$h^{k-1}(t) = \frac{g^{(k-1)}(0)}{k} + R^{(k-1)}(t) \quad \text{and} \quad h^{(k)} = R^{(k)}. \tag{3.8}$$

The claim that $2 \Rightarrow 3$ follows if both $R^{(k-1)}(t) \to 0$ and $uR^{(k)}(t) \to 0$ as $t \to 0$.

Applying Lemma 3.4.1 to R and r yields that $R^{(k-1)}(t) \to 0$ as $t \to 0$. To see the second condition holds, note that $uR^{(k)} = r^{(k-1)} - (k)R^{(k-1)}$ which has limit 0 as $t \to 0$ as required.

To see that $(3 \Rightarrow 1)$, suppose that h has the properties in the third statement. Let $f \in J_k$ and compute $(hf)^{(k)}(x)$ away from 0. We have that

$$(hf)^{(k)}(x) = \sum_{m=0}^{k} \begin{pmatrix} k \\ m \end{pmatrix} h^{(k-m)}(x) f^{(m)}(x)$$

Now $h^{(k-m)}(0)f^{(m)}(0)$ clearly exists for $m \in \{1, \ldots, k\}$, leaving us with the last term $h^{(k)}(x)f(x)$. However, since $f(0) = 0$, the Mean Value Theorem gives us that $|f(x)| \leq |x|\|f'\|_\infty$ and hence that $\lim\limits_{x\to 0} f(x)h^{(k)}(x) \leq \lim\limits_{x\to 0} xh^{(k)}(x)\|f'\|_\infty = 0$. Now we compute the difference quotient for $(hf)^{(k-1)}$ at 0. That is

$$\frac{(hf)^{(k-1)}(x) - (hf)^{(k-1)}(0)}{x}.$$

We can expand $(hf)^{(k-1)}(x)$ and $(hf)^{(k-1)}(x)$ to give

$$\frac{\sum\limits_{m=0}^{k-1} \begin{pmatrix} k-1 \\ m \end{pmatrix} h^{(k-1-m)}(x) f^{(m)}(x) - \sum\limits_{m=0}^{k-1} \begin{pmatrix} k-1 \\ m \end{pmatrix} h^{(k-1-m)}(0) f^{(m)}(0)}{x}.$$

We can add

$$\sum_{m=1}^{k-1} \begin{pmatrix} k-1 \\ m \end{pmatrix} (h^{(k-1-m)}(0)f^{(m)}(x) - h^{(k-1-m)}(0)f^{(m)}(x))$$

from the numerator, noting especially that we do not include $h^{(k-1)}(0)f(x)$ in this sum. This simplifies to $h^{(k-1)}(x)\frac{f(x)-f(0)}{x} + S$ where S is

$$\sum_{m=1}^{k-1} \begin{pmatrix} k-1 \\ m \end{pmatrix} \left(h^{(k-1-m)}(0)\frac{f^{(m)}(x)-f^{(m)}(0)}{x} + f^{(m)}(x)\frac{h^{(k-1-m)}(x)-h^{(k-1-m)}(0)}{x}\right).$$

Taking limits as $x \to 0$ gives us that the difference quotient has limit

$$h^{(k-1)}(0)f'(0) + \sum_{m=1}^{k-1} \begin{pmatrix} k-1 \\ m \end{pmatrix} \left(h^{(k-1-m)}(0)f^{(m+1)}(0) + h^{(k-m)}(0)f^{(m)}(0)\right).$$

Hence we have that $hf \in J_k$ and that $h \in M_{J_k}$ as required. We therefore have that 1, 2 and 3 are equivalent. □

Once again, we see that there are similarities with the C^* case, despite C_{J_k} not being a Banach algebra. Interestingly, these J_k diverge even further from being C^*-algebras in a different way. Whereas all C^*-algebras have bounded approximate identities, and are hence closed in their multiplier algebras, J_k fails to even have an approximate identity, as the following result shows.

Proposition 3.4.3. *If J_k is as in Theorem 3.4.2 then J_k has no approximate identity.*

Proof. Suppose that J_k has an approximate identity $(e_\lambda)_{\lambda\in\Lambda}$ and aim for a contradiction. Since (e_λ) is an approximate identity, $\|u - e_\lambda u\|_{J_k} = \|u - e_\lambda u\|_\infty + \|(u - e_\lambda u)'\|_\infty + \ldots + \|(u - e_\lambda u)^{(k)}\|_\infty$ must go to 0 as $\lambda \to \infty$. However, we see that $|(u - e_\lambda u)'(0)| =$

$|u'(0) - e_\lambda(0)u'(0) + e'_\lambda(0)u(0)| = 1$, so $\|u - e_\lambda u\|_{J_k} \geq 1$ for all $\lambda \in \Lambda$ and e_λ is not an approximate identity. □

Lemma 3.4.4. *Let J_k and M_{J_k} be as in Theorem 3.4.2 and let $\|h\|_{\mathcal{M}_k} = \|h\|_\infty + \ldots + \|h^{(k-1)}\|_\infty + \|uh^{(k)}\|_\infty$. There exists an $M > 0$ such that $\|h\|_{M_{J_k}} \leq M\|h\|_{\mathcal{M}_k}$.*

Proof. Take $\|h\|_{M_{J_k}} = \sup\{\|hf\|_{J_k} : \|f\|_{J_k} = 1\}$. Similarly to Proposition 3.3.7, $\sup\{\|hf\|_\infty : \|f\|_{J_k} = 1\} \leq \|h\|_\infty$ so

$$\|h\|_{M_{J_k}} \leq \|h\|_\infty + \sup\{\|(hf)'\|_\infty + \ldots + \|(hf)^{(k)}\|_\infty : \|f\|_{J_k} = 1\}.$$

Next we see that $\|(hf)'\|_\infty \leq \|h\|_\infty + \|h'\|_\infty$ and extrapolate to see that for $j \in \{1, \ldots, k-1\}$, that

$$\|(hf)^{(j)}\|_\infty = \left\| \sum_{i=0}^{j} \begin{pmatrix} j \\ i \end{pmatrix} h^{(j-i)} f^{(i)} \right\|_\infty \leq \sum_{i=0}^{j} \begin{pmatrix} j \\ i \end{pmatrix} \left\| h^{(j-i)} \right\|_\infty \|f^{(i)}\|_\infty.$$

Lastly we use the Mean Value Theorem as we did in proposition 3.3.7 to give us $\|h^{(k)} f\| \leq \|uh^{(k)}\|_\infty$. Now it is clear that

$$\begin{aligned} \|h\|_{M_{J_k}} &\leq k \sup\{\|(hf)^{(k-1)}\|_\infty : \|f\|_{J_k} = 1\} + \|uh^{(k)}\|_\infty \\ &\leq k \sum_{i=0}^{k-1} \begin{pmatrix} k-1 \\ i \end{pmatrix} \left\| h^{(k-1-i)} \right\|_\infty + \|uh^{(k)}\|_\infty \\ &\leq k \sum_{i=0}^{k-1} \begin{pmatrix} k-1 \\ i \end{pmatrix} \|h\|_{\mathcal{M}_k}. \end{aligned}$$

□

Proposition 3.4.5. *Let J_k and M_{J_k} be as in Theorem 3.4.2. Then J_k is not closed in M_{J_k}.*

Proof. Consider the family of functions

$$f_n(x) = \begin{cases} 0 & x < 0 \\ x^{k-1}x^{\frac{n+1}{n}} & x > 0 \end{cases}$$

for $n \in \mathbb{N}$. Clearly, $f_n \in J_k$ for each n but (f_n) has limit

$$f(x) = \begin{cases} 0 & x < 0 \\ x^k & x > 0 \end{cases}$$

with respect to $\|\cdot\|_{\mathcal{M}_k}$ as $n \to \infty$. Hence we have by Lemma 3.4.4 that f_n converges to f in M_{J_k}. It is easily seen that f is not k-times differentiable at 0 so J_k is not closed in M_{J_k} as required. □

3.5 Vanishing at infinity

If we were to consider instead the C^*-algebra $C_0(\mathbb{R})$ we would see that its multiplier algebra is $C_b(\mathbb{R})$. The last Banach algebra we will analyse here will be the functions in $C_0(\mathbb{R})$ which also have bounded derivative.

Proposition 3.5.1. *Let $J_\infty = \{f \in C_0(\mathbb{R}) : f' \in C_b(\mathbb{R})\}$. The multiplier algebra of J_∞ is $\{h \in C_b(\mathbb{R}) : h' \in C_b(\mathbb{R})\}$.*

Proof. Let $h \in M_{J_\infty}$ and take the following family of functions.

$$f_n(x) = \begin{cases} 1 & -n \le x \le n \\ \frac{1}{2}(\cos(x+n)+1) & -n-\pi \le x \le -n \\ \frac{1}{2}(\cos(x-n)+1) & n \le x \le n+\pi \\ 0 & x \le -n-\pi, x \ge n+\pi \end{cases} \tag{3.9}$$

We can see that $f_n \in J_\infty$ for all $n \in \mathbb{N}$ so $hf_n \in J_\infty$ since h is a multiplier. However, we now have that for all $[-a,a] \subset \mathbb{R}$, there exists $n_a \in \mathbb{N}$ such that $|h(x)f_{n_a}(x)| = |h(x)|$ for all $x \in [-a,a]$. Therefore $h \in C_b(\mathbb{R})$ and is also differentiable everywhere.

Similarly, we see that for all $[-a,a] \subseteq \mathbb{R}$, there exists $n_a \in \mathbb{N}$ such that $|(hf_{n_a})'(x)| = |h(x)f'_{n_a}(x) + h'(x)f_{n_a}(x)| = |h'(x)|$. Hence we have that $h' \in C_b(\mathbb{R})$ as required.

Now we suppose that $h \in C_b(\mathbb{R})$ and that $h' \in C_b(\mathbb{R})$, let $f \in J_\infty$. We see that hf is bounded and differentiable since both h and f are, and that $hf \in C_0(\mathbb{R})$ since $f \in C_0(\mathbb{R})$. Lastly, we observe that $|(hf)'(x)| = |h(x)f'(x) + h'(x)f(x)| \leq \|h\|_\infty \| \|f'\|_\infty + \|h'\|_\infty \|f\|_\infty$. Therefore $hf \in J_\infty$ and hence $h \in M_{J_\infty}$.

□

Once again, we retain a similar feel to the C^* case. However, since our vanishing point is at infinity, and not a fixed point as in Proposition 3.3.2 and Theorem 3.4.2, we have a different result concerning idempotents.

Proposition 3.5.2. *Let J_∞ and M_{J_∞} be as in Proposition 3.5.1. There are at least four idempotents in C_{J_∞}*

Proof. We have that $q_{J_\infty}(0)$ and $q_{J_\infty}(1)$ are both idempotents. Now take the following two functions.

$$f_1(x) = \begin{cases} 1 & x < -\pi/2 \\ \frac{1}{2}(1 - \sin x) & -\pi/2 \leq x \leq \pi/2 \\ 0 & x \geq \pi/2 \end{cases} \qquad f_2(x) = \begin{cases} 1 & x < -\pi/2 \\ \frac{1}{2}(\sin x + 1) & -\pi/2 \leq x \leq \pi/2 \\ 0 & x \geq \pi/2 \end{cases}$$

It is easy to see that $\|f_1\|_\infty = \|f_2\|_\infty = 1$. We now calculate the derivatives.

$$f_1'(x) = \begin{cases} 0 & x < -\pi/2 \\ -\frac{1}{2}\cos x & -\pi/2 \leq x \leq \pi/2 \\ 0 & x \geq \pi/2 \end{cases} \qquad f_2'(x) = \begin{cases} 0 & x < -\pi/2 \\ \frac{1}{2}\cos x & -\pi/2 \leq x \leq \pi/2 \\ 0 & x \geq \pi/2 \end{cases}$$

Here we have that $\|f_1'\|_\infty = \|f_2'\|_\infty = 1/2$ and hence $f_1, f_2 \in M_{J_\infty}$. Now we need to show that $g_i = f_i^2 - f_i \in J_\infty$ for $i \in \{1, 2\}$. We provide g_1 and g_2.

$$g_1(x) = \begin{cases} 0 & x < -\pi/2 \\ \frac{1}{4}\cos^2 x & -\pi/2 \leq x \leq \pi/2 \\ 0 & x \geq \pi/2 \end{cases} \qquad g_2(x) = \begin{cases} 0 & x < -\pi/2 \\ -\frac{1}{4}\cos^2 x & -\pi/2 \leq x \leq \pi/2 \\ 0 & x \geq \pi/2 \end{cases}$$

Clearly both g_1 and g_2 are vanishing at infinity. Moreover,

$$\|g_2'\|_\infty = \|g_1'\|_\infty = \sup\{|\frac{1}{2}\cos x \sin x| : x \in [-\pi/2, \pi/2]\}$$

which is bounded above by 1 so $g_1, g_2 \in J_\infty$. Finally, $f_1 - f_2$, $f_1 - 1$ and $f_2 - 1$ are all not in J_∞ so $q_{J_\infty}(f_1)$, $q_{J_\infty}(f_2)$, $q_{J_\infty}(1)$ and $q_{J_\infty}(0)$ are distinct idempotents in C_{J_∞}. □

We therefore see in this section that there is still structure in these cases where our J is not closed in M_J.

Chapter 4

Building the correct category for studying extensions in Balg

When studying these extensions of Banach algebras, Busby's method of analysing them does not hold when ι_J does not have closed range. This, as we have seen, is because the corona algebra of J, $M_J/\iota_J(J)$ fails to be a Banach algebra.

However, [16] describes a machinery for studying quotients of Banach spaces where the subspace need not be closed. This is subtler than looking at seminormed spaces, and it draws its inspiration from the world of derived categories. Further into this series of papers, in [17], Waelbroeck applies this theory to Banach algebras, where it would be of particular use to us.

However, the paper [16] occasionally oversteps when trying to simplify the subtle derived category theory machinery, often making mistakes and omitting details. We therefore seek corroborating material to back this up. These mistakes are pointed out in Section 4 of [19].

4.1 Mon(**Ban**) and hMon(**Ban**)

Fortunately [19] also provides the assurance we need and we follow Waelbroeck's general setup for constructing a category of formal quotients, which we apply to the category **Ban** of Banach spaces and bounded linear maps in this section.

The main benefit of going through this construction with Banach spaces, and not general objects in an additive category, is that we gain a real understanding for the morphisms involved. Moreover, Wegner [19] does not explore the algebra structure which we hope to study, so an in-depth knowledge of the maps will be beneficial. This will further enhance our ability to classify extensions when the time comes.

The following results will be adaptations of those in [19]. Occasionally, the proofs will differ from the source material due to the nature of specifying a category. We begin with the category of monics, where we will base our theory.

Definition 4.1.1 (Mon(**Ban**))**.** Define objects in Mon(**Ban**) to be the monomorphisms in **Ban**. To be precise: objects of Mon(**Ban**) are triples (X', f_X, X) where $X', X \in$ **Ban** and $f_X : X' \to X$ is an injective bounded linear map.

Let $f_X : X' \to X, f_Y : Y' \to Y \in$ Mon(**Ban**). A morphism from f_X to f_Y will be a pair of bounded linear maps (α', α) which makes diagram 4.1 commute.

$$\begin{array}{ccc} X' & \xrightarrow{\alpha'} & Y' \\ \big\downarrow{\scriptstyle f_X} & & \big\downarrow{\scriptstyle f_Y} \\ X & \xrightarrow{\alpha} & Y \end{array} \tag{4.1}$$

Remark 4.1.2 (Important notational convention)**.** We frequently use the following convention. Whenever we refer to $f_X \in$ Mon(**Ban**), it is understood that f_X has domain X' and codomain X.

