

Quantum backreaction in laser-driven plasma

A. Conroy¹, C. Fiedler¹, A. Noble², and D. A. Burton¹

¹Department of Physics, Lancaster University, Lancaster, LA1 4YB, UK

²Department of Physics, SUPA and University of Strathclyde, Glasgow, G4 0NG, UK

June 22, 2019

Abstract

We present a new approach for investigating quantum effects in laser-driven plasma. Unlike the modelling strategies underpinning particle-in-cell codes that include the effects of quantum electrodynamics, our new field theory incorporates multi-particle effects from the outset. Our approach is based on the path-integral quantisation of a classical bi-scalar field theory describing the behaviour of a laser pulse propagating through an underdense plasma. Results established in the context of quantum field theory on curved spacetime are used to derive a non-linear, non-local, effective field theory that describes the evolution of the laser-driven plasma due to quantum fluctuations. As the first application of our new theory, we explore the behaviour of perturbations to fields describing a uniform, monochromatic, laser beam propagating through a uniform plasma. Our results suggest that quantum fluctuations could play a significant role in the evolution of an underdense plasma, with plasma frequency 10 THz, driven by a laser pulse with wavelength $\lesssim 150$ nm, width $30 \mu\text{m}$ and duration > 100 fs. Such parameters should be realisable in an experiment using a facility such as the European XFEL.

1 Introduction

The new generation of high-power laser systems will drive the experimental study of high-intensity laser-matter interactions into novel territory. Forthcoming facilities [1] are expected to allow experimental investigations of uncharted parameter regimes using laser pulses with unprecedented peak intensities (greater than 10^{22} Wcm^{-2}). In such regimes, the relativistic quantum-mechanical aspects of laser-matter interactions must be included [2, 3], and this requirement has driven the development of particle-in-cell codes, such as EPOCH [4], that incorporate the effects of quantum electrodynamics. In such codes, the matter is represented by a large number of classical macro-particles whose electromagnetic fields, and the laser field, serve as a background in the perturbative calculation of single-particle matrix elements associated with each macro-particle. Each quantum process is assumed to be active only within a spacetime region whose size is negligible in comparison to the classical length and time scales of the laser and matter variables. Interactions between macro-particles in this model are implemented through their contributions to Maxwell equations as classical sources; as a consequence, the multi-particle effects calculated using this approach are classical, rather than quantum, in origin. Furthermore, tools such as EPOCH require high-performance computing facilities to be of greatest utility. Even without quantum effects, the 3-dimensional calculation of the propagation of micrometre-length laser pulses through many centimetres of underdense plasma is a considerable computational challenge. Such considerations have motivated the development of models of classical laser-driven plasma with reduced degrees of freedom, such as those underpinning INF&RNO [5], leading to a reduction of the computational burden by several orders of magnitude. The vigorous effort devoted to the development of such computational tools shows no sign of abating; however, the role of quantum theory in the context of reduced models is yet to be thoroughly investigated.

This article offers a new theory for investigating the quantum effects exhibited by a laser pulse propagating through an underdense plasma. Rather than modelling the microscopic degrees of freedom by appealing to quantum electrodynamics, our approach is based on the quantisation of a fluid description of the laser-driven

plasma. Hence, multi-particle considerations are included from the outset. The underpinning ingredients rely on standard approximations in classical laser-plasma theory; the slowly-varying envelope approximation for the laser pulse, and the ponderomotive approximation for the force exerted by the laser on the plasma electrons [6]. We exploit results established in the context of quantum field theory on curved spacetime to obtain the quantum corrections to the classical field equations. As the first application of our new theory, we show that the quantum fluctuations lead to the relationship

$$\omega = \sqrt{\frac{3}{128 \pi^{3/2}}} \sqrt{\frac{\lambda_e}{w_0}} \left[\frac{\alpha a_0^2}{(a_0^2 + 1)^3} \right]^{\frac{1}{4}} \frac{c^2 \kappa^2}{\sqrt{\omega_0 \omega_p}} \quad (1)$$

between the angular frequency ω and wavenumber κ of a longitudinal perturbation to the laser-plasma variables in the rest frame of the plasma ions. The quantity a_0 is the dimensionless amplitude of the laser pulse, ω_0 is the frequency of the laser pulse, ω_p is the plasma frequency in the unperturbed state, λ_e is the Compton wavelength of the electron, α is the fine-structure constant, and w_0 is the width of the laser pulse. Equation (1) is valid for perturbations along the axis of the pulse whose wavelength is shorter than the length of the pulse, and the perturbation is static in the underlying classical theory (recovered in the limit $\hbar \rightarrow 0$).

Using (1), the characteristic time scale τ over which the length of a Gaussian disturbance approximately doubles due to quantum fluctuations, in the rest frame of the plasma ions, satisfies

$$\tau \sim \sqrt{\frac{128 \pi^{3/2}}{3}} \sqrt{\frac{w_0}{\lambda_e}} \left[\frac{(a_0^2 + 1)^3}{\alpha a_0^2} \right]^{\frac{1}{4}} \frac{\sqrt{\omega_0 \omega_p}}{c^2} \sigma^2, \quad (2)$$

where σ is the initial length of the disturbance. The implications of (2) are readily appreciated by expressing it in terms of the quantities $\tilde{\tau}$, $\tilde{\sigma}$ normalised with respect to the laser period $2\pi/\omega_0$ and laser wavelength $\lambda_0 = 2\pi c/\omega_0$, respectively. The approximate inequality

$$\tilde{\tau} \gtrsim 534 \sqrt{\frac{w_0 \lambda_0}{\lambda_p \lambda_e}} \tilde{\sigma}^2 \quad (3)$$

results from noting that $a_0 = 1/\sqrt{2}$ minimises the term within square parentheses in (2), where the length $\lambda_p = 2\pi c/\omega_p$ has been introduced. Moreover, the theory underpinning (1) requires the laser wavelength λ_0 to be the shortest classical length scale in the analysis; thus, $\tilde{\sigma} > 1$ and

$$\tilde{\tau} \gtrsim 534 \sqrt{\frac{w_0 \lambda_0}{\lambda_p \lambda_e}} \quad (4)$$

follows immediately.

A comparison of (4) with the number of oscillations $\pi w_0^2/\lambda_0^2$ corresponding to the Rayleigh length of a laser beam with waist w_0 yields the upper bound

$$\lambda_0 \lesssim 0.128 (\lambda_e \lambda_p w_0^3)^{1/5} \quad (5)$$

on the laser wavelength. If (5) is significantly violated then it is likely that classical diffraction will ensure the laser pulse disperses before the effects of quantum fluctuations become significant. For example, the representative parameter choice $w_0 = \lambda_p = 30 \mu\text{m}$ describing a matched laser pulse yields 146 nm for the upper bound on λ_0 . Thus, a comprehensive experimental investigation of our new results should be possible using an x-ray laser (e.g. the European XFEL [8]) that produces pulses with duration greater than 100 fs.

Section 2 introduces the classical theory underpinning our approach. Section 3 details the 1-loop effective action that arises from a path-integral quantisation of the underlying classical theory, and Section 4 summarises the non-linear field equations that emerge. By construction, the field equations include the quantum backreaction of the laser-plasma system. Section 5 is a perturbative analysis of the field equations, applicable when the wavelength of the perturbation to the laser-plasma variables is much shorter than the length of the laser pulse. Equation (1) emerges as a result.

2 Classical theory of laser-driven plasma

Our particular interest here is in the interaction of electrons with an intense laser pulse propagating through an underdense plasma, where the internal oscillations of the pulse determine the shortest significant classical length and time scales of the system. The large difference between the scales associated with the internal oscillations of the laser pulse and the behaviour of the wake behind the front of the pulse permits approximations to be introduced that greatly simplify the analysis. The precise details of the plasma electron motion due to the fast oscillations of the fields within the laser pulse are sacrificed to obtain an efficient model of the electron dynamics over distances that are much greater than the wavelength of the laser. Computationally efficient models in this context typically exploit the ponderomotive approximation for calculating the effect of the laser pulse on the plasma electrons in tandem with a slowly-varying envelope approximation for determining the influence of the matter on the laser pulse [5].

The total electric field and total magnetic field are each expressed a sum of two terms. The first term can be understood as the local average of the respective total field over the fast oscillations of the laser pulse, and the second term is the field of the laser itself. In particular, the total electric field \mathbf{E}_{tot} satisfies $\mathbf{E}_{\text{tot}} = \mathbf{E} - \partial_t \mathbf{A}_0$ where, for notational convenience, the local average of \mathbf{E}_{tot} is denoted \mathbf{E} and the laser pulse is encoded by the vector potential \mathbf{A}_0 . Likewise, the total magnetic field \mathbf{B}_{tot} is $\mathbf{B}_{\text{tot}} = \mathbf{B} + \nabla \times \mathbf{A}_0$.

The electrons in a cold fluid model of a laser-plasma satisfy

$$\partial_t \mathbf{p} + (\mathbf{v} \cdot \nabla) \mathbf{p} = -\frac{e^2}{2m_e \gamma} \nabla \langle \mathbf{A}_0^2 \rangle - e(\mathbf{E} + \mathbf{v} \times \mathbf{B}), \quad \gamma = \sqrt{1 + \frac{\mathbf{p}^2}{m_e^2 c^2} + \frac{e^2 \langle \mathbf{A}_0^2 \rangle}{m_e^2 c^2}}, \quad \mathbf{p} = m_e \gamma \mathbf{v} \quad (6)$$

in the relativistic ponderomotive approximation, where m_e is the rest mass of the electron, e is the elementary charge and c is the speed of light in vacuo. The effect of the laser pulse on the electrons is determined by $\langle \mathbf{A}_0^2 \rangle$, the square of the magnitude of the vector potential \mathbf{A}_0 of the pulse averaged over its fast internal oscillations. The vector fields \mathbf{v} , \mathbf{p} are understood as the averaged velocity and averaged momentum, respectively, of the plasma electrons. The contribution to γ^2 (the square of the Lorentz factor γ) proportional to $\langle \mathbf{A}_0^2 \rangle$ is due to the fast oscillatory motion of the electrons induced by the laser pulse.

The fields \mathbf{E} , \mathbf{B} are produced from the averaged properties of the electrons; in particular,

$$\nabla \cdot \mathbf{D} = -en + \rho_i, \quad \nabla \times \mathbf{H} = \partial_t \mathbf{D} - en\mathbf{v} + \mathbf{j}_i, \quad \nabla \times \mathbf{E} = -\partial_t \mathbf{B}, \quad \nabla \cdot \mathbf{B} = 0, \quad (7)$$

with $\mathbf{D} = \varepsilon_0 \mathbf{E}$, $\mathbf{H} = \mathbf{B}/\mu_0$, where ε_0 , μ_0 are the permittivity and permeability of the vacuum, respectively, and n is the averaged electron number density. The fields ρ_i , \mathbf{j}_i are the ion charge density and ion current density, respectively, and are specified as data.

