Algebras of operators on Banach spaces, and homomorphisms thereof

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„S en-sorsom legsötétebb, mert színe s alja poltron, Élet s mű között bolyong, mint aggály-aszú lélek: Életben mű riongat, a műben lét a sorsom — S a sír-írásom ez: se égimű, se földi élet.”

— Szentkuthy Miklós

Declaration

The research presented in this thesis has not been submitted for a higher degree elsewhere and is, to the best of my knowledge and belief, original and my own work, except as acknowledged herein.

- Chapter 2 contains joint work with Y. Choi and N. J. Laustsen [12].
- Chapter 3 contains work submitted for publication [34].
- Chapter 4 contains work accepted for publication [33].

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Abstract

This thesis concerns the theory of Banach algebras of operators on Banach spaces. The emphasis for most of the thesis is on the homomorphisms and perturbations of homomorphisms of such algebras.

Chapter 2 of this thesis is devoted to the study of perturbations of homomorphisms between Banach algebras. We say that a bounded linear map $\phi : A \to B$ between Banach algebras A and B is *δ-multiplicative*, if $\delta > 0$ and $\|\phi(ab) - \phi(a)\phi(b)\| \leq \delta\|a\|\|b\|$ for every $a, b \in A$. Intuitively, if ϕ is δ-multiplicative for a small δ then ϕ may be considered to be almost homomorphic. Another way of measuring how perturbed a bounded linear map is to see how near it is to the closed set of multiplicative bounded linear maps between A and B. One would like to explore, of course, the connection between these two notions. If a bounded linear map is ϵ-close to a genuine homomorphism in norm, then it is δ-multiplicative for some δ depending on ϵ and the norm of the map. More precisely:

Let A, B be Banach algebras and let $\epsilon, K > 0$. It is easy to check (see [38, Proposition 1.1]) that there is a $\delta > 0$ such that for any bounded linear map $\phi : A \to B$ with $\|\phi\| \leq K$, if there is a bounded linear multiplicative map $\psi : A \to B$ with $\|\phi - \psi\| < \delta$, then ϕ is ϵ-multiplicative.

The converse direction is far from being true, in general. Pairs of Banach algebras (A, B) which satisfy the converse direction, are said to have the *AMNM property*, see Definition 2.1.1 for a precise formulation. Although this topic has its roots in the

classical Hyers–Ulam stability theory, it was really K. Jarosz who initiated the study of this field in [36]. Later, B. E. Johnson in [37] studied the AMNM property of Banach algebras of the form $(A, \mathbb{K})$, where A is a commutative Banach algebra over some scalar field $\mathbb{K}$. He substantially extended the scope of his study in his seminal paper [38], where he now turned his attention to general pairs of Banach algebras. Arguably, the most important result of his paper is [38, Theorem 3.1], which states that if A is an amenable Banach algebra and B is a dual Banach algebra then (A, B) has the AMNM property. By carefully studying the proof of Johnson's theorem, we extend this result in Theorem 2.3.3. This allows us to prove the main result (Theorem 2.1.2) of Chapter 2, namely, that $(\mathcal{B}(E), \mathcal{B}(F))$ has the AMNM property whenever $E \in \{\ell_1, L_p[0,1] : 1 \leq p < \infty\}$ or E is a Banach space with a subsymmetric, shrinking Schauder basis, and F is a separable, reflexive Banach space. (Here $\mathcal{B}(E)$ denotes the Banach algebra of bounded linear operators on E.) This extends another result of Johnson; it was shown in [38, Proposition 6.3] that $(\mathcal{B}(\mathcal{H}), \mathcal{B}(\mathcal{H}))$ has the AMNM property, where $\mathcal{H}$ is a separable Hilbert space. We would like to draw the reader's attention to the fact that rather non-trivial methods are needed to obtain this latter result. The reason behind this is twofold: Firstly, $\mathcal{B}(E)$ is very rarely amenable for infinite-dimensional Banach spaces E, see [69]. Secondly, when considering bounded linear maps between $\mathcal{B}(E)$ and $\mathcal{B}(F)$, they are not assumed to be weak* - to - weak* continuous, even if both E and F are reflexive.

In the penultimate section of Chapter 2 we study δ-multiplicative maps $\varphi : \mathcal{A}(E) \to \mathcal{B}(F)$, where E and F are Banach spaces and $\mathcal{A}(E)$ denotes the Banach algebra of approximable operators on E. Inspired by a result of M. D. P. Daws ([17, Lemma 3.3.14]), we show that if E has a Schauder basis, φ is a δ-multiplicative, norm one linear map from $\mathcal{A}(E)$ to $\mathcal{B}(F)$ for a sufficiently small $\delta > 0$, then F has a closed subspace isomorphic to E.

In the last section of Chapter 2 we establish a connection between the second bounded Hochschild cohomology group of a Banach algebra A with coefficients in a Banach A-bimodule X, and the AMNM property of the pair $(A, A \ltimes X)$, where $A \ltimes X$ denotes the semi-direct product of A with X.

In Chapter 3 we study representations of Banach algebras of the form $\mathcal{B}(E)$. More precisely, we are interested in the following question:
Suppose E is a Banach space. If F is a non-zero Banach space and $\psi : \mathcal{B}(E) \to \mathcal{B}(F)$ is a surjective algebra homomorphism, is ψ necessarily injective? Positive answer to this question would imply — by a classical result of Eidelheit, see e.g. [15, Theorem 2.5.7] — that E and F are isomorphic. If E satisfies the property above for every non-zero Banach space F and every surjective algebra homomorphism $\psi : \mathcal{B}(E) \to \mathcal{B}(F)$, then we say that E has the *SHAI property* (Definition 3.1.1). As we shall see in Chapter 3, many of the classical Banach spaces and Hilbert spaces of arbitrary density character (that is, of arbitrarily large dimension) satisfy this property (see Example 3.3.8 and Theorem 3.4.8), as well as some more exotic ones (Corollary 3.3.12 and Theorem 3.4.6). We demonstrate in Proposition 3.5.3 that the SHAI property is stable under taking finite sums. We also give many examples for Banach spaces which fail to have this property, see Examples 3.2.2.
In the last section of Chapter 3, we study the question in the "opposite direction": Can we find infinite-dimensional Banach spaces E and F and a surjective algebra homomorphism $\psi : \mathcal{B}(E) \to \mathcal{B}(F)$ which is not injective? We shall see in Theorem 3.6.13 that for every separable, reflexive Banach space F it is possible to find a Banach space E and a surjective algebra homomorphism $\psi : \mathcal{B}(E) \to \mathcal{B}(F)$ which is not injective.

In Chapter 4 we study certain ring-theoretic properties of $\mathcal{B}(E)$, characterised by their idempotents. In the classical (non-commutative) ring-theoretic context the notions of *Dedekind-finiteness, Dedekind-infiniteness* and *proper infiniteness* are very well know. In a Banach algebraic context, there is a stronger notion than Dedekind-finiteness; this is having *stable rank one*, in the sense of Rieffel, see Definition 4.1.3 and Lemma 4.2.1. The study of these ring-theoretic properties of $\mathcal{B}(E)$ was laid out by N. J. Laustsen in [47]. In particular, it was shown in the aforementioned paper that $\mathcal{B}(E)$ is Dedekind-finite whenever E is a complex hereditarily indecomposable Banach space. We shall see in Corollary 4.2.9 that in fact more is true, for such a Banach space E, its algebra of operators $\mathcal{B}(E)$ has stable rank one. To complement this, we show in Theorems 4.2.16 and 4.3.5 that there are examples for both complex and real Banach spaces such that their algebras of operators are Dedekind-finite but they do not have stable rank one. The complex example is provided by the complexification of Tarbard's indecomposable but not hereditarily indecomposable space X_∞, and the real example is $C(K, \mathbb{R})$, where K is a connected Koszmider space.

In the last section of Chapter 4 we explore the connection between the existence of a certain unique maximal ideal in a Banach algebra and Dedekind-finiteness.

Table of contents

1 Preliminaries **1**

1.1 General background 1

1.2 Background material on Banach spaces and Banach algebras 5

2 The Johnson AMNM property and perturbations of homomorphisms **19**

2.1 Introduction and statement of the main results 19

2.2 Summary of background material for the AMNM property 20

2.2.1 Dual Banach algebras 20

2.2.2 Amenable Banach algebras, approximate identities 25

2.3 The main technical result 26

2.4 Proof of the main result 52

2.5 An approximate version of a lemma of Daws 57

2.6 The AMNM property and bounded Hochschild cohomology 63

3 The SHAI property for Banach spaces **71**

3.1 Introduction and preliminaries 71

3.2 Examples of Banach spaces without the SHAI property 76

3.3 The SHAI property for Banach spaces X where $\mathcal{E}(X)$ is a maximal ideal 78

3.4 The SHAI property for Banach spaces X where $\mathcal{E}(X)$ is not a maximal ideal 85

3.5 The SHAI property is stable under finite sums 92

3.6 Constructing surjective, non-injective homomorphisms from $\mathcal{B}(Y_X)$ to $\mathcal{B}(X)$ 96

3.6.1 First remarks 96

3.6.2 The construction 103

4 Finiteness and stable rank of algebras of operators on Banach spaces 113

4.1 Introduction and basic terminology 113

4.2 When $\mathcal{B}(X)$ is DF or it has stable rank one 115

4.3 A real $C(K)$-space example 124

4.4 On the existence of a certain maximal ideal in Banach algebras 128

5 List of Symbols and Index 135

List of Symbols 136

Index 141

References 145

Chapter 1

Preliminaries

In this chapter we introduce the necessary terminology and background, later to be used throughout this thesis.

1.1 General background

Numbers and sets

The symbol $\mathbb{N}$ stands for the natural numbers, excluding zero. We put $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. The first infinite cardinal is denoted by $\aleph_0$ and we refer to the cardinal $2^{\aleph_0}$ as the *continuum*. The symbols $\mathbb{R}$ and $\mathbb{C}$ stand for the fields of real and complex numbers, respectively. The usual Euclidean topologies on $\mathbb{R}$ and $\mathbb{C}$ will be denoted by $\mathcal{E}_{\mathbb{C}}$ and $\mathcal{E}_{\mathbb{R}}$, respectively.

If X is a set then $\mathcal{P}(X)$ denotes its power set, and $|X|$ denotes the cardinality of X. If X, Y are sets then Y^X is the set of functions from X to Y. If Y is a vector space and $f : X \to Y$ is a function we put $\operatorname{supp}(f) := \{x \in X : f(x) \neq 0\}$.

If I is a set and for every $i \in I$, X_i is a set then the Cartesian product will be denoted by $\prod_{i \in I} X_i$. An element of $\prod_{i \in I} X_i$ will be called a *system*, and when I is directed it will be called a *net*. If X is a set and for all $i \in I$, $X_i = X$ then $\prod_{i \in I} X_i$ will be identified with X^I. If $I = \{1, \ldots, n\}$ for some $n \in \mathbb{N}$ then we may write $\prod_{i \in I} X_i$ as $X_1 \times X_2 \times \cdots \times X_n$. Let Γ be a set. A family $\mathcal{F} \subseteq \mathcal{P}(\Gamma)$ is called *almost disjoint* if for any distinct $A, B \in \mathcal{F}$ the set $A \cap B$ is finite. The following lemma is standard, see [2, Lemma 2.5.3] for a proof that is different to the one given below.

Lemma 1.1.1. *There exists an almost disjoint family of continuum cardinality consisting of infinite subsets of the natural numbers.*

Proof. It is clear that the set $\Gamma := \{0, 1\}^{\mathbb{N}}$ has continuum cardinality. Let us fix $w := (w_i)_{i \in \mathbb{N}} \in \Gamma$. The set $Z_w := \{(w_i)_{i=1}^n : n \in \mathbb{N}\}$ is countably infinite and clearly $Z_w \subseteq \Gamma_<$, where $\Gamma_< := \bigcup_{n \in \mathbb{N}} \{0, 1\}^n$. Let $w, w' \in \Gamma$ be distinct, then there is a $j \in \mathbb{N}$ with $w_j \neq w'_j$. Hence, for every $n \geq j$, $(w_i)_{i=1}^n \neq (w'_i)_{i=1}^n$, showing that $Z_w \cap Z_{w'}$ is finite. In particular, let $\mathcal{P}_\infty(\Gamma_<)$ denote the set consisting of infinite subsets of $\Gamma_<$, then

$$Z : \Gamma \to \mathcal{P}_\infty(\Gamma_<); \quad w \mapsto Z_w \tag{1.1}$$

is a well-defined injection. Thus $\operatorname{Ran}(Z)$ is an almost disjoint family of continuum cardinality consisting of infinite subsets of $\Gamma_<$.
Since $|\Gamma_<| = \aleph_0$, there is a bijection $\phi : \Gamma_< \to \mathbb{N}$. The set $\mathcal{D} := \{\phi[A] : A \in \operatorname{Ran}(Z)\}$ is therefore as required. □

Ultrafilters, ultralimits

Let X be a set. We say that a non-empty set $\mathcal{F} \subseteq \mathcal{P}(X)$ is a *filter* on X if the empty set does not belong to $\mathcal{F}$, and $\mathcal{F}$ is closed under finite intersections and under supersets. This latter means that whenever $A \in \mathcal{F}$ and $B \in \mathcal{P}(X)$ are such that $A \subseteq B$ then $B \in \mathcal{F}$. An *ultrafilter* $\mathcal{U}$ on X is a filter on X such that for every $A \in \mathcal{P}(X)$ either $A \in \mathcal{U}$ or $X \backslash A \in \mathcal{U}$. If $\mathcal{F}$ is a filter on a set X and $\mathcal{U}$ is an ultrafilter on X with

$\mathcal{F} \subseteq \mathcal{U}$, then we say that $\mathcal{U}$ *extends* $\mathcal{F}$. The following result is well-known, its proof relies on Zorn's Lemma.

Lemma 1.1.2. *Let X be a set and let $\mathcal{F}$ be a filter on X. There exists an ultrafilter $\mathcal{U}$ on X such that $\mathcal{F} \subseteq \mathcal{U}$.*

Let X be a non-empty set and let $a \in X$ be arbitrary, it is easy to see that $U_a := \{A \in \mathcal{P}(X) : a \in A\}$ is an ultrafilter on X. We say that an ultrafilter $\mathcal{U}$ is *fixed* on a non-empty set X if there is $a \in X$ such that $\mathcal{U} = U_a$. If X is a finite set then every ultrafilter on X is fixed. An ultrafilter $\mathcal{U}$ on X is *free* if it is not fixed.
On every infinite set it is possible to "find" a free ultrafilter. To see this, let X be an infinite set and define $\mathcal{F}_f := \{A \in \mathcal{P}(X) : |X \backslash A| < \infty\}$. It is easy to see that $\mathcal{F}_f$ is a filter on X, called the *Fréchet filter*. It is not an ultrafilter however since X has an infinite subset such that its complement is also infinite. From Lemma 1.1.2 it follows that there exists an ultrafilter $\mathcal{U}$ on X with $\mathcal{F}_f \subseteq \mathcal{U}$. Now $\mathcal{U}$ is a free ultrafilter. For assume towards a contradiction that $\mathcal{U}$ is fixed, this is, there exists $a \in X$ such that $\mathcal{U} = U_a$, so in particular $\mathcal{F}_f \subseteq U_a$. Since $X \backslash \{a\} \in \mathcal{F}_f$, it follows that $X \backslash \{a\} \in U_a$, equivalently $a \in X \backslash \{a\}$, which is nonsense.
Let X be a topological space and let $x \in X$. Let $(x_i)_{i \in I}$ be a system in X and let $\mathcal{U}$ be an ultrafilter on I. We say that $(x_i)_{i \in I}$ *converges to x along $\mathcal{U}$* if for every neighbourhood $V \subseteq X$ of x we have $\{i \in I : x_i \in V\} \in \mathcal{U}$. We will denote this by $x = \lim_{i \to \mathcal{U}} x_i$, and we will say that *$x$ is an ultralimit of $(x_i)_{i \in I}$ along $\mathcal{U}$*. Let X be a Hausdorff topological space, let $(x_i)_{i \in I}$ be a system in X and let $\mathcal{U}$ be an ultrafilter on I such that $\lim_{i \to \mathcal{U}} x_i$ exists. Then $\lim_{i \to \mathcal{U}} x_i$ is unique.
Let X, Y be topological spaces and let $f : X \to Y$ be a continuous function. If $(x_i)_{i \in I}$ is a system in X and $\mathcal{U}$ is an ultrafilter on I such that $(x_i)_{i \in I}$ converges to $x \in X$ along $\mathcal{U}$, then it is immediate that $(f(x_i))_{i \in I}$ converges to $f(x) \in Y$ along $\mathcal{U}$.

The following basic lemma is of paramount importance for us in the thesis, for a proof we refer the reader to [1, Lemma 1.5.9].

Lemma 1.1.3. *Let X be a compact topological space, and let $(x_i)_{i\in I}$ be a net in X. If $\mathcal{U}$ is an ultrafilter on I then the ultralimit $\lim_{i\to\mathcal{U}} x_i$ exists.*

We shall now connect convergence of a net in a topological space in the usual sense to convergence along ultrafilters. To establish this, we will need the following:
Let I be a directed set. For any $i \in I$ we define $A_i := \{j \in I : j \geq i\}$. Then the set

$$\mathcal{F}_{\text{ord}} := \{S \in \mathcal{P}(I) : (\exists i \in I)(A_i \subseteq S)\} \tag{1.2}$$

is easily seen to be a filter on I, called the *order filter*.
Let X be a topological space and let $(x_i)_{i\in I}$ be a net in X converging to $x \in X$. Let $V \subseteq X$ be a neighbourhood of x, then there is $i \in I$ such that $x_j \in V$ for all $j \in I$ with $j \geq i$. With the notation above, this is equivalent to saying that $A_i \subseteq \{j \in I : x_j \in V\}$, thus $\{j \in I : x_j \in V\} \in \mathcal{F}_{\text{ord}}$. In particular, if $\mathcal{U}$ is an ultrafilter on I with $\mathcal{F}_{\text{ord}} \subseteq \mathcal{U}$ then $\{j \in I : x_j \in V\} \in \mathcal{U}$. By definition this is equivalent to $x = \lim_{i\to\mathcal{U}} x_i$.

Idempotents in rings

In a ring R, $p \in R$ is called *idempotent* if $p^2 = p$. It is easy to see that $p \in R$ is an idempotent in a unital ring if and only if $(2p-1)^2 = 1$. Two idempotents $p, q \in R$ are *equivalent*, denoted by $p \sim q$, if there exist $a, b \in R$ such that $ab = p$ and $ba = q$. It is easy to see that $\sim$ is an equivalence relation on the set of idempotent elements of R.

1.2 Background material on Banach spaces and Banach algebras

Basic geometry of normed spaces

All topological vector spaces and algebras are assumed to be over $\mathbb{C}$, unless explicitly stated otherwise. If E is a normed space, $r > 0$ and $x \in E$ then we set $B_r(x; E) := \{y \in E : \|x - y\| < r\}$. Since E is a normed space, it follows that the norm-closure of $B_r(x; E)$, denoted by $\overline{B}_r(x; E)$, equals to $\{y \in E : \|x - y\| \leq r\}$. It is well-known that in a normed space E, for any $r > 0$, the ball $\overline{B}_r(0; E)$ is compact if and only if E is finite-dimensional.

Let E, F be normed spaces. A map $\phi : E \to F$ is an *isometry* if $\|\phi(x)\| = \|x\|$ for every $x \in E$.

Quotients of Banach spaces and algebras

Let E be a normed space, let F be a linear subspace of E. Let $\pi : E \to E/F$ be the quotient map. Then the vector space E/F is a semi-normed space endowed with the semi-norm

$$\|\pi(x)\|_{E/F} := \inf\{\|x - y\| : y \in F\} \quad (x \in E). \tag{1.3}$$

It is well-known that $\|\cdot\|_{E/F}$ is a norm on E/F if and only if F is closed. If E is a normed space and F is a closed linear subspace of E then $\pi : E \to E/F$ is an open mapping and $\|\pi\| = 1$. Moreover, E is a Banach space if and only if E/F and F are Banach spaces.

If A is a Banach algebra and I is a closed, two-sided ideal in A then A/I is a Banach algebra with respect to the norm $\| \cdot \|_{A/I}$. The following result is standard (see [32, Proposition 21.3] or [45, Theorem 2.3.3]), we shall use it frequently in this thesis.

Theorem 1.2.1 (The Fundamental Isomorphism Theorem)**.** *Let E, F be normed spaces [algebras] and let $\psi : E \to F$ be a continuous linear map [algebra homomorphism]. Let G be a closed linear subspace [two-sided ideal] of E with $G \subseteq \mathrm{Ker}(\psi)$. Then there exists a unique continuous linear mapping [algebra homomorphism] $\varphi : E/G \to F$ such that $\varphi \circ \pi = \psi$ (where $\pi : E \to E/G$ is the quotient map) and $\|\varphi\| = \|\psi\|$. Moreover, φ is invertible if and only if $G = \mathrm{Ker}(\psi)$ and ψ is an open mapping.*

The dual space; weak- and weak* topologies

If E is a normed space, then for its *dual space* we write E^*; that is, the normed space of norm - to - $\mathcal{E}_{\mathbb{C}}$ continuous linear maps on E with values in $\mathbb{C}$. In line with the usual convention, we introduce the notation $E^{**} := (E^*)^*$. In the following $\langle \cdot, \cdot \rangle$ denotes the duality pairing; that is, $\langle x, f \rangle := f(x)$ whenever $x \in E$ and $f \in E^*$.

Let E be a normed space, let $x \in E$ be fixed. Then let $\kappa_E(x) : E^* \to \mathbb{C}$; $f \mapsto \langle x, f \rangle$, it is easy to see that $\kappa_E(x) \in E^{**}$. Thus it follows that $\kappa_E : E \to E^{**}$; $x \mapsto \kappa_E(x)$ is a bounded linear operator. In fact from the Hahn–Banach Extension Theorem we obtain that κ_E is an isometry. A Banach space E is *reflexive* if κ_E is surjective.

Let E be a complex vector space and let F be a linear subspace of the vector space of linear functionals on E. Then $\sigma(E, F)$ will denote the smallest linear topology τ on E such that every $f \in F$ is τ - to - $\mathcal{E}_{\mathbb{C}}$ continuous.

In particular, when E is a normed space, $\sigma(E, E^*)$ is called the *weak topology of E.* A net $(x_i)_{\in I}$ in E converges to $x \in E$ in the weak topology and only if $\lim_{i \in I} \langle x_i, f \rangle = \langle x, f \rangle$ for all $f \in E^*$.

If E is a normed space, then $\sigma(E^*, \mathrm{Ran}(\kappa_E))$ is called the *weak* topology of E^**, and following the usual convention we will simply denote this by $\sigma(E^*, E)$.

A net $(f_i)_{i\in I}$ converges to $f \in E^*$ in the weak* topology if and only if $\lim_{i\in I}\langle x, f_i\rangle = \langle x, f\rangle$ for all $x \in E$.

Operators on Banach spaces

The identity operator on a vector space E is denoted by I_E. If A is a Banach algebra, then the identity operator on A is denoted by id_A.

If E, F are normed spaces then $\mathcal{B}(E, F)$ denotes the normed space of bounded linear operators from E to F. We denote $\mathcal{B}(E, E)$ simply by $\mathcal{B}(E)$. For $T \in \mathcal{B}(E, F)$ its adjoint is denoted by T^*. If G, H are linear subspaces of E and F, respectively, then for $T \in \mathcal{B}(E, F)$ we denote the restriction of T to G by $T|_G$, clearly $T|_G \in \mathcal{B}(G, F)$. If $\mathrm{Ran}(T) \subseteq H$ then $T|^H$ denotes T considered as a bounded linear operator between E and H, that is, $T|^H \in \mathcal{B}(E, H)$.

If E, F are normed spaces and $T \in \mathcal{B}(E, F)$, we say that T *is bounded below* if there exists $c > 0$ such that for all $x \in E$, $c\|x\| \leq \|Tx\|$. If E, F are Banach spaces, it follows from the Banach Isomorphism Theorem that T is bounded below if and only if $\mathrm{Ran}(T)$ is closed and T is injective; consequently, $T|^{\mathrm{Ran}(T)} \in \mathcal{B}(E, \mathrm{Ran}(T))$ is a linear homeomorphism of Banach spaces.

If E, F are normed spaces and $x \in F$, $\varphi \in E^*$ then we define $x \otimes \varphi :\ E \to F;\ y \mapsto \langle y, \varphi\rangle x$. It is clear that $x \otimes \varphi \in \mathcal{B}(E, F)$ is rank-one with $\|x \otimes \varphi\| = \|x\|\|\varphi\|$, whenever $x \in X$ and $\varphi \in X^*$ are non-zero.

If E is a Banach space then a linear subspace F of E is *complemented* if there exists an idempotent $P \in \mathcal{B}(E)$ such that $F = \mathrm{Ran}(P)$. In particular, a complemented subspace is necessarily closed. It follows from the Hahn–Banach Extension Theorem that a finite-dimensional subspace of a Banach space is automatically complemented.

If E is a Banach space then there exist closed linear subspaces and F, G of E with $E = F + G$ and $F \cap G = \{0\}$ if and only if F and G are complements of each other in

E in the sense that there exists an idempotent $P \in \mathcal{B}(E)$ such that $\mathrm{Ran}(P) = F$ and $\mathrm{Ker}(P) = G$.

Two Banach spaces E and F are said to be *isomorphic* if there is a linear homeomorphism between E and F, it will be denoted by $E \simeq F$.

By an *isomorphism of Banach algebras* A and B we understand that there is an algebra homomorphism between A and B which is also a homeomorphism. This will also be denoted by $A \simeq B$. Throughout this thesis, whenever two Banach spaces [algebras] E and F are isometrically isomorphic, we shall freely identify them when it does not cause any confusion.

For a unital algebra A, the group of invertible elements in A will be denoted by $\mathrm{inv}(A)$. As is well-known, $\mathrm{inv}(A)$ is an open subset of A whenever A is a Banach algebra.

A *character* on a complex unital Banach algebra A is a unit-preserving algebra homomorphism from A to $\mathbb{C}$. Any such character is necessarily of norm one.

Finite sums of Banach spaces

The finite *direct sum* of Banach spaces $\{E_i\}_{i=1}^n$ will be denoted by $\bigoplus_{i=1}^n E_i$, and is defined as follows. As a vector space, it is the set $\prod_{i=1}^n E_i$ equipped with the pointwise addition and pointwise scalar multiplication. For $1 \leq p < \infty$, one can define $\|(x_i)_{i=1}^n\|_p := \left(\sum_{i=1}^n \|x_i\|^p\right)^{1/p}$ and $\|(x_i)_{i=1}^n\|_\infty := \max_{1 \leq i \leq n} \|x_i\|$ (where $(x_i)_{i=1}^n \in \bigoplus_{i=1}^n E_i$), they are easily seen to be equivalent complete norms on $\bigoplus_{i=1}^n E_i$. When we want to specify the norm, we write $\left(\bigoplus_{i=1}^n E_i\right)_p$, where $p \in [1, \infty]$.

Let E be a Banach space, let F be a finite-dimensional subspace of E and let G be a closed subspace of E with $F \cap G = \{0\}$. Then $F + G$ is a closed linear subspace of E, and in particular, $F + G \simeq F \oplus G$ as Banach spaces.

If E is a Banach space with $E \simeq E \oplus E$ we say that E *is isomorphic to its square.*

An infinite-dimensional Banach space X is *indecomposable*, if there are no closed, infinite-dimensional subspaces Y, Z of X such that $X \simeq Y \oplus Z$. A Banach space X is *hereditarily indecomposable* or *HI* if every closed, infinite-dimensional subspace of X is indecomposable.

Unitisation of Banach algebras

Let A be a Banach algebra over $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. The *(forced) unitisation of* A, denoted by $A^\sharp$, is the Banach space $\mathbb{K} \oplus_1 A$ endowed with the algebra product

$$(\lambda, a) \cdot (\mu, b) := (\lambda\mu, \lambda b + \mu a + ab) \quad (\lambda, \mu \in \mathbb{K},\ a, b \in A). \tag{1.4}$$

Now $A^\sharp$ is easily seen to be a unital Banach algebra.
If A is a unital Banach algebra and I is a proper, closed, two-sided ideal of A then $I^\sharp \simeq \mathbb{K}.1_A + I$ as Banach algebras.

Ideals of operators on Banach spaces

Definition 1.2.2. An *operator ideal* is an assignment $\mathcal{J}$ which associates to each pair of Banach spaces (E, F) a linear subspace $\mathcal{J}(E, F)$ of $\mathcal{B}(E, F)$ such that

- There exist Banach spaces E_0 and F_0 such that $\mathcal{J}(E_0, F_0)$ is non-zero;
- for any E, F, G, H Banach spaces, any $T \in \mathcal{B}(E, F)$, $S \in \mathcal{J}(F, G)$, $R \in \mathcal{B}(G, H)$ it follows that $R \circ S \circ T \in \mathcal{J}(E, H)$.

If E is a Banach space and $\mathcal{J}$ is an operator ideal we write $\mathcal{J}(E) := \mathcal{J}(E, E)$. Clearly $\mathcal{J}(E)$ is a two-sided ideal in $\mathcal{B}(E)$. If $\mathcal{J}_1, \mathcal{J}_2$ are operator ideals, we write $\mathcal{J}_1 \subseteq \mathcal{J}_2$ if for every pair of Banach spaces (E, F) the containment $\mathcal{J}_1(E, F) \subseteq \mathcal{J}_2(E, F)$ holds. If $\mathcal{J}$ is an operator ideal and E, F are Banach spaces and $\overline{\mathcal{J}}(E, F)$ denotes the (operator)norm-closure of $\mathcal{J}(E, F)$ then one can easily check that the assignment $\overline{\mathcal{J}}$ defines an operator ideal. We say that an operator ideal $\mathcal{J}$ is *closed* if $\mathcal{J} = \overline{\mathcal{J}}$.

Let E, F be Banach spaces, let $T \in \mathcal{B}(E,F)$. Then T is a *finite-rank operator* if $\mathrm{Ran}(T)$ is finite-dimensional. The symbol $\mathcal{F}(E,F)$ stands for the set of finite-rank operators on E. It is well-known that $\mathcal{F}$ is the smallest operator ideal, see for example [63, Theorem 1.2.2]. Moreover for a Banach space E, we have that $\mathcal{F}(E)$ is proper if and only if E is infinite-dimensional if and only if $\mathcal{F}(E)$ is non-closed.

Let E, F be infinite-dimensional Banach spaces. The symbol $\mathcal{A}(E,F)$ stands for the (operator)norm-closure of $\mathcal{F}(E,F)$. It is clear that $\mathcal{A}$ is the smallest closed operator ideal. An element of $\mathcal{A}(E,F)$ is called an *approximable operator*.

We say that $T \in \mathcal{B}(E,F)$ is a *compact operator* if $T[\overline{B}_1(0;E)]$ is a relatively compact subset of F with respect to the operator norm; the set of compact operators from E to F is denoted by $\mathcal{K}(E,F)$. It is known that $\mathcal{K}$ is a closed operator ideal, see [63, Theorem 1.4.2].

We say that $T \in \mathcal{B}(E,F)$ is *strictly singular* if there is no infinite-dimensional subspace W of E such that $T|_W \in \mathcal{B}(W,F)$ is bounded below. The set of strictly singular operators from E to F is denoted by $\mathcal{S}(E,F)$.

We say that $T \in \mathcal{B}(E,F)$ is *inessential* if for every $S \in \mathcal{B}(F,E)$ it follows that $I_E + ST$ is a *Fredholm operator*; that is, $\dim(\mathrm{Ker}(I_E+ST)) < \infty$ and $\mathrm{codim}_E(\mathrm{Ran}(I_E+ST)) < \infty$. The set of inessential operators from E to F is denoted by $\mathcal{E}(E,F)$. We remark in passing that it was shown by Pietsch in [62] that only one of the above conditions in needed for an operator to be inessential: $T \in \mathcal{E}(E,F)$ if and only if for every $S \in \mathcal{B}(F,E)$, $\dim(\mathrm{Ker}(I_E + ST)) < \infty$ holds.

We say that $T \in \mathcal{B}(E,F)$ is *weakly compact* if $T[\overline{B}_1(0;E)]$ is a precompact subset of F in the relative weak topology. The set of weakly compact operators from E to F is denoted by $\mathcal{W}(E,F)$.

Lastly, the set of operators with *separable range* from E to F is denoted by $\mathcal{X}(E,F)$. It is well-known that $\mathcal{S}, \mathcal{E}, \mathcal{W}, \mathcal{X}$ are closed operator ideals, and the containments

$\mathcal{A} \subseteq \mathcal{K} \subseteq \mathcal{S} \subseteq \mathcal{E}$ and $\mathcal{K} \subseteq \mathcal{W} \cap \mathcal{X}$ hold. We refer the interested reader to [63] and [10].
A Banach space E has the *approximation property* if for every $\epsilon > 0$ and for every compact set $K \subseteq E$ there exists $T \in \mathcal{F}(E)$ such that $\sup_{x\in K} \|Tx - x\| < \epsilon$. A Banach space E has the *bounded approximation property* if there exists $C > 0$, independently of ϵ and K, such that T can be chosen with the property $\|T\| \leq C$. A Banach space has the *metric approximation property* if it has the bounded approximation property with $C = 1$. A Banach space F has the approximation property if and only if for every Banach space E we have $\mathcal{A}(E, F) = \mathcal{K}(E, F)$, see for example [70, Proposition 4.12].

Schauder bases in Banach spaces

In a Banach space E, a sequence of vectors $(b_n)_{n\in\mathbb{N}}$ is called a *Schauder basis* or *basis*, if for every $x \in E$ there exists a unique sequence $(\alpha_n)_{n\in\mathbb{N}}$ in $\mathbb{C}$ such that the series $\sum_{n\in\mathbb{N}} \alpha_n b_n$ converges in E with limit x. The basis $(b_n)_{n\in\mathbb{N}}$ is *normalised* if $\|b_n\| = 1$ for all $n \in \mathbb{N}$. We remark in passing that if a Banach space has a basis then it also has the approximation property, see [56, Theorem 4.1.33].
We say that a sequence $(y_n)_{n\in\mathbb{N}}$ in a Banach space E is a *basic sequence* if it is a basis for the closed linear span of $\{y_n\}_{n\in\mathbb{N}}$.
Let E be a Banach space with Schauder basis $(b_n)_{n\in\mathbb{N}}$. For all $n \in \mathbb{N}$

$$f_n : E \to \mathbb{C}; \quad \sum_{i\in\mathbb{N}} \alpha_i b_i \mapsto \alpha_n \tag{1.5}$$

is called the n^{th} *coordinate functional* and

$$P_n : E \to E; \quad \sum_{i\in\mathbb{N}} \alpha_i b_i \mapsto \sum_{i=1}^{n} \alpha_i b_i \tag{1.6}$$

is called the n^{th} *coordinate projection associated to* $(b_n)_{n\in\mathbb{N}}$. It is clear that for every $n \in \mathbb{N}$ the map P_n is a linear idempotent map on E. Moreover, for every $n \in \mathbb{N}$, it follows that $P_n \in \mathcal{B}(E)$ (see [56, Theorem 4.1.15]) and since $(P_n)_{n\in\mathbb{N}}$ converges to I_E in the strong operator topology by its very definition, it follows from the Uniform Boundedness Principle that $K_b := \sup\{\|P_n\| \,:\, n \in \mathbb{N}\} < \infty$. For all $n \in \mathbb{N}$ one immediately obtains $\|f_n\| \leq 2K_b/\|b_n\|$, see for example [56, Corollary 4.1.16], thus $f_n \in E^*$.

The basis $(b_n)_{n\in\mathbb{N}}$ is *monotone* if $K_b = 1$, or equivalently, $\|P_n\| = 1$ for all $n \in \mathbb{N}$. By passing to an equivalent norm it can be always arranged that the basis is monotone, see for example, [56, Theorem 4.1.14].

We recall that a series $\sum\limits_{n\in\mathbb{N}} x_n$ in a Banach space E *converges unconditionally*, if for every σ permutation of $\mathbb{N}$ the series $\sum\limits_{n\in\mathbb{N}} x_{\sigma(n)}$ also converges in E. We remark here (see [56, Proposition 4.2.1]) that if $\sum\limits_{n\in\mathbb{N}} x_n$ is an unconditionally convergent series with sum $s \in E$ then for every σ permutation of $\mathbb{N}$ the series $\sum\limits_{n\in\mathbb{N}} x_{\sigma(n)}$ converges to s. For a Banach space E we say that a Schauder basis $(b_n)_{n\in\mathbb{N}}$ is *unconditional* if for every $x \in E$ the series $\sum\limits_{n\in\mathbb{N}} \langle x, f_n\rangle b_n$ converges unconditionally in E.

The following lemma is standard, we refer the reader to the discussion after [53, Proposition 1.c.6].

Lemma 1.2.3. *Let E be a Banach space with an unconditional Schauder basis $(b_n)_{n\in\mathbb{N}}$ and let $(f_n)_{n\in\mathbb{N}}$ be the sequence of coordinate functionals associated to $(b_n)_{n\in\mathbb{N}}$. Then for any $A \subseteq \mathbb{N}$*

$$P_A :\ E \to E; \quad x \mapsto \sum_{n\in A} \langle x, f_n\rangle b_n \tag{1.7}$$

defines a bounded linear idempotent operator on E. Also,

$$K_{ub} := \sup\{\|P_A\| \,:\, A \in \mathcal{P}(\mathbb{N})\} < \infty. \tag{1.8}$$

Remark 1.2.4. If E is a Banach space with an unconditional basis $(b_n)_{n\in\mathbb{N}}$ then for any $h \in \{-1,1\}^{\mathbb{N}}$

$$M_h : E \to E; \quad x \mapsto \sum_{n\in\mathbb{N}} h_n \langle x, f_n \rangle b_n \tag{1.9}$$

defines a bounded linear operator. By [56, Corollary 4.2.27] it follows that $K_u := \sup\{\|M_h\| : h \in \{-1,1\}^{\mathbb{N}}\} < \infty$ holds. If $K_u = 1$ then the basis is called *1-unconditional*. By [56, Theorem 4.2.16 and Proposition 4.2.31] the inequality $1 \leq K_{ub} \leq K_u$ holds in general and $1 = K_{ub} = K_u$ can always be arranged by passing to an equivalent norm.

Let E, F be Banach spaces and let $(b_n)_{n\in\mathbb{N}}$ and $(d_n)_{n\in\mathbb{N}}$ be basic sequences in E and F, respectively. We say that $(b_n)_{n\in\mathbb{N}}$ and $(d_n)_{n\in\mathbb{N}}$ are *equivalent* if for any sequence of scalars $(\alpha_n)_{n\in\mathbb{N}}$ the sum $\sum_{n\in\mathbb{N}} \alpha_n b_n$ converges in norm in E if and only if $\sum_{n\in\mathbb{N}} \alpha_n d_n$ converges in norm in F.