Remark 4.1.3. Given (α', α) as in Definition 4.1.1, there is a unique linear map $T_\alpha : X/f_X(X') \to Y/f_Y(Y')$ making the following diagram commute.

$$\begin{array}{ccc} X' & \xrightarrow{\quad\alpha'\quad} & Y' \\ \downarrow{\scriptstyle f_X} & & \downarrow{\scriptstyle f_Y} \\ X & \xrightarrow{\quad\alpha\quad} & Y \\ \downarrow{\scriptstyle q_X} & & \downarrow{\scriptstyle q_Y} \\ X/f_X(X') & \xrightarrow{T_\alpha} & Y/f_Y(Y') \end{array} \tag{4.2}$$

where q_X and q_Y are the natural quotient maps of vector spaces. If we write $\widehat{x}$ for $q_X(x)$ and $\widehat{y}$ for $q_Y(y)$, then T_α is defined by the formula $T_\alpha(\widehat{x}) = \widehat{\alpha(x)}$.

Proposition 4.1.4. *The collection of objects* Mon(***Ban***)*, together with the morphisms described above, forms a category (see Definition A.1.1).*

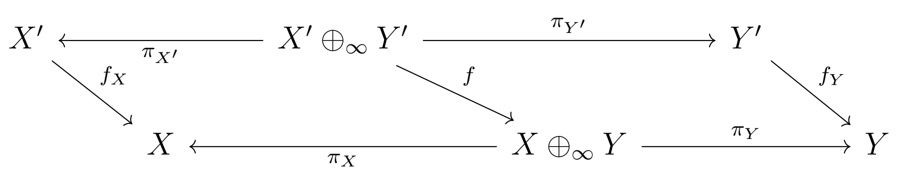
Proof. Clearly, for an $f_X : X' \to X \in \text{Mon}(\mathbf{Ban})$ the pair $(\text{id}_{X'}, \text{id}_X)$ forms an identity morphism.

For any three $f_X, f_Y, f_Z \in \text{Mon}(\mathbf{Ban})$ with morphisms $(\alpha', \alpha) \in \hom_{\text{Mon}(\mathbf{Ban})}(f_X, f_Y)$ and $(\beta', \beta) \in \hom_{\text{Mon}(\mathbf{Ban})}(f_Y, f_Z)$, we have that $f_Z\beta'\alpha' = \beta f_Y \alpha' = \beta\alpha f_X$. Therefore, $(\beta'\alpha', \beta\alpha) \in \hom_{\text{Mon}(\mathbf{Ban})}(f_X, f_Z)$ as required. Moreover, this composition is associative since composition in **Ban** is associative, so Mon(**Ban**) forms a category. □

It is important to note that Mon(**Ban**) has a zero object (see Definition A.1.7), namely $\text{id}_0 : 0 \to 0$ where 0 is the zero Banach space. Consequently, for any two objects $f_X, f_Y \in \text{Mon}(\mathbf{Ban})$, the morphism $(0_{X'Y'}, 0_{XY}) \in \hom_{\text{Mon}(\mathbf{Ban})}(f_X, f_Y)$ is a zero morphism (See Definition A.1.7). With this in mind, we can now describe a group operation on these hom-sets in order to show Mon(**Ban**) is an additive category (see Definition A.1.8).

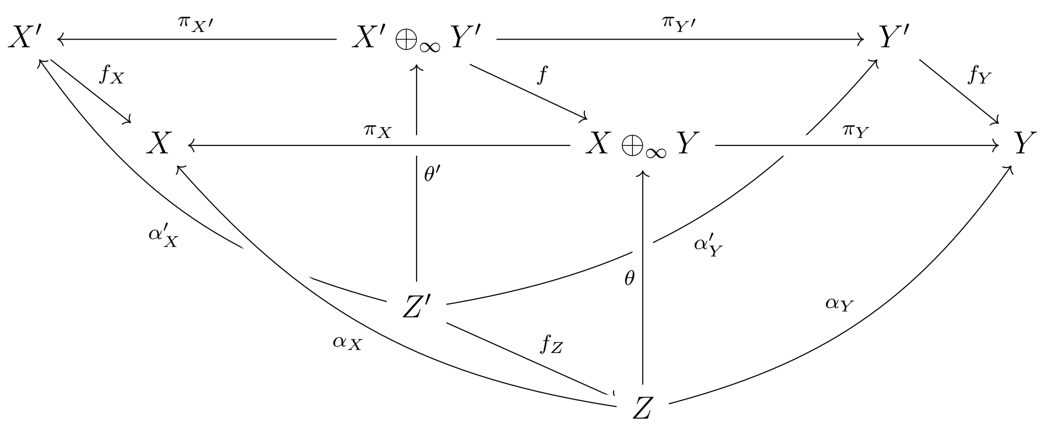
Proposition 4.1.5. *The category* Mon(***Ban***) *is an additive category.*

Proof. We will first show that for any two objects in Mon(**Ban**), a product of these objects can be formed. Recall that in the category **Ban**, the product of two Banach spaces X and Y is the infinity sum $X \oplus_\infty Y$, the product space with the max norm. Since this is possible in **Ban**, we are given a natural candidate for our product. Once again, let $f_X, f_Y \in$ Mon(**Ban**) and consider diagram 4.3.



(4.3)

Here f is the map $(x', y') \mapsto (f_X(x'), f_Y(y'))$. It is easily checked to be injective bounded linear and is therefore in Mon(**Ban**). Moreover, the pairs (π'_X, π_X) and (π'_Y, π_Y) trivially make their respective squares commute so they are morphisms in Mon(**Ban**). Now suppose we had a third object in $f_Z \in$ Mon(**Ban**) together with a pair of morphisms $(\alpha'_X, \alpha_X) : f_Z \to f_X$ and $(\alpha'_Y, \alpha_Y) : f_Z \to f_Y$. By the universal property of $X' \oplus_\infty Y'$ there exists a unique bounded linear map $\theta' : Z' \to X' \oplus_\infty Y'$ such that $\alpha'_X = \pi'_X \theta'$ and $\alpha'_Y = \pi'_Y \theta'$, specifically $z' \mapsto (\alpha'_X(z'), \alpha'_Y(z'))$. Similarly, there exists a unique bounded linear map $\theta : Z \to X \oplus_\infty Y$ such that $\alpha_X = \pi_X \theta$ and $\alpha_Y = \pi_Y \theta$, specifically $z \mapsto (\alpha_X(z), \alpha_Y(z))$. The fact that Diagram 4.4 commutes is now trivial.



(4.4)

Therefore products exist in Mon(**Ban**). Now we have to show that for all $f_X, f_Y \in$ Mon(**Ban**), the hom set $\hom_{\mathrm{Mon}(\mathbf{Ban})}(f_X, f_Y)$ forms an abelian group and that composition respects the group operation. To this end, we notice that a canonical binary operation exists, namely coordinate-wise point-wise addition of bounded linear maps. We must show it is a closed binary operation.

Let $(\alpha', \alpha), (\beta', \beta) \in \hom_{\mathrm{Mon}(\mathbf{Ban})}(f_X, f_Y)$ and take $(\alpha'+\beta', \alpha+\beta)$. Both coordinates are bounded linear maps, so we check that $f_Y(\alpha' + \beta') = f_Y\alpha' + f_Y\beta' = \alpha f_X + \beta f_X = (\alpha+\beta)f_X$ as required. Associativity follows from the associativity of point-wise addition of bounded linear maps and $(0_{X'}, 0_X)$ is the identity morphism. We have that the inverse of (α', α) is $(-\alpha', -\alpha)$ and the operation is commutative since point-wise addition of bounded linear maps is commutative. Therefore, hom sets in Mon(**Ban**) form abelian groups. Lastly, it is easy to check that composition respects the group operation, since composition of bounded linear maps respects pointwise addition. Hence Mon(**Ban**) is an additive category. □

In order to generate Mon(**Ban**) morphisms in later proofs, it will be necessary to provide the following lemma concerning incomplete squares.

Lemma 4.1.6. *Let $f_X, f_Y \in$ Mon(**Ban**) and let $\alpha \in \mathcal{B}(X, Y)$. If $\alpha f_X(X') \subseteq f_Y(Y')$, then there exists $\alpha' \in \mathcal{B}(X', Y')$ such that (α', α) makes Diagram 4.1 commute and hence $(\alpha', \alpha) \in \hom_{\mathrm{Mon}(\mathbf{Ban})}(f_X, f_Y)$.*

Proof. Take $x' \in X'$. We can construct a linear map α' from X' to Y' by sending x' to the preimage of $\alpha f_X(x')$ under f_Y. We need only check that it is bounded, so we seek to apply the Closed Graph Theorem to the dotted map in Diagram 4.5.

$$\begin{array}{ccc} X' & \overset{\alpha'}{\dashrightarrow} & Y' \\ \downarrow{\scriptstyle f_X} & & \downarrow{\scriptstyle f_Y} \\ X & \xrightarrow{\alpha} & Y \end{array} \tag{4.5}$$

To this end let (x'_n) be a sequence in X' and suppose that both (x'_n) converges to an element $x' \in X'$ and $\alpha'(x'_n)$ converges to an element $y' \in Y'$. We aim to show that $\alpha'(x') = y'$. Now applying f_Y to both sides we have that this is the same as showing $\alpha f_X(x') = f_Y(y')$ since f_Y is monic. However, since y' is the limit of $\alpha'(x'_n)$, we have that $f_Y(y')$ must be the limit of $\alpha f_X(x'_n)$, but this is precisely $\alpha f_X(x')$ as required. $\square$

Since our aim is to study quotients, it would be wise to equate certain morphisms to suitable zero morphisms. These will be the morphisms $(\alpha', \alpha) \in \hom_{\mathrm{Mon}(\mathbf{Ban})}(f_X, f_Y)$ in which α factors through f_Y.

Definition 4.1.7 (The ideal of null-homotopic morphisms)**.** Let $f_X, f_Y \in \mathrm{Mon}(\mathbf{Ban})$. We will call the collection of morphisms

$$I(f_X, f_Y) = \{(\alpha', \alpha) \in \hom_{\mathrm{Mon}(\mathbf{Ban})}(f_X, f_Y) : \exists \rho \in \hom_{\mathbf{Ban}}(X, Y'), \alpha = f_Y \rho\}$$

the null-homotopic morphisms from f_X to f_Y and

$$\mathcal{I} = \{I(f_X, f_Y)\}_{f_X, f_Y \in \mathrm{Mon}(\mathbf{Ban})}$$

will be the null-homotopic morphisms in Mon(**Ban**). We illustrate this concept with Diagram 4.6.

$$\begin{array}{ccc} X' & \xrightarrow{\alpha'} & Y' \\ \downarrow{\scriptstyle f_X} & \overset{\rho}{\nearrow} & \downarrow{\scriptstyle f_Y} \\ X & \xrightarrow{\alpha} & Y \end{array} \tag{4.6}$$

Formally these are the null-homotopic morphisms in a category of chain complexes. If we show that $\mathcal{I}$ forms an ideal in Mon(**Ban**) (see Definition A.1.10), then by Theorem A.1.12 we can form a new category. This new category would have the same objects as Mon(**Ban**), but its morphisms would be equivalence classes of morphisms.

Theorem 4.1.8. *The collection of null-homotopic morphisms, $\mathcal{I}$ in* Mon(***Ban***) *forms an ideal.*

Proof. First we check whether for all $f_X, f_Y \in \text{Mon}(\textbf{Ban})$, that $I(f_X, f_Y)$ forms an abelian subgroup of $\hom_{\text{Mon}(\textbf{Ban})}(f_X, f_Y)$. We have that $(0_{X'Y'}, 0_{XY}) \in I(f_X, f_Y)$ since $0_{XY} = f_Y 0_{XY'}$. Now let $(\alpha_1', \alpha_1), (\alpha_2', \alpha_2) \in I(f_X, f_Y)$ and let ρ_1 and ρ_2 be the bounded linear maps such that $\alpha_i = f_Y \rho_i$ for $i = \{1, 2\}$. Then $\alpha_1 + \alpha_2 = f_Y(\rho_1 + \rho_2)$, giving us that $I(f_X, f_Y)$ is closed under addition. Moreover, it is clear that if $\alpha_1 = f_Y \rho_1$, then $-\alpha_1 = f_Y(-\rho_1)$ so $I(f_X, f_Y)$ is closed under additive inverses and is also therefore a subgroup of $\hom_{\text{Mon}(\textbf{Ban})}(f_X, f_Y)$. The fact that it is abelian follows from it being a subgroup of an abelian group.

Next we need to check that $\mathcal{I}$ is stable under left and right composition with morphisms in Mon(**Ban**). To this end, let $(\alpha', \alpha) \in \hom_{\text{Mon}(\textbf{Ban})}(f_W, f_X)$, $(\beta', \beta) \in I(f_X, f_Y)$, $(\gamma', \gamma) \in \hom_{\text{Mon}(\textbf{Ban})}(f_Y, f_Z)$ and let $\rho : X \to Y'$ be the bounded linear map such that $\beta = f_Y \rho$.

Since $\beta\alpha = f_Y \rho \alpha$, $(\beta'\alpha', \beta\alpha) \in I(f_W, f_Y)$. Moreover, as $\gamma\beta = \gamma f_Y \rho = f_Z \gamma' \rho$, we also have that $(\gamma'\beta', \gamma\beta) \in I(f_X, f_Z)$. Therefore, $\mathcal{I}$ forms an ideal in Mon(**Ban**) as required. □

Definition 4.1.9 (hMon(**Ban**))**.** We can now apply Theorem A.1.12 with $\mathcal{I}$, the ideal of null-homotopic maps.

We defined the category hMon(**Ban**) as follows: the objects of hMon(**Ban**) are monics in **Ban**; and given $f_X, f_Y \in \text{hMon}(\textbf{Ban})$, define $\hom_{\text{hMon}(\textbf{Ban})}(f_X, f_Y)$ to be set equivalence classes of morphisms in Mon(**Ban**).

Two morphisms (α', α) and (β', β) will be in the same class if the difference $(\alpha' - \beta', \alpha - \beta) \in \mathcal{I}$. When referring to one of these classes, we will denote it with square brackets, for example $[\alpha', \alpha] \in \hom_{\text{hMon}(\textbf{Ban})}(f_X, f_Y)$. However, it may suit us to make specific choices of representative when illustrating concepts with commutative diagrams or proving certain results.

Although hMon(**Ban**) describes what can be thought of as formal quotients of Banach spaces quite well, there is one desirable property which fails to hold. To illustrate this, let $f_X \in \text{hMon}(\mathbf{Ban})$ such that $f_X(X')$ is closed in X, let $E \cong X/f_X(X')$ with $q : X \to E$ being the canonical quotient map and let $0_{0E} : 0 \to E$ be the unique monic from the zero Banach space into the Banach space E. If we want this category to describe formal quotients sufficiently, it would be desirable for the morphism $[0_{X'0}, q] : f_X \to 0_{0E}$ in Diagram 4.7 to be an isomorphism in hMon(**Ban**).

$$\begin{array}{ccc} X' & \xrightarrow{0_{X'0}} & 0 \\ \Big\downarrow{\scriptstyle f_X} & & \Big\downarrow{\scriptstyle 0_{0E}} \\ X & \xrightarrow{q} & E \end{array} \tag{4.7}$$

However, as the following proposition demonstrates, this is not always the case. The statement of Proposition 4.1.10 can be found in [16], we have provided a proof for the benefit of the reader.

Proposition 4.1.10. *Let $f_X \in \text{hMon}(\boldsymbol{Ban})$ such that $f_X(X')$ is closed in X. Let $E \cong X/f_X(X')$ with quotient map $q : X \to E$.*

The class of morphisms $[0_{X'0}, q]$ is an isomorphism in hMon(***Ban***) *if and only if $f_X(X')$ is topologically complemented in X.*

Proof. Suppose to begin with that $f_X(X')$ is topologically complemented in X with complementary subspace Y. We know $X \cong f_X(X') \oplus Y$ and as a consequence $E \cong Y$. We therefore have a bounded linear map $p : E \to Y$ where clearly $p0_{0E} = f_X 0_{0X'}$ and we aim to show that $[p, 0_{0X'}] : 0_{0E} \to f_X$ is the inverse of $[0_{X'0}, q]$ in hMon(**Ban**). Trivially, $qp = \text{id}_E$. We note that $\text{id}_X - pq$ is a bounded linear map whose range is $f_X(X')$. Since f_X is injective with closed range, it is possible to construct a bounded linear $\rho : E \to X'$ such that $f_X(X')\rho = \mathrm{id}_X - pq$. Therefore $[0_{X'0}, q]$ is an isomorphism in hMon(**Ban**).