To proceed further, the properties of the laser potential must be specified. A popular strategy used in many studies of laser-plasma accelerators [6] is to solve

$$\partial_t^2 \mathbf{A}_0 - c^2 \nabla^2 \mathbf{A}_0 = -\omega_p^2 \mathbf{A}_0, \quad \omega_p = \sqrt{\frac{e^2 n}{\varepsilon_0 m_e \gamma}} \quad (8)$$

for the vector potential \mathbf{A}_0 , where ω_p is the plasma frequency given by the local electron number density n . For practical purposes, slowly-varying envelope, or eikonal, approximations are commonly used to remove the fast oscillations from (8) before further analysis. In addition to the separation of scales, a complete justification of the model (6), (7), (8) requires the dominant component of the electron momentum to be parallel to the direction of propagation of the laser pulse. We will exploit this facet of the classical model when developing the quantum theory in Section 3.

It can be shown [7] that applying the eikonal approximation to (8) yields the conservation of wave action

$$\partial_t (\langle \mathbf{A}_0^2 \rangle \omega_0) + c^2 \nabla \cdot (\langle \mathbf{A}_0^2 \rangle \mathbf{k}_0) = 0, \quad (9)$$

where the local frequency ω_0 and local wave vector \mathbf{k}_0 of the laser pulse satisfy $\partial_t \mathbf{k}_0 = -\nabla \omega_0$ and the local dispersion relation

$$\omega_0^2 - c^2 \mathbf{k}_0^2 = \omega_p^2. \quad (10)$$

Equations (6), (7), (9), (10) constitute a closed system of classical field equations for a laser-driven plasma.

2.1 Reduction of the classical theory

There are numerous ways of developing effective quantum theories from the above system of classical field equations. A simple strategy is to focus on a regime in which the effects due to the interaction between the laser field and matter dominate over those directly connected to the averaged electromagnetic fields; thus, \mathbf{E} , \mathbf{B} are treated as negligible. To lowest order, the field equation for \mathbf{p} is (6) with \mathbf{E} , \mathbf{B} set to zero:

$$\partial_t \mathbf{p} + (\mathbf{v} \cdot \nabla) \mathbf{p} = -\frac{e^2}{2m_e \gamma} \nabla \langle \mathbf{A}_0^2 \rangle. \quad (11)$$

Further simplification is achieved by focussing on potential flow. Equation (11) can be expressed as

$$\partial_t \mathbf{p} - \mathbf{v} \times (\nabla \times \mathbf{p}) = -m_e c^2 \nabla \gamma, \quad (12)$$

which is solved by $\mathbf{p} = \nabla \tilde{\Psi}$ with the momentum potential $\tilde{\Psi}$ satisfying $\partial_t \tilde{\Psi} = -m_e c^2 \gamma$. The latter can be rearranged to give

$$e^2 c^2 \langle \mathbf{A}_0^2 \rangle = (\partial_t \tilde{\Psi})^2 - c^2 (\nabla \tilde{\Psi})^2 - m_e^2 c^4. \quad (13)$$

Although \mathbf{E} , \mathbf{B} feature in the lowest order Maxwell equations (7), their precise forms are not needed; the only necessary consequence of (7) is charge conservation. The electron number density n is required to obtain the plasma frequency ω_p , and charge conservation provides a suitable field equation for n . Since the ion charge density is locally conserved, the electron number density must satisfy

$$\partial_t n + \nabla \cdot (n \mathbf{v}) = 0 \quad (14)$$

which, using the expression for ω_p given in (8), yields

$$\partial_t (\omega_p^2 \partial_t \tilde{\Psi}) - c^2 \nabla \cdot (\omega_p^2 \nabla \tilde{\Psi}) = 0. \quad (15)$$

where $m_e \mathbf{v} = \nabla \tilde{\Psi} / \gamma$, $\partial_t \tilde{\Psi} = -m_e c^2 \gamma$ have been employed.

The quantity $\varepsilon_0 \omega_p^2 \langle \mathbf{A}_0^2 \rangle / 2$ has the physical dimensions of an energy density, and it can serve as the Lagrangian density in an action principle for the field equations (9), (15). Expressing $\varepsilon_0 \omega_p^2 \langle \mathbf{A}_0^2 \rangle / 2$ in terms of $\tilde{\Psi}$ and the phase $\tilde{\Phi}$, where $\omega_0 = -\partial_t \tilde{\Phi}$, $\mathbf{k} = \nabla \tilde{\Phi}$, suggests the action

$$\mathcal{S}[\tilde{\Phi}, \tilde{\Psi}] = \frac{\varepsilon_0 \sigma_*}{e^2 c^2} \int dt d^3 \mathbf{x} \frac{1}{2} \{ (\partial_t \tilde{\Phi})^2 - c^2 (\nabla \tilde{\Phi})^2 \} \{ (\partial_t \tilde{\Psi})^2 - c^2 (\nabla \tilde{\Psi})^2 - m_e^2 c^4 \} \quad (16)$$

where (10), (13) have been used to substitute ω_p , $\langle \mathbf{A}_0^2 \rangle$, respectively. Stationary variations of (16) with respect to $\tilde{\Phi}$, $\tilde{\Psi}$ yield (9), (15), respectively. The dimensionless constant σ_* has been introduced because (16) has been obtained using dimensional reasoning and, although σ_* is inert in the classical theory, it will scale the quantum corrections to the classical field equations.

Although substantial simplifications have been made to obtain (16), it is highly beneficial, from the perspective of quantum theory, to replace (16) with its counterpart theory in one spatial dimension. This strategy is physically justified by noting that the laser-plasma variables $\tilde{\Phi}$, $\tilde{\Psi}$ change over a much shorter distance parallel to the direction of propagation of the laser pulse than transverse to it. This property of the laser-plasma system is closely connected to the justification behind the introduction of (6), (8). Furthermore, within this approximation, it is reasonable to choose $\omega_p^2 \langle \mathbf{A}_0^2 \rangle$ to be expressible as a product of a function of (t, z) and a function of (x, y) . Thus, the quantity L_* given by

$$L_* = \sqrt{\frac{\sigma_* \int dx dy \omega_p^2 \langle \mathbf{A}_0^2 \rangle}{\omega_p^2 \langle \mathbf{A}_0^2 \rangle |_{x=y=0}}} \quad (17)$$

is constant, and we arrive at the estimate

$$\mathcal{S}[\tilde{\Phi}, \tilde{\Psi}] \approx \frac{\varepsilon_0 L_*^2}{e^2 c^2} \int dt dz \frac{1}{2} \{ (\partial_t \tilde{\Phi})^2 - c^2 (\partial_z \tilde{\Phi})^2 \} \{ (\partial_t \tilde{\Psi})^2 - c^2 (\partial_z \tilde{\Psi})^2 - m_e^2 c^4 \} |_{x=y=0} \quad (18)$$

for the action (16). The line $x = y = 0$ has been chosen to lie along the centre of the laser pulse.

The above considerations suggest a relativistic bi-scalar field theory on 2-dimensional spacetime given by the action

$$S[\Phi, \Psi] = \hbar \int d^2x \sqrt{-\eta} \frac{1}{2} \eta^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi (\eta^{\sigma\tau} \partial_\sigma \Psi \partial_\tau \Psi + 1) \quad (19)$$

where $\eta_{\mu\nu}$ is the Minkowski metric with signature $(-, +)$, and $\mu, \nu = 0, 1$. The fields $\Phi, \Psi, \eta_{\mu\nu}$, and coordinates x^0, x^1 are dimensionless. The action (19) can be obtained from (18) using the substitutions

$$x^0 = \frac{ct}{l_*}, \quad x^1 = \frac{z}{l_*}, \quad \tilde{\Phi}|_{x=y=0} = \sqrt{\frac{\hbar e^2}{\epsilon_0 m_e^2 c^3 L_*^2}} \Phi, \quad \tilde{\Psi}|_{x=y=0} = m_e c l_* \Psi. \quad (20)$$

Note that the length scales l_* , L_* are unrelated; the former is inert (it has no direct physical meaning) and is introduced solely for mathematical elegance, whereas the latter is proportional to the transverse size (width) of the laser pulse.

Unless otherwise indicated, for convenience we will henceforth adopt units in which the reduced Planck constant \hbar is unity.

3 Quantum considerations

A perturbative exploration of some of the quantum implications of

$$S[\Phi, \Psi] = \int d^2x \sqrt{-\eta} \frac{1}{2} \eta^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi (\eta^{\sigma\tau} \partial_\sigma \Psi \partial_\tau \Psi + 1) \quad (21)$$

can be undertaken using the 1-loop effective action Γ given by

$$\Gamma[\vec{\Phi}] = S[\vec{\Phi}] - i \ln \left\{ \int \mathcal{D}\vec{f} \exp(i\Lambda[\vec{f}]) \right\} \quad (22)$$

where

$$\Lambda[\vec{f}] = \frac{1}{2} \int d^2x \int d^2y \frac{\delta^2 S}{\delta \Phi_A(x) \delta \Phi_B(y)} f_A(x) f_B(y) \quad (23)$$

with the indices A, B ranging over 1, 2 and $\Phi_1 = \Phi, \Phi_2 = \Psi$. For convenience, the notation $\vec{\Phi} = (\Phi_1 \ \Phi_2)^T$, $\vec{f} = (f_1 \ f_2)^T$ has been introduced, with T denoting matrix transposition. The functional Λ can be expressed as

$$\Lambda[\vec{f}] = \int d^2x \sqrt{-\eta} \frac{1}{2} \left\{ (\eta^{\mu\nu} \partial_\mu \Psi \partial_\nu \Psi + 1) \eta^{\sigma\tau} \partial_\sigma f_1 \partial_\tau f_1 + \eta^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi \eta^{\sigma\tau} \partial_\sigma f_2 \partial_\tau f_2 \right. \\ \left. + 4\eta^{\mu\nu} \partial_\mu \Phi \partial_\nu f_1 \eta^{\sigma\tau} \partial_\sigma \Psi \partial_\tau f_2 \right\} \quad (24)$$

or, equivalently,

$$\Lambda[\vec{f}] = \int d^2x \sqrt{-\eta} \frac{1}{2} \vec{f}^\dagger \mathcal{O} \vec{f} \quad (25)$$

using integration by parts, where \vec{f}^\dagger is the Hermitian conjugate of \vec{f} . In the above, and throughout the following, we adopt the minimal approach in which \vec{f} is regarded as a map on an arbitrarily large torus; hence, boundary terms do not arise when integration by parts is used. The operator \mathcal{O} is given by

$$\mathcal{O} \vec{f} = -\nabla_\mu^{(\eta)} \left\{ \begin{pmatrix} (\eta^{\sigma\tau} \partial_\sigma \Psi \partial_\tau \Psi + 1) \eta^{\mu\nu} & 2\eta^{\mu\sigma} \partial_\sigma \Phi \eta^{\nu\tau} \partial_\tau \Psi \\ 2\eta^{\mu\sigma} \partial_\sigma \Psi \eta^{\nu\tau} \partial_\tau \Phi & \eta^{\sigma\tau} \partial_\sigma \Phi \partial_\tau \Phi \eta^{\mu\nu} \end{pmatrix} \begin{pmatrix} \partial_\nu f_1 \\ \partial_\nu f_2 \end{pmatrix} \right\} \quad (26)$$

with $\nabla_\mu^{(\eta)}$ the Levi-Civita covariant derivative induced from $\eta_{\mu\nu}$. Since \mathcal{O} is Hermitian with respect to the inner product $(\vec{a}, \vec{b}) = \int d^2x \sqrt{-\eta} \vec{a}^\dagger \vec{b}$, \mathcal{O} has real eigenvalues. By definition, $\int \mathcal{D}\vec{f} \exp(i\Lambda[\vec{f}]) = 1/\sqrt{\det(-i\mathcal{O})}$ where the functional determinant $\det(-i\mathcal{O})$ is formally equal to the product $\prod_n(-i\lambda_n)$ of the non-zero eigenvalues $\{-i\lambda_n\}$ of $-i\mathcal{O}$.

