# Scaling limits and fluctuations for random growth under capacity rescaling

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**Abstract**

We evaluate a strongly regularised version of the Hastings-Levitov model HL($\alpha$) for $0 \leq \alpha < 2$. Previous results have concentrated on the small-particle limit where the size of the attaching particle approaches zero in the limit. However, we consider the case where we rescale the whole cluster by its capacity before taking limits, whilst keeping the particle size fixed. We first consider the case where $\alpha = 0$ and show that under capacity rescaling, the limiting structure of the cluster is not a disk, unlike in the small-particle limit. Then we consider the case where $0 < \alpha < 2$ and show that under the same rescaling the cluster approaches a disk. We also evaluate the fluctuations and show that, when represented as a holomorphic function, they behave like a Gaussian field dependent on $\alpha$. Furthermore, this field becomes degenerate as $\alpha$ approaches 0 and 2, suggesting the existence of phase transitions at these values.

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# 1 Introduction

Random growth occurs in many real world settings, for example we see it exhibited in the growth of tumours and bacterial growth. As such we would like to be able to model such processes to determine their behaviour in their scaling limits. Since the the 1960's, models have been built in order to describe individual processes. Perhaps the most famous examples of such models are the Eden model [1] and Diffusion Limited Aggregation (DLA) [2]. The Eden model is used to describe bacterial colony growth, whereas, DLA describes mineral aggregation (see for example [3]).

In their 1998 paper [4], Hastings and Levitov introduced a one parameter family of conformal maps HL($\alpha$) which can be used to model Laplacian growth processes and allows us to vary between the previous models by varying the parameter $\alpha$. In contrast to many well studied lattice based models, HL($\alpha$) is formed by using conformal mappings [4]. We can then use complex analysis techniques to evaluate the growth. We consider a regularised version of this model and show that at certain values of $\alpha$ a phase transition on the scaling limits occurs.

## 1.1 Outline of the model

In order to define our model we start by defining the single particle map. Define $\Delta$ as the exterior of the unit disk in the complex plane, $\Delta = \{|z| > 1\}$. For any conformal map $f : \Delta \to \mathbb{C}$ we define the logarithmic capacity of the map to be,

$$\lim_{z\to\infty} \log\big(f'(z)\big) := \log f'(\infty).$$

For each $c > 0$, we then choose a general single particle mapping $f_c : \Delta \to \mathbb{C}\backslash K$ which takes the exterior of the unit disk to itself minus a particle of logarithmic capacity $c > 0$ at $z = 1$. Note that we can then rescale and rotate the mapping $f_c(z)$ to allow any attaching point on the boundary of the unit disk by letting $f_n(z) = e^{i\theta_n} f_{c_n}(ze^{-i\theta_n})$ where $\theta_n$ is the attaching angle and $c_n$ is the logarithmic capacity of the n$^{\text{th}}$ particle map $f_{c_n}(z)$.

We can now form the cluster by composing the single particle maps. Let $K_0 = \Delta^c = \{|z| \leq 1\}$. Suppose that we have some compact set $K_n$ made up of $n$ particles. We can find a bi-holomorphic map which fixes $\infty$ and takes the exterior of the unit disk to the complement of $K_n$ in the complex plane, $\phi_n : \Delta \to \mathbb{C}\backslash K_n$. We then define the map $\phi_{n+1}$ inductively;

$$\phi_{n+1} = \phi_n \circ f_{n+1} = f_1 \circ f_2 \circ .... \circ f_{n+1}.$$

There are several possible choices for the particle map $f_c$. The choice we make is dependent on what shape we would like the attaching particle to have. Hastings and Levitov introduce both the strike and bump mappings in [4]. The strike map attaches a single slit onto the boundary at $z = 1$ whereas the bump map attaches a particle with non-empty interior. We would like our model to map a general particle so that we can easily recover all of our results for all of the classical maps, including those mentioned above. In [5], Norris et al show that the following mapping suffices,

$$f_c(z) = e^c z \exp\left(\frac{2c}{z-1} + \delta_c(z)\right) \tag{1}$$

where $\delta_c(z)$ is some function of $z$ with $|\delta_c(z)| < \frac{\tilde{\lambda}c^{\frac{3}{2}}|z|}{|z-1|(|z|-1)}$ and $\tilde{\lambda} \in [0, \infty)$ is some constant. Therefore, we take our single particle mappings from a class of particles satisfying (1) for fixed $\tilde{\lambda}$.

Now it just remains is to define how the attaching points $\theta_n$ and capacities $c_n$ are randomly distributed. We want to model Laplacian growth and so we choose the $\theta_n$ to be uniformly distributed on the circle. This choice is made because after renormalisation of $\phi_n$, the Lebesgue measure of the unit circle under the image of $\phi_n$ is harmonic measure as seen from infinity [3], but the harmonic measure of a portion of the unit circle is just the arclength of that portion rescaled by $2\pi$.

Finally, we must choose how the capacities $c_n$ are distributed. Hastings and Levitov [4] introduced a parameter $\alpha$ in order to distinguish between the various individual models they would like to encode within this one model for Laplacian growth. They choose,

$$c_n = c|\phi'_{n-1}(e^{i\theta_n})|^{-\alpha}$$

for some $c > 0$. This gives the Eden model when $\alpha = 1$ and DLA when $\alpha = 2$. In [4], the authors argue that, to leading order the total capacity, $\log(\phi'_n(\infty))$ is given by $(1+\alpha cn)^{\frac{1}{\alpha}}$. Therefore, if we define our version of HL($\alpha$) using the very strong regularisation $c_n = c|\phi'_{n-1}(\infty)|^{-\alpha}$, $c_n$ is approximately given by

$$c_n^* = \frac{c}{1 + \alpha c(n-1)}. \tag{2}$$

In what follows, we denote $\phi_n = f_1 \circ ... \circ f_n$ where $f_n(z) = e^{i\theta_n} f_{c_n^*}(ze^{-i\theta_n})$ with $\theta_n$ i.i.d uniform on $[0, 2\pi]$. We then keep $c$ fixed and rescale the cluster by its total capacity and evaluate the shape of the rescaled cluster $e^{-\sum_{i=1}^n c_i^*}\phi_n$ as $n \to \infty$.

### 1.2 Previous work

With the model now defined we can outline the work already done in this area. Most work has been done in the small-particle limit. This method involves evaluating the limiting cluster $\phi_n$ as we send the particle capacity $c \to 0$ while sending $n \to \infty$ with $nc \sim t$ for some $t$. Using this method Turner and Norris show that for $\alpha = 0$ the limiting cluster in the small particle case behaves like a growing disk [6]. Furthermore, Turner, Viklund and Sola show that in the small particle limit the shape of the cluster in a regularised setting approaches a circle for all $\alpha \geq 0$ provided the regularisation is sufficient [7]. Moreover, Silvestri [8] shows that the fluctuations on the boundary, for HL(0), in this small particle limit can be characterised by a log-correlated Gaussian field.

A different approach to that of the small-particle limit is to not let $c \to 0$ as $n \to \infty$, but instead, the limit of the cluster is found by rescaling the whole cluster by the capacity of the cluster at time $n$, before taking limits as the number of particles tends to infinity. Rhode and Zinsmeister introduce a regularisation to the Hastings-Levitov model and show that in the case of $\alpha = 0$ the limiting cluster under capacity rescaling exists and has finite length [3].

Our work will follow the second approach. We will use results and ideas from the papers listed above, and in particular methods from [5], in order to characterise the limiting shape of the cluster in a regularised setting for $0 \leq \alpha < 2$ and then evaluate the fluctuations. Our results break down for $\alpha \geq 2$, this will be the subject of future work.

### 1.3 Statement of results

We first consider the case where $\alpha = 0$ and show that under capacity rescaling, the limiting structure of the rescaled cluster is not a disk. This comes in the form of the following theorem.

**Theorem 1.1.** *Given any sequence $\{\theta_k\}_{1\leq k\leq n}$ of angles between $0$ and $2\pi$ and $c > 0$, set $\Psi_n = f_1 \circ ... \circ f_n$ where $f_k(z) = e^{i\theta_k} f_c(e^{-i\theta_k} z)$ and $f_c(z)$ is any fixed capacity map in the class of particles given by (1). There exists some $c_0 > 0$, which depends only on $\tilde{\lambda}$ such that for all $0 < c < c_0$, there exists an $\epsilon > 0$ such that for all $r > 0$,*

$$\limsup_{n\to\infty} \sup_{|z|>r} |e^{-cn}\Psi_n(z) - z| > \epsilon.$$

*In particular if $\{\theta_k\}_{1\leq k\leq n}$ are i.i.d uniform on $[0, 2\pi]$ then $\Psi_n$ is the HL$(0)$ and the statement above shows that HL$(0)$ does not converge to a disk under capacity rescaling.*

This result is particularly interesting because it is independent of our choice of angles. If we have a constant capacity map of the right form then there is no possible way to choose the angles so that under capacity rescaling the limiting cluster looks like a disk.

Next we consider the case where $0 < \alpha < 2$ and show that under capacity rescaling the $HL(\alpha)$ cluster approaches a disk. We then evaluate the fluctuations and show that they behave like a Gaussian field dependent on $\alpha$. Our two main results are stated as follows.

**Theorem 1.2.** *For $0 < \alpha < 2$, let the map $\phi_n$ be defined as above with $c_n^*$ as defined in (2) and $\theta_n$ i.i.d. Then for any $r > 1$,*

$$\mathbb{P}\left(\limsup_{n\to\infty}\left\{\sup_{|z|\geq r}|e^{-\sum_{i=1}^n c_i^*}\phi_n(z) - z| > \frac{\log n}{\sqrt{n}}\right\}\right) = 0.$$

This result tells us we have uniform convergence of our cluster in the exterior disk to a disk. The following result shows that the fluctuations behave like a Gaussian field.

**Theorem 1.3.** *Let $0 < \alpha < 2$ and $\phi_n$ be defined as in Theorem 1.2. Then as $n \to \infty$,*

$$\sqrt{n}\left(e^{-\sum_{i=1}^n c_i^*}\phi_n(z) - z\right) \to \mathcal{F}(z)$$

*in distribution on $\mathcal{H}$, where $\mathcal{H}$ is the space of holomorphic functions on $|z| > 1$, equipped with a suitable metric $\boldsymbol{d}_{\mathcal{H}}$ defined later, and where*

$$\mathcal{F}(z) = \sum_{m=0}^{\infty}(A_m + iB_m)z^{-m}$$

*with $A_m$, $B_m \sim \mathcal{N}\left(0, \frac{2}{\alpha(2m+2-\alpha)}\right)$ and $A_m$, $B_k$ independent for all choices of $m$ and $k$.*

Notice that it is clear this result does not hold for $\alpha = 0$ or $\alpha = 2$. This is in contrast to [7] where results hold for all $\alpha \geq 0$ suggesting a phase transition at these values.

### 1.4 Outline of the paper

The outline of the paper is as follows. In Section 2 we will show that for clusters formed by composing maps of constant capacity and of a certain form, we can not pick a sequence of angles so that the limiting cluster under capacity rescaling approaches a disk. In particular, under capacity rescaling HL(0) is not a growing disk. Then in Section 3 we will provide estimates that will allow us to set up our problem for $0 < \alpha < 2$. In Section 4, we show that the pointwise limit of the cluster for $0 < \alpha < 2$ is a disk and then in Section 5 we will use a Borel-Cantelli argument to show we have uniform convergence on the exterior disk. Finally, in Section 6 we will evaluate the fluctuations for $0 < \alpha < 2$ and show that they are distributed according to a Gaussian field dependent on $\alpha$.

## 2 The case where $\alpha = 0$

We want to evaluate the limiting shape of our random cluster. We first deal with the case where $\alpha = 0$. We will show in this section that in the limit HL(0) does not approach a disk. Furthermore, we will prove a stronger statement that for clusters formed by composing maps of constant capacity, in the class of particles defined in (1), we can not approach a disk under capacity rescaling. We note that in the case where $\alpha = 0$ our regularisation does not effect the model, so this result holds for HL(0) under no regularisation. Our proof is reliant on the fact that under capacity rescaling the limit for HL(0) exists, this result was proved by Rhode and Zinsmeister in [3]. One might expect, given this result, that the scaling limit is a growing disk, this would agree with the result in the small particle limit [6]. However, the following theorem proves this does not hold.

**Theorem 1.1.** *Given any sequence* $\{\theta_k\}_{1\leq k\leq n}$ *of angles between* $0$ *and* $2\pi$ *and* $c>0$*, set* $\Psi_n = f_1 \circ ... \circ f_n$ *where* $f_k(z) = e^{i\theta_k} f_c(e^{-i\theta_k} z)$ *and* $f_c(z)$ *is any fixed capacity map in the class of particles given by (1). There exists some* $c_0 > 0$*, which depends only on* $\tilde{\lambda}$ *such that for all* $0 < c < c_0$*, there exists an* $\epsilon > 0$ *such that for all* $r > 0$*,*

$$\limsup_{n\to\infty} \sup_{|z|>r} |e^{-cn}\Psi_n(z) - z| > \epsilon.$$

*In particular if* $\{\theta_k\}_{1\leq k\leq n}$ *are i.i.d uniform on* $[0, 2\pi]$ *then* $\Psi_n$ *is the HL*$(0)$ *and the statement above shows that HL*$(0)$ *does not converge to a disk under capacity rescaling.*

*Proof.* First suppose this does not hold. Then for any $\epsilon > 0$,

$$\limsup_{n\to\infty} \sup_{|z|>r} |e^{-cn}\Psi_n(z) - z| < \epsilon.$$

Then we can write,

$$|e^{-cn}\Psi_n(z) - z| = \left|\left(e^{-c}f_n(z) - z\right) + e^{-c}\left(e^{-c(n-1)}\Psi_{n-1}(f_n(z)) - f_n(z)\right)\right|$$

which we can bound below for all $|z| > r$ as follows,

$$|e^{-cn}\Psi_n(z) - z| \geq |e^{-c}f_n(z) - z| - \sup_{|z|>r} |e^{-c}||e^{-c(n-1)}\Psi_{n-1}(f_n(z)) - f_n(z)|.$$

We can then take the supremum of both sides, and use that $|f_n(z)| > r$ for all $|z| > r$, to reach the following bound on the supremum,

$$\sup_{|z|>r} |e^{-cn}\Psi_n(z) - z| \geq \sup_{|z|>r} |e^{-c}f_n(z) - z| - \sup_{|z|>r} |e^{-c}||e^{-c(n-1)}\Psi_{n-1}(z) - z|.$$

Taking the limit supremum and using our initial assumption we have,

$$\limsup_{n\to\infty} \sup_{|z|>r} |e^{-cn}\Psi_n(z) - z| \geq \limsup_{n\to\infty} \sup_{|z|>r} |e^{-c}f_n(z) - z| - e^{-c}\epsilon.$$