Now suppose that $[0_{X'0}, q]$ is an isomorphism, which implies that there exists $[p, 0_{0X'}] : 0_{0E} \to f_X$ and a bounded linear $\rho : X \to X'$ such that $pq - \mathrm{id}_X = f_X\rho$. Note that for $x' \in f_X(X')$, applying our map gives $f_X\rho(x') = x'$ so this map is a projection with range $f_X(X')$, which implies $f_X(X')$ is topologically complemented. □

Since these objects will not necessarily be isomorphic, this indicates that our category, hMon(**Ban**), might not have enough morphisms in it. We therefore seek to enlarge our category, via localisation.

4.2 The subcategory of pulation morphisms

Recall from Definition A.1.2 that a subcategory of a given category $\mathcal{C}$ is wide if it contains all objects of $\mathcal{C}$. In this section, we seek a collection of morphisms in hMon(**Ban**) which contains id_{f_X} for every $f_X \in$ hMon(**Ban**) and which is closed under composition. We would then localise hMon(**Ban**) at that wide subcategory. If the proposed subcategory has certain desirable properties, our localised category may be surprisingly friendly to work with.

This is the route taken in [19], though it is taken at an abstract level where we will specifically be focusing on Banach spaces. For a far more in-depth look at localisation of a category, see [10, chap. 7]. It should be pointed out that this is where Waelbroeck starts in [16], though he makes an incorrect choice of subcategory to localise at (see Remark 4.2.5).

To begin with, we define what we mean by a pulation morphism in Mon(**Ban**).

Definition 4.2.1 (Pulation morphism)**.** Let $(\alpha', \alpha) \in \hom_{\mathrm{Mon}(\mathbf{Ban})}(f_X, f_Y)$. We say (α', α) is a pulation morphism if the commuting square in Diagram 4.8 is a pulation square. Recall from Definition A.1.9 that a pulation square is one which is both a pullback and a pushout.

$$\begin{array}{ccc} X' & \xrightarrow{\alpha'} & Y' \\ \Big\downarrow{\scriptstyle f_X} & & \Big\downarrow{\scriptstyle f_Y} \\ X & \xrightarrow{\alpha} & Y \end{array} \tag{4.8}$$

In fact, this property is stable up to homotopy, which we will prove in the following lemma.

Lemma 4.2.2. *Let $(\alpha', \alpha) \in \hom_{\mathrm{Mon}(\boldsymbol{Ban})}(f_X, f_Y)$ be a pulation morphism. If there exists $(\beta', \beta) \in \hom_{\mathrm{Mon}(\boldsymbol{Ban})}(f_X, f_Y)$ such that $(\alpha', \alpha) - (\beta', \beta)$ is null-homotopic, then Diagram 4.9 is also a pulation square.*

$$\begin{array}{ccc} X' & \xrightarrow{\beta'} & Y' \\ \Big\downarrow{\scriptstyle f_X} & & \Big\downarrow{\scriptstyle f_Y} \\ X & \xrightarrow{\beta} & Y \end{array} \tag{4.9}$$

Proof. Since (β', β) is homotopic to (α', α), we must have that there exists a bounded linear $\rho : X \to Y'$ such that $\beta = \alpha + f_Y\rho$. We now compute that $f_Y\beta' = \beta f_X = \alpha f_X + f_Y\rho f_X = f_Y(\alpha' + \rho f_X)$. Since f_Y is monic, we also have that $\beta' = \alpha' + \rho f_X$.

Therefore, we first aim to show that our square is a pullback. To this end, let E be a Banach space with bounded linear maps $g_X : E \to X$ and $g_{Y'} : E \to Y'$ such that $\beta g_X = f_Y g_{Y'}$. Substituting $\alpha + f_Y\rho$ for β we get that $\alpha g_X + f_Y\rho g_X = f_Y g_{Y'}$, which we can rearrange to $\alpha g_X = f_Y(g_{Y'} - \rho g_X)$. Since (α', α) is a pullback square, there exists a unique bounded linear map $g : E \to X'$ such that $g_X = f_X g$ and $g_{Y'} - \rho g_X = \alpha' g$. However, we can substitute the first into the second to give $g_{Y'} = (\alpha' + \rho f_X)g = \beta' g$. So g is also the the unique map that gives us $\beta' g = g_{Y'}$ and $f_X g = g_X$ and hence our square is a pullback square.

Next we show that our commuting square is a pushout square. Let F be a Banach space with bounded linear maps $h_X : X \to F$ and $h_{Y'} : Y' \to F$ such that

$h_X f_X = h_{Y'}\beta'$. Substituting $\alpha' + \rho f_X$ for β' we get that $(h_X - h_{Y'}\rho)f_X = h_{Y'}\alpha'$. Since (α', α) is a pushout square, there exists a unique bounded linear map $h : Y \to F$ such that $hf_Y = h_{Y'}$ and $h\alpha = h_X - h_{Y'}\rho$. Substituting the first into the second, we have that $h_X = h(\alpha + f_Y\rho) = h\beta$. So h is also the unique map that gives us $h\beta = h_X$ and $hf_Y = h_{Y'}$ and hence our square is also a pushout square. Since it is both a pullback and a pushout, it is consequently a pulation. □

Thanks to this lemma, we can now work with equivalence classes of morphisms in hMon(**Ban**) since it will not matter what morphism we choose as a representative for our equivalence class. In a slight abuse of terminology, we will refer to a class of morphisms in hMon(**Ban**) as a pulation morphism if its representatives are pulation morphisms in Mon(**Ban**). As it turns out, these morphisms form the subcategory of hMon(**Ban**) that we will localise at. However, before we localise, we will aim to understand specifically what these morphisms are in the Banach space case, beginning with a definition of a property we will repeatedly use.

Definition 4.2.3 (Surjective modulo)**.** Let V and W be vector spaces and let U be a subspace of V. A linear map $L : W \to V$ is surjective modulo U if for all $v \in V$, there exists $w \in W$ such that $L(w) - v \in U$.

Theorem 4.2.4. *Let* $[\alpha', \alpha] \in \hom_{\mathrm{hMon}(\boldsymbol{Ban})}(f_X, f_Y)$. *The following are equivalent:*

1. $[\alpha', \alpha]$ *is a pulation morphism,*
2. $[\alpha', \alpha]$ *has the following two properties:*
   - *If* $\alpha(x) \in f_Y(Y')$, *then* $x \in f_X(X')$,
   - α *is surjective modulo* $f_Y(Y')$,
3. *The vector space map* $T_\alpha : X/f_X(X') \to Y/f_Y(Y')$ *which takes* $\widehat{x} \mapsto \widehat{\alpha(x)}$ *is a bijection.*

Proof. (1. $\implies$ 2.) Begin by assuming $[\alpha', \alpha]$ is a pulation morphism. Lemma 4.2.2 allows us to work with a specific representative (α', α) without loss of generality. Take the pullback of f_Y and α, which we may recall is the space $PB = \{(x, y') \in X \oplus_\infty Y' : \alpha(x) = f_Y(y)\}$ together with projection maps π_X and $\pi_{Y'}$. By the pullback property of X', there exists a morphism $\theta_1 : PB \to X'$ such that $\pi_X = f_X\theta_1$ and $\pi_Y = \alpha'\theta_1$. Since PB is also a pullback, θ_1 is an isomorphism. As f_Y is monic, the preimage of $f_Y(Y')$ under α is $\pi_X(PB)$. Since $\pi_X = f_X\theta_1$, we have that if $\alpha(x) \in f_Y(Y')$ then $x \in f_X(X')$.

Next take the pushout of α' and f_X. We recall that this is the space $PO = X \oplus_1 Y'/\overline{F}$ where $F = \{(\alpha'(x'), -f_X(x')) \subseteq X \oplus_1 Y'\}$ together with maps q_X and $q_{Y'}$. These maps are the canonical subspace inclusions ι_X and $\iota_{Y'}$ of X and Y' into $X \oplus_1 Y'$ respectively, followed by q_F, the quotient by $\overline{F}$. We observe that ι_X is surjective modulo Y' and that q_F is surjective, so q_X will be surjective modulo $q_F(Y')$. By the pushout property of Y, there exists a unique morphism $\varphi_1 : Y \to PO$ such that $q_X = \varphi_1\alpha$ and $q_{Y'} = \varphi_1 f_Y$. Since PO is also a pushout, φ_1 is an isomorphism. Finally, let $y \in Y$ and take $\varphi_1(y)$. There exists $x \in X$ and $r \in q_{Y'}(Y')$ such that $q_X(x) - \varphi_1(y) = r$. We compute that $\alpha(x) = y + \varphi_1^{-1}(r)$ and since $r \in q_{Y'}(Y')$, we have that $\varphi_1^{-1}(r) \in f_Y(Y')$. Hence α is surjective modulo $f_Y(Y')$ as required.

(2. $\implies$ 1.) Now assume that if $\alpha(x) \in f_Y(Y')$ then $x \in f_X(X')$, and that α is surjective modulo $f_Y(Y')$ with the aim of showing that (α', α) is a pulation morphism. We begin by showing it forms a pullback square. It will suffice to show that X' is isomorphic to the pullback of f_Y and α. Since $f_Y\alpha' = \alpha f_X$, there is a unique bounded linear map $\theta_2 : X' \to PB$ and we shall aim to show θ_2 is a bijection. We observe that θ_2 is injective because if $\theta_2(x') = (f_X(x'), \alpha'(x')) = (0, 0)$ then $f_X(x') = 0$ and hence $x' = 0$ since f_X is monic. Now we check surjectivity. Let $(x, y') \in PB$ and notice that $\alpha(x) = f_Y(y')$, which implies that $x \in f_X(x')$ for an $x' \in X'$. Now $\alpha'(x') = y'$

by construction, so we have that $\theta_2(x') = (x, y')$ as required. Hence we have that our commuting square forms a pullback square.

Next, to check that it forms a pushout square, it will suffice to show that it is isomorphic to the pushout of f_X and α'. Since $\alpha f_X = f_Y \alpha'$, we have that there is a unique bounded linear map $\varphi_2 : PO \to Y$, which takes $(\widehat{x}, \widehat{y'}) \mapsto \alpha(x) + f_Y(y')$. We note that $(\widehat{x}, \widehat{y'}) \in \ker \varphi_2$ if and only if $\alpha(x) = -f_Y(y')$. Since we seek to show that $(\widehat{x}, \widehat{y'}) = (\widehat{0}, \widehat{0})$, it suffices to find an $x' \in X'$ such that $f_X(x') = x$ and $-\alpha(x') = y'$. Note that since $\alpha(x) = -f_Y(y')$, we have that $x \in f_X(X')$ and we therefore choose our x' to be such that $f_X(x') = x$.

We now only need to show that $-\alpha'(x') = y'$, but $f_Y \alpha'(x') = -f_Y(y') = f_Y(-y')$ so this holds.

Lastly we check that φ_2 is surjective. Let $y \in Y$ and using the fact that α is surjective modulo $f_Y(Y')$, choose $x \in X$ and $f_Y(y') \in f_Y(Y')$ such that $y = \alpha(x) + f_Y(y')$. Then we can see that $\varphi_2((\widehat{x}, \widehat{y'})) = y$ as required. Therefore our commuting square is a pushout and hence also a pulation.

(2. $\implies$ 3.) We wish to show T_α is a bijection. Let $\widehat{y} \in Y/f_Y(Y')$, there exists $x \in X$ such that $\alpha(x) - y \in f_Y(Y')$ so we have that $T_\alpha(\widehat{x}) = \widehat{y}$. This gives us that T_α is surjective. Further to this, $\ker T_\alpha = \{\widehat{x} \in X/f_X(X') : \alpha(x) \in f_Y(Y')\} = \{\widehat{x} \in X/f_X(X') : x \in f_X(X')\} = \{\widehat{0}\}$. This gives us that T_α is injective and hence bijective.

(3. $\implies$ 2.) Clearly if T_α is surjective, then α is surjective modulo $f_Y(Y')$. Lastly, assume $\ker T_\alpha = 0$. If $\alpha(x) \in f_Y(Y')$ then $T_\alpha(\widehat{x}) = \widehat{0}$ which implies $\widehat{x} = \widehat{0}$ and hence $x \in f_X(X')$ as required.

□

Remark 4.2.5. In Definition 3 [16, p. 554], Waelbroeck defines what he calls pseudo-isomorphisms. These are similar to what we, in our category theory setting, have been calling pulation morphisms. However, he places a stronger condition on them, this being that he requires full surjectivity, instead of surjectivity modulo the range of a monic. If the aim is to carry out a derived category construction, then pulation morphisms are the correct choice.

Our next step will be to show that these pulation morphisms form a wide subcategory of hMon(**Ban**). Refer to Definition A.1.2 for the definition of a wide subcategory.

Proposition 4.2.6. *Let $\mathcal{W}$ be the category whose objects are the objects of* hMon(***Ban***) *and whose morphisms are the pulation morphisms in* hMon(***Ban***)*, then $\mathcal{W}$ is a wide subcategory of* hMon(***Ban***)*.*

Proof. For any $f_X \in \mathrm{hMon}(\mathbf{Ban})$, take the identity morphism $[\mathrm{id}_{X'}, \mathrm{id}_X]$. It is trivial that id_X is surjective modulo $f_X(X')$ and $\mathrm{id}_X^{-1}(f_X(X')) = f_X(X')$ so by Theorem 4.2.4 it must be a pulation morphism.

We need only check that the composition of pulation morphisms is a pulation morphism. To this end, let $[\alpha', \alpha] \in \hom_{\mathrm{hMon}(\mathbf{Ban})}(f_X, f_Y)$ and $[\beta', \beta] \in \hom_{\mathrm{hMon}(\mathbf{Ban})}(f_Y, f_Z)$ be pulation morphisms and take $[\beta'\alpha', \beta\alpha]$. Observe that

$$(\beta\alpha)^{-1}[f_Z(Z')] = \alpha^{-1}[\beta^{-1}[f_Z(Z')]] = \alpha^{-1}[f_Y(Y')] = f_X(X').$$

Moreover, since β is surjective modulo $f_Z(Z')$ and α is surjective modulo $f_Y(Y')$, we have that for all $z \in Z$ there exists $y \in Y$ and $z' \in f_Z(Z')$ such that $\beta(y) = z + f_Z(z')$. However, we also have that there exists $x \in X$ and $f_Y(y') \in f_Y(Y')$ such that $\alpha(x) = y + f_Y(y')$. Now $\beta\alpha(x) = \beta(y + f_Y(y')) = z + f_Z(z') + \beta f_Y(y') = z + f_Z(z') + f_Z\beta'(y')$ so $\beta\alpha$ is surjective modulo $f_Z(Z')$. We now have that by Theorem 4.2.4 that $[\beta'\alpha', \beta\alpha]$ is a pulation morphism and hence the pulation morphisms form a wide subcategory. □

Definition 4.2.7 (Notational convention for pulation morphisms)**.** For the remainder of this thesis, we shall refer to the wide subcategory of pulation morphisms as $\mathcal{W}$.