In general, it is not straightforward to analytically compute $\det(-i\mathcal{O})$. However, the calculation is trivial when Φ, Ψ are linear functions of Minkowski coordinates adapted to $\eta_{\mu\nu}$ because, in that case, the matrix within the curly brackets in (26) is constant when expressed in those coordinates. Thus, the eigenfunctions of \mathcal{O} have the form $(a \ b)^\top \exp(il_\mu x^\mu)$, where the components of the wave 2-vector l_μ and the coefficients a, b are constant, and the pair of eigenvalues λ_l^+, λ_l^- corresponding to each l_μ satisfies

$$\lambda_l^+ \lambda_l^- = (\eta^{\sigma\tau} \partial_\sigma \Psi \partial_\tau \Psi + 1) \eta^{\gamma\delta} \partial_\gamma \Phi \partial_\delta \Phi (\eta^{\mu\nu} l_\mu l_\nu)^2 - 4(\eta^{\mu\sigma} \partial_\sigma \Phi \eta^{\nu\tau} \partial_\tau \Psi l_\mu l_\nu)^2. \quad (27)$$

For subsequent analysis, the fact that the right-hand side of (27) can be readily factorised is key. It follows that

$$\lambda_l^+ \lambda_l^- = (\mathcal{A}_+^{\mu\nu} l_\mu l_\nu) (\mathcal{A}_-^{\sigma\tau} l_\sigma l_\tau) \quad (28)$$

where the symmetric tensors $\mathcal{A}_+^{\mu\nu}, \mathcal{A}_-^{\mu\nu}$ are

$$\mathcal{A}_+^{\mu\nu} = \sqrt{(\eta^{\sigma\tau} \partial_\sigma \Psi \partial_\tau \Psi + 1) \eta^{\gamma\delta} \partial_\gamma \Phi \partial_\delta \Phi} \eta^{\mu\nu} + \eta^{\mu\sigma} \eta^{\nu\tau} \partial_\sigma \Phi \partial_\tau \Psi + \eta^{\mu\sigma} \eta^{\nu\tau} \partial_\sigma \Psi \partial_\tau \Phi, \quad (29)$$

$$\mathcal{A}_-^{\mu\nu} = \sqrt{(\eta^{\sigma\tau} \partial_\sigma \Psi \partial_\tau \Psi + 1) \eta^{\gamma\delta} \partial_\gamma \Phi \partial_\delta \Phi} \eta^{\mu\nu} - \eta^{\mu\sigma} \eta^{\nu\tau} \partial_\sigma \Phi \partial_\tau \Psi - \eta^{\mu\sigma} \eta^{\nu\tau} \partial_\sigma \Psi \partial_\tau \Phi. \quad (30)$$

Thus, $\det(-i\mathcal{O}) = \prod_l (i\lambda_l^+ i\lambda_l^-) = \prod_l (-i\lambda_l^+) \prod_{\bar{q}} (-i\lambda_{\bar{q}}^-) = \det(-i\mathcal{O}_+) \det(-i\mathcal{O}_-)$ where the second-order differential operators $\mathcal{O}_+, \mathcal{O}_-$ are

$$\mathcal{O}_+ f = -\nabla_\mu^{(\eta)} (\mathcal{A}_+^{\mu\nu} \partial_\nu f), \quad \mathcal{O}_- f = -\nabla_\mu^{(\eta)} (\mathcal{A}_-^{\mu\nu} \partial_\nu f). \quad (31)$$

Hence, one can factorise $\int \mathcal{D}\vec{f} \exp(i\Lambda[\vec{f}])$ as

$$\int \mathcal{D}\vec{f} \exp(i\Lambda[\vec{f}]) = \left\{ \int \mathcal{D}f \exp(i\Lambda_+[f]) \right\} \left\{ \int \mathcal{D}f \exp(i\Lambda_-[f]) \right\} \quad (32)$$

where

$$\Lambda_+[f] = \int d^2x \sqrt{-\eta} \frac{1}{2} \mathcal{A}_+^{\mu\nu} \partial_\mu f \partial_\nu f, \quad \Lambda_-[f] = \int d^2x \sqrt{-\eta} \frac{1}{2} \mathcal{A}_-^{\mu\nu} \partial_\mu f \partial_\nu f. \quad (33)$$

The above considerations are strictly only applicable to the cases where Φ, Ψ are linear functions of Minkowski coordinates adapted to $\eta_{\mu\nu}$. Fields Φ, Ψ of this type describe a non-evolving monochromatic laser beam propagating through a uniform plasma. However, it is plausible that (29), (30), (32), (33) hold to a reasonable approximation when Φ, Ψ are more general. This assertion can be justified by appealing to the WKB approximation; seeking solutions to $\mathcal{O}\vec{f} = \lambda\vec{f}$ of the form $\vec{f} = \sum_{n=0}^{\infty} \varepsilon^n \vec{a}_n \exp(i\chi/\varepsilon)$, where \vec{a}_n, χ are fields and ε is the WKB expansion parameter, leads to a pair of eigenvalues that satisfy (27) with l_μ substituted by $\partial_\mu \chi/\varepsilon$. Those eigenvalues are identical to the eigenvalues of $\mathcal{O}_+, \mathcal{O}_-$ in the WKB approximation and, hence, (32), (33) hold. This approach is analogous to using the Euler-Heisenberg action to describe QED vacuum polarisation even when the electromagnetic invariants are not constant. Although the derivation of the Euler-Heisenberg action requires the electromagnetic fields to be constant (the potentials are linear in Minkowski coordinates), it is not uncommon to use the result in more general circumstances.

Although one can replace (31) with more complicated second-order linear operators and, via the WKB approximation, motivate more complicated expressions for Λ_+, Λ_- , the choice (33) is perhaps the most natural. This conclusion is supported by the special case in which $\eta^{\sigma\lambda} \partial_\sigma \Psi \partial_\lambda \Psi \ll -1$ and $\partial_\mu \Phi \approx \partial_\mu \Psi$. In this particular situation

$$\Lambda[\vec{f}] \approx \int d^2x \sqrt{-\eta} \frac{1}{2} \{ \eta^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi (\eta^{\sigma\tau} \partial_\sigma f_1 \partial_\tau f_1 + \eta^{\sigma\tau} \partial_\sigma f_2 \partial_\tau f_2) + 4\eta^{\mu\nu} \partial_\mu \Phi \partial_\nu f_1 \eta^{\sigma\tau} \partial_\sigma \Phi \partial_\tau f_2 \} \quad (34)$$

follows from (24). Introducing the new variables $\check{f}_1 = (f_1 + f_2)/\sqrt{2}$, $\check{f}_2 = (-f_1 + f_2)/\sqrt{2}$ immediately yields

$$\Lambda \approx \int d^2x \sqrt{-\eta} \frac{1}{2} (\mathcal{A}_+^{\mu\nu} \partial_\mu \check{f}_1 \partial_\nu \check{f}_1 + \mathcal{A}_-^{\mu\nu} \partial_\mu \check{f}_2 \partial_\nu \check{f}_2) \quad (35)$$

where $\mathcal{A}_+^{\mu\nu}$, $\mathcal{A}_-^{\mu\nu}$ are given by (29), (30) with the substitutions $\partial_\mu \Psi \rightarrow \partial_\mu \Phi$, $\eta^{\sigma\tau} \partial_\sigma \Psi \partial_\tau \Psi + 1 \rightarrow \eta^{\sigma\tau} \partial_\sigma \Phi \partial_\tau \Phi$. The functional measure satisfies $\mathcal{D}\vec{f} = \mathcal{D}\vec{\check{f}}$ because the transformation between \vec{f} and $\vec{\check{f}}$ is a constant rotation, and we obtain

$$\int \mathcal{D}\vec{f} \exp(i\Lambda[\vec{f}]) \approx \left\{ \int \mathcal{D}\check{f}_1 \exp(i\Lambda_+[\check{f}_1]) \right\} \left\{ \int \mathcal{D}\check{f}_2 \exp(i\Lambda_-[\check{f}_2]) \right\} \quad (36)$$

as required. Unfortunately, when considered as a metric, $\mathcal{A}_+^{\mu\nu}$ does not have a Lorentzian signature in this case and the classical states in this regime are not perturbatively stable. Hence, the validity of the effective action Γ is questionable in this regime. Nevertheless, the above considerations suggest the use of (29), (30), (32), (33) in cases where $\mathcal{A}_+^{\mu\nu}$, $\mathcal{A}_-^{\mu\nu}$ both have Lorentzian signatures.