We have assumed the maps have constant capacity and so the absolute value of the single particle map $f$ has no dependence on $n$ so we can remove the limit supremum from the lower bound to leave,

$$\limsup_{n\to\infty} \sup_{|z|>r} |e^{-cn}\Psi_n(z) - z| \geq \sup_{|z|>r} |e^{-c}f_c(z) - z| - e^{-c}\epsilon.$$

But using the definition of $f_c(z) = e^c z \exp\left(\frac{2c}{z-1} + \delta_c(z)\right)$ we can rewrite this as

$$\limsup_{n\to\infty} \sup_{|z|>r} |e^{-cn}\Psi_n(z) - z| \geq \sup_{|z|>r} |z| \left|\exp\left(\frac{2c}{z-1} + \delta_c(z)\right) - 1\right| - e^{-c}\epsilon.$$

Then by using the integral form of Taylor's remainder formula we see that for any complex $x$, $|e^x - (1+x)| \leq |x|^2 e^{|x|}$ and therefore,

$$|e^x - 1| \geq |x| - |x|^2 e^{|x|}.$$

Hence, we can find a lower bound on the expression above,

$$\begin{aligned}&\limsup_{n\to\infty} \sup_{|z|>r} |e^{-cn}\Psi_n(z) - z| \\ &\geq \sup_{|z|>r} |z| \left(\left|\frac{2c}{z-1} + \delta_c(z)\right| - \left|\frac{2c}{z-1} + \delta_c(z)\right|^2 \exp\left(\left|\frac{2c}{z-1} + \delta_c(z)\right|\right)\right) - e^{-c}\epsilon.\end{aligned}$$

Then by Proposition 2.1 in [5] and taking $z \to \infty$ we bound below by a constant term,

$$\limsup_{n\to\infty} \sup_{|z|>r} |e^{-cn}\Psi_n(z) - z| \geq 2c|\beta| - e^{-c}\epsilon$$

where $\beta$ is non-zero for $0 < c < c_0 = \frac{4}{1+\tilde{\lambda}}$ and $\tilde{\lambda}$ is defined as in (1). So choose $\epsilon = \frac{2c\beta}{(1+e^{-c})} > 0$ then

$$\limsup_{n\to\infty} \sup_{|z|>r} |e^{-cn}\Psi_n(z) - z| \geq \epsilon$$

a contradiction. □

This is a strong result because it proves that if we have a cluster which is composed of functions of the right form, no matter how we pick our sequence of attaching angles $\{\theta_n\}$ the limiting structure of the cluster, when rescaled by its capacity, does not approach a disk.

# 3 Estimates

In this section we will provide estimates for several variables which we will then call on throughout the rest of the paper. Whilst this work is an essential part of the analysis, we advise that the reader may skip the proofs of this section if they are only interested the main results of the paper.

### 3.1 Notation

We start by providing some notation used throughout the remainder of the paper. Let $\phi_k$ and $c_i^*$ be defined as above, then we denote $C_{n,k}^* = \sum_{i=k}^n c_i^*$ and for any $z \in \mathbb{C}$ we define our increments $X_{k,n}(z)$ as;

$$X_{k,n}(z) := e^{-C_{n,1}^*} \left( \phi_k \left( e^{-C_{n,k+1}^*} z \right) - \phi_{k-1} \left( e^{-C_{n,k}^*} z \right) \right).$$

Furthermore, we also define the sum,

$$M_n(z) := \sum_{k=1}^n X_{k,n}(z) = e^{-C_{n,1}^*} \phi_n(z) - z$$

and the bounded variation

$$T_n(z) := \sum_{k=1}^n \mathbb{E}(|X_{k,n}(z)|^2 \mid \mathcal{F}_{k-1})$$

where $\mathcal{F}_{k-1}$ is the $\sigma$-algebra generated by the set $\{\theta_i \ : 1 \le i \le k-1\}$.

Our first aim is to show that we approach a disk pointwise, equivalently, for a fixed value $z$, we want to show $|M_n(z)| \to 0$ as $n \to \infty$. Throughout we use $\lambda$ to denote strictly positive, unless stated otherwise, constants which may change from line to line. Where these constants depend on parameters from the model we indicate these explicitly.

### 3.2 Error term evaluation

In order to more easily apply complex analysis methods to our cluster we would like to write the sum $C_{n,1}^*$ in a simplified form. We do so by providing the following approximation on the sum, subject to an error term which we will evaluate in the preceeding lemmas.

**Lemma 3.1.** *For $c_n^* = \frac{c}{1+\alpha c(n-1)}$ we have the following equality;*

$$C_{n,k}^* = C_{n,k}(1 + \epsilon_{n,k})$$

*where $C_{n,k} = \frac{1}{\alpha} \log \left( \frac{1+\alpha cn}{1+\alpha c(k-1)} \right)$ and*

$$0 < \epsilon_{n,k} < \frac{\alpha^2 c^2 (n-k+1)}{(1+\alpha c(k-1))(1+\alpha cn) \log \left( \frac{1+\alpha cn}{1+\alpha c(k-1)} \right)}.$$

*Proof.* We will approximate the sum with

$$C_{n,k} = \frac{1}{\alpha} \log \left( \frac{1+\alpha cn}{1+\alpha c(k-1)} \right) = \int_k^{n+1} \frac{c}{1+\alpha c(x-1)} dx$$

Then

$$\begin{aligned}
C^*_{n,k} - C_{n,k} &= \sum_{i=k}^{n} \left( c_i^* - \int_k^{i+1} \frac{c}{1+\alpha c(x-1)} dx \right) \\
&\leq \sum_{i=k}^{n} \left( c_i^* - c_{i+1}^* \right) \\
&= \frac{\alpha c^2 (n-k+1)}{(1+\alpha c(k-1))(1+\alpha cn)}.
\end{aligned}$$

Thus,

$$0 < \epsilon_{n,k} < \frac{\alpha^2 c^2 (n-k+1)}{(1+\alpha c(k-1))(1+\alpha cn)\log\left(\frac{1+\alpha cn}{1+\alpha c(k-1)}\right)}.$$

□

We now claim that errors get small as $n$ tends to infinity and as such we will be able to ignore them later on. Furthermore, we can find a nice bound on $(1+\alpha ck)^{1+\epsilon_{n,k}}$ which will make computations in later sections easier.

**Lemma 3.2.** *With $\epsilon_{n,k}$ defined as above, we have $\epsilon_{n,k} \to 0$ as $n \to \infty$. Furthermore, for $1 \leq k \leq n$ and $\alpha \geq 0$ the following bound holds,*

$$(1+\alpha ck)^{1+\epsilon_{n,k}} \leq (1+\alpha c e^{\alpha c})(1+\alpha ck).$$

*Proof.* We have shown

$$0 < \epsilon_{n,k} < \frac{\alpha^2 c^2 (n-k+1)}{(1+\alpha c(k-1))(1+\alpha cn)\log\left(\frac{1+\alpha cn}{1+\alpha c(k-1)}\right)}.$$

So we consider,

$$\begin{aligned}
&\sup_{k\leq n} \frac{\alpha^2 c^2 (n-k+1)}{(1+\alpha c(k-1))(1+\alpha cn)\log\left(\frac{1+\alpha cn}{1+\alpha c(k-1)}\right)} \\
&= \frac{\alpha^2 c^2}{1+\alpha cn} \sup_{k\leq n} \frac{n-k+1}{(1+\alpha c(k-1))\log\left(\frac{1+\alpha cn}{1+\alpha c(k-1)}\right)}
\end{aligned}$$

So let us find

$$\sup_{k\leq n} \frac{n-k+1}{(1+\alpha c(k-1))\log\left(\frac{1+\alpha cn}{1+\alpha c(k-1)}\right)}.$$

Let $x = 1+\alpha c(k-1)$ and find the derivative

$$\frac{d}{dx}\left(\frac{1+\alpha cn - x}{x\log\left(\frac{1+\alpha cn}{x}\right)}\right) = \frac{(1+\alpha cn) - (1+\alpha cn)\log\left(\frac{1+\alpha cn}{x}\right) - x}{x^2\left(\log\left(\frac{1+\alpha cn}{x}\right)\right)^2}.$$

The numerator in this fraction is increasing and from this it is clear that the derivative is negative. Therefore the maximum occurs when $k = 1$. Thus,

$$0 \leq \epsilon_{n,k} \leq \frac{\alpha^2 c^2}{1+\alpha c n} \frac{n}{\log(1+\alpha c n)}.$$

Here it is clear to see $\epsilon_{n,k} < \alpha c$. Furthermore, taking the limit as $n \to \infty$ we have $\epsilon_{n,k} \to 0$ as claimed. Finally, we see we can write

$$(1+\alpha c k)^{1+\epsilon_{n,k}} = (1+\alpha c k)(1+\alpha c k)^{\epsilon_{n,k}} = (1+\alpha c k)(1+(1+\alpha c k)^{\epsilon_{n,k}} - 1).$$

So let $\delta_{n,k} = (1+\alpha c k)^{\epsilon_{n,k}} - 1$, then

$$\delta_{n,k} = (e^{\epsilon_{n,k}\log(1+\alpha c k)} - 1) \leq \epsilon_{n,k}\log(1+\alpha c k) e^{\epsilon_{n,k}\log(1+\alpha c k)}.$$

We have just shown that

$$|\epsilon_{n,k}| \leq \frac{\alpha^2 c^2}{1+\alpha c n} \frac{n}{\log(1+\alpha c n)}.$$

So,

$$0 \leq |\delta_{n,k}| \leq e^{\alpha c} \frac{\alpha^2 c^2 n}{1+\alpha c n} \leq \alpha c e^{\alpha c}.$$

Therefore,

$$(1+\alpha c k)^{1+\epsilon_{n,k}} \leq (1+\alpha c k)(1+\alpha c e^{\alpha c}).$$

□

Thus from now on we can use the approximation $e^{C^*_{n,k}} \approx \left(\frac{1+\alpha c n}{1+\alpha c(k-1)}\right)^{\frac{1}{\alpha}}$.

### 3.3 Pointwise estimates for $0 < \alpha < 2$

The aim of this section is to find pointwise bounds on $X_{k,n}(z)$ and $T_n(z)$ defined in Section 3.1. By definition;

$$\begin{aligned} |X_{k,n}(z)| &= e^{-C^*_{n,1}} |\phi_k(e^{C^*_{n,k+1}} z) - \phi_{k-1}(e^{C^*_{n,k}} z)| \\ &= e^{-C^*_{n,1}} |\phi_{k-1}(e^{i\theta_k} f_{c^*_k}(e^{-i\theta_k} e^{C^*_{n,k+1}} z)) - \phi_{k-1}(e^{C^*_{n,k}} z)| \end{aligned}$$

So we introduce the following parameterisation.

**Definition 3.3.** *For each $n \in \mathbb{N}$, $z \in \mathbb{C}$, $k \leq n$ and $\delta_c(z)$ defined as in (1), we define the following parameterisation for $0 < s < 1$,*

$$\eta_{k,n}(s,z) = e^{C^*_{n,k}} z \exp\left( s \left( \frac{2c^*_k}{e^{-i\theta_k} e^{C^*_{n,k+1}} z - 1} + \delta_c\left(e^{-i\theta_k} e^{C^*_{n,k+1}} z\right)\right)\right).$$

We first show that for $|z| > r$, for some $r > 1$, we can bound $\delta\left(e^{-i\theta_k} e^{C^*_{n,k+1}} z\right)$ by a constant via the following lemma.

**Lemma 3.4.** *For $C^*_{n,k}$ and $\delta_c(z)$ defined as above, and for $|z| > r$ for some $r > 1$, the following bound holds,*

$$|\delta_{c_k^*}\left(e^{-i\theta_k}e^{C^*_{n,k+1}}z\right)| < \lambda(\alpha,c,r)k^{\frac{1}{\alpha}-\frac{3}{2}}n^{\frac{-1}{\alpha}} \leq \lambda(\alpha,c,r)k^{-\frac{3}{2}} < \lambda(\alpha,c,r)$$

*where $\lambda(\alpha,c,r)$ is a positive constant dependent on $\alpha$, $c$ and $r$.*

*Proof.* From equation (1) we know

$$|\delta_{c_k^*}(z)| \leq \frac{\tilde{\lambda}(c_k^*)^{\frac{3}{2}}|z|}{|z-1|(|z|-1)}$$

where $\tilde{\lambda}$ is some constant. Therefore,

$$\left|\delta_{c_k^*}\left(e^{-i\theta_k}e^{C^*_{n,k+1}}z\right)\right| \leq \frac{\tilde{\lambda}(c_k^*)^{\frac{3}{2}}|e^{C^*_{n,k+1}}z|}{|e^{-i\theta_k}e^{C^*_{n,k+1}}z-1|(|e^{C^*_{n,k+1}}z|-1)}.$$

Since $|z| > r$,

$$\left|\delta_{c_k^*}\left(e^{-i\theta_k}e^{C^*_{n,k+1}}z\right)\right| \leq \frac{\tilde{\lambda}(c_k^*)^{\frac{3}{2}}e^{C^*_{n,k+1}}r}{(e^{C^*_{n,k+1}}r-1)^2}.$$

Note that $\tilde{\lambda}$ could equal zero here. So using the estimates on $e^{C^*_{n,k+1}}$ and $\epsilon_{n,k}$ from Lemmas 3.1 and 3.2 respectively we have the following bound,

$$\begin{aligned}\left|\delta_{c_k^*}\left(e^{-i\theta_k}e^{C^*_{n,k+1}}z\right)\right| &\leq \lambda(\alpha,c,r)k^{\frac{1}{\alpha}-\frac{3}{2}}n^{\frac{-1}{\alpha}} \\ &\leq \lambda(\alpha,c,r)k^{-\frac{3}{2}} < \lambda(\alpha,c,r)\end{aligned}$$

where $\lambda(\alpha,c,r)$ is a constant dependent on $\alpha$, $c$ and $r$. □

Now using Definition 3.3 we see,

$$\eta(0) = e^{C^*_{n,k}}z, \qquad \eta(1) = e^{i\theta_k}f_{c_k^*}(e^{-i\theta_k}e^{C^*_{n,k+1}}z)$$

where $f_{c_k^*}(z)$ is defined as in Section 1. Therefore,

$$|X_{k,n}(z)| = e^{-C^*_{n,1}}|\phi_{k-1}(\eta(1)) - \phi_{k-1}(\eta(0))|.$$

Before finding pointwise bounds on $X_{k,n}(z)$ and $T_n(z)$, we first find pointwise bounds on elements of $\eta_{k,n}(s,z)$ and its derivative.