4.3 Localisation

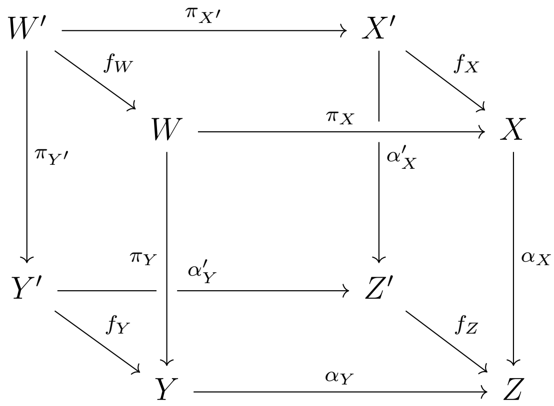
Our next task before we localise hMon(**Ban**) at $\mathcal{W}$ will be to show that the pair (hMon(**Ban**), $\mathcal{W}$) admits a calculus of right fractions (see Definition A.1.3). In some of the literature, including [19], this is referred to as a multiplicative system. However, we will be adopting the convention used in [10, chap. 7].

Theorem 4.3.1. *The pair* (hMon(***Ban***), $\mathcal{W}$) *admits a calculus of right fractions.*

Proof. We have from Proposition 4.2.6 that $\mathcal{W}$ forms a wide subcategory of hMon(**Ban**). Next we show that the right Ore condition (Definition A.1.3) holds.

Let $f_X, f_Y, f_Z \in$ hMon(**Ban**) with morphisms $[\alpha'_X, \alpha_X] \in \hom_{\mathrm{hMon}(\mathbf{Ban})}(f_X, f_Z)$ and $[\alpha'_Y, \alpha_Y] \in \hom_{\mathcal{W}}(f_Y, f_Z)$. Let $W = \{(x, y) \in X \times Y : \alpha_X(x) - \alpha_Y(y) \in f_Z(Z')\}$, which is a Banach space when endowed with the norm $\|(x, y)\|_W = \|x\|_X + \|y\|_Y + \|\alpha_X(x) - \alpha_Y(y)\|$. Further to this, let $W' = X' \times Y'$, which is also a Banach space with norm $\|(x', y')\|_{W'} = \|x'\|_{X'} + \|y'\|_{Y'}$. Note that we can construct the monic $f_W : W' \to W, (x', y') \mapsto (f_X(x'), f_Y(y'))$ since $\alpha_X f_X(x') - \alpha_Y f_Y(y') = f_Z(\alpha'_X(x') - \alpha'_Y(y'))$ so we have that $f_W \in$ hMon(**Ban**). We can also define the canonical coordinate maps to be $\pi_X : W \to X$, $\pi_Y : W \to Y$, $\pi_{X'} : W' \to X'$ and $\pi_{Y'} : W' \to Y'$. Trivially, we have that $f_X \pi_{X'} = \pi_X f_W$ and $f_Y \pi_{Y'} = \pi_Y f_W$ so both $[\pi_{X'}, \pi_X]$ and $[\pi_{Y'}, \pi_Y]$ are hMon(**Ban**) morphisms.

This setup is illustrated in Diagram 4.10 with specific choices of representatives of classes of morphisms.



(4.10)

We now need to check whether $[\pi_{X'}, \pi_X]$ is a pulation morphism, which will be true if π_X is surjective modulo $f_X(X')$, and $\pi_X^{-1}(f_X(X')) = f_W(W')$. Let $x \in X$, take $\alpha_X(x)$, and observe that since α_Y is surjective modulo $f_Z(Z')$, that there exists $y \in Y$ and $f_Z(z') \in f_Z(Z')$ such that $\alpha_Y(y) = \alpha_X(x) + f_Z(z')$. Now we have that $(x, y) \in W$ so π_X is surjective and hence surjective modulo $f_X(X')$.

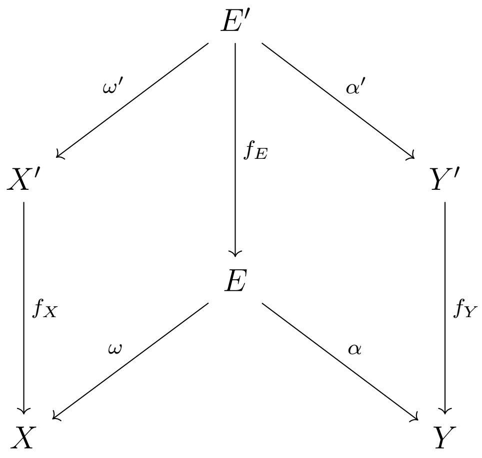
Next let $f_X(x') \in f_X(X')$. In particular, this means that there exists a $y \in Y$ such that the pair $(f_X(x'), y) \in W$ exists. We have that $\alpha_Y \pi_Y((f_X(x'), y)) = \alpha_X f_X(x') = f_Z \alpha'_X(x') \in f_Z(Z')$ so since $\alpha_Y^{-1}(f_Z(Z')) = f_Y(Y')$, we have that $y \in f_Y(Y')$. From this, it is evident that $(f_X(x'), y) \in f_W(W')$ and hence $\pi_X^{-1}(f_X(X')) = f_W(W')$ as required which implies that $[\pi_{X'}, \pi_X]$ is a pulation morphism.

We now check the third and final condition of Definition A.1.3. Let $f_X, f_Y, f_Z \in$ hMon(**Ban**), let $[\alpha'_1, \alpha_1], [\alpha'_2, \alpha_2] \in \hom_{\text{hMon}(\mathbf{Ban})}(f_X, f_Y)$ and let $[\beta', \beta] \in \hom_{\mathcal{W}}(f_Y, f_Z)$ such that $[\beta'\alpha'_1, \beta\alpha_1] = [\beta'\alpha'_2, \beta\alpha_2]$. Taking the difference, we have that $[\beta'(\alpha'_1 - \alpha'_2), \beta(\alpha_1 - \alpha_2)] = [0, 0]$ and hence $\beta(\alpha_1 - \alpha_2)(X) \subseteq f_Z(Z')$. Now $\beta^{-1}(f_Z(Z')) = f_Y(Y')$ so $\alpha_1 - \alpha_2$ is a bounded linear map with range contained in $f_Y(Y')$. Using the Open Mapping Theorem and the fact that f_Y is monic, we deduce that $\alpha_1 - \alpha_2 = f_Y \rho$ where $\rho : X \to Y'$ is bounded linear. This gives us that $[\alpha'_1, \alpha_1] = [\alpha'_2, \alpha_2]$. Consequently it is now easy to find a pulation morphism since $[\mathrm{id}_X, \mathrm{id}_{X'}] \in \hom_{\mathcal{W}}(f_X, f_X)$ will suffice. Therefore, the pair $(\mathrm{hMon}(\mathbf{Ban}), \mathcal{W})$ admits a calculus of right fractions as required. □

We can now localise hMon(**Ban**) at $\mathcal{W}$. Informally, this means that we adjoin the inveses of all morphisms in $\mathcal{W}$ to hMon(**Ban**). As motivation, recall that the morphisms in $\mathcal{W}$ are exactly these $[\alpha', \alpha]$ for which T_α is a vector space isomorphism, see Theorem 4.2.4.

Since Theorem 4.3.1 holds and $(\mathrm{hMon}(\mathbf{Ban}), \mathcal{W})$ admits a calculus of right fractions, we can construct the localisation at $\mathcal{W}$ quite explicitly. Our new category,which we will denote as Q(**Ban**) due to its ability to describe quotients in the category **Ban**, will have the same objects as hMon(**Ban**). However, morphisms in Q(**Ban**) will be equivalence classes of $\mathcal{W}$-spans in hMon(**Ban**). Definition 4.3.2 below details what a $\mathcal{W}$-span in hMon(**Ban**) is, before we describe how to compose two of them. Lastly, we provide the equivalence relation which makes Q(**Ban**) into a category. Both the definition and the proof of the equivalence relation are covered in [10, p. 155]. We will also provide the proof that we have an equivalence relation in the general case in Proposition A.1.5 for the ease of the reader.

Definition 4.3.2 ($\mathcal{W}$-span)**.** Let $f_X, f_Y \in \mathrm{hMon}(\mathbf{Ban})$. A $\mathcal{W}$-span from f_X to f_Y will be a triple $([\omega', \omega], f_E, [\alpha', \alpha])$ such that $f_E \in \mathrm{hMon}(\mathbf{Ban})$, $[\omega', \omega] \in \hom_{\mathcal{W}}(f_E, f_X)$ and $[\alpha', \alpha] \in \hom_{\mathrm{hMon}(\mathbf{Ban})}(f_E, f_Y)$. We provide Diagram 4.11 to clarify this below with specific choices of representatives.

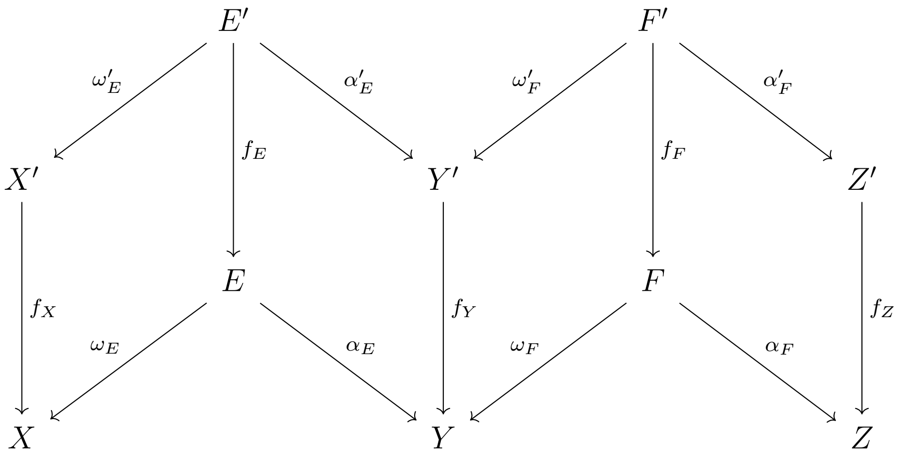


(4.11)

When we refer to a $\mathcal{W}$-span, and later a Q(**Ban**) morphism, we may use the notation $[\alpha', \alpha][\omega', \omega]^{-1}$.

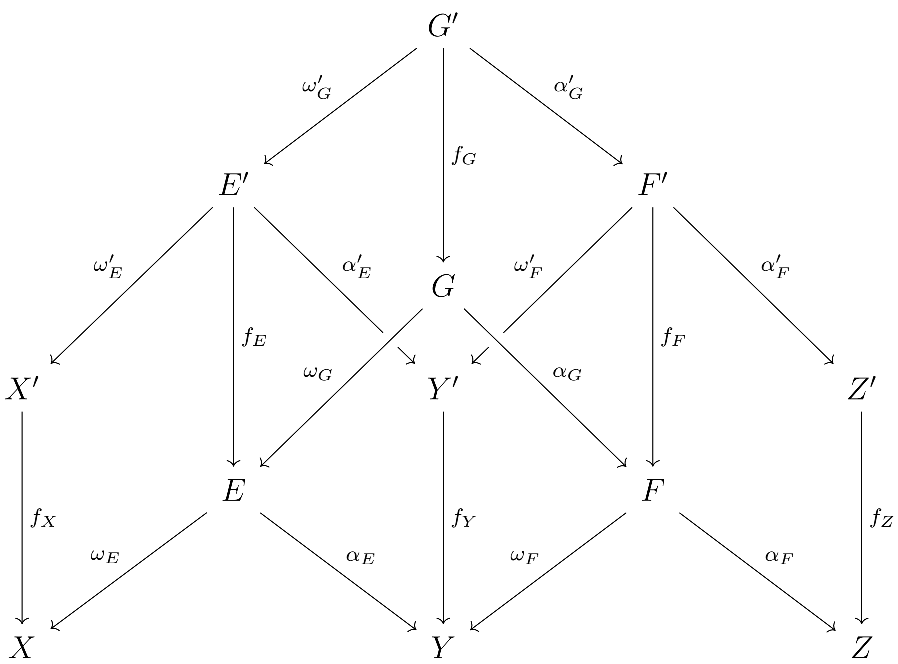
With this definition in mind, it is now important to make sense of the claim that all morphisms in Q(**Ban**) can be represented by $\mathcal{W}$-spans. In particular, if two $\mathcal{W}$-spans are concatenated, there should be a way of expressing this as a $\mathcal{W}$-span in its own right.

Let $f_X, f_Y, f_Z \in \text{hMon}(\textbf{Ban})$ and let $([\omega'_E, \omega_E], f_E, [\alpha'_E, \alpha_E])$ and $([\omega'_F, \omega_F], f_F, [\alpha'_F, \alpha_F])$ be $\mathcal{W}$-spans from f_X to f_Y and f_Y to f_Z respectively. We illustrate this situation in Diagram 4.12 with specific choices of representatives.



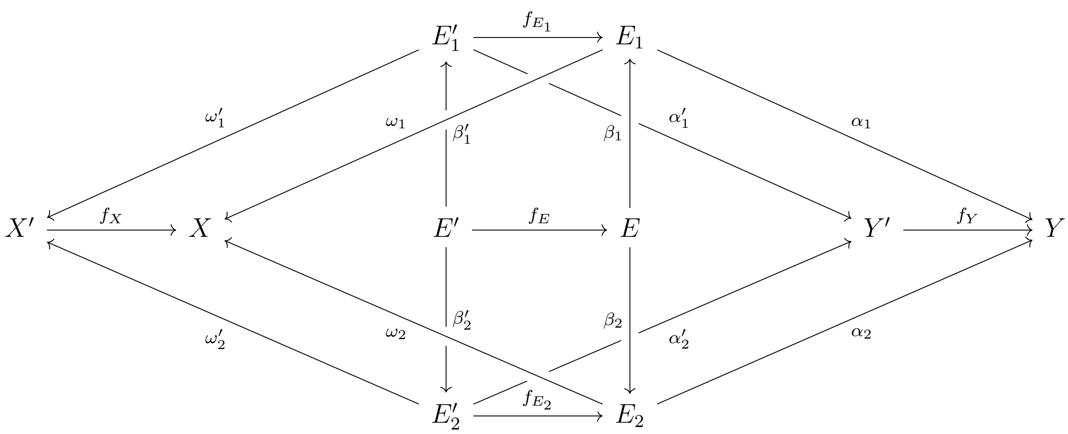
(4.12)

We can apply the right Ore condition to diagram 4.12, that is there exists a monic $f_G \in \mathrm{hMon}(\mathbf{Ban})$ together with morphisms $[\omega'_G, \omega_G] \in \hom_{\mathcal{W}}(f_G, f_E)$ and $[\alpha'_G, \alpha_G] \in \hom_{\mathrm{hMon}(\mathbf{Ban})}(f_G, f_F)$ such that $[\alpha'_E\omega'_G, \alpha_E\omega_G] = [\omega'_F\alpha'_G, \omega_F\alpha_G]$. Our new, composed $\mathcal{W}$-span will be $([\omega'_E\omega'_G, \omega_E\omega_G], f_G, [\alpha'_F\alpha'_G, \alpha_F\alpha_G])$, which will be the top edge of diagram 4.13.



(4.13)

Proposition 4.3.3. *The following relation on $\mathcal{W}$-spans is an equivalence relation. Let* $([\omega_1', \omega_1], f_{E_1}, [\alpha_1', \alpha_1])$ *and* $([\omega_2', \omega_2], f_{E_2}, [\alpha_2', \alpha_2])$ *be $\mathcal{W}$-spans from f_X to f_Y. Then* $([\omega_1', \omega_1], f_{E_1}, [\alpha_1', \alpha_1])$ *is related to* $([\omega_2', \omega_2], f_{E_2}, [\alpha_2', \alpha_2])$ *if there exists a monic f_E together with morphisms* $[\beta_1', \beta_1] \in \mathrm{hom}_{\mathrm{hMon}(\boldsymbol{Ban})}(f_E, f_{E_1})$ *and* $[\beta_2', \beta_2] \in \mathrm{hom}_{\mathrm{hMon}(\boldsymbol{Ban})}(f_E, f_{E_2})$ *such that* $[\alpha_1'\beta_1', \alpha_1\beta_1] = [\alpha_2'\beta_2', \alpha_2\beta_2]$ *and* $[\omega_1'\beta_1', \omega_1\beta_1] =$
$[\omega_2'\omega_2', \omega_2\beta_2] \in \mathrm{hom}_{\mathcal{W}}(f_E, f_X)$. *We provide Diagram 4.14 for clarity.*



(4.14)

Proof. This is a direct application of A.1.5. □

From now on, when we refer to a morphism in Q(**Ban**), we are really referring an equivalence class of $\mathcal{W}$-spans in hMon(**Ban**).