The metric signatures of $\mathcal{A}_{\mu\nu}^+$, $\mathcal{A}_{\mu\nu}^-$ can be deduced from the signs of the eigenvalues of the tensors $M_{+\nu}^\mu$, $M_{-\nu}^\mu$ given by

$$M_{+\nu}^\mu = \delta_\nu^\mu + B \eta^{\mu\omega} (X_\omega Y_\nu + Y_\omega X_\nu), \quad M_{-\nu}^\mu = \delta_\nu^\mu - B \eta^{\mu\omega} (X_\omega Y_\nu + Y_\omega X_\nu), \quad (37)$$

where the timelike unit normalised covector fields X_μ , Y_ν and scalar field B (satisfying $B > 1$) are

$$X_\mu = \frac{\partial_\mu \Phi}{\sqrt{-\eta^{\sigma\tau} \partial_\sigma \Phi \partial_\tau \Phi}}, \quad Y_\mu = \frac{\partial_\mu \Psi}{\sqrt{-\eta^{\sigma\tau} \partial_\sigma \Psi \partial_\tau \Psi}}, \quad B = \sqrt{\frac{\eta^{\sigma\tau} \partial_\sigma \Psi \partial_\tau \Psi}{\eta^{\gamma\delta} \partial_\gamma \Psi \partial_\delta \Psi + 1}}. \quad (38)$$

The eigenvalues of $M_{+\nu}^\mu$, $M_{-\nu}^\mu$ are $\{1 - B(\cosh \chi + 1), 1 - B(\cosh \chi - 1)\}$ and $\{1 + B(\cosh \chi + 1), 1 + B(\cosh \chi - 1)\}$, respectively, where $\cosh \chi = -\eta^{\mu\nu} X_\mu Y_\nu$. By inspection, the eigenvalues of $M_{-\nu}^\mu$ are always positive and, since $M_{-\nu}^\mu = \eta_{\nu\sigma} \mathcal{A}_-^{\sigma\mu} / C$ where C is a positive scalar field, it follows that $\mathcal{A}_-^{\mu\nu}$ has the same signature as $\eta^{\mu\nu}$. However, one of the eigenvalues of $M_{+\nu}^\mu = \eta_{\nu\sigma} \mathcal{A}_+^{\sigma\mu} / C$ is always negative and the sign of the remaining eigenvalue depends on the properties of the fields. The remaining eigenvalue must be negative for $\mathcal{A}_+^{\mu\nu}$ to be Lorentzian, albeit of opposite signature to $\eta^{\mu\nu}$. Thus, $\cosh \chi > (1 + B)/B$, which can be expressed as the condition

$$-\eta^{\mu\nu} \partial_\mu \Phi \partial_\nu \Psi > \sqrt{\eta^{\sigma\tau} \partial_\sigma \Phi \partial_\tau \Phi (\eta^{\mu\nu} \partial_\mu \Psi \partial_\nu \Psi + 1)} + \sqrt{\eta^{\sigma\tau} \partial_\sigma \Phi \partial_\tau \Phi \eta^{\mu\nu} \partial_\mu \Psi \partial_\nu \Psi}. \quad (39)$$

Henceforth, we will only consider the regime in which (39) is satisfied.

An explicit expression for the effective action Γ is readily obtained by appealing to studies of the behaviour of the quantum vacuum in curved spacetimes. The required results emerge when (32) is expressed in terms of a massless field theory on a dilatonic curved background. The pair of metrics $g_{\mu\nu}^+$, $g_{\mu\nu}^-$ and the pair of dilatons φ^+ , φ^- are

$$g_+^{\mu\nu} = \frac{\mathcal{A}_+^{\mu\nu}}{\sqrt{A_+}}, \quad \varphi_+ = -\frac{1}{4} \ln(A_+), \quad g_-^{\mu\nu} = \frac{\mathcal{A}_-^{\mu\nu}}{\sqrt{A_-}}, \quad \varphi_- = -\frac{1}{4} \ln(A_-) \quad (40)$$

where A_+ , A_- are the determinants of the tensors $A_{+\mu}^\nu = \eta_{\mu\sigma} \mathcal{A}_+^{\sigma\nu}$, $A_{-\mu}^\nu = \eta_{\mu\sigma} \mathcal{A}_-^{\sigma\nu}$, respectively. It follows that (33) can be expressed as

$$\Lambda_+[f] = \int d^2x \sqrt{-g^+} \frac{1}{2} \exp(-2\varphi_+) g_+^{\mu\nu} \partial_\mu f \partial_\nu f, \quad \Lambda_-[f] = \int d^2x \sqrt{-g^-} \frac{1}{2} \exp(-2\varphi_-) g_-^{\mu\nu} \partial_\mu f \partial_\nu f \quad (41)$$

with g^+ , g^- the determinants of $g_{\mu\nu}^+$, $g_{\mu\nu}^-$, respectively, and the effective action (22) decomposes as

$$\Gamma = S - i \ln \left\{ \int \mathcal{D}f \exp(i\Lambda_+[f]) \right\} - i \ln \left\{ \int \mathcal{D}f \exp(i\Lambda_-[f]) \right\}. \quad (42)$$

The effective action W given by

$$\exp(iW[g_{\mu\nu}, \varphi]) = \int \mathcal{D}f \exp \left\{ \frac{i}{2} \int d^2x \sqrt{-g} \exp(-2\varphi) (\nabla f)^2 \right\} \quad (43)$$

describes the coupling of a dilaton φ to a Lorentzian metric $g_{\mu\nu}$, and their self-couplings, due to the vacuum fluctuations of a massless scalar field f . Its exact renormalised form is [9, 10]

$$W[g_{\mu\nu}, \varphi] = \frac{1}{24\pi} \int d^2x \sqrt{-g} \left\{ \frac{1}{4} R \square^{-1} R + R(\psi + 2\varphi) - 3(\nabla\varphi)^2 \square^{-1} R - (\nabla\varphi)^2 - 4\nabla\varphi \cdot \nabla\psi - (\nabla\psi)^2 \right\} \\ - \frac{1}{4\pi} \ln \mu \int d^2x \sqrt{-g} (\nabla\varphi)^2 \quad (44)$$

where $(\nabla f)^2 = \nabla f \cdot \nabla f$, $\nabla\varphi \cdot \nabla\psi = g^{\mu\nu} \nabla_\mu \varphi \nabla_\nu \psi$, $\square = g^{\mu\nu} \nabla_\mu \nabla_\nu$, ∇_μ is the Levi-Civita covariant derivative given by $g_{\mu\nu}$, and R is the scalar curvature of $g_{\mu\nu}$. The conventions used for the Riemann tensor and Ricci tensor underpinning R are given in Ref. [11]. The scalar field ψ that appears in (44) captures some of the freedom in the choice of the measure $\mathcal{D}f$. In particular, the quantity $\int \mathcal{D}f \exp(i\langle f, f \rangle)$ is chosen to be a field-independent constant, where the inner product $\langle \cdot, \cdot \rangle$ is given by $\langle a, b \rangle = \int d^2x \sqrt{-g} \exp(-2\psi) a^* b$. The constant μ in (44) emerges from the zeta-function regularisation technique used to derive (44) and, in general, must be fixed using additional information.

The result of each functional integral in (42) follows immediately from (44) using the respective substitutions $g_{\mu\nu} = g_{\mu\nu}^+$, $\varphi = \varphi^+$, $\mu = \mu^+$ and $g_{\mu\nu} = g_{\mu\nu}^-$, $\varphi = \varphi^-$, $\mu = \mu^-$. The natural inner product in both cases is $\int d^2x \sqrt{-\eta} a^* b$, and since $g^+ = g^- = \eta$ follows from (40), we set $\psi = 0$ in $\langle \cdot, \cdot \rangle$. In summary, an effective theory describing the self-interaction of a laser-driven plasma due to quantum vacuum fluctuations is

$$\Gamma[\Phi, \Psi] = S[\Phi, \Psi] + w[g_{\mu\nu}^+, \varphi_+, \mu_+] + w[g_{\mu\nu}^-, \varphi_-, \mu_-] \quad (45)$$

where

$$w[g_{\mu\nu}, \varphi, \mu] = \frac{1}{24\pi} \int d^2x \sqrt{-g} \left\{ \frac{1}{4} R \square^{-1} R + 2R\varphi - 3(\nabla\varphi)^2 \square^{-1} R - (\nabla\varphi)^2 \right\} - \frac{1}{4\pi} \ln \mu \int d^2x \sqrt{-g} (\nabla\varphi)^2 \quad (46)$$

and $g_{\mu\nu}^+$, $g_{\mu\nu}^-$, φ^+ , φ^- depend on Φ , Ψ according to (29), (30), (40). The fields must satisfy $\eta^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi < 0$, $\eta^{\mu\nu} \partial_\mu \Psi \partial_\nu \Psi < -1$, and the condition (39).

4 Field equations for Φ and Ψ

Stationary variations of the action (45) with respect to Φ , Ψ lead to field equations describing a laser-driven plasma that include the backreaction of the quantum fluctuations. The field equations arising from the Φ variation and Ψ variation can be expressed as

$$\nabla_\sigma^{(\eta)} (\varepsilon \beta^\sigma + \mathcal{B}_+^\sigma + \mathcal{B}_-^\sigma) = 0, \quad \nabla_\sigma^{(\eta)} (\alpha \zeta^\sigma + \mathcal{C}_+^\sigma + \mathcal{C}_-^\sigma) = 0, \quad (47)$$

respectively, where

$$\alpha = \eta^{\sigma\tau} \partial_\sigma \Phi \partial_\tau \Phi, \quad \beta^\mu = \eta^{\mu\nu} \partial_\nu \Phi, \quad \varepsilon = \eta^{\sigma\tau} \partial_\sigma \Psi \partial_\tau \Psi + 1, \quad \zeta^\mu = \eta^{\mu\nu} \partial_\nu \Psi, \quad (48)$$

with

$$\mathcal{B}_+^\sigma = \frac{1}{\sqrt{-\eta}} \frac{\delta w_+}{\delta \mathcal{A}_+^{\mu\nu}} \left(\eta^{\mu\nu} \sqrt{\frac{\varepsilon}{\alpha}} \beta^\sigma + 2\zeta^\mu \eta^{\nu\sigma} \right), \quad \mathcal{B}_-^\sigma = \frac{1}{\sqrt{-\eta}} \frac{\delta w_-}{\delta \mathcal{A}_-^{\mu\nu}} \left(\eta^{\mu\nu} \sqrt{\frac{\varepsilon}{\alpha}} \beta^\sigma - 2\zeta^\mu \eta^{\nu\sigma} \right), \quad (49)$$

$$\mathcal{C}_+^\sigma = \frac{1}{\sqrt{-\eta}} \frac{\delta w_+}{\delta \mathcal{A}_+^{\mu\nu}} \left(\eta^{\mu\nu} \sqrt{\frac{\alpha}{\varepsilon}} \zeta^\sigma + 2\beta^\mu \eta^{\nu\sigma} \right), \quad \mathcal{C}_-^\sigma = \frac{1}{\sqrt{-\eta}} \frac{\delta w_-}{\delta \mathcal{A}_-^{\mu\nu}} \left(\eta^{\mu\nu} \sqrt{\frac{\alpha}{\varepsilon}} \zeta^\sigma - 2\beta^\mu \eta^{\nu\sigma} \right), \quad (50)$$

where $w_+ = w[g_{\mu\nu}^+, \varphi_+, \mu_+]$, $w_- = w[g_{\mu\nu}^-, \varphi_-, \mu_-]$.