**Lemma 3.5.** *For $\eta_{k,n}(s,z)$ defined in (3.3), for each $z \in \mathbb{C}$ with $|z| > r$ and each $0 \leq s \leq 1$, the following pointwise bound holds,*

$$\left|\exp\left(s\left(\frac{2c_k^*}{e^{-i\theta_k}e^{C^*_{n,k+1}}z-1} + \delta_{c_k^*}\left(e^{-i\theta_k}e^{C^*_{n,k+1}}z\right)\right)\right)\right| \leq \lambda(\alpha,c,r)$$

*where $\lambda(\alpha,c,r)$ is a constant dependent on $\alpha$, $c$ and $r$. Furthermore,*

$$|\dot{\eta}_{k,n}(s)| \leq \lambda(\alpha,c,r)\left|\frac{c_k^*e^{C^*_{n,k}}z}{e^{-i\theta_k}e^{C^*_{n,k+1}}z-1}\right| \leq \lambda(\alpha,c,r)\frac{c_k^*e^{C^*_{n,k}}}{e^{C^*_{n,k+1}}r-1}.$$

*Proof.* Let $\lambda(\alpha, c, r)$ be some constant that we allow to vary throughout the proof. First notice that since $c_k^* < c$ and $e^{C_{n,k+1}^*}|z| > r$ it follows that

$$\left| s\left( \frac{2c_k^*}{e^{-i\theta_k} e^{C_{n,k+1}^*} z - 1} \right) \right| \leq \frac{2c}{r-1}.$$

Therefore as,

$$\begin{aligned} &\left| \exp\left( s\left( \frac{2c_k^*}{e^{-i\theta_k} e^{C_{n,k+1}^*} z - 1} + \delta_{c_k^*}\left( e^{-i\theta_k} e^{C_{n,k+1}^*} z \right) \right) \right) \right| \\ &\leq \exp\left( \left| \frac{2c_k^*}{e^{-i\theta_k} e^{C_{n,k+1}^*} z - 1} \right| + \left| \delta_{c_k^*}\left( e^{-i\theta_k} e^{C_{n,k+1}^*} z \right) \right| \right) \end{aligned}$$

we use the bound above along with Lemma 3.4 to reach the following bound

$$\begin{aligned} \left| \exp\left( s\left( \frac{2c_k^*}{e^{-i\theta_k} e^{C_{n,k+1}^*} z - 1} + \delta\left( e^{-i\theta_k} e^{C_{n,k+1}^*} z \right) \right) \right) \right| &\leq \exp\left( \frac{2c}{r-1} + \lambda(\alpha, c, r) \right) \\ &= \lambda(\alpha, c, r). \end{aligned}$$

Now consider $\dot{\eta}_{k,n}(s)$. Recalling that

$$\eta_{k,n}(s, z) = e^{C_{n,k}^*} z \exp\left( s\left( \frac{2c_k^*}{e^{-i\theta_k} e^{C_{n,k+1}^*} z - 1} + \delta_{c_k^*}\left( e^{-i\theta_k} e^{C_{n,k+1}^*} z \right) \right) \right)$$

we see that

$$|\dot{\eta}_{k,n}(s)| \leq \left| \left( \frac{2c_k^*}{e^{-i\theta_k} e^{C_{n,k+1}^*} z - 1} + \delta_{c_k^*}\left( e^{-i\theta_k} e^{C_{n,k+1}^*} z \right) \right) \right| |\eta_{k,n}(s, z)|.$$

Then using the bound we found above,

$$|\dot{\eta}_{k,n}(s)| \leq \lambda(\alpha, c, r) |e^{C_{n,k}^*} z| \left( \left| \frac{2c_k^*}{e^{-i\theta_k} e^{C_{n,k+1}^*} z - 1} \right| + \left| \delta_{c_k^*}\left( e^{-i\theta_k} e^{C_{n,k+1}^*} z \right) \right| \right)$$

where $\lambda(\alpha, c, r)$ is some constant. Now using the fact that $|z| > r$ and the bound from Lemma 3.4 we see that

$$|\dot{\eta}_{k,n}(s)| \leq \lambda(\alpha, c, r) \left| \frac{2c_k^* e^{C_{n,k}^*} z}{e^{-i\theta_k} e^{C_{n,k+1}^*} z - 1} \right| \leq \lambda(\alpha, c, r) \frac{2c_k^* e^{C_{n,k}^*}}{e^{C_{n,k+1}^*} r - 1}.$$

where the second inequality follows by using that $|z| > r$ again. ∎

Now we can use the bounds above to give us a pointwise bound on $X_{k,n}(z)$. We will use the following distortion theorem in the proof [9].

**Theorem 3.6.** *For a function from the exterior disc into the complex plane $F : \Delta \to \mathbb{C}$ that is univalent except for a simple pole at $\infty$ and Laurent expansion of the form*

$$F(z) = z + a_0 + \sum_{n=1}^{\infty} a_n z^{-n}$$

*we have the estimate*

$$\frac{|z|^2 - 1}{|z|^2} \leq |F'(z)| \leq \frac{|z|^2}{|z|^2 - 1} \leq \frac{|z|}{|z| - 1} \qquad z \in \Delta.$$

Our bound on $X_{k,n}(z)$ is given by the following lemma.

**Lemma 3.7.** *For the sequence $\{X_{k,n}(z)\}_{k=0}^{n}$ and corresponding filtration $\mathcal{F}_k$ defined as above, and for a fixed $|z| > r$, the following property is satified for all $0 < k \leq n$;*

$$|X_{k,n}(z)| < \lambda(\alpha, c, r) \frac{c_k^*}{e^{C_{n,k+1}^*} r - 1}$$

*where $\lambda(\alpha, c, r)$ is a constant dependent on $\alpha$, $c$ and $r$. Furthermore, for $0 < \alpha \leq 1$,*

$$\sup_{k \leq n} |X_{k,n}(z)| < \lambda(\alpha, c, r) \frac{1}{n}$$

*and for $\alpha > 1$,*

$$\sup_{k \leq n} |X_{k,n}(z)| < \lambda(\alpha, c, r) \frac{1}{n^{\frac{1}{\alpha}}}.$$

*Proof.* By definition

$$|X_{k,n}(z)| = e^{-C_{n,1}^*} |\phi_{k-1}(\eta(1)) - \phi_{k-1}(\eta(0))|.$$

Hence,

$$\begin{aligned} |X_{k,n}(z)| &< e^{-C_{n,1}^*} \left| \int_0^1 \phi_{k-1}'(\eta_{k,n}(s,z)) \, \dot{\eta}_{k,n}(s) ds \right| \\ &< e^{-C_{n,1}^*} \left| \int_0^1 \phi_{k-1}'(\eta_{k,n}(s,z)) \, ds \right| |\dot{\eta}_{k,n}(s)| \end{aligned}$$

Using Lemma 3.5 we have,

$$|\dot{\eta}_{k,n}(s)| \leq \lambda(\alpha, c, r) \frac{2 c_k^* e^{C_{n,k}^*} r}{e^{C_{n,k+1}^*} r - 1}.$$

where $\lambda(\alpha, c, r)$ is a non-zero constant that will vary throughout this proof. Moreover, we can find a bound on $\left| \int_0^1 \phi_{k-1}'(\eta_{k,n}(s,z)) \, ds \right|$ using Theorem 3.6,

$$\left| \int_0^1 \phi_{k-1}'(\eta_{k,n}(s,z)) \, ds \right| < e^{C_{k-1,1}^*} \sup_{0<s<1} \frac{|\eta_{k,n}(s,z)|}{|\eta_{k,n}(s,z)| - 1}.$$

Note that in order to apply the distortion theorem to our function $\phi_{k-1}$ we had to rescale by a factor of $e^{C^*_{k-1,1}}$. It is easy to show that $\inf_{0\le s\le 1}|\eta_{k,n}(s,z)| \ge |z|$ and therefore for $|z| > r$,

$$\Big|\int_0^1 \phi'_{k-1}(\eta_{k,n}(s,z))\,ds\Big| < e^{C^*_{k-1,1}}\frac{r}{r-1}.$$

Thus, by compiling the bounds above,

$$\begin{aligned}|X_{k,n}| &< \lambda(\alpha,c,r)e^{-C^*_{n,1}}\frac{e^{C^*_{k-1,1}}r}{r-1}\frac{2c^*_k e^{C^*_{n,k}}r}{e^{C^*_{n,k+1}}r-1}\\ &< \lambda(\alpha,c,r)\frac{c^*_k}{e^{C^*_{n,k+1}}r-1}.\end{aligned}$$

Using the estimates in Lemma 3.1 and 3.2 we have,

$$|X_{k,n}| < \lambda(\alpha,c,r)k^{\frac{1}{\alpha}-1}n^{-\frac{1}{\alpha}}.$$

First consider the case where $0 < \alpha \le 1$. Then $\frac{1-\alpha}{\alpha} \ge 0$. Hence, it is clear that the maximum occurs when $k = n$ and thus

$$\sup_{k\le n}|X_{k,n}(z)| < \lambda(\alpha,c,r)\frac{1}{n}$$

However, when $\alpha > 1$, $k^{\frac{1-\alpha}{\alpha}} < 1$ , so

$$\sup_{k\le n}|X_{k,n}(z)| < \lambda(\alpha,c,r)\frac{1}{n^{\frac{1}{\alpha}}}$$

where $\lambda(\alpha,c,r)$ is a constant dependent on $\alpha$, $c$ and $r$. □

It is now clear to see that as $n$ approaches infinity the bound on $X_{k,n}(z)$ approaches zero pointwise.

**Corollary 3.8.** *For $X_{k,n}(z)$ defined as above;*

$$\lim_{n\to\infty}\sup_{k\le n}|X_{k,n}(z)| = 0$$

Now we want to calculate a bound on the variation $T_n(z) = \sum_{k=1}^n \mathbb{E}(|X_{k,n}(z)|^2|\mathcal{F}_{k-1})$. This is given by the following lemma.

**Lemma 3.9.** *The following ineqaulity holds for sufficiently large $n$. If $0 < \alpha < 2$,*

$$T_n(z) \le \lambda(\alpha,c,r)\frac{1}{n}$$

*where $\lambda(\alpha,c,r) > 0$ is some constant.*

*Proof.* First let us look at $|X_{k,n}(z)|^2$. As before we can bound

$$|X_{k,n}(z)|^2 < e^{-2C^*_{n,1}}\Big|\int_0^1 \phi'_{k-1}(\eta_{k,n}(s,z))\,ds\Big|^2 |\dot{\eta_{k,n}}(s)|^2.$$

Therefore,

$$\mathbb{E}(|X_{k,n}(z)|^2 \mid \mathcal{F}_{k-1}) \leq e^{-2C^*_{n,1}}\mathbb{E}\left(\Big|\int_0^1 \phi'_{k-1}(\eta_{k,n}(s,z))\,ds\Big|^2 |\dot{\eta_{k,n}}(s)|^2 \mid \mathcal{F}_{k-1}\right).$$

We can find an upper bound on the integral using a distortion theorem again and then remove it from the expectation. By above,

$$\Big|\int_0^1 \phi'_{k-1}(\eta_{k,n}(s,z))\,ds\Big|^2 < e^{2C^*_{k-1,1}}\frac{r^2}{(r-1)^2}.$$

So all that remains to calculate is $\mathbb{E}(|\dot{\eta_{k,n}(s,z)}|^2|\mathcal{F}_{k-1})$. Firstly by Lemma 3.5,

$$|\dot{\eta_{k,n}}(s)| \leq \lambda(\alpha,c,r)\frac{c^*_k e^{C^*_{n,k}}}{e^{C^*_{n,k+1}}r-1}.$$

Then let $w = e^{C^*_{n,k+1}}r$ and so

$$|\dot{\eta_{k,n}}(s,z)| \leq \lambda(\alpha,c,r)\frac{c^*_k\; e^{c^*_k}w}{e^{-i\theta_k}w-1}.$$

Moreover, since the $c^*_k$ are predetermined, the only randomness here comes from the $\theta_k$ and thus,

$$\mathbb{E}(|\dot{\eta_{k,n}}(s)|^2 \mid \mathcal{F}_{k-1}) \leq 4(c^*_k)^2 e^{2c^*_k}\int_0^{2\pi}\frac{|w|^2}{|e^{-i\theta}w-1|^2}d\theta.$$

It is easily shown that for $w \in \mathbb{C}$,

$$\int_0^{2\pi}\frac{|w|^2}{|e^{-i\theta}w-1|^2}d\theta \leq \frac{6|w|}{\gamma|w|-1}.$$

Therefore,

$$\mathbb{E}(|\dot{\eta_{k,n}}(s)|^2|\mathcal{F}_{k-1}) \leq 24(c^*_k)^2 e^{2c^*_k}\frac{re^{C^*_{n,k+1}}}{re^{C^*_{n,k+1}}-1}$$

It is clear for all $k \leq n$, $c^*_k < c$, therefore,

$$\mathbb{E}(|\dot{\eta_{k,n}}(s)|^2 \mid \mathcal{F}_{k-1}) \leq 24e^c(c^*_k)^2\frac{re^{C^*_{n,k}}}{re^{C^*_{n,k+1}}-1}.$$

Finally we can use the bound $\frac{1}{re^{C^*_{n,k+1}}-1} \leq \frac{1}{re^{C^*_{n,k}}-1}\frac{e^{c}r}{r-1}$ and bring together the previous bounds to reach the following bound on $T_n(z)$. Let $\lambda(\alpha, c, r) > 0$ be some constant that will vary throughout. Then,

$$\begin{aligned}
T_n(z) &\leq \lambda(\alpha, c, r)\sum_{k=1}^{n}\left(e^{-2C^*_{n,1}}\ \ e^{2C^*_{k-1,1}}\ \ (c^*_k)^2\frac{e^{C^*_{n,k}}}{re^{C^*_{n,k}}-1}\right) \\
&\leq \lambda(\alpha, c, r)\sum_{k=1}^{n}(c^*_k)^2\frac{e^{-C^*_{n,k}}}{\left(e^{C^*_{n,k}}r-1\right)}.
\end{aligned}$$

We can evaluate this bound to reach an upper bound for $T_n(z)$. We will manipulate the $c^*_k$ term. Recall,

$$c^*_k = \frac{c}{1+\alpha c(k-1)}.$$

This can be rewritten as

$$c^*_k = \frac{c}{\left(e^{\alpha\sum_{i=1}^{k-1}c^*_i}\right)^{\frac{1}{1+\epsilon_{k-1,1}}}}$$

where $\epsilon_{k-1,1}$ is the error term from Lemma 3.1. Furthermore,

$$e^{\alpha\sum_{i=1}^{k-1}c^*_i} = e^{\alpha(\sum_{i=1}^{n}c^*_i-\sum_{i=k}^{n}c^*_i)} = \frac{(1+\alpha cn)^{1+\epsilon_{n,1}}}{e^{\alpha(\sum_{i=k}^{n}c^*_i)}}.$$