Let E be a Banach space with a Schauder basis $(b_n)_{n\in\mathbb{N}}$. We recall that $(b_n)_{n\in\mathbb{N}}$ is *subsymmetric* if it is an uncounditional basis and for every strictly monotone increasing function $\sigma : \mathbb{N} \to \mathbb{N}$, $(b_{\sigma(n)})_{n\in\mathbb{N}}$ is equivalent to $(b_n)_{n\in\mathbb{N}}$. We note that the natural bases for c_0 and ℓ_p $(1 \leq p < \infty)$ are subsymmetric, see [2, Section 9.2]. (In fact, the natural basis for any of these spaces is *symmetric*, which is a stronger property, but we do not need this in the present thesis.) For $p \in [1,\infty)\setminus\{2\}$ the space $L_p[0,1]$ does not have a subsymmetric basis, see [75, Theorem 21.1]. In fact, $L_1[0,1]$ does not even have an unconditional basis by [2, Theorem 6.3.3].

We summarise some well-known facts about subsymmetric bases here, see for example the paragraph after [53, Definition 3.a.2].

Proposition 1.2.5. *Let E be a Banach space with a subsymmetric basis $(b_n)_{n\in\mathbb{N}}$. For any strictly monotone increasing function $\sigma : \mathbb{N} \to \mathbb{N}$ the map*

$$S_\sigma : E \to E; \quad x \mapsto \sum_{n\in\mathbb{N}} \langle x, f_n\rangle b_{\sigma(n)} \tag{1.10}$$

is an isomorphism onto its range. Also, $K_{sub} := \sup_{\sigma,h} \|M_h S_\sigma\| < \infty$, where the supremum is taken over all $h \in \{-1,1\}^{\mathbb{N}}$ and for all strictly monotone increasing function $\sigma : \mathbb{N} \to \mathbb{N}$. The formula

$$\|x\|_{sub} := \sup_{\sigma,h} \left\| \sum_{n\in\mathbb{N}} \langle x, f_n\rangle h_n b_{\sigma(n)} \right\| \quad (x \in E) \tag{1.11}$$

defines an equivalent norm on E. For any $x \in E$, $h \in \{-1,1\}^{\mathbb{N}}$ and any strictly monotone increasing function $\sigma : \mathbb{N} \to \mathbb{N}$, $\|M_h S_\sigma x\|_{sub} = \|x\|_{sub}$ holds.

Remark 1.2.6. If E is a Banach space with a subsymmetric basis $(b_n)_{n\in\mathbb{N}}$, then if $K_{sub} = 1$ the basis is called *1-subsymmetric*. Since $K_u \leq K_{sub}$ clearly holds by the definitions, a 1-subsymmetric basis is in particular 1-unconditional. It is immediate that for any strictly monotone increasing function $\sigma : \mathbb{N} \to \mathbb{N}$ the operator $S_\sigma \in \mathcal{B}(E)$ is an isometry in the $\|\cdot\|_{sub}$-norm. Also, $(b_n)_{n\in\mathbb{N}}$ is 1-subsymmetric in the $\|\cdot\|_{sub}$-norm.

We recall that a Schauder basis $(b_n)_{n\in\mathbb{N}}$ for a Banach space E is *shrinking* if the sequence of coordinate functionals $(f_n)_{n\in\mathbb{N}}$ associated with $(b_n)_{n\in\mathbb{N}}$ is a Schauder basis for E^*. As is well-known, (see [75, Example 4.3]) any Schauder basis in a reflexive Banach space is shrinking. Clearly ℓ_1 and $L_1[0,1]$ cannot have shrinking bases since their dual spaces are non-separable.

Idempotent operators on Banach spaces

Lemma 1.2.7. *Let E, F be Banach spaces, let $T \in \mathcal{K}(E,F)$. Then T has closed range if and only if T has finite-dimensional range.*

Proof. For the non-trivial direction, suppose $H := \mathrm{Ran}(T)$ is closed, thus it is a Banach space in its own right. On the one hand T is a compact operator and therefore $\overline{T[\overline{B}_1(0;E)]}$ is a compact subset of H. On the other hand $T|^H \in \mathcal{B}(E,H)$ is surjective thus by the Open Mapping Theorem $T[B_1(0;E)]$ is an open subset of H. Let $r \in (0,1)$ be such that $B_r(0;H) \subseteq T[B_1(0;E)]$, so in particular $\overline{B}_r(0;H) \subseteq \overline{T[\overline{B}_1(0;E)]}$. Since the right-hand side is compact it follows that $\overline{B}_r(0;H)$ is compact; thus $H = \mathrm{Ran}(T)$ must be finite-dimensional. □

Corollary 1.2.8. *If E is a Banach space and $P \in \mathcal{B}(E)$ is an idempotent then* $\mathrm{Ran}(P)$ *is finite-dimensional if and only if $P \in \mathcal{K}(E)$.*

Definition 1.2.9. Let E be a Banach space. Two idempotents $P, Q \in \mathcal{B}(E)$ are said to be *almost orthogonal* if $PQ, QP \in \mathcal{F}(E)$.

The following result is surely well known but we were unable to locate it in the literature. We are therefore including a proof.

Lemma 1.2.10. *Let E be a Banach space with a 1-unconditional Schauder basis. Then $\mathcal{B}(E)$ admits a family $\mathcal{Q}$ of norm one, commuting, non-compact, almost orthogonal idempotents such that $|\mathcal{Q}| = 2^{\aleph_0}$.*

Proof. By Lemma 1.1.1 we can take an almost disjoint family $\mathcal{D}$ of continuum cardinality consisting of infinite subsets of $\mathbb{N}$. By Lemma 1.2.3, for any $N \in \mathcal{D}$ the formula (1.7) defines a norm one idempotent $P_N \in \mathcal{B}(E)$. Since the set N is countably infinite it follows that $\mathrm{Ran}(P_N)$ is infinite-dimensional hence by Corollary 1.2.8, $P_N \in \mathcal{B}(E)$ cannot be a compact operator. Also, for distinct $N, M \in \mathcal{D}$ the operator $P_N P_M = P_{N \cap M} = P_M P_N$ has finite rank, since $N \cap M$ is a finite set. Thus the family $\mathcal{Q} := \{P_N\}_{N \in \mathcal{D}}$ has the required properties. □

The following lemma is well-known (see for example [47, Lemma 1.4]), but since we use it very often in this thesis we shall present a proof here.

Lemma 1.2.11. *Let* X_1, X_2 *be Banach spaces and let* $P \in \mathcal{B}(X_1)$ *and* $Q \in \mathcal{B}(X_2)$ *be idempotents. Then* $\mathrm{Ran}(P) \simeq \mathrm{Ran}(Q)$ *as Banach spaces if and only if there exist* $U \in \mathcal{B}(X_2, X_1)$ *and* $V \in \mathcal{B}(X_1, X_2)$ *with* $P = U \circ V$ *and* $Q = V \circ U$*. Moreover, if* $\mathrm{Ran}(P)$ *and* $\mathrm{Ran}(Q)$ *are isometrically isomorphic and* $\|P\| = 1 = \|Q\|$ *then we can assume* $\|U\|, \|V\| = 1$.

Proof. Let $Y := \mathrm{Ran}(P)$, $Z := \mathrm{Ran}(Q)$, then $P|_Y \circ P|^Y = P$, $Q|_Z \circ Q|^Z = Q$ and $P|^Y \circ P|_Y = I_Y$, $Q|^Z \circ Q|_Z = I_Z$. Suppose that there exist $U \in \mathcal{B}(X_2, X_1)$ and $V \in \mathcal{B}(X_1, X_2)$ such that $P = U \circ V$ and $Q = V \circ U$. One immediately obtains that $P = P^2 = U \circ V \circ U \circ V = U \circ Q \circ V$ and $Q = Q^2 = V \circ U \circ V \circ U = V \circ P \circ U$. Let $T := Q|^Z \circ V \circ P|_Y$ and $S := P|^Y \circ U \circ Q|_Z$; it is clear that $T \in \mathcal{B}(Y, Z)$ and $S \in \mathcal{B}(Z, Y)$ and

$$\begin{aligned} T \circ S = Q|^Z \circ V \circ P|_Y \circ P|^Y \circ U \circ Q|_Z = Q|^Z \circ V \circ P \circ U \circ Q|_Z \\ = Q|^Z \circ Q \circ Q|_Z = I_Z \end{aligned} \tag{1.12}$$

and similarly $S \circ T = I_Y$. This proves $\mathrm{Ran}(P) \simeq \mathrm{Ran}(Q)$. In the other direction, suppose $\mathrm{Ran}(P) \simeq \mathrm{Ran}(Q)$, and let $T \in \mathcal{B}(Y, Z)$ and $S \in \mathcal{B}(Z, Y)$ be such that $T \circ S = I_Z$ and $S \circ T = I_Y$. With $U := P|_Y \circ T \circ Q|^Z$ and $V := Q|_Z \circ S \circ P|^Y$ we clearly have $U \in \mathcal{B}(X_2, X_1)$ and $V \in \mathcal{B}(X_1, X_2)$. Also,

$$\begin{aligned} U \circ V = P|_Y \circ T \circ Q|^Z \circ Q|_Z \circ S \circ P|^Y = P|_Y \circ T \circ S \circ P|^Y \\ = P|_Y \circ P^Y = P, \end{aligned} \tag{1.13}$$

and similarly $V \circ U = Q$. If $\mathrm{Ran}(P) \simeq \mathrm{Ran}(Q)$ isometrically, then both $T \in \mathcal{B}(Y, Z)$ and $S \in \mathcal{B}(Z, Y)$ can be taken to be isometries. If both P and Q are of norm one, it follows from the definitions of U and V that $\|U\|, \|V\| \leq 1$. Also, $1 = \|Q\| = \|V \circ U\| \leq \|V\|\|U\|$, thus $\|U\| = 1 = \|V\|$ as required. □

Tensor products of Banach spaces

Let E and F be Banach spaces, then

$$\|u\|_\pi := \inf \left\{ \sum_{i=1}^{n} \|x_i\| \|y_i\| \ : \ u = \sum_{i=1}^{n} x_i \otimes y_i \right\} \quad (u \in E \otimes F) \tag{1.14}$$

denotes the *projective tensor norm on* $E \otimes F$. The vector space $E \otimes F$ endowed with the norm $\|\cdot\|_\pi$ is denoted by $E \otimes_\pi F$. The completion of $E \otimes_\pi F$ with respect to $\|\cdot\|_\pi$ is called the *projective tensor product of* E *and* F and it is denoted by $E \hat{\otimes}_\pi F$.

The projective tensor product enjoys the following useful property, to which we shall refer as the *universal property of the projective tensor product* (see [70, Theorem 2.9]): If E, F, G are Banach spaces and $\varphi : E \times F \to G$ is a bounded bilinear map, then there exists a unique bounded linear map $\psi : E \hat{\otimes}_\pi F \to G$ such that $\|\psi\|_\pi = \|\varphi\|$ and $\psi(x \otimes y) = \varphi(x, y)$ for all $x \in E$, $y \in F$.

It is well-known (see e.g. [70, Proposition 2.3]) that for Banach spaces E, F, G, H if $S \in \mathcal{B}(E, G)$ and $T \in \mathcal{B}(F, H)$ then there exists a unique $S \otimes_\pi T \in \mathcal{B}(E \hat{\otimes}_\pi F, G \hat{\otimes}_\pi H)$ such that for every $x \in E$, $y \in F$ the identity $(S \otimes_\pi T)(x \otimes y) = (Sx) \otimes (Ty)$ holds. Moreover $\|S \otimes_\pi T\| = \|S\| \|T\|$.

Another important property of the projective tensor product is the following:

Proposition 1.2.12. *([70, Chapter 2, page 24]) Let* E, F *be Banach spaces. Then there exists an isometric isomorphism* $\psi : (E \hat{\otimes}_\pi F)^* \to \mathcal{B}(E, F^*)$ *such that for any* $\chi \in (E \hat{\otimes}_\pi F)^*$, $x \in E$ *and* $y \in F$*:*

$$\langle y, (\psi(\chi))(x) \rangle = \langle x \otimes y, \chi \rangle. \tag{1.15}$$

We shall now turn our attention towards the injective tensor product of Banach spaces. For Banach spaces E and F,

$$\|u\|_\epsilon := \sup\left\{\left\|\sum_{i=1}^{n}\langle x_i,\varphi\rangle y_i\right\| \,:\, u=\sum_{i=1}^{n} x_i\otimes y_i,\, \varphi\in E^*,\, \|\varphi\|\le 1\right\} \quad (u\in E\otimes F) \tag{1.16}$$

denotes the *injective tensor norm on* $E\otimes F$. The vector space $E\otimes F$ endowed with the norm $\|\cdot\|_\epsilon$ is denoted by $E\otimes_\epsilon F$. The completion of $E\otimes_\epsilon F$ with respect to $\|\cdot\|_\epsilon$ is called the *injective tensor product of* E *and* F and it is denoted by $E\hat{\otimes}_\epsilon F$. Analogously to the projective tensor product, the injective tensor product also has the property (see e.g. [70, Proposition 3.2]) that for Banach spaces E, F, G, H if $S\in\mathcal{B}(E,G)$ and $T\in\mathcal{B}(F,H)$ then there exists a unique $S\otimes_\epsilon T\in\mathcal{B}(E\hat{\otimes}_\epsilon F, G\hat{\otimes}_\epsilon H)$ such that for every $x\in E$, $y\in F$ the identity $(S\otimes_\epsilon T)(x\otimes y)=(Sx)\otimes(Ty)$ holds. Moreover $\|S\otimes_\epsilon T\|=\|S\|\|T\|$.

Lemma 1.2.13. *([18, Example 4.2(1)]) Let* E, F *be Banach spaces. Then there is an isometric isomorphism* $u:\mathcal{A}(F,E)\to E\hat{\otimes}_\epsilon F^*$ *of Banach spaces, such that for all* $x\in E$ *and* $g\in F^*$, $u(x\otimes g)=x\otimes g$. *On the left-hand side of this equation* $x\otimes g$ *denotes the rank-one operator* $F\to E$; $y\mapsto\langle y,g\rangle x$; *and on the right-hand side it denotes the elementary tensor.*

Chapter 2

The Johnson AMNM property and perturbations of homomorphisms

2.1 Introduction and statement of the main results

Let A and B be Banach algebras, then $\mathrm{Mult}(A, B)$ denotes the set of bounded linear multiplicative maps from A to B. It is clear that $\mathrm{Mult}(A, B)$ is a closed subset of $\mathcal{B}(A, B)$.

For any $T \in \mathcal{B}(A, B)$ we can define

$$\mathrm{dist}(T) := \inf\{\|T - S\| \,:\, S \in \mathrm{Mult}(A, B)\}, \tag{2.1}$$

the *distance of the map* T *from the set* $\mathrm{Mult}(A, B)$. Since $\mathrm{Mult}(A, B)$ is closed $\mathrm{dist}(T) = 0$ holds if and only if $T \in \mathrm{Mult}(A, B)$. Let us also define the *multiplicative defect* of a $T \in \mathcal{B}(A, B)$:

$$\mathrm{def}(T) := \sup\{\|T(ab) - T(a)T(b)\| \,:\, a, b \in A,\ \|a\|, \|b\| \leq 1\}. \tag{2.2}$$

Definition 2.1.1. ([38, Definition 1.2]) Let A, B be Banach algebras. Then (A, B) is called an *AMNM pair*, or (A, B) has the *AMNM property* if the following holds:
For any $\epsilon, K > 0$ there exists $\delta > 0$ such that for all $T \in \mathcal{B}(A, B)$ if $\|T\| \leq K$ and $\mathrm{def}(T) < \delta$ hold then $\mathrm{dist}(T) < \epsilon$.

The acronym AMNM stands for Approximately Multiplicative map is Near a Multiplicative one. The main results of this chapter are the following.

Theorem 2.1.2. *Let E be a Banach space and let F be a separable, reflexive Banach space such that one of the following three conditions is satisfied.*

1. *E has a subsymmetric, shrinking Schauder basis;*
2. *$E = \ell_1$; or*
3. *$E = L_p[0,1]$ $(1 \leq p < \infty)$.*

Then $(\mathcal{B}(E), \mathcal{B}(F))$ is an AMNM pair.

Corollary 2.1.3. *Let E be a Banach space satisfying the conditions of Theorem 2.1.2. Let F be a separable, reflexive Banach space such that F has the bounded approximation property. Then $(\mathcal{B}(E), \mathcal{A}(F))$ is an AMNM pair, where $\mathcal{A}(E)$ denotes the approximable operators on E.*

2.2 Summary of background material for the AMNM property

2.2.1 Dual Banach algebras

If A is a Banach algebra, then a *Banach left A-module* is a left A-module X which is a Banach space and satisfies $\|a \cdot x\| \leq \|a\|\|x\|$, whenever $x \in X$ and $a \in A$. One can similarly introduce the notion of a *Banach right A-module* and a *Banach A-bimodule*. We will drop the notation for the module multiplication whenever it does not cause any confusion.

The following notation will be useful for us: Let A be a Banach algebra and let X be a Banach A-bimodule. For $a \in A$ we define the maps

$$\begin{aligned} \lambda_a^{\mathrm{mod}} &: X \to X; \quad x \mapsto ax, \\ \rho_a^{\mathrm{mod}} &: X \to X; \quad x \mapsto xa. \end{aligned} \tag{2.3}$$

It is easy to see that $\lambda_a^{\mathrm{mod}}, \rho_a^{\mathrm{mod}} \in \mathcal{B}(X)$, for all $a \in A$.

If A is a Banach algebra and X is a Banach A-bimodule then X^* is also a Banach A-bimodule; if $a \in A$ and $f \in X^*$ then $a \cdot f$ and $f \cdot a$ are defined by

$$\langle x, a \cdot f \rangle := \langle xa, f \rangle, \quad \langle x, f \cdot a \rangle := \langle ax, f \rangle \quad (x \in X), \tag{2.4}$$

see [15, Examples 2.6.2(v)].

Let A be a Banach algebra, and let $\Delta \in A\hat{\otimes}_\pi A$. We define the maps

$$\begin{aligned} \iota_\Delta &: A \to A\hat{\otimes}_\pi A\hat{\otimes}_\pi A; \quad a \mapsto \Delta \otimes a; \\ \sigma_\Delta &: A \to A\hat{\otimes}_\pi A\hat{\otimes}_\pi A; \quad a \mapsto a \otimes \Delta; \\ \pi_A &: A\hat{\otimes}_\pi A \to A; \quad a \otimes b \mapsto ab. \end{aligned} \tag{2.5}$$

It is not hard to show (see [15, Theorem 2.6.4]) that $A\hat{\otimes}_\pi A$ and $A\hat{\otimes}_\pi A\hat{\otimes}_\pi A$ are Banach A-bimodules, thus ι_Δ is a continuous right A-module homomorphism and σ_Δ is a continuous left A-module homomorphism both of norms $\|\Delta\|$. The map π_A is a continuous A-bimodule homomorphism of norm at most 1.

A Banach algebra B is a *dual Banach algebra with predual* (B_*, φ), if B_* is a Banach

B-bimodule and $\varphi : B \to (B_*)^*$ is an isomorphism of Banach B-bimodules such that the maps

$$\begin{aligned} l_a &:= \varphi \circ \lambda_a \circ \varphi^{-1} \quad (a \in B) \\ r_a &:= \varphi \circ \rho_a \circ \varphi^{-1} \quad (a \in B) \end{aligned} \tag{2.6}$$

are $\sigma((B_*)^*, B_*)$ - to - $\sigma((B_*)^*, B_*)$ continuous; here λ_a and ρ_a denote the multiplication on B by the element a from the left and right, respectively.

If E is a Banach space, then $E \hat{\otimes}_\pi E^*$ is easily seen to be a Banach $\mathcal{B}(E)$-bimodule with the multiplication defined pointwise for $A \in \mathcal{B}(E)$, $x \in E$, and $\varphi \in E^*$ as

$$A \cdot (x \otimes \varphi) := (Ax) \otimes \varphi, \quad (x \otimes \varphi) \cdot A := x \otimes (A^* \varphi) \tag{2.7}$$

and then extended by linearity and continuity.

In the following, if E is a Banach space, $(f_i)_{i \in I}$ is a system in the topological space $(E^*, \sigma(E^*, E))$ and $\mathcal{U}$ is an ultrafilter on I such that the ultralimit of $(f_i)_{i \in I}$ along $\mathcal{U}$ with respect to the topology $\sigma(E^*, E)$ exists in E^*, then this limit will be denoted by $\operatorname{w^*-lim}_{i \to \mathcal{U}} f_i$.

Remark 2.2.1. Let B be a dual Banach algebra with predual (B_*, φ). Let $(a_i)_{i \in I}$ be a system in B such that $\operatorname{w^*-lim}_{i \to \mathcal{U}} \varphi(a_i) \in (B_*)^*$ exists, where $\mathcal{U}$ is an ultrafilter on I. Then for any $b \in B$ we have

$$\begin{aligned} \varphi\left(b \varphi^{-1}\left(\operatorname{w^*-lim}_{i \to \mathcal{U}} \varphi(a_i)\right)\right) &= l_b\left(\operatorname{w^*-lim}_{i \to \mathcal{U}} \varphi(a_i)\right) \\ &= \operatorname{w^*-lim}_{i \to \mathcal{U}} \left(l_b(\varphi(a_i))\right) \\ &= \operatorname{w^*-lim}_{i \to \mathcal{U}} \varphi(b a_i). \end{aligned} \tag{2.8}$$

Introducing the notation

$$\operatorname*{d-lim}_{i\to\mathcal{U}} a_i := \varphi^{-1}\left(\operatorname*{w^*-lim}_{i\to\mathcal{U}} \varphi(a_i)\right) \in B, \tag{2.9}$$

and by an analogous argument for r_b we obtain the identities

$$\operatorname*{d-lim}_{i\to\mathcal{U}}(ba_i) = b\left(\operatorname*{d-lim}_{i\to\mathcal{U}} a_i\right), \quad \operatorname*{d-lim}_{i\to\mathcal{U}}(a_i b) = \left(\operatorname*{d-lim}_{i\to\mathcal{U}} a_i\right) b \quad (b \in A). \tag{2.10}$$

Proposition 2.2.2. *([59, Proposition 2.4.12])*
Let E, F be normed spaces and let $T : F^* \to E^*$ be a linear operator. Then the following are equivalent:

1. *T is $\sigma(F^*, F)$ - to - $\sigma(E^*, E)$ continuous;*
2. *there exists $S \in \mathcal{B}(E, F)$ with $T = S^*$.*

In particular, if any of the above is satisfied then $T \in \mathcal{B}(F^*, E^*)$.

Lemma 2.2.3. *Let E be a reflexive Banach space. Then there is an isometric isomorphism $\varphi : \mathcal{B}(E) \to (E\hat{\otimes}_\pi E^*)^*$ such that for any $x \in E, f \in E^*$ and $A \in \mathcal{B}(E)$:*

$$\langle x \otimes f, \varphi(A)\rangle = \langle Ax, f\rangle, \tag{2.11}$$

and $((E\hat{\otimes}_\pi E^*)^*, \varphi)$ is a predual for $\mathcal{B}(E)$.

Proof. Let κ_E denote the canonical isomorphism, let $\psi : (E\hat{\otimes}_\pi E^*)^* \to \mathcal{B}(E, E^{**})$ be the isometric isomorphism from Proposition 1.2.12. Let us observe that the map

$$\varphi : \mathcal{B}(E) \to (E\hat{\otimes}_\pi E^*)^*; \quad A \mapsto \psi^{-1}(\kappa_E \circ A) \tag{2.12}$$

is an isometric isomorphism of Banach $\mathcal{B}(E)$-bimodules with inverse

$$\varphi^{-1} : (E\hat{\otimes}_\pi E^*)^* \to \mathcal{B}(E); \quad \chi \mapsto \kappa_E^{-1} \circ \psi(\chi). \tag{2.13}$$

If $x \in E, f \in E^*$ and $A \in \mathcal{B}(E)$ then one immediately obtains

$$\langle x \otimes f, \varphi(A)\rangle = \langle x \otimes f, \psi^{-1}(\kappa_E \circ A)\rangle = \langle f, \kappa_E(Ax)\rangle = \langle Ax, f\rangle. \tag{2.14}$$

For $((E\hat{\otimes}_\pi E^*)^*, \varphi)$ to be a predual of $\mathcal{B}(E)$ we must show that for any $A \in \mathcal{B}(E)$ the maps $\varphi \circ \lambda_A \circ \varphi^{-1}$ and $\varphi \circ \rho_A \circ \varphi^{-1}$ are weak* - to - weak* continuous. (Here λ_A and ρ_A denote the multiplication on $\mathcal{B}(E)$ by A from the left and right, respectively.) In view of Proposition 2.2.2 it is enough to show that $\varphi \circ \lambda_A \circ \varphi^{-1} = (\rho_A^{\text{mod}})^*$ and $\varphi \circ \rho_A \circ \varphi^{-1} = (\lambda_A^{\text{mod}})^*$ hold. (Here λ_A^{mod} and ρ_A^{mod} denote the multiplication on $E\hat{\otimes}_\pi E^*$ by A from the left and right, respectively.) To see the former, let us fix $x \in E$, $f \in E^*$ and $\chi \in (E\hat{\otimes}_\pi E^*)^*$. Hence we obtain

$$\begin{aligned}
\langle x \otimes f, (\varphi \circ \lambda_A \circ \varphi^{-1})(\chi)\rangle &= \langle x \otimes f, (\varphi \circ \lambda_A)(\kappa_E^{-1} \circ \psi(\chi))\rangle \\
&= \langle x \otimes f, \psi^{-1}(\kappa_E \circ A \circ \kappa_E^{-1} \circ \psi(\chi))\rangle \\
&= \langle f, \kappa_E(A(\kappa_E^{-1}(\psi(\chi)x)))\rangle \\
&= \langle A(\kappa_E^{-1}(\psi(\chi)x)), f\rangle \\
&= \langle \kappa_E^{-1}(\psi(\chi)x), A^* f\rangle \\
&= \langle A^* f, \psi(\chi)x\rangle \\
&= \langle x \otimes (A^* f), \chi\rangle \\
&= \langle (x \otimes f) \cdot A, \chi\rangle \\
&= \langle x \otimes f, (\rho_A^{\text{mod}})^*(\chi)\rangle.
\end{aligned} \tag{2.15}$$

By linearity and continuity the result follows. □

2.2.2 Amenable Banach algebras, approximate identities

Let A be a Banach algebra. A bounded net $(\Delta_\gamma)_{\gamma\in\Gamma}$ in $A\hat{\otimes}_\pi A$ is called a *bounded approximate diagonal for* A if for every $a \in A$

$$\lim_{\gamma\in\Gamma}(a\cdot\Delta_\gamma - \Delta_\gamma\cdot a) = 0, \quad \text{and}$$
$$\lim_{\gamma\in\Gamma} a\pi_A(\Delta_\gamma) = a, \tag{2.16}$$

where the limits are taken in the norm topology. A Banach algebra A is *amenable* if there is a bounded approximate diagonal for A. It is well-known (see for example [15, Proposition 2.8.58(i)]) that a Banach algebra A is amenable if and only if its unitisation $A^\sharp$ is.

Let $C > 0$, a net $(e_\gamma)_{\gamma\in\Gamma}$ in a Banach algebra is a *bounded left [right] approximate identity with bound* C if $\sup_{\gamma\in\Gamma}\|e_\gamma\| \le C$ and $\lim_{\gamma\in\Gamma} e_\gamma a = a$ $\left[\lim_{\gamma\in\Gamma} ae_\gamma = a\right]$ for every $a \in A$. A net $(e_\gamma)_{\gamma\in\Gamma}$ is a *bounded approximate identity with bound* C if it is a bounded left- and right approximate identity with bound C, and a *contractive approximate identity* if it is a bounded approximate identity with bound 1.

We recall the following result, see [15, Theorem 2.9.37].

Theorem 2.2.4. *Let* E *be a non-zero Banach space.*

- *The Banach algebra* $\mathcal{A}(E)$ *has a bounded left approximate identity if and only if* E *has the bounded approximation property.*
- *The Banach algebra* $\mathcal{A}(E)$ *has a bounded right approximate identity if and only if* E^* *has the bounded approximation property.*

In fact we shall be interested in the following special cases.

Corollary 2.2.5. *Let E be a Banach space with a Schauder basis. Then the corresponding sequence of coordinate projections $(P_n)_{n\in\mathbb{N}}$ is a bounded left approximate identity for $\mathcal{K}(E)$. If E has a shrinking basis then $(P_n)_{n\in\mathbb{N}}$ is a bounded approximate identity for $\mathcal{K}(E)$.*

Proposition 2.2.6. *Let $E \in \{\ell_1, L_p[0,1] : 1 \leq p \leq \infty\}$. Then $\mathcal{K}(E)$ has a contractive approximate identity.*

Proof. In view of [31, Thereom 3.3] it is enough to show that ℓ_∞ and $L_p[0,1]$ (where $1 < p < \infty$) have the metric approximation property. Since $\ell_\infty \simeq C(\beta\mathbb{N})$, where $\beta\mathbb{N}$ denotes the Čech–Stone compactification of $\mathbb{N}$, these follow from [70, Examples 4.2 and 4.5], respectively. □

2.3 The main technical result

In this section we develop the machinery which allows us to prove the main results of Chapter 2.

If A, B are both unital Banach algebras with multiplicative identities 1_A and 1_B, respectively, then we we define

$$\mathcal{B}_1(A,B) := \{T \in \mathcal{B}(A,B) \,:\, T(1_A) = 1_B\}, \tag{2.17}$$

which is clearly a closed subset of $\mathcal{B}(A,B)$.

Remark 2.3.1. Let A, B be Banach algebras and let $T \in \mathcal{B}(A,B)$. We define

$$T^\vee : A \times A \to B; \quad (a,b) \mapsto T(ab) - T(a)T(b). \tag{2.18}$$

It is easy to see that $T^\vee$ is a bounded bilinear map with $\operatorname{def}(T) = \|T^\vee\| \leq \|T\| + \|T\|^2$. Therefore by the universal property of the projective tensor product, there exists a unique bounded linear map $\tilde{T} : A\hat{\otimes}_\pi A \to B$ such that for any $a, b \in A$ we have $\tilde{T}(a \otimes b) = T^\vee(a, b)$ and $\operatorname{def}(T) = \|T^\vee\| = \|\tilde{T}\|$.

Lemma 2.3.2. *Let A be a Banach algebra and let B be a dual Banach algebra with predual (B_*, φ). If $(S_i)_{i\in I}$ is a system in $\mathcal{B}(A, B)$ bounded by $K > 0$ and $\mathcal{U}$ is an ultrafilter on I, then there is a unique $S \in \mathcal{B}(A, B)$ such that*

$$S(a) = \varphi^{-1}(\text{w}^*\text{-}\lim_{i\to\mathcal{U}} \varphi(S_i(a))) = \text{d-}\lim_{i\to\mathcal{U}} S_i(a) \quad (a \in A). \tag{2.19}$$

Moreover, $\|S\| \leq \|\varphi^{-1}\|\|\varphi\|K$.

Proof. We recall that by Lemma 1.2.12 there is an isometric isomorphism of Banach spaces $\theta : (A\hat{\otimes}_\pi B_*)^* \to \mathcal{B}(A, (B_*)^*)$ such that for any $\chi \in (A\hat{\otimes}_\pi B_*)^*$, $a \in A$ and $x \in B_*$

$$\langle x, (\theta(\chi))(a)\rangle = \langle a \otimes x, \chi\rangle. \tag{2.20}$$

Clearly, for any $R \in \mathcal{B}(A, (B_*)^*)$, $a \in A$ and $x \in B_*$ we have

$$\langle a \otimes x, \theta^{-1}(R)\rangle = \langle x, R(a)\rangle. \tag{2.21}$$

Now let $(S_i)_{i\in I}$ be a bounded system in $\mathcal{B}(A, B)$ and let us fix an ultrafilter $\mathcal{U}$ on I. The net $(\theta^{-1}(\varphi \circ S_i))_{i\in I}$ is contained in a closed ball of $(A\hat{\otimes}_\pi B_*)^*$ centred at zero. This set is Hausdorff and compact with respect to the relative weak* topology by the Banach–Alaoglu Theorem, thus by Lemma 1.1.3 the system $(\theta^{-1}(\varphi \circ S_i))_{i\in I}$ has a unique ultralimit along $\mathcal{U}$ with respect to the relative weak* topology. Let it be denoted by $\text{w}^*\text{-}\lim_{i\to\mathcal{U}} \theta^{-1}(\varphi \circ S_i) \in (A\hat{\otimes}_\pi B_*)^*$. This allows us to define

$$S := \varphi^{-1} \circ \theta\Big(\text{w}^*\text{-}\lim_{i\to\mathcal{U}} \theta^{-1}(\varphi \circ S_i)\Big) \in \mathcal{B}(A, B). \tag{2.22}$$

Now we observe that for any $a \in A$ and $x \in B_*$:

$$\begin{aligned}\left\langle x, \left(\theta\Big(\operatorname*{w^*\text{-}lim}_{i\to\mathcal{U}} \theta^{-1}(\varphi\circ S_i)\Big)\right)(a)\right\rangle &= \left\langle a\otimes x, \operatorname*{w^*\text{-}lim}_{i\to\mathcal{U}} \theta^{-1}(\varphi\circ S_i)\right\rangle \\ &= \lim_{i\to\mathcal{U}} \left\langle a\otimes x, \theta^{-1}(\varphi\circ S_i)\right\rangle \\ &= \lim_{i\to\mathcal{U}} \langle x, \varphi(S_i(a))\rangle . \end{aligned} \tag{2.23}$$

This shows that the net $(\varphi(S_i(a)))_{i\in I}$ converges along $\mathcal{U}$ in the weak* topology of $(B_*)^*$ and

$$\left(\theta\Big(\operatorname*{w^*\text{-}lim}_{i\to\mathcal{U}} \theta^{-1}(\varphi\circ S_i)\Big)\right)(a) = \operatorname*{w^*\text{-}lim}_{i\to\mathcal{U}} \varphi(S_i(a)). \tag{2.24}$$

Now (2.19) follows from (2.24) and the estimate on the norm of S is immediate. □

The following theorem is the main technical tool of this section. The result is essentially a modification of Johnson's result [38, Theorem 3.1]. We would like to take this opportunity to expand the steps behind Johnson's brilliant idea, since we found his proof in its original form very dense.

Theorem 2.3.3. *Let A, B be unital Banach algebras such that B is a dual Banach algebra. Assume C is a closed, amenable subalgebra of A with $1_A \in C$. Then the following holds:*
For every $\epsilon > 0$ and $K \geq 1$ there exists $\delta > 0$ such that whenever $T \in \mathcal{B}_1(A, B)$ satisfies $\|T\| \leq K$ and $\operatorname{def}(T) < \delta$, there exists $\psi \in \mathcal{B}_1(A, B)$ such that $\|T - \psi\| < \epsilon$ and

$$\psi(cad) = \psi(c)\psi(a)\psi(d) \quad (a \in A,\ c, d \in C). \tag{2.25}$$

Proof. Let us first fix a bounded approximate diagonal $(\Delta_\gamma)_{\gamma\in\Gamma}$ in $C\hat{\otimes}_\pi C$, let $M > \sup_{\gamma\in\Gamma} \|\Delta_\gamma\|$. Since $(\pi_A(\Delta_\gamma))_{\gamma\in\Gamma}$ converges to 1_A in norm, necessarily $M \geq 1$. For all $\gamma \in \Gamma$, $\|\Delta_\gamma\| < M$, thus it follows from [70, Proposition 2.8] that for all $\gamma \in \Gamma$ there exist $(c_n^{(\gamma)})_{n\in\mathbb{N}}$ and $(d_n^{(\gamma)})_{n\in\mathbb{N}}$ in C such that

$$\Delta_\gamma = \sum_{n\in\mathbb{N}} c_n^{(\gamma)} \otimes d_n^{(\gamma)}, \tag{2.26}$$

where the sum converges in the projective tensor norm and

$$\sum_{n\in\mathbb{N}} \|c_n^{(\gamma)}\|\|d_n^{(\gamma)}\| < M. \tag{2.27}$$

Let (B_*, φ) be a predual of B. For the sake of readability, we assume $\|\varphi\|, \|\varphi^{-1}\| \leq 1$. The proof carries over trivially to the more general case. Let us fix an ultrafilter $\mathcal{U}$ on Γ which extends the order filter.