The structure of (49), (50) follows because the effective metrics $g_{\mu\nu}^+$, $g_{\mu\nu}^-$ and dilatons φ_+ , φ_- can be expressed solely in terms of $\mathcal{A}_+^{\mu\nu}$, $\mathcal{A}_-^{\mu\nu}$, respectively. The details follow using

$$\delta w_+ = \int d^2x \frac{\delta w_+}{\delta \mathcal{A}_+^{\mu\nu}} \delta \mathcal{A}_+^{\mu\nu}, \quad \delta w_- = \int d^2x \frac{\delta w_-}{\delta \mathcal{A}_-^{\mu\nu}} \delta \mathcal{A}_-^{\mu\nu}, \quad (51)$$

with

$$\delta \mathcal{A}_+^{\mu\nu} = \left(\eta^{\mu\nu} \sqrt{\frac{\varepsilon}{\alpha}} \beta^\sigma + 2\zeta^{(\mu} \eta^{\nu)\sigma} \right) \partial_\sigma \delta \Phi + \left(\eta^{\mu\nu} \sqrt{\frac{\alpha}{\varepsilon}} \zeta^\sigma + 2\beta^{(\mu} \eta^{\nu)\sigma} \right) \partial_\sigma \delta \Psi, \quad (52)$$

$$\delta \mathcal{A}_-^{\mu\nu} = \left(\eta^{\mu\nu} \sqrt{\frac{\varepsilon}{\alpha}} \beta^\sigma - 2\zeta^{(\mu} \eta^{\nu)\sigma} \right) \partial_\sigma \delta \Phi + \left(\eta^{\mu\nu} \sqrt{\frac{\alpha}{\varepsilon}} \zeta^\sigma - 2\beta^{(\mu} \eta^{\nu)\sigma} \right) \partial_\sigma \delta \Psi, \quad (53)$$

which emerge from (29), (30). The parentheses enclosing indices denote symmetrisation with the standard weighting; e.g. $2\beta^{(\mu} \eta^{\nu)\sigma} = \beta^\mu \eta^{\nu\sigma} + \beta^\nu \eta^{\mu\sigma}$. As usual, the variations $\delta\Phi$, $\delta\Psi$ are chosen to have compact support; thus, no boundary terms arise during the derivation of the field equations.

The remainder of this section is focussed on determining the functional derivatives of w_+ , w_- with respect to $\mathcal{A}_+^{\mu\nu}$, $\mathcal{A}_-^{\mu\nu}$, respectively. To achieve this goal, it is fruitful to briefly return to the most natural variables for expressing w_+ , w_- ; the effective metrics and dilatons.

4.1 Variations of w with respect to $g^{\mu\nu}$ and φ

Since w_+ (or w_-) is simply w evaluated at particular values of its arguments, we can capture the variations of w_+ with respect to $g^{\mu\nu}$, φ_+ (or $g^{\mu\nu}$, φ_-) by appealing solely to the variations of w with respect to $g^{\mu\nu}$, φ .

Taking care of the inverse D'Alembertian operator \square^{-1} using the techniques given in Ref. [12], we find that the functional derivatives of w with respect to $g^{\mu\nu}$, φ are

$$\begin{aligned} \frac{48\pi}{\sqrt{-g}} \frac{\delta w}{\delta g^{\mu\nu}} &= -\nabla_\mu \nabla_\nu \square^{-1} R + g_{\mu\nu} R - \frac{1}{4} g_{\mu\nu} \nabla^\sigma \square^{-1} R \nabla_\sigma \square^{-1} R + \frac{1}{2} \nabla_\nu \square^{-1} R \nabla_\mu \square^{-1} R \\ &\quad + 3g_{\mu\nu} (\nabla\varphi)^2 \square^{-1} R + 3g_{\mu\nu} \nabla^\sigma \square^{-1} (\nabla\varphi)^2 \nabla_\sigma \square^{-1} R - 6\nabla_{(\nu} \square^{-1} (\nabla\varphi)^2 \nabla_{\mu)} \square^{-1} R \\ &\quad - 6(\nabla_\mu \varphi \nabla_\nu \varphi) \square^{-1} R + 6\nabla_\mu \nabla_\nu \square^{-1} (\nabla\varphi)^2 - 4\nabla_\mu \nabla_\nu \varphi + 4g_{\mu\nu} \square\varphi \\ &\quad + g_{\mu\nu} (-5 + 6 \ln \mu) (\nabla\varphi)^2 - 2(\nabla_\mu \varphi \nabla_\nu \varphi) (1 + 6 \ln \mu), \end{aligned} \quad (54)$$

$$\frac{12\pi}{\sqrt{-g}} \frac{\delta w}{\delta \varphi} = 3\nabla_\mu \square^{-1} R \nabla^\mu \varphi + (3\square^{-1} R + 1 + 6 \ln \mu) \square\varphi + R. \quad (55)$$

For convenience, indices have been lowered (or raised) using the metric tensor $g_{\mu\nu}$ (or its inverse $g^{\mu\nu}$) in (54), (55).

4.2 Variation of w with respect to $\mathcal{A}^{\mu\nu}$

The appropriate combination of (54), (55) that appears in the field equations (47) emerges upon introducing the variable $\mathcal{A}^{\mu\nu} = e^{-2\varphi} g^{\mu\nu}$, where $4\varphi = -\ln(A)$ with A the determinant of $A^\nu{}_\mu = \eta_{\mu\sigma} \mathcal{A}^{\sigma\nu}$. Hence, $\delta g^{\mu\nu} = e^{2\varphi} \delta \mathcal{A}^{\mu\nu} + 2g^{\mu\nu} \delta\varphi$ and $4\delta\varphi = -e^{2\varphi} g_{\mu\nu} \delta \mathcal{A}^{\mu\nu}$. It follows

$$\delta w = \int d^2x \left(\frac{\delta w}{\delta g^{\mu\nu}} \delta g^{\mu\nu} + \frac{\delta w}{\delta \varphi} \delta \varphi \right) = \int d^2x \left[\frac{\delta w}{\delta g^{\mu\nu}} - \frac{1}{4} \left(2 \frac{\delta w}{\delta g^{\sigma\tau}} g^{\sigma\tau} + \frac{\delta w}{\delta \varphi} \right) g_{\mu\nu} \right] e^{2\varphi} \delta \mathcal{A}^{\mu\nu} \quad (56)$$

and we obtain

$$\frac{1}{\sqrt{-\eta}} \frac{\delta w}{\delta \mathcal{A}^{\mu\nu}} = \frac{1}{\sqrt{-g}} \left[\frac{\delta w}{\delta g^{\mu\nu}} - \frac{1}{4} \left(2 \frac{\delta w}{\delta g^{\sigma\tau}} g^{\sigma\tau} + \frac{\delta w}{\delta \varphi} \right) g_{\mu\nu} \right] e^{2\varphi}. \quad (57)$$

The equality of the determinants of $\eta_{\mu\nu}$ and $g_{\mu\nu}$ has been used to express (57) in a convenient form.

Hence, the field equations (47) for Φ , Ψ are specified by substituting the functional derivatives found in (49), (50) with

$$\frac{1}{\sqrt{-\eta}} \frac{\delta w_+}{\delta \mathcal{A}_+^{\mu\nu}} = \frac{1}{\sqrt{-g^+}} \left[\frac{\delta w_+}{\delta g_+^{\mu\nu}} - \frac{1}{4} \left(2 \frac{\delta w_+}{\delta g_+^{\sigma\tau}} g_+^{\sigma\tau} + \frac{\delta w_+}{\delta \varphi_+} \right) g_{\mu\nu}^+ \right] e^{2\varphi_+}, \quad (58)$$

$$\frac{1}{\sqrt{-\eta}} \frac{\delta w_-}{\delta \mathcal{A}_-^{\mu\nu}} = \frac{1}{\sqrt{-g^-}} \left[\frac{\delta w_-}{\delta g_-^{\mu\nu}} - \frac{1}{4} \left(2 \frac{\delta w_-}{\delta g_-^{\sigma\tau}} g_-^{\sigma\tau} + \frac{\delta w_-}{\delta \varphi_-} \right) g_{\mu\nu}^- \right] e^{2\varphi_-}. \quad (59)$$

The right-hand side of (58) (or (59)) is given by substituting $g_{\mu\nu}^+$, φ_+ , μ_+ (or $g_{\mu\nu}^-$, φ_- , μ_-) into (54), (55).

5 Linearised field equations

The simplest exact solutions to (47) describe a uniform, monochromatic, laser beam propagating through a uniform plasma. In this case, β^μ , ζ^ν are covariantly constant with respect to the Levi-Civita connection $\nabla^{(\eta)}$ of the flat spacetime metric $\eta_{\mu\nu}$. Using (48), it is clear that the classical terms in (47) immediately vanish. The dilatons φ_+ , φ_- and effective metrics $g_{\mu\nu}^+$, $g_{\mu\nu}^-$ are constructed solely from tensors that are covariantly constant with respect to $\nabla^{(\eta)}$; thus, they are also covariantly constant with respect to $\nabla^{(\eta)}$. It follows that the components of the effective metrics are constant in a Minkowski coordinate system adapted to $\eta_{\mu\nu}$; thus, their Christoffel symbols are zero. In addition to the fact that the dilatons are constant, we conclude that the curvatures of the effective metrics are zero. Inspection of (54), (55) shows that the quantum corrections to the classical field equations are zero as required.

We will now uncover the impact of the quantum backreaction on perturbations to the exact solutions describing a uniform, monochromatic, laser beam propagating through a uniform plasma. Throughout the following, we will use a ‘bar’ to denote fields and operators associated with the unperturbed exact solutions. For simplicity, we will use Minkowski coordinates adapted to $\eta_{\mu\nu}$; thus, for the reasons given above, all components of ‘bar’ tensors are constant.