But by the bound found in Lemma 3.2,

$$c^*_k = \frac{c(e^{\alpha(\sum_{i=k}^{n}c^*_i)})^{\frac{1}{1+\epsilon_{k-1,1}}}}{(1+\alpha cn)^{\frac{1+\epsilon_{n,1}}{1+\epsilon_{n,k}}}} \leq \frac{\lambda(\alpha, c, r)(e^{\alpha(\sum_{i=k}^{n}c^*_i)})}{(1+\alpha cn)}$$

for some constant $\lambda(\alpha, c, r)$. We can substitute this into the bound on $T_n(z)$ to give

$$T_n(z) \leq \lambda(\alpha, c, r)\frac{1}{n}\sum_{k=1}^{n}(c^*_k)\frac{e^{C^*_{n,k}(\alpha-1)}}{\left(e^{C^*_{n,k}}r-1\right)}.$$

We can now approximate this with a Riemann integral on intervals of length $c^*_k$, letting $x = C^*_{n,k}$ we have

$$T_n(z) \leq \lambda(\alpha, c, r)\frac{1}{n}\int_0^{C^*_{n,1}}\frac{e^{x(\alpha-1)}}{(e^x r-1)}dx.$$

Using a substitution $u = e^x r - 1$ gives

$$T_n(z) \leq \lambda(\alpha, c, r)\frac{1}{n}\int_{r-1}^{e^{C^*_{n,1}}r-1}\frac{(u+1)^{\alpha-2}}{u}du.$$

We use that $u < u + 1$ and $0 < \alpha < 2$,

$$T_n(z) \leq \lambda(\alpha, c, r)\frac{1}{n}\int_{r-1}^{(1+\alpha cn)^{\frac{1}{\alpha}}r-1} u^{\alpha-3}du.$$

Now since $\alpha \neq 2$,

$$\begin{aligned} T_n(z) &\leq \lambda(\alpha, c, r)\frac{1}{n}\left[u^{\alpha-2}\right]_{r-1}^{(1+\alpha cn)^{\frac{1}{\alpha}}r-1} \\ &= \lambda(\alpha, c, r)\frac{1}{n}\left(((1+\alpha cn)^{\frac{1}{\alpha}}r-1)^{\alpha-2} - (r-1)^{\alpha-2}\right). \end{aligned}$$

Since $0 < \alpha < 2$, then $\alpha - 2 < 0$ so

$$T_n(z) \leq \lambda(\alpha, c, r)\frac{1}{n}\left((r-1)^{\alpha-2} - ((1+\alpha cn)^{\frac{1}{\alpha}}r-1)^{\alpha-2}\right).$$

which is positive and we can bound above by

$$T_n(z) \leq \lambda(\alpha, c, r)\frac{1}{n}.$$

□

Moreover since $T_n(z) \geq 0$, we have the following corollary.

**Corollary 3.10.** *For $0 < \alpha < 2$,*

$$\lim_{n\to\infty} T_n(z) = 0$$

Note that throughout this section many of the bounds hold for $\alpha \geq 2$, however, in later sections our methods will not hold. Hence, we will focus on the case where $0 < \alpha < 2$ and the $\alpha \geq 2$ case is left for future work.

# 4 Pointwise results for $0 < \alpha < 2$

We are now in a position to analyse the limiting structure of the map $\phi_n$ as $n \to \infty$ for $0 < \alpha < 2$. Our aim is to use the bounds on the increments $X_{k,n}(z)$ and $T_n(z)$ found in the previous section to produce a pointwise estimate on the difference between the cluster map and the disk of capacity $e^{C^*_{n,k}}$. In order to do so we will apply the following theorem of Freedman [10].

**Theorem 4.1** (Freedman)**.** *Suppose $X_{k,n}$ is $\mathcal{F}$-measurable and $\mathbb{E}\{X_{k,n} \mid \mathcal{F}_{k-1}\} = 0$ and define $M_n$ and $T_n$ as above. Let $M$ be a positive real number and suppose $\mathbb{P}\{|X_{k,n}| \leq M \mid k \leq n\} = 1$. Then for all positive numbers $a$ and $b$,*

$$\mathbb{P}\{M_n \geq a \ \text{ and } \ T_n(z) \leq b \text{ for some } n > 0\} \leq \exp\left[\frac{-a^2}{2(Ma+b)}\right].$$

Thus, if we can show our bounds on $X_{k,n}$ and $T_n$ are sufficient we will be able to use this theorem to bound the difference, $M_n$, between the cluster and the disk of radius $e^{\sum_{i=1}^n c_i^*}$ provided that we satisfy the conditions of the theorem. Our aim for the remainder of this section will be to show that the hypotheses of Theorem 4.1 are satisfied and then apply the theorem. First we need to show $\mathbb{E}(X_{k,n}(z)|\mathcal{F}_{k-1}) = 0$. We do so via the following lemma.

**Lemma 4.2.** *For each fixed $z \in \mathbb{C}$, and the sequence $\{X_{k,n}(z)\}_{k=0}^n$ and corresponding filtration $\mathcal{F}_k$ defined as above, the following property is satified for all $0 < k \leq n$,*

$$\mathbb{E}(X_{k,n}(z)|\mathcal{F}_{k-1}) = 0.$$

*Proof.* We first show;

$$\int_0^{2\pi} \phi_{k-1}(e^{i\theta} f_{c_k^*}(e^{-i\theta} z))\frac{d\theta}{2\pi} = \phi_{k-1}(e^{c_k^*} z).$$

Let $w = e^{i\theta}$, then the integral can be rewritten as

$$\int_0^{2\pi} \phi_{k-1}(e^{i\theta} f_{c_k^*}(e^{-i\theta} z))\frac{d\theta}{2\pi} = \frac{1}{2\pi i}\int_C \frac{\phi_{k-1}(w f_{c_k^*}(z/w))}{w} dw$$

where $C$ is the unit circle centered at 0. The map $\phi_{k-1}(w f_{c_k^*}(z/w)) : \Delta \to \Delta$ is analytic with a removable singularity at 0 and so by Cauchy's integral formula,

$$\begin{aligned}\frac{1}{2\pi i}\int_C \frac{\phi_{k-1}(w f_{c_k^*}(z/w))}{w} dw &= \lim_{w\to 0} \phi_{k-1}(w f_{c_k^*}(z/w)) \\ &= \phi_{k-1}(\lim_{w\to 0} w f_{c_k^*}(z/w)) \\ &= \phi_{k-1}(\lim_{w\to 0}(e^{c_k^*} z + a_0 w + a_1 \frac{w^2}{z^2} + ...))\end{aligned}$$

for some complex number sequence of $a_i$'s. Thus,

$$\int_0^{2\pi} \phi_{k-1}(e^{i\theta} f_{c_k^*}(e^{-i\theta} z))\frac{d\theta}{2\pi} = \phi_{k-1}(e^{c_k^*} z)$$

as required. So now let us consider $\mathbb{E}(\phi_k(z)|\mathcal{F}_{k-1})$. This can be rewritten as

$$\mathbb{E}(\phi_k(z)|\mathcal{F}_{k-1}) = \mathbb{E}(\phi_{k-1}(e^{i\theta} f_{c_k^*}(e^{-i\theta} z))|\mathcal{F}_{k-1}).$$

The only randomness here comes from $\theta_k$, the $c_k^*$ are pre-determined, and so,

$$\mathbb{E}(\phi_k(z)|\mathcal{F}_{k-1}) = \int_0^{2\pi} \phi_{k-1}(e^{i\theta} f_{c_k^*}(e^{-i\theta} z))\frac{d\theta}{2\pi} = \phi_{k-1}(e^{c_k^*} z).$$

Therefore,

$$\mathbb{E}(\phi_k(e^{C_{n,k+1}^*} z)|\mathcal{F}_{k-1}) = \phi_{k-1}(e^{C_{n,k}^*} z).$$

Thus,

$$\mathbb{E}(X_{k,n}|\mathcal{F}_{k-1}) = e^{-C_{n,1}^*}\left(\mathbb{E}(\phi_k(e^{C_{n,k+1}^*} z)|\mathcal{F}_{k-1}) - \phi_{k-1}(e^{C_{n,k}^*} z)\right) = 0$$

as required.

□

Hence, we can now apply Theorem 4.1 to our cluster.

**Theorem 4.3.** *Let $c_i^*$ and $\phi_k$ be defined as above. Then for $0 < \alpha < 2$, and any positive real number $a \leq \frac{\log(n)}{\sqrt{n}}$,*

$$\mathbb{P}\left(|e^{-C^*_{n,1}}\phi_n(z) - z| > a\right) \leq e^{\frac{-a^2 n}{\lambda(\alpha,c,r)}}$$

*for some strictly positive constant $\lambda(\alpha, c, r)$. Therefore, for all $0 < \alpha < 2$ and $\frac{1}{\sqrt{n}} \ll a \leq \frac{\log(n)}{\sqrt{n}}$,*

$$\lim_{n\to\infty} \mathbb{P}\left(|e^{-C^*_{n,1}}\phi_n(z) - z| > a\right) = 0$$

*Proof.* First note, we have shown above $\mathbb{E}(X_{k,n}|\mathcal{F}_{k-1}) = 0$ where

$$X_{k,n}(z) = e^{-C^*_{n,1}}\left(\phi_k\left(e^{C^*_{n,k+1}}z\right) - \phi_{k-1}\left(e^{C^*_{n,k}}z\right)\right)$$

Now, let $M_n(z) = \sum_{k=1}^n X_{k,n}(z)$ and note that we can split $M_n$ into real and imaginary parts, thus,

$$\mathbb{P}\left(\big|M_n\big| > a\right) \leq \mathbb{P}\left(\Re(M_n) > a\right) + \mathbb{P}\left(\Im(M_n) > a\right)$$

Moreover,

$$\sup_{k\leq n}\Re(X_{k,n}(z)) < \sup_{k\leq n}|X_{k,n}(z)|$$

$$\sup_{k\leq n}\Im(X_{k,n}(z)) < \sup_{k\leq n}|X_{k,n}(z)|$$

It is easy to see that both $\Re(X_{k,n}(z))$ $\Im(X_{k,n}(z))$ both satisfy the same property that the expectation with respect to the filtration is zero and so by Theorem 4.1, for any positive real number $a$,

$$\begin{aligned}\mathbb{P}\left(\big|\sum_{k=1}^n X_{k,n}(z)\big| \geq a\right) &\leq \mathbb{P}\left(\Re\left(\sum_{k=1}^n X_{k,n}(z)\right) \geq a\right) + \mathbb{P}\left(\Im\left(\sum_{k=1}^n X_{k,n}(z)\right) \geq a\right)\\ &\leq 2\exp\left[\frac{-a^2}{2(b_X(k,n)a + b_T(n))}\right]\end{aligned}$$

where $b_X(k,n)$, $b_T(n)$ are the bounds on $|X_{k,n}(z)|$ and $T_n(z)$ respectively. We first deal with the case that $0 < \alpha \leq 1$. In Section 3 we have seen

$$\sup_{k\leq n}|X_{k,n}(z)| < \lambda^1(\alpha,c,r)\frac{1}{n}$$

for some positive constant $\lambda^1(\alpha, c, r)$ and

$$T_n(z) \leq \lambda^2(\alpha,c,r)\frac{1}{n}$$

for some positive constant $\lambda^2(\alpha, c, r)$. Therefore,

$$\mathbb{P}\left(|e^{-\sum_{i=1}^n c_i^*}\phi_n(z) - z| > a\right) \leq 2e^{\frac{-a^2 n}{2(\lambda^1(\alpha,c,r)a + \lambda^2(\alpha,c,r))}}.$$

But for $n$ sufficiently large, $\lambda^1(\alpha, c, r)a \leq \lambda^2(\alpha, c, r)$ so let $\lambda(\alpha, c, r) = \frac{4\lambda^2(\alpha,c,r)}{\log(2)}$ then

$$\mathbb{P}\left(|e^{-\sum_{i=1}^n c_i^*}\phi_n(z) - z| > a\right) \leq e^{\frac{-a^2 n}{\lambda(\alpha,c,r)}}.$$

Now for $1 < \alpha < 2$,

$$\sup_{k \leq n} |X_{k,n}(z)| < \lambda(\alpha, c, r)^1 \frac{1}{n^{\frac{1}{\alpha}}}$$

for some positive constant $\lambda^1(\alpha, c, r)$ and

$$T_n(z) \leq \lambda^2(\alpha, c, r)\frac{1}{n}$$

for some positive constant $\lambda^2(\alpha, c, r)$. Therefore,

$$\mathbb{P}\left(|e^{-\sum_{i=1}^n c_i^*}\phi_n(z) - z| > a\right) \leq 2e^{\frac{-a^2 n^{\frac{1}{\alpha}}}{2(\lambda^1(\alpha,c,r)a + \lambda^2(\alpha,c,r)n^{\frac{1-\alpha}{\alpha}})}}.$$

But for $a \leq \frac{\log(n)}{\sqrt{n}}$, and $n$ sufficiently large, $\lambda^1(\alpha, c, r)a \leq \lambda^2(\alpha, c, r)n^{\frac{1-\alpha}{\alpha}}$. Therefore, using the same $\lambda(\alpha, c, r)$ as above,

$$\mathbb{P}\left(|e^{-\sum_{i=1}^n c_i^*}\phi_n(z) - z| > a\right) \leq e^{\frac{-a^2 n}{\lambda(\alpha,r,c)}}.$$

So for all $0 < \alpha < 2$,

$$\mathbb{P}\left(|e^{-\sum_{i=1}^n c_i^*}\phi_n(z) - z| > a\right) \leq e^{\frac{-a^2 n}{\lambda(\alpha,r,c)}}.$$

Therefore for $\frac{1}{\sqrt{n}} \ll a \leq \frac{\log(n)}{\sqrt{n}}$,

$$\lim_{n\to\infty} \mathbb{P}\left(|e^{-C_{n,1}^*}\phi_n(z) - z| > a\right) = 0.$$

□

## 5 Uniform convergence in the exterior disk

So far we have seen that when evaluated at a fixed point our map looks like a disk. Our aim now is to show that if we map from a disc of fixed radius then all points on the exterior disc will satisfy the same property. Our aim of this section will be to prove the following theorem.