We show that for any $\eta > 0$ and $K \geq 1$ there exists $\delta > 0$ such that whenever $T \in \mathcal{B}_1(A,B)$ satisfies $\|T\| \leq K$ and $\operatorname{def}(T) < \delta$, there exists $R \in \mathcal{B}_1(A,B)$ such that $\|T - R\| < \eta$, $\|R\| \leq 2K$ and

$$R(ca) = R(c)R(a) \quad (a \in A,\ c \in C). \tag{2.28}$$

For the moment let us fix $K, \delta > 0$ and $T \in \mathcal{B}_1(A,B)$ such that $\|T\| \leq K$ and $\operatorname{def}(T) < \delta$. We recall that $\tilde{T}$ denotes the unique bounded linear map from $A\hat{\otimes}_\pi A$ to B such that for any $a, b \in A$ the identities $\tilde{T}(a\otimes b) = T(ab) - T(a)T(b)$ and $\|\tilde{T}\| = \operatorname{def}(T)$ hold. In the notations of (2.5) we see that $\sup_{\gamma\in\Gamma} \|\pi_B \circ (T \otimes_\pi \tilde{T}) \circ \iota_{\Delta_\gamma}\| \leq \delta K M$, thus Lemma 2.3.2 ensures that the map

$$S:\ A \to B; \quad a \mapsto \operatorname*{d-lim}_{\gamma\to\mathcal{U}} \sum_j T(c_j^{(\gamma)}) T^\vee(d_j^{(\gamma)}, a) \tag{2.29}$$

defines an operator in $\mathcal{B}(A,B)$ with $\|S\| \leq \|T|_C\| \cdot \|T^\vee|_{C\times A}\| \leq \delta K M$. From now on we will omit the "$\gamma \to \mathcal{U}$" symbol when it does not cause any confusion. Let us observe that for any $x, y \in A$ we have

$$(T+S)^{\vee}(x,y) = T^{\vee}(x,y) - S(x)S(y) + S(xy) - T(x)S(y) - S(x)T(y). \tag{2.30}$$

Applying the definition (2.29) of S to the last three terms and then the definition (2.18) of $T^{\vee}$ we obtain

$$\begin{aligned} &S(xy) - T(x)S(y) - S(x)T(y) \\ &\quad = \text{d-lim} \sum_j \Big(T(c_j^{(\gamma)})T^{\vee}(d_j^{(\gamma)}, xy) - T(x)T(c_j^{(\gamma)})T^{\vee}(d_j^{(\gamma)}, y) \\ &\qquad - T(c_j^{(\gamma)})T^{\vee}(d_j^{(\gamma)}, x)T(y) \Big) \qquad (2.31) \\ &\quad = \text{d-lim} \sum_j \Big(T(c_j^{(\gamma)}))T(d_j^{(\gamma)} xy) - T(c_j^{(\gamma)})T(d_j^{(\gamma)})T(xy) \\ &\qquad - T(x)T(c_j^{(\gamma)})T(d_j^{(\gamma)} y) + T(x)T(c_j^{(\gamma)})T(d_j^{(\gamma)})T(y) \\ &\qquad - T(c_j^{(\gamma)})T(d_j^{(\gamma)} x)T(y) + T(c_j^{(\gamma)})T(d_j^{(\gamma)})T(x)T(y) \Big) \qquad (2.32) \\ &\quad = \text{d-lim} \sum_j \Big(- T(c_j^{(\gamma)} d_j^{(\gamma)})T^{\vee}(x,y) + T^{\vee}(c_j^{(\gamma)}, d_j^{(\gamma)})T^{\vee}(x,y) \\ &\qquad + T(c_j^{(\gamma)})T^{\vee}(d_j^{(\gamma)} x, y) - T(x c_j^{(\gamma)})T^{\vee}(d_j^{(\gamma)}, y) + T^{\vee}(x, c_j^{(\gamma)})T^{\vee}(d_j^{(\gamma)}, y) \Big). \qquad (2.33) \end{aligned}$$

Expanding the third and fourth term of (2.33) we observe that for all $j \in \mathbb{N}$ and $\gamma \in \Gamma$

$$\begin{aligned} &T(c_j^{(\gamma)})T^{\vee}(d_j^{(\gamma)} x, y) - T(x c_j^{(\gamma)})T^{\vee}(d_j^{(\gamma)}, y) \\ &\qquad = T(c_j^{(\gamma)})T(d_j^{(\gamma)} xy) - T(x c_j^{(\gamma)})T(d_j^{(\gamma)} y) \\ &\qquad\quad + \Big(T(x c_j^{(\gamma)})T(d_j^{(\gamma)}) - T(c_j^{(\gamma)})T(d_j^{(\gamma)} x) \Big) T(y). \qquad (2.34) \end{aligned}$$

Substituting (2.33) and (2.34) into (2.30) we obtain that for any $x \in C$ and $y \in A$ with $\|x\|, \|y\| \leq 1$

$$
\begin{aligned}
(T+S)^\vee(x,y) = &- S(x)S(y) && (2.35)\\
&+ \left(1_B - \text{d-lim} \sum_j T(c_j^{(\gamma)} d_j^{(\gamma)})\right) T^\vee(x,y) && (2.36)\\
&+ \text{d-lim} \sum_j T^\vee(c_j^{(\gamma)}, d_j^{(\gamma)}) T^\vee(x,y) && (2.37)\\
&+ \text{d-lim} \sum_j \Big(T(c_j^{(\gamma)}) T(d_j^{(\gamma)} xy) - T(x c_j^{(\gamma)}) T(d_j^{(\gamma)} y)\Big) && (2.38)\\
&+ \text{d-lim} \sum_j \Big(T(x c_j^{(\gamma)}) T(d_j^{(\gamma)}) - T(c_j^{(\gamma)}) T(d_j^{(\gamma)} x)\Big) T(y) && (2.39)\\
&+ \text{d-lim} \sum_j T^\vee(x, c_j^{(\gamma)}) T^\vee(d_j^{(\gamma)}, y). && (2.40)
\end{aligned}
$$

To justify that we can actually take the "d-limits" individually above, it is enough to show that the "d-limits" exist individually. Let us consider the terms line by line.

- Recall that $(\Delta_\gamma)_{\gamma\in\Gamma}$ is a bounded approximate diagonal for the unital subalgebra C and therefore $\lim\limits_{\gamma\to\mathcal{U}} \pi_A(\Delta_\gamma) = 1_A$. Since $T \in \mathcal{B}_1(A,B)$, it follows that $\lim\limits_{\gamma\to\mathcal{U}} T(\pi_A(\Delta_\gamma)) = 1_B$. In other words, $\lim\limits_{\gamma\to\mathcal{U}} \sum\limits_j T(c_j^{(\gamma)} d_j^{(\gamma)}) = 1_B$ and thus $\text{d-lim} \sum\limits_j T(c_j^{(\gamma)} d_j^{(\gamma)}) = 1_B$. This shows that (2.36) is zero.
- For any $\gamma \in \Gamma$, $\|\sum\limits_j T^\vee(c_j^{(\gamma)}, d_j^{(\gamma)})\| \leq \|T^\vee|_{C\times C}\| \cdot M$ holds, thus (2.37) exists by Lemma 2.3.2 and its norm is bounded by $\|T^\vee|_{C\times C}\| \cdot \|T^\vee|_{C\times A}\| \cdot M$.
- First let us observe that regarding $A\hat{\otimes}_\pi A$ as a Banach A-bimodule, we have for all $\gamma \in \Gamma$:

$$
\begin{aligned}
&\sum_j \Big(T(c_j^{(\gamma)}) T(d_j^{(\gamma)} xy) - T(x c_j^{(\gamma)}) T(d_j^{(\gamma)} y)\Big)\\
=&\sum_j (\pi_B \circ (T \otimes_\pi T))\Big(c_j^{(\gamma)} \otimes (d_j^{(\gamma)} xy) - (x c_j^{(\gamma)}) \otimes (d_j^{(\gamma)} y)\Big)
\end{aligned}
$$

$$
\begin{aligned}
&=(\pi_B \circ (T \otimes_\pi T)) \left(\left(\sum_j c_j^{(\gamma)} \otimes d_j^{(\gamma)} \right) \cdot xy - x \cdot \left(\sum_j c_j^{(\gamma)} \otimes d_j^{(\gamma)} \right) \cdot y \right) \\
&=(\pi_B \circ (T \otimes_\pi T)) \big((\Delta_\gamma \cdot x - x \cdot \Delta_\gamma) \cdot y \big). \qquad (2.41)
\end{aligned}
$$

Recall that $(\Delta_\gamma)_{\gamma\in\Gamma}$ is a bounded approximate diagonal for C and thus $x \in C$ implies $\lim_{\gamma\to\mathcal{U}} (\Delta_\gamma \cdot x - x \cdot \Delta_\gamma) = 0$. Consequently $\lim_{\gamma\to\mathcal{U}} (\pi_B \circ (T \otimes T))((\Delta_\gamma \cdot x - x \cdot \Delta_\gamma) \cdot y) = 0$, showing that (2.38) is zero.

- Applying the argument in the previous bullet point in the case where $y = 1_A$ we obtain that (2.39) is zero as well.

- Lastly, (2.40) exists by Lemma 2.3.2 since for any $\gamma \in \Gamma$

$$
\left\| \sum_j T^\vee(x, c_j^{(\gamma)}) T^\vee(d_j^{(\gamma)}, y) \right\| \le \|T^\vee|_{C\times C}\| \cdot \|T^\vee|_{C\times A}\| \cdot M. \qquad (2.42)
$$

Thus we conclude that

$$
\begin{aligned}
\|(T+S)^\vee|_{C\times A}\| &\le 2\|T^\vee|_{C\times C}\| \cdot \|T^\vee|_{C\times A}\| \cdot M + \|S\|^2 \\
&\le 2\delta^2 M + \|S\|^2. \qquad (2.43)
\end{aligned}
$$

Now let us fix $\eta \in (0,1)$ and $K \ge 1$; we define $\delta := (4M + 8K^2M^2)^{-1}\eta$. Let $T \in \mathcal{B}_1(A, B)$ be such that $\|T\| \le K$ and $\operatorname{def}(T) < \delta$. We will show that there exists $R \in \mathcal{B}_1(A, B)$ such that $\|T - R\| < \eta$ and for any $c \in C$ and $a \in A$, $R(ca) = R(c)R(a)$ holds.

Firstly, for all $n \in \mathbb{N}_0$ let us define $K_n := (2 - 2^{-n})K$ and $\delta_n := 2^{-n}\delta$. Then recursively, we shall define sequences $(T_n)_{n\in\mathbb{N}_0}$ in $\mathcal{B}_1(A, B)$ and $(S_n)_{n\in\mathbb{N}_0}$ in $\mathcal{B}(A, B)$

such that $T_0 := T$ and for all $n \in \mathbb{N}_0$

$$\begin{aligned} S_n(a) &:= \text{d-lim} \sum_j T_n(c_j^{(\gamma)}) T_n^{\vee}(d_j^{(\gamma)}, a) \quad (a \in A) \\ T_{n+1} &:= T_n + S_n. \end{aligned} \tag{2.44}$$

Then $S_n(1_A) = 0$, $\|S_n\| \leq K_n \delta_n M$, $\|T_n\| \leq K_n$ and $\|T_n^{\vee}|_{C\times A}\| \leq \delta_n$ hold.

We show now why the sequences $(T_n)_{n\in\mathbb{N}_0}$ and $(S_n)_{n\in\mathbb{N}_0}$ have the required properties. Due to (2.29), the operator S_0 is well-defined with $S_0(1_A) = 0$ and the norms of T_0, $T_0^{\vee}|_{C\times A}$ and S_0 satisfy the required estimates. Now suppose $n \in \mathbb{N}$ is fixed and $T_n \in \mathcal{B}_1(A, B)$ and $S_n \in \mathcal{B}(A, B)$ have the required properties. Clearly from $T_n(1_A) = 1_B$ and $S_n(1_A) = 0$, the equality $T_{n+1}(1_A) = 1_B$ follows. Thus in particular, we obtain that

$$\begin{aligned} \|T_{n+1}\| &\leq \|T_n\| + \|S_n\| \\ &\leq K_n(1 + M2^{-n}\delta) \\ &= K_n \left(1 + \frac{M2^{-n}\eta}{4M + 8K^2M^2}\right) \\ &\leq K_n(1 + 2^{-n-2}) \\ &= K(2 - 2^{-n})(1 + 2^{-n-2}) \\ &\leq K(2 - 2^{-n-1}) \\ &= K_{n+1}. \end{aligned} \tag{2.45}$$

Also, by (2.43) we obtain that

$$\|(T_n + S_n)^{\vee}|_{C\times A}\| \leq 2\delta_n^2 M + K_n^2 \delta_n^2 M^2, \tag{2.46}$$

and therefore

$$\begin{aligned}
\|T_{n+1}^{\vee}|_{C\times A}\| &\leq (2M + K_n^2 M^2)\delta_n^2 \\
&= (2M + (4 - 2^{2-n} + 2^{-2n})K^2M^2)\delta_n^2 \\
&\leq \frac{1}{2}(4M + 8K^2M^2)\delta_n^2 \\
&= \delta_{n+1}(4M + 8K^2M^2)\delta_n \\
&= \delta_{n+1}(4M + 8K^2M^2)\delta 2^{-n} \\
&= \delta_{n+1}\eta 2^{-n} \\
&\leq \delta_{n+1}. \qquad (2.47)
\end{aligned}$$

Consequently we obtain that for any $a \in A$ with $\|a\| \leq 1$

$$\begin{aligned}
\sup_{\gamma\in\Gamma}\left\|\sum_j T_{n+1}(c_j^{(\gamma)})T_{n+1}^{\vee}(d_j^{(\gamma)}, a)\right\| &\leq M \cdot \|T_{n+1}\| \cdot \|T_{n+1}^{\vee}|_{C\times A}\| \\
&\leq MK_{n+1}\delta_{n+1} \qquad (2.48)
\end{aligned}$$

thus from Lemma 2.3.2 it follows that $S_{n+1} \in \mathcal{B}(A, B)$ is well-defined with the required upper bound on its norm. The above shows the existence of sequences $(T_n)_{n\in\mathbb{N}_0}, (S_n)_{n\in\mathbb{N}_0}$ with the specified properties.

Now by (2.44) we obtain for all $n, m \in \mathbb{N}_0$ with $m > n$

$$\|T_m - T_n\| = \left\|\sum_{i=n}^{m-1} S_i\right\| \leq \sum_{i=n}^{m-1}\|S_i\| \leq 2KM\sum_{i=n}^{m-1} 2^{-i}, \qquad (2.49)$$

showing that $(T_n)_{n\in\mathbb{N}_0}$ is a Cauchy sequence in $\mathcal{B}_1(A, B)$. Let $R \in \mathcal{B}_1(A, B)$ be the limit of this sequence. For any $c \in C$ and $a \in A$ with $\|a\|, \|c\| \leq 1$ we have

$$\|R(ca) - R(c)R(a)\| = \|\lim_{n\to\infty} T_n(ca) - \lim_{n\to\infty} T_n(c)T_n(a)\|$$

$$\begin{aligned} &= \lim_{n\to\infty} \|T_n^\vee(c,a)\| \\ &\leq \lim_{n\to\infty} \delta_n = 0, \end{aligned} \tag{2.50}$$

and consequently $R(ca) = R(c)R(a)$. Finally applying (2.49) for $n = 0$ we obtain

$$\|T - R\| \leq 2KM \sum_{i=0}^{\infty} \delta_i = 4KM\delta < \eta \tag{2.51}$$

hence

$$\|R\| \leq \eta + \|T\| \leq \eta + K \leq 2K, \tag{2.52}$$

which concludes the proof of the claim.

We are now ready to prove the full theorem.

Let $\epsilon > 0$ and $K \geq 1$ be fixed. We set $\eta := \epsilon(1 + 2KM(2 + 3K))^{-1}$ and fix $\delta \in (0, \eta)$. Let $T \in \mathcal{B}_1(A, B)$ be such that $\|T\| \leq K$ and $\operatorname{def}(T) < \delta$. Choose $R \in \mathcal{B}_1(A, B)$ which satisfies the conditions that $\|R\| \leq 2K$, $\|T - R\| < \eta$, and $R(c)R(a) = R(ca)$ for all $a \in A, c \in C$. Let us observe that we have an upper bound on $\operatorname{def}(R)$ depending on K, η, δ. Indeed, for any $x, y \in A$ with $\|x\|, \|y\| \leq 1$

$$\begin{aligned} \|R(xy) - R(x)R(y)\| &\leq \|R(xy) - T(xy)\| + \|T(xy) - T(x)T(y)\| \\ &\quad + \|T(x)T(y) - T(x)R(y)\| + \|T(x)R(y) - R(x)R(y)\| \\ &\leq \|R - T\| + \operatorname{def}(T) + \|T\| \cdot \|T - R\| + \|T - R\| \cdot \|R\| \\ &\leq \eta + \delta + K\eta + 2K\eta \\ &= (1 + 3K)\eta + \delta, \end{aligned} \tag{2.53}$$

thus $\operatorname{def}(R) \leq (1 + 3K)\eta + \delta$.

We observe that $\sup_{\gamma\in\Gamma} \|\pi_B \circ (\tilde{R} \otimes_\pi R) \circ \rho_{\Delta_\gamma}\| \leq \mathrm{def}(R) \cdot \|R\| \cdot M$ holds, thus by Lemma 2.3.2,

$$Q: \, A \to B; \quad a \mapsto \text{d-lim} \sum_j R^\vee(a, c_j^{(\gamma)}) R(d_j^{(\gamma)}) \tag{2.54}$$

defines an operator in $\mathcal{B}(A, B)$ with $\|Q\| \leq \mathrm{def}(R) \cdot \|R\| \cdot M$. By the properties of $R \in \mathcal{B}_1(A, B)$ we immediately see that for any $j \in \mathbb{N}$, $\gamma \in \Gamma$ and $c \in C$, $R^\vee(c, c_j^{(\gamma)}) = 0$ and therefore $Q|_C = 0$. Using the same properties we also obtain that for any $c \in C$ and $a \in A$

$$\begin{aligned} R^\vee(ca, c_j^{(\gamma)}) &= R(cac_j^{(\gamma)}) - R(ca)R(c_j^{(\gamma)}) \\ &= R(c)R(ac_j^{(\gamma)}) - R(c)R(a)R(c_j^{(\gamma)}) \\ &= R(c)R^\vee(a, c_j^{(\gamma)}) \end{aligned} \tag{2.55}$$

and therefore by (2.10)

$$\begin{aligned} Q(ca) &= \text{d-lim} \sum_j R^\vee(ca, c_j^{(\gamma)}) R(d_j^{(\gamma)}) \\ &= R(c) \left(\text{d-lim} \sum_j R^\vee(a, c_j^{(\gamma)}) R(d_j^{(\gamma)}) \right) \\ &= R(c)Q(a). \end{aligned} \tag{2.56}$$

Consequently

$$\begin{aligned} (R+Q)^\vee(c, a) &= (R+Q)(ca) - (R+Q)(c)(R+Q)(a) \\ &= R(ca) + Q(ca) - R(c)R(a) - R(c)Q(a) - Q(c)R(a) - Q(c)Q(a) \\ &= 0 \end{aligned} \tag{2.57}$$

or equivalently, $(R+Q)^\vee|_{C\times A} = 0$.

The aim of this paragraph is to establish that $(R+Q)^{\vee}|_{A\times C}=0$ also holds. To this end, let us first observe that for any $a\in A$ and $c\in C$:

$$\begin{aligned}(R+Q)^{\vee}(a,c)&=R(ac)+Q(ac)-R(a)R(c)-R(a)Q(c)-Q(a)R(c)-Q(a)Q(c)\\&=R^{\vee}(a,c)+Q(ac)-Q(a)R(c).\end{aligned}\tag{2.58}$$

Thus, we must show that $Q(ac)-Q(a)R(c)=-R^{\vee}(a,c)$. To verify this we observe that by the multiplicativity of R on the subalgebra C, for any $\gamma\in\Gamma$ and $j\in\mathbb{N}$

$$\begin{aligned}R^{\vee}(ac,c_j^{(\gamma)})R(d_j^{(\gamma)})&-R^{\vee}(a,c_j^{(\gamma)})R(d_j^{(\gamma)})R(c)\\&=R(acc_j^{(\gamma)})R(d_j^{(\gamma)})-R(ac)R(c_j^{(\gamma)})R(d_j^{(\gamma)})\\&\quad-R(ac_j^{(\gamma)})R(d_j^{(\gamma)})R(c)+R(a)R(c_j^{(\gamma)})R(d_j^{(\gamma)})R(c)\\&=R(a)R(c_j^{(\gamma)}d_j^{(\gamma)})R(c)-R(ac)R(c_j^{(\gamma)}d_j^{(\gamma)})\\&\quad+R(acc_j^{(\gamma)})R(d_j^{(\gamma)})-R(ac_j^{(\gamma)})R(d_j^{(\gamma)}c).\end{aligned}\tag{2.59}$$

Consequently by (2.54) we obtain

$$Q(ac)-Q(a)R(c)=\text{d-lim}\sum_j\Big(R^{\vee}(ac,c_j^{(\gamma)})R(d_j^{(\gamma)})-R^{\vee}(a,c_j^{(\gamma)})R(d_j^{(\gamma)})R(c)\Big)\tag{2.60}$$

$$=\text{d-lim}\sum_j\Big(R(a)R(c_j^{(\gamma)}d_j^{(\gamma)})R(c)-R(ac)R(c_j^{(\gamma)}d_j^{(\gamma)})\Big)\tag{2.61}$$

$$+\,\text{d-lim}\sum_j\Big(R(acc_j^{(\gamma)})R(d_j^{(\gamma)})-R(ac_j^{(\gamma)})R(d_j^{(\gamma)}c)\Big).\tag{2.62}$$

Let us consider the two terms in the last equation separately.

- Recall that $(\Delta_\gamma)_{\gamma\in\Gamma}$ is a bounded approximate diagonal for the unital subalgebra C and therefore $\lim_{\gamma\to\mathcal{U}}\pi_A(\Delta_\gamma)=1_A$. Since $R\in\mathcal{B}_1(A,B)$, it follows that $\lim_{\gamma\to\mathcal{U}}R(\pi_A(\Delta_\gamma))=1_B$, where convergence is in the norm topologies of A and B, respectively. Thus $\lim_{\gamma\to\mathcal{U}} \sum_j R(c_j^{(\gamma)} d_j^{(\gamma)}) = 1_B$ and consequently d-lim $\sum_j R(c_j^{(\gamma)} d_j^{(\gamma)}) = 1_B$. This shows that (2.61) is equal to $-R^\vee(a,c)$.

- Since R is multiplicative on C, for each $\gamma \in \Gamma$ and $j \in \mathbb{N}$

$$
\begin{aligned}
&R(acc_j^{(\gamma)})R(d_j^{(\gamma)}) - R(ac_j^{(\gamma)})R(d_j^{(\gamma)}c) \\
&\qquad = (\pi_B \circ (R \otimes_\pi R))(acc_j^{(\gamma)} \otimes d_j^{(\gamma)}) - (\pi_B \circ (R \otimes_\pi R))(ac_j^{(\gamma)} \otimes d_j^{(\gamma)}c) \\
&\qquad = (\pi_B \circ (R \otimes_\pi R))(a \cdot (cc_j^{(\gamma)} \otimes d_j^{(\gamma)} - c_j^{(\gamma)} \otimes d_j^{(\gamma)}c)), \qquad (2.63)
\end{aligned}
$$

and

$$
\begin{aligned}
&\sum_j \Big(R(acc_j^{(\gamma)})R(d_j^{(\gamma)}) - R(ac_j^{(\gamma)})R(d_j^{(\gamma)})R(c)\Big) \\
&= (\pi_B \circ (R \otimes_\pi R)) \left(a \cdot \sum_j (cc_j^{(\gamma)} \otimes d_j^{(\gamma)} - c_j^{(\gamma)} \otimes d_j^{(\gamma)}c)\right) \\
&= (\pi_B \circ (R \otimes_\pi R))(a \cdot (c \cdot \Delta_\gamma - \Delta_\gamma \cdot c)). \qquad (2.64)
\end{aligned}
$$

Since $(\Delta_\gamma)_{\gamma\in\Gamma}$ is a bounded approximate diagonal for C it follows that (2.64) converges to zero in norm along $\mathcal{U}$, thus (2.62) is zero.

This proves the required identity $Q(ac) - Q(a)R(c) = -R^\vee(a,c)$. Therefore, for any $a \in A$ and $c \in C$, $(R+Q)^\vee(a,c) = 0$, as required.

Now let us define $\psi := R+Q$. The identities $(R+Q)^\vee|_{A\times C} = 0$ and $(R+Q)^\vee|_{C\times A} = 0$ imply that for any $c \in C$ and $a \in A$, $\psi(ac) = \psi(a)\psi(c)$ and $\psi(ca) = \psi(c)\psi(a)$. Also, $\psi(1_A) = 1_B$ follows from the facts that $R(1_A) = 1_B$ and $Q(1_A) = 0$. Recall that $\|Q\| \le \operatorname{def}(R) \cdot \|R\| \cdot M$ and therefore by the estimates on the norm and defect of R we obtain that $\|Q\| \le 2KM((1+3K)\eta + \delta) < 2KM(2+3K)\eta$. This implies that

$$\|\psi - T\| \le \|R - T\| + \|Q\| < \eta + 2KM(2+3K)\eta = \epsilon, \qquad (2.65)$$

which concludes the proof of Theorem 2.3.3. □

Definition 2.3.4. Let A, B be unital Banach algebras and let C be a closed subalgebra of A. Then the triple $(A, C; B)$ has the *AMNM-bootstrap property* if:

For every $\epsilon, K > 0$ there exists $\delta > 0$ such that if $T \in \mathcal{B}_1(A, B)$ satisfies $\|T\| \leq K$ and $\operatorname{def}(T) < \delta$ then there exists $\psi \in \mathcal{B}_1(A, B)$ such that for every $a \in A$ and $c \in C$ the identities $\psi(ac) = \psi(a)\psi(c)$ and $\psi(ca) = \psi(c)\psi(a)$ hold and $\|T - \psi\| < \epsilon$.
If $(A, C; B)$ has the AMNM-bootstrap property and $\epsilon, K > 0$ are fixed, then $\delta' > 0$ is called an *AMNM-bootstrap constant of* $(A, C; B)$ *for* (ϵ, K) if it is the supremum of all $\delta > 0$ constants satisfying the property defined above.

We note that in the definition above we did *not* require that the subalgebra C of A contains the multiplicative identity 1_A of A.

Corollary 2.3.5. *Let A, B be unital Banach algebras, let $I \trianglelefteq A$ be a closed, amenable, two-sided ideal and suppose B is a dual Banach algebra. Then the triple $(A, I; B)$ has the AMNM-bootstrap property.*

Proof. We recall that the amenability of I is equivalent to the amenability of $I^\sharp$. Also, $I^\sharp$ is isomorphic to the closed unital subalgebra $\mathbb{C}1_A + I$; therefore $\mathbb{C}1_A + I$ is amenable. Thus Theorem 2.3.3 immediately implies that the triple $(A, I; B)$ has the AMNM-bootstrap property. □

Lemma 2.3.6. *Let A be a Banach algebra, let $I \trianglelefteq A$ be a closed, amenable, two-sided ideal and suppose B is a unital, dual Banach algebra. Let $\epsilon, K > 0$. There exists $\delta > 0$ such that for every $\phi \in \mathcal{B}(A, B)$ with $\|\phi\| \leq K$ and $\operatorname{def}(\phi) < \delta$ there exists $\psi \in \mathcal{B}(A, B)$ with $\|\phi - \psi\| < \epsilon$ such that*

$$\psi(a)\psi(i) = \psi(ai), \quad \psi(i)\psi(a) = \psi(ia) \quad (a \in A,\, i \in I). \tag{2.66}$$

Proof. Since the map $\iota : A \to A^\sharp$; $\iota(a) := (0, a)$ is an isometric algebra homomorphism, it follows that $\iota[I]$ is a closed, amenable, two-sided ideal in $A^\sharp$. Thus the triple $(A^\sharp, \iota[I]; B)$ has the AMNM-bootstrap property by Corollary 2.3.5. Let $\delta > 0$ be the AMNM-bootstrap constant for $(\epsilon, \max\{1, K\})$. Let $\phi \in \mathcal{B}(A, B)$ be such that $\|\phi\| \leq K$ and $\operatorname{def}(\phi) < \delta$. We define the map

$$\tilde{\phi} : A^\sharp \to B; \quad (\lambda, a) \mapsto \lambda 1_B + \phi(a), \tag{2.67}$$

this is easily seen to be a bounded linear map with $\|\tilde{\phi}\| \leq \max\{1, K\}$ and $\tilde{\phi}(1_{A^\sharp}) = \tilde{\phi}(1, 0) = 1_B$. We now observe that $\operatorname{def}(\tilde{\phi}) = \operatorname{def}(\phi)$. This readily follows from the fact that for any $\lambda, \mu \in \mathbb{C}$ and $a, b \in A$

$$\begin{aligned}
\tilde{\phi}((\lambda, a)(\mu, b)) &= \tilde{\phi}(\lambda\mu, \lambda b + \mu a + ab) \\
&= \lambda\mu 1_B + \phi(\lambda b + \mu a + ab) \\
&= \lambda\mu 1_B + \lambda\phi(b) + \mu\phi(a) + \phi(ab); && (2.68) \\
\tilde{\phi}(\lambda, a)\tilde{\phi}(\mu, b) &= (\lambda 1_B + \phi(a))(\mu 1_B + \phi(b)) \\
&= \lambda\mu 1_B + \lambda\phi(b) + \mu\phi(a) + \phi(a)\phi(b). && (2.69)
\end{aligned}$$

Consequently $\operatorname{def}(\tilde{\phi}) < \delta$, thus there exists a $\theta \in \mathcal{B}_1(A, B)$ such that $\|\theta - \tilde{\phi}\| < \epsilon$ and the identities $\theta(bc) = \theta(b)\theta(c)$ and $\theta(cb) = \theta(c)\theta(b)$ hold for every $b \in A^\sharp$ and $c \in \iota[I]$. We define the map

$$\psi : A \to B; \quad a \mapsto \theta(0, a), \tag{2.70}$$

this is clearly a bounded linear map with the property that for every $a \in A$ and $i \in I$ the identity $\psi(ai) = \psi(a)\psi(i)$ holds. Indeed,

$$\psi(a)\psi(i) = \theta(0, a)\theta(0, i) = \theta((0, a)(0, i)) = \theta(0, ai) = \psi(ai). \tag{2.71}$$

An analogous argument shows $\psi(ia) = \psi(i)\psi(a)$. It remains to show $\|\phi - \psi| < \epsilon$. To see this let $a \in A$ be fixed with $\|a\| \leq 1$, then

$$\|\phi(a) - \psi(a)\| = \|\tilde{\phi}(0, a) - \theta(0, a)\| \leq \|\tilde{\phi} - \theta\| < \epsilon, \tag{2.72}$$

concluding the claim. □

Definition 2.3.7. Let A, B be Banach algebras and let I be a closed two-sided ideal of A. Then the triple $(A, I; B)$ has the *pre-AMNM property* if:

For every $\epsilon, K > 0$ there exists $\delta > 0$ such that if $\phi \in \mathcal{B}(A, B)$ satisfies $\|\phi\| \leq K$ and $\operatorname{def}(\phi) < \delta$ then there exists $\psi \in \mathcal{B}(A, B)$ such that for every $a \in A$ and $i \in I$ the identities $\psi(ai) = \psi(a)\psi(i)$ and $\psi(ia) = \psi(i)\psi(a)$ hold and $\|\phi - \psi\| < \epsilon$.

If $(A, I; B)$ has the pre-AMNM property and $\epsilon, K > 0$ are fixed, then $\delta' > 0$ is called an *pre-AMNM constant of* $(A, I; B)$ *for* (ϵ, K) if it is the supremum of all $\delta > 0$ constants satisfying the property defined above.

In view of the above definition, we can reformulate Lemma 2.3.6 as follows:

Lemma 2.3.8. *Let A be a Banach algebra and let $I \trianglelefteq A$ be a closed, amenable, two-sided ideal. Let B be a unital, dual Banach algebra. Then the triple $(A, I; B)$ has the pre-AMNM property.*

Lemma 2.3.9. *Let B be a dual Banach algebra with predual (B_*, φ). Let $(q_\gamma)_{\gamma \in \Gamma}$ be a net in B bounded by $M > 0$ such that for any $\gamma \in \Gamma$, $\lim_{\omega \in \Gamma} q_\omega q_\gamma = q_\gamma$ in norm. Then* *for any ultrafilter $\mathcal{U}$ on Γ extending the order filter, $p := \varphi^{-1}(\text{w*-}\lim_{\gamma\to\mathcal{U}} \varphi(q_\gamma)) \in B$ exists and defines an idempotent with $\|p\| \le M\|\varphi^{-1}\|\|\varphi\|$.*

Proof. Let $\mathcal{U}$ be an ultrafilter on Γ extending the order filter. By the Banach–Alaoglu Theorem $p := \varphi^{-1}(\text{w*-}\lim_{\gamma\to\mathcal{U}} \varphi(q_\gamma)) \in B$ is well-defined. It is also clear that $\|p\| \le M\|\varphi^{-1}\|\|\varphi\|$ holds. It remains to show that $p \in B$ is idempotent. We recall that for any $b \in B$ the maps $\varphi \circ \lambda_b \circ \varphi^{-1}$ and $\varphi \circ \rho_b \circ \varphi^{-1}$ are weak*-continuous on $(B_*)^*$ and therefore for any $\gamma \in \Gamma$

$$\begin{aligned}
\varphi(pq_\gamma) &= (\varphi \circ \rho_{q_\gamma} \circ \varphi^{-1})(\text{w*-}\lim_{\omega\to\mathcal{U}} \varphi(q_\omega)) \\
&= \text{w*-}\lim_{\omega\to\mathcal{U}} \varphi(q_\omega q_\gamma) \\
&= \varphi(q_\gamma)
\end{aligned} \tag{2.73}$$

because $\lim_{\omega\to\mathcal{U}} q_\omega q_\gamma = q_\gamma$. Consequently,

$$\begin{aligned}
\varphi(p^2) &= \varphi\left(p\varphi^{-1}(\text{w*-}\lim_{\gamma\to\mathcal{U}} \varphi(q_\gamma))\right) \\
&= (\varphi \circ \lambda_p \circ \varphi^{-1})(\text{w*-}\lim_{\gamma\to\mathcal{U}} \varphi(q_\gamma)) \\
&= \text{w*-}\lim_{\gamma\to\mathcal{U}} \varphi(pq_\gamma) \\
&= \text{w*-}\lim_{\gamma\to\mathcal{U}} \varphi(q_\gamma) \\
&= \varphi(p).
\end{aligned} \tag{2.74}$$

This shows that $p^2 = p$, proving the claim. □

Lemma 2.3.10. *Let A be a Banach algebra, let $J \trianglelefteq A$ be a closed, two-sided ideal with a bounded approximate identity $(e_\gamma)_{\gamma\in\Gamma}$ with bound $K > 0$. Let B be a unital, dual Banach algebra with predual (B_*, φ). Suppose $\psi : A \to B$ is a bounded linear map such* *that*

$$\psi(ac) = \psi(a)\psi(c) \quad \textit{and} \quad \psi(ca) = \psi(c)\psi(a) \quad (a \in A, c \in J). \tag{2.75}$$

If $\mathcal{U}$ is an ultrafilter on Γ which extends the order filter, then:

(1) $p := \varphi^{-1}\left(\operatorname{w*-lim}_{\gamma\to\mathcal{U}} \varphi(\psi(e_\gamma))\right) \in B$ *is idempotent and* $\|p\| \leq K\|\psi\|\|\varphi^{-1}\|\|\varphi\|$*;*

(2) For any $c \in J$*,* $p\psi(c) = \psi(c) = \psi(c)p$*;*

(3) For any $a \in A$*,* $p\psi(a) = p\psi(a)p = \psi(a)p$*;*

(4) For any $a, b \in A$*,* $p\psi(ab) = p\psi(a)\psi(b)$ *and* $\psi(a)\psi(b)p = \psi(ab)p$*.*

Moreover

$$\psi_1 : \ A \to B; \quad a \mapsto p\psi(a)p \tag{2.76}$$

$$\psi_2 : \ A \to B; \quad a \mapsto (1_B - p)\psi(a)(1_B - p) \tag{2.77}$$

are bounded linear maps such that

(a) $\psi = \psi_1 + \psi_2$*;*

(b) ψ_1 *is an algebra homomorphism;*

(c) $\psi_2|_J = 0$*; and*

(d) $\psi_2(ab) - \psi_2(a)\psi_2(b) = \psi(ab) - \psi(a)\psi(b) \quad (a, b \in A)$*.*

Proof. (1) Since $(e_\gamma)_{\gamma\in\Gamma}$ is a bounded approximate identity in J, from (2.75) it follows that for any $\omega \in \Gamma$, $\lim_{\gamma\in\Gamma} \psi(e_\gamma)\psi(e_\omega) = \lim_{\gamma\in\Gamma} \psi(e_\gamma e_\omega) = \psi(e_\omega)$ and $\lim_{\gamma\in\Gamma} \psi(e_\omega)\psi(e_\gamma) = \lim_{\gamma\in\Gamma} \psi(e_\omega e_\gamma) = \psi(e_\omega)$ hold. Thus the statement follows from Lemma 2.3.9.
Before we proceed we observe that for any $a \in A$

$$\varphi(p\psi(a)) = \varphi\left(\varphi^{-1}\left(\operatorname{w*-lim}_{\gamma\to\mathcal{U}} \varphi(\psi(e_\gamma))\right)\psi(a)\right)$$

$$
\begin{aligned}
&= (\varphi \circ \rho_{\psi(a)} \circ \varphi^{-1})(\text{w*-}\lim_{\gamma\to\mathcal{U}} \varphi(\psi(e_\gamma))) \\
&= \text{w*-}\lim_{\gamma\to\mathcal{U}} \varphi(\psi(e_\gamma)\psi(a)) \\
&= \text{w*-}\lim_{\gamma\to\mathcal{U}} \varphi(\psi(e_\gamma a)).
\end{aligned} \tag{2.78}
$$

(2) Let us fix $c \in J$. Then from (2.78) and the fact that $(e_\gamma)_{\gamma\in\Gamma}$ is a bounded approximate identity for J we obtain

$$
\varphi(p\psi(c)) = \text{w*-}\lim_{\gamma\to\mathcal{U}} \varphi(\psi(e_\gamma c)) = \varphi(\psi(c)), \tag{2.79}
$$

proving $p\psi(c) = \psi(c)$. An analogous argument shows $\psi(c)p = \psi(c)$.

(3) Let us fix $a \in A$. Since for any $\gamma \in \Gamma$, $e_\gamma a \in J$, it follows from (2) that $\psi(e_\gamma a) = \psi(e_\gamma a)p = \psi(e_\gamma)\psi(a)p = \rho_{\psi(a)p}\psi(e_\gamma)$. From this and (2.78) we obtain

$$
\begin{aligned}
\varphi(p\psi(a)) &= \text{w*-}\lim_{\gamma\to\mathcal{U}} \varphi(\psi(e_\gamma a)) \\
&= \text{w*-}\lim_{\gamma\to\mathcal{U}}(\varphi \circ \rho_{\psi(a)p} \circ \varphi^{-1})(\varphi(\psi(e_\gamma))) \\
&= (\varphi \circ \rho_{\psi(a)p} \circ \varphi^{-1}) \left(\text{w*-}\lim_{\gamma\to\mathcal{U}} \varphi(\psi(e_\gamma))\right) \\
&= \varphi(p\psi(a)p).
\end{aligned} \tag{2.80}
$$

Consequently $p\psi(a) = p\psi(a)p$ holds. A similar argument shows $\psi(a)p = p\psi(a)p$.

(4) Let us fix $a, b \in A$. For any $\gamma \in \Gamma$ we have $\psi(e_\gamma ab) = \psi(e_\gamma a)\psi(b) = \psi(e_\gamma)\psi(a)\psi(b)$. From this and (2.78) the identity

$$
\begin{aligned}
\varphi(p\psi(ab)) &= \text{w*-}\lim_{\gamma\to\mathcal{U}} \varphi(\psi(e_\gamma)\psi(a)\psi(b)) \\
&= \text{w*-}\lim_{\gamma\to\mathcal{U}}(\varphi \circ \rho_{\psi(a)\psi(b)} \circ \varphi^{-1})(\varphi(\psi(e_\gamma))) \\
&= (\varphi \circ \rho_{\psi(a)\psi(b)} \circ \varphi^{-1}) \left(\text{w*-}\lim_{\gamma\to\mathcal{U}}(\varphi(\psi(e_\gamma)))\right)
\end{aligned}
$$

$$= \varphi(p\psi(a)\psi(b)) \tag{2.81}$$

follows. Consequently $p\psi(ab) = p\psi(a)\psi(b)$ holds as required. An analogous argument shows the identity $\psi(a)\psi(b)p = \psi(ab)p$.

(a) Let us fix $a \in A$. By the definitions of ψ_1, ψ_2 and (3) we obtain

$$\begin{aligned} \psi_1(a) + \psi_2(a) &= p\psi(a)p + \psi(a) - \psi(a)p - p\psi(a) + p\psi(a)p \\ &= \psi(a). \end{aligned} \tag{2.82}$$

(b) By the definition of ψ_1 and (3), (4) we obtain that for any $a, b \in A$

$$\psi_1(ab) = p\psi(ab)p = p\psi(a)\psi(b)p = (p\psi(a)p)(p\psi(b)p) = \psi_1(a)\psi_1(b), \tag{2.83}$$

thus proving that ψ_1 is a homomorphism.