Introducing the substitutions

$$g^{\mu\nu} = \bar{g}^{\mu\nu} + g_{(1)}^{\mu\nu}, \quad \varphi = \bar{\varphi} + \varphi_{(1)} \quad (60)$$

in (54), (55) gives

$$\frac{48\pi}{\sqrt{-g}} \frac{\delta w}{\delta g^{\mu\nu}} = -\partial_\mu \partial_\nu \bar{\square}^{-1} R_{(1)} + \bar{g}_{\mu\nu} R_{(1)} - 4\partial_\mu \partial_\nu \varphi_{(1)} + 4\bar{g}_{\mu\nu} \bar{\square} \varphi_{(1)}, \quad (61)$$

$$\frac{12\pi}{\sqrt{-g}} \frac{\delta w}{\delta \varphi} = R_{(1)} + (1 + 6 \ln \mu) \bar{\square} \varphi_{(1)} \quad (62)$$

to first order in the perturbations $g_{(1)}^{\mu\nu}$, $\varphi_{(1)}$. The scalar curvature perturbation $R_{(1)}$ is

$$R_{(1)} = -\partial_\mu \partial_\nu g_{(1)}^{\mu\nu} + \bar{g}_{\mu\nu} \bar{\square} g_{(1)}^{\mu\nu}, \quad (63)$$

and $\bar{\square} = \bar{g}^{\mu\nu} \partial_\mu \partial_\nu$.

Hence, (47), (49), (50), (58), (59), (61), (62) together give

$$\begin{aligned} 0 = & \bar{\varepsilon} \bar{\square}_{(\eta)} \Phi_{(1)} + 2\bar{\beta}^\mu \bar{\zeta}^\nu \partial_\mu \partial_\nu \Psi_{(1)} \\ & + \frac{1}{96\pi} \sqrt{\frac{\bar{\varepsilon}}{\bar{\alpha}}} \bar{\beta}^\sigma \partial_\sigma \left\{ e^{2\bar{\varphi}^+} \left[-2(\bar{\square}_{(\eta)}/\bar{\square}^+) R_{(1)}^+ - c^+ R_{(1)}^+ - 8\bar{\square}_{(\eta)} \varphi_{(1)}^+ + 2c^+ (1 - 6 \ln \mu_+) \bar{\square}^+ \varphi_{(1)}^+ \right] \right. \\ & \quad \left. + e^{2\bar{\varphi}^-} \left[-2(\bar{\square}_{(\eta)}/\bar{\square}^-) R_{(1)}^- - c^- R_{(1)}^- - 8\bar{\square}_{(\eta)} \varphi_{(1)}^- + 2c^- (1 - 6 \ln \mu_-) \bar{\square}^- \varphi_{(1)}^- \right] \right\} \\ & + \frac{1}{48\pi} \bar{\zeta}^\mu \left\{ e^{2\bar{\varphi}^+} \left[-2(\bar{\square}_{(\eta)}/\bar{\square}^+) \partial_\mu R_{(1)}^+ - \bar{g}_{\mu\nu}^+ \partial^\nu R_{(1)}^+ - 8\partial_\mu \bar{\square}_{(\eta)} \varphi_{(1)}^+ + 2(1 - 6 \ln \mu_+) \bar{g}_{\mu\nu}^+ \partial^\nu \varphi_{(1)}^+ \right] \right. \\ & \quad \left. - e^{2\bar{\varphi}^-} \left[-2(\bar{\square}_{(\eta)}/\bar{\square}^-) \partial_\mu R_{(1)}^- - \bar{g}_{\mu\nu}^- \partial^\nu R_{(1)}^- - 8\partial_\mu \bar{\square}_{(\eta)} \varphi_{(1)}^- + 2(1 - 6 \ln \mu_-) \bar{g}_{\mu\nu}^- \partial^\nu \varphi_{(1)}^- \right] \right\}, \quad (64) \end{aligned}$$

and

$$\begin{aligned}
0 &= \bar{\alpha}\square_{(\eta)}\Psi_{(1)} + 2\bar{\beta}^{\mu}\bar{\zeta}^{\nu}\partial_{\mu}\partial_{\nu}\Phi_{(1)} \\
&+ \frac{1}{96\pi}\sqrt{\frac{\bar{\alpha}}{\bar{\varepsilon}}}\bar{\zeta}^{\sigma}\partial_{\sigma}\left\{e^{2\bar{\varphi}^+}\left[-2(\square_{(\eta)}/\bar{\square}^+)R_{(1)}^+ - c^+R_{(1)}^+ - 8\square_{(\eta)}\varphi_{(1)}^+ + 2c^+(1-6\ln\mu_+)\bar{\square}^+\varphi_{(1)}^+\right]\right. \\
&\quad \left.+ e^{2\bar{\varphi}^-}\left[-2(\square_{(\eta)}/\bar{\square}^-)R_{(1)}^- - c^-R_{(1)}^- - 8\square_{(\eta)}\varphi_{(1)}^- + 2c^-(1-6\ln\mu_-)\bar{\square}^-\varphi_{(1)}^-\right]\right\} \\
&+ \frac{1}{48\pi}\bar{\beta}^{\mu}\left\{e^{2\bar{\varphi}^+}\left[-2(\square_{(\eta)}/\bar{\square}^+)\partial_{\mu}R_{(1)}^+ - \bar{g}_{\mu\nu}^+\partial^{\nu}R_{(1)}^+ - 8\partial_{\mu}\square_{(\eta)}\varphi_{(1)}^+ + 2(1-6\ln\mu_+)\bar{g}_{\mu\nu}^+\partial^{\nu}\varphi_{(1)}^+\right]\right. \\
&\quad \left.- e^{2\bar{\varphi}^-}\left[-2(\square_{(\eta)}/\bar{\square}^-)\partial_{\mu}R_{(1)}^- - \bar{g}_{\mu\nu}^-\partial^{\nu}R_{(1)}^- - 8\partial_{\mu}\square_{(\eta)}\varphi_{(1)}^- + 2(1-6\ln\mu_-)\bar{g}_{\mu\nu}^-\partial^{\nu}\varphi_{(1)}^-\right]\right\}, \quad (65)
\end{aligned}$$

where $\partial^{\mu} = \eta^{\mu\nu}\partial_{\nu}$, $\square_{(\eta)} = \eta^{\mu\nu}\partial_{\mu}\partial_{\nu}$, $c^+ = \bar{g}_{+}^{\mu\nu}\eta_{\mu\nu}$, $c^- = \bar{g}_{-}^{\mu\nu}\eta_{\mu\nu}$, and

$$R_{(1)}^+ = -\partial_{\mu}\partial_{\nu}g_{+(1)}^{\mu\nu} + \bar{g}_{\mu\nu}^+\bar{\square}^+g_{+(1)}^{\mu\nu}, \quad R_{(1)}^- = -\partial_{\mu}\partial_{\nu}g_{-(1)}^{\mu\nu} + \bar{g}_{\mu\nu}^-\bar{\square}^-g_{-(1)}^{\mu\nu}. \quad (66)$$

The perturbations to the effective metrics and dilatons are given in terms of $\Phi_{(1)}$, $\Psi_{(1)}$ by

$$g_{+(1)}^{\mu\nu} = 2\varphi_{(1)}^+\bar{g}_{+}^{\mu\nu} + e^{2\bar{\varphi}^+}\mathcal{A}_{+(1)}^{\mu\nu}, \quad g_{-(1)}^{\mu\nu} = 2\varphi_{(1)}^-\bar{g}_{-}^{\mu\nu} + e^{2\bar{\varphi}^-}\mathcal{A}_{-(1)}^{\mu\nu}, \quad (67)$$

$$4\varphi_{(1)}^+ = -e^{2\bar{\varphi}^+}\bar{g}_{\mu\nu}^+\mathcal{A}_{+(1)}^{\mu\nu}, \quad 4\varphi_{(1)}^- = -e^{2\bar{\varphi}^-}\bar{g}_{\mu\nu}^-\mathcal{A}_{-(1)}^{\mu\nu}, \quad (68)$$

with

$$\mathcal{A}_{+(1)}^{\mu\nu} = \left(\eta^{\mu\nu}\sqrt{\frac{\bar{\varepsilon}}{\bar{\alpha}}}\bar{\beta}^{\sigma} + 2\bar{\zeta}^{(\mu}\eta^{\nu)\sigma}\right)\partial_{\sigma}\Phi_{(1)} + \left(\eta^{\mu\nu}\sqrt{\frac{\bar{\alpha}}{\bar{\varepsilon}}}\bar{\zeta}^{\sigma} + 2\bar{\beta}^{(\mu}\eta^{\nu)\sigma}\right)\partial_{\sigma}\Psi_{(1)}, \quad (69)$$

$$\mathcal{A}_{-(1)}^{\mu\nu} = \left(\eta^{\mu\nu}\sqrt{\frac{\bar{\varepsilon}}{\bar{\alpha}}}\bar{\beta}^{\sigma} - 2\bar{\zeta}^{(\mu}\eta^{\nu)\sigma}\right)\partial_{\sigma}\Phi_{(1)} + \left(\eta^{\mu\nu}\sqrt{\frac{\bar{\alpha}}{\bar{\varepsilon}}}\bar{\zeta}^{\sigma} - 2\bar{\beta}^{(\mu}\eta^{\nu)\sigma}\right)\partial_{\sigma}\Psi_{(1)}. \quad (70)$$

5.1 Plane-wave perturbations and their dispersion relations

Inspection of (64), (65) shows that the classical behaviour of the perturbations $\Phi_{(1)}$, $\Psi_{(1)}$ is determined by the linear equations

$$\bar{\varepsilon}\square_{(\eta)}\Phi_{(1)} + 2\bar{\beta}^{\mu}\bar{\zeta}^{\nu}\partial_{\mu}\partial_{\nu}\Psi_{(1)} = 0, \quad \bar{\alpha}\square_{(\eta)}\Psi_{(1)} + 2\bar{\beta}^{\mu}\bar{\zeta}^{\nu}\partial_{\mu}\partial_{\nu}\Phi_{(1)} = 0. \quad (71)$$