**Theorem 1.2.** *For $0 < \alpha < 2$, let the map $\phi_n$ be defined as above with $c_n^*$ as defined in (2) and $\theta_n$ i.i.d. Then for any $r > 1$ we have the following inequality*

$$\mathbb{P}\left(\sup_{|z|\geq r} |e^{-\sum_{i=1}^n c_i^*}\phi_n(z) - z| > \frac{\log(n)}{\sqrt{n}}\right) < \lambda^1(\alpha, c, r)e^{-\frac{\log(n)^2}{\lambda^2(\alpha,c,r)}}$$

*where $\lambda^1(\alpha, c, r),\ \lambda^2(\alpha, c, r) > 0$ are constants. Hence, by Borel Cantelli,*

$$\mathbb{P}\left(\limsup_{n\to\infty}\left\{\sup_{|z|\geq r} |e^{-\sum_{i=1}^n c_i^*}\phi_n(z) - z| > \frac{\log n}{\sqrt{n}}\right\}\right) = 0.$$

The proof of the theorem will be constructed as follows. We will show that for a finite number of equally spaced points along the circle $|z| = r$ the inequality holds. Then we will show that between these points the probability that the difference between the maps when evaluated at these points is sufficiently small. First define

$$M_n(z, w) := M_n(z) - M_n(w)$$

with $M_n(z)$ defined as above. Then we must choose the spacing between the finite set of points. With the choice of $\alpha$ and $c$ fixed we choose points, on a radius $|z| = r$, to be equally spaced at angles $\frac{2\pi}{L_{r,n}}$ where

$$L_{r,n} = \gamma(\alpha, c, r)n^{\frac{3}{2}}$$

and $\gamma(\alpha, c, r)$ is a constant,

$$\gamma(\alpha, c, r) = 4\pi r\frac{1}{c}(e^c + 1)(1 + \alpha c)(1 + \alpha e^{\alpha c})\left(\log\left(\frac{r}{r-1}\right) + 1\right)(\log(1 + \alpha c) + 1)\,. \quad (3)$$

The reason for this choice of spacing will become clear in the proof of the lemmas that follow. We start by proving that we can find a finite number of equally spaced points, with the above spacing along the circle $|z| = r$, such that the inequality in Theorem 1.2 holds.

**Lemma 5.1.** *Let $\{z_i\}_{i=1}^{L_{r,n}}$ be defined as finite set of points on the boundary of the unit circle of radius $|z| = r$ with equally spaced at angles $\frac{2\pi}{L_{r,n}}$ and $L_{r,n}$ defined as above. Then, for sufficiently large $n$, we have the following inequality*

$$\mathbb{P}\left(\exists i : |M_n(z_i)| > \frac{1}{2}\sqrt{\frac{(\log(1 + \alpha cn))^2}{(1 + \alpha cn)}}\right) < \lambda^1(\alpha, c, r)e^{\frac{-(\log(1+\alpha cn))^2}{\lambda^2(\alpha,c,r)}}$$

*where $\lambda^1(\alpha, c, r), \lambda^2(\alpha, c, r) > 0$ are constants.*

*Proof.* We have shown using Theorem 4.3 that for $0 < \alpha < 2$ and $a \leq \frac{\log n}{\sqrt{n}}$ ,

$$\mathbb{P}\left(\exists i : |M_n(z_i)| > \frac{a}{2}\right) \leq 2e^{\frac{-a^2 n}{\lambda(\alpha,c,r)}}$$

for some constant $\lambda(\alpha, c, r) > 0$. Therefore,

$$\mathbb{P}\left(\exists i : |M_n(z_i)| > \frac{a}{2}\right) < 2\sum_{k=1}^{L_{r,n}} e^{\frac{-a^2 n}{\lambda(\alpha,c,r)}}.$$

So let $a^2 = \frac{\log(n)^2}{n}$. Then,

$$\mathbb{P}\left(\exists i : |M_n(z_i)| > \frac{\log n}{2\sqrt{n}}\right) \leq 2\sum_{k=1}^{L_{r,n}} e^{\frac{-\log(1+\alpha cn)^2}{\lambda(\alpha,c,r)}}.$$

The terms in the sum have no dependence on $k$ and as such we can find an upper bound,

$$\begin{aligned}\mathbb{P}\left(\exists i : |M_n(z_i)| > \frac{\log n}{2\sqrt{n}}\right) &\leq 2L_{r,n} e^{\frac{-\log(n)^2}{\lambda(\alpha,c,r)}} \\ &= \gamma(\alpha, c, r) n^{\frac{3}{2}} e^{\frac{-\log(n)^2}{\lambda(\alpha,c,r)}}\end{aligned}$$

where $\gamma(\alpha, c, r) > 0$ is the constant defined in equation (3). Let $\lambda^1(\alpha, c, r) = \gamma(\alpha, c, r)$, then

$$\mathbb{P}\left(\exists i : |M_n(z_i)| > \frac{\log n}{2\sqrt{n}}\right) \leq \lambda^1(\alpha, c, r) e^{\frac{3}{2}\log n - \frac{\log(n)^2}{\lambda(\alpha,c,r)}}.$$

For sufficiently large $n > e^{3\lambda(\alpha,c,r)}$,

$$\frac{\frac{3}{2}\log n}{\frac{\log(n)^2}{\lambda(\alpha,c,r)}} \leq \frac{1}{2}.$$

Therefore, let $\lambda^2(\alpha, c, r) = 2\lambda(\alpha, c, r)$ and then for $n$ sufficiently large,

$$\mathbb{P}\left(\exists i : |M_n(z_i)| > \frac{\log n}{2\sqrt{n}}\right) \leq \lambda^1(\alpha, c, r) e^{\frac{-(\log(n))^2}{\lambda^2(\alpha,c,r)}}$$

with $\lambda^1(\alpha, c, r), \lambda^2(\alpha, c, r) > 0$. □

We now prove that for points $w \in \mathbb{C}$ inbetween the points in the set $\{z_i\}_{i=1}^{L_{r,n}}$ the difference $M_n(z_i, w)$ is negligible.

**Lemma 5.2.** *For* $|z| = |w| = r$ *with* $\arg(z) = \theta_z$, $\arg(w) = \theta_w$ *and* $|\theta_z - \theta_w| < \frac{2\pi}{L_{r,n}}$ *and* $L_{r,n}$ *defined as above we have the following bound;*

$$|M_n(z, w)| \leq \frac{\log(n)}{\sqrt{n}}$$

*and hence,*

$$\mathbb{P}\left(\exists w, z \in \mathbb{C} : |\theta_z - \theta_w| < \frac{2\pi}{L_{r,n}},\ M_n(z, w) > \frac{\log(n)}{2\sqrt{n}}\right) = 0.$$

*Proof.* We want to find a bound on $|M_n(z,w)|$ so we first find a bound on $|X_{k,n}(z,w)| = |X_{k,n}(z) - X_{k,n}(w)|$.

$$\begin{aligned}
&|X_{k,n}(z,w)| \\
&= e^{-\sum_{k=1}^{n} c_i^*} \left| \left( \phi_k \left( e^{C^*_{n,k+1}} z \right) - \phi_{k-1} \left( e^{C^*_{n,k}} z \right) \right) - \left( \phi_k \left( e^{C^*_{n,k+1}} w \right) - \phi_{k-1} \left( e^{C^*_{n,k}} w \right) \right) \right| .
\end{aligned}$$

Let $0 \leq s, t \leq 1$ and then

$$\begin{aligned}
\tau_{k,n}(s) =& e^{C^*_{n,k+1}} |z| e^{i(\theta_z s + \theta_w (1-s))} \\
\rho_{k,n}(t) =& e^{C^*_{n,k}} |z| e^{i(\theta_z t + \theta_w (1-t))} .
\end{aligned}$$

Thus,

$$|X_{k,n}(z,w) \leq |\phi_k(\tau_{k,n}(1)) - \phi_k(\tau_{k,n}(0))| + |\phi_{k-1}(\rho_{k,n}(1)) - \phi_{k-1}(\rho_{k,n}(0))|.$$

If we consider the $\tau$ terms in the upper bound, we have

$$|\phi_k(\tau_{k,n}(1)) - \phi_k(\tau_{k,n}(0))| \leq \left| \int_0^1 \phi_k^{'}(\tau_{k,n}(s)) ds \right| |\dot{\tau_{k,n}}(s)|.$$

Using the distortion theorem [9],

$$|\phi_k(\tau_{k,n}(1)) - \phi_k(\tau_{k,n}(0))| \leq e^{C^*_{k,1}} \sup_{0 \leq s \leq 1} \frac{|\tau_{k,n}(s)|}{|\tau_{k,n}(s)| - 1} e^{C^*_{n,k+1}} |\theta_z - \theta_w| |z|.$$

Therefore,

$$|\phi_k(\tau_{k,n}(1)) - \phi_k(\tau_{k,n}(0))| \leq e^{C^*_{n,1}} |z|^2 |\theta_z - \theta_w| \frac{e^{C^*_{n,k+1}}}{e^{C^*_{n,k+1}} |z| - 1}.$$

By a similar argument

$$|\phi_{k-1}(\rho_{k,n}(1)) - \phi_{k-1}(\rho_{k,n}(0))| \leq e^{C^*_{n,1}} |z|^2 |\theta_z - \theta_w| \frac{e^c e^{C^*_{n,k+1}}}{e^{C^*_{n,k+1}} |z| - 1}.$$

Therefore using the fact $|z| = r$,

$$|X_{k,n}(z,w)| \leq 2r^2 (e^c + 1) |\theta_z - \theta_w| \frac{e^{C^*_{n,k+1}}}{e^{C^*_{n,k+1}} r - 1}.$$

We can therefore use the approximation $e^{C^*_{n,k}} \approx \left( \frac{1+\alpha c n}{1 + \alpha c (k-1)} \right)^{\frac{1}{\alpha}}$ and take the sum to write

$$|M_n(z,w)| \leq 2r^2 (e^c + 1) |\theta_z - \theta_w| \left| \sum_{k=1}^{n} \left( \frac{\left( \frac{1+\alpha c n}{1+\alpha c k} \right)^{\frac{1+\epsilon_{n,k}}{\alpha}}}{r \left( \frac{1+\alpha c n}{1+\alpha c k} \right)^{\frac{1+\epsilon_{n,k}}{\alpha}} - 1} \right) \right|$$

where $\epsilon_{n,k}$ is the same error term from Section 2. We can use the bounds from Lemma 3.2 to remove the $\epsilon_{n,k}$ term,

$$\left(\frac{1+\alpha cn}{1+\alpha ck}\right)^{\frac{1}{\alpha}} < \left(\frac{1+\alpha cn}{1+\alpha ck}\right)^{\frac{1+\epsilon_{n,k}}{\alpha}} \leq (1+\alpha ce^{\alpha c})\left(\frac{1+\alpha cn}{1+\alpha ck}\right)^{\frac{1}{\alpha}}$$

Then $x = \left(\frac{1+\alpha cn}{1+\alpha ck}\right)^{\frac{1}{\alpha}}$ and integrating between $x = \left(\frac{1+\alpha cn}{1+\alpha c}\right)^{\frac{1}{\alpha}}$ and $x = 1$ gives

$$\begin{aligned}\left|\sum_{k=1}^{n}\left(\frac{\left(\frac{1+\alpha cn}{1+\alpha ck}\right)^{\frac{1+\epsilon_{n,k}}{\alpha}}}{r\left(\frac{1+\alpha cn}{1+\alpha ck}\right)^{\frac{1+\epsilon_{n,k}}{\alpha}}-1}\right)\right| &\leq \frac{1}{c}\left|\int_1^{\left(\frac{1+\alpha cn}{1+\alpha c}\right)^{\frac{1}{\alpha}}}\frac{1+\alpha ck}{rx-1}dx\right| \\ &\leq \frac{1}{c}(1+\alpha cn)\left|\int_1^{\left(\frac{1+\alpha cn}{1+\alpha c}\right)^{\frac{1}{\alpha}}}\frac{1}{rx-1}dx\right|.\end{aligned}$$

Thus,

$$\begin{aligned}\left|\sum_{k=1}^{n}\left(\frac{\left(\frac{1+\alpha cn}{1+\alpha ck}\right)^{\frac{1+\epsilon_{n,k}}{\alpha}}}{r\left(\frac{1+\alpha cn}{1+\alpha ck}\right)^{\frac{1+\epsilon_{n,k}}{\alpha}}-1}\right)\right| &\leq \frac{1}{cr}(1+\alpha cn)\left|\log\left(\frac{r-1}{r\left(\frac{1+\alpha cn}{1+\alpha c}\right)^{\frac{1}{\alpha}}-1}\right)\right| \\ &\leq \frac{1}{cr}(1+\alpha cn)\log\left(\frac{r\left(1+\alpha cn\right)^{\frac{1}{\alpha}}}{r-1}\right).\end{aligned}$$

Therefore,

$$|M_n(z,w)| \leq \frac{\gamma(\alpha,c,r)}{2\pi}|\theta_z-\theta_w| n\log n$$

where $\gamma(\alpha,c,r)$ is the constant defined in equation (3). Then we use the fact that $|\theta_z - \theta_w| = \frac{2\pi}{L_{r,n}}$ and write

$$|M_n(z,w)| \leq \frac{\log n}{\sqrt{n}}.$$

So,

$$\mathbb{P}\left(\exists w,z\in\mathbb{C} : |\theta_z-\theta_w| < \frac{2\pi}{L_{r,n}},\ M_n(z,w) > \frac{\log n}{2\sqrt{n}}\right) = 0.$$

□

So we can combine these two lemmas to give our proof of Theorem 1.2.

*Proof of Theorem 1.2.* As in the previous two lemmas we seperate the circle into points $\frac{2\pi}{L_{r,n}}$ apart. We can then form the following bound;

$$\begin{aligned}
&\mathbb{P}\left(\sup_{|z|=r}|e^{-C^*_{n,1}}\phi_n(z)-z|>\frac{\log n}{\sqrt{n}}\right)\\
&\leq \mathbb{P}\left(\exists i:|M_n(z_i)|>\frac{1}{2}\frac{\log n}{\sqrt{n}}\right)\\
&+\mathbb{P}\left(\exists w,z\in\mathbb{C}:|\theta_z-\theta_w|<\frac{2\pi}{L_{r,n}},\ M_n(z,w)>\frac{1}{2}\frac{\log n}{\sqrt{n}}\right).
\end{aligned}$$

Using Lemmas 5.1 and 5.2 we see,

$$\mathbb{P}\left(\sup_{|z|=r}|e^{-C^*_{n,1}}\phi_n(z)-z|>\frac{\log n}{\sqrt{n}}\right)\leq\lambda^1(\alpha,c,r)e^{\frac{-(\log(n))^2}{\lambda^2(\alpha,c,r)}}$$

where $\lambda^1(\alpha,c,r),\lambda^2(\alpha,c,r)>0$ are constants. Then using the maximum modulus principle we see that that the maxiumum occurs on the boundary and so,

$$\mathbb{P}\left(\sup_{|z|\geq r}|e^{-C^*_{n,1}}\phi_n(z)-z|>\frac{\log n}{\sqrt{n}}\right)\leq\lambda^1(\alpha,c,r)e^{\frac{-(\log(n))^2}{\lambda^2(\alpha,c,r)}}.$$

It is clear to see the upper bound is summable and hence by a Borel Cantelli argument,

$$\mathbb{P}\left(\limsup_{n\to\infty}\sup_{|z|\geq r}|e^{-C^*_{n,1}}\phi_n(z)-z|>\frac{\log n}{\sqrt{n}}\right)=0.$$

□

# 6 Fluctuations for $0<\alpha<2$

## 6.1 Discarding the lower order terms

In the previous sections we have seen that, by using the result of Freedman [10], we have convergence to a disk in the exterior disk. Now we would like to see how much we fluctuate from this disc. To do so we aim to produce a central limit theorem that will tell us what the distribution of the fluctuations is. Up until this point we have used

$$X_{k,n}(z)=e^{-C^*_{n,1}}\left(\phi_k\left(e^{-C^*_{n,k+1}}z\right)-\phi_{k-1}\left(e^{-C^*_{n,k}}z\right)\right).$$

We aim to prove that the fluctuations are of order $\sqrt{n}$. Furthermore, we want to show we can discard the lower order terms of the increments $X_{k,n}(z)$ in order to simplify the calculation of the fluctuations. Therefore, we introduce the rescaled increment,

$$\mathcal{X}_{k,n}(z)=\frac{2c^*_k\sqrt{n}z}{e^{-i\theta_k}e^{C^*_{n,k+1}}z-1}.$$

The following lemma shows that we can discard the lower order terms.