(c) For any $c \in J$, we immediately obtain from (2) that

$$\begin{aligned} \psi_2(c) &= (1_B - p)\psi(c)(1_B - p) \\ &= (1_B - p)p\psi(c)(1_B - p) \\ &= 0. \end{aligned} \tag{2.84}$$

(d) Let us fix $a, b \in A$. Then (4) implies that

$$\begin{aligned} \psi_2(ab) &= \psi(ab) - \psi(ab)p - p\psi(ab) + p\psi(ab)p \\ &= \psi(ab) - \psi(a)\psi(b)p - p\psi(a)\psi(b) + p\psi(a)\psi(b)p. \end{aligned} \tag{2.85}$$

Also, by (3) it follows that

$$\begin{aligned}\psi_2(a)\psi_2(b) &= (1_B - p)\psi(a)(1_B - p)\psi(b)(1_B - p)\\ &= (1_B - p)\psi(a)\psi(b)(1_B - p) - (1_B - p)\psi(a)p\psi(b)(1_B - p)\\ &= (1_B - p)\psi(a)\psi(b)(1_B - p) - (1_B - p)p\psi(a)p\psi(b)(1_B - p)\\ &= \psi(a)\psi(b) - \psi(a)\psi(b)p - p\psi(a)\psi(b) + p\psi(a)\psi(b)p. \qquad (2.86)\end{aligned}$$

Consequently $\psi_2(ab) - \psi_2(a)\psi_2(b) = \psi(ab) - \psi(a)\psi(b)$ as required. □

Before we proceed let us recall some basic probability-theoretic background and terminology. In the brief exposition below we follow Fremlin's book [24, Sections 254J-254R].

Remark 2.3.11. We consider the the probability space $(\{0,1\}, \mathcal{P}(\{0,1\}), \mu)$ where μ is the "fair-coin" probability measure; thus $\mu(\{0\}) = 1/2 = \mu(\{1\})$. Let $(\{0,1\}^{\mathbb{N}}, \Lambda, \nu)$ denote the product of the system $\Big((\{0,1\}, \mathcal{P}(\{0,1\}), \mu)\Big)_{n\in\mathbb{N}}$ of probability spaces. The measure space $(\{0,1\}^{\mathbb{N}}, \Lambda, \nu)$ is isomorphic to $([0,1], \mathcal{A}, \lambda)$, where λ is the Lebesgue measure restricted to $[0,1]$. Consequently for all $1 \leq p < \infty$ the spaces $L_p(\{0,1\}^{\mathbb{N}}, \Lambda, \nu)$ and $L_p([0,1], \mathcal{A}, \lambda)$ are isometrically isomorphic as Banach spaces (see also [2, page 125]). For any $S \subseteq \mathbb{N}$ let us define

$$\pi_S : \ \{0,1\}^{\mathbb{N}} \to \{0,1\}^S; \quad (x_n)_{n\in\mathbb{N}} \mapsto (x_n)_{n\in S} \qquad (2.87)$$

and

$$\Lambda_S := \Big\{A \in \Lambda : \ A = \pi_S^{-1}[\pi_S[A]]\Big\}. \qquad (2.88)$$

The set Λ_S is a σ-subalgebra of Λ. In the case when S is an infinite subset of $\mathbb{N}$, it follows that $(\{0,1\}^{\mathbb{N}}, \Lambda_S, \nu|_{\Lambda_S})$ is isomorphic to $([0,1], \mathcal{A}, \lambda)$, thus for any $1 \leq p < \infty$

the Banach spaces $L_p(\{0,1\}^{\mathbb{N}}, \Lambda_S, \nu|_{\Lambda_S})$ and $L_p([0,1], \mathcal{A}, \lambda)$ are isometrically isomorphic. On the other hand, if S is a finite subset of $\mathbb{N}$ then $L_p(\{0,1\}^{\mathbb{N}}, \Lambda_S, \nu|_{\Lambda_S})$ is a finite-dimensional Banach space. To see this, it is enough to show that Λ_S is a finite set, since in this case the aforementioned Banach space is the linear span of the indicator functions of the sets $A \in \Lambda_S$. To see that Λ_S is finite, just observe that the function

$$\pi_S^{-1} : \mathcal{P}(\{0,1\}^S) \to \mathcal{P}(\{0,1\}^{\mathbb{N}}); \quad B \mapsto \pi_S^{-1}[B] \tag{2.89}$$

has finite range simply because $\{0,1\}^S$ is finite; and $\Lambda_S \subseteq \mathrm{Ran}(\pi_S^{-1})$ holds by definition.

The above technique is well know among experts in Banach space theory, we refer the interested reader to [40] for a more sophisticated approach.

Proposition 2.3.12. *Let E be a Banach space such that one of the following two conditions is satisfied.*

(1) E has a 1-subsymmetric Schauder basis; or

(2) $E = L_p[0,1]$ where $1 \leq p < \infty$.

Then $\mathcal{B}(E)$ admits a family $\mathcal{Q}$ of norm one, commuting, non-compact, almost orthogonal idempotents such that $|\mathcal{Q}| = 2^{\aleph_0}$ and for every $P \in \mathcal{Q}$, $\mathrm{Ran}(P) \simeq E$. Moreover, for every $P \in \mathcal{Q}$ there exist $U, V \in \mathcal{B}(E)$ with $P = UV$, $I_E = VU$ and $\|U\|, \|V\| = 1$.

Proof. By Lemma 1.1.1 we can take an almost disjoint family $\mathcal{D}$ of continuum cardinality consisting of infinite subsets of $\mathbb{N}$.
(1) Suppose E has a 1-subsymmetric Schauder basis $(b_n)_{n\in\mathbb{N}}$ with coordinate functionals $(f_n)_{n\in\mathbb{N}}$, then in particular $K_{sub} = K_u = K_{ub} = 1$. Let $\mathcal{Q} := \{P_N\}_{N\in\mathcal{D}}$, where for $N \in \mathcal{D}$

$$P_N x := \sum_{n\in N} \langle x, f_n\rangle b_n \quad (x \in E) \tag{2.90}$$

defines an idempotent in $\mathcal{B}(E)$, as in the proof of Lemma 1.2.10. Let us fix $N \in \mathcal{D}$, there is strictly monotone increasing function $\sigma_N : \mathbb{N} \to \mathbb{N}$ with $N = \mathrm{Ran}(\sigma_N)$. Let $S_{\sigma_N} \in \mathcal{B}(E)$ be as defined by (1.10), then clearly $\mathrm{Ran}(P_N) = \mathrm{Ran}(S_{\sigma_N})$, thus by Proposition 1.2.5 and Remark 1.2.6 the operator S_{σ_N} is an isometric isomorphism onto $\mathrm{Ran}(P_N)$. Thus the claim follows from Lemma 1.2.10 and Lemma 1.2.11.

(2) In the notation of Remark 2.3.11, for every $N \in \mathcal{D}$ we consider the conditional expectation operator

$$\mathbb{E}(\cdot|\Lambda_N) : \ L_p(\{0,1\}^{\mathbb{N}}, \Lambda, \mu) \to L_p(\{0,1\}^{\mathbb{N}}, \Lambda, \mu); \quad f \mapsto \mathbb{E}(f|\Lambda_N). \tag{2.91}$$

By [2, Lemma 6.1.1], for any $N \in \mathcal{D}$ the bounded linear operator $\mathbb{E}(\cdot|\Lambda_N)$ is a norm one idempotent with range $L_p(\{0,1\}^{\mathbb{N}}, \Lambda_N, \mu|_{\Lambda_N})$, so in particular $\mathrm{Ran}(\mathbb{E}(\cdot|\Lambda_N))$ is isometrically isomorphic to $L_p([0,1], \mathcal{A}, \lambda)$. It follows from [24, Theorem 254Ra] that for any two distinct $N, M \in \mathcal{D}$

$$\mathbb{E}(\cdot|\Lambda_N)\mathbb{E}(\cdot|\Lambda_M) = \mathbb{E}(\cdot|\Lambda_{N\cap M}). \tag{2.92}$$

Thus we conclude that

$$\mathrm{Ran}\left(\mathbb{E}(\cdot|\Lambda_N)\mathbb{E}(\cdot|\Lambda_M)\right) = \mathrm{Ran}\left(\mathbb{E}(\cdot|\Lambda_{N\cap M})\right) = L_p(\{0,1\}^{\mathbb{N}}, \Lambda_{N\cap M}, \mu|_{\Lambda_{N\cap M}}) \tag{2.93}$$

is finite-dimensional. For all $N \in \mathcal{D}$ let $P_N := \mathbb{E}(\cdot|\Lambda_N)$. Let $T : \ L_p([0,1], \mathcal{A}, \lambda) \to L_p(\{0,1\}^{\mathbb{N}}, \Lambda, \mu)$ be an isometric isomorphism, and define $Q_N := T^{-1} \circ P_N \circ T$ for all $N \in \mathcal{D}$, clearly $\|Q_N\| \leq 1$. From the identities $T^{-1} \circ P_N \circ P_N \circ T = Q_N$ and $P_N \circ T \circ T^{-1} \circ P_N = P_N$ we immediately obtain that $Q_N \in \mathcal{B}(L_p[0,1])$ is an idempotent with $\|Q_N\| = 1$ and $\mathrm{Ran}(Q_N) \simeq \mathrm{Ran}(P_N)$, where the isomorphism is isometric. Thus for all $N \in \mathcal{D}$ we obtain the isometric isomorphism

$$\mathrm{Ran}(Q_N) \simeq \mathrm{Ran}(P_N) = L_p(\{0,1\}^{\mathbb{N}}, \Lambda_N, \mu|_{\Lambda_N}) \simeq L_p([0,1], \mathcal{A}, \lambda). \tag{2.94}$$

The existence of the required $U, V \in \mathcal{B}(L_p[0,1])$ follow from Lemma 1.2.11. It is clear that $\mathrm{Ran}(Q_N Q_M)$ is finite-dimensional for distinct $N, M \in \mathcal{D}$. Setting $\mathcal{Q} := \{Q_N\}_{N \in \mathcal{D}}$ finishes the proof. □

Remark 2.3.13. For a fixed $M > 0$ we define the function

$$f_M : [0, 1/4) \to \mathbb{R}; \quad f_M(x) = (M + 1/2)((1 - 4x)^{-1/2} - 1). \tag{2.95}$$

It is clear that f_M is a non-negative function such that $f_M(x) \to 0$ as $x \to 0$.

The following lemma is well-known, it can be found for example in [38, Lemma 2.1] without a proof. For the convenience of the reader we include a proof here.

Proposition 2.3.14. *Let A be a unital Banach algebra, and let $a \in A$ be such that $\nu := \|a^2 - a\| < 1/4$. Then there is an idempotent $p \in A$ such that $\|p - a\| \leq f_{\|a\|}(\nu)$ holds.*

Proof. Because $\nu < 1/4$, it follows that the series $\sum_{n=0}^{\infty} \binom{2n}{n} \nu^n$ converges in $[0, \infty)$ with sum $(1 - 4\nu)^{-1/2}$, consequently $s := \sum_{n=0}^{\infty} \binom{2n}{n} (a - a^2)^n$ is absolutely convergent and therefore convergent in A. Clearly s commutes with any polynomial in a. Let $p := (a - 1/2)s + 1/2$. We show that $p \in A$ is an idempotent, which is equivalent to showing that $(2p - 1)^2 = 1$. We first observe that by the Cauchy product formula

$$s^2 = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{2k}{k} \binom{2(n-k)}{n-k} \right) (a - a^2)^n = \sum_{n=0}^{\infty} 4^n (a - a^2)^n. \tag{2.96}$$

Secondly, by $\nu < 1/4$ it follows that $1 - 4a + 4a^2$ is invertible in A with inverse $(1 - 4a + 4a^2)^{-1} = \sum_{n=0}^{\infty} (4(a - a^2))^n$ by the Carl Neumann series. Thus $s^2 = (1 - 4a + 4a^2)^{-1}$

and consequently $(2p-1)^2 = ((2a-1)s)^2 = (2a-1)^2 s^2 = (4a^2-4a+1)(1-4a+4a^2)^{-1} = 1$. Moreover, we have that

$$\begin{aligned}
\|p-a\| &= \|(a-1/2)s + 1/2 - a\| \\
&= \|(a-1/2)(s-1)\| \\
&\leq (\|a\| + 1/2)\|s-1\| \\
&\leq (\|a\| + 1/2)\sum_{n=1}^{\infty} \binom{2n}{n} \|a-a^2\|^n \\
&= (\|a\| + 1/2)((1-4\nu)^{-1/2} - 1) \\
&= f_{\|a\|}(\nu) \qquad (2.97)
\end{aligned}$$

by the definition of $f_{\|a\|}$. □

We remark in passing that a slicker proof of Proposition 2.3.14 can be given with the holomorphic functional calculus. We think however that it is beneficial to record a more concrete proof too.

Lemma 2.3.15. *Let E be a Banach space such that $\mathcal{B}(E)$ admits an uncountable family $\mathcal{Q}$ of norm one, commuting, non-compact, almost orthogonal idempotents, and let F be a separable Banach space. Let $\alpha : \mathcal{B}(E) \to \mathcal{B}(F)$ be a bounded linear map such that $\alpha|_{\mathcal{F}(E)} = 0$ and choose $M \in (0, \infty)$ such that $\|\alpha\| \leq M$ and $\delta \in (0, 1/4)$ such that $(f_M(\delta) + M)f_M(\delta) + f_M(\delta)M + \delta < 1$. If $\operatorname{def}(\alpha) < \delta$, then the set*

$$\{P \in \mathcal{Q} : \|\alpha(P)\| \leq f_M(\delta)\} \qquad (2.98)$$

is uncountable.

Proof. Set $\eta := (f_M(\delta) + M)f_M(\delta) + f_M(\delta)M + \delta$. Let us observe that for any $P \in \mathcal{Q}$, $\|\alpha(P) - \alpha(P)\alpha(P)\| < \delta$ holds, and therefore by Lemma 2.3.14 there is an idempotent

$Q_P \in \mathcal{B}(F)$ such that $\|\alpha(P) - Q_P\| \leq f_M(\delta)$. Clearly, either $\|Q_P\| \geq 1$ or $Q_P = 0$. We show that for uncountably many $P \in \mathcal{Q}$, $Q_P = 0$ holds. In order to see this, we shall show that the set $\Omega := \{P \in \mathcal{Q} :\, Q_P \neq 0\}$ is at most countable. To this end, we choose a family $(x_P)_{P\in\Omega}$ such that $x_P \in \mathrm{Ran}(Q_P)$ is a unit vector for each $P \in \Omega$. Let us now successively deduce the following estimates for distinct $P, R \in \Omega$:

$$\|Q_P\| \leq \|Q_P - \alpha(P)\| + \|\alpha(P)\| \leq f_M(\delta) + M, \tag{2.99}$$

$$\begin{aligned} \|Q_P Q_R - \alpha(P)\alpha(R)\| &\leq \|Q_P\|\|Q_R - \alpha(R)\| + \|Q_P - \alpha(P)\|\|\alpha(R)\| \\ &\leq (f_M(\delta) + M) f_M(\delta) + f_M(\delta) M. \end{aligned} \tag{2.100}$$

Hence, by $PR \in \mathcal{F}(E)$ and thus $\alpha(PR) = 0$, we obtain

$$\begin{aligned} \|Q_P Q_R\| &\leq \|Q_P Q_R - \alpha(P)\alpha(R)\| + \|\alpha(P)\alpha(R) - \alpha(PR)\| \\ &\leq (f_M(\delta) + M) f_M(\delta) + f_M(\delta) M + \delta \\ &= \eta < 1. \end{aligned} \tag{2.101}$$

Also,

$$\begin{aligned} 1 = \|x_P\| = \|Q_P x_P\| &\leq \|Q_P x_P - Q_P x_R\| + \|Q_P Q_R x_R\| \\ &\leq \|Q_P\|\|x_P - x_R\| + \|Q_P Q_R\| \\ &\leq (f_M(\delta) + M)\|x_P - x_R\| + \eta. \end{aligned} \tag{2.102}$$

This latter inequality amounts to

$$\frac{1-\eta}{f_M(\delta) + M} \leq \|x_P - x_R\|, \tag{2.103}$$

whenever $P, R \in \Omega$ are distinct. Notice that the left-hand side of (2.103) is strictly greater than zero. Therefore, since F is separable, it follows that $\{x_P : P \in \Omega\}$ is countable.

Since $x_P \neq x_Q$ for distinct $P, Q \in \Omega$, we conclude that Ω is countable. Thus $\mathcal{Q}\backslash\Omega$ is uncountable, equivalently, for uncountably many $P \in \mathcal{Q}$, $Q_P = 0$. Consequently for uncountably many $P \in \mathcal{Q}$, $\|\alpha(P)\| \leq f_M(\delta)$ holds. □

2.4 Proof of the main result

Lemma 2.4.1. *Let E be a Banach space and let F be a reflexive Banach space such that one of the following three conditions is satisfied.*

1. *E has a subsymmetric, shrinking Schauder basis;*
2. *$E = \ell_1$; or*
3. *$E = L_p[0,1]$ $(1 \leq p \leq \infty)$.*

Then the triple $(\mathcal{B}(E), \mathcal{K}(E); \mathcal{B}(F))$ has the pre-AMNM property.

Proof. By reflexivity of F it follows from Lemma 2.2.3 that $\mathcal{B}(F)$ is a dual Banach algebra with isometric predual $(F\hat{\otimes}_\pi F^*, \varphi)$. Thus, by Lemma 2.3.8 it is enough to show that $\mathcal{K}(E)$ is amenable.

If E has a subsymmetric, shrinking Schauder basis then by [30, Theorems 4.5 and 4.2] it follows that $\mathcal{K}(E)$ is amenable. If $E = \ell_1$ or $E = L_p[0,1]$ for $1 \leq p \leq \infty$ then this follows from [30, Theorems 4.7 and 4.2]. □

We are now ready to prove our main result, which we restate for the convenience of the reader.

Theorem 2.4.2. *Let E be a Banach space and let F be a separable, reflexive Banach space such that one of the following three conditions is satisfied.*

1. *E has a subsymmetric, shrinking Schauder basis;*

2. *$E = \ell_1$; or*

3. *$E = L_p[0,1]$ ($1 \leq p < \infty$).*

Then $(\mathcal{B}(E), \mathcal{B}(F))$ *is an AMNM pair.*

Proof. In view of Lemma 2.2.3 we can take the canonical, isometric predual $(F \hat{\otimes}_\pi F^*, \theta)$ of $\mathcal{B}(F)$.

First suppose E has a subsymmetric, shrinking basis. We recall (see Remark 1.2.6) that there is an equivalent renorming of E such that $K_{sub} = K_u = K_{ub} = 1$. Since the AMNM property is an isomorphism invariant, we may suppose that E is endowed with this equivalent renorming. Consequently, by Corollary 2.2.5, it follows that the sequence of associated coordinate projections $(P_n)_{n \in \mathbb{N}}$ is a contractive approximate identity for $\mathcal{K}(E)$.
Now suppose $E \in \{\ell_1, L_p[0,1] : 1 \leq p < \infty\}$, then $\mathcal{K}(E)$ has a contractive approximate identity by Corollary 2.2.6.

In any case, let $(e_\gamma)_{\gamma \in \Gamma}$ be a contractive approximate identity in $\mathcal{K}(E)$, and let $\mathcal{U}$ be an ultrafilter on Γ extending the order filter.

Let $\epsilon, K > 0$ be arbitrary. We introduce the auxiliary constants $M_{\epsilon,K} := \epsilon + K$ and $M := (1 + M_{\epsilon,K})^2 M_{\epsilon,K}$. By Lemma 2.4.1 we can pick a sufficiently small $\delta \in (0, 1/4)$ satisfying the following properties:

- The number δ is at most the pre-AMNM constant of $(\mathcal{B}(E), \mathcal{K}(E); \mathcal{B}(F))$ for $(\epsilon/2, K)$;

- $(f_M(\delta) + M) f_M(\delta) + f_M(\delta) M + \delta < 1$; and

- $\delta + M(\delta + M\delta + M^2 f_M(\delta)) < \epsilon/2$.

Let $\phi : \mathcal{B}(E) \to \mathcal{B}(F)$ be a bounded linear map with $\|\phi\| \leq K$ and $\operatorname{def}(\phi) < \delta$. Thus there is a bounded linear map $\psi : \mathcal{B}(E) \to \mathcal{B}(F)$ such that

- $\|\phi - \psi\| < \epsilon/2$; and
- $\psi(SR) = \psi(S)\psi(R)$ and $\psi(RS) = \psi(R)\psi(S)$ $(S \in \mathcal{B}(E),\, R \in \mathcal{K}(E))$.

Clearly $\|\psi\| \leq \epsilon/2 + K < \epsilon + K = M_{\epsilon,K}$ holds. By Lemma 2.3.10 we can define the idempotent $Q := \theta^{-1}(\text{w*-}\lim_{\gamma\to\mathcal{U}} \theta(\psi(e_\gamma))) \in \mathcal{B}(F)$, where $\|Q\| \leq M_{\epsilon,K}$ since θ is an isometry and $\sup_{\gamma\in\Gamma} \|e_\gamma\| \leq 1$. Also by Lemma 2.3.10 the maps

$$\psi_1 : \mathcal{B}(E) \to \mathcal{B}(F); \quad S \mapsto Q\psi(S)Q \tag{2.104}$$

$$\psi_2 : \mathcal{B}(E) \to \mathcal{B}(F); \quad S \mapsto (I_F - Q)\psi(S)(I_F - Q) \tag{2.105}$$

are bounded linear maps such that

1. $\psi = \psi_1 + \psi_2$;
2. ψ_1 is an algebra homomorphism;
3. $\psi_2|_{\mathcal{K}(E)} = 0$; and
4. for any $S, T \in \mathcal{B}(E)$, $\psi_2(ST) - \psi_2(S)\psi_2(T) = \psi(ST) - \psi(S)\psi(T)$.

It is immediate from the above that $\operatorname{def}(\psi_2) = \operatorname{def}(\psi) < \delta$ and
$\|\psi_2\| \leq (1 + M_{\epsilon,K})^2 M_{\epsilon,K} = M$. By Proposition 2.3.12 and Lemma 2.3.15 we can take a norm one idempotent $P \in \mathcal{B}(E)$ such that $\|\psi_2(P)\| \leq f_M(\delta)$, and there exist $U, V \in \mathcal{B}(E)$ such that $P = UV$, $I_E = VU$ and $\|U\|, \|V\| = 1$. (In fact, we can take uncountably many such idempotents, but we shall not need this here.) Therefore $I_E = VPU$ and thus

$$\|\psi_2(I_E)\| \leq \|\psi_2(VPU) - \psi_2(V)\psi_2(PU)\|$$

$$
\begin{aligned}
&+ \|\psi_2(V)\psi_2(PU) - \psi_2(V)\psi_2(P)\psi_2(U)\| \\
&+ \|\psi_2(V)\psi_2(P)\psi_2(U)\| \\
&\leq \delta + M\delta + M^2 f_M(\delta). \qquad (2.106)
\end{aligned}
$$

We observe that for any $A \in \mathcal{B}(A)$, if $\|A\| \leq 1$ then $\|\psi_2(A)\psi_2(I_E) - \psi_2(A)\| < \delta$. Consequently,

$$
\begin{aligned}
\|\psi_2(A)\| &\leq \|\psi_2(A) - \psi_2(A)\psi_2(I_E)\| + \|\psi_2(A)\|\|\psi_2(I_E)\| \\
&\leq \delta + M(\delta + M\delta + M^2 f_M(\delta)) < \epsilon/2, \qquad (2.107)
\end{aligned}
$$

thus $\|\psi_2\| \leq \epsilon/2$. Consequently,

$$\|\phi - \psi_1\| \leq \|\phi - \psi\| + \|\psi_2\| < \epsilon/2 + \epsilon/2 = \epsilon. \qquad (2.108)$$

Since ψ_1 is a continuous algebra homomorphism, this shows that $(\mathcal{B}(E), \mathcal{B}(F))$ has the AMNM property. □

The main idea of the following lemma is implicitly contained in the proof of [38, Corollary 3.4].

Lemma 2.4.3. *Let A be a Banach algebra with a bounded right approximate identity. Let B be a Banach algebra and let I be a closed, two-sided ideal of B. If (A, B) has the AMNM property then so does (A, I).*

Proof. Let $(e_\gamma)_{\gamma \in \Gamma}$ be a bounded right approximate identity in A with $\sup_{\gamma \in \Gamma} \|e_\gamma\| \leq M$, where $M > 0$. Let $K > 0$ and $\epsilon \in (0, 1/2M)$ be arbitrary. Since (A, B) has the AMNM property, there exists $\delta > 0$ such that for any $\psi \in \mathcal{B}(A, B)$ with $\|\psi\| < K$ and $\operatorname{def}(\psi) < \delta$ it follows that $\operatorname{dist}(\psi) < \epsilon$. Let us fix $\psi \in \mathcal{B}(A, I)$ with $\|\psi\| < K$ and $\operatorname{def}(\psi) < \delta$. Clearly, $\psi \in \mathcal{B}(A, B)$ thus there exists $\varphi \in \operatorname{Mult}(A, B)$ such that $\|\psi - \varphi\| < \epsilon$. Let

$\pi : B \to B/I$ denote the quotient map. Since $\mathrm{Ran}(\psi) \subseteq I$, it follows that for any $c \in A$, $\pi(\varphi(c)) = \pi(\varphi(c)) - \pi(\psi(c))$ and thus $\|\pi(\varphi(c))\| \leq \|\varphi - \psi\| \cdot \|c\| \leq \epsilon\|c\|$. We will now show that $\mathrm{Ran}(\varphi) \subseteq I$. Assume towards a contradiction that there is $a \in A$ such that $\pi(\varphi(a)) \neq 0$. Then we obtain

$$\begin{aligned}\|\pi(\varphi(a))\| &= \lim_{\gamma\in\Gamma} \|\pi(\varphi(ae_\gamma))\| \\ &= \lim_{\gamma\in\Gamma} \|\pi(\varphi(a))\pi(\varphi(e_\gamma))\| \\ &\leq \|\pi(\varphi(a))\|\Big(\sup_{\gamma\in\Gamma} \|\pi(\varphi(e_\gamma))\|\Big) \\ &\leq \|\pi(\varphi(a))\| \cdot \epsilon \cdot \sup_{\gamma\in\Gamma} \|e_\gamma\| \\ &\leq \|\pi(\varphi(a))\|\epsilon M \\ &< \frac{1}{2}\|\pi(\varphi(a))\|, \end{aligned} \tag{2.109}$$

a contradiction. Therefore $\mathrm{Ran}(\varphi) \subseteq I$ and thus $\varphi \in \mathrm{Mult}(A, I)$, proving that (A, I) has the AMNM property. □

We restate the key corollary for the convenience of the reader.

Corollary 2.4.4. *Let E be a Banach space satisfying the conditions of Theorem 2.1.2. Let F be a separable, reflexive Banach space such that F has the bounded approximation property. Then $(\mathcal{B}(E), \mathcal{A}(F))$ is an AMNM pair, where $\mathcal{A}(E)$ denotes the approximable operators on E.*

Proof. Since F is reflexive and $F \simeq F^{**}$ has the bounded approximation property it follows from [15, Proposition A.3.60(iv)] that F^* also has the bounded approximation property. By Theorem 2.2.4 this is equivalent to $\mathcal{A}(F)$ having a bounded right approximate identity. Thus by Theorem 2.1.2 and Lemma 2.4.3 the claim follows. □

2.5 An approximate version of a lemma of Daws

In this section we prove a "δ-perturbation" analogue of the following result, observed by Daws in [17]:

Lemma 2.5.1. *([17, Lemma 3.3.14]) Let E, F be Banach spaces and suppose there exists a non-zero, continuous algebra homomorphism $\varphi : \mathcal{A}(E) \to \mathcal{B}(F)$. Then there exist $T \in \mathcal{B}(E, F)$ and $S \in \mathcal{B}(E^*, F^*)$ such that $I_{E^*} = T^*S$.*

Although Daws in [17] does not state it in this form, he actually proves the above result. In fact, with a bit of extra work, we can say slightly more.

Proposition 2.5.2. *Let E, F be Banach spaces, then the following are equivalent:*

1. *There exists a non-zero, continuous algebra homomorphism $\varphi : \mathcal{A}(E) \to \mathcal{B}(F)$,*

2. *there exist $T \in \mathcal{B}(E, F)$ and $S \in \mathcal{B}(E^*, F^*)$ such that $I_{E^*} = T^*S$.*

Moreover, suppose the following stronger version of condition 2 holds: There exist $T \in \mathcal{B}(E, F)$ and $S \in \mathcal{B}(E^*, F^*)$ such that $I_{E^*} = T^*S$ and S is $\sigma(E^*, E)$ - to - $\sigma(F^*, F)$ continuous. Then there is a continuous, injective algebra homomorphism $\theta : \mathcal{B}(E) \to \mathcal{B}(F)$.

Proof. $(1 \Rightarrow 2)$. Assume $\varphi : \mathcal{A}(E) \to \mathcal{B}(F)$ is a non-zero, continuous algebra homomorphism. In the proof of [17, Lemma 3.3.14] it is shown that there exist $T \in \mathcal{B}(E, F)$ and $S \in \mathcal{B}(E^*, F^*)$ such that $S^* \circ \kappa_F \circ T = \kappa_E$. Thus $I_{E^*} = T^*S$, since for all $x \in E$ and $f \in E^*$:

$$\langle x, T^*Sf \rangle = \langle Tx, Sf \rangle = \langle Sf, \kappa_F(Tx) \rangle = \langle f, (S^* \circ \kappa_F \circ T)x \rangle = \langle f, \kappa_E(x) \rangle = \langle x, f \rangle. \tag{2.110}$$

$(2 \Rightarrow 1)$. Assume there exist $T \in \mathcal{B}(E, F)$ and $S \in \mathcal{B}(E^*, F^*)$ such that $I_{E^*} = T^*S$. Let $u_E : \mathcal{A}(E) \to E \hat{\otimes}_\epsilon E^*$ and $u_F : \mathcal{A}(F) \to F \hat{\otimes}_\epsilon F^*$ be isometric isomorphisms from

Lemma 1.2.13. It is clear that $\varphi := u_F^{-1} \circ (T \otimes_\epsilon S) \circ u_E$ is a bounded linear map from $\mathcal{A}(E)$ to $\mathcal{A}(F)$ with $\|\varphi\| = \|T\|\|S\|$. In particular φ is a non-zero, continuous, linear map; we show that it is multiplicative. To this end, let $x, y, z \in E$ and $f, g \in E^*$ be arbitrary. Then

$$\begin{aligned}\varphi(x \otimes f)\varphi(y \otimes g)z &= \varphi(x \otimes f)(Ty \otimes Sg)z = \varphi(x \otimes f)\langle z, Sg\rangle Ty \\ &= \langle z, Sg\rangle (Tx \otimes Sf)Ty = \langle z, Sg\rangle\langle Ty, Sf\rangle Tx \\ &= \langle z, Sg\rangle\langle y, T^*Sf\rangle Tx = \langle z, Sg\rangle\langle y, f\rangle Tx \\ &= \langle y, f\rangle\langle z, Sg\rangle Tx = \langle y, f\rangle (Tx \otimes Sg)z \\ &= \langle y, f\rangle \varphi(x \otimes g)z = \varphi\left(\langle y, f\rangle x \otimes g\right) z \\ &= \varphi\left((x \otimes f)(y \otimes g)\right) z. \end{aligned} \tag{2.111}$$

By continuity and linearity of φ it follows that it is multiplicative on $\mathcal{A}(E)$.

For the last part, assume there exists $T \in \mathcal{B}(E, F)$ and $S \in \mathcal{B}(E^*, F^*)$ such that $I_{E^*} = T^*S$ and S is $\sigma(E^*, E)$ - to - $\sigma(F^*, F)$ continuous. By Proposition 2.2.2 this is equivalent to saying that there exists $R \in \mathcal{B}(F, E)$ with $R^* = S$. This immediately yields $I_E = RT$, since for any $x \in E$ and $f \in E^*$

$$\langle RTx, f\rangle = \langle Tx, R^*f\rangle = \langle Tx, Sf\rangle = \langle x, T^*Sf\rangle = \langle x, f\rangle. \tag{2.112}$$

We define

$$\theta : \ \mathcal{B}(E) \to \mathcal{B}(F); \quad A \mapsto TAR, \tag{2.113}$$

this is clearly a linear map with $\|\theta\| \leq \|T\|\|S\|$. It is multiplicative, since for all $A, B \in \mathcal{B}(E)$

$$\theta(A)\theta(B) = TARTBR = TABR = \theta(AB). \tag{2.114}$$

Since $P := TR \in \mathcal{B}(F)$ is a non-zero idempotent and $\theta(I_E) = P$, in fact $\|\theta\| \geq 1$ follows. We now show that θ extends $\varphi := u_F^{-1} \circ (T \otimes_\epsilon S) \circ u_E$. To see this, we fix $x \in E$, $f \in E^*$ and $y \in F$. Then

$$\begin{aligned}\theta(x \otimes f)y = T(x \otimes f)Ry = T\left(\langle Ry, f\rangle x\right) = \langle Ry, f\rangle Tx = \langle y, Sf\rangle Tx \\ = (Tx \otimes Sf)y = \varphi(x \otimes f)y,\end{aligned} \tag{2.115}$$

thus $\theta(x \otimes f) = \varphi(x \otimes f)$. By linearity and continuity of θ and φ, it follows that $\theta|_{\mathcal{A}(E)} = \varphi$.

It remains to show that θ is injective. Assume towards a contradiction that it is not, then $\mathcal{A}(E) \subseteq \operatorname{Ker}(\theta)$, so $\varphi = \theta|_{\mathcal{A}(E)} = 0$, which is impossible. □

In particular, the result above yields a cheap way of obtaining continuous, injective homomorphism on the algebra of bounded linear operators on reflexive Banach space, provided that there is a non-zero, continuous algebra homomorphism on the algebra of approximable operators:

Corollary 2.5.3. *Let E, F be Banach spaces such that E is reflexive. Let $\varphi : \mathcal{A}(E) \to \mathcal{B}(F)$ be a non-zero, continuous algebra homomorphism. Then there is a continuous, injective algebra homomorphism $\theta : \mathcal{B}(E) \to \mathcal{B}(F)$.*

Proof. By the first part of Proposition 2.5.2 there exist $T \in \mathcal{B}(E, F)$ and $S \in \mathcal{B}(E^*, F^*)$ with $I_{E^*} = T^*S$. Since E is reflexive, S is $\sigma(E^*, E)$ - to - $\sigma(F^*, F)$ continuous thus the result immediately follows from the second part of Proposition 2.5.2. □

We recall that if A and B are Banach algebras, $\delta > 0$ and $\varphi : A \to B$ is a bounded linear map, then φ is *δ-multiplicative* if $\operatorname{def}(\varphi) \leq \delta$, or equivalently, $\|\varphi(ab) - \varphi(a)\varphi(b)\| \leq \delta$ whenever $a, b \in A$ satisfy $\|a\|, \|b\| \leq 1$.

Lemma 2.5.4. *Let A and B be Banach algebras. Let $\epsilon \in (0,1)$ and let $\varphi : A \to B$ be a contractive, δ-multiplicative linear map, where $0 < \delta < \frac{1}{4}(1 - (\frac{2}{3}\epsilon + 1)^{-2})$. Then for any norm one idempotent $p \in A$ either $\|\varphi(p)\| < \epsilon$ or $\|\varphi(p)\| > 1 - \epsilon$.*

Proof. Since $\delta < 1/4$ and $\nu := \|\varphi(p) - \varphi(p)\varphi(p)\| < \delta$, it follows from Proposition 2.3.14 that there exists an idempotent $q \in B$ with

$$\|\varphi(p) - q\| \leq f_1(\nu) = \frac{3}{2}((1 - 4\delta)^{-1/2} - 1) < \epsilon. \tag{2.116}$$

If $q = 0$ then $\|\varphi(p)\| < \epsilon$. Otherwise, since q is an idempotent, it follows that $\|q\| \geq 1$ and therefore

$$\|\varphi(p)\| \geq \|q\| - \|q - \varphi(p)\| > 1 - \epsilon. \tag{2.117}$$

□

Proposition 2.5.5. *Let A and B be Banach algebras such that A has a bounded left approximate identity $(p_\gamma)_{\gamma\in\Gamma}$ consisting of norm one idempotents. Let $\epsilon \in (0, 1/2)$ be fixed. Let $\varphi : A \to B$ be a linear, norm one, δ-multiplicative map, where $0 < \delta < \frac{1}{4}(1 - (\frac{2}{3}\epsilon + 1)^{-2})$. Then there exists $\gamma \in \Gamma$ such that $\|\varphi(p_\gamma)\| > 1 - \epsilon$.*

Proof. Assume towards a contradiction that there is no $\gamma \in \Gamma$ such that $\|\varphi(p_\gamma)\| > 1-\epsilon$ holds. Then by Lemma 2.5.4 it follows that for every $\gamma \in \Gamma$, $\|\varphi(p_\gamma)\| < \epsilon$. Let $a \in A$ be such that $\|a\| \leq 1$. Since $\lim_{\gamma\in\Gamma} p_\gamma a = a$, from the continuity of φ it follows that $\|\varphi(a)\| = \lim_{\gamma\in\Gamma} \|\varphi(p_\gamma a)\| \leq \sup_{\gamma\in\Gamma} \|\varphi(p_\gamma a)\|$. Also, by the δ-multiplicativity of φ, we obtain for any $\gamma \in \Gamma$:

$$\|\varphi(p_\gamma a)\| \leq \|\varphi(p_\gamma a) - \varphi(p_\gamma)\varphi(a)\| + \|\varphi(p_\gamma)\varphi(a)\| \leq \delta + \epsilon. \tag{2.118}$$

Consequently $\|\varphi\| \leq \delta + \epsilon$, which by $\delta + \epsilon < 1/4 + 1/2 < 1$ yields a contradiction. □

Lemma 2.5.6. *Let E be a Banach space with a monotone, normalised Schauder basis $(b_n)_{n\in\mathbb{N}}$, and let $(f_n)_{n\in\mathbb{N}}$ denote the sequence of coordinate functionals. Let $\epsilon \in (0, 1/4)$ be fixed. Let F be a Banach space and assume $\varphi : \mathcal{K}(E) \to \mathcal{B}(F)$ is a linear, norm one, δ-multiplicative map, where $0 < \delta < \frac{1}{4}(1 - (\frac{2}{3}\epsilon + 1)^{-2})$. Then there exists $n \in \mathbb{N}$ such that $2\delta < \|\varphi(b_n \otimes f_n)\|$.*

Proof. Let us first observe that the choices of ϵ and δ guarantee $1 - \epsilon > 1 - 2\epsilon > 2\delta$. By Corollary 2.2.5 the sequence of coordinate projections $(P_n)_{n\in\mathbb{N}}$ is a bounded left approximate identity for $\mathcal{K}(E)$ consisting of norm one idempotents. By Proposition 2.5.5 there exists $n \in \mathbb{N}$ such that $\|\varphi(P_n)\| > 1 - \epsilon$. Let $N \in \mathbb{N}$ be the smallest such n. If $N = 1$ then $P_N = P_1 = b_1 \otimes f_1$ and the claim follows. Otherwise $N \geq 2$. Then $\|\varphi(P_{N-1})\| \leq 1 - \epsilon$ by the definition of N, therefore Lemma 2.5.4 implies that $\|\varphi(P_{N-1})\| < \epsilon$. Since $b_N \otimes f_N = P_N - P_{N-1}$, we have

$$\|\varphi(b_N \otimes f_N)\| = \|\varphi(P_N) - \varphi(P_{N-1})\| \geq \|\varphi(P_N)\| - \|\varphi(P_{N-1})\| > 1 - 2\epsilon, \tag{2.119}$$

as required. □

We are now ready to state and prove the main result of this section. Having done the necessary preparations, the proof is just a straightforward modification of Daws's Lemma 2.5.1.