Using the plane-wave ansätze $\Phi_{(1)} \propto \exp(ikx)$, $\Psi_{(1)} \propto \exp(ikx)$ in (71), where $kx \equiv k_{\mu}x^{\mu}$, leads to the dispersion relation $\bar{\mathcal{A}}_{+}^{\mu\nu}k_{\mu}k_{\nu}\bar{\mathcal{A}}^{\sigma\omega}k_{\sigma}k_{\omega} = 0$ where $\bar{\mathcal{A}}_{+}^{\mu\nu} = \sqrt{\bar{\alpha}\bar{\varepsilon}}\eta^{\mu\nu} + 2\bar{\beta}^{(\mu}\bar{\zeta}^{\nu)}$, $\bar{\mathcal{A}}^{\mu\nu} = \sqrt{\bar{\alpha}\bar{\varepsilon}}\eta^{\mu\nu} - 2\bar{\beta}^{(\mu}\bar{\zeta}^{\nu)}$. Furthermore, the presence of $1/\bar{\square}^{\pm}$ within the quantum corrections in (64), (65) suggests that the terms denoted “...” inside the equations

$$\begin{aligned}
0 &= \bar{\varepsilon}\square_{(\eta)}\Phi_{(1)} + 2\bar{\beta}^{\mu}\bar{\zeta}^{\nu}\partial_{\mu}\partial_{\nu}\Psi_{(1)} \\
&+ \frac{1}{96\pi}\sqrt{\frac{\bar{\varepsilon}}{\bar{\alpha}}}\bar{\beta}^{\sigma}\partial_{\sigma}\left\{e^{2\bar{\varphi}^+}\left[-2(\square_{(\eta)}/\bar{\square}^+)R_{(1)}^+ + \dots\right]\right\} + \frac{1}{48\pi}\bar{\zeta}^{\mu}\left\{e^{2\bar{\varphi}^+}\left[-2(\square_{(\eta)}/\bar{\square}^+)\partial_{\mu}R_{(1)}^+ + \dots\right]\right\}, \quad (72)
\end{aligned}$$

$$\begin{aligned}
0 &= \bar{\alpha}\square_{(\eta)}\Psi_{(1)} + 2\bar{\beta}^{\mu}\bar{\zeta}^{\nu}\partial_{\mu}\partial_{\nu}\Phi_{(1)} \\
&+ \frac{1}{96\pi}\sqrt{\frac{\bar{\alpha}}{\bar{\varepsilon}}}\bar{\zeta}^{\sigma}\partial_{\sigma}\left\{e^{2\bar{\varphi}^+}\left[-2(\square_{(\eta)}/\bar{\square}^+)R_{(1)}^+ + \dots\right]\right\} + \frac{1}{48\pi}\bar{\beta}^{\mu}\left\{e^{2\bar{\varphi}^+}\left[-2(\square_{(\eta)}/\bar{\square}^+)\partial_{\mu}R_{(1)}^+ + \dots\right]\right\} \quad (73)
\end{aligned}$$

are negligible close to the classical solution satisfying $\bar{A}_+^{\mu\nu} k_\mu k_\nu = 0$. Likewise, the presence of $1/\bar{\square}^-$ within the quantum corrections in (64), (65) suggests that the terms denoted “...” inside

$$0 = \bar{\varepsilon} \square_{(\eta)} \Phi_{(1)} + 2\bar{\beta}^\mu \bar{\zeta}^\nu \partial_\mu \partial_\nu \Psi_{(1)} + \frac{1}{96\pi} \sqrt{\frac{\bar{\varepsilon}}{\bar{\alpha}}} \bar{\beta}^\sigma \partial_\sigma \{ e^{2\bar{\varphi}^-} [-2(\square_{(\eta)}/\bar{\square}^-) R_{(1)}^- + \dots] \} + \frac{1}{48\pi} \bar{\zeta}^\mu \{ -e^{2\bar{\varphi}^-} [-2(\square_{(\eta)}/\bar{\square}^-) \partial_\mu R_{(1)}^- + \dots] \}, \quad (74)$$

$$0 = \bar{\alpha} \square_{(\eta)} \Psi_{(1)} + 2\bar{\beta}^\mu \bar{\zeta}^\nu \partial_\mu \partial_\nu \Phi_{(1)} + \frac{1}{96\pi} \sqrt{\frac{\bar{\alpha}}{\bar{\varepsilon}}} \bar{\zeta}^\sigma \partial_\sigma \{ e^{2\bar{\varphi}^-} [-2(\square_{(\eta)}/\bar{\square}^-) R_{(1)}^- + \dots] \} + \frac{1}{48\pi} \bar{\beta}^\mu \{ -e^{2\bar{\varphi}^-} [-2(\square_{(\eta)}/\bar{\square}^-) \partial_\mu R_{(1)}^- + \dots] \} \quad (75)$$

are negligible close to the classical solution satisfying $\bar{A}_-^{\mu\nu} k_\mu k_\nu = 0$. Note that the unknown constants μ_+ , μ_- in (64), (65) do not contribute in this regime.

Focussing on (72), (73), the above considerations suggest a perturbative analysis of

$$0 = \bar{g}_+^{\mu\nu} k_\mu k_\nu (\bar{\varepsilon} k \cdot k \Phi_{(1)} + 2\bar{\beta} k \bar{\zeta} k \Psi_{(1)}) + \epsilon^2 \frac{e^{2\bar{\varphi}^+}}{48\pi} \left(\sqrt{\frac{\bar{\varepsilon}}{\bar{\alpha}}} i \bar{\beta} k + 2i \bar{\zeta} k \right) k \cdot k R_{(1)}^+ + \mathcal{O}(\epsilon^3), \quad (76)$$

$$0 = \bar{g}_+^{\mu\nu} k_\mu k_\nu (\bar{\alpha} k \cdot k \Psi_{(1)} + 2\bar{\beta} k \bar{\zeta} k \Phi_{(1)}) + \epsilon^2 \frac{e^{2\bar{\varphi}^+}}{48\pi} \left(\sqrt{\frac{\bar{\alpha}}{\bar{\varepsilon}}} i \bar{\zeta} k + 2i \bar{\beta} k \right) k \cdot k R_{(1)}^+ + \mathcal{O}(\epsilon^3), \quad (77)$$

where $\bar{g}_+^{\mu\nu} k_\mu k_\nu = \mathcal{O}(\epsilon)$, $k \cdot k \equiv \eta^{\mu\nu} k_\mu k_\nu$, $\bar{\beta} k \equiv \bar{\beta}^\mu k_\mu$, and $\bar{\zeta} k \equiv \bar{\zeta}^\mu k_\mu$. The perturbation parameter ϵ has been introduced for clarity of exposition, and the ϵ -orders of the terms have been allocated *a posteriori* so that the working is self-consistent. The parameter ϵ is merely a device for capturing perturbative orders, and can be set to unity at the end of the calculation. Thus, $R_{(1)}^+ = k_\mu k_\nu g_{(1)}^{\mu\nu} + \mathcal{O}(\epsilon)$ follows from (66), and so

$$\bar{g}_+^{\mu\nu} k_\mu k_\nu (\bar{\varepsilon} k \cdot k \Phi_{(1)} + 2\bar{\beta} k \bar{\zeta} k \Psi_{(1)}) - \epsilon^2 \frac{1}{48\pi} a (a \Phi_{(1)} + b \Psi_{(1)}) = \mathcal{O}(\epsilon^3), \quad (78)$$

$$\bar{g}_+^{\mu\nu} k_\mu k_\nu (\bar{\alpha} k \cdot k \Psi_{(1)} + 2\bar{\beta} k \bar{\zeta} k \Phi_{(1)}) - \epsilon^2 \frac{1}{48\pi} b (a \Phi_{(1)} + b \Psi_{(1)}) = \mathcal{O}(\epsilon^3) \quad (79)$$

emerge from (76), (77) using (69), where the quantities a , b are

$$a = \left(\sqrt{\frac{\bar{\varepsilon}}{\bar{\alpha}}} \bar{\beta} k + 2\bar{\zeta} k \right) k \cdot k e^{2\bar{\varphi}^+}, \quad b = \left(\sqrt{\frac{\bar{\alpha}}{\bar{\varepsilon}}} \bar{\zeta} k + 2\bar{\beta} k \right) k \cdot k e^{2\bar{\varphi}^+}. \quad (80)$$

Equations (78), (79) together form a homogeneous linear system for $\Phi_{(1)}$, $\Psi_{(1)}$. For a non-zero solution to exist, the condition

$$\left| \bar{g}_+^{\mu\nu} k_\mu k_\nu \begin{pmatrix} \bar{\varepsilon} k \cdot k & 2\bar{\beta} k \bar{\zeta} k \\ 2\bar{\beta} k \bar{\zeta} k & \bar{\alpha} k \cdot k \end{pmatrix} - \epsilon^2 \frac{1}{48\pi} \begin{pmatrix} a^2 & ab \\ ab & b^2 \end{pmatrix} \right| = \mathcal{O}(\epsilon^6) \quad (81)$$

on the matrix determinant of the coefficients of the linear system must be satisfied. Equation (81) is the dispersion relation

$$e^{2\bar{\varphi}^+} (\bar{A}_+^{\mu\nu} k_\mu k_\nu)^2 \bar{A}_-^{\sigma\omega} k_\sigma k_\omega - \epsilon^2 \left[\frac{k \cdot k}{48\pi} (a^2 \bar{\alpha} + b^2 \bar{\varepsilon}) - \frac{ab}{12\pi} \bar{\beta} k \bar{\zeta} k \right] = \mathcal{O}(\epsilon^6) \quad (82)$$

where $\bar{A}_+^{\mu\nu} k_\mu k_\nu = \sqrt{\bar{\alpha} \bar{\varepsilon}} k \cdot k + 2\bar{\beta} k \bar{\zeta} k$ and $\bar{A}_-^{\mu\nu} k_\mu k_\nu = \sqrt{\bar{\alpha} \bar{\varepsilon}} k \cdot k - 2\bar{\beta} k \bar{\zeta} k$ have been used. Introducing the substitution $2\bar{\beta} k \bar{\zeta} k = \bar{A}_+^{\mu\nu} k_\mu k_\nu - \sqrt{\bar{\alpha} \bar{\varepsilon}} k \cdot k$ in the final term of (82), and introducing $\bar{A}_-^{\mu\nu} k_\mu k_\nu = 2\sqrt{\bar{\alpha} \bar{\varepsilon}} k \cdot k - \bar{A}_+^{\mu\nu} k_\mu k_\nu$ in the first term, yields

$$e^{2\bar{\varphi}^+} (\bar{A}_+^{\mu\nu} k_\mu k_\nu)^2 - \epsilon^2 \frac{1}{96\pi} \left[a \left(\frac{\bar{\alpha}}{\bar{\varepsilon}} \right)^{\frac{1}{4}} + b \left(\frac{\bar{\varepsilon}}{\bar{\alpha}} \right)^{\frac{1}{4}} \right]^2 = \mathcal{O}(\epsilon^3) \quad (83)$$

since $\bar{\mathcal{A}}_+^{\mu\nu} k_\mu k_\nu = \mathcal{O}(\epsilon)$. The latter follows from $\bar{g}_+^{\mu\nu} k_\mu k_\nu = \mathcal{O}(\epsilon)$. Hence, substituting a, b using (80) gives

$$|\sqrt{\bar{\alpha}\bar{\epsilon}} k \cdot k + 2\bar{\beta}k \bar{\zeta}k| = \epsilon \frac{3}{4} \frac{1}{\sqrt{6\pi}} \frac{|k \cdot k|}{[(\sqrt{\bar{\alpha}\bar{\epsilon}} + \bar{\beta} \cdot \bar{\zeta})^2 - \bar{\beta} \cdot \bar{\beta} \bar{\zeta} \cdot \bar{\zeta}]^{1/4}} \left| \left(\frac{\bar{\alpha}}{\bar{\epsilon}} \right)^{\frac{1}{4}} \bar{\zeta}k + \left(\frac{\bar{\epsilon}}{\bar{\alpha}} \right)^{\frac{1}{4}} \bar{\beta}k \right| + \mathcal{O}(\epsilon^2) \quad (84)$$

where $e^{-4\bar{\varphi}_+} = (\sqrt{\bar{\alpha}\bar{\epsilon}} + \bar{\beta} \cdot \bar{\zeta})^2 - \bar{\beta} \cdot \bar{\beta} \bar{\zeta} \cdot \bar{\zeta}$ has been used to eliminate $\bar{\varphi}_+$, which follows because $e^{-4\bar{\varphi}_+}$ is equal to the determinant of the tensor $\bar{A}_+^{\nu\mu} = \eta_{\mu\sigma} \bar{\mathcal{A}}_+^{\sigma\nu}$.