**Lemma 6.1.** *Let $Y_{k,n}(z) = \sqrt{n}X_{k,n}(z) - \mathcal{X}_{k,n}(z)$. Then if $0 < \alpha < 2$, for any $\epsilon > 0$ and $r > 1$,*

$$\mathbb{P}\left(\lim_{n\to\infty}\sup_{|z|>r}\left|\sum_{k=1}^{n} Y_{k,n}(z)\right| > \epsilon\right) = 0$$

*Proof.* Fix some $r > 1$. Then in Theorem 5.1 we showed that for $|z| > r$,

$$\mathbb{P}\left(\limsup_{n\to\infty}\left\{\sup_{|z|\geq r}|e^{-C^*_{n,1}}\phi_n(z) - z| > \frac{\log n}{\sqrt{n}}\right\}\right) = 0.$$

Denote the event,

$$\omega(r) = \left\{\limsup_{n\to\infty}\left\{\sup_{|z|\geq r}|e^{-C^*_{n,1}}\phi_n(z) - z| \leq \frac{\log n}{\sqrt{n}}\right\}\right\}.$$

Now choose $r' = \frac{r+1}{2}$, then,

$$\begin{aligned}\mathbb{P}\left(\lim_{n\to\infty}\sup_{|z|>r}\left|\sum_{k=1}^{n} Y_{k,n}(z)\right| < \epsilon\right) &= \mathbb{P}\left(\lim_{n\to\infty}\sup_{|z|>r}\left|\sum_{k=1}^{n} Y_{k,n}(z)\right| < \epsilon \,\middle|\, \omega(r')\right)\mathbb{P}\left(\omega(r')\right) \\ &+ \mathbb{P}\left(\lim_{n\to\infty}\sup_{|z|>r}\left|\sum_{k=1}^{n} Y_{k,n}(z)\right| < \epsilon \,\middle|\, \omega(r')^c\right)\mathbb{P}\left(\omega(r')^c\right).\end{aligned}$$

We have shown that $\mathbb{P}\left(\omega(r')\right) = 1$. Therefore,

$$\mathbb{P}\left(\lim_{n\to\infty}\sup_{|z|>r}\left|\sum_{k=1}^{n} Y_{k,n}(z)\right| < \epsilon\right) = \mathbb{P}\left(\lim_{n\to\infty}\sup_{|z|>r}\left|\sum_{k=1}^{n} Y_{k,n}(z)\right| < \epsilon \,\middle|\, \omega(r')\right).$$

We first calculate a bound on $|Y_{k,n}(z)|$. Let

$$\widetilde{X}_{k,n}(z) = \sqrt{n}e^{-C^*_{n,k}}\int_0^1 \dot{\eta}_{k,n}(s,z)ds$$

where $\eta_{k,n}(s,z)$ is defined as in Section 3. Then,

$$\sqrt{n}X_{k,n}(z) - \widetilde{X}_{k,n}(z) = \sqrt{n}e^{-C^*_{n,1}}\left(\int_0^1 \dot{\eta}_{k,n}(s,z)\left(\phi_{k-1}(\eta_{k,n}(s,z)) - e^{C^*_{k-1,1}}\right)ds\right).$$

But for $|z| > r'$ on the event $\omega(r')$,

$$|e^{-C^*_{k-1,1}}\phi_n(z) - z| < \frac{\log(k-1)}{\sqrt{k-1}}.$$

Then let $g(z) = e^{-C^*_{k-1,1}}\phi_n(z) - z$. The map $g$ is holomorphic on the closed disc $|\zeta - z| < R := |z| - r'$. So by Cauchy's theorem, for $0 < \alpha < 2$,

$$g'(z) = \frac{1}{2\pi i}\int_{C_R}\frac{g(\zeta)}{(\zeta - z)^2}d\zeta$$

where $C_R$ is the circle of radius $R$ centred at $z$. Therefore,

$$|g'(z)| \le \frac{1}{(|z|-r')}\frac{\log(k-1)}{\sqrt{k-1}}.$$

So on $\omega(r')$,

$$|X_{k,n}(z)-\widetilde{X}_{k,n}(z)| \le \sqrt{n}e^{-C^*_{n,1}}\left(\int_0^1 \dot{\eta}_{k,n}(s,z)\left(\frac{1}{(|\eta_{k,n}(s,z)|-r')}\frac{e^{C^*_{k-1,1}}\log(k-1)}{\sqrt{k-1}}\right)ds\right).$$

Then since, $\inf_{0\le k\le n}|\eta_{k,n}(s,z)| \ge |z|$,

$$\begin{aligned}|X_{k,n}(z)-\widetilde{X}_{k,n}(z)| &\le \sqrt{n}\frac{1}{r-r'}e^{-C^*_{n,k}}\frac{\log(k-1)}{\sqrt{k-1}}\int_0^1|\dot{\eta}_{k,n}(s,z)|ds\\ &\le \lambda(\alpha,c,r)\sqrt{n}e^{-C^*_{n,k}}\frac{\log(k-1)}{\sqrt{k-1}}\frac{c^*_k e^{C^*_{n,k}}}{e^{C^*_{n,k+1}}r-1}\\ &\le \lambda(\alpha,c,r)\frac{\sqrt{n}}{n^{\frac{1}{\alpha}}}\frac{\log(k)k^{\frac{1}{\alpha}}}{k^{\frac{3}{2}}}.\end{aligned}$$

Where the second inequality follows from Lemma 3.5. Now consider,

$$\begin{aligned}&|\widetilde{X}_{k,n}(z)-\mathcal{X}_{k,n}(z)|\\ &\le \sqrt{n}\left|\left(\frac{2c^*_k}{e^{-i\theta_k}e^{C^*_{n,k+1}}z-1}\right)\left(e^{-C^*_{n,k}}\int_0^1\eta_{k,n}(s,z)ds-z\right)\right|\\ &+\sqrt{n}\left|\left(e^{-C^*_{n,k}}\int_0^1\eta_{k,n}(s,z)ds\right)\delta_{c^*_k}\left(e^{-i\theta_k}e^{C^*_{n,k+1}}z\right)\right|\\ &\le \sqrt{n}\left(\left(\frac{2c^*_k}{e^{C^*_{n,k+1}}r-1}\right)\left(r\int_0^1\left|e^{x_{k,n}(s)}-1\right|ds\right)+\lambda(\alpha,c,r)\left|\delta_{c^*_k}\left(e^{-i\theta_k}e^{C^*_{n,k+1}}z\right)\right|\right)\end{aligned}$$

where $\lambda(\alpha,c,r)$ is some positive constant that we will vary and

$$x_{k,n}(s) = s\left(\frac{2c^*_k}{e^{-i\theta_k}e^{C^*_{n,k+1}}z-1}+\delta_{c^*_k}\left(e^{-i\theta_k}e^{C^*_{n,k+1}}z\right)\right).$$

Furthermore,

$$|e^{x_{k,n}(s)}-1| \le \lambda(\alpha,c,r)|x_{k,n}(s)| \le \lambda(\alpha,c,r)k^{\frac{1}{\alpha}-1}n^{\frac{-1}{\alpha}}$$

where the second inequality follows from Lemmas 3.1, 3.2 and 3.4. Hence by using the bound on $\delta_c$ from Lemma 3.4 we see that,

$$|\widetilde{X}_{k,n}(z)-\mathcal{X}_{k,n}(z)| \le \lambda(\alpha,c,r')\sqrt{n}\left(\left(k^{\frac{1}{\alpha}-1}n^{\frac{-1}{\alpha}}\right)^2+k^{\frac{1}{\alpha}-\frac{3}{2}}n^{\frac{-1}{\alpha}}\right).$$

Since $k^{\frac{1}{\alpha}} \le n^{\frac{1}{\alpha}}$ we have

$$|\widetilde{X}_{k,n}(z)-\mathcal{X}_{k,n}(z)| \le \lambda(\alpha,c,r)k^{\frac{1}{\alpha}-\frac{3}{2}}n^{\frac{1}{2}-\frac{1}{\alpha}}.$$

Therefore,

$$|Y_{k,n}(z)| \leq \lambda(\alpha, c, r) \log(n) n^{\frac{1}{2}-\frac{1}{\alpha}} k^{\frac{1}{\alpha}-\frac{3}{2}}.$$

Then we split into cases, if $0 < \alpha < \frac{2}{3}$,

$$\sup_{k \leq n} |Y_{k,n}(z)| \leq \lambda(\alpha, c, r) \frac{\log(n)}{n} \to 0$$

as $n \to \infty$. However, if $\frac{2}{3} < \alpha < 2$ then

$$\sup_{k \leq n} |Y_{k,n}(z)| \leq \lambda(\alpha, c, r) \log(n) n^{\frac{1}{2}-\frac{1}{\alpha}} \to 0$$

as $n \to \infty$. Moreover,

$$\mathbb{E}(Y_{k,n}(z)|^2 \; |\mathcal{F}_{k-1}) \leq \lambda(\alpha, c, r) \frac{n}{n^{\frac{2}{\alpha}}} \frac{\log(n)^2 k^{\frac{2}{\alpha}}}{k^3}.$$

thus if $0 < \alpha < 1$,

$$\sum_{k=1}^{n} \mathbb{E}(|Y_{k,n}(z)|^2 \; |\mathcal{F}_{k-1}) \leq \lambda(\alpha, c, r) \frac{\log(n)^3}{n} \to 0$$

as $n \to \infty$. If $1 < \alpha < 2$,

$$\sum_{k=1}^{n} \mathbb{E}(|Y_{k,n}(z)|^2 \; |\mathcal{F}_{k-1}) \leq \lambda(\alpha, c, r) \frac{\log(n)^2 n}{n^{\frac{2}{\alpha}}} \to 0$$

as $n \to \infty$. Therefore, since $Y_{k,n}(z)$ is also a martingale difference array we can use these bounds to apply the same methods to the difference $Y_{k,n}(z)$ as we did to $X_{k,n}(z)$ in Sections 4 and 5 along with a Borel Cantelli argument to show that

$$\mathbb{P}\left( \lim_{n \to \infty} \sup_{|z|>r} \sum_{k=1}^{n} |Y_{k,n}(z)| > \epsilon \right) = 0.$$

□

## 6.2 Laurent Coefficients

In the previous section we showed that we could discard the lower order terms of $X_{k,n}(z)$. We now wish to calculate the Laurent coefficients of the remaining higher order terms $\mathcal{X}_{k,n}(z)$ and hence evaluate the fluctuations of the cluster. We first notice that

$$\mathbb{E}(\mathcal{X}_{k,n}(z)|\mathcal{F}_{k-1}) = 0$$

and therefore $\mathcal{X}_{k,n}(z)$ is also a martingale difference array. We aim to use the following result of Mcleish [11] to produce a central limit theorem.

**Theorem 6.2** (McLeish). *Let $(X_{k,n})_{1\leq k\leq n}$ be a martingale difference array with respect to the filtration $\mathcal{F}_{k,n} = \sigma(X_{1,N}, X_{2,N}, ..., X_{k,n})$. Let $M_n = \sum_{i=1}^{n} X_{i,n}$ and assume that;*

(1) *for all $\rho > 0$, $\sum_{k=1}^{n} X_{k,n}^2 \ \mathbb{1}(|X_{k,n}| > \rho) \to 0$ in probability as $n \to \infty$.*

*(2) $\sum_{k=1}^{n} X_{k,n}^2 \to s^2$ in probability as $n \to \infty$ for some $s^2 > 0$.*

*Then $M_n$ converges in distribution to $\mathcal{N}(0, s^2)$.*

Note that condition (1) in Theorem 6.2 combines two conditions in [11] as a result of the Lindberg condition [12]. Theorem 6.2 only applies to real valued random variables and as such we will split $\mathcal{X}_{k,n}(z)$ into real and imaginary parts. We start by calculating the Laurent Coefficients so that we can apply the theorem to these coefficients.

$$\mathcal{X}_{k,n}(z) = \frac{2c_k^*\sqrt{n}}{e^{-i\theta_k} e^{C_{n,k+1}^*}} \left( \frac{1}{1 - \frac{1}{e^{-i\theta_k} e^{C_{n,k+1}^*} z}} \right).$$

We can choose $|z| > r$ such that $\left| \frac{1}{e^{-i\theta_k} e^{C_{n,k+1}^*} z} \right| < 1$, then

$$\mathcal{X}_{k,n}(z) = \sum_{m=0}^{\infty} \frac{2c_k^*\sqrt{n}}{(e^{-i\theta_k} e^{C_{n,k+1}^*})^{m+1}} \frac{1}{z^m}.$$

So the $m^{\text{th}}$ coefficient is dependent on $n$ and $k$ and we can rewrite $\mathcal{X}_{k,n}$ as

$$\mathcal{X}_{k,n}(z) = \sum_{m=0}^{\infty} a_{n,k}(m) \frac{1}{z^m}$$

where $a_{n,k}(m) = \frac{2c_k^*\sqrt{n}}{(e^{C_{n,k+1}^*})^{m+1}} e^{i\theta_k(m+1)}$. So we can calculate real and imaginary parts of these coefficients,

$$\Re(a_{n,k}(m)) = \frac{2c_k^*\sqrt{n}}{(e^{C_{n,k+1}^*})^{m+1}} \cos(\theta_k(m+1)),$$

$$\Im(a_{n,k}(m)) = \frac{2c_k^*\sqrt{n}}{(e^{C_{n,k+1}^*})^{m+1}} \sin(\theta_k(m+1)).$$