4.4 Standard representatives for Q(Ban) morphisms

Notably, we can always refine our choice of representative span to aid with proofs. First, however, we need to define what we can refine to.

Definition 4.4.1 (Free Banach space and standard monics)**.** A Banach space is free if it is isomorphic to $\ell_1(S)$ for some indexing set S. A monic $f_X : X' \to X$ is standard if X is free.

The following Lemma will be used repeatedly in order to simplify proofs involving monics in Q(**Ban**).

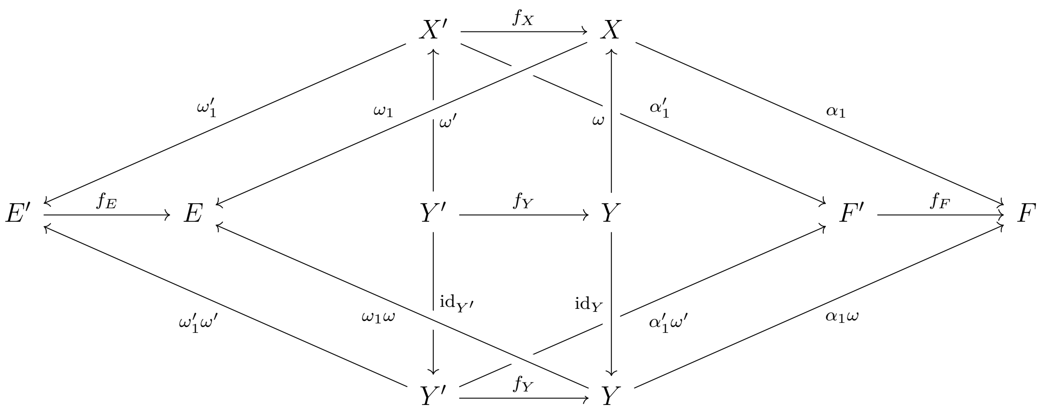
Remark 4.4.2. The notions of a free Banach space and a standard monic are adapted from Definition 4 [16, p. 554], where they are used to simplify proofs. Though Lemmas 4.4.3 and 4.4.5 are analogous to Propositions 1 [16, p. 554] and 2 [16, p. 555], new work has been done since we are working with pulation morphisms and not pseudo-isomorphisms (see Remark 4.2.5).

Lemma 4.4.3. *For every $f_X \in \mathrm{Q}(\boldsymbol{Ban})$ there exists a standard monic $f_Y \in \mathrm{Q}(\boldsymbol{Ban})$ and a pulation morphism $[\omega', \omega] \in \hom_{\mathcal{W}}(f_Y, f_X)$.*

Proof. Let $f_X : X' \to X$ be a monic. We know from [4, p. 9] that for some indexing set S, there exists a bounded linear surjection $\omega : \ell_1(S) \to X$. By the first isomorphism theorem, we therefore have that $X \cong \ell_1(S)/\ker\omega$. Now take the subspace Y' to be $\omega^{-1}[f_X(X')] \subseteq \ell_1(S)$ with monic f_Y to be subspace inclusion. Applying Lemma 4.1.6 to ω gives us our pulation morphism $[\omega', \omega]$ as required. □

Corollary 4.4.4. *Let $([\omega_1', \omega_1], f_X, [\alpha_1', \alpha_1])$ be a $\mathcal{W}$-span from f_E to f_F. We can always find an equivalent span $([\omega_2', \omega_2], f_Y, [\alpha', \alpha])$ where f_Y is standard.*

Proof. By Lemma 4.4.3, there exists a standard monic f_Y and a pulation morphism $[\omega', \omega] \in \hom_{\mathcal{W}}(f_Y, f_X)$ such that diagram 4.15 commutes.



(4.15)

Moreover, since $[\omega_1'\omega', \omega_1\omega]$ is clearly a pulation morphism, the spans $([\omega', \omega], f_X, [\alpha', \alpha])$ and $([\omega_1'\omega', \omega_1\omega], f_Y, [\alpha_1'\alpha', \alpha_1\alpha])$ are equivalent. $\square$

Lemma 4.4.5. *Let f_X be a standard monic and let $[\omega', \omega] \in \hom_{\mathcal{W}}(f_E, f_X)$ be a pulation morphism. Then $[\omega', \omega]$ is invertible in* hMon(***Ban***).

Proof. Since f_X is standard, we have that $f_X : X' \to \ell_1(S)$ for some indexing set S. Recall that $PO = (X' \oplus_1 E)/\overline{\Delta}$ where $\Delta = \{(\omega'(e'), -f_E(e')) : e' \in E'\}$ is the pushout of ω' and f_E together with inclusion maps $\iota_{X'} : X' \to X' \oplus_1 E$ and $\iota_E : E \to X' \oplus_1 E$. Since $\omega f_E = f_X \omega'$, we know there exists a unique bounded linear map $\theta : PO \to \ell_1(S)$ such that $\theta q_\Delta \iota_E = \omega$ and $\theta q_\Delta \iota_{X'} = f_X$ where q_Δ is quotienting out by $\overline{\Delta}$. Moreover, θ is an isomorphism by the argument in Theorem 4.2.4.

Let δ_s be the element of $\ell_1(S)$ which takes the value 1 on $s \in S$ and 0 elsewhere, we seek to use these indicator elements to construct a Mon(**Ban**) morphism (α', α).

We will define α by how it acts on these indicator elements and then extend it linearly to the rest of $\ell_1(S)$. Since θq_Δ is surjective, we know by Corollary A.2.5 that there exists a $K > 0$ such that for all $\delta_s \in \ell_1(S)$, we can find $(x_s', e_s) \in X' \oplus_1 E$ with $\|x_s'\|_{X'} + \|e_s\|_E < K$ and $\theta q_\Delta((x_s', e_s)) = \delta_s$. Define α by the prescription $\alpha(\delta_s) = e_s$

and extend this linearly to all of $\ell_1(S)$. Moreover, we get that the norm of α is bounded by K by construction.

We now need to show that $\alpha f_X(X') \subseteq f_E(E')$ in order to apply Lemma 4.1.6. To this end, let $x' \in X'$ and note that since $\theta q_\Delta \iota_{X'} = f_X$, we have that $\theta^{-1} f_X(x') = q_\Delta \iota_{X'}(x') = \widehat{(x', 0)}$. Now from the construction of α, there exists $(x'', e) \in X' \oplus_1 E$ with $\theta q_\Delta((x'', e)) = f_X(x')$ such that $\alpha f_X(x') = e$. Since $[\omega', \omega]$ is a pulation morphism, we move to check that $\omega(e) \in f_X(X')$. Since $\theta q_\Delta((x'', e)) = \theta q_\Delta((x', 0)) = f_X(x')$ we know that $\widehat{(x', 0)} = \widehat{(x'', e)}$ and hence that $\widehat{(x' - x'', 0)} = \widehat{(0, e)}$. Now $\omega(e) = \theta q_\Delta \iota_E(e) = \theta(\widehat{(x' - x'', 0)}) = f_X(x' - x'')$ which is clearly in the image of f_X and hence $e \in f_E(E')$ as required. By Lemma 4.1.6, we can construct an $\omega' : X' \to E'$ such that $[\omega', \omega] \in \hom_{\mathrm{hMon}(\mathbf{Ban})}(f_X, f_E)$.

Lastly, we need only check that $[\alpha', \alpha] = [\omega', \omega]^{-1}$. Since we are working up to homotopy, this amounts to checking both that $\mathrm{Ran}(\omega\alpha - \mathrm{id}_{\ell_1(S)}) \subseteq f_X(X')$ and that $\mathrm{Ran}(\alpha\omega - \mathrm{id}_E) \subseteq f_E(E')$. Let $x \in \ell_1(S)$ and by the construction in α, choose an $(x', e) \in X' \oplus_1 E$ with $\theta q_\Delta((x', e)) = x$ so that $\alpha(x) = e$. Note that $\widehat{(0, e)} = \widehat{(x', e)} - \widehat{(x', 0)}$, so $\omega\alpha(x) - x = \theta(\widehat{(x', e)} - \widehat{(x', 0)}) - x = f_X(-x')$ as required.

Now let $e \in E$ and take $\omega(\alpha\omega(e) - e) = \omega\alpha(\omega(e)) - \mathrm{id}_{\ell_1(S)}(\omega(e))$. But $\omega\alpha(\omega(e)) - \mathrm{id}_{\ell_1(S)}(\omega(e)) \in f_X(X')$ by the previous argument which implies $\alpha\omega(e) - \mathrm{id}_E(e) \in f_E(E')$ since ω is a pulation. Therefore $[\alpha', \alpha] = [\omega', \omega]^{-1}$. $\square$

Chapter 5

Studying extensions in Balg using Q(Ban) morphisms

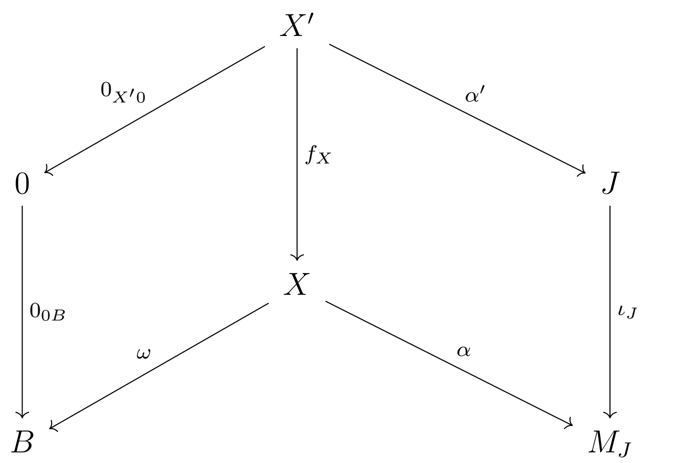
Now that we have this theory of Q(**Ban**), our immediate goal is to put it to use, building on our study of Banach algebra extensions from Chapter 2. Let J be a faithful Banach algebra, and recall from Theorem 2.4.3 that given an algebra homomorphism φ from a Banach algebra B into $M_J/\iota_J(J)$, that we could norm the pullback of φ and q_J, with the extra assumption that $\varphi = q_J h$ for some $h \in \hom_{\mathbf{Ban}}(B, M_J)$. Note that we did not require h to be an algebra homomorphism. Moreover, we can choose our norm so that P_φ is a Banach algebra.

In this chapter we shall use the framework of Q(**Ban**) morphisms and $\mathcal{W}$-spans to establish an analogue of Theorem 2.4.3 which does not require the existence of such a map h.

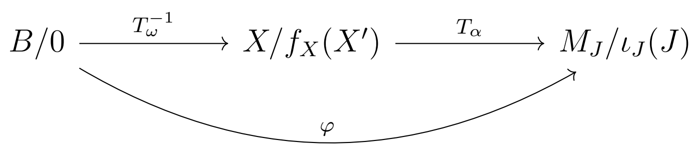
5.1 Using Q(Ban) maps

Before we attempt to look at the case where we wish to norm the pullback, we must first re-establish some of the theory from Chapter 2. However, for this to work we must relax our definition of a Busby map to include our new Q(**Ban**) morphisms.

Definition 5.1.1 (Q-Busby map)**.** Let B and J be Banach algebras where J is faithful but not necessarily closed in its multiplier algebra. A Q-Busby map will be any algebra homomorphism $\varphi : B \to M_J/\iota_J(J)$ that can be written as $T_\alpha T_\omega^{-1}$ for some Q(**Ban**) morphism $[\alpha', \alpha][0_{X'0}, \omega]^{-1}$ from $0_{0B} : 0 \to B$ to $\iota_J : J \to M_J$. We provide Diagram 5.1 for further clarification.



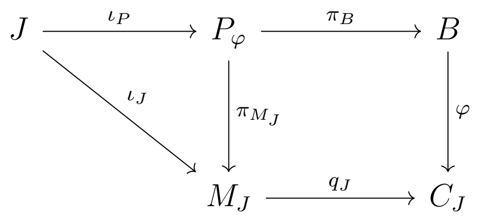
(5.1)



Before taking this to be our new definition, we must first check that our Busby maps in Chapter 2 are still Q-Busby maps under our new definition.

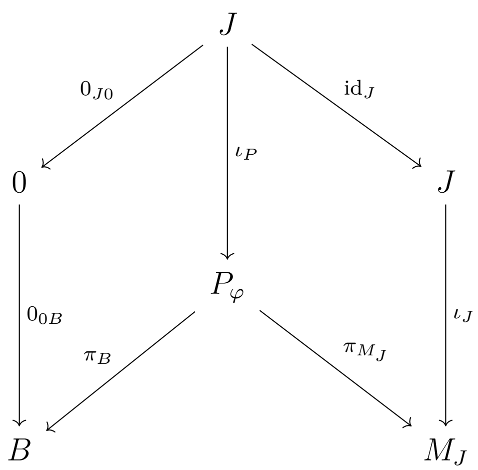
Proposition 5.1.2. *Let J and B be Banach algebras, let M_J be the multiplier algebra of J, and suppose that $\iota_J : J \to M_J$ is injective with closed range. Let $\varphi : B \to C_J$ be a continuous algebra homomorphism. Then φ is a Q-Busby map.*

Proof. Recall from Chapter 2 that we can take the pullback of φ and q_J to form an extension of B by J. This extension is the top edge of Diagram 5.2.



(5.2)

We note that π_B is surjective with $\ker \pi_B = \operatorname{Ran} \iota_P$, so Diagram 5.3 forms a $\mathcal{W}$-span.



(5.3)

We wish to show that $\varphi = T_{\pi_{M_J}} T_{\pi_B}^{-1}$ as linear maps. Postcomposing with the bijection T_{π_B}, we see that we need to check that $\varphi T_{\pi_B} = T_{\pi_{M_J}}$. To this end, let $[m, b] \in P_\varphi / \iota_P(J)$ and take $\varphi T_{\pi_B}([m, b]) = \varphi \pi_B((m, b))$. Now $\varphi \pi_B((m, b)) = q_J \pi_{M_J}((m, b)) = T_{M_J}([m, b])$ so $[\mathrm{id}_J, \pi_J][0_{J0}, \pi_B]^{-1}$ is a Q(**Ban**) morphism which makes φ into a Q-Busby map. $\square$

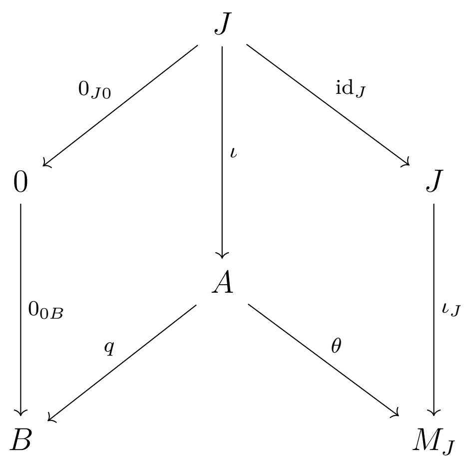
Proposition 5.1.3. *Let J and B be as in Proposition 5.1.2. Let $\varphi : B \to C_J$ be a Q-Busby map in the sense of Definition 5.1.1. Then φ is continuous as a map between Banach spaces.*

Proof. Since φ is a Q-Busby map, there exists a Q(**Ban**) morphism $[\alpha', \alpha][0_{X'0}, \omega]^{-1} : 0_{0B} \to \iota_J$ with $\varphi = T_\alpha T_\omega^{-1}$ (refer to Diagram 5.1). We have that $\omega f_X = 0_{X'B}$ and since $[0_{X'0}, \omega]$ is a pulation morphism, we can see that $\ker \omega = \operatorname{Ran} f_X$. Therefore, f_X is closed and $X/f_X(X')$ is a Banach space. Since T_ω is a bounded linear bijection between Banach spaces, we have that T_ω^{-1} is bounded by Theorem A.2.4. Now as both $X/f_X(X')$ and $M_J/\iota_J(J)$ are Banach spaces, T_α is also continuous and hence φ is continuous. □

From Propositions 5.1.2 and 5.1.3 we know that our new definition for a Q-Busby map is consistent with Definition 2.2.5. What remains to be shown is that we can construct Q-Busby maps from extensions and vice versa.