Theorem 2.5.7. *Let E and F be Banach spaces where E has a monotone, normalised Schauder basis. Let $\varphi : \mathcal{K}(E) \to \mathcal{B}(F)$ be a linear, norm one, δ-multiplicative map, where $0 < \delta < 13/196$. Then E is isomorphic to a closed subspace of F.*

Proof. Note that $\frac{1}{4}(1 - (\frac{7}{6})^{-2}) = \frac{13}{196}$. Let $(b_n)_{n\in\mathbb{N}}$ be a monotone, normalised basis for E, let $(f_n)_{n\in\mathbb{N}}$ denote the sequence of coordinate functionals. By Lemma 2.5.6, there exists $n \in \mathbb{N}$ such that $2\delta < \|\varphi(b_n \otimes f_n)\|$, let $b := b_n$ and $f := f_n$. Thus there exists $\tilde{y} \in F$ such that $\|\tilde{y}\| = 1$ and $2\delta < \|(\varphi(b \otimes f))(\tilde{y})\|$. Now let us define $y := \|(\varphi(b \otimes f))(\tilde{y})\|^{-1}\tilde{y} \in F$. Then clearly $\gamma := \|y\| = \|(\varphi(b \otimes f))(\tilde{y})\|^{-1} < 1/2\delta$ and $\|(\varphi(b \otimes f))(y)\| = 1$. We observe that in particular $2\gamma\delta < 1$ holds. Let $\lambda \in F^*$ be such that $\|\lambda\| = 1$ and $\langle(\varphi(b \otimes f))(y), \lambda\rangle = 1$. Now let us define the following maps:

$$\begin{aligned} T :\ & E \to F; \quad x \mapsto (\varphi(x \otimes f))(y) \\ S :\ & E^* \to F^*; \quad \mu \mapsto (\varphi(b \otimes \mu))^*(\lambda). \end{aligned} \tag{2.120}$$

We immediately see that both S and T are linear with $\|T\| \leq 2\gamma$, $\|S\| \leq 1$ therefore $T \in \mathcal{B}(E, F)$ and $S \in \mathcal{B}(E^*, F^*)$. Now let us observe that for any $x \in E$ and $\mu \in E^*$ with $\|x\|, \|\mu\| \leq 1$ the following identities hold:

$$\begin{aligned} \langle(\varphi((b \otimes \mu)(x \otimes f)))(y), \lambda\rangle &= \langle(\varphi(\langle x, \mu\rangle(b \otimes f)))(y), \lambda\rangle \\ &= \langle x, \mu\rangle\langle(\varphi(b \otimes f))(y), \lambda\rangle \\ &= \langle \mu, \kappa_E(x)\rangle \end{aligned} \tag{2.121}$$

and

$$\begin{aligned} \langle(\varphi(b \otimes \mu)\varphi(x \otimes f))(y), \lambda\rangle &= \langle(\varphi(x \otimes f))(y), (\varphi(b \otimes \mu))^*(\lambda)\rangle \\ &= \langle T(x), S(\mu)\rangle \end{aligned}$$

$$= \langle \mu, (S^* \circ \kappa_F \circ T)(x) \rangle. \tag{2.122}$$

Since $\|b \otimes \mu\| \leq 1$ and $\|x \otimes f\| \leq 2$, the δ-multiplicativity of φ yields

$$\begin{aligned} &|\langle \mu, (S^* \circ \kappa_F \circ T - \kappa_E)(x) \rangle| \\ &= |\langle (\varphi(b \otimes \mu)\varphi(x \otimes f) - \varphi((b \otimes \mu)(x \otimes f)))\,(y), \lambda \rangle| \\ &\leq \|\varphi(b \otimes \mu)\varphi(x \otimes f) - \varphi((b \otimes \mu)(x \otimes f))\| \cdot \|y\| \cdot \|\lambda\| \\ &\leq 2\gamma\delta. \end{aligned} \tag{2.123}$$

Therefore $\|S^* \circ \kappa_F \circ T - \kappa_E\| \leq 2\gamma\delta$ holds, consequently, since κ_E is an isometry, it follows that $S^* \circ \kappa_F \circ T$ is bounded below by $1 - 2\gamma\delta > 0$. Thus T is bounded below, equivalently $T|^{\mathrm{Ran}(T)} : E \to \mathrm{Ran}(T)$ is an isomorphism of Banach spaces, proving the claim. □

2.6 The AMNM property and bounded Hochschild cohomology

Let A be a Banach algebra and let X be a Banach A-bimodule. We define the *semi-direct product* $A \ltimes X$ to be the Banach space $A \oplus_1 X$ endowed with the product

$$(a, x)(b, y) := (ab, ay + xb) \quad (a, b \in A\ x, y \in X). \tag{2.124}$$

It is easy to see that $A \ltimes X$ is a Banach algebra.

In the following, for a Banach algebra A and a Banach A-bimodule X, the symbol $\mathrm{Bil}(A, A; X)$ stands for the Banach space of bounded bilinear maps from $A \times A$ to X.

A map $T \in \mathrm{Bil}(A, A; X)$ is called a *2-cocycle* if

$$aT(b,c) - T(ab,c) + T(a,bc) - T(a,b)c = 0 \quad (a,b,c \in A). \tag{2.125}$$

The set of 2-cocycles is denoted by $\mathcal{Z}^2(A, X)$, and it is a closed linear subspace of $\mathrm{Bil}(A, A; X)$. We define the map $\delta : \mathcal{B}(A, X) \to \mathrm{Bil}(A, A; X)$ by

$$\delta(S)(a,b) := S(ab) - S(a)b - aS(b) \quad (a,b \in A), \tag{2.126}$$

it is easy to see that δ is a bounded linear map. We now define $\mathcal{N}^2(A, X) := \mathrm{Ran}(\delta)$, elements of which are called *2-coboundaries*. It follows that $\mathcal{N}^2(A, X)$ is a linear subspace of $\mathrm{Bil}(A, A; X)$, however, in general $\mathcal{N}^2(A, X)$ is not closed. The following result is standard, see for example [15].

Lemma 2.6.1. *Let A be a Banach algerba and let X be a Banach A-bimodule. Then $\mathcal{N}^2(A, X) \subseteq \mathcal{Z}^2(A, X)$.*

Proof. Let $T \in \mathcal{N}^2(A, X)$, then there exists $S \in \mathcal{B}(A, X)$ with $T = \delta(S)$, thus

$$T(a,b) = S(ab) - S(a)b - aS(b) \quad (a,b \in A). \tag{2.127}$$

We need to show that for any $c, d, e \in A$ the identity

$$cT(d,e) - T(cd,e) + T(c,de) - T(c,d)e = 0 \tag{2.128}$$

holds. This however readily follows from the identities

$$cT(d,e) = cS(de) - cS(d)e - cdS(e) \tag{2.129}$$

$$T(cd,e) = S(cde) - S(cd)e - cdS(e) \tag{2.130}$$

$$T(c, de) = S(cde) - S(c)de - cS(de) \tag{2.131}$$

$$T(c, d)e = S(cd)e - S(c)de - cS(d)e. \tag{2.132}$$

□

The *second bounded Hochschild cohomoloy group* is defined as

$$\mathcal{H}^2(A, X) := \mathcal{Z}^2(A, X)/\mathcal{N}^2(A, X), \tag{2.133}$$

it is clear from our discussion that $\mathcal{H}^2(A, X)$ is a semi-normed vector space which is a Banach space if and only if $\mathcal{N}^2(A, X)$ is closed.

Theorem 2.6.2. *Let A be a Banach algebra and let X be a Banach A-bimodule such that the pair $(A, A \ltimes X)$ has the AMNM property. Then the second bounded Hochschild cohomology group $\mathcal{H}^2(A, X)$ is a Banach space.*

Proof. We prove the contrapositive. Assume A is a Banach algebra and X is a Banach A-bimodule such that $\mathcal{H}^2(A, X)$ is not a Banach space, or equivalently, $\mathcal{N}^2(A, X)$ is not closed. We show that the pair $(A, A \ltimes X)$ does not have the AMNM property. By the Fundamental Isomorphism Theorem, there exists a unique bounded linear injective map

$$\tilde{\delta} : \mathcal{B}(A, X)/\operatorname{Ker}(\delta) \to \operatorname{Bil}(A, A; X) \tag{2.134}$$

such that $\tilde{\delta} \circ \pi = \delta$ and $\|\delta\| = \|\tilde{\delta}\|$, where

$$\pi : \mathcal{B}(A, X) \to \mathcal{B}(A, X)/\operatorname{Ker}(\delta) \tag{2.135}$$

denotes the quotient map. Since $\mathcal{N}^2(A, X) = \operatorname{Ran}(\delta) = \operatorname{Ran}(\tilde{\delta})$ is not closed, it follows that $\tilde{\delta}$ cannot be bounded below.

We need to show that there exist $\epsilon, K > 0$ such that for all $\nu \in (0,1)$ there exists $R \in \mathcal{B}(A, A \ltimes X)$ with $\|R\| < K$, $\operatorname{def}(R) < \nu$ and $\operatorname{dist}(R) \geq \epsilon$. Let $K := 3$ and let $\epsilon \in (0, 1/5)$ be fixed. Let $\nu \in (0,1)$ be arbitrary. Since $\tilde{\delta}$ is not bounded below, there exists $S \in \mathcal{B}(A, X)$ such that $\|\pi(S)\| = 1$ and $\|\tilde{\delta}(\pi(S))\| < \nu$, or equivalently, $\|\delta(S)\| < \nu$. We can assume without loss of generality that $\|S\| < 1 + \nu < 2$.
We define

$$R: \ A \to A \ltimes X; \quad a \mapsto (a, -S(a)). \tag{2.136}$$

Clearly, R is a bounded linear operator with $\|R\| \leq 1 + \|S\| < 3$. We show that $\operatorname{def}(R) = \|\delta(S)\|$. In order to see this, let us fix $a, b \in A$, by definition of the product on $A \ltimes X$ we obtain

$$R(a)R(b) = (a, -S(a))(b, -S(b)) = (ab, -S(a)b - aS(b)) \tag{2.137}$$

and therefore

$$\|R(ab) - R(a)R(b)\| = \|S(ab) - S(a)b - aS(b)\| = \|\delta(S)(a, b)\|, \tag{2.138}$$

proving $\operatorname{def}(R) = \|\delta(S)\|$, as required. This yields $\operatorname{def}(R) < \nu$.
It remains to show that $\operatorname{dist}(R) \geq \epsilon$. Assume towards a contradiction that $\operatorname{dist}(R) < \epsilon$, that is, there exists $\phi \in \operatorname{Mult}(A, A \ltimes X)$ with $\|R - \phi\| < \epsilon$. In particular, $\|\phi\| \leq \|\phi - R\| + \|R\| < \epsilon + 3$. Let us define

$$\begin{aligned} \pi_A : \ & A \ltimes X \to A; \quad (a, x) \mapsto a, \\ \pi_X : \ & A \ltimes X \to X; \quad (a, x) \mapsto x, \end{aligned} \tag{2.139}$$

it is clear that both π_A and π_X are bounded linear maps with norms at most 1. Introducing the bounded linear maps $\phi_A := \pi_A \circ \phi \in \mathcal{B}(A)$ and $\phi_X := \pi_X \circ \phi \in \mathcal{B}(A, X)$, we can write $\phi(a) = (\phi_A(a), \phi_X(a))$ for all $a \in A$. Since ϕ is multiplicative, for any $a, b \in A$ the identity $\phi(ab) = \phi(a)\phi(b)$ amounts to

$$\phi_A(ab) = \phi_A(a)\phi_A(b) \tag{2.140}$$

and

$$\phi_X(ab) = \phi_A(a)\phi_X(b) + \phi_X(a)\phi_A(b). \tag{2.141}$$

Equation (2.140) shows that ϕ_A is multiplicative. Since $A \ltimes X$ is endowed with the $\|\cdot\|_1$-norm, we immediately have that for any $a \in A$

$$\begin{aligned} \|a - \phi_A(a)\| + \| - S(a) - \phi_X(a)\| &= \|(a - \phi_A(a), -S(a) - \phi_X(a))\| \\ &= \|(a, -S(a)) - (\phi_A(a), \phi_X(a))\| \\ &= \|R(a) - \phi(a)\|, \end{aligned} \tag{2.142}$$

thus

$$\|\mathrm{id}_A - \phi_A\|, \|S + \phi_X\| \leq \|R - \phi\|. \tag{2.143}$$

In particular, $\|\mathrm{id}_A - \phi_A\| < \epsilon < 1$, consequently by the Carl Neumann series there exists the continuous algebra homomorphism $\phi_A^{-1} : A \to A$ with

$$\|\mathrm{id}_A - \phi_A^{-1}\| \leq \frac{\|\mathrm{id}_A - \phi_A\|}{1 - \|\mathrm{id}_A - \phi_A\|}. \tag{2.144}$$

Let $T := -\phi_X \circ \phi_A^{-1} \in \mathcal{B}(A, X)$. We show that $T \in \operatorname{Ker}(\delta)$. Indeed, for any $a, b \in A$, using (2.141) and the multiplicative property of ϕ_A^{-1} a direct calculation shows:

$$\begin{aligned}\delta(T)(a,b) &= -aT(b) + T(ab) - T(a)b \\ &= a\phi_X(\phi_A^{-1}(b)) - \phi_X(\phi_A^{-1}(ab)) + \phi_X(\phi_A^{-1}(a))b \\ &= a\phi_X(\phi_A^{-1}(b)) - \phi_A(\phi_A^{-1}(a))\phi_X(\phi_A^{-1}(b)) \\ &\quad - \phi_X(\phi_A^{-1}(a))\phi_A(\phi_A^{-1}(b)) + \phi_X(\phi_A^{-1}(a))b \\ &= 0. \end{aligned} \tag{2.145}$$

Now we observe that by Equations (2.143) and (2.144) we obtain

$$\begin{aligned}\|S - T\| &\leq \|S + \phi_X\| + \| - \phi_X - T\| \\ &\leq \|S + \phi_X\| + \|\phi_X\|\|\mathrm{id}_A - \phi_A^{-1}\| \\ &\leq \|S + \phi_X\| + \|\phi\|\|\mathrm{id}_A - \phi_A\|(1 - \|\mathrm{id}_A - \phi_A\|)^{-1} \\ &\leq \|R - \phi\| + \|\phi\|\|R - \phi\|(1 - \|R - \phi\|)^{-1} \\ &< \epsilon + (\epsilon + 3)\epsilon(1 - \epsilon)^{-1} = 4\epsilon(1 - \epsilon)^{-1}. \end{aligned} \tag{2.146}$$

Since $\epsilon \in (0, 1/5)$ and $T \in \operatorname{Ker}(\delta)$, it follows that

$$\|\pi(S)\| \leq \|S - T\| \leq 4\epsilon(1 - \epsilon)^{-1} < 1, \tag{2.147}$$

contradicting $\|\pi(S)\| = 1$. Thus $\operatorname{dist}(S) \geq \epsilon$ must hold, as required. □

In the following we show that the converse of Theorem 2.6.2 is not true. To see this, we recall two key results of Johnson:

Lemma 2.6.3. *([37, Corollary 3.5]) If A is a commutative Banach algebra and J is a closed ideal in A such A/J is finite-dimensional then $(A, \mathbb{C})$ has the AMNM* *property if and only if $(A/J, \mathbb{C})$ has the AMNM property. In particular $(A, \mathbb{C})$ has the AMNM property if and only if $(A^\sharp, \mathbb{C})$ has the AMNM property, where $A^\sharp$ denotes the unitisation of A.*

Proposition 2.6.4. *([38, Proposition 5.1]) Let A and B be Banach algebras and suppose that B is commutative and contains a non-zero idempotent. If (A, B) has the AMNM property then $(B, \mathbb{C})$ has the AMNM property.*

Remark 2.6.5. We are now ready to show that the converse of Theorem 2.6.2 is not true. Indeed, if we let A be a Banach algebra such that (A, A) does not have the AMNM property and we let X be the trivial module, then clearly A and $A \ltimes X$ are isomorphic, thus $(A, A \ltimes X)$ does not have the AMNM property, but $\mathcal{H}^2(A, X) = \{0\}$ is obviously a Banach space. To see that the above is not vacuous, we need to show that there exists a Banach algebra A such that (A, A) does not have the AMNM property. Let $\mathcal{V}$ denote the Volterra algebra. As was shown by Johnson in [37, Example 8.6], the pair $(\mathcal{V}, \mathbb{C})$ does not have the AMNM property. Equivalently, in view of Lemma 2.6.3, the pair $(\mathcal{V}^\sharp, \mathbb{C})$ does not have the AMNM property, thus Proposition 2.6.4 yields that the pair $(\mathcal{V}^\sharp, \mathcal{V}^\sharp)$ does not have the AMNM property either. (We remark in passing that in order to apply Proposition 2.6.4 it was necessary to consider the unitisation $\mathcal{V}^\sharp$, as $\mathcal{V}$ itself is a *radical* Banach algebra, that is, $\lim_{n\to\infty} \|a^n\|^{1/n} = 0$ for every $a \in \mathcal{V}$ and therefore it does not have non-trivial idempotents.)

Remark 2.6.6. The significance of Theorem 2.6.2 is that it allows one to construct pairs of Banach algebras lacking the AMNM property by appealing to known examples of Banach algebras with their second bounded Hochschild cohomology group being non-Banach for some coefficient module X. More precisely, if A is a Banach algebra such that there exists a Banach A-bimodule X such that $\mathcal{H}^2(A, X)$ is not a Banach space then by Theorem 2.6.2 the pair $(A, A \ltimes X)$ does not have the AMNM property.

Chapter 3

The SHAI property for Banach spaces

3.1 Introduction and preliminaries

A classical result of Eidelheit (see for example [15, Theorem 2.5.7]) asserts that if X, Y are Banach spaces then they are isomorphic if and only if their algebras of operators $\mathcal{B}(X)$ and $\mathcal{B}(Y)$ are isomorphic as Banach algebras, in the sense that there exists a continuous bijective algebra homomorphism $\psi : \mathcal{B}(X) \to \mathcal{B}(Y)$. It is natural to ask whether for some class of Banach spaces X this theorem can be strengthened in the following sense: If Y is a non-zero Banach space and $\psi : \mathcal{B}(X) \to \mathcal{B}(Y)$ is a continuous, surjective algebra homomorphism, is ψ automatically injective?

It is easy to find an example of a Banach space with this property. Indeed, let X be a finite-dimensional Banach space, let Y be a non-zero Banach space and let $\psi : \mathcal{B}(X) \to \mathcal{B}(Y)$ be a surjective algebra homomorphism. Since $\mathcal{B}(X) \simeq M_n(\mathbb{C})$ for some $n \in \mathbb{N}$, simplicity of $M_n(\mathbb{C})$ implies that $\operatorname{Ker}(\psi) = \{0\}$. One can also obtain an infinite-dimensional example: Let $\mathcal{H}$ be a separable, infinite-dimensional Hilbert space, let Y be a non-zero Banach space and let $\psi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(Y)$ be a continuous, surjective algebra homomorphism. Since $\mathrm{Ker}(\psi)$ is a non-trivial, closed, two-sided ideal in $\mathcal{B}(\mathcal{H})$, by the well-known ideal classification result due to Calkin ([9]), $\mathrm{Ker}(\psi) = \{0\}$ or $\mathrm{Ker}(\psi) = \mathcal{K}(\mathcal{H})$ must hold. In the latter case, $\mathrm{Cal}(\mathcal{H}) := \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H}) \simeq \mathcal{B}(Y)$. Clearly $\mathrm{Cal}(\mathcal{H})$ is simple and infinite-dimensional. If Y is infinite-dimensional, then $\mathcal{B}(Y)$ is not simple, which is impossible; if Y is finite-dimensional then so is $\mathcal{B}(Y)$, a contradiction. Thus ψ must be injective. This simple observation ensures that the following definition is not vacuous.

Definition 3.1.1. A Banach space X has the *SHAI property* (Surjective Homomorphisms Are Injective) if for every non-zero Banach space Y every surjective algebra homomorphism $\psi : \mathcal{B}(X) \to \mathcal{B}(Y)$ is automatically injective.

We will show in this chapter that all of the following Banach spaces have the SHAI property:

(1) c_0 and ℓ_p for all $1 \leq p \leq \infty$ (Example 3.3.8);

(2) Schlumprecht's arbitrarily distortable Banach space $\mathbf{S}$ (Corollary 3.3.12);

(3) the spaces $\left(\bigoplus_{n \in \mathbb{N}} \ell_2^n \right)_Y$ where $Y \in \{c_0, \ell_1\}$ (Theorem 3.4.6);

(4) Hilbert spaces of arbitrary density character (Theorem 3.4.8);

(5) $F \oplus G$ if F and G are Banach spaces with the SHAI property (Proposition 3.5.3).

We remark in passing that the stability of the SHAI property under finite sums is of interest to us since $\mathcal{B}(F \oplus G)$ can have a very complicated lattice of closed two-sided ideals even if $\mathcal{B}(F)$ and $\mathcal{B}(G)$ themselves have the simplest possible ideal structure, we refer the reader to [23] and [73].

In the last part of this chapter, we show that for every separable, reflexive Banach space X, there exists a Banach space Y_X and a surjective, non-injective algebra homomorphism $\Theta : \mathcal{B}(Y_X) \to \mathcal{B}(X)$. Details of the construction, and some of its extra properties are given in Section 3.6.

Our notations and terminology are the same as in Chapters 1-2.

General

In what follows, all Banach spaces and algebras are assumed to be complex. If X is a Banach space, then the *density character of* X, denoted by $\text{dens}(X)$, is the smallest cardinal κ such that X has a dense subset of cardinality κ.

Infinite sums of Banach spaces

Let A be a (possibly infinite) subset of $\mathbb{N}$, for every $n \in A$ let X_n be a non-zero Banach space. Then the ℓ_1*-sum of* $(X_n)_{n\in A}$ is the set

$$\left(\bigoplus_{n\in A} X_n\right)_{\ell_1} := \left\{(x_n)_{n\in A} \in \prod_{n\in A} X_n : \sum_{n\in A} \|x_n\| < \infty\right\}, \tag{3.1}$$

which is a vector space endowed with pointwise addition and pointwise scalar product, and it is a Banach space with the norm $\|(x_n)_{n\in A}\| := \sum\limits_{n\in A} \|x_n\|$.
Similarly, the c_0*-sum of* $(X_n)_{n\in A}$ is the set

$$\left(\bigoplus_{n\in A} X_n\right)_{c_0} := \left\{(x_n)_{n\in A} \in \prod_{n\in A} X_n : (\forall \epsilon > 0)\, (\{n \in A : \|x_n\| \geq \epsilon\} \text{ is finite})\right\}, \tag{3.2}$$

which is a vector space endowed with pointwise addition and pointwise scalar product, and it is a Banach space with the norm $\|(x_n)_{n\in A}\| := \sup\limits_{n\in A} \|x_n\|$.

Idempotents, projections

If $p, q \in R$ are idempotents in a ring R then we say that they are *mutually orthogonal* and write $p \perp q$ if $pq = 0 = qp$. We recall that for idempotents $p, q \in R$ we write $p \sim q$

if there exist $a, b \in R$ such that $p = ab$ and $q = ba$, in this case we say that p and q are *equivalent*. If $p, q \in R$ are idempotents, then we write $q \leq p$ whenever $pq = q$ and $qp = q$ hold. This is a partial order on the set of idempotents of R. We say that an idempotent $p \in R$ is *minimal* if it is minimal in the set of non-zero idempotents of R with respect to this partial order. We write $q < p$ if both $q \leq p$ and $q \neq p$ hold.
In a C^*-algebra A an idempotent $p \in A$ is called a *projection* if it is self-adjoint.

Lemma 3.1.2. *Let X be a Banach space. Then $\mathcal{B}(X)$ has minimal idempotents.*

Proof. Let $x \in X$ be such that $\|x\| = 1$. By the Hahn-Banach Extension Theorem there is $\varphi \in X^*$ such that $\|x\| = 1 = \langle x, \varphi \rangle$. We show that the rank-one idempotent $x \otimes \varphi \in \mathcal{B}(X)$ is minimal. To see this, let $P \in \mathcal{B}(X)$ be a non-zero idempotent with $P \leq x \otimes \varphi$. This is equivalent to $(x \otimes \varphi)P = P = P(x \otimes \varphi)$, or equivalently

$$\langle Py, \varphi \rangle x = Py \quad (y \in X); \tag{3.3}$$

$$\langle y, \varphi \rangle Px = Py \quad (y \in X). \tag{3.4}$$

From (3.3) we obtain with $y := x$ that $\langle Px, \varphi \rangle x = Px$, and from (3.4) with $y := Px$ we get $\langle Px, \varphi \rangle Px = Px$. Consequently $\langle Px, \varphi \rangle = 0$ or $Px = x$. If the former, then $Px = 0$, thus by (3.4) it follows that $Py = 0$ for every $y \in X$, contradicting that P is non-zero. Thus $Px = x$, so from (3.4) again we obtain $\langle y, \varphi \rangle x = Py$ for all $y \in X$, so $x \otimes \varphi = P$. Thus $x \otimes \varphi \in \mathcal{B}(X)$ is minimal. □

Lemma 3.1.3. *Let A be an algebra and let $J \trianglelefteq A$ be a two-sided ideal. If $p, q \in A$ are idempotents with $p \sim q$, then $p \in J$ if and only if $q \in J$.*

Proof. Let $a, b \in A$ be such that $p = ab$ and $q = ba$. Then $p = p^2 = abab = aqb$ and similarly $q = bpa$, thus the claim follows. □

Simple and semisimple algebras

We say that a unital algebra A is *simple* if the only non-trivial, two-sided ideal in A is A. A unital Banach algebra A is *topologically simple* if the only non-trival, closed, two-sided ideal in A is A.

Lemma 3.1.4. *A unital Banach algebra A is topologically simple if and only if it is simple.*

Proof. For the non-trivial direction suppose A is topologically simple and let $J \trianglelefteq A$ be a proper, two-sided ideal of A. Thus $J \subseteq A \setminus \mathrm{inv}(A)$, and since $\mathrm{inv}(A)$ is open, $\overline{J} \subseteq A \setminus \mathrm{inv}(A)$. So $\overline{J}$ is a proper, closed, two-sided ideal of A and thus $\overline{J}$ and therefore J must be $\{0\}$. Thus A is simple. $\square$

Henceforth we shall not distinguish between these two notions of simplicity in unital Banach algebras.

If A is a unital algebra, the *Jacobson radical* of A, denoted by $\mathrm{rad}(A)$, is the intersection of all maximal left ideals in A, and it is a two-sided ideal in A. If there are no proper left ideals in A we put $\mathrm{rad}(A) := A$. A unital algebra is *semisimple* if its Jacobson radical is trivial. The Jacobson radical has the following characterisation, for a convenient proof we refer the reader to [44, Lemma 4.3]:

Lemma 3.1.5. *If A is a unital algebra, then*

$$\mathrm{rad}(A) = \{a \in A : \ (\forall b, c \in A)(1_A - bac \in \mathrm{inv}(A))\}. \tag{3.5}$$

On the one hand, the above result allows us to deduce the following well-known useful fact:

Lemma 3.1.6. *For a Banach space X, the Banach algebra $\mathcal{B}(X)$ is semisimple.*

Proof. Let $T \in \mathcal{B}(X)$ be non-zero. By Lemma 3.1.5 it is enough to show that there is an $S \in \mathcal{B}(X)$ such that $I_X - ST \notin \mathrm{inv}(\mathcal{B}(X))$.

Since T is non-zero, there is $x_0 \in X$ (necessarily non-zero) such that $Tx_0 \neq 0$. By the Hahn-Banach Extension Theorem there is $\varphi \in X^*$ such that $\langle Tx_0, \varphi \rangle = \|Tx_0\|$. Then the map

$$S: \ X \to X; \quad x \mapsto \frac{\langle x, \varphi \rangle}{\|Tx_0\|} x_0 \tag{3.6}$$

is easily seen to be a bounded linear map with $STx_0 = x_0$. Consequently $(I_X - ST)x_0 = 0$, showing $I_X - ST \notin \mathrm{inv}(\mathcal{B}(X))$ as required. □

On the other hand, for a Banach space X the Banach algebra $\mathcal{B}(X)$ is simple if and only if X is infinite-dimensional, since $\mathcal{A}(X)$ is a proper non-trivial closed two-sided ideal in $\mathcal{B}(X)$ whenever X is infinite-dimensional.

A classical deep result of B. E. Johnson asserts the following.

Theorem 3.1.7 (Johnson)**.** *If A, B are Banach algebras such that B is semisimple, then every surjective algebra homomorphism $\psi: A \to B$ is automatically continuous.*

For a modern discussion of this result we refer the reader to [15, Theorem 5.1.5]. In what follows we shall use this fundamental result without explicitly mentioning it.

3.2 Examples of Banach spaces without the SHAI property

We first observe that there is a large class of Banach spaces which obviously lack the SHAI property.

Lemma 3.2.1. *Let X be an infinite-dimensional Banach space such that $M_n(\mathbb{C})$ is a quotient of $\mathcal{B}(X)$ for some $n \in \mathbb{N}$. Then X does not have the SHAI property.*

Proof. Let $\varphi : \mathcal{B}(X) \to M_n(\mathbb{C})$ be a surjective algebra homomorphism. Since $\mathcal{B}(\mathbb{C}^n) \simeq M_n(\mathbb{C})$ we immediately obtain that that there is a surjective algebra homomorphism $\psi : \mathcal{B}(X) \to \mathcal{B}(\mathbb{C}^n)$ which cannot be injective, since X is infinite-dimensional. □

We recall that an infinite-dimensional Banach space X is *indecomposable*, if there are no closed, infinite-dimensional subspaces Y, Z of X such that $X \simeq Y \oplus Z$. A Banach space X is *hereditarily indecomposable* if every closed, infinite-dimensional subspace of X is indecomposable.

In each of the following examples, $\mathcal{B}(X)$ has a character, so X does not have the SHAI property by Lemma 3.2.1. In examples (1)–(3) this character is shown explicitly and in examples (4)–(8) the character is obtained from a commutative quotient on $\mathcal{B}(X)$.

Example 3.2.2. None of the following spaces X have the SHAI property:

(1) X is a hereditarily indecomposable Banach space, since by [28, Theorem 18] $\mathcal{B}(X)$ has a character whose kernel is $\mathcal{S}(X)$;

(2) $X = \mathcal{J}_p$ where $1 < p < \infty$ and $\mathcal{J}_p$ is the p^{th} James space, since by [22, Paragraph 8] $\mathcal{B}(X)$ has a character whose kernel is $\mathcal{W}(X)$, see also [46, Theorem 4.16];

(3) $X = C[0, \omega_1]$, where ω_1 is the first uncountable ordinal, since by [22, Paragraph 9] $\mathcal{B}(X)$ has a character, see also [54, Proposition 3.1];

(4) $X = X_\infty$, where X_∞ is the indecomposable but not hereditarily indecomposable Banach space constructed by Tarbard in [78, Chapter 4], since $\mathcal{B}(X)/\mathcal{K}(X) \simeq \ell_1(\mathbb{N}_0)$, where the right-hand side is endowed with the convolution product;

(5) $X = X_K$, where K is a countable compact Hausdorff space and X_K is the Banach space construced by Motakis, Puglisi and Zisimopoulou in [57, Theorem B], since $\mathcal{B}(X)/\mathcal{K}(X) \simeq C(K)$;

(6) $X = C(K_0)$, where is K_0 is a Koszmider space without isolated points, since $\mathcal{B}(X)/\mathcal{W}(X) \simeq C(K_0)$, as shown in [16, Theorem 6.5(i)];

(7) $X = \mathcal{G}$, where $\mathcal{G}$ is the Banach space constructed by Gowers in [27], since $\mathcal{B}(X)/\mathcal{S}(X) \simeq \ell_\infty/c_0$, as shown in [46, Corollary 8.3];

(8) $X = X_M$, where X_M is the separable, superreflexive Banach space constructed by Mankiewicz in [55, Theorem 1.1], since there exists a surjective algebra homomorphism from $\mathcal{B}(X_M)$ to ℓ_∞.

3.3 The SHAI property for Banach spaces X where $\mathcal{E}(X)$ is a maximal ideal

We shall start this section by proving a theorem which allows us to show that all the classical sequence spaces and Schlumprecht's arbitrarily distortable Banach space possess the SHAI property. We remind the reader that the definitions of strictly singular and inessential operators can be found in the Preliminaries.

Theorem 3.3.1. *Let X be a Banach space such that $\mathcal{E}(X)$ is a maximal ideal in $\mathcal{B}(X)$ and X has a complemented subspace isomorphic to $X \oplus X$. Then X has the SHAI property.*

The proof of this theorem requires some lemmas.

The following result is an immediate corollary of [47, Propositions 1.9 and 2.3] and [15, Proposition 1.3.34], for the convenience of the reader we give a direct proof here.

Proposition 3.3.2. *Let X be a Banach space such that it has a complemented subspace isomorphic to $X \oplus X$. Then $\mathcal{B}(X)$ does not have finite-codimensional proper two-sided ideals.*

Proof. Let $P \in \mathcal{B}(X)$ be an idempotent such that $\mathrm{Ran}(P) \simeq X \oplus X$ holds. By Lemma 1.2.11 we can take $T \in \mathcal{B}(X, X \oplus X)$ and $S \in \mathcal{B}(X \oplus X, X)$ with $T \circ S = I_{X \oplus X}$ and $S \circ T = P$. We will show that there is a set of mutually orthogonal idempotents $\{Q_n\}_{n \in \mathbb{N}}$ in $\mathcal{B}(X)$ such that $Q_n \sim I_X$ for every $n \in \mathbb{N}$. To this end we consider the auxiliary operators

$$\begin{aligned} pr_1 :\ & X \oplus X \to X; \quad (x, y) \mapsto x, \\ pr_2 :\ & X \oplus X \to X; \quad (x, y) \mapsto y, \\ \iota_1 :\ & X \to X \oplus X; \quad x \mapsto (x, 0), \\ \iota_2 :\ & X \to X \oplus X; \quad y \mapsto (0, y). \end{aligned} \tag{3.7}$$

It is clear that for $i, j \in \{1, 2\}$ if $i \neq j$ then $pr_i \circ \iota_i = I_X$ and $pr_i \circ \iota_j = 0$. For every $n \in \mathbb{N}_0$ we define $Q_n := (S \circ \iota_2)^n \circ (S \circ \iota_1) \circ (pr_1 \circ T) \circ (pr_2 \circ T)^n$, clearly $Q_n \in \mathcal{B}(X)$. By induction, for all $n \in \mathbb{N}_0$ the identity $(pr_2 \circ T)^n \circ (S \circ \iota_2)^n = I_X$ holds. Since for every $n \in \mathbb{N}_0$ clearly $(pr_1 \circ T) \circ (pr_2 \circ T)^n \in \mathcal{B}(X)$ and $(S \circ \iota_2)^n \circ (S \circ \iota_1) \in \mathcal{B}(X)$, the identity

$$(pr_1 \circ T) \circ (pr_2 \circ T)^n \circ (S \circ \iota_2)^n \circ (S \circ \iota_1) = pr_1 \circ T \circ S \circ \iota_1 = I_X \tag{3.8}$$

shows that $Q_n \in \mathcal{B}(X)$ is an idempotent with $Q_n \sim I_X$. To see that they are mutually orthogonal, let $k, l \in \mathbb{N}_0$ be distinct. First suppose $k < l$, then

$$\begin{aligned} (pr_2 \circ T)^l \circ (S \circ \iota_2)^k \circ (S \circ \iota_1) &= (pr_2 \circ T)^{l-k} \circ (pr_2 \circ T)^k \circ (S \circ \iota_2)^k \circ (S \circ \iota_1) \\ &= (pr_2 \circ T)^{l-k} \circ (S \circ \iota_1) \\ &= (pr_2 \circ T)^{l-k-1} \circ (pr_2 \circ T) \circ (S \circ \iota_1) \\ &= (pr_2 \circ T)^{l-k-1} \circ pr_2 \circ \iota_1 \end{aligned}$$

$$= 0 \tag{3.9}$$

If $l < k$ then with a similar argument one obtains

$$(pr_1 \circ T) \circ (pr_2 \circ T)^l \circ (S \circ \iota_2)^k = 0. \tag{3.10}$$

Consequently, for every distinct $k, l \in \mathbb{N}_0$, we obtain from the above and the definitions of Q_k and Q_l that $Q_k Q_l = 0$, thus proving that $\{Q_n\}_{n \in \mathbb{N}}$ has the required properties. Now let $\mathcal{J} \trianglelefteq \mathcal{B}(X)$ be a proper two-sided ideal in $\mathcal{B}(X)$, and let $\pi : \mathcal{B}(X) \to \mathcal{B}(X)/\mathcal{J}$ be the quotient map. Then $\{\pi(Q_n)\}_{n \in \mathbb{N}}$ is a set of mutually orthogonal non-zero idempotents in $\mathcal{B}(X)/\mathcal{J}$. Indeed, the for every $n \in \mathbb{N}$, $\pi(Q_n) \neq 0$ by Remark 3.1.3, since $I_X \sim Q_n$ and $I_X \notin \mathcal{J}$; the rest is trivial. Thus $\{\pi(Q_n)\}_{n \in \mathbb{N}}$ is linearly independent in $\mathcal{B}(X)/\mathcal{J}$. To see this, let $(\alpha_i)_{i=1}^N$ be a family of scalars with $\sum\limits_{i=1}^{N} \alpha_i \pi(Q_i) = 0$. Then for every $j \in \{1, \ldots, N\}$

$$0 = \pi(Q_j) \left(\sum_{i=1}^{N} \alpha_i \pi(Q_i) \right) = \alpha_j \pi(Q_j^2) + \sum_{i \in \{1, \ldots, j-1, j+1, \ldots, N\}} \alpha_i \pi(Q_j Q_i) = \alpha_j \pi(Q_j), \tag{3.11}$$

thus $\alpha_j = 0$ by $\pi(Q_j) \neq 0$. Hence $\mathcal{B}(X)/\mathcal{J}$ cannot be finite-dimensional, as required. □

Corollary 3.3.3. *Let X be a Banach space such that X contains a complemented subspace isomorphic to $X \oplus X$. Then the following are equivalent:*

1. *X has the SHAI property,*
2. *for any infinite-dimensional Banach space Y any surjective algebra homomorphism $\psi : \mathcal{B}(X) \to \mathcal{B}(Y)$ is automatically injective.*

Proof. Let Y be a non-zero Banach space and let $\psi : \mathcal{B}(X) \to \mathcal{B}(Y)$ be a surjective algebra homomorphism, we show that Y must be infinite-dimensional. For assume towards a contradiction it is not; then clearly $\mathcal{B}(Y)$ is finite-dimensional, thus by $\mathcal{B}(X)/\operatorname{Ker}(\psi) \simeq \mathcal{B}(Y)$ we have that $\operatorname{Ker}(\psi)$ is finite-codimensional in $\mathcal{B}(X)$. This contradicts Proposition 3.3.2. □

We recall the following well-known elementary fact.