In order to use McLeish we need to calculate the second moments of the coefficients. We will just consider the case of the real coefficients here but the imaginary coefficients give the same results. Thus, we calculate,

$$\begin{aligned}\mathbb{E}((\Re(a_{n,k}(m)))^2|\mathcal{F}_{k-1}) &= \frac{4(c_k^*)^2 n}{(e^{C_{n,k+1}^*})^{2(m+1)}} \frac{1}{2\pi} \int_0^{2\pi} \cos^2(\theta(m+1))d\theta \\ &= \frac{2(c_k^*)^2 n}{(e^{C_{n,k+1}^*})^{2(m+1)}}.\end{aligned}$$

It is clear to see here why we have the same expected value of the imaginary coefficients. So now we can take the sum over $n$,

$$\lim_{n\to\infty}\sum_{k=1}^{n}\mathbb{E}((\Re(a_{n,k}(m)))^2|\mathcal{F}_{k-1}) = \lim_{n\to\infty}\left(2n\sum_{k=1}^{n}\frac{(c_k^*)^2}{\left(e^{C^*_{n,k+1}}\right)^{2(m+1)}}\right)$$

Recall that $c_k^* = \frac{c}{1+\alpha c(k-1)}$ and we have shown we can approximate the term in the denominator in the following way;

$$e^{C^*_{n,k+1}} = \left(\frac{1+\alpha cn}{1+\alpha ck}\right)^{\frac{1+\epsilon_{n,k+1}}{\alpha}}$$

where $\epsilon_{n,k+1}$ is the error we found a bound on in Lemma 3.1. Therefore, we can write

$$\lim_{n\to\infty}\sum_{k=1}^{n}\mathbb{E}((\Re(a_{n,k}(m)))^2|\mathcal{F}_{k-1}) = \lim_{n\to\infty}\left(2nc^2\sum_{k=1}^{n}\frac{(1+\alpha ck)^{\left(\frac{(1+\epsilon_{n,k+1})(2(m+1))}{\alpha}\right)-2}}{(1+\alpha cn)^{\left(\frac{(1+\epsilon_{n,k+1})(2(m+1))}{\alpha}\right)}}\right)$$

We know $\epsilon_{n,k+1}\to 0$ so our aim is to show that this term in the sum is insignificant. We define the function $h:\mathbb{R}\to\mathbb{R}$ as the term inside the sum;

$$h(x) := \frac{(1+\alpha ck)^{\left(\frac{(1+x)(2(m+1))}{\alpha}\right)-2}}{(1+\alpha cn)^{\left(\frac{(1+x)(2(m+1))}{\alpha}\right)}}.$$

Our aim is to show,

$$\left|\lim_{n\to\infty}2nc^2\sum_{k=1}^{n}(h(\epsilon_{n,k+1})-h(0))\right| = 0.$$

If we can show this then we can just ignore the $\epsilon_{n,k}$ and find the limit,

$$\lim_{n\to\infty}2nc^2\sum_{k=1}^{n}h(0)$$

which we will show converges to a real number. We provide this in the form of the following lemma.

**Lemma 6.3.** *With $h:\mathbb{R}\to\mathbb{R}$ defined as above we have*

$$\left|\lim_{n\to\infty}2nc^2\sum_{k=1}^{n}(h(\epsilon_{n,k+1})-h(0))\right| = 0$$

*Proof.* Consider

$$|h(\epsilon_{n,k+1})-h(0)| = \left|\frac{(1+\alpha ck)^{\left(\frac{(1+\epsilon_{n,k+1})(2(m+1))}{\alpha}\right)-2}}{(1+\alpha cn)^{\left(\frac{(1+\epsilon_{n,k+1})(2(m+1))}{\alpha}\right)}} - \frac{(1+\alpha ck)^{\left(\frac{(2(m+1))}{\alpha}\right)-2}}{(1+\alpha cn)^{\left(\frac{(2(m+1))}{\alpha}\right)}}\right|.$$

Then let $y_{n,k} = \left(\frac{1+\alpha ck}{1+\alpha cn}\right)^{\frac{2m+2}{\alpha}}$, thus we can write

$$|h(\epsilon_{n,k+1}) - h(0)| = \frac{1}{(1+\alpha ck)^2}|y_{n,k}|\left|y_{n,k}^{\epsilon_{n,k+1}} - 1\right|.$$

Furthermore, since $\log(y_{n,k}) < 1$,

$$\left|y_{n,k}^{\epsilon_{n,k+1}} - 1\right| = \left|e^{\epsilon_{n,k+1}\log y_{n,k}} - 1\right| \leq |\epsilon_{n,k+1}||\log y_{n,k}|.$$

So using the bound on $\epsilon_{n,k}$ from Section 3 we have

$$\begin{aligned}|h(\epsilon_{n,k+1}) - h(0)| &\leq \frac{1}{(1+\alpha ck)^2}\left(\frac{1+\alpha ck}{1+\alpha cn}\right)^{\frac{2m+2}{\alpha}} \frac{\alpha\left(\alpha c^2(n-k)\right)\left|\log\left(\left(\frac{1+\alpha ck}{1+\alpha cn}\right)^{\frac{2m+2}{\alpha}}\right)\right|}{(1+\alpha ck)(1+\alpha cn)\log\left(\frac{1+\alpha cn}{1+\alpha ck}\right)} \\ &\leq (2m+2)\alpha c^2 n\frac{(1+\alpha ck)^{\frac{2m+2}{\alpha}-3}}{(1+\alpha cn)^{\frac{2m+2}{\alpha}+1}}.\end{aligned}$$

Now we take the sum over $k$,

$$2nc^2\sum_{k=1}^{n}|h(\epsilon_{n,k+1}) - h(0)| \leq 4n^2(m+1)\alpha c^4\frac{1}{(1+\alpha cn)^{\frac{2m+2}{\alpha}+1}}\sum_{k=1}^{n}(1+\alpha ck)^{\frac{2m+2}{\alpha}-3}.$$

Which we can approximate with a Riemann integral;

$$2nc^2\sum_{k=1}^{n}|h(\epsilon_{n,k+1}) - h(0)| \leq 4n^2(m+1)\alpha c^4\frac{1}{(1+\alpha cn)^{\frac{2m+2}{\alpha}+1}}\int_0^n(1+\alpha cx)^{\frac{2m+2}{\alpha}-3}dx.$$

Now we need to consider cases, firstly in the case where we have $\frac{2m+2}{\alpha} - 3 \neq -1$ and so

$$\begin{aligned}&\left|\lim_{n\to\infty} 2nc^2\sum_{k=1}^{n}(h(\epsilon_{n,k+1}) - h(0))\right| \\ &\leq \lim_{n\to\infty} 4n^2(m+1)\alpha c^4\frac{1}{(1+\alpha cn)^{\frac{2m+2}{\alpha}+1}}\left[\frac{1}{\alpha c\left(\frac{2m+2}{\alpha}-2\right)}(1+\alpha cx)^{\frac{2m+2}{\alpha}-2}\right]_0^n \\ &= \lim_{n\to\infty}\left(\frac{2(m+1)\alpha c^3}{m+1-\alpha}\left(\frac{n^2}{(1+\alpha cn)^3} - \frac{n^2}{(1+\alpha cn)^{\frac{2m+2}{\alpha}+1}}\right)\right).\end{aligned}$$

Hence, since $0 < \alpha < 2$,

$$\left|\lim_{n\to\infty} 2nc^2\sum_{k=1}^{n}(h(\epsilon_{n,k+1}) - h(0))\right| = 0$$

Now consider the case where $\frac{2m+2}{\alpha} - 3 = -1$ and so

$$\begin{aligned}
&\left|\lim_{n\to\infty} 2nc^2 \sum_{k=1}^{n} (h(\epsilon_{n,k+1}) - h(0))\right| \\
&\leq \lim_{n\to\infty} 4n^2(m+1)\alpha c^4 \frac{1}{(1+\alpha cn)^{\frac{2m+2}{\alpha}+1}} \left[\frac{1}{\alpha c}\log(1+\alpha cx)\right]_0^n \\
&= \lim_{n\to\infty} 4n^2 c^3 \frac{\log(1+\alpha cn)}{(1+\alpha cn)^3} \\
&= 0.
\end{aligned}$$

Therefore in all cases we have

$$\left|\lim_{n\to\infty} 2nc^2 \sum_{k=1}^{n} (h(\epsilon_{n,k+1}) - h(0))\right| = 0$$

□

Hence by using the above lemma we can ignore the $\epsilon_{n,k+1}$ term in our summation. All that remains to calculate is the limit of the summation without the error term.

**Lemma 6.4.** *Assume $m > 0$ and $0 < \alpha < 2$. Then*

$$\lim_{n\to\infty} \sum_{k=1}^{n} \mathbb{E}((\Re(a_{n,k}(m)))^2 | \mathcal{F}_{k-1}) = \lim_{n\to\infty} \sum_{k=1}^{n} \mathbb{E}((\Im(a_{n,k}(m)))^2 | \mathcal{F}_{k-1}) = \frac{2}{\alpha(2m+2-\alpha)}.$$

*Proof.* We have shown above that, in the case of the real coefficients, calculating $\lim_{n\to\infty} \sum_{k=1}^{n} \mathbb{E}((\Re(a_{n,k}(m)))^2 | \mathcal{F}_{k-1})$ reduces to calculating the expression

$$\lim_{n\to\infty} \left( 2nc^2 \sum_{k=1}^{n} \frac{(1+\alpha ck)^{\left(\frac{(2(m+1))}{\alpha}\right)-2}}{(1+\alpha cn)^{\left(\frac{(2(m+1))}{\alpha}\right)}} \right).$$

The imaginary coefficients follow by the same argument. We can approximate this with a Riemann integral.

$$2nc^2 \sum_{k=1}^{n} \frac{(1+\alpha ck)^{\left(\frac{(2(m+1))}{\alpha}\right)-2}}{(1+\alpha cn)^{\left(\frac{(2(m+1))}{\alpha}\right)}} \approx \frac{2nc^2}{(1+\alpha cn)^{\left(\frac{(2(m+1))}{\alpha}\right)}} \int_0^n (1+\alpha cx)^{\left(\frac{(2(m+1))}{\alpha}\right)-2} dx$$

Since for all $m > 0$ and $0 < \alpha < 2$, $\frac{(2(m+1))}{\alpha} - 2 > -1$, we have,

$$\begin{aligned}
&= \frac{2nc^2}{(1+\alpha cn)^{\left(\frac{(2(m+1))}{\alpha}\right)}} \left[\frac{1}{2c(m+1)-\alpha c}(1+\alpha cx)^{\left(\frac{(2(m+1))}{\alpha}\right)-1}\right]_0^n \\
&= \frac{2c^2}{2c(m+1)-\alpha c} \left[\frac{n}{(1+\alpha cn)} - \frac{n}{(1+\alpha cn)^{\left(\frac{(2(m+1))}{\alpha}\right)}}\right].
\end{aligned}$$

We know for all $m > 0$ and $0 < \alpha < 2$, $\frac{(2(m+1))}{\alpha} > 1$ and so when we take the limit as $n \to \infty$ we have,

$$\lim_{n\to\infty}\left(2nc^2\sum_{k=1}^{n}\frac{(1+\alpha ck)^{\left(\frac{(2(m+1))}{\alpha}\right)-2}}{(1+\alpha cn)^{\left(\frac{(2(m+1))}{\alpha}\right)}}\right)=\frac{2}{\alpha(2(m+1)-\alpha)}.$$

Therefore,

$$\lim_{n\to\infty}\sum_{k=1}^{n}\mathbb{E}((\Re(a_{n,k}(m)))^2|\mathcal{F}_{k-1})=\lim_{n\to\infty}\sum_{k=1}^{n}\mathbb{E}((\Im(a_{n,k}(m)))^2|\mathcal{F}_{k-1})=\frac{2}{\alpha(2m+2-\alpha)}.$$

□

So we have shown that sum of the second moments converge. Note that it is clear to see that letting $\alpha = 2$ will not provide a finite limit using the above lemma. To apply Theorem 6.2 we need to show that $\sum_{k=1}^{n}(\Re(a_{n,k}(m)))^2$ also converges to the same value. We prove this with the following lemma, using a similar method to that of Silvestri in [8].

**Lemma 6.5.** *Let $0 < \alpha < 2$ and assume for each $m > 0$,*

$$\lim_{n\to\infty}\sum_{k=1}^{n}\mathbb{E}((\Re(a_{n,k}(m)))^2|\mathcal{F}_{k-1})=\lim_{n\to\infty}\sum_{k=1}^{n}\mathbb{E}((\Im(a_{n,k}(m)))^2|\mathcal{F}_{k-1})=s^2$$

*for some $s^2 > 0$. Then for each $m > 0$,*

$$\lim_{n\to\infty}\sum_{k=1}^{n}\Re(a_{n,k}(m)))^2=\lim_{n\to\infty}\sum_{k=1}^{n}\Im(a_{n,k}(m)))^2=s^2.$$

*Proof.* First we note that since $(\mathcal{X}_{k,n})_{k\le n}(z)$ is a martingale difference array with respect to the filtration $(\mathcal{F}_{k,n})_{k\le n}$, so too is

$$\mathcal{Y}_k(z)=(\Re(a_{n,k}(m)))^2-\mathbb{E}((\Re(a_{n,k}(m)))^2|\mathcal{F}_{k-1})$$

a martingale difference array with respect to the same filtration. We need to show $\mathbb{P}(|\sum_{k=1}^{n}\mathcal{Y}_k(z)| > \eta) \to 0$ as $n \to \infty$. So we first notice that by Markov's inequality,

$$\mathbb{P}\left(|\sum_{k=1}^{n}\mathcal{Y}_k|>\eta\right)\le\frac{1}{\eta^2}\mathbb{E}\left(|\sum_{k=1}^{n}\mathcal{Y}_k|^2\right)=\frac{1}{\eta^2}\sum_{k=1}^{n}\mathbb{E}(\mathcal{Y}_k^2).$$

and so finally by using the property that for a random variable $X$, $\mathbb{E}((X-\mathbb{E}(X))^2)\le\mathbb{E}(X^2)$ we see

$$\mathbb{P}\left(|\sum_{k=1}^{n}\mathcal{Y}_k|>\eta\right)\le\frac{1}{\eta^2}\sum_{k=1}^{n}\mathbb{E}(\Re(a_{n,k}(m)))^4).$$

We have shown,

$$\Re(a_{n,k}(m))=\frac{2c_k^*\sqrt{n}}{(e^{C^*_{n,k+1}})^{m+1}}\cos(\theta_k(m+1)).$$

So using the property that $c_k^* = \frac{c}{1+\alpha c(k-1)}$ and $e^{-C^*_{n,k+1}} \leq \left(\frac{1+\alpha ck}{1+\alpha cn}\right)^{1/\alpha}$ we reach the upper bound,

$$\Re(a_{n,k}(m)) \leq 2c(1+\alpha c)\sqrt{n}\frac{(1+\alpha ck)^{\frac{m+1}{\alpha}-1}}{(1+\alpha cn)^{\frac{m+1}{\alpha}}}. \tag{4}$$

Thus,

$$\Re(a_{n,k}(m))^4 \leq (2c(1+\alpha c))^4 \, \frac{n^2(1+\alpha ck)^{\frac{4(m+1)}{\alpha}-4}}{(1+\alpha cn)^{\frac{4(m+1)}{\alpha}}}.$$

Then we consider cases, if $0 < \alpha \leq \frac{4}{3}(m+1)$ then when we sum over $k$ we reach the following bound,

$$\frac{1}{\eta^2}\left(\sum_{k=1}^{n}\mathbb{E}\left((\Re(a_{n,k}(m)))^4\right)\right) \leq \lambda(\alpha, c)\frac{1}{n}.$$

where $\lambda(\alpha, c)$ is some constant. This converges to zero as $n \to \infty$. Moreover, if $\frac{4}{3}(m+1) < \alpha < 2$ then when we sum over $k$ we reach the following bound,

$$\frac{1}{\eta^2}\left(\sum_{k=1}^{n}\mathbb{E}\left((\Re(a_{n,k}(m)))^4\right)\right) \leq \lambda(\alpha, c)\frac{n}{n^{\frac{4(m+1)}{\alpha}}}.$$

where $\lambda(\alpha, c)$ is some constant. This converges to zero as $n \to \infty$. Therefore in both cases we have convergence to zero. The proof of the imaginary case holds by the same argument. ☐

Therefore, we have shown, in the form of the following corollary, that the condition (2) of Theorem 6.2 is satified.