Theorem 5.1.4. *Let B and J be Banach algebras and let (ι, A, q) be an extension of B by J. Let $\varphi : B \to C_J$ be the algebra homomorphism defined as in Diagram 2.5. Then φ is a Q-Busby map.*

Proof. Recall from Chapter 2 that we can map A into M_J with the injective continuous homomorphism $\theta(a) = (L_a^A, R_a^A)$ where $L_a^A(j) = \iota^{[-1]}(a\iota(j))$ and $R_a^A(j) = \iota^{[-1]}(\iota(j)a)$. We now need to check that the $\mathcal{W}$-span in Diagram 5.4 is a Q(**Ban**) morphism.



(5.4)

The fact that $T_\theta T_q^{-1}$ is an algebra homomorphism follows from the fact that $\iota(J)$ is an ideal in A, so all that remains is to check whether $[0_{J0}, q]$ forms a pulation morphism. This holds since q is surjective and $\ker q = \iota(J)$. □

The next step will be to craft an extension from a Q-Busby map. Recall that in Chapter 2, we did this by constructing the pullback of our Q-Busby map and the quotient map q_J. Since this Q(**Ban**) construction will have to coincide with the original theory in the case where J is closed in its multiplier algebra, it makes sense to try and make the algebraic pullback into a Banach algebra. First, however, we will show it is a Banach space under a suitable norm.

Proposition 5.1.5. *Let J be a faithful Banach algebra, let B be a Banach algebra and let $\varphi : B \to M_J/\iota_J(J)$ be a Q-Busby map with representative $[\alpha', \alpha][0_{X'0}, \omega]^{-1}$. Further to this, let $P_\varphi = \{(m, b) \in M_J \times B : q_J(m) = \varphi(b)\}$. Then P_φ is a linear subspace of $M_J \oplus B$, and the formula*

$$\|(m, b)\|_{P_\varphi} = \inf\{\|\iota_J{}^{[-1]}(m - \alpha(x))\|_J + \|x\|_X : \omega(x) = b\}$$

defines a norm on P_φ.

Proof. We have that P_φ is a vector space since it is the algebraic pullback of q_J and φ.

Note that $m - \alpha(x) \in \iota_J(J)$ since $q_J(m - \alpha(x)) = \varphi(b) - \varphi(b) = 0$. We shall check each of the three conditions on $\|\cdot\|_{P_\varphi}$ for it to define a norm. To this end, let $(m, b) \in P_\varphi$ and suppose $\|(m, b)\|_{P_\varphi} = 0$. For any $\varepsilon > 0$, we can choose an $x \in X$ such that $\|\iota_J{}^{[-1]}(m - \alpha(x))\|_J + \|x\|_X < \varepsilon$. This gives us that $\|x\|_X < \varepsilon$ and hence$\|b\|_B = \|\omega(x)\| < \|\omega\|\varepsilon$. Moreover, $\|m\|_{M_J} - \|\alpha(x)\|_{M_J} \leq \|m - \alpha(x)\|_{M_J} \leq \|\iota_J{}^{[-1]}(m - \alpha(x))\|_J < \varepsilon$, hence we have that $\|m\|_{M_J} \leq (\|\alpha\| + 1)\varepsilon$. From here it is easy to infer that $(m, b) = (0, 0)$.

Now let $\lambda \in \mathbb{C}$. We have that $\inf\{\|{\iota_J}^{[-1]}(\lambda m - \alpha(x))\|_J + \|x\|_X : \omega(x) = \lambda b\} = \inf\{\|{\iota_J}^{[-1]}(\lambda m - \lambda\alpha(x/\lambda))\|_J + \|x/\lambda\|_X : \omega(x/\lambda) = b\} = |\lambda| \inf\{\|{\iota_J}^{[-1]}(m - \alpha(x))\|_J + \|x\|_X : \omega(x) = b\}$, hence $\|\lambda(m,b)\|_{P_\varphi} = |\lambda|\|(m,b)\|_{P_\varphi}$.

Lastly, let $(m_1, b_1), (m_2, b_2) \in P_\varphi$ and take $\|(m_1 + m_2, b_1 + b_2)\|_{P_\varphi}$. We have that since $m_1 - \alpha(x_1)$ and $m_2 - \alpha(x_2)$ are both in $\iota_J(J)$, that

$$\inf\{\|{\iota_J}^{[-1]}(m_1 + m_2 - \alpha(x_1 + x_2))\|_J + \|x_1 + x_2\|_X : \omega(x_1 + x_2) = b_1 + b_2\}$$

can be rewritten as

$$\inf\{\|{\iota_J}^{[-1]}(m_1 - \alpha(x_1)) + {\iota_J}^{[-1]}(m_2 - \alpha(x_2))\|_J + \|x_1 + x_2\|_X : \omega(x_1) = b_1, \omega(x_2) = b_2\}.$$

Now this is less than or equal to

$$\inf\{\|{\iota_J}^{[-1]}(m_1 - \alpha(x_1))\|_J + \|{\iota_J}^{[-1]}(m_2 - \alpha(x_2))\|_J + \|x_1\|_X + \|x_2\|_X : \omega(x_1) = b_1, \omega(x_2) = b_2\}$$

which is then equal to $\inf\{\|{\iota_J}^{[-1]}(m_1 - \alpha(x_1))\|_J + \|x_1\|_X : \omega(x_1) = b_1\} + \inf\{\|{\iota_J}^{[-1]}(m_2 - \alpha(x_2))\|_J + \|x_2\|_X : \alpha(x_2) = b_2\}$. We therefore have that $\|(m_1 + m_2, b_1 + b_2)\|_{P_\varphi} \leq \|(m_1, b_1)\|_{P_\varphi} + \|(m_2, b_2)\|_{P_\varphi}$ and hence that $\|\cdot\|_{P_\varphi}$ forms a norm making P_φ into a normed space. □

Lemma 5.1.6. *Let B, J and P_φ be as in Proposition 5.1.5. The normed space P_φ is complete with respect to $\|\cdot\|_{P_\varphi}$.*

Proof. Since P_φ is a normed space by Proposition 5.1.5, we need only check that for a sequence $(m_n, b_n) \in P_\varphi$ with $\sum_{n=1}^{\infty} \|(m_n, b_n)\| < \infty$, that $\sum_{n=1}^{\infty} (m_n, b_n)$ converges.

To this end, take such a sequence $(m_n, b_n) \in P_\varphi$. By the definition of an infimum, for each $n \in \mathbb{N}$, we can choose an $x_n \in X$ such that

$$\|{\iota_J}^{[-1]}(m_n - \alpha(x_n))\|_J + \|x_n\|_X \leq 2\|(m_n, b_n)\|_{P_\varphi}.$$

Therefore, we have that

$$\sum_{n=1}^{\infty}(\|{\iota_J}^{[-1]}(m_n - \alpha(x_n))\|_J + \|x_n\|_X) \leq 2\sum_{n=1}^{\infty}\|(m_n, b_n)\|_{P_\varphi}.$$

We can now see that both $\sum_{n=1}^{\infty}\|\iota^{[-1]}(m_n - \alpha(x_n))\|_J < \infty$ and $\sum_{n=1}^{\infty}\|x_n\|_X < \infty$. Since X is a Banach space, we have that $\sum_{n=1}^{\infty} x_n$ converges to an $x \in X$, and by the continuity of ω, we have that $b_n = \omega(x_n)$ converges to $\omega(x) \in B$.

Now since ι_J is continuous, we can see that

$$\sum_{n=1}^{\infty}\|m_n\|_{M_J} - \|\alpha\|\sum_{n=1}^{\infty}\|x_n\|_X \leq \sum_{n=1}^{\infty}\|m_n - \alpha(x_n)\|_{M_J} \leq \sum_{n=1}^{\infty}\|{\iota_J}^{[-1]}(m_n - \alpha(x_n))\|_J < \infty.$$

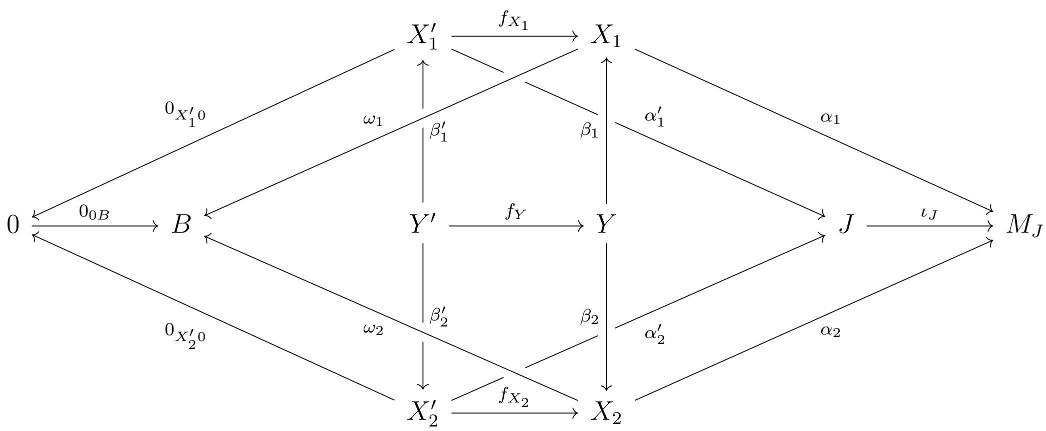
This means that $\sum_{n=1}^{\infty} m_n$ converges to $m \in M_J$. So (m_n, b_n) has limit $(m, \omega(x))$ and if this is in P_φ then we have shown P_φ is a Banach space.

To see that $(m, \omega(x)) \in P_\varphi$, we need $q_J(m) = \varphi\omega(x) = q_J\alpha(x)$, which is true if and only if $m - \alpha(x) \in \iota_J(J)$. Recall that $\sum_{n=1}^{\infty}\|{\iota_J}^{[-1]}(m_n - \alpha(x_n))\|_J < \infty$ so $\sum_{n=1}^{\infty}{\iota_J}^{[-1]}(m_n - \alpha(x_n))$ has limit $z \in J$. Continuity of ι_J and uniqueness of limits gives us that $\iota(z) = m - \alpha(x)$ as required. □

We notice that in the construction of our norm on P_φ, we make a choice of representative for φ. We would hope that choosing an equivalent $\mathcal{W}$-span, in the sense of Proposition A.1.5 would yield an equivalent norm on P_φ. We show that this is the case in the following Lemma.

Lemma 5.1.7. *Let B and φ be as in Proposition 5.1.5. Different choices of representative for φ yield equivalent norms on P_φ.*

Proof. Consider the following diagram of two $\mathcal{W}$-spans from 0_{0B} to ι_J, which are equivalent by the equivalence described in Proposition 4.3.3. Recall that $[0_{Y'0}, \omega_i\beta_i]$ is a pulation morphism.



(5.5)

Let $\|\cdot\|^{(i)}_{P_\varphi}$ for $i \in \{1,2\}$ be the norm constructed by choosing the $\mathcal{W}$-span $[\alpha'_i, \alpha_i][0_{X'_i0}, \omega_i]^{-1}$. Our aim is to show that these two norms are equivalent.

However, we note that the $\mathcal{W}$-span $[\alpha'_1\beta'_1, \alpha_1\beta_1][0_{Y'0}, \omega_1\beta_1]^{-1}$ is trivially equivalent to both $\mathcal{W}$-spans and therefore generates a norm $\|\cdot\|^{(Y)}_{P_\varphi}$ on P_φ shown in equation 5.6.

$$\|(m,b)\|^{(Y)}_{P_\varphi} = \inf\{\|{\iota_J}^{[-1]}(m - \alpha_1\beta_1(y))\|_J + \|y\|_Y : \omega_1\beta_1(y) = b\} \tag{5.6}$$

We will therefore show that $\|\cdot\|^{(Y)}_{P_\varphi}$ is equivalent to $\|\cdot\|^{(1)}_{P_\varphi}$ and appeal to the symmetry of Diagram 5.5 to arrive at the required conclusion.

We begin by showing that there exists $C > 0$ such that $\|\cdot\|^{(1)}_{P_\varphi} \leq C\|\cdot\|^{(Y)}_{P_\varphi}$. Since these are infima, for all $\varepsilon > 0$, we can choose $y \in Y$ such that $\|{\iota_J}^{[-1]}(m - \alpha_1\beta_1(y))\|_J + \|y\|_Y < \|(m,b)\|^{(Y)}_{P_\varphi} + \varepsilon$. Now we observe that $\beta_1(y) \in X$ and

$$\|{\iota_J}^{[-1]}(m - \alpha_1\beta_1(y))\|_J + \|\beta_1(y)\|_{X_1} \leq (1 + \|\beta_1\|)(\|{\iota_J}^{[-1]}(m - \alpha_1\beta_1(y))\|_J + \|y\|_Y)$$

We note that the left of the inequality is always greater than or equal to $\|(m,b)\|^{(1)}_{P_\varphi}$ and that we can form this inequality for any given ε. Therefore we have that $\|(m,b)\|^{(1)}_{P_\varphi} \leq (1+\|\beta\|)\|(m,b)\|^{(Y)}_{P_\varphi}$.

But this gives us that the identity map from $(P_\varphi, \|\cdot\|^{(Y)}_{P_\varphi})$ to $(P_\varphi, \|\cdot\|^{(1)}_{P_\varphi})$ is continuous. Applying the Banach Isomorphism Theorem gives us that its inverse is also continuous and hence that $\|\cdot\|^{(Y)}_{P_\varphi}$ is equivalent to $\|\cdot\|^{(1)}_{P_\varphi}$.

Since Diagram 5.5 is symmetrical, we also have that the norms $\|\cdot\|^{(Y)}_{P_\varphi}$ and $\|\cdot\|^{(2)}_{P_\varphi}$ are equivalent. Finally, this shows that $\|\cdot\|^{(1)}_{P_\varphi}$ and $\|\cdot\|^{(2)}_{P_\varphi}$ are equivalent as required. $\square$

Lemma 5.1.8. *Let $\iota_V \in \mathrm{Mon}(\boldsymbol{Ban})$, suppose that V is a Banach algebra with its given norm and suppose that $\iota_V(V')$ is a not necessarily closed subalgebra of V. Define the map $m : V' \times V' \to V'$ to be*

$$m(x', y') = {\iota_V}^{[-1]}(\iota_V(x')\iota_V(y')). \tag{5.7}$$

Then m is an associative bounded bilinear map on V'.

Proof. We first check that m is bilinear. Let $x'_1, x'_2, y' \in V'$ and let $\lambda \in \mathbb{C}$. We have that $m(x'_1 + \lambda x'_2, y') = {\iota_V}^{[-1]}(\iota_V(x'_1 + \lambda x'_2)\iota_V(y'))$ which, due to the linearity of ι_V, is equal to ${\iota_V}^{[-1]}(\iota_V(x'_1)\iota_V(y') + \lambda\iota_V(x'_2)\iota_V(y'))$. Since ${\iota_V}^{[-1]}$ is linear from $\iota_V(V') \to V'$, we have that $m(x'_1 + \lambda x'_2, y') = m(x'_1, y') + \lambda m(x'_2, y')$. A similar argument yields that m is linear in the second variable.