Remark 3.3.4. If A, B are unital algebras and $\theta : A \to B$ is a surjective algebra homomorphism then $\theta[\operatorname{rad}(A)] \subseteq \operatorname{rad}(B)$. Indeed, let M be a maximal left ideal in B, then by surjectivity of θ it follows that $\theta^{-1}[M]$ is a maximal left ideal in A. Consequently $\operatorname{rad}(A) \subseteq \theta^{-1}[M]$ and thus $\theta[\operatorname{rad}(A)] \subseteq M$. Since M is an arbitrary maximal left ideal in B, the result readily follows.

We recall the following classical result about inessential operators:

Theorem 3.3.5. *(Kleinecke's Theorem) Let X be a Banach space, and let $\pi : \mathcal{B}(X) \to \mathcal{B}(X)/\mathcal{A}(X)$ denote the quotient map. Then*

$$\mathcal{E}(X) = \{T \in \mathcal{B}(X) : \pi(T) \in \operatorname{rad}(\mathcal{B}(X)/\mathcal{A}(X))\}. \tag{3.12}$$

A proof of the above theorem can be found, for example, in [10, Theorem 5.5.9] or [45, Theorem 5.3.1].

Lemma 3.3.6. *Let X be a Banach space, let B be a unital Banach algebra and let $\psi : \mathcal{B}(X) \to B$ be a continuous, surjective, non-injective algebra homomorphism. Then $\psi[\mathcal{E}(X)] \subseteq \operatorname{rad}(B)$. In particular, if B is semisimple then $\mathcal{E}(X) \subseteq \operatorname{Ker}(\psi)$.*

Proof. Since ψ is not injective $\mathcal{A}(X) \subseteq \operatorname{Ker}(\psi)$ holds and therefore there exists a unique surjective algebra homomorphism $\theta : \mathcal{B}(X)/\mathcal{A}(X) \to B$ with $\theta \circ \pi = \psi$ and $\|\psi\| = \|\theta\|$, where $\pi : \mathcal{B}(X) \to \mathcal{B}(X)/\mathcal{A}(X)$ is the quotient map. Thus $\theta[\operatorname{rad}(\mathcal{B}(X)/\mathcal{A}(X))] \subseteq$

rad(B), which by Kleinecke's Theorem 3.3.5 is equivalent to $\theta[\pi[\mathcal{E}(X)]] \subseteq \text{rad}(B)$. This is equivalent to $\psi[\mathcal{E}(X)] \subseteq \text{rad}(B)$, as required. □

Proof of Theorem 3.3.1. Let Y be an infinite-dimensional Banach space and let $\psi : \mathcal{B}(X) \to \mathcal{B}(Y)$ be a surjective algebra homomorphism. Assume towards a contradiction that ψ in not injective. Since $\mathcal{B}(Y)$ is semisimple in view of Lemma 3.3.6 it follows that $\mathcal{E}(X) \subseteq \text{Ker}(\psi)$ must hold. Since ψ is surjective, Ker(ψ) is a proper ideal thus by maximality of $\mathcal{E}(X)$ in $\mathcal{B}(X)$ it follows that Ker$(\psi) = \mathcal{E}(X)$. Thus $\mathcal{B}(X)/\mathcal{E}(X) \simeq \mathcal{B}(Y)$, where the left-hand side is simple, due to maximality of $\mathcal{E}(X)$ in $\mathcal{B}(X)$, which is a contradiction. Therefore ψ must be injective thus by Corollary 3.3.3 the claim is proven. □

Remark 3.3.7. We observe that the condition "X has a complemented subspace isomorphic to $X \oplus X$" in Theorem 3.3.1 cannot be dropped in general. Indeed, let X be a hereditarily indecomposable Banach space, then $\mathcal{E}(X) = \mathcal{S}(X)$ is a maximal ideal in $\mathcal{B}(X)$ but by Example 3.2.2 (1) the space X does not have the SHAI property.

Example 3.3.8.

(1) Let X be c_0 or ℓ_p where $1 \leq p < \infty$. Then $X \simeq X \oplus X$, and by the results of Markus–Gohberg–Feldman in [25], $\mathcal{A}(\ell_p) = \mathcal{K}(\ell_p) = \mathcal{S}(\ell_p) = \mathcal{E}(\ell_p)$ is the only closed, non-trivial, proper two-sided ideal in $\mathcal{B}(X)$. Thus X satisfies the conditions of Theorem 3.3.1 and hence it has the SHAI property.

(2) In [48, page 253], Loy and Laustsen showed that $\mathcal{W}(\ell_\infty) = \mathcal{X}(\ell_\infty) = \mathcal{S}(\ell_\infty) = \mathcal{E}(\ell_\infty)$ is the unique maximal ideal in $\mathcal{B}(\ell_\infty)$. Therefore, since $\ell_\infty \simeq \ell_\infty \oplus \ell_\infty$, we deduce from Theorem 3.3.1 that ℓ_∞ has the SHAI property.

There are more exotic examples which satisfy the conditions of Theorem 3.3.1. To explain this, we require some preliminary results.

Definition 3.3.9. An infinite-dimensional Banach space is *complementably homogenous* if for every closed linear subspace Y of X with $Y \simeq X$ there exists a complemented subspace Z of X with $Z \simeq X$ and $Z \subseteq Y$. An infinite-dimensional Banach space X is called *complementably minimal* if every closed, infinite-dimensional subspace of X contains a subspace which is complemented in X and isomorphic to X.

It is immediate therefore that every complementably minimal Banach space is complementably homogenous.
Let X, Y be Banach spaces and let $T \in \mathcal{B}(X)$. We say that T *is* Y*-singular* if there is no closed linear subspace W of X with $W \simeq Y$ such that $T|_W$ is bounded below. The set of Y-singular operators on X is denoted by $\mathcal{S}_Y(X)$, clearly $0 \in \mathcal{S}_Y(X)$. This set is closed under multiplying elements of it from the left and right by elements of $\mathcal{B}(X)$. However $\mathcal{S}_Y(X)$ need not be closed under addition, consider, for example, $X = Y := \ell_p \oplus \ell_q$, where $1 \leq q < p < \infty$. Nevertheless, we immediately have $\mathcal{S}(X) \subseteq \mathcal{S}_Y(X)$ for infinite-dimensional Banach spaces X and Y.

Lemma 3.3.10.

1. *Let X be a complementably homogenous Banach space. Then $\mathcal{S}_X(X)$ contains every proper, two-sided ideal in $\mathcal{B}(X)$.*

2. *Let X be a complementably minimal Banach space. Then $\mathcal{E}(X) = \mathcal{S}(X) = \mathcal{S}_X(X)$ is the largest proper two-sided ideal in $\mathcal{B}(X)$.*

Proof. Suppose X is complementably homogenous. Let $\mathcal{J} \trianglelefteq \mathcal{B}(X)$ be a two-sided ideal such that $\mathcal{J} \nsubseteq \mathcal{S}_X(X)$. Take $T \in \mathcal{J}$ such that $T \notin \mathcal{S}_X(X)$, then there exists a closed linear subspace W of X such that $W \simeq X$ and $T|_W$ is bounded below. Let $T_1 := T|_W^{\mathrm{Ran}(T|_W)}$, then $T_1 \in \mathcal{B}(W, \mathrm{Ran}(T|_W))$ is an isomorphism. In particular, $\mathrm{Ran}(T|_W) \simeq W$ hence $\mathrm{Ran}(T|_W) \simeq X$. Since X is complementably homogenous, there exists an idempotent $P \in \mathcal{B}(X)$ with $\mathrm{Ran}(P) \simeq X$ and $\mathrm{Ran}(P) \subseteq \mathrm{Ran}(T|_W)$.

Let $S \in \mathcal{B}(\mathrm{Ran}(P), X)$ be an isomorphism, let $\iota : W \to X$ denote the canonical embedding. Since $\mathrm{Ran}(P) \subseteq \mathrm{Ran}(T|_W)$, clearly $T_1^{-1}|_{\mathrm{Ran}(P)} \in \mathcal{B}(\mathrm{Ran}(P), W)$. It is therefore immediate that

$$(S \circ P|^{\mathrm{Ran}(P)}) \circ T \circ (\iota \circ T_1^{-1}|_{\mathrm{Ran}(P)} \circ S^{-1}) = S \circ P|^{\mathrm{Ran}(P)} \circ P|_{\mathrm{Ran}(P)} \circ S^{-1} = I_X. \tag{3.13}$$

Thus from $T \in \mathcal{J}$ it follows that $I_X \in \mathcal{J}$, equivalently, $\mathcal{J} = \mathcal{B}(X)$.

Suppose X is complementably minimal. We recall that $\mathcal{S}(X) \subseteq \mathcal{S}_X(X)$ automatically holds. Now let $T \in \mathcal{B}(X)$ with $T \notin \mathcal{S}(X)$. Hence there is an infinite-dimensional subspace W of X such that $T|_W$ is bounded below. There exists a complemented subspace Z of X with $Z \simeq X$ and $Z \subseteq W$. Clearly $T|_Z$ is bounded below, proving $T \notin \mathcal{S}_X(X)$. This shows $\mathcal{S}_X(X) = \mathcal{S}(X)$. We recall that $\mathcal{S}(X) \subseteq \mathcal{E}(X)$ automatically holds. Now by the first part of the theorem it follows that $\mathcal{S}(X)$ is the largest proper two-sided ideal in $\mathcal{B}(X)$, thus $\mathcal{E}(X) = \mathcal{S}(X) = \mathcal{S}_X(X)$ must hold. □

We remark here that the second part of Lemma 3.3.10 was first observed by Whitley in [79, Theorem 6.2].

Proposition 3.3.11. *Let X be a complementably minimal Banach space isomorphic to its square. Then X has the SHAI property.*

Proof. Since X is complementably minimal, we have by Lemma 3.3.10 that $\mathcal{S}(X) = \mathcal{E}(X)$ is the largest proper two-sided ideal in $\mathcal{B}(X)$. In particular $\mathcal{E}(X)$ is maximal in $\mathcal{B}(X)$, thus Theorem 3.3.1 yields the claim. □

Corollary 3.3.12. *Schlumprecht's space* **S** *has the SHAI property.*

Proof. We recall that **S** satisfies the properties of Proposition 3.3.11. Indeed, it is isomorphic to it is square and it is complementably minimal, as shown, for example, in [72]. □

Remark 3.3.13. Let us remark here that Lemma 3.3.11 yields another proof of the facts that c_0 and ℓ_p have the SHAI property for $1 \leq p < \infty$. Indeed, these spaces are complementably minimal by [61, Lemma 2] and clearly isomorphic to their squares.

3.4 The SHAI property for Banach spaces X where $\mathcal{E}(X)$ is not a maximal ideal

We recall a folklore lifting result for Calkin algebras of Banach spaces, this will be essential in the proof of Theorem 3.4.6. A convenient reference for the proof of this lemma is [6, Lemma 2.6]. It also follows from the much more general result [4, Theorem C].

Theorem 3.4.1. *Let X be a Banach space and let $p \in \mathcal{B}(X)/\mathcal{K}(X)$ be an idempotent. Then there exists an idempotent $P \in \mathcal{B}(X)$ with $p = \pi(P)$ where $\pi : \mathcal{B}(X) \to \mathcal{B}(X)/\mathcal{K}(X)$ is the quotient map.*

Lemma 3.4.2. *Let X be a Banach space and suppose $Q \in \mathcal{B}(X)$ is an idempotent such that* $\operatorname{Ran}(Q)$ *is isomorphic to its square. Then there exist mutually orthogonal idempotents $Q_1, Q_2 \in \mathcal{B}(X)$ with $Q_1, Q_2 \sim Q$ and $Q_1 + Q_2 = Q$. Furthermore, if $\mathcal{J} \trianglelefteq \mathcal{B}(X)$ is a closed, two-sided ideal with $Q \notin \mathcal{J}$, then $Q_1, Q_2 \notin \mathcal{J}$ and $\pi(Q_1), \pi(Q_2) < \pi(Q)$, where $\pi : \mathcal{B}(X) \to \mathcal{B}(X)/\mathcal{J}$ is the quotient map.*

Proof. Let $Y := \operatorname{Ran}(Q)$, clearly $Q|_Y \circ Q|^Y = Q$ and $Q|^Y \circ Q|_Y = I_Y$. We consider the bounded linear maps

$$\begin{aligned} pr_1 &: Y \oplus Y \to Y; \quad (x, y) \mapsto x, \qquad (3.14) \\ pr_2 &: Y \oplus Y \to Y; \quad (x, y) \mapsto y, \\ \iota_1 &: Y \to Y \oplus Y; \quad x \mapsto (x, 0), \end{aligned}$$

$$\iota_2 : \; Y \to Y \oplus Y; \quad y \mapsto (0, y).$$

We observe that for $i, j \in \{1, 2\}$ if $i \neq j$ then $pr_i \circ \iota_i = I_Y$ and $pr_i \circ \iota_j = 0$. Clearly ι_i is an isometry and $\|pr_i\| \leq 1$. Also $\iota_1 \circ pr_1 + \iota_2 \circ pr_2 = I_{Y \oplus Y}$. By the assumption on Y we can take an isomorphism $T : \; Y \to Y \oplus Y$. For $i \in \{1, 2\}$ we define

$$Q_i := Q|_Y \circ T^{-1} \circ \iota_i \circ pr_i \circ T \circ Q|^Y. \tag{3.15}$$

It is clear from the identities above that $Q_1, Q_2 \in \mathcal{B}(X)$ are idempotents with $Q_1 + Q_2 = Q$ and $Q_1 \perp Q_2$. Let $i \in \{1, 2\}$ be fixed. It is not hard to see that $Q_i \sim Q$. Indeed, it is enough to observe that

$$\begin{aligned} (Q|_Y \circ T^{-1} \circ \iota_i \circ Q|^Y) \circ (Q|_Y \circ pr_i \circ T \circ Q|^Y) &= Q_i; \\ (Q|_Y \circ pr_1 \circ T \circ Q|^Y) \circ (Q|_Y \circ T^{-1} \circ \iota_1 \circ Q|^Y) &= Q. \end{aligned} \tag{3.16}$$

For $i \in \{1, 2\}$ we immediately get $Q_i \leq Q$ and thus $\pi(Q_i) \leq \pi(Q)$. Since $Q_i \sim Q$, in view of Remark 3.1.3 the condition $Q \notin \mathcal{J}$ is equivalent to $Q_i \notin \mathcal{J}$. Also, for $i, j \in \{1, 2\}$ if $i \neq j$ then $Q_j = Q - Q_i$ thus $\pi(Q_i) \neq \pi(Q)$. □

Proposition 3.4.3. *Let X be a Banach space such that every infinite-dimensional complemented subspace of X is isomorphic to its square. Then $\mathcal{B}(X)/\mathcal{K}(X)$ does not have minimal idempotents.*

Proof. Let $p \in \mathcal{B}(X)/\mathcal{K}(X)$ be a non-zero idempotent. By Lemma 3.4.1 there exists an idempotent $P \in \mathcal{B}(X)$ with $p = \pi(P)$, where $\pi : \; \mathcal{B}(X) \to \mathcal{B}(X)/\mathcal{K}(X)$ is the quotient map. Clearly $P \notin \mathcal{K}(X)$, equivalently $\operatorname{Ran}(P)$ is infinite-dimensional. Thus by the hypothesis it is isomorphic to its square, consequently Lemma 3.4.2 implies that there exists an idempotent $Q \in \mathcal{B}(X)$ such that $Q \notin \mathcal{K}(X)$ and $\pi(Q) < \pi(P)$. □

We recall that ℓ_2^n denotes the Banach space $\mathbb{C}^n$ endowed with the ℓ_2-norm, whenever $n \in \mathbb{N}$.

Example 3.4.4. For the following (non-Hilbertian) Banach spaces X every infinite-dimensional complemented subspace of X is isomorphic to its square therefore by Proposition 3.4.3 the Calkin algebra $\mathcal{B}(X)/\mathcal{K}(X)$ does not have minimal idempotents:

(1) $X = c_0(\lambda)$, where λ is an infinite cardinal, since by [3, Proposition 2.8] every infinite-dimensional complemented subspace of $c_0(\lambda)$ is isomorphic to $c_0(\kappa)$ for some infinite cardinal $\kappa \leq \lambda$, and $c_0(\kappa) \simeq c_0(\kappa) \oplus c_0(\kappa)$,

(2) $X = \ell_p$ where $p \in [1, \infty)\backslash\{2\}$, since by Pełczyński's theorem ([61]) every infinite-dimensional complemented subspace of ℓ_p is isomorphic to ℓ_p and $\ell_p \simeq \ell_p \oplus \ell_p$,

(3) $X = \ell_\infty$, since by Lindenstrauss' theorem ([52]) every infinite-dimensional complemented subspace of ℓ_∞ is isomorphic to ℓ_∞ and $\ell_\infty \simeq \ell_\infty \oplus \ell_\infty$,

(4) $X = \ell_\infty^c(\lambda)$, where λ is an infinite cardinal, since by [39, Theorem 1.4] every infinite-dimensional complemented subspace of $\ell_\infty^c(\lambda)$ is isomorphic to ℓ_∞ or $\ell_\infty^c(\kappa)$ for some infinite cardinal $\kappa \leq \lambda$, and $\ell_\infty^c(\kappa) \simeq \ell_\infty^c(\kappa) \oplus \ell_\infty^c(\kappa)$,

(5) $X = C[0, \omega^\omega]$, where ω is the first infinite ordinal, since by [5, Theorem 3] every infinite-dimensional complemented subspace of $C[0, \omega^\omega]$ is isomorphic to c_0 or $C[0, \omega^\omega]$ and $C[0, \omega^\omega] \simeq C[0, \omega^\omega] \oplus C[0, \omega^\omega]$ by [67, Remark 2.25 and Lemma 2.26],

(6) $X = \left(\bigoplus_{n \in \mathbb{N}} \ell_2^n \right)_Y$ where $Y \in \{c_0, \ell_1\}$, since by [8, Corollary 8.4 and Theorem 8.3] every infinite-dimensional complemented subspace of X is isomorphic to Y or X and $X \simeq X \oplus X$ by [11, Corollary 7(i)].

Before we recall two important results of Laustsen–Loy–Read, and Laustsen–Schlumprecht–Zsák, let us remind the reader of the following terminology. For Banach spaces X and Y the symbol $\overline{\mathscr{G}}_Y(X)$ denotes the closed, two-sided ideal of operators on X which *factor through Y approximately*, that is, the closed linear span of the set $\{ST : S \in \mathcal{B}(Y,X), T \in \mathcal{B}(X,Y)\}$.

Theorem 3.4.5. *([49, Corollary 5.6], [50, Theorem 2.12])*
Let $X = \left(\bigoplus_{n\in\mathbb{N}} \ell_2^n\right)_Y$*, where* $Y \in \{c_0, \ell_1\}$*. Then the lattice of closed, two-sided ideals in* $\mathcal{B}(X)$ *is given by*

$$\{0\} \subsetneq \mathcal{K}(X) \subsetneq \overline{\mathscr{G}}_Y(X) \subsetneq \mathcal{B}(X). \tag{3.17}$$

Theorem 3.4.6. *Let X be either* $\left(\bigoplus_{n\in\mathbb{N}} \ell_2^n\right)_{c_0}$ *or* $\left(\bigoplus_{n\in\mathbb{N}} \ell_2^n\right)_{\ell_1}$*. Then* $\mathcal{A}(X) = \mathcal{E}(X)$ *but X has the SHAI property.*

Proof. The equality $\mathcal{A}(X) = \mathcal{E}(X)$ is given by [49, Corollary 3.8].
Let Z be a Banach space and let $\psi : \mathcal{B}(X) \to \mathcal{B}(Z)$ be a surjective algebra homomorphism. Since $X \simeq X \oplus X$, by Lemma 3.3.3 we may suppose that Z is infinite-dimensional. Since $\mathcal{B}(X)/\operatorname{Ker}(\psi) \simeq \mathcal{B}(Z)$, by Theorem 3.4.5 it is enough to show that neither $\operatorname{Ker}(\psi) = \mathcal{K}(X)$ nor $\operatorname{Ker}(\psi) = \overline{\mathscr{G}}_Y(X)$ can hold. The case $\operatorname{Ker}(\psi) = \overline{\mathscr{G}}_Y(X)$ is not possible, since $\overline{\mathscr{G}}_Y(X)$ is a maximal two-sided ideal in $\mathcal{B}(X)$ by Theorem 3.4.5 and therefore $\mathcal{B}(X)/\overline{\mathscr{G}}_Y(X)$ is simple as a Banach algebra whereas $\mathcal{B}(Z)$ is not, since Z is infinite-dimensional. To see that $\operatorname{Ker}(\psi) = \mathcal{K}(X)$ cannot hold we observe that $\mathcal{B}(X)/\mathcal{K}(X)$ does not have minimal idempotents by Example 3.4.4 (6) whereas $\mathcal{B}(Z)$ clearly does by Lemma 3.1.2. Consequently $\operatorname{Ker}(\psi) = \{0\}$ must hold, thus proving the claim. □

We now consider non-separable Hilbert spaces, where the ideal of inessential operators is too small to be maximal (see Remark 3.4.9 for a precise statement). In the following we show that for a Hilbert space H of arbitrary density character, the projections lift from any quotient of $\mathcal{B}(H)$. In what follows, if (X, μ) is a measure space and $f \in L_\infty(X,\mu)$ then

$$M_f : L_2(X,\mu) \to L_2(X,\mu); \quad g \mapsto fg \tag{3.18}$$

is called the *multiplication operator by* f and is clearly a bounded linear operator.

Lemma 3.4.7. *Let H be a Hilbert space and let $\mathcal{J}$ be a closed, two-sided ideal in $\mathcal{B}(H)$. For any projection $p \in \mathcal{B}(H)/\mathcal{J}$ there exists a projection $P \in \mathcal{B}(H)$ such that $p = \pi(P)$, where $\pi : \mathcal{B}(H) \to \mathcal{B}(H)/\mathcal{J}$ denotes the quotient map.*

Proof. Let $p \in \mathcal{B}(H)/\mathcal{J}$ be a projection. There exists a self-adjoint $A \in \mathcal{B}(H)$ such that $p = \pi(A)$. By the Spectral Theorem for bounded self-adjoint operators [13, Chapter IX., Theorem 4.6] there exists a measure space (X,μ), a μ-almost everywhere bounded, real-valued function f on X and an isometric isomorphism $U : H \to L_2(X,\mu)$ such that $A = U^{-1} \circ M_f \circ U$. Consequently

$$\pi(U^{-1} \circ M_f \circ U) = \pi(A) = p = p^2 = \pi(A^2) = \pi(U^{-1} \circ M_{f^2} \circ U), \tag{3.19}$$

which is equivalent to

$$U^{-1} \circ M_{f-f^2} \circ U = U^{-1} \circ (M_f - M_{f^2}) \circ U \in \mathcal{J}. \tag{3.20}$$

Let $\tilde{f}$ be a representative of the class f and let h be the class of $\mathbf{1}_{[\tilde{f} \geq 1/2]}$, the indicator function of the set $[\tilde{f} \geq 1/2] := \{x \in X : \tilde{f}(x) \geq 1/2\}$. Clearly $h \in L_\infty(X,\mu)$ is well-defined and $P := U^{-1} \circ M_h \circ U \in \mathcal{B}(H)$ is a projection. We show that $p = \pi(P)$, which is equivalent to showing that $U^{-1} \circ M_{f-h} \circ U \in \mathcal{J}$. We first observe that it is enough to find $g \in L_\infty(X,\mu)$ such that $g(f - f^2) = h - f$. Indeed, if such a function g

were to exist then $M_g \circ M_{f-f^2} = M_{h-f}$ and consequently

$$U^{-1} \circ M_{h-f} \circ U = U^{-1} \circ M_g \circ M_{f-f^2} \circ U = (U^{-1} \circ M_g \circ U) \circ (U^{-1} \circ M_{f-f^2} \circ U) \in \mathcal{J} \tag{3.21}$$

holds by Equation (3.20) and the fact that $\mathcal{J}$ is an ideal in $\mathcal{B}(H)$.

Thus let $\tilde{g} : X \to \mathbb{R}$ be the following function:

$$\tilde{g}(x) := \begin{cases} 1/(\tilde{f}(x) - 1) & \text{if } \tilde{f}(x) < 1/2 \\ 1/\tilde{f}(x) & \text{otherwise.} \end{cases} \tag{3.22}$$

Let g be the class of $\tilde{g}$, clearly g is μ-almost everywhere bounded by 2. A simple calculation shows that

$$\tilde{g}(x)(\tilde{f}(x) - \tilde{f}^2(x)) = \begin{cases} (\tilde{f}(x) - \tilde{f}^2(x)/(\tilde{f}(x) - 1) = -\tilde{f}(x) & \text{if } \tilde{f}(x) < 1/2 \\ (\tilde{f}(x) - \tilde{f}^2(x)/\tilde{f}(x) = 1 - \tilde{f}(x) & \text{otherwise,} \end{cases} \tag{3.23}$$

so $\tilde{g}(x)(\tilde{f}(x) - \tilde{f}^2(x)) = \mathbf{1}_{[\tilde{f} \geq 1/2]}(x) - \tilde{f}(x)$ holds for every $x \in X$. Consequently $g(f - f^2) = h - f$, which proves the claim. □

Theorem 3.4.8. *Let H be a Hilbert space (not necessarily separable).*

1. *If $\mathcal{J}$ is a proper, closed, two-sided ideal in $\mathcal{B}(H)$, then $\mathcal{B}(H)/\mathcal{J}$ has no minimal idempotents.*

2. *Consequently, H has the SHAI property.*

Proof. For the first part of the proof, let $e \in \mathcal{B}(H)/\mathcal{J}$ be a non-zero idempotent. Since $\mathcal{B}(H)/\mathcal{J}$ is a C^*-algebra, by [71, Exercise 3.11(i)] there exists a projection $p \in A$ with $p \sim e$. Thus there exist $a, b \in A$ such that $ab = p$ and $ba = e$, consequently

$ae = pa$ and $bp = eb$. By Lemma 3.4.7 there exists a projection $P \in \mathcal{B}(H)$ with $p = \pi(P)$, where $\pi : \mathcal{B}(H) \to \mathcal{B}(H)/\mathcal{J}$ is the quotient map. Clearly $P \notin \mathcal{J}$, otherwise $p = \pi(P) = 0$ and thus $0 = bpa = e$, a contradiction. In particular, $\mathcal{F}(H) \subseteq \mathcal{J}$ implies that $\operatorname{Ran}(P)$ is infinite-dimensional. Since H is a Hilbert space, it follows that $\operatorname{Ran}(P)$ is (isometrically) isomorphic to its square. Hence, by Lemma 3.4.2 there exists an idempotent $Q \in \mathcal{B}(H)$ such that $Q \notin \mathcal{J}$ and $\pi(Q) < \pi(P)$. We define $q := \pi(Q)$, then $q \in \mathcal{B}(H)/\mathcal{J}$ is a non-zero idempotent with $q < p$. We define $f := bqa$, and observe that $f \in A$ is a non-zero idempotent. Indeed, $f^2 = bqabqa = bqpqa = bqa = f$ and $f \neq 0$ otherwise $0 = afb = abqab = pqp = q = \pi(Q)$ which is impossible. Also, $f \leq e$ since $ef = ebqa = bpqa = bqa = f$ and similarly $fe = f$ hold. We now show that $e \neq f$. Assume towards a contradiction that $e = f$, then $aeb = afb$, equivalently $pab = abqab$. This is equivalent to $p = pqp$ which in turn is equivalent to $p = q$, a contradiction. Thus $f < e$, which shows that e is not a minimal idempotent.

For the second part of the proof, let Y be a Banach space and assume towards a contradiction that there exists a surjective, non-injective algebra homomorphism $\psi : \mathcal{B}(H) \to \mathcal{B}(Y)$. Then $\operatorname{Ker}(\psi)$ is non-trivial and $\mathcal{B}(H)/\operatorname{Ker}(\psi)$ is isomorphic to $\mathcal{B}(X)$. This is a contradiction since $\mathcal{B}(H)/\operatorname{Ker}(\psi)$ has no minimal idempotents by the first part of the theorem, whereas $\mathcal{B}(X)$ clearly does by Lemma 3.1.2. ☐

Remark 3.4.9. We recall the well-known result that for a Hilbert space H of arbitrary density character the equality $\mathcal{K}(H) = \mathcal{E}(H)$ holds. Indeed,

1. $\mathcal{E}(H)/\mathcal{K}(H) = \operatorname{rad}(\mathcal{B}(H)/\mathcal{K}(H))$ by Kleinecke's Theorem, and
2. $\mathcal{B}(H)/\mathcal{K}(H)$ is semisimple, since it is a C^*-algebra.

Consequently, if H is non-separable, then $\mathcal{E}(H) = \mathcal{K}(H)$ is properly contained in $\mathcal{X}(H)$, the ideal of operators with separable range on H. Clearly $\mathcal{X}(H)$ itself is a proper closed two-sided ideal in $\mathcal{B}(H)$.

3.5 The SHAI property is stable under finite sums

Finally in this section we shall establish some permanence properties of Banach spaces with the SHAI property. We recall two trivial observations:

Remark 3.5.1. Let A and B be unital algebras and let $\psi : A \to B$ be a surjective algebra homomorphism. Then $\psi(1_A) = 1_B$. Indeed, there exists $a \in A$ such that $\psi(a) = 1_B$, thus

$$\psi(1_A) = \psi(1_A)1_B = \psi(1_A)\psi(a) = \psi(1_A a) = \psi(a) = 1_B. \tag{3.24}$$

Remark 3.5.2. If X is an infinite-dimensional Banach space and $\mathcal{J}$ is a closed, two-sided ideal in $\mathcal{B}(X)$ such that $A^2 = 0$ for all $A \in \mathcal{J}$ then $\mathcal{J} = \{0\}$. This follows from the fact that $\mathcal{A}(X)$ is the smallest non-trivial, closed, two-sided ideal in $\mathcal{B}(X)$ and $\mathcal{A}(X)$ has an abundance of non-zero rank-one idempotents.

Proposition 3.5.3. *Let E be a Banach space and let F, G be closed subspaces of E with trivial intersection and $E = F + G$. If both F and G have the SHAI property then E has the SHAI property.*

Proof. Let $P, Q \in \mathcal{B}(E)$ be idempotents with $F = \operatorname{Ran}(P)$ and $G = \operatorname{Ran}(Q)$. Then $P + Q = I_E$ and $PQ = 0 = QP$. Now let X be a non-zero Banach space and let $\psi : \mathcal{B}(E) \to \mathcal{B}(X)$ be a surjective algebra homomorphism. Then $Y := \operatorname{Ran}(\psi(P))$ and $Z := \operatorname{Ran}(\psi(Q))$ are closed (complemented) subspaces of X. Let us fix $T \in \mathcal{B}(F)$, we observe that $\psi(P|_F \circ T \circ P|^F)|_Y \in \mathcal{B}(Y)$ holds. The only thing we need to check is that the range of $\psi(P|_F \circ T \circ P|^F)|_Y$ is contained in Y which is clearly true since $\psi(P) \circ \psi(P|_F \circ T \circ P|^F) \circ \psi(P) = \psi(P|_F \circ T \circ P|^F)$. Consequently the map

$$\varphi : \ \mathcal{B}(F) \to \mathcal{B}(Y); \quad T \mapsto \psi(P|_F \circ T \circ P|^F)|_Y \tag{3.25}$$

is well-defined. It is immediate to see that φ is a linear map. To see that it is multiplicative, it is enough to observe that $P|^F \circ P|_F = I_F$ thus by multiplicativity of ψ, for any $T, S \in \mathcal{B}(F)$ we obtain $\varphi(T) \circ \varphi(S) = \varphi(T \circ S)$.

We show that φ is surjective. To see this we fix an $R \in \mathcal{B}(Y)$. Then $\psi(P)|_Y \circ R \circ \psi(P)|^Y \in \mathcal{B}(X)$ so by surjectivity of ψ it follows that there exists $A \in \mathcal{B}(E)$ such that $\psi(A) = \psi(P)|_Y \circ R \circ \psi(P)|^Y$. Consequently $\psi(P \circ A \circ P) = \psi(P) \circ \psi(A) \circ \psi(P) = \psi(P)|_Y \circ R \circ \psi(P)|^Y$ and thus by the definition of φ we obtain

$$\begin{aligned} \varphi(P|^F \circ A \circ P|_F) &= \psi(P|_F \circ P|^F \circ A \circ P|_F \circ P|^F)|_Y = \psi(P \circ A \circ P)|_Y \\ &= \Big(\psi(P)|_Y \circ R \circ \psi(P)|^Y\Big)\Big|_Y = R. \end{aligned} \tag{3.26}$$

This proves that φ is a surjective algebra homomorphism. Similarly we can show that

$$\theta : \mathcal{B}(G) \to \mathcal{B}(Z); \quad T \mapsto \psi(Q|_G \circ T \circ Q|^G)\Big|_Z \tag{3.27}$$

is a well-defined, surjective algebra homomorphism. Assume first that Y and Z are both non-trivial subspaces of X. Since both F and G have the SHAI property it follows that φ and θ are injective. Now let $A \in \operatorname{Ker}(\psi)$ be arbitrary. Then $\psi(A) = 0$ implies

$$\begin{aligned} \varphi(P|^F \circ A \circ P|_F) &= \psi(P|_F \circ P|^F \circ A \circ P|_F \circ P|^F)\Big|_Y = \psi(P \circ A \circ P)|_Y \\ &= \psi(P) \circ \psi(A) \circ \psi(P)|_Y = 0. \end{aligned} \tag{3.28}$$

Since φ is injective it follows that $P|^F \circ A \circ P|_F = 0$. Using the injectivity of θ a similar argument shows that $Q|^G \circ A \circ Q|_G = 0$. We recall that $E \simeq F \oplus G$ and thus every

$A \in \mathcal{B}(E)$ can be represented as the (2×2)-matrix

$$\begin{bmatrix} P|^F \circ A \circ P|_F & P|^F \circ A \circ Q|_G \\ Q|^G \circ A \circ P|_F & Q|^G \circ A \circ Q|_G \end{bmatrix}. \tag{3.29}$$

From the previous we obtain that whenever $A \in \text{Ker}(\psi)$ then A has the off-diagonal matrix form

$$A = \begin{bmatrix} 0 & P|^F \circ A \circ Q|_G \\ Q|^G \circ A \circ P|_F & 0 \end{bmatrix}. \tag{3.30}$$

On the one hand, since $\text{Ker}(\psi)$ is an ideal in $\mathcal{B}(X)$, we obviously have that $A^2 \in \text{Ker}(\psi)$ whenever $A \in \text{Ker}(\psi)$, thus A^2 also has the off-diagonal form

$$A^2 = \begin{bmatrix} 0 & P|^F \circ A^2 \circ Q|_G \\ Q|^G \circ A^2 \circ P|_F & 0 \end{bmatrix}. \tag{3.31}$$

On the other hand, the product of two (2×2) off-diagonal matrices is diagonal and therefore by Equation (3.30)

$$\begin{aligned} P|^F \circ A^2 \circ Q|_G &= 0, \\ Q|^G \circ A^2 \circ P|_F &= 0 \end{aligned} \tag{3.32}$$

must also hold. Consequently $A^2 = 0$, thus by Remark 3.5.2 the equality $\text{Ker}(\psi) = \{0\}$ must hold, equivalently, ψ is injective.

Let us observe that both $Y = \{0\}$ and $Z = \{0\}$ cannot hold. Indeed, if both $\psi(Q)$ and $\psi(P)$ were zero, then by Remark 3.5.1 we had $0 = \psi(P + Q) = \psi(I_E) = I_X$, contradicting that X is non-zero. Thus without loss of generality we may assume $Y = \{0\}$ and $Z \neq \{0\}$. Hence $\psi(P) = 0$, thus $\psi(Q) = \psi(P) + \psi(Q) = \psi(P + Q) =$

$\psi(I_E) = I_X$. This is equivalent to $Z = \text{Ran}(\psi(Q)) = X$, and thus $\mathcal{B}(Z) = \mathcal{B}(X)$. Therefore $\theta : \mathcal{B}(G) \to \mathcal{B}(X)$, defined in Equation (3.27) is a surjective algebra homomorphism. Since G has the SHAI property and X is non-zero, it follows that θ is injective. Let $A \in \mathcal{B}(E)$ be such that $A \in \text{Ker}(\psi)$. Then

$$\begin{aligned}\theta(Q|^G \circ A \circ Q|_G) &= \psi(Q|_G \circ Q|^G \circ A \circ Q|_G \circ Q|^G) \\ &= \psi(Q \circ A \circ Q) = \psi(Q) \circ \psi(A) \circ \psi(Q) = 0. \end{aligned} \tag{3.33}$$

Since θ is injective, this is equivalent to $Q|^G \circ A \circ Q|_G = 0$ which in turn is equivalent to $Q \circ A \circ Q = 0$. We observe that $Q \neq 0$, otherwise $I_X = \psi(Q) = 0$ which contradicts the fact that X is non-zero. Hence we can choose $x \in \text{Ran}(Q)$ and $\xi \in E^*$ norm one vectors with $\langle x, \xi \rangle = 1$. Assume towards a contradiction that ψ is not injective. Then in particular $x \otimes \xi \in \mathcal{F}(E) \subseteq \text{Ker}(\psi)$, consequently $Q \circ (x \otimes \xi) \circ Q = 0$. Thus $0 = (Q \circ (x \otimes \xi) \circ Q)x = \langle Qx, \xi \rangle Qx = \langle x, \xi \rangle x = x$, a contradiction. Hence ψ is injective, and therefore we conclude that E has the SHAI property. □

From Proposition 3.5.3 we immediately obtain the following corollary.