The equivalent calculation using (74), (75) instead of (72), (73) yields

$$|\sqrt{\bar{\alpha}\bar{\epsilon}} k \cdot k - 2\bar{\beta}k \bar{\zeta}k| = \epsilon \frac{3}{4} \frac{1}{\sqrt{6\pi}} \frac{|k \cdot k|}{[(\sqrt{\bar{\alpha}\bar{\epsilon}} - \bar{\beta} \cdot \bar{\zeta})^2 - \bar{\beta} \cdot \bar{\beta} \bar{\zeta} \cdot \bar{\zeta}]^{1/4}} \left| \left(\frac{\bar{\alpha}}{\bar{\epsilon}} \right)^{\frac{1}{4}} \bar{\zeta}k - \left(\frac{\bar{\epsilon}}{\bar{\alpha}} \right)^{\frac{1}{4}} \bar{\beta}k \right| + \mathcal{O}(\epsilon^2) \quad (85)$$

since $\bar{\mathcal{A}}_-^{\mu\nu} k_\mu k_\nu = \mathcal{O}(\epsilon)$, and $e^{-4\bar{\varphi}_-} = (\sqrt{\bar{\alpha}\bar{\epsilon}} - \bar{\beta} \cdot \bar{\zeta})^2 - \bar{\beta} \cdot \bar{\beta} \bar{\zeta} \cdot \bar{\zeta}$ has been used to eliminate $\bar{\varphi}_-$. The latter is obtained because $e^{-4\bar{\varphi}_-}$ is equal to the determinant of the tensor $\bar{A}_-^{\nu\mu} = \eta_{\mu\sigma} \bar{\mathcal{A}}_-^{\sigma\nu}$. Finally, (39) yields the condition

$$-\bar{\beta} \cdot \bar{\zeta} > \sqrt{\bar{\alpha}\bar{\epsilon}} + \sqrt{\bar{\beta} \cdot \bar{\beta} \bar{\zeta} \cdot \bar{\zeta}} \quad (86)$$

on the unperturbed fields.

Henceforth, for notational convenience, we will drop $\mathcal{O}(\epsilon^2)$ from (84) and (85), set $\epsilon = 1$ in the remaining terms, and treat the results as equalities. The physical consequences of those results readily emerge by introducing an ultrarelativistic approximation for $\bar{\beta}^\mu$ via the asymptotic series $\bar{\beta}^\mu = \bar{\beta}_{[-1]}^\mu / \chi + \chi \bar{\beta}_{[1]}^\mu + \mathcal{O}(\chi^2)$. Like ϵ , the sole reason for introducing the parameter χ is to track the perturbative orders of terms, and it will be set to unity after use. Note that the condition $\bar{\beta}_{[-1]} \cdot \bar{\beta}_{[-1]} = 0$ is a key ingredient in our approach, and so $|\bar{\beta} \cdot \bar{\beta}| = \mathcal{O}(1)$, $|\bar{\beta} \cdot \bar{\zeta}| = \mathcal{O}(1/\chi)$. We find that the same result

$$|\bar{\zeta}k| = \sqrt{\chi} \frac{3}{8} \frac{1}{\sqrt{6\pi}} \left(\frac{\bar{\zeta} \cdot \bar{\zeta} + 1}{\bar{\alpha}_{[0]}} \right)^{\frac{1}{4}} \frac{|k \cdot k|}{\sqrt{|\bar{\zeta} \cdot \bar{\beta}_{[-1]}|}} + \mathcal{O}(\chi^{3/2}), \quad (87)$$

where $\bar{\alpha} = \bar{\alpha}_{[0]} + \mathcal{O}(\chi^2)$, emerges from both (84) and (85). Equating the coefficients of equal powers of χ throughout the condition (86) yields the requirement $|\bar{\zeta} \cdot \bar{\beta}_{[-1]}| > 0$, which is trivially satisfied because $\bar{\zeta} \cdot \bar{\zeta} < 0$ and $\bar{\beta}_{[-1]} \cdot \bar{\beta}_{[-1]} = 0$.

Equation (87) can be reduced further by orthogonally decomposing the spacetime metric $\eta^{\mu\nu}$ with respect to the timelike, unit normalised, vector $n^\mu = \bar{\zeta}^\mu / \sqrt{-\bar{\zeta} \cdot \bar{\zeta}}$. It follows $\eta^{\mu\nu} = -n^\mu n^\nu + n_\perp^\mu n_\perp^\nu$ where the spacelike vector n_\perp^μ is orthogonal to n^μ , i.e. $n_\perp \cdot n = 0$. Thus, (87) gives

$$|nk| = \sqrt{\chi} \frac{3}{8} \frac{1}{\sqrt{6\pi}} \left(\frac{\bar{\zeta} \cdot \bar{\zeta} + 1}{|\bar{\zeta} \cdot \bar{\zeta}|^3 \bar{\alpha}_{[0]}} \right)^{\frac{1}{4}} \frac{(n_\perp k)^2}{\sqrt{|n \cdot \bar{\beta}_{[-1]}|}} + \mathcal{O}(\chi^{3/2}). \quad (88)$$

Equation (88) has been simplified using $nk = \mathcal{O}(\chi^{1/2})$. The quantities $|nk|$ and $|n_\perp k|$ are the dimensionless frequency and dimensionless wavenumber, respectively, of the perturbations to the laser-plasma variables in the rest frame of the plasma electrons. Furthermore, the introduction of n^μ makes the physical content of the ultrarelativistic approximation used in passing from (84), (85) to (87) straightforward to appreciate. The quantity $|n \cdot \bar{\beta}|$ is the dimensionless frequency of the laser in the rest frame of the plasma electrons, and the ultrarelativistic approximation is admissible because the frequency of the laser is much larger than the plasma frequency.

Finally, we will now express (88) in terms of the dimensionful variables that were introduced in the context of the underlying 3-dimensional classical theory in Section 2. Dropping $\mathcal{O}(\chi^{3/2})$ from (88), and setting χ to unity, yields the dispersion relation

$$\omega = \frac{3}{8} \frac{1}{\sqrt{6\pi}} \left(\frac{\hbar e^2}{\epsilon_0 m_e^2 c^3 L_*^2} \right)^{\frac{1}{4}} \left[\frac{a_0^2}{(a_0^2 + 1)^3} \right]^{\frac{1}{4}} \frac{c^2 \kappa^2}{\sqrt{\omega_0 \omega_p}} \Big|_{x=y=0}. \quad (89)$$

The angular frequency ω and wavenumber κ of the perturbation in the rest frame of the plasma electrons are given by $\omega = c|nk|/l_*$ and $\kappa = |n_\perp k|/l_*$, respectively, where l_* is the inert length scale used in the construction of (19). The laser frequency ω_0 , plasma frequency ω_p , and dimensionless laser amplitude

$$a_0 = \frac{e\sqrt{\langle \mathbf{A}_0^2 \rangle}}{m_e c} \quad (90)$$

emerge using the substitutions

$$|n \cdot \bar{\beta}_{[-1]}| = \frac{l_* \omega_0}{c} \Big|_{x=y=0}, \quad \bar{a}_{[0]} = -\frac{\varepsilon_0 m_e^2 c^3 L_*^2 l_*^2 \omega_p^2}{\hbar e^2 c^2} \Big|_{x=y=0}, \quad \bar{\zeta} \cdot \bar{\zeta} = -(a_0^2 + 1)|_{x=y=0}, \quad (91)$$

which hold to lowest order in perturbation theory. The details of (91) follow from (20), $\beta_\mu = \partial_\mu \Phi$, $\zeta_\mu = \partial_\mu \Psi$,

$$(\partial_t \tilde{\Phi})^2 - c^2 (\nabla \tilde{\Phi})^2 = \omega_p^2, \quad (\partial_t \tilde{\Psi})^2 - c^2 (\nabla \tilde{\Psi})^2 - m_e^2 c^4 = e^2 c^2 \langle \mathbf{A}_0^2 \rangle, \quad (92)$$

and $\partial_x \tilde{\Phi}|_{x=y=0} = \partial_y \tilde{\Phi}|_{x=y=0} = 0$, $\partial_x \tilde{\Psi}|_{x=y=0} = \partial_y \tilde{\Psi}|_{x=y=0} = 0$. Equation (1) follows immediately from (89), upon replacing L_* with the width of the laser pulse.

Acknowledgements

We thank Robin W Tucker for useful discussions. This work was supported by the UK Engineering and Physical Sciences Research Council grant EP/N028694/1 (D.A.B., A.C., A.N.), the Cockcroft Institute (A.C.), and the Lancaster University Faculty of Science and Technology (C.F.). All of the results can be fully reproduced using the methods described in the paper.

References

- [1] <https://www.eli-beams.eu/>
- [2] A. Di Piazza, C. Müller, K.Z. Hatsagortsyan, C.H. Keitel, Rev. Mod. Phys. 84 (2012) 1177.
- [3] D.A. Burton, A. Noble, Contemp. Phys. 55 (2014) 110.
- [4] T.D. Arber, *et al.*, Plasma Phys. Control. Fusion 57 (2015) 113001.
- [5] C. Benedetti, *et al.*, Plasma Phys. Control. Fusion 60 (2018) 014002.
- [6] E. Esarey, C.B. Schroeder, W.P. Leemans, Rev. Mod. Phys. 81 (2009) 1229.
- [7] R.A. Cairns, A. Reitsma, R. Bingham, Phys. Plasmas 11 (2004) 766.
- [8] <https://www.xfel.eu/>
- [9] W. Kummer, D.V. Vassilevich, Phys. Rev. D 60 (1999) 084021.
- [10] W. Kummer, D.V. Vassilevich, Ann. Phys. (Leipzig) 8 (1999) 10 801.
- [11] D.V. Vassilevich, Phys. Rep. 388 (2003) 279.
- [12] A. Conroy, T. Koivisto, A. Mazumdar, A. Teimouri, Class. Quant. Grav. 32 (2015) 015024.