We now show m is associative. Let $x', y', z' \in V'$. By definition,

$$\begin{aligned} \iota_V m(x', m(y', z')) &= \iota_V(x')\iota_V m(y, z') \\ &= \iota_V(x')(\iota_V(y')\iota_V(z')) \end{aligned}$$

and $\iota_V m(m(x',y'),z') = \iota_V m(x',y')\iota_V(z') = (\iota_V(x')\iota_V(y'))\iota_V(z')$. Since multiplication is associative in V, and ι_V is injective, it follows that $m(x',m(y',z')) = m(m(x',y'),z')$ as required.

Our final step is to show that m is bounded, which will follow from showing it is separately continuous and applying Banach-Steinhaus. To this end, fix $x' \in V'$ and suppose we have a sequence $(y'_n) \in V'$ with limit $y' \in V'$, together with a $z' \in V'$ such that $m(x',y'_n) \to z'$ as $n \to \infty$. By the definition of m and using the continuity of ι_V, we have that $\iota_V(x')\iota_V(y'_n) \to \iota_V(z')$ as $n \to \infty$. The continuity of ι_V and the continuity of multiplication in V gives us that $\iota_V(x')\iota_V(y') = \iota_V(z')$. Now, since ι_V is injective, $m(x',y') = z'$, and hence m is continuous in the second variable by the Closed Graph Theorem. A similar argument gives us that m is continuous in the first variable. Hence m is separately continuous and so it is bounded by Banach-Steinhaus.

We have shown that m is an associative bounded bilinear map on V'. □

Proposition 5.1.9. *Let B and φ be as in Proposition 5.1.5. Then $\|\cdot\|_{P_\varphi}$ is submultiplicative up to a constant.*

Proof. Take the Banach algebra $M_J \oplus B$ equipped with the norm $\|(m,b)\| = \|m\|_{M_J} + \|b\|_B$. Define $\iota_\varphi : P_\varphi \to M_J \oplus B$ to be the inclusion map $\iota_\varphi((m,b)) = (m,b)$. We aim to show that $\iota_\varphi \in \mathrm{Mon}(\mathbf{Ban})$, and $\iota_V(P_\varphi)$ is a subalgebra of $M_J \oplus B$. Our claim would then follow from Lemma 5.1.8.

We know that $\iota_\varphi(P_\varphi)$ is a subalgebra of $M_J \oplus B$ and that ι_φ is injective. We only need to show ι_φ is continuous with respect to the norm $\|\cdot\|_{P_\varphi}$.

Let $(m,b) \in P_\varphi$ and recall that

$$\|(m,b)\|_{P_\varphi} = \inf\{\|{\iota_J}^{[-1]}(m-\alpha(x))\|_J + \|x\|_X : \omega(x) = b\}.$$

Note that $\|\iota_\varphi((m,b))\| = \|m\|_{M_J} + \|b\|_B$. Let $\varepsilon > 0$ and choose $x \in X$ such that

$$\|\iota_J{}^{[-1]}(m - \alpha(x))\|_J + \|x\|_X < \|(m,b)\|_{P_\varphi} + \varepsilon$$

Since $b = \omega(x)$,

$$\|b\|_B \leq \|\omega\|\|x\|_X < \|\omega\|(\|(m,b)\|_{P_\varphi} + \varepsilon). \tag{5.8}$$

The triangle inequality that $\|m\|_{M_J} \leq \|m - \alpha(x)\|_{M_J} + \|\alpha(x)\|_{M_J}$. Since $m - \alpha(x) = \iota_J(\iota_J{}^{[-1]}(m - \alpha(x)))$, we can then see that

$$\|m\|_{M_J} \leq \|\iota_J\|\|\iota_J{}^{[-1]}(m - \alpha(x))\|_J + \|\alpha\|\|x\|_X \leq (\|\iota_J\| + \|\alpha\|)(\|(m,b)\|_{P_\varphi} + \varepsilon). \tag{5.9}$$

We therefore have from Equations (5.8) and (5.9) that $\|\iota_\varphi((m,b))\| \leq (\|\iota_J\| + \|\alpha\| + \|\omega\|)\|(m,b)\|_{P_\varphi}$, giving us that ι_φ is continuous and hence in Mon(**Ban**).

Applying Lemma 5.1.8, we deduce that the multiplication on P_φ is bounded bilinear as a map $P_\varphi \times P_\varphi \to P_\varphi$ with respect to $\|\cdot\|_{P_\varphi}$. Hence there is some $K > 0$ such that $\|(m_1,b_1)(m_2,b_2)\|_{P_\varphi} \leq K\|(m_1,b_1)\|_{P_\varphi}\|(m_2,b_2)\|_{P_\varphi}$ for all (m_1,b_1) and (m_2,b_2) in P_φ. □

Theorem 5.1.10. *Let $\varphi : B \to C_J$ be a Q-Busby map. There is a Banach algebra norm on the algebraic pullback P_φ, such that*

$$0 \longrightarrow J \xrightarrow{\iota_\varphi} P_\varphi \xrightarrow{\pi_B} B \longrightarrow 0 \tag{5.10}$$

is an extension of Banach algebras (see Definition 2.3.3 for the notation).

Proof. We already know that ι_φ and π_B are algebra homomorphisms, from the calculations in Section 2.4.

Fix a representative for the Q-Busby map φ, and equip P_φ with the norm $\|\cdot\|_{P_\varphi}$ defined in Proposition 5.1.5. Note that for all j

inJ

$$\|\iota_\varphi(j)\|_{P_\varphi} = \|(\iota_J(j), 0)\|_{P_\varphi} \leq \|\iota_J{}^{[-1]}\iota_J(j)\|_J = \|j\|_J$$

and so ι_φ is continuous. Also note that if $(m, b) \in P_\varphi$ and we choose $x \in X$ such that $\omega(x) = b$, then

$$\|\pi_B(m, b)\|_B = \|\omega(x)\|_B \leq \|\omega\|\|x\|_X$$

and by taking the infimum over all such x, we obtain $\|\pi_B((m, b))\|_B \leq \|\omega\|\|(m, b)\|_{P_\varphi}$. So π_B is also continuous.

Finally, by Lemma 5.1.6 and Proposition 5.1.9, together with Proposition 2.1.9 in [7, p.156], there is a Banach algebra norm on P_φ that is equivalent to $\|\cdot\|_{P_\varphi}$. When we equip P_φ with this new norm, ι_φ and π_B remain continuous. □

5.2 Classifying extensions with Q-Busby maps

Having seen how to go from Banach algebra extensions to Q-Busby maps, and back again, it is natural to hope for a full generalisation of Theorem 2.3.9 to cases where J is faithful but ι_J need not have closed range.

This runs into several technical details which are unclear. For instance, should an object of the “Q-Busby category” be a Q-Busby map in the sense of Definition 5.1.1, or should it be a Q-Busby map together with a particular choice of representative $\mathcal{W}$-span? If we try to define a Pull functor from some "Q-Busby category" to $\mathbf{Ext}(J)$, this functor must be well-defined on objects. Currently, given a Q-Busby map φ we have only defined the Banach algebra structure on P_φ up to non-unique isomorphism.

Also, in Theorem 2.3.9 we were allowing the base algebra B to vary, and even when two extensions share the same base algebra B we allowed an isomorphism between these two extensions (in the sense of Definition 2.2.2) to induce a non-trivial automorphism of B. In such situations the two extensions could induce different Q-Busby maps

$B \to C_J$, which would be related by this automorphism of B. Hence, we would need to pin down an appropriate notion of isomorphism between two Q-Busby maps.

Therefore, we limit ourselves to a more restricted form of classification. This is similar to the choice made in Definition 1.4 of [3, p.42], which is then later used in Theorem 1.1 of [3, p.45].

Fix $B, J \in \mathbf{Balg}$ and assume that J is faithful. Consider the proper class $\mathcal{S}$ of all Banach algebra extensions of B by J, in the usual sense of this thesis (Definition 2.1.3).

Definition 5.2.1. Define the following equivalence relation between two such extensions: $(\iota_1, A_1, q_1) \sim (\iota_2, A_2, q_2)$ if there is an extension morphism (θ, id_B) from the first extension to the second one, in the sense of Definition 2.2.2. Recall that by Lemma 2.3.6, θ will then be an isomorphism in $\mathbf{Balg}$.

Remark 5.2.2.

1. If two extensions are equivalent in this sense, they are isomorphic in the category $\mathbf{Ext}(J)$ that was introduced in Section 2.2.
2. If two extensions are equivalent in this sense, then when we apply Theorem 5.1.4, we obtain the same Q-Busby map φ.

Let $\mathbf{Ext}(B, J)$ denote the quotient of the class $\mathcal{S}$ by this equivalence relation. By Theorem 5.1.4 and 5.2.2, the function Bus from Definition 2.3.1 which sends (ι, A, q) to the homomorphism $\varphi_A : B \to C_J$ induces a well-defined function

$$\mathrm{qBus} : \mathbf{Ext}(B, J) \longrightarrow \{\text{Q-Busby maps } B \to C_J\} \tag{5.11}$$

Theorem 5.2.3. *The function* qBus *in Equation 5.11 is a bijection. Informally: Banach algebra extensions of B by J are classified, up to equivalence as defined above, by Q-Busby maps from B to C_J.*

Proof. Surjectivity of qBus follows immediately from Theorem 5.1.10. To prove injectivity, let (ι_i, A_i, q_i) be extensions of B by J for $i = 1, 2$, which both give rise to the same homomorphism $\varphi : B \to C_J$ as in Theorem 5.1.4. Form the extension $(\iota_\varphi, P_\varphi, \pi_B)$ as given by the Theorem 5.1.10. Then as in the proof of Theorem 2.4.6, we can show that $(\iota_i, A_i, q_i) \sim (\iota_\varphi, P_\varphi, \pi_B)$ for $i = 1, 2$. Since $\sim$ is an equivalence relation, it follows that $(\iota_1, A_1, q_1) \sim (\iota_2, A_2, q_2)$, and so both extensions define the same element of $\mathbf{Ext}(B, J)$. $\square$

Appendix A

A.1 Category theory definitions and results

Definition A.1.1 (Category)**.** A category will be a set of objects, denoted $\text{obj}(\mathcal{C})$, and for each $A, B \in \text{obj}(\mathcal{C})$, a set of morphisms from A to B, denoted by $\hom_{\mathcal{C}}(A, B)$, satisfying the following conditions.

- For any $A, B, C \in \mathcal{C}$ together with $f \in \hom(A, B)$ and $g \in \hom(B, C)$, there is a composition $gf \in \hom(A, C)$.
- For every object $A \in \mathcal{C}$, there exists an identity morphism id_A which acts as an identity with respect to the composition.
- Composition of morphisms is associative.

To simplify notation we will frequently write $A \in \mathcal{C}$ as shorthand for $A \in \text{obj}(\mathcal{C})$, and we will sometimes abbreviate $\hom_{\mathcal{C}}(A, B)$ to $\hom(A, B)$ when there is no risk of confusion.

Definition A.1.2 (Subcategory and wide subcategory)**.** A subcategory $\mathcal{S}$ of a category $\mathcal{C}$ will be collection of objects and morphisms, taken from the objects and morphisms of $\mathcal{C}$, that form a category in their own right.

A wide subcategory of $\mathcal{C}$ will be a subcategory which contains all identity morphisms in $\mathcal{C}$. Equivalently, a wide subcategory of $\mathcal{C}$ is a subcategory $\mathcal{W}$ with $\text{obj}(\mathcal{W}) = \text{obj}(\mathcal{C})$.

Definition A.1.3 (Calculus of right fractions)**.** A category $\mathcal{C}$ with a subcategory $\mathcal{W}$ admits a calculus of right fractions if the following holds:

- $\mathcal{W}$ forms a wide subcategory.
- For $B, C, D \in \mathcal{C}$ with morphisms $w_1 \in \hom_{\mathcal{W}}(B, C)$ and $f_1 \in \hom_{\mathcal{C}}(C, D)$, there exists $A \in \mathcal{C}$, $w_2 \in \hom_{\mathcal{W}}(A, C)$ and $f_2 \in \hom_{\mathcal{C}}(A, B)$ such that diagram A.1 commutes.

$$\begin{array}{ccc} A & \xrightarrow{f_2} & B \\ \downarrow{\scriptstyle w_2} & & \downarrow{\scriptstyle w_1} \\ C & \xrightarrow{f_1} & D \end{array} \tag{A.1}$$

  This is known as the right Ore condition.
- Whenever we have a pair of parallel morphisms $f, g : \hom_{\mathcal{C}}(B, C)$, and $w_1 \in \hom_{\mathcal{W}}(C, D)$ such that $w_1 f = w_1 g$. Then we can find a second morphism $w_2 \in \hom_{\mathcal{W}}(A, C)$ such that $f w_2 = g w_2$.

Definition A.1.4 (Spans)**.** Let $\mathcal{C}$ be a category and let $\mathcal{D}$ be a subcategory. Consider Diagram A.2, which is a diagram in $\mathcal{C}$.

$$\begin{array}{ccccc} & & X & & \\ & \swarrow^{f} & & \searrow^{g} & \\ A & & & & B \end{array} \tag{A.2}$$

We say the above is a $\mathcal{D}$-span from A to B in $\mathcal{C}$ if $f \in \hom_{\mathcal{D}}(X, A)$.

Proposition A.1.5. *Let $\mathcal{C}$ be a category with a wide subcategory $\mathcal{W}$. If $\mathcal{W}$ admits a calculus of right fractions, then the following relation is an equivalence relation on $\mathcal{W}$-spans.*

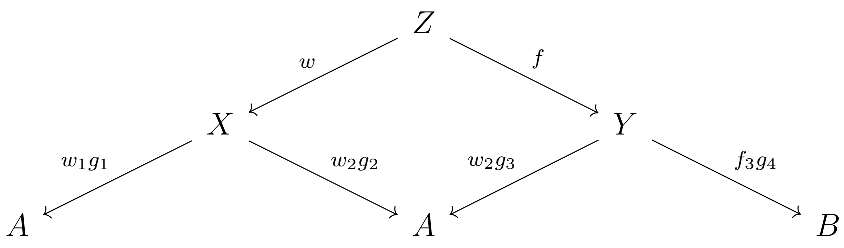
Let (w_1, X_1, f_1) and (w_2, X_2, f_2) be $\mathcal{W}$-spans from A to B. Then (w_1, X_1, f_1) is related to (w_2, X_2, f_2) if there exists a $X \in \mathcal{C}$ together with morphisms $g_1 \in \hom_{\mathcal{C}}(X, X_1)$ and $g_2 \in \hom_{\mathcal{C}}(X, X_2)$ such that $f_1 g_1 = f_2 g_2$ and $w_1 g_1 = w_2 g_2 \in \hom_{\mathcal{W}}(X, A)$.

Proof. The relation is obviously reflexive. If (w, X, f) is a $\mathcal{W}$-span from A to B, then we can always choose X together with the morphism id_X as our relation requirement.

Now suppose we have two $\mathcal{W}$-spans, (w_1, X_1, f_1) and (w_2, X_2, f_2) such that (w_1, X_1, f_1) is related to (w_2, X_2, f_2) with object and morphisms X, g_1 and g_2. Then we have that (w_2, X_2, f_2) is related to (w_1, X_1, f_1) with the same object and morphisms. Hence the relation is symmetric.

Lastly we check transitivity. Suppose we have three $\mathcal{W}$-spans (w_1, X_1, f_1), (w_2, X_2, f_2) and (w_3, X_3, f_3) from A to B. Now suppose the first and second spans are related with monic X and morphisms $g_1 \in \hom(X, X_1)$ and $g_2 \in \hom(X, X_2)$ and that the second and third spans are related with object Y and morphisms $g_3 \in \hom(Y, X_2)$ and $g_4 \in \hom(Y, X_3)$.