Corollary 3.5.4. *If $N \in \mathbb{N}$ and $\{E_i\}_{i=1}^N$ is set of Banach spaces with the SHAI property then $\bigoplus_{i=1}^{N} E_i$ has the SHAI property.*

3.6 Constructing surjective, non-injective homomorphisms from $\mathcal{B}(Y_X)$ to $\mathcal{B}(X)$

3.6.1 First remarks

Ordinals

In the following we give the von Neumann definition of ordinals and we record some of their fundamental properties. We refer the reader to [77], [51] or [35] for further details.

A set S is called *transitive* if every element of S is also a subset of S. Let S be a set and let $a, b \in S$, we define $a < b$ by $a \in b$. A transitive set S is an *ordinal* if $(S, <)$ is well-ordered. If α and β are ordinals then $\alpha < \beta$ denotes $\alpha \in \beta$. If α, β are ordinals such that either $\alpha < \beta$ or $\alpha = \beta$ holds then we denote this by $\alpha \leq \beta$. In particular, if α and β are ordinals with $\alpha < \beta$ then $\alpha \subsetneq \beta$. If α is an ordinal and $u \in \alpha$, then u is an ordinal. We set $0 := \emptyset$, this is the smallest ordinal. If α is a non-zero ordinal then $\alpha = \{\beta : \beta < \alpha\}$.

If A is a non-empty set of ordinals, then $\bigcup\limits_{\alpha \in A} \alpha$ is an ordinal and it is the supremum of A, see [77, Theorem 7.19, Theorem 7.20 and Theorem 7.21].

If α is an ordinal, then $\alpha^+ := \alpha \cup \{\alpha\}$ is called the *ordinal successor* or *successor* of α. If α is an ordinal then α^+ is an ordinal, moreover, it is the smallest ordinal β with the property $\alpha < \beta$. An ordinal β is called a *successor ordinal* if there is an ordinal α with $\alpha^+ = \beta$, otherwise β is called a *limit ordinal*. By convention, we consider 0 a limit ordinal. Non-zero limit ordinals exist, and the smallest non-zero limit ordinal is denoted by ω. The symbol ω_1 denotes the set of all countable ordinals, it can be shown that this is the smallest uncountable ordinal.

The order topology on ordinals

In the following we recall some of the basic properties of the order topology on ordinals. We mostly follow [76, Sections 39–44].

We first introduce some notation. For ordinals α and β, we define $[\alpha, \beta) := \{\gamma : \alpha \leq \gamma < \beta\}$ and $[\alpha, \beta] := \{\gamma : \alpha \leq \gamma \leq \beta\}$. Let γ be a non-zero ordinal, clearly $[0, \gamma) = \gamma$ and $[0, \gamma] = \gamma^+$ hold. If α is an ordinal then the *order topology* on $[0, \alpha)$ is the topology generated by the base

$$\mathscr{B} := \{[0, \beta),\, (\beta, \delta),\, (\delta, \alpha) : \beta, \delta < \alpha\}. \tag{3.34}$$

When we want to emphasise the fact that we are considering a non-zero ordinal α as a topological space endowed with the order topology, we will write it as $[0, \alpha)$. The order topology on $[0, \alpha)$ is Hausdorff and locally compact. The space $[0, \alpha)$ is compact if and only if α is a successor ordinal, that is, $[0, \alpha) = [0, \beta^+) = [0, \beta]$ for some ordinal β. It is well-known that the one-point (or Alexandroff) compactification of α is α^+. (Moreover, it can be shown that the one-point and Čech–Stone compactifications of $[0, \omega_1)$ coincide.)

Let α be a non-zero ordinal. Then a point in $[0, \alpha)$ is isolated if and only if it is 0 or a successor ordinal less then α, and it is an accumulation point if and only if it is a non-zero limit ordinal less then α.

The space $[0, \alpha)$ is scattered: Indeed, let S be a non-empty subset of $[0, \alpha)$ and let β be the smallest element of S. Then β is an isolated point in the subspace topology of S. To see this, we just observe that $S \subseteq [\beta, \alpha)$ and thus $\{\beta\} = [\beta, \beta^+) = [0, \beta^+) \cap [\beta, \alpha) = [0, \beta^+) \cap S$.

We recall the following useful property of closed subsets of ordinals, see also [51, page 144, after Definition 4.11(ii)].

Lemma 3.6.1. *Let κ be an ordinal and let $C \subseteq [0, \kappa)$ be a closed subset. If $\alpha < \kappa$ is a non-zero ordinal with $\alpha = \sup(C \cap \alpha)$ then $\alpha \in C$.*

Proof. Let $\alpha < \kappa$ be a non-zero ordinal with $\alpha = \sup(C \cap \alpha)$. Since C is closed, it is enough to show that α is an accumulation point of C. To this end, let U be a neighbourhood of α. Take a basic open set I with $\alpha \in I \subseteq U$. By the definition of the order topology on $[0, \kappa)$, one of the following must hold:

1. There exist ordinals $\beta < \delta \leq \kappa$ with $I = (\beta, \delta)$, or

2. there is an ordinal $\delta < \kappa$ with $I = [0, \delta)$.

Since α is non-zero, it is in fact sufficient to consider (1). Consequently $\alpha > \beta$, or equivalently, $\sup(C \cap \alpha) > \beta$, thus there exists $\epsilon \in C \cap \alpha$ with $\epsilon > \beta$. So $\epsilon \in C$ is an ordinal with $\beta < \epsilon < \alpha$. Thus in particular there is $\epsilon \in (\beta, \delta) \subseteq U$ is distinct from α. Since U is an arbitrary neighbourhood of α it follows that α is an accumulation point of C as required. □

Corollary 3.6.2. *Let $C \subseteq [0, \omega_1)$ be a closed subset and let $(\alpha_n)_{n \in \mathbb{N}}$ be a strictly monotone increasing sequence in C. Then $\sup\{\alpha_n : n \in \mathbb{N}\} \in C$.*

Proof. Let $\alpha := \sup\{\alpha_n : n \in \mathbb{N}\}$, clearly α is non-zero and $\alpha < \omega_1$. In view of Lemma 3.6.1 it suffices to show that $\alpha = \sup(C \cap \alpha)$. Since $\alpha_n < \alpha$ and $\alpha_n \in C$ for every $n \in \mathbb{N}$, or equivalently, $\{\alpha_n : n \in \mathbb{N}\} \subseteq C \cap \alpha$, it follows that $\alpha \leq \sup(C \cap \alpha)$. But $\alpha \geq \sup(C \cap \alpha)$ must hold thus $\alpha = \sup(C \cap \alpha)$ as required. □

A subset $D \subseteq [0, \omega_1)$ is called a *club subset* if D is a closed and unbounded subset of $[0, \omega_1)$.

The following elementary lemma plays a crucial role in the main theorem of this section, it can be found, for example, in [35, Lemma 3.4]. For the sake of completeness we include a proof here.

Lemma 3.6.3. *A countable intersection of club subsets is a club subset.*

Proof. Let $(C_n)_{n\in\mathbb{N}}$ be a sequence of club subsets of $[0,\omega_1)$, and let $C := \bigcap_{n\in\mathbb{N}} C_n$. It is immediate that C is a closed subset of $[0,\omega_1)$. To see that C is unbounded, let us fix $\alpha_0 \in [0,\omega_1)$, we need to find $\alpha \in C$ such that $\alpha > \alpha_0$.
We will recursively construct sequences $(\alpha_n)_{n\in\mathbb{N}_0}$ and $(\alpha^m_{n+1})_{n\in\mathbb{N}_0}$ in $[0,\omega_1)$ for all $m \in \mathbb{N}$ such that $\alpha^m_{n+1} \in C_m$ and $\alpha_{n+1} \geq \alpha^m_{n+1} > \alpha_n$ for all $n \in \mathbb{N}_0$ and $m \in \mathbb{N}$. Suppose $\alpha_n \in [0,\omega_1)$ has already been defined. For each $m \in \mathbb{N}$, since C_m is unbounded, we can choose $\alpha^m_{n+1} \in C_m$ such that $\alpha^m_{n+1} > \alpha_n$. Let $\alpha_{n+1} := \sup\{\alpha^m_{n+1} : m \in \mathbb{N}\}$, clearly $\alpha_{n+1} \in [0,\omega_1)$. This shows the existence of such sequences.
Now let $\alpha := \sup\{\alpha_n : n \in \mathbb{N}_0\}$, clearly $\alpha \in [0,\omega_1)$. Let us fix $m \in \mathbb{N}$. We observe that $\alpha = \sup\{\alpha^m_{n+1} : n \in \mathbb{N}_0\}$. Indeed, let $n \in \mathbb{N}_0$ be arbitrary, then $\alpha \geq \alpha_{n+1} \geq \alpha^m_{n+1}$, which shows that α is an upper bound for $(\alpha^m_{n+1})_{n\in\mathbb{N}_0}$. Also, if $\beta \in [0,\omega_1)$ is such that for every $n \in \mathbb{N}_0$, $\beta \geq \alpha^m_{n+1}$ holds, then $\beta > \alpha_n$, so $\beta \geq \alpha$. Hence α is the least upper bound for $(\alpha^m_{n+1})_{n\in\mathbb{N}_0}$. Since for every $m \in \mathbb{N}$, C_m is closed, it follows from Corollary 3.6.2 that $\alpha \in C_m$, thus $\alpha \in C$. It remains to show that $\alpha > \alpha_0$, which trivially follows, for example, from $\alpha \geq \alpha_1 > \alpha_0$. □

Banach spaces of continuous functions on ordinals

If K is a compact Hausdorff space then $C(K)$ denotes the Banach space of complex-valued functions on K, with respect to the supremum norm. The Banach space $C[0,\omega_1]$ is called the *Semadeni space*, since he showed in [74] that $C[0,\omega_1]$ is not isomorphic to its square. If L is a locally compact Hausdorff space, and $\tilde{L} := L \cup \{\infty\}$ is its one-point compactification, then we introduce $C_0(L) := \{g \in C(\tilde{L}) : g(\infty) = 0\}$, the Banach space of continuous functions vanishing at infinity, with respect to the supremum norm. In this notation $C_0[0,\omega_1) = \{g \in C[0,\omega_1] : g(\omega_1) = 0\}$.
For a countable ordinal α let $\mathbf{1}_{[0,\alpha]}$ denote the indicator function of the interval $[0,\alpha]$.

Since $[0,\alpha]$ is clopen, it follows that $\mathbf{1}_{[0,\alpha]} \in C_0[0,\omega_1)$. Also, by a theorem of Rudin [68, Theorem 6], the Banach space $C[0,\omega_1]^*$ is isometrically isomorphic to the Banach space

$$\ell_1(\omega_1^+) = \left\{ f : [0,\omega_1] \to \mathbb{C} : \sum_{\alpha \leq \omega_1} |f(\alpha)| < \infty \right\}. \tag{3.35}$$

In the following if K is a compact Hausdorff space and X is a non-zero Banach space then $C(K;X)$ denotes the Banach space of continuous functions from K to X, endowed with the supremum norm.

Definition 3.6.4. Let X be a non-zero Banach space. We define

$$Y_X := \{F \in C([0,\omega_1];X) : F(\omega_1) = 0\}. \tag{3.36}$$

Lemma 3.6.5. *Let X be a non-zero Banach space. Then Y_X is a complemented subspace of $C([0,\omega_1];X)$.*

Proof. For a fixed $x_0 \in X$ let us define the constant function

$$c_{x_0} : [0,\omega_1] \to X; \quad \alpha \mapsto x_0, \tag{3.37}$$

obviously $c_{x_0} \in C([0,\omega_1];X)$. Thus we can define the map

$$Q : C([0,\omega_1];X) \to C([0,\omega_1];X); \quad F \mapsto F - c_{F(\omega_1)}. \tag{3.38}$$

It is clear that Q is a bounded linear map with $\|Q(F)\| \leq 2\|F\|$. Now we observe that for any $F \in C([0,\omega_1];X)$ we clearly have $Q(F)(\omega_1) = 0$, showing that $Q(F) \in Y_X$. Also, for any $F \in Y_X$ and any $\alpha \in [0,\omega_1]$ we have $(Q(F))(\alpha) = F(\alpha)$, consequently Q is an idempotent with $\mathrm{Ran}(Q) = Y_X$ thus proving the claim. $\square$

With the notations of the proof of Lemma 3.6.5, we define

$$P:\ C[0,\omega_1] \to C[0,\omega_1], \quad g \mapsto g - c_{g(\omega_1)}. \tag{3.39}$$

In particular, $\mathrm{Ran}(P) = C_0[0,\omega_1)$.

Remark 3.6.6. Clearly for any $g \in C[0,\omega_1]$, $x \in X$ and $\alpha \in [0,\omega_1]$ we have $(Q(g \otimes x))(\alpha) = (Pg \otimes x)(\alpha)$. From this it follows that $(P \otimes_\epsilon I_X)Q(g \otimes x) = Pg \otimes x = Q(g \otimes x)$, thus by linearity and continuity we obtain

$$I_{Y_X} = (P \otimes_\epsilon I_X)|_{Y_X}. \tag{3.40}$$

Lemma 3.6.7. *Let X be a non-zero Banach space and suppose $\mu, \xi \in (Y_X)^*$ satisfy $\langle f \otimes x, \xi \rangle = \langle f \otimes x, \mu \rangle$ for all $f \in C_0[0,\omega_1)$ and $x \in X$. Then $\xi = \mu$.*

Proof. The definition of P and the hypothesis of the lemma ensure that for any $x \in X$ and $g \in C[0,\omega_1]$ the equality $\langle Pg \otimes x, \xi \rangle = \langle Pg \otimes x, \mu \rangle$ holds. By Remark 3.6.6 we have $\langle Q(g \otimes x), \xi \rangle = \langle Q(g \otimes x), \mu \rangle$, equivalently, $\langle g \otimes x, (Q|^{Y_X})^* \xi \rangle = \langle g \otimes x, (Q|^{Y_X})^* \mu \rangle$ and thus by linearity and continuity of $(Q|^{Y_X})^* \mu$ and $(Q|^{Y_X})^* \xi$ we obtain that for all $u \in C([0,\omega_1];X)$ the identity $\langle u, (Q|^{Y_X})^* \xi \rangle = \langle u, (Q|^{Y_X})^* \mu \rangle$ holds. Thus for any $u \in C([0,\omega_1];X)$ we have $\langle Qu, \xi \rangle = \langle Qu, \mu \rangle$ consequently by Lemma 3.6.5 for all $v \in Y_X$ we have that $\langle v, \xi \rangle = \langle v, \mu \rangle$, proving the claim. □

Proposition 3.6.8. *([70, Section 3.2]) For a compact Hausdorff space K and a non-zero Banach space X there is an isometric isomorphism*

$$J:\ C(K)\hat{\otimes}_\epsilon X \to C(K;X) \tag{3.41}$$

such that for every $f \in C(K)$, $x \in X$ and $k \in K$

$$(J(f \otimes x))(k) = f(k)x. \tag{3.42}$$

Although we do not need this, we shall remark in passing that it follows from Proposition 3.6.8, Lemma 3.6.7 and the Hahn-Banach Separation Theorem that $C_0[0, \omega_1)\hat{\otimes}_\epsilon X$ and Y_X are isometrically isomorphic.

Remark 3.6.9. Let X be a non-zero Banach space. It is easy to see that Y_X is not separable. Indeed, let $x_0 \in X$ be such that $\|x_0\| = 1$ and let us define the map

$$\iota : \ C_0[0, \omega_1) \to Y_X; \quad f \mapsto f \otimes x_0. \tag{3.43}$$

This is clearly a linear isometry, thus, since separability passes to subsets it follows that Y_X cannot be separable.

In the following, if $\alpha \leq \omega_1$ is an ordinal, then $\delta_\alpha \in C[0, \omega_1]^*$ denotes the *Dirac measure* centred at α; that is, the bounded linear functional defined by $\delta_\alpha(g) := g(\alpha)$ for $g \in C[0, \omega_1]$.

Remark 3.6.10. Let X be a non-zero Banach space and let $\alpha \in [0, \omega_1]$ and $\psi \in X^*$ be fixed. We can define a map by

$$\delta_\alpha \otimes \psi : \ C([0, \omega_1]; X) \to \mathbb{C}; \quad u \mapsto \langle u(\alpha), \psi \rangle, \tag{3.44}$$

clearly $\delta_\alpha \otimes \psi \in C([0, \omega_1]; X)^*$.

We recall that $C[0, \omega_1]$ has the approximation property (see [70, Example 4.2]). By [68, Theorem 6] we know that $C[0, \omega_1]^*$ is isometrically isomorphic to $\ell_1(\omega_1^+)$, which has the Radon-Nikodým property (see [70, Example 5.14]), consequently by [70, Theorem 5.33], the Banach space $(C[0, \omega_1]\hat{\otimes}_\epsilon X)^*$ is isometrically isomorphic to $C[0,\omega_1]^*\hat{\otimes}_\pi X^*$. Equivalently, by Proposition 3.6.8, $C([0,\omega_1];X)^*$ is isometrically isomorphic to $\ell_1(\omega_1^+;X^*)$, the Banach space of summable transfinite sequences on ω_1^+ with entries in X^*. This justifies the tensor notation in the definition of the functional $\delta_\alpha \otimes \psi$.

3.6.2 The construction

Our main theorem relies on the following result of Kania–Koszmider–Laustsen:

Theorem 3.6.11. *([41, Theorem 1.5]) For every $T \in \mathcal{B}(C_0[0,\omega_1))$ there exists a unique $\varphi(T) \in \mathbb{C}$ such that there exists a club subset $D \subseteq [0,\omega_1)$ such that for all $f \in C_0[0,\omega_1)$ and $\alpha \in D$:*

$$(Tf)(\alpha) = \varphi(T)f(\alpha). \tag{3.45}$$

Moreover, $\varphi : \mathcal{B}(C_0[0,\omega_1)) \to \mathbb{C};\ T \mapsto \varphi(T)$ is a character.

Remark 3.6.12. We remark here that if $T \in \mathcal{B}(C_0[0,\omega_1))$ then the club subset D in Theorem 3.6.11 corresponding to T is not unique. To see this, let $\alpha_0 \in D$ be fixed and let $D_0 := D \cap [\alpha_0^+,\omega_1)$. Then D_0 is a closed subset of $[0,\omega_1)$, we show that it is unbounded. To this end we pick an arbitrary $\gamma \in [0,\omega_1)$. Since D is unbounded, there is $\beta \in D$ such that $\beta > \max\{\alpha_0,\gamma\}$. In particular, $\beta > \gamma$ and $\beta \in D_0$ as required. It is clear that $D_0 \subsetneq D$ and Equation (3.45) holds for any $f \in C_0[0,\omega_1)$ and $\alpha \in D_0$.

In [41] the character $\varphi : \mathcal{B}(C_0[0,\omega_1)) \to \mathbb{C}$ of the previous Theorem 3.6.11 is termed the *Alspach–Benyamini character* and its kernel the *Loy–Willis ideal* of $\mathcal{B}(C_0[0,\omega_1))$, and is denoted by $\mathcal{M}_{LW}$. Partial structure of the lattice of closed two-sided ideals of $\mathcal{B}(C_0[0,\omega_1))$ is given in [42], in particular $\mathcal{E}(C_0[0,\omega_1)) = \mathcal{K}(C_0[0,\omega_1)) \subsetneq \mathcal{M}_{LW}$.

Theorem 3.6.13. *Let X be a non-zero, separable, reflexive Banach space. For every $S \in \mathcal{B}(Y_X)$ there exists a unique $\Theta(S) \in \mathcal{B}(X)$ such that there exists a club subset*

$D \subseteq [0, \omega_1)$ such that for all $\alpha \in D$ and all $\psi \in X^*$:

$$S^*(\delta_\alpha \otimes \psi) = \delta_\alpha \otimes \Theta(S)^*\psi. \tag{3.46}$$

Moreover, the map $\Theta : \mathcal{B}(Y_X) \to \mathcal{B}(X)$; $S \mapsto \Theta(S)$ is a non-injective algebra homomorphism of norm one; and there exists an algebra homomorphism $\Lambda : \mathcal{B}(X) \to \mathcal{B}(Y_X)$ of norm one with $\Theta \circ \Lambda = \mathrm{id}_{\mathcal{B}(X)}$. In particular Θ is surjective.

Proof. Fix $S \in \mathcal{B}(Y_X)$, $x \in X$ and $\psi \in X^*$. For any $f \in C_0[0, \omega_1)$ we can define the map

$$S_x^\psi f : [0, \omega_1] \to \mathbb{C}; \quad \alpha \mapsto \langle (S(f \otimes x))(\alpha), \psi \rangle. \tag{3.47}$$

It is clear that $S_x^\psi f$ is a continuous map, moreover by $S(f \otimes x) \in Y_X$ we also have $(S_x^\psi f)(\omega_1) = 0$, consequently $S_x^\psi f \in C_0[0, \omega_1)$. This allows us to define the map

$$S_x^\psi : C_0[0, \omega_1) \to C_0[0, \omega_1); \quad f \mapsto S_x^\psi f. \tag{3.48}$$

It is clear that S_x^ψ is a linear map with $\|S_x^\psi\| \leq \|S\|\|x\|\|\psi\|$. Consequently, by Theorem 3.6.11 there exists a club subset $D_{x,\psi} \subseteq [0, \omega_1)$ such that for all $\alpha \in D_{x,\psi}$ the equality

$$(S_x^\psi)^*\delta_\alpha = \varphi(S_x^\psi)\delta_\alpha \tag{3.49}$$

holds. We also have $|\varphi(S_x^\psi)| \leq \|S\|\|x\|\|\psi\|$, since $\|\varphi\| = 1$. This allows us to define the map

$$\tilde{\Theta}_S : X \times X^* \to \mathbb{C}; \quad (x, \psi) \mapsto \varphi(S_x^\psi), \tag{3.50}$$

and we have for any $x \in X$ and $\psi \in X^*$ that $|\tilde{\Theta}_S(x, \psi)| \leq \|S\|\|x\|\|\psi\|$. Now we show that $\tilde{\Theta}_S$ is bilinear. We only check that it is linear in the first variable, linearity in the second variable follows by an analogous argument. Let $x, y \in X$, $\psi \in X^*$ and $\lambda \in \mathbb{C}$ be arbitrary. Fix $f \in C_0[0, \omega_1)$ and $\alpha \in [0, \omega_1]$, then using linearity of the tensor product in the second variable, of S and of the functional ψ it follows that

$$\begin{aligned}(S^{\psi}_{x+\lambda y} f)(\alpha) &= \langle (S(f \otimes (x + \lambda y)))(\alpha), \psi \rangle \\ &= \langle (S(f \otimes x))(\alpha), \psi \rangle + \lambda \langle (S(f \otimes y))(\alpha), \psi \rangle \\ &= (S^{\psi}_x f)(\alpha) + \lambda (S^{\psi}_y f)(\alpha), \end{aligned} \tag{3.51}$$

proving $S^{\psi}_{x+\lambda y} = S_x + \lambda S^{\psi}_y$. Since φ is linear, $\tilde{\Theta}_S(x + \lambda y, \psi) = \varphi(S^{\psi}_{x+\lambda y}) = \varphi(S^{\psi}_x) + \lambda \varphi(S^{\psi}_y) = \tilde{\Theta}_S(x, \psi) + \lambda \tilde{\Theta}_S(y, \psi)$ readily follows, proving linearity of $\tilde{\Theta}_S$ in the first variable. Consequently $\tilde{\Theta}_S$ is a bounded bilinear form on $X \times X^*$. If $\kappa_X : X \to X^{**}$ denotes the canonical embedding then by reflexivity of X the map

$$\Theta_S : \; X \to X; \quad x \mapsto \kappa_X^{-1}(\tilde{\Theta}_S(x, \cdot)) \tag{3.52}$$

defines a bounded linear operator on X with $\|\Theta_S\| = \|\tilde{\Theta}_S\|$ and $\langle \Theta_S(x), \psi \rangle = \tilde{\Theta}_S(x, \psi) = \varphi(S^{\psi}_x)$ for all $x \in X$, $\psi \in X^*$. Thus we can define the map

$$\Theta : \; \mathcal{B}(Y_X) \to \mathcal{B}(X); \quad S \mapsto \Theta_S. \tag{3.53}$$

Since X is separable and reflexive it follows that X^* is separable too. Let $\mathcal{Q} \subseteq X$ and $\mathcal{R} \subseteq X^*$ be countable dense subsets. Let us fix $S \in \mathcal{B}(Y_X)$, $x \in \mathcal{Q}$ and $\psi \in \mathcal{R}$. As above, there exists a club subset $D^S_{x,\psi} \subseteq [0, \omega_1)$ such that for any $\alpha \in D^S_{x,\psi}$ and any

$f \in C_0[0,\omega_1)$: $(S_x^\psi f)(\alpha) = \varphi(S_x^\psi) f(\alpha)$ and hence

$$\begin{aligned} \langle S(f\otimes x), \delta_\alpha \otimes \psi\rangle &= \langle (S(f\otimes x))(\alpha), \psi\rangle = (S_x^\psi f)(\alpha) \\ &= f(\alpha)\varphi(S_x^\psi) = \langle f(\alpha)\Theta(S)x, \psi\rangle \\ &= \langle f \otimes (\Theta(S)x), \delta_\alpha \otimes \psi\rangle. \end{aligned} \tag{3.54}$$

By Lemma 3.6.3 it follows that

$$D^S := \bigcap_{(x,\psi)\in \mathcal{Q}\times\mathcal{R}} D^S_{x,\psi} \tag{3.55}$$

is a club subset of $[0,\omega_1)$. Consequently for any $\alpha \in D^S$, any $f \in C_0[0,\omega_1)$ and any $x \in \mathcal{Q}$, $\psi \in \mathcal{R}$, Equation (3.54) holds. It is clear that for a fixed $S \in \mathcal{B}(Y_X)$, $f \in C_0[0,\omega_1)$ and $\alpha \in D^S$ the maps

$$\begin{aligned} &X \times X^* \to \mathbb{C}; \quad (x,\psi) \mapsto \langle S(f\otimes x), \delta_\alpha \otimes \psi\rangle, \\ &X \times X^* \to \mathbb{C}; \quad (x,\psi) \mapsto \langle f \otimes (\Theta(S)x), \delta_\alpha \otimes \psi\rangle \end{aligned} \tag{3.56}$$

are continuous functions between metric spaces and thus by density of $\mathcal{Q}\times\mathcal{R}$ in $X \times X^*$, Equation (3.54) holds everywhere on $X \times X^*$. In other words, for any $S \in \mathcal{B}(Y_X)$ there exists a club subset $D^S \subseteq [0,\omega_1)$ such that for any $\alpha \in D^S$, $f \in C_0[0,\omega_1)$ and $x \in X$, $\psi \in X^*$

$$\langle f\otimes x, S^*(\delta_\alpha \otimes \psi)\rangle = \langle f \otimes x, \delta_\alpha \otimes (\Theta(S)^*\psi)\rangle \tag{3.57}$$

holds. Therefore by Lemma 3.6.7 we obtain that for all $\alpha \in D^S$ and $\psi \in X^*$:

$$S^*(\delta_\alpha \otimes \psi) = \delta_\alpha \otimes (\Theta(S)^*\psi). \tag{3.58}$$

We show that for any $S \in \mathcal{B}(Y_X)$ the operator $\Theta(S)$ is determined by this property. Indeed, suppose $\Theta_1(S), \Theta_2(S) \in \mathcal{B}(X)$ are such that there exist club subsets $D_1^S, D_2^S \subseteq [0, \omega_1)$ such that for $i \in \{1, 2\}$, all $\alpha \in D_i^S$ and all $\psi \in X^*$

$$S^*(\delta_\alpha \otimes \psi) = \delta_\alpha \otimes (\Theta_i(S)^*\psi). \tag{3.59}$$

Let $\alpha \in D_1^S \cap D_2^S$, $x \in X$ and $\psi \in X^*$ be fixed. Then

$$\begin{aligned} \langle \Theta_1(S)x, \psi \rangle &= \langle \mathbf{1}_{[0,\alpha]} \otimes x, \delta_\alpha \otimes (\Theta_1(S)^*\psi) \rangle \\ &= \langle \mathbf{1}_{[0,\alpha]} \otimes x, S^*(\delta_\alpha \otimes \psi) \rangle \\ &= \langle \mathbf{1}_{[0,\alpha]} \otimes x, \delta_\alpha \otimes (\Theta_2(S)^*\psi) \rangle \\ &= \langle \Theta_2(S)x, \psi \rangle \end{aligned} \tag{3.60}$$

and thus $\Theta_1(S) = \Theta_2(S)$. We are now prepared to prove that Θ is an algebra homomorphism. To see this let $S, T \in \mathcal{B}(Y_X)$ be fixed. Let $D^T, D^S, D^{TS} \subseteq [0, \omega_1)$ be club subsets satisfying Equation (3.58). To see multiplicativity, let $\alpha \in D^T \cap D^S \cap D^{TS}$, $x \in X$ and $\psi \in X^*$ be arbitrary. Then we obtain:

$$\begin{aligned} \delta_\alpha \otimes (\Theta(TS)^*\psi) &= (TS)^*(\delta_\alpha \otimes \psi) = S^*T^*(\delta_\alpha \otimes \psi) \\ &= S^*(\delta_\alpha \otimes (\Theta(T)^*\psi)) \\ &= \delta_\alpha \otimes (\Theta(S)^*\Theta(T)^*\psi) \\ &= \delta_\alpha \otimes ((\Theta(T)\Theta(S))^*\psi), \end{aligned} \tag{3.61}$$

hence $\Theta(TS)^*\psi = (\Theta(T)\Theta(S))^*\psi$, so $\Theta(TS)^* = (\Theta(T)\Theta(S))^*$, equivalently $\Theta(TS) = \Theta(T)\Theta(S)$.

Linearity can be shown with analogous reasoning.

For any $S \in \mathcal{B}(Y_X)$ we have $\|\Theta(S)\| = \|\tilde{\Theta}_S\| \le \|S\|$, thus $\|\Theta\| \le 1$.

We now show that Θ has a right inverse. Let $P \in \mathcal{B}(C[0,\omega_1])$ be the idempotent operator as in Equation (3.39). Let us fix an $A \in \mathcal{B}(X)$. We observe that $S := (P \otimes_\epsilon A)|_{Y_X}$ belongs to $\mathcal{B}(Y_X)$. Indeed, for any $g \in C[0,\omega_1]$ and $x \in X$ the identity $((P \otimes_\epsilon A)(g \otimes x))(\omega_1) = (Pg)(\omega_1)Ax = 0$ holds plainly because $Pg \in C_0[0,\omega_1)$; thus by linearity and continuity of $P \otimes_\epsilon A$ in fact $((P \otimes_\epsilon A)u)(\omega_1) = 0$ for all $u \in C[0,\omega_1]\hat{\otimes}_\epsilon X$. This shows that $S \in \mathcal{B}(Y_X)$ and therefore there exists a club subset $D^S \subseteq [0,\omega_1)$ such that Equation (3.58) is satisfied for all $\alpha \in D^S$ and all $\psi \in X^*$. Fix $\alpha \in D^S$. For any $x \in X$ and $\psi \in X^*$

$$\begin{aligned}
\langle Ax, \psi \rangle &= \langle \mathbf{1}_{[0,\alpha]} \otimes (Ax), \delta_\alpha \otimes \psi \rangle = \langle (P \otimes_\epsilon A)(\mathbf{1}_{[0,\alpha]} \otimes x), \delta_\alpha \otimes \psi \rangle \\
&= \langle \mathbf{1}_{[0,\alpha]} \otimes x, S^*(\delta_\alpha \otimes \psi) \rangle \\
&= \langle \mathbf{1}_{[0,\alpha]} \otimes x, \delta_\alpha \otimes (\Theta(S)^* \psi) \rangle \\
&= \langle x, \Theta(S)^* \psi \rangle \\
&= \langle \Theta(S)x, \psi \rangle,
\end{aligned} \tag{3.62}$$

and thus $\Theta(S) = A$. In particular, we obtain $\Theta(I_{Y_X}) = I_X$, with $\|\Theta\| \leq 1$ this yields $\|\Theta\| = 1$. Also, the above shows that the map

$$\Lambda : \ \mathcal{B}(X) \to \mathcal{B}(Y_X); \quad A \mapsto (P \otimes_\epsilon A)|_{Y_X} \tag{3.63}$$

satisfies $\Theta \circ \Lambda = \mathrm{id}_{\mathcal{B}(X)}$. It is immediate that Λ is linear with $\|\Lambda\| \leq 1$. Also, $\Lambda(I_X) = I_{Y_X}$ holds by Equation (3.40), consequently $\|\Lambda\| = 1$. The map Λ is an algebra homomorphism plainly because $P \in \mathcal{B}(C_0[0,\omega_1))$ is an idempotent, therefore $(P \otimes_\epsilon A)(P \otimes_\epsilon B) = P \otimes_\epsilon (AB)$ for every $A, B \in \mathcal{B}(X)$.

It remains to prove that Θ is not injective. For assume towards a contradiction it is; then $\mathcal{B}(Y_X)$ and $\mathcal{B}(X)$ are isomorphic as Banach algebras. By Eidelheit's Theorem this is equivalent to saying that Y_X and X are isomorphic as Banach spaces. This is clearly nonsense, since for example, X is separable whereas by Remark 3.6.9 the Banach space Y_X is not. □

Remark 3.6.14. With the notations established in the proof of Theorem 3.6.13 we clearly have in fact

$$\operatorname{Ker}(\Theta) = \{S \in \mathcal{B}(Y_X) \,:\, (\forall x \in X)(\forall \psi \in X^*)(S_x^\psi \in \mathcal{M}_{LW})\}, \tag{3.64}$$

where S_x^ψ is defined by (3.47).

If X is an infinite-dimensional Banach space then $\operatorname{Ker}(\Theta)$ is of course not maximal in $\mathcal{B}(Y_X)$, however, it is not the smallest possible ideal in $\mathcal{B}(Y_X)$. To see this, we need some preliminary observations.

In the following, let $P \in \mathcal{B}(C[0,\omega_1])$ be as in Equation (3.39). If X is a non-zero Banach space, we fix $x_0 \in X$ and $\xi \in X^*$ such that $\|x_0\| = \|\xi\| = \langle x_0, \xi \rangle = 1$ and consider the linear isometry

$$\iota :\; C_0[0,\omega_1) \to Y_X; \quad f \mapsto f \otimes x_0. \tag{3.65}$$

We also consider the norm one linear map

$$\rho :\; C[0,\omega_1]\hat{\otimes}_\epsilon X \to C[0,\omega_1] \tag{3.66}$$

which is unique with the property that for any $g \in C[0,\omega_1]$ and $x \in X$ the identity $\rho(g \otimes x) = \langle x, \xi \rangle g$ holds. With this we obtain the following:

Lemma 3.6.15. *Let X be a non-zero Banach space. Then*

$$\Xi :\; \mathcal{B}(C_0[0,\omega_1)) \to \mathcal{B}(Y_X); \quad S \mapsto \Big((P|_{C_0[0,\omega_1)} \circ S \circ P|^{C_0[0,\omega_1)}) \otimes_\epsilon I_X\Big)|_{Y_X} \tag{3.67}$$

$$\Upsilon :\; \mathcal{B}(Y_X) \to \mathcal{B}(C_0[0,\omega_1)); \quad T \mapsto P|^{C_0[0,\omega_1)} \circ \rho|_{Y_X} \circ T \circ \iota \tag{3.68}$$

define norm one linear maps with $\Upsilon \circ \Xi = \mathrm{id}_{\mathcal{B}(C_0[0,\omega_1))}$*. Moreover,* Ξ *is an algebra homomorphism such that* $(\Xi(S))_x^\psi = \langle x, \psi \rangle S$ *for every* $x \in X$ *and* $\psi \in X^*$*.*

Proof. It is clear that $\Big((P|_{C_0[0,\omega_1)} \circ S \circ P|^{C_0[0,\omega_1)}) \otimes_\epsilon I_X \Big)|_{Y_X} \in \mathcal{B}(Y_X)$ holds for any $S \in \mathcal{B}(C_0[0,\omega_1))$, thus Ξ is well-defined. It is easy to see that Ξ is linear with $\|\Xi\| \leq 1$. From Equation (3.40) it follows that $\Xi(I_{C_0[0,\omega_1)}) = I_{Y_X}$, thus $\|\Xi\| = 1$. The map Ξ is multiplicative simply by the defining property of injective tensor products of operators and by $P|^{C_0[0,\omega_1)} \circ P|_{C_0[0,\omega_1)} = I_{C_0[0,\omega_1)}$. Let $S \in \mathcal{B}(C_0[0,\omega_1))$, $x \in X$ and $\psi \in X^*$ be fixed. Then for any $f \in C_0[0,\omega_1)$ and $\alpha \leq \omega_1$ ordinal

$$\Big((\Xi(S))_x^\psi f \Big)(\alpha) = \langle (\Xi(S)(f \otimes x))(\alpha), \psi \rangle = \langle (Sf)(\alpha) x, \psi \rangle = (Sf)(\alpha) \langle x, \psi \rangle, \tag{3.69}$$

thus $(\Xi(S))_x^\psi = \langle x, \psi \rangle S$ indeed.

Linearity of Υ is immediate, so is $\|\Upsilon\| \leq 1$. Since $\Upsilon(I_{Y_X}) = I_{C_0[0,\omega_1)}$ follows from the definition of Υ, we obtain $\|\Upsilon\| = 1$ as required.

It remains to show that $\Upsilon \circ \Xi = \mathrm{id}_{\mathcal{B}(C_0[0,\omega_1))}$. For any $S \in \mathcal{B}(C_0[0,\omega_1))$ and $f \in C_0[0,\omega_1)$

$$\begin{aligned} \Upsilon(\Xi(S))f &= (P|^{C_0[0,\omega_1)} \circ \rho|_{Y_X} \circ \Xi(S) \circ \iota) f \\ &= (P|^{C_0[0,\omega_1)} \circ \rho|_{Y_X} \circ \Xi(S))(f \otimes x_0) \\ &= (P|^{C_0[0,\omega_1)} \circ \rho|_{Y_X})(Sf \otimes x_0) \\ &= P|^{C_0[0,\omega_1)}(\langle x_0, \xi \rangle Sf) \\ &= Sf, \end{aligned} \tag{3.70}$$

consequently $\Upsilon(\Xi(S)) = S$, which proves the claim. □

Corollary 3.6.16. *The proper containment* $\mathcal{E}(Y_X) \subsetneq \mathrm{Ker}(\Theta)$ *holds.*

Proof. By Lemma 3.3.6 it follows that $\mathcal{E}(Y_X) \subseteq \text{Ker}(\Theta)$, we show that the containment is proper. For assume towards a contradiction that $\text{Ker}(\Theta) = \mathcal{E}(Y_X)$. If $S \in \mathcal{M}_{LW}$ then by Lemma 3.6.15 for all $x \in X$ and $\psi \in X^*$ in fact $(\Xi(S))_x^\psi = \langle x, \psi \rangle S \in \mathcal{M}_{LW}$, thus by Remark 3.6.14 then $\Xi(S) \in \text{Ker}(\Theta)$ follows. Thus $\Xi(S) \in \mathcal{E}(Y_X)$ by the indirect assumption and since $\mathcal{E}$ is an operator ideal, it follows from Lemma 3.6.15 that

$$S = \Upsilon(\Xi(S)) = P|^{C_0[0,\omega_1)} \circ \rho|_{Y_X} \circ \Xi(S) \circ \iota \in \mathcal{E}(C_0[0,\omega_1)). \tag{3.71}$$

This yields $\mathcal{M}_{LW} = \mathcal{E}(C_0[0,\omega_1))$, which is a contradiction. $\square$

Chapter 4

Finiteness and stable rank of algebras of operators on Banach spaces

4.1 Introduction and basic terminology

Let us first recall the following ring-theoretic definition:

Definition 4.1.1. Let A be a unital ring with identity 1_A. Then A is called

1. *Dedekind-finite* or *directly finite* or *DF* for short, if the only idempotent $p \in A$ with $p \sim 1_A$ is the identity 1_A,

2. *Dedekind-infinite* if A is not Dedefind-finite,

3. *properly infinite* if there exist orthogonal idempotents $p, q \in A$ such that $p, q \sim 1_A$.