**Corollary 6.6.** *For $a_{n,k}(m)$ defined as above, the following expression holds*

$$\lim_{n\to\infty}\sum_{k=1}^{n}\Re(a_{n,k}(m)))^2 = \lim_{n\to\infty}\sum_{k=1}^{n}\Im(a_{n,k}(m)))^2 = \frac{2}{\alpha(2m+2-\alpha)}$$

So now we just need show the first part of Theorem 6.2 holds in order to apply it. We will again use a similar method to Silvestri [8].

**Lemma 6.7.** *Let $0 < \alpha < 2$ and let $\rho > 0$ then for each $m > 0$ it holds that*

$$\sum_{k=1}^{n}(\Re(a_{n,k}(m)))^2\mathbb{1}(|\Re(a_{n,k}(m))| > \rho) \to 0$$

*and*

$$\sum_{k=1}^{n}(\Im(a_{n,k}(m)))^2\mathbb{1}(|\Im(a_{n,k}(m))| > \rho) \to 0$$

*as $n \to \infty$.*

*Proof.* We will only prove the real case. We use a similar method as [8]. Let $\delta > 0$ then

$$\mathbb{P}\left(\sum_{k=1}^{n}(\Re(a_{n,k}(m)))^2\mathbb{1}(|\Re(a_{n,k}(m))| > \rho) > \delta\right) \leq \mathbb{P}\left(\max_{1\leq k\leq n}|\Re(a_{n,k}(m))| > \rho\right)$$
$$\leq \frac{1}{\rho}\mathbb{E}\left(\max_{1\leq k\leq n}|\Re(a_{n,k}(m))|\right)$$

with the second inequality following by Markov's inequality. As in the proof of Lemma 6.5, we have shown that

$$|\Re(a_{n,k}(m))| \leq 2c(1+\alpha c)\sqrt{n}\frac{(1+\alpha ck)^{\frac{m+1}{\alpha}-1}}{(1+\alpha cn)^{\frac{m+1}{\alpha}}}.$$

So if $m+1 \geq \alpha$,

$$\max_{0\leq k\leq n}|\Re(a_{n,k}(m))| \leq 2c(1+\alpha c)\sqrt{n}\frac{1}{(1+\alpha cn)}.$$

Then if $m+1 < \alpha$,

$$\max_{0\leq k\leq n}|\Re(a_{n,k}(m))| \leq 2c(1+\alpha c)\sqrt{n}\frac{1}{(1+\alpha cn)^{\frac{m+1}{\alpha}}}.$$

In both cases $\max_{0\leq k\leq n}\Re(a_{n,k}(m))$ converges to zero as $n\to\infty$. Thus taking the limit gives

$$\sum_{k=1}^{n}(\Re(a_{n,k}(m)))^2\mathbb{1}(|\Re(a_{n,k}(m))| > \rho) \to 0.$$

The imaginary case follows by the same argument. □

So now we have all we need in order to apply Theorem 6.2. This leads to the following result on the distribution of the Laurent coefficients.

**Theorem 6.8.** *Let $0 < \alpha < 2$ then for each $m \geq 0$, it holds that*

$$\lim_{n\to\infty}\sum_{k=1}^{n}a_{n,k}(m) = A_m + iB_m$$

*with $A_m$, $B_m \sim \mathcal{N}(0, \frac{2}{\alpha(2m+2-\alpha)})$ and $A_m$, $B_m$ independent.*

*Proof.* We have shown that both $\Re(a_{n,k}(m)$ and $\Im(a_{n,k}(m)$ are martingale difference arrays in $k$. Furthermore, Lemma 6.7 and Corollary 6.6 prove that the conditions of Theorem 6.2 are satisfied. Therefore, applying Theorem 6.2 to both $\Re(a_{n,k}(m)$ and $\Im(a_{n,k}(m)$ we have

$$\lim_{n\to\infty}\sum_{k=1}^{n}\Re(a_{n,k}(m) = A_m, \quad \lim_{n\to\infty}\sum_{k=1}^{n}\Im(a_{n,k}(m) = B_m$$

where $A_m,\ B_m \sim \mathcal{N}(0, \frac{2}{\alpha(2m+2-\alpha)})$. Hence, for a fixed $m \geq 0$,

$$\lim_{n\to\infty} \sum_{k=1}^{n} (\Re(a_{n,k}(m) + i\Im(a_{n,k}(m)) = A_m + iB_m.$$

Furthermore, calculating the covariance pairwise of each combination of the random variables we see that for any $m_1,\ m_2$

$$\begin{aligned}\mathrm{Cov}(\Re(a_{n,k}(m_1), \Im(a_{n,k}(m_2)) &= \mathbb{E}(\Re(a_{n,k}(m_1)\Im(a_{n,k}(m_2)) \\ &= \frac{4(c_k^*)^2}{(\gamma e^{C^*_{n,k+1}})^{m_1+m_2+2}} \int_0^{2\pi} \cos(\theta(m_1+1))\sin(\theta(m_2+1))d\theta \\ &= 0.\end{aligned}$$

Moreover for $m_1 \neq m_2$,

$$\begin{aligned}\mathrm{Cov}(\Re(a_{n,k}(m_1), \Re(a_{n,k}(m_2)) &= \mathbb{E}(\Re(a_{n,k}(m_1)\Re(a_{n,k}(m_2)) \\ &= \frac{4(c_k^*)^2}{(\gamma e^{C^*_{n,k+1}})^{m_1+m_2+2}} \int_0^{2\pi} \cos(\theta(m_1+1))\cos(\theta(m_2+1))d\theta \\ &= 0.\end{aligned}$$

Finally for $m_1 \neq m_2$,

$$\begin{aligned}\mathrm{Cov}(\Im(a_{n,k}(m_1), \Im(a_{n,k}(m_2)) &= \mathbb{E}(\Im(a_{n,k}(m_1)\Im(a_{n,k}(m_2)) \\ &= \frac{4(c_k^*)^2}{(\gamma e^{C^*_{n,k+1}})^{m_1+m_2+2}} \int_0^{2\pi} \sin(\theta(m_1+1))\sin(\theta(m_2+1))d\theta \\ &= 0.\end{aligned}$$

Therefore, the covariance and hence the correlation between each pairwise combination of random variables is equal to zero. Therefore we have a mulivariate Gaussian process as required. Independence also follows quickly from the fact that the correlation is zero.

□

### 6.3 Convergence as a holomorphic function

Now that we have proved that the Laurent coefficients converge, we wish to show that we also have the convergence of the fluctuations as a holomorphic function. We first define the functions,

$$\tilde{\mathcal{F}}(n,z) = \sqrt{n}\left(e^{-C^*_{n,1}}\phi_n(z) - z\right)$$

and

$$\mathcal{F}(z) = \sum_{m=0}^{\infty} (A_m + iB_m) z^{-m}$$

where $A_m,\, B_m \sim \mathcal{N}(0, \frac{2}{\alpha(2m+2-\alpha)})$ and each $A_{m_1},\, B_{m_2}$ independent of each other for any $m_1,\, m_2$. Our aim is to show that $\tilde{\mathcal{F}}(n,z) \to \mathcal{F}(z)$ in distribution as $n \to \infty$ on the space of holomorphic functions, $\mathcal{H}$, equipped with the metric,

$$\mathbf{d}_{\mathcal{H}}(f,g) = \sum_{m\geq 0} 2^{-m} \left( 1 \wedge \sup_{|z|\geq 1+2^{-m}} |f(z)-g(z)| \right).$$

We use a similar method as in [5] by defining,

$$\mathbf{d}_r(f,g) = \sup_{|z|>r} |f(z)-g(z)|.$$

To make notation easier, we also define $M(n,m) = \sum_{k=1}^{n} a_{n,k}(m)$. We first need the following lemma used to discard the tail terms.

**Lemma 6.9.** *Let $r > 1$ and $N > 0$ then for any $\epsilon > 0$*

$$\lim_{T\to\infty} \sup_{n>N} \mathbb{P}\left( \boldsymbol{d}_r \left( \sum_{m=T}^{\infty} M(n,m) z^{-m}, 0 \right) > \epsilon \right) = 0.$$

*Proof.* Using the definition of $\mathbf{d}_r(f,g)$ we see that,

$$\mathbf{d}_r \left( \sum_{m=T}^{\infty} M(n,m) z^{-m}, 0 \right) = \sup_{|z|>r} \left| \sum_{m=T}^{\infty} M(n,m) z^{-m} \right|.$$

By Markov's inequality,

$$\begin{aligned} \mathbb{P}\left( \mathbf{d}_r \left( \sum_{m=T}^{\infty} M(n,m) z^{-m}, 0 \right) > \epsilon \right) &\leq \frac{1}{\epsilon^2} \mathbb{E}\left( \sup_{|z|>r} \left| \sum_{m=T}^{\infty} M(n,m) z^{-m} \right|^2 \right) \\ &\leq \frac{1}{\epsilon^2} \mathbb{E}\left( \sup_{|z|>r} \left( \sum_{m=T}^{\infty} |M(n,m)||z|^{-m} \right)^2 \right) \\ &\leq \frac{1}{\epsilon^2} \mathbb{E}\left( \left( \sum_{m=T}^{\infty} |M(n,m)| r^{-m} \right)^2 \right). \end{aligned}$$

Using the Cauchy-Schwartz inequality we have,

$$\begin{aligned} \mathbb{P}\left( \mathbf{d}_r \left( \sum_{m=T}^{\infty} M(n,m) z^{-m}, 0 \right) > \epsilon \right) &\leq \frac{1}{\epsilon^2} \mathbb{E}\left( \left( \sum_{m=T}^{\infty} |M(n,m)|^2 r^{-m} \right) \left( \sum_{m=T}^{\infty} r^{-m} \right) \right) \\ &\leq \frac{\lambda(r)}{\epsilon^2} \mathbb{E}\left( \sum_{m=T}^{\infty} |M(n,m)|^2 r^{-m} \right) \end{aligned}$$

where $\lambda(r)$ is some constant dependent on r. Then we can take the expectation inside the sum, thus,

$$\mathbb{P}\left(\mathbf{d}_r\left(\sum_{m=T}^{\infty} M(n,m)z^{-m}, 0\right) > \epsilon\right) \leq \frac{1}{\epsilon^2}\sum_{m=T}^{\infty}\mathbb{E}\left(|M(n,m)|^2\right) r^{-m}.$$

Now notice that,

$$\mathbb{E}\left(|M(n,m)|^2\right) = \mathbb{E}\left(\left|\sum_{k=1}^{n} a_{n,k}(m)\right|^2\right) \leq \mathbb{E}\left(\sum_{k=1}^{n}(\Re(a_{n,k}(m))^2 + (\Im(a_{n,k}(m))^2\right).$$

But in equation (4) we show that

$$(\Re(a_{n,k}(m))^2 + (\Im(a_{n,k}(m))^2 \leq \lambda(\alpha,c,r)n^{1-\frac{2(m+1)}{\alpha}}k^{\frac{2(m+1)}{\alpha}-2}$$

where $\lambda(\alpha,c,r)$ is some constant. Taking the sum over $k$ we see that,

$$\mathbb{E}\left(|M(n,m)|^2\right) \leq \lambda(\alpha,c,r).$$

Therefore,

$$\begin{aligned}\lim_{T\to\infty}\sup_{n>N}\mathbb{P}\left(\mathbf{d}_r\left(\sum_{m=T}^{\infty} M(n,m)z^{-m}, 0\right) > \epsilon\right) &\leq \lim_{T\to\infty}\frac{1}{\epsilon^2}\lambda(\alpha,c,r)\sum_{m=T}^{\infty} r^{-m} \\ &\to 0 \text{ as } T\to\infty.\end{aligned}$$

□

Therefore, through Theorem 6.8 we have shown that we have convergence of the Laurent coefficients. Moreover, Lemma 6.9 shows that the tails of the Laurent series tend to zero in the limit. We can then combine these two results with a result of Billingsley [13] to show that we have convergence as a holomorphic function and therefore the fluctuations behave like a Gaussian field.

**Theorem 1.3.** *Let $0 < \alpha < 2$ and $\phi_n$ be defined as in Theorem 1.2. Then as $n \to \infty$,*

$$\sqrt{n}\left(e^{-\sum_{i=1}^{n} c_i^*}\phi_n(z) - z\right) \to \mathcal{F}(z)$$

*in distribution on $\mathcal{H}$, where $\mathcal{H}$ is the space of holomorphic functions on $|z| > 1$, equipped with metric $\boldsymbol{d}_{\mathcal{H}}$ defined above, and where*

$$\mathcal{F}(z) = \sum_{m=0}^{\infty}(A_m + iB_m)z^{-m}$$

*and $A_m$, $B_m \sim \mathcal{N}\left(0, \frac{2}{\alpha(2m+2-\alpha)}\right)$ and $A_m$, $B_k$ independent for all choices of $m$ and $k$.*

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