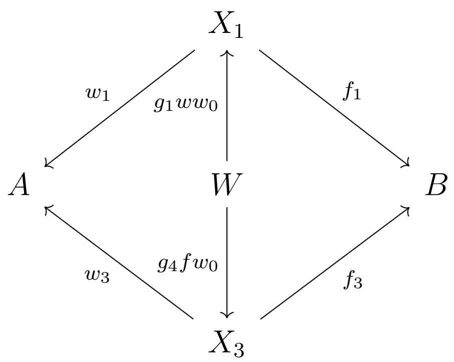
By definition, we have that $w_1 g_1, w_2 g_2, \in \hom_{\mathcal{W}}(X, A)$ so $(w_1 g_1, X, w_2 g_2)$ is a $\mathcal{W}$-span from A to A. Similarly, $(w_2 g_3, Y, f_3 g_4)$ is a $\mathcal{W}$-span from A to B and since these spans are adjacent, we can compose them. That is, there exists an object Z, together with morphisms $w \in \hom_{\mathcal{W}}(Z, A)$ and $f \in \hom_{\mathcal{W}}(Z, B)$ such that diagram A.3 commutes.



(A.3)

We now have that $w_1 g_1 w = w_2 g_2 w$ since the first and second spans are related and by the construction of our composition $w_2 g_2 w = w_2 g_3 f$. Lastly $w_2 g_2 f = w_3 g_4 f$ since the second and third spans are related and moreover, both $w_1 g_1 w, w_3 g_4 f \in \hom_{\mathcal{W}}(Z, A)$.

Further to this, we have that $f_3 g_4 f = f_2 g_3 f$ since the second and third spans are related. Referring to our composition above, we note that $w_2 g_2 w = w_2 g_3 f$, but we also note that w_2 is a morphism in $\mathcal{W}$ so by the third condition in Definition A.1.3, there exists an object $W \in \mathcal{C}$ together with a morphism $w_0 \in \hom_{\mathcal{W}}(W, Z)$ such that $g_2 w w_0 = g_3 f w_0$ so we now know that $f_2 g_2 f w_0 = f_2 g_4 w w_0$. Lastly we use the fact that the first and second spans are related to deduce that $f_2 g_2 w w_0 = f_1 g_1 w w_0$, which makes Diagram A.4 commute and hence show that the first and third spans are related.



(A.4)

Hence we have an equivalence relation on $\mathcal{W}$-spans. □

Proposition A.1.6. *Let $\mathcal{C}$ be a category with a wide subcategory $\mathcal{W}$. If $\mathcal{W}$ admits a calculus of right fractions, then composition of $\mathcal{W}$-spans using the right Ore condition is well defined and associative.*

Proof. This is stated in [10] but left as a routine exercise for the reader. This is due to the size of the diagrams which arise in this proof. We too shall omit this proof. □

Definition A.1.7 (Initial, terminal and zero objects)**.** Let $\mathcal{C}$ be a category. An object $I \in \mathcal{C}$ is initial if for all $A \in \mathcal{C}$, there exists a unique morphism $f_A : I \to A$. Similarly, an object $T \in \mathcal{C}$ is terminal if for all $A \in \mathcal{C}$, there exists a unique morphism $g_A : A \to T$.

An object $0 \in \mathcal{C}$ is a zero object if it is both initial and terminal. For two objects $A, B \in \mathcal{C}$, we may refer to the zero morphism $0_{AB} : A \to B$ as the unique morphism from A to B factoring through 0.

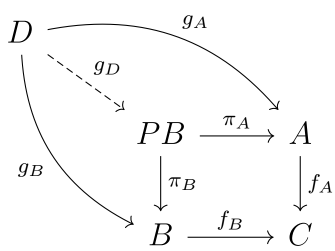
Definition A.1.8 (Preadditive and additive category)**.** A category $\mathcal{C}$ will be preadditive if for all $A, B \in \mathcal{C}$, either $\hom(A, B) = \emptyset$ or $\hom(A, B)$ forms an abelian group. Moreover, composition of morphisms has to be bilinear with respect to the group operation.

A category is additive if it is preadditive and it has all finite products.

It should be noted that a ring is a one object preadditive category. Composition of morphisms describes the multiplication and the group structure on the single hom set describes the addition. Since composition is bilinear with respect to the group operation, we have that multiplication distributes over addition. Because of this, it makes sense to generalise some ring theory to preadditive categories.

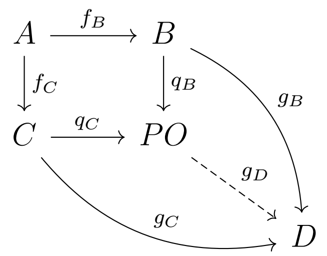
Definition A.1.9 (Pullback, pushout and pulation)**.** Let $\mathcal{C}$ be a category and let $A, B, C \in \mathcal{C}$.

Let $f_A \in \hom(A, C)$ and $f_B \in \hom(B, C)$. The pullback of f_A and f_B will be a fourth object $PB \in \mathcal{C}$ together with morphisms $\pi_A : PB \to A$ and $\pi_B : PB \to B$ such that $f_A \pi_A = f_B \pi_B$. Moreover, if there is a $D \in \mathcal{C}$ together with morphisms $g_A : D \to A$ and $g_B : D \to B$ where $f_A g_A = f_B g_B$, then there exists a unique morphism $g_D : D \to PB$ such that $g_A = \pi_A g_D$ and $g_B = \pi_B g_D$. We illustrate this concept with diagram A.5.



(A.5)

Once again, let $A, B, C \in \mathcal{C}$ and this time let $f_B \in \hom(A, B)$ and $f_C \in \hom(A, C)$. The pushout of f_B and f_C will be a fourth object $PO \in \mathcal{C}$ together with morphisms $q_B : B \to PO$ and $q_C : C \to PO$ such that $q_B f_B = q_C f_C$. Moreover, if there exists a $D \in \mathcal{C}$ together with morphisms $g_B : B \to D$ and $g_C : C \to D$ where $g_B f_B = g_C f_C$, then there exists a unique morphism $g_D : PO \to D$ such that $g_B = g_D q_B$ and $g_C = g_D q_C$. We illustrate this with diagram A.6.



(A.6)

Lastly, a commuting square is a pulation if it is both a pullback and a pushout.

Definition A.1.10 (Ideals in a preadditive category)**.** Let $\mathcal{C}$ be a preadditive category. A (left) right ideal will be a collection $\{I(A, B)\}_{A,B \in \mathcal{C}}$ of abelian subgroups $I(A, B) \subseteq \hom(A, B)$ which is stable under (left) right composition.

We will refer to such a collection as an ideal if it is both a left and a right ideal.

Lemma A.1.11. *Let $\mathcal{C}$ be a preadditive category and let $\mathcal{I}$ be an ideal in $\mathcal{C}$. The relation on morphisms in $\mathcal{C}$ given by $f \sim g$ if $f - g \in \mathcal{I}$ is an equivalence relation.*

Proof. Let $A, B \in \mathcal{C}$ such that $\hom(A, B)$ is non-empty. We have that $I(A, B)$ contains the zero morphism of $\hom(A, B)$ since it is a subgroup. Now for all $f \in \hom(A, B)$, $f - f = 0_{A,B} \in I(A, B) \subseteq \mathcal{I}$. Therefore the relation is reflexive.

Now suppose for $f, g \in \hom(A, B)$, that $f - g \in I(A, B)$. Since $I(A, B)$ is an abelian subgroup, it is closed under inverses and so $-(f - g) = g - f \in I(A, B) \subseteq \mathcal{I}$. Therefore the relation is symmetric.

Lastly, suppose for $f, g, h \in I(A, B)$, that $f - g, g - h \in I(A, B)$. We have that $I(A, B)$ is closed under the group operation so $f - g + g - h = f - h \in I(A, B) \subseteq \mathcal{I}$. Therefore the relation is transitive and hence an equivalence relation as required. □

Theorem A.1.12 (Quotienting). *Let $\mathcal{C}$ be a preadditive category, let $\mathcal{I}$ be an ideal in $\mathcal{C}$ and let $\sim$ be the relation described in Lemma A.1.11. We can construct a new preadditive category $\widehat{\mathcal{C}}$ which has the same objects as $\mathcal{C}$ and whose morphisms are equivalence classes of morphisms in $\mathcal{C}$.*

Proof. First we check that composition of classes of morphisms is well defined. Let $A, B \in \widehat{\mathcal{C}}$ and let $f_1, f_2 \in \mathcal{C}(A, B)$ and let $g_1, g_2 \in \mathcal{C}(B, C)$ such that $\widehat{f_1} = \widehat{f_2}$ and $\widehat{g_1} = \widehat{g_2}$. We need $\widehat{g_1 f_1} = \widehat{g_2 f_2}$. There exists $h \in I(B, C)$ such that $g_2 = g_1 + h$. Now $g_1 f_1 - g_2 f_2 = g_1 f_1 - (g_1 + h) f_2 = g_1(f_1 - f_2) - h f_2 \in \mathcal{I}$ as required. As composition is well defined, the $\widehat{\mathrm{id}_A}$ is the identity morphism for A in $\widehat{\mathcal{C}}$ and associativity of composition in $\widehat{\mathcal{C}}$ follows from associativity in $\mathcal{C}$. Therefore, $\widehat{\mathcal{C}}$ is a category.

Note that since $\hom_{\widehat{\mathcal{C}}}(A, B) = \hom_{\mathcal{C}}(A, B)/I(A, B)$ we have that morphism addition will be well defined. This is because we have taken the quotient of an abelian group by an abelian, and hence normal, subgroup. Next we tackle preadditivity. If $\hom_{\mathcal{C}}(A, B)$ is empty, then $\hom_{\widehat{\mathcal{C}}}(A, B)$ is empty. If the original hom-set were not empty, it is now $\hom_{\mathcal{C}}(A, B)/I(A, B)$, which is an abelian group.

Finally, we check composition is bilinear with respect to the group operation in $\widehat{\mathcal{C}}$. Let $\widehat{k}, \widehat{f} \in \hom_{\widehat{\mathcal{C}}}(A, B)$ and let $\widehat{g}, \widehat{h} \in \hom_{\widehat{\mathcal{C}}}(B, C)$. First we check $(\widehat{k} + \widehat{f})\widehat{g} = \widehat{k + f}\widehat{g} = \widehat{kg + fg} = \widehat{kg} + \widehat{fg} = \widehat{k}\widehat{g} + \widehat{f}\widehat{g}$. Similarly, we have that $\widehat{f}(\widehat{g} + \widehat{h}) = \widehat{f}\widehat{g} + \widehat{f}\widehat{h}$. We conclude that $\widehat{\mathcal{C}}$ is a preadditive category. □

Definition A.1.13 (Functor). Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A functor $F : \mathcal{C} \to \mathcal{D}$ will be a map which takes an object $A \in \mathcal{C}$ to an object $F(A) \in \mathcal{D}$. Moreover, F takes morphisms $f \in \hom_{\mathcal{C}}(A, B)$ to morphisms $F(f) \in \hom_{\mathcal{D}}(F(A), F(B))$. Lastly, F needs to satisfy the following:

- $F(\text{id}_A) = \text{id}_{F(A)}$ for all $A \in \mathcal{C}$.
- $F(gf) = F(g)F(f)$ for all $f \in \hom_{\mathcal{C}}(A, B)$ and $g \in \hom_{\mathcal{C}}(B, C)$.

Definition A.1.14 (Natural transformation/isomorphism)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories with functors $F, G : \mathcal{C} \to \mathcal{D}$. A natural transformation $\eta : F \to G$ is a family of morphisms in $\mathcal{D}$ such that for each object $A \in \mathcal{C}$, there exists a morphism $\eta_A : F(A) \to G(A)$. For all morphisms $f : A \to B$, these morphisms must make Diagram A.7 commute.

$$\begin{array}{ccc} F(A) & \xrightarrow{F(f)} & F(B) \\ \big\downarrow{\scriptstyle \eta_A} & & \big\downarrow{\scriptstyle \eta_B} \\ G(A) & \xrightarrow{G(f)} & G(B) \end{array} \tag{A.7}$$

If η_A is an isomorphism for all $A \in \mathcal{C}$, then we say η is a natural isomorphism.

Definition A.1.15 (Equivalence of categories)**.** We say two categories, $\mathcal{C}$ and $\mathcal{D}$ are equivalent if there exists functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ and natural isomorphisms $\eta : GF \to \text{id}_{\mathcal{C}}$ and $\varepsilon : FG \to \text{id}_{\mathcal{D}}$.

A.2 Consequences of the Open Mapping Theorem

Definition A.2.1 (Open map)**.** A map $T : X \to Y$ is open if for all open sets $U \subseteq X$, we have that $T(U)$ is open in Y.

Theorem A.2.2 (Open Mapping Theorem)**.** *Let X and Y be Banach spaces and let $T \in B(X; Y)$. If T is surjective, then it is an open map.*

Theorem A.2.3 (Closed Graph Theorem)**.** *Let X and Y be Banach spaces, let $T : X \to Y$ be a linear map, and let the graph of T be the space $\{(x, y) \in X \times Y : T(x) = y\}$. Then T is continuous if and only if the graph of T is a closed subspace of $X \times Y$.*

Theorem A.2.4 (Banach Isomorphism Theorem)**.** *Let X and Y be Banach spaces and let $T \in B(X; Y)$. If T is bijective, then it has a bounded inverse.*

Proofs of the Open Mapping Theorem and Closed Graph Theorem will not be given here, see [15, p. 236, p. 238]. The proof of the Banach Isomorphism Theorem is left as an exercise and so we provide a proof.

Proof. If T is bijective, then the graph of T is the graph of T^{-1}. The fact that T is continuous implies the graph of T is closed. Hence we have that the graph of T^{-1} is closed and that T^{-1} must be continuous. □

Corollary A.2.5. *Let X and Y be Banach spaces and let $T \in B(X; Y)$. If T is surjective, then there exists $K > 0$ such that for all $y \in Y$ with $\|y\| = 1$ we can find $x \in X$ with $T(x) = y$ and $\|x\| < K$.*

Proof. Let $\tilde{T} : X/\ker T \to Y$ be the map associated with T by the First Isomorphism Theorem. Since $\tilde{T}$ is bijective, $\tilde{T}^{-1}$ is bounded by the Banach Isomorphism Theorem. Let $y \in Y$ with $\|y\| = 1$, we have that $\|\tilde{T}^{-1}(y)\| \leq \|\tilde{T}^{-1}\|$. Now $\|\tilde{T}^{-1}(y)\| = \inf\{\|x\|_X : \tilde{T}^{-1}(y) - x \in \ker T\} \leq \|\tilde{T}^{-1}\|$ so there must exist an $x \in X$ such that $T(x) = y$ and $\|x\| < 2\|\tilde{T}^{-1}\|$. □

Proposition A.2.6. *Let X be an arbitrary indexing set and consider $\ell_1(X)$. Let Y be a Banach space with a surjective bounded linear $S : Y \to \ell_1(X)$. We can construct a bounded linear $T : \ell_1(X) \to Y$ such that $ST = \mathrm{id}_{\ell_1(X)}$.*

Proof. Define δ_x to be the element which takes value 1 on $x \in X$ and 0 elsewhere. Let $a \in \ell_1(X)$, we can write $a = \sum_{x\in X} a_x\delta_x$ where $\sum_{x\in X} |a_x| < \infty$. Since S is surjective and $\|\delta_x\| = 1$ we can find a $K > 0$ and $\gamma_x \in Y$ such that $\|\gamma_x\| < K$ and $S(\gamma_x) = \delta_x$. This is true for all $x \in X$ and the bound K works for all $x \in X$. Set $T(\delta_x) = \gamma_x$ and extend this linearly, we aim to show T is bounded.

Let $b = \sum_{x\in X} b_x\delta_x \in \ell_1(X)$ with $\|b\| = \sum_{x\in X} |b_x| = 1$. We compute $\|T(b)\| = \|\sum_{x\in X} b_x\gamma_x\| \leq \sum_{x\in X} \|b_x\gamma_x\|$. Now $\|b_x\gamma_x\| \leq |b_x|\|\gamma_x\| < |b_X|K$ for all $x \in X$, so $\|T(b)\| < K \sum_{x\in X} |b_x| = K$. The fact that $ST = \mathrm{id}_{\ell_1(X)}$ is obvious by construction. □

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