It is easy to see that a properly infinite ring is Dedekind-infinite. Clearly every commutative, unital ring is Dedekind-finite. Another easy example is the matrix ring $M_n(\mathbb{C})\,(n \geq 1)$ since an $(n \times n)$ complex matrix is left-invertible if and only if it is right-invertible.

Therefore it is natural to examine the unital Banach algebra $\mathcal{B}(X)$ of bounded linear operators from this perspective, where X is an infinite-dimensional Banach space. In this chapter all Banach spaces and algebras are assumed to be complex, unless explicitly stated otherwise. The systematic study of Dedekind-(in)finiteness of $\mathcal{B}(X)$ was laid out by Laustsen in [47], where the author characterises Dedekind-finiteness and proper infiniteness of $\mathcal{B}(X)$ in terms of the complemented subspaces of X. For our purposes the former is of greater importance, therefore we recall this result here:

Lemma 4.1.2. *([47, Corollary 1.5]) Let X be a Banach space. Then $\mathcal{B}(X)$ is Dedekind-finite if and only if no proper, complemented subspace of X is isomorphic to X as a Banach space.*

Proof. Suppose $\mathcal{B}(X)$ is DF and let $P \in \mathcal{B}(X)$ be an idempotent such that $\operatorname{Ran}(P) \simeq X$. By Lemma 1.2.11 this is equivalent to $P \sim I_X$. Thus $P = I_X$ or equivalently $\operatorname{Ran}(P) = X$.

We show the other direction by proving the contrapositive; suppose $\mathcal{B}(X)$ is not DF. Then there exists $Q \in \mathcal{B}(X)$ idempotent such that $Q \sim I_X$ and $Q \neq I_X$. By Lemma 1.2.11 this is equivalent to $\operatorname{Ran}(Q) \simeq X$ and $\operatorname{Ran}(Q) \neq X$, as required. □

As is observed in [47, Corollary 1.7], every hereditarily indecomposable Banach space X satisfies the conditions of Lemma 4.1.2. However, as we shall demonstrate in Corollary 4.2.9, if X is an HI space, then $\mathcal{B}(X)$ in fact possesses the stronger property of having stable rank one. This definition was introduced by Rieffel in [66]:

Definition 4.1.3. A unital Banach algebra A has *stable rank one* if the group of invertible elements $\operatorname{inv}(A)$ is dense in A with respect to the norm topology.

4.2 Algebras of operators with stable rank one and their connection to Dedekind-finiteness

The following observation is an immediate corollary of [66, Proposition 3.1]. We include a short proof for the reader's convenience.

Lemma 4.2.1. *A unital Banach algebra with stable rank one is Dedekind-finite.*

Proof. Let A be a Banach algebra with stable rank one. Assume $p \in A$ is an idempotent such that $p \sim 1_A$. Then there exist $a, b \in A$ such that $p = ab$ and $1_A = ba$. Let $u \in \operatorname{inv}(A)$ be such that $\|a-u\| < \|b\|^{-1}$. Then $\|1_A - bu\| = \|ba - bu\| \leq \|b\|\|a-u\| < 1$, so in particular $bu \in \operatorname{inv}(A)$ holds, and consequently $b = buu^{-1} \in \operatorname{inv}(A)$. From this and $1_A = ba$ we get $a = b^{-1}$, consequently $p = ab = 1_A$. Thus A is Dedekind-finite. $\square$

Let us recall that in a Banach algebra A an element $a \in A$ is a *topological zero divisor* if $\inf\{\|xa\| + \|ax\| : x \in A, \|x\| = 1\} = 0$. It follows for example from [7, Section 2, Theorem 14] that for a unital Banach algebra A the *topological boundary of* $\operatorname{inv}(A)$, denoted by $\partial(\operatorname{inv}(A))$ is contained in the set of topological zero divisors of A.

We note that the converse of Lemma 4.2.1 is clearly false. We demonstrate this with an example which will be essential in the proof of our main result, Theorem 4.2.16.

Example 4.2.2. The complex unital Banach algebra $\ell_1(\mathbb{N}_0) := \ell_1(\mathbb{N}_0; \mathbb{C})$ (endowed with the convolution product) is Dedekind-finite but does not have stable rank one. The former is trivial since $\ell_1(\mathbb{N}_0)$ is commutative. Now let us show that it does not have stable rank one. This in fact is contained in the proof of [21, Proposition 4.7], we include the argument here for the sake of completeness. Let $(\delta_n)_{n \in \mathbb{N}_0}$ stand for the canonical basis of $\ell_1(\mathbb{N}_0)$. Observe that δ_1 is a non-invertible element in $\ell_1(\mathbb{N}_0)$. For assume towards a contradiction that there is $x = (x_n)_{n \in \mathbb{N}_0} \in \ell_1(\mathbb{N}_0)$ with $\delta_0 = x * \delta_1$. Then $\delta_0 = x_n \delta_{n+1}$ for some $n \in \mathbb{N}_0$, consequently $n = -1$, which is impossible. We now show that δ_1 is not a topological zero divisor. To see this, let $x = (x_n)_{n\in\mathbb{N}_0} \in \ell_1(\mathbb{N}_0)$ be arbitrary. Then

$$\|x * \delta_1\| = \left\| \sum_{n\in\mathbb{N}_0} x_n \delta_n * \delta_1 \right\| = \left\| \sum_{n\in\mathbb{N}_0} x_n \delta_{n+1} \right\| = \sum_{n\in\mathbb{N}_0} |x_n| = \|x\|. \tag{4.1}$$

Thus by the discussion preceding the example we see that $\delta_1 \notin \partial(\mathrm{inv}(\ell_1(\mathbb{N}_0)))$. Hence we conclude that $\delta_1 \notin \overline{\mathrm{inv}(\ell_1(\mathbb{N}_0))}$, therefore $\ell_1(\mathbb{N}_0)$ cannot have stable rank one.

As we shall see in Corollary 4.2.9, all the examples given in [47] such that $\mathcal{B}(X)$ is Dedekind-finite have stable rank one. Thus the following question naturally arises:

Question 4.2.3. Does there exist a Banach space X such that $\mathcal{B}(X)$ is Dedekind-finite but it does not have stable rank one?

The purpose of the following is to answer this question in the positive.
Recall that if A is a unital algebra over a field K and C is a unital subalgebra then $\mathrm{inv}(C) \subseteq \mathrm{inv}(A) \cap C$ holds but there is not equality in general. In the following, if J is a two-sided ideal of A we introduce the notation $\tilde{J} := K1_A + J$.

Lemma 4.2.4. *Let A be a unital algebra over a field K and let $J \trianglelefteq A$ be a proper, two-sided ideal. Then for the unital subalgebra $\tilde{J}$ the equality* $\mathrm{inv}(\tilde{J}) = \mathrm{inv}(A) \cap \tilde{J}$ *holds.*

Proof. It is clear that $\tilde{J}$ is a unital subalgebra of A. Thus we only need to show the inclusion $\mathrm{inv}(A) \cap \tilde{J} \subseteq \mathrm{inv}(\tilde{J})$. To see this let us pick an arbitrary $\lambda \in K$ and $j \in J$ such that $\lambda 1_A + j \in \mathrm{inv}(A)$. Clearly $\lambda \neq 0$ otherwise $j \in \mathrm{inv}(A)$ which contradicts J being a proper ideal of A. Now it is clear that $a := \lambda^{-1} 1_A - \lambda^{-1}(\lambda 1_A + j)^{-1} j \in K1_A + J$, and a simple calculation shows that $a(\lambda 1_A + j) = 1_A = (\lambda 1_A + j)a$ holds, proving the claim. □

Remark 4.2.5. If A is a complex unital Banach algebra and $J \trianglelefteq A$ is a proper, closed, two-sided ideal of A then $\tilde{J} := \mathbb{C}1_A + J$ is a closed, unital subalgebra of A. (Closedness follows from the fact that $\mathbb{C}1_A$ and J are respectively finite-dimensional and closed subspaces of the Banach space A.) Also, $\tilde{J}$ is equal to the closed unital subalgebra of A generated by the set $\{1_A\} \cup J$ and clearly $\tilde{J} \simeq J^\sharp$.

Lemma 4.2.6. *Let A be a complex, unital Banach algebra and let $a \in A$ be such that $0 \in \mathbb{C}$ is not in the interior of the spectrum $\sigma_A(a)$. Then $a \in \overline{\mathrm{inv}(A)}$.*

Proof. By the hypothesis it follows that $0 \notin \mathrm{int}(\sigma_A(a)) = \mathbb{C} \backslash \left(\overline{\mathbb{C} \backslash \sigma_A(a)}\right)$. Thus there exists a sequence $(\lambda_n)_{n \in \mathbb{N}}$ in the resolvent set of the element a converging to $0 \in \mathbb{C}$. Therefore $(a - \lambda_n 1_A)_{n \in \mathbb{N}}$ is a sequence of invertible elements in A converging to a. $\square$

Proposition 4.2.7. *Let X be a Banach space, and let $J \trianglelefteq \mathcal{B}(X)$ be a closed, two-sided ideal with $J \subseteq \mathcal{E}(X)$. Then for any $\alpha \in \mathbb{C}$ and $T \in J$, $\alpha I_X + T \in \overline{\mathrm{inv}(\tilde{J})}$ holds, and therefore $\tilde{J}$ has stable rank one.*

Proof. Let us pick $\alpha \in \mathbb{C}$ and $T \in J$. It is an immediate corollary of Lemma 4.2.4 that $\sigma_{\tilde{J}}(T) = \sigma_{\mathcal{B}(X)}(T)$. Now by the Spectral Mapping Theorem $\sigma_{\tilde{J}}(\alpha I_X + T) = \alpha + \sigma_{\tilde{J}}(T)$, putting this together with the previous observation we conclude that

$$\sigma_{\tilde{J}}(\alpha I_X + T) = \alpha + \sigma_{\mathcal{B}(X)}(T). \tag{4.2}$$

Since $T \in J \subseteq \mathcal{E}(X)$, it follows from [10, Lemma 5.6.1] that T is a *Riesz operator* (see [10, Definition 3.1.1]), thus $\sigma_{\mathcal{B}(X)}(T) \backslash \{0\}$ has no accumulation point, hence $\sigma_{\mathcal{B}(X)}(T)$ must be countable. Consequently $\sigma_{\tilde{J}}(\alpha I_X + T)$ must be countable, thus it has empty interior, so in particular Lemma 4.2.6 yields $\alpha I_X + T \in \overline{\mathrm{inv}(\tilde{J})}$. $\square$

Remark 4.2.8. Let us note that in the previous proposition the assumption that the ideal is contained in the inessential operators cannot be dropped in general. To see this we consider the p^{th} quasi-reflexive James space $\mathcal{J}_p$, where $1 < p < \infty$. Since the closed, two-sided ideal $\mathcal{W}(\mathcal{J}_p)$ of weakly compact operators is one-codimensional in $\mathcal{B}(\mathcal{J}_p)$, it is in particular a complemented subspace of $\mathcal{B}(\mathcal{J}_p)$ and therefore $\mathcal{B}(\mathcal{J}_p) = \mathbb{C}I_{\mathcal{J}_p} + \mathcal{W}(\mathcal{J}_p)$ holds. On the other hand, as observed in [47, Propostition 1.13], the Banach algebra $\mathcal{B}(\mathcal{J}_p)$ is Dedekind-infinite so by Lemma 4.2.1 it cannot have stable rank one.

Corollary 4.2.9. *For a complex hereditarily indecomposable Banach space X the Banach algebra $\mathcal{B}(X)$ has stable rank one.*

Proof. As was proven by Gowers and Maurey in [28, Theorem 18], for any complex HI space X, $\mathcal{B}(X) = \mathbb{C}I_X + \mathcal{S}(X)$ holds. Together with Proposition 4.2.7 the result immediately follows. □

The result above is known, we refer the interested reader to [21], see the text preceeding Theorem 4.16. However, their proof differs from the one presented here.

The following simple algebraic lemma is the key step in the proof our main result.

Lemma 4.2.10. *Let A be a unital algebra over a field K and let $J \trianglelefteq A$ be a two-sided ideal such that both $\tilde{J}$ and A/J are Dedekind-finite. Let $\pi : A \to A/J$ denote the quotient map. If $\pi\left[\mathrm{inv}(A)\right] = \mathrm{inv}\left(A/J\right)$ holds then A is Dedekind-finite.*

Proof. Let $p \in A$ be an idempotent such that $p \sim 1_A$. Then there exist $a, b \in A$ such that $ab = 1_A$ and $ba = p$. The identities $\pi(a)\pi(b) = \pi(1_A)$ and $\pi(b)\pi(a) = \pi(p)$ show that $\pi(p)$ is an idempotent in A/J such that $\pi(p) \sim \pi(1_A)$. Since A/J is DF it follows that $\pi(p) = \pi(1_A)$, equivalently $\pi(b)\pi(a) = \pi(1_A)$ and consequently $\pi(a) \in \mathrm{inv}\left(A/J\right)$. By the assumption there exists $c \in \mathrm{inv}(A)$ such that $\pi(a) = \pi(c)$, equivalently $a - c \in J$. Thus $c^{-1} - b = c^{-1}ab - c^{-1}cb = c^{-1}(a-c)b \in J$. Let us define $a' := (a-c)c^{-1}$ and $b' := c(b - c^{-1})$, it is clear from the previous that $a', b' \in J$. Now we show that the following identities hold:

- $(1_A + a')(1_A + b') = 1_A$ or equivalently $a' + b' + a'b' = 0$,
- $(1_A + b')(1_A + a') = cpc^{-1}$ or equivalently $a' + b' + b'a' = cpc^{-1} - 1_A$.

To see these, we observe that from the definitions of a' and b' we obtain

$$a' + b' = (a - c)c^{-1} + c(b - c^{-1}) = ac^{-1} + cb - 2 \cdot 1_A, \tag{4.3}$$

$$b'a' = c(b - c^{-1})(a - c)c^{-1} = c(ba - bc - c^{-1}a + 1_A)c^{-1} = cpc^{-1} - cb - ac^{-1} + 1_A, \tag{4.4}$$

$$a'b' = (a - c)c^{-1}c(b - c^{-1}) = ab - ac^{-1} - cb + 1_A = 2 \cdot 1_A - ac^{-1} - cb. \tag{4.5}$$

The above immediately yield the required identities. Thus we have obtained that cpc^{-1} is an idempotent in $\tilde{J}$ equivalent to 1_A. Since $\tilde{J}$ is DF it follows that $cpc^{-1} = 1_A$. That is, $p = 1_A$ which concludes the proof. □

In what follows, if K is a compact Hausdorff space then $C(K)$ denotes the Banach algebra of complex valued continuous functions on K.

Remark 4.2.11. Let us note here that in the previous lemma, the condition that the invertible elements in A surject onto the invertible elements in A/J is not superfluous. To see this, we recall some basic properties of the *Toeplitz algebra*, see [71, Example 9.4.4] for full details of the construction. Let $\mathcal{H}$ be a separable Hilbert space, let us fix an orthonormal basis in $\mathcal{H}$ and let $S \in \mathcal{B}(\mathcal{H})$ be the right shift operator with respect to this basis. Let $S^* \in \mathcal{B}(\mathcal{H})$ denote the adjoint of S. The unital sub-C^*-algebra of $\mathcal{B}(\mathcal{H})$ generated by S is called the Toeplitz algebra $\mathcal{T}$. We recall that $\mathcal{K}(\mathcal{H}) \subseteq \mathcal{T}$ and that $\mathcal{T}/\mathcal{K}(\mathcal{H})$ is isomorphic to $C(\mathbb{T})$, where $\mathbb{T}$ is the unit circle. Since $C(\mathbb{T})$ is commutative, it is clearly Dedekind-finite. As is well-known (see [14, Corollary 5] or Proposition 4.2.7 above), $\tilde{\mathcal{K}}(\mathcal{H})$ has stable rank one thus by Lemma 4.2.1 it is also Dedekind-finite. On the other hand, $S^*S = I_{\mathcal{H}}$ and $SS^* \neq I_{\mathcal{H}}$, thus $\mathcal{T}$ is Dedekind-infinite.

For a unital Banach algebra A let $\exp(A) := \{\exp(a) : a \in A\}$. Recall that $\exp(A) \subseteq \mathrm{inv}(A)$ and when A is commutative, $\exp(A)$ is both a subgroup and the connected component of the identity in $\mathrm{inv}(A)$, see for example [15, Corollary 2.4.27].

Lemma 4.2.12. *Let A be a unital Banach algebra and suppose $J \trianglelefteq A$ is a closed, two-sided ideal in A such that A/J is commutative. Let $\pi : A \to A/J$ denote the quotient map. If* $\mathrm{inv}(A/J)$ *is connected then* $\pi\left[\exp(A)\right] = \mathrm{inv}(A/J)$ *holds. In particular* $\pi\left[\mathrm{inv}(A)\right] = \mathrm{inv}(A/J)$.

Proof. Since A/J is commutative and $\mathrm{inv}(A/J)$ is connected it follows that $\mathrm{inv}(A/J) = \exp(A/J)$. We now observe that $\exp(A/J) = \pi[\exp(A)]$ holds, since for any $a \in A$, the series expansion of $\exp(a)$ converges (absolutely) in A and π is a continuous surjective algerba homomorphism; thus it readily follows that $\pi(\exp(a)) = \exp(\pi(a))$. The second part of the claim follows from $\pi[\mathrm{inv}(A)] \subseteq \mathrm{inv}(A/J)$. □

Lemma 4.2.13. *The group* $\mathrm{inv}(\ell_1(\mathbb{N}_0))$ *is connected.*

Proof. Let $A := \ell_1(\mathbb{N}_0)$. It is known (see for example [15, Theorem 4.6.9]) that the character space Γ_A of A is homeomorphic to the closed unit disc $\overline{\mathbb{D}}$. Thus by the Arens–Royden Theorem (see [58, Theorem 3.5.19] and the text preceding it) we obtain the following isomorphism of groups:

$$\mathrm{inv}(A)/\exp(A) \simeq \mathrm{inv}(C(\overline{\mathbb{D}}))/\exp(C(\overline{\mathbb{D}})) \simeq \pi^1(\overline{\mathbb{D}}), \tag{4.6}$$

where $\pi^1(\overline{\mathbb{D}})$ denotes the first fundamental group of $\overline{\mathbb{D}}$. Since $\overline{\mathbb{D}}$ is simply connected we obtain $\mathrm{inv}(A) = \exp(A)$ proving that $\mathrm{inv}(A)$ is connected as required. □

Remark 4.2.14. In the proof of the previous lemma we do not use the surjective part of the Arens–Royden Theorem, only the much weaker statement that $\mathrm{inv}(A)/\exp(A)$ injects into $\mathrm{inv}(C(\Gamma_A))/\exp(C(\Gamma_A))$.

Let us recall the properties of Tarbard's ingenious indecomposable Banach space construction that are relevant to our purposes, we refer the interested reader to [78, Chapter 4] to see the following theorem in its full might.

Theorem 4.2.15. *([78, Theorems 4.1.1 and 4.1.5]) There exists a real indecomposable Banach space X_∞ such that the real unital Banach algebras $\mathcal{B}(X_\infty)/\mathcal{K}(X_\infty)$ and $\ell_1(\mathbb{N}_0;\mathbb{R})$ are isometrically isomorphic.*

Before we state the main theorem of this chapter, we recall some facts about the complexification of real Banach spaces and real Banach algebras. For further details we refer the reader to [7, Section 13] and [65, Chapter I, Section 3].
Let X be a real Banach space. The *complexification* of X, denoted by $\hat{X}$, is the set $X \times X$ endowed with coordinate-wise addition, the scalar product defined as

$$(\alpha + i\beta) \cdot (x, y) := (\alpha x - \beta y, \alpha y + \beta x) \quad (x, y \in X;\, \alpha, \beta \in \mathbb{R}), \tag{4.7}$$

and the norm

$$\|(x, y)\| := \max\{(\|\alpha x - \beta y\|^2 + \|\alpha y + \beta x\|^2)^{1/2} : \alpha, \beta \in \mathbb{R},\, \alpha^2 + \beta^2 = 1\}. \tag{4.8}$$

One can show that endowed with the operations above, $\hat{X}$ becomes a complex Banach space.
If $T \in \mathcal{B}(X)$, then

$$\hat{T} : \hat{X} \to \hat{X}; \quad (x, y) \mapsto (Tx, Ty) \tag{4.9}$$

is a bounded linear operator on the complex Banach space $\hat{X}$ such that $\|\hat{T}\| = \|T\|$. It is easy to see that if $T \in \mathcal{K}(X)$ then $\hat{T} \in \mathcal{K}(\hat{X})$.
Let A be a real Banach algebra. Let $\hat{A}$ be the Banach space complexification of A. Let us endow the Banach space $\hat{A}$ with the algebra product

$$(a, b) \cdot (c, d) := (ac - bd, ad + bc) \quad (a, b, c, d \in A). \tag{4.10}$$

Then $\hat{A}$ becomes a complex Banach algebra with these operations. It is elementary to see that if A and B are real Banach algebras such that they are isomorphic, then $\hat{A}$ and $\hat{B}$ are isomorphic as complex Banach algebras. Let A be a real Banach algebra and let $I \trianglelefteq A$ be a closed two-sided ideal in A. Then $\hat{I}$ is a closed two-sided ideal in $\hat{A}$. Let $\pi_1 : A \to A/I$ and $\pi_2 : \hat{A} \to \hat{A}/\hat{I}$ denote the quotient maps. The map

$$\theta : \widehat{A/I} \to \hat{A}/\hat{I}; \quad (\pi_1(a), \pi_1(b)) \mapsto \pi_2(a, b) \tag{4.11}$$

is easily seen to be an isomorphism of the complex Banach algebras $\widehat{A/I}$ and $\hat{A}/\hat{I}$. Let X be a real Banach space. The map

$$\psi : \widehat{\mathcal{B}(X)} \to \mathcal{B}(\hat{X}); \quad (S, T) \mapsto \hat{S} + i\hat{T} \tag{4.12}$$

is an isomorphism of complex Banach algebras. The only non-trivial part is surjectivity of ψ. This however immediately follows from the fact that if $R \in \mathcal{B}(\hat{X})$, then complex linearity of R implies $R(-x, 0) = R(i(0, x)) = iR(0, x)$ for every $x \in X$. We refer the reader to [26, Proposition 2.2] for further details. From the above one can easily show that ψ restricts to an isomorphism between $\widehat{\mathcal{K}(X)}$ and $\mathcal{K}(\hat{X})$, see [26, Proposition 2.4]. In particular this implies the following isomorphism of complex Banach algebras:

$$\mathcal{B}(\hat{X})/\mathcal{K}(\hat{X}) \simeq \widehat{\mathcal{B}(X)}/\widehat{\mathcal{K}(X)} \simeq \mathcal{B}(\widehat{X)/\mathcal{K}}(X). \tag{4.13}$$

We are now ready to state and prove the main result of this chapter.

Theorem 4.2.16. *The complex Banach algebra $\mathcal{B}(\hat{X}_\infty)$ is Dedekind-finite but does not have stable rank one.*

Proof. We first show that $B(\hat{X}_\infty)$ does not have stable rank one. Assume towards a contradiction that it does. Then it immediately follows that $\mathcal{B}(\hat{X}_\infty)/\mathcal{K}(\hat{X}_\infty) \simeq$

$\widehat{\mathcal{B}(X_\infty)/\mathcal{K}(X_\infty)}$ also have stable rank one, which in view of Theorem 4.2.15 is equivalent to $\ell_1(\mathbb{N}_0;\mathbb{C}) =: \ell_1(\mathbb{N}_0)$ having stable rank one. This is impossible by Example 4.2.2. Now we show that $\mathcal{B}(\hat{X}_\infty)$ is Dedekind-finite. By Proposition 4.2.7 we obtain that $\tilde{\mathcal{K}}(\hat{X}_\infty)$ has stable rank one so by Lemma 4.2.1 it is Dedekind-finite. By Example 4.2.2 we have that $\ell_1(\mathbb{N}_0)$ and thus $\mathcal{B}(\hat{X}_\infty)/\mathcal{K}(\hat{X}_\infty)$ is also Dedekind-finite. Thus applying Lemmas 4.2.13, 4.2.12 and 4.2.10 successively, we obtain that $\mathcal{B}(\hat{X}_\infty)$ is Dedekind-finite, which completes the proof. □

With the aid of Lemma 4.1.2 we observe the following:

Corollary 4.2.17. *No proper, complemented subspace of $\hat{X}_\infty$ is isomorphic to $\hat{X}_\infty$.*

At the time of writing this thesis, we do not know whether the complexification $\hat{X}_\infty$ of the real indecomposable space X_∞ is complex indecomposable or not. It is very likely however that Tarbard's original construction of the real Banach space X_∞ carries over to the complex numbers, and provides a complex indecomposable Banach space with the same properties that X_∞ possesses.
However, we would like to draw the reader's attention to the fact that Corollary 4.2.17 does not hold in general for indecomposable Banach spaces. This follows directly from the following deep result of Gowers and Maurey:

Theorem 4.2.18. *([29, Section (4.2) and Theorem 13]) There exists an indecomposable, prime Banach space.*

We recall that an infinite-dimensional Banach space X is *prime* if it is isomorphic to all its infinite-dimensional, complemented subspaces.

Lemma 4.2.19. *Let X be an indecomposable Banach space. Then $\mathcal{B}(X)$ cannot be properly infinite.*

Proof. Assume towards a contradiction that $\mathcal{B}(X)$ is properly infinite. Then there exist $P, Q \in \mathcal{B}(X)$ orthogonal idempotents such that $P, Q \sim I_X$. By Lemma 1.2.11 this is equivalent to $\mathrm{Ran}(P) \simeq X \simeq \mathrm{Ran}(Q)$. Clearly $X \simeq \mathrm{Ran}(P) \oplus \mathrm{Ran}(I_X - P)$ and since $\mathrm{Ran}(P)$ is infinite-dimensional and X is indecomposable we obtain that $\mathrm{Ran}(I_X - P)$ must be finite-dimensional. Consequently, the range of $Q = Q(I_X - P)$ is finite-dimensional, contradicting $\mathrm{Ran}(Q) \simeq X$. □

An infinite-dimensional Banach space X is *primary* if for every $P \in \mathcal{B}(X)$ idempotent either $\mathrm{Ker}(P)$ or $\mathrm{Ran}(P)$ is isomorphic to X. A prime Banach space is clearly primary.

Lemma 4.2.20. *Let X be a primary Banach space. Then $\mathcal{B}(X)$ is Dedekind-infinite.*

Proof. Let $P \in \mathcal{B}(X)$ be an idempotent with $\dim(\mathrm{Ker}(P)) = 1$. Since X is primary, $\mathrm{Ran}(P) \simeq X$ holds. By Lemma 1.2.11 this is equivalent to $P \sim I_X$. If $\mathcal{B}(X)$ were DF then $P = I_X$ which is impossible. □

Theorem 4.2.18 ensures that the following corollary of Lemmas 4.2.19 and 4.2.20 is not vacuous:

Corollary 4.2.21. *For an indecomposable, primary Banach space X the algebra of operators $\mathcal{B}(X)$ is Dedekind-infinite but not properly infinite.*

4.3 A real $C(K)$-space such that its algebra of operators is Dedekind finite but it does not have stable rank one

In the following, if K is a compact Hausdorff space, we can consider $C(K)$ as either a *real* or *complex* Banach space. When we want to emphasise that we take the underlying scalar field to be the real numbers, we write $C(K, \mathbb{R})$ for $C(K)$. Let K be a compact Hausdorff space, and let $g \in C(K)$. Then

$$M_g : C(K) \to C(K); \quad f \mapsto fg \tag{4.14}$$

is the *multiplication operator corresponding to* g. An operator $T \in \mathcal{B}(C(K))$ is called a *weak multiplication* if there is a $g \in C(K)$ and $S \in \mathcal{W}(C(K))$ such that $T = M_g + S$. We define the map

$$\mu : C(K) \to \mathcal{B}(C(K)); \quad g \mapsto M_g. \tag{4.15}$$

It is clear that μ is an isometric algebra homomorphism. In particular, $\mathrm{Ran}(\mu)$ is a closed unital subalgebra of $\mathcal{B}(C(K))$, isometrically isomorphic to $C(K)$.
We say that an infinite compact Hausdorff space is a *Koszmider space* if every bounded linear operator on $C(K)$ is a weak multiplication. In [43, Theorem 6.1] Koszmider showed that assuming the Continuum Hypothesis, there exist connected Koszmider spaces and there exist zero-dimensional Koszmider spaces. In [64, Theorem 1.3] Plebanek showed the existence of a connected Koszmider space without any assumptions beyond ZFC. We recall a result of Dales–Kania–Kochanek–Koszmider–Laustsen here. Since its proof is relatively simple but probably not well-known, we repeat it here. In the following $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$.

We first recall that by a standard result of Pełczyński, $\mathcal{W}(C(K)) = \mathcal{S}(C(K))$ holds for any compact Hausdorff space K; see [60, Theorem 1].

Theorem 4.3.1. *([16, Theorem 6.5(i)]) Let K be a Koszmider space without isolated points. Then* $\mathcal{B}(C(K)) = \mathrm{Ran}(\mu) + \mathcal{W}(C(K))$ *and* $\mathrm{Ran}(\mu) \cap \mathcal{W}(C(K)) = \{0\}$.

Proof. Since every bounded linear operator is a weak multiplication on a $C(K)$, it clearly suffices to show that $\mathrm{Ran}(\mu) \cap \mathcal{S}(C(K)) = \{0\}$.

Let $g \in C(K)$ be non-zero, and take $k_0 \in K$ such that $g(k_0) \neq 0$. Let $\epsilon \in (0, |g(k_0)|/2)$; since $g(k_0) \notin \overline{B}_\epsilon(0; \mathbb{K})$ we can take a neighbourhood Ω of $g(k_0) \in \mathbb{K}$ such that $\Omega \cap \overline{B}_\epsilon(0; \mathbb{K}) = \emptyset$. By continuity of g there is an open neighbourhood V of $k_0 \in K$ with $g[V] \subseteq \Omega$, thus clearly $g[V] \cap \overline{B}_\epsilon(0; \mathbb{K}) = \emptyset$. In other words, for all $k \in V$, $|g(k)| > \epsilon$. We *claim* that to following closed subspace of $C(K)$ is infinite dimensional:

$$F := \{f \in C(K) : (\forall k \in K \backslash V)(f(k) = 0)\} \tag{4.16}$$

To see this, let us fix $N \in \mathbb{N}$. Since K does not have isolated points, there exist distinct $x_1, x_2, \ldots, x_N \in V$. Let $(U_i)_{i=1}^N$ be a system of mutually disjoint open subsets in K such that for every $i \in \{1, \ldots, N\}$, $x_i \in U_i$. Clearly $(V \cap U_i)_{i=1}^N$ is a system of mutually disjoint open subsets in K such that for every $i \in \{1, \ldots, N\}$, $x_i \in V \cap U_i$. By Urysohn's Lemma there exists a system $(f_i)_{i=1}^N$ of real-valued continuous functions on K such that for every $i \in \{1, \ldots, N\}$, $f_i(x_i) = 1$ and $\overline{\text{supp}}(f_i) \subseteq V \cap U_i$. Thus it immediately follows that $\{f_i\}_{i=1}^N$ is linearly independent in $C(K)$. Moreover, for any $i \in \{1, \ldots, N\}$ and any $k \in K \backslash V$: $k \notin \overline{\text{supp}}(f_i)$, thus $f_i(k) = 0$. So $\{f_i\}_{i=1}^N$ is a subset of F and it is linearly independent. Since $N \in \mathbb{N}$ was arbitrary, F is infinite-dimensional.

For any $f \in F$ we have

$$\begin{aligned} \|M_g f\| &= \sup_{k \in K} |g(k)f(k)| = \sup_{k \in V} |g(k)||f(k)| \\ &\geq |g(k_0)| \left(\sup_{k \in V} |f(k)| \right) = |g(k_0)| \left(\sup_{k \in K} |f(k)| \right) \\ &\geq \epsilon \|f\|. \end{aligned} \tag{4.17}$$

So M_g is bounded below on F, thus $\mu(g) = M_g \notin \mathcal{S}(C(K))$, concluding the proof. $\square$

The following result is an immediate corollary of Theorems 2.3 and 5.2 in Koszmider's paper [43].

Theorem 4.3.2. *([43]) Let K be a connected Koszmider space. Then $C(K)$ is not isomorphic to any of its proper, closed subspaces.*

Lemma 4.3.3. *Let K be a compact connected Hausdorff space with at least two points. Then $C(K,\mathbb{R})$ does not have stable rank one.*

Proof. Let $x, y \in K$ be distinct, and take U, V disjoint open subsets of K such that $x \in U$ and $y \in V$. By Urysohn's Lemma there exist $f, g \in C(K,\mathbb{R})$ such that $\mathrm{supp}(f) \subseteq U$, $\mathrm{supp}(g) \subseteq V$ and $f(x) = 1$, $g(y) = 1$. Let $h := f - g$, clearly $h \in C(K,\mathbb{R})$ is such that $h(x) = 1$ and $h(y) = -1$. Suppose $k \in C(K,\mathbb{R})$ is such that $\|k - h\| < 1/2$, thus $|k(x) - 1| < 1/2$ and $|k(y) + 1| < 1/2$. In particular, $1/2 < k(x)$ and $-1/2 > k(y)$, thus by continuity of k and connectedness of K we obtain that there is $z \in K$ such that $k(z) = 0$. Thus $k \in C(K,\mathbb{R})$ cannot be invertible. This shows that $\mathrm{inv}(C(K,\mathbb{R}))$ is not dense in $C(K,\mathbb{R})$, as required. □

Remark 4.3.4. We note however that Lemma 4.3.3 is not true in general for *complex* $C(K)$ spaces, where K is a connected compact Hausdorff space. For example, the complex Banach algebra $C[0,1] := C([0,1],\mathbb{C})$ has stable rank one. Indeed, $[0,1]$ is well-known to have covering dimension 1, therefore by [66, Proposition 1.7] we have that $C[0,1]$ has stable rank one.

We recall that a connected T_1 space with at least two points does not have isolated points. Indeed, if K is T_1 with at least two points and $x \in K$ is isolated in K, then $\{x\}$ is a clopen set in K thus K cannot be connected.

Theorem 4.3.5. *Let K be a connected Koszmider space. Then $\mathcal{B}(C(K,\mathbb{R}))$ is Dedekind-finite but it does not have stable rank one.*

Proof. If $\mathcal{B}(C(K,\mathbb{R}))$ had stable rank one, then so would $\mathcal{B}(C(K,\mathbb{R}))/\mathcal{W}(C(K,\mathbb{R}))$. By Theorem 4.3.1

$$\mathcal{B}(C(K,\mathbb{R}))/\mathcal{W}(C(K,\mathbb{R})) \simeq \mathrm{Ran}(\mu) \simeq C(K,\mathbb{R}), \tag{4.18}$$

thus $C(K,\mathbb{R})$ has stable rank one; this contradicts Lemma 4.3.3. The fact that $\mathcal{B}(C(K,\mathbb{R}))$ is DF follows from Lemma 4.1.2 and Theorem 4.3.2. □

Remark 4.3.6. We remark in passing that Tarbard's space X_∞ and $C(K,\mathbb{R})$ (where K is a connected Koszmider space) are both indecomposable, but not hereditarily indecomposable Banach spaces. Indeed, X_∞ is indecomposable by [78, Proposition 4.1.5] but not hereditarily indecomposable by [78, Proposition 4.1.4]. If K is a connected Koszmider space then by [43, Theorem 2.5] it follows that $C(K)$ is indecomposable. On the other hand, it is well-known that for any infinite compact Hausdorff space K, $C(K)$ cannot be hereditarily indecomposable. This follows from the fact that if K is such then $C(K)$ has a closed subspace (isometrically) isomorphic to c_0, see for example, [67, Lemma 2.5(d)].

4.4 On the existence of a certain maximal ideal in Banach algebras

If A is a unital algebra and $a \in A$ is fixed, we say that 1_A *factors through* a, if there exist $b, c \in A$ such that $1_A = bac$. In a unital algebra A let $\mathrm{linv}(A)$ and $\mathrm{rinv}(A)$ denote the set of left- and right invertible elements, respectively.

Notation 4.4.1. Let A be a unital algebra. We define the set

$$M_A := \{a \in A : 1_A \text{ does not factor through } a\}. \tag{4.19}$$

Remark 4.4.2. Clearly, the zero element of A is always contained in M_A. We immediately see that the identity is never contained in M_A therefore it is a proper subset of A. Moreover, $M_A \subseteq A \setminus (\operatorname{rinv}(A) \cup \operatorname{linv}(A)) \subseteq A \setminus \operatorname{inv}(A)$. Indeed, if $a \in \operatorname{linv}(A)$ then there exists $b \in A$ such that $1_A = ba$, thus the identity $1_A = ba1_A$ shows that $a \notin M_A$. The containment $M_A \subseteq A \setminus \operatorname{rinv}(A)$ follows similarly.

Lemma 4.4.3. *Let A be a unital algebra over a field K. Then M_A is a proper subset of A such that $A \cdot M_A \cdot A \subseteq M_A$.*

Proof. Assume towards a contradiction that there exist $a \in M_A$ and $r, q \in A$ are such that $qar \notin M_A$. Then there exist $b, c \in A$ such that $1_A = b(qar)c = (bq)a(rc)$. In particular $a \notin M_A$ follows, a contradiction. □

Lemma 4.4.4. *Let A be a unital algebra. Then every proper two-sided ideal of A is contained in M_A.*

Proof. Let $I \trianglelefteq A$ be a two-sided ideal in A. Let $i \in I$ be such that $i \notin M_A$. Then there exist $b, c \in A$ such that $1_A = bic$. Thus $1_A \in I$ and consequently $I = A$ holds. □

From the previous lemmas we immediately obtain the following:

Proposition 4.4.5. *For a unital algebra A, the set M_A is the largest proper (and therefore unique maximal) two-sided ideal in A if and only if M_A is closed under addition.*

In the following we give a necessary and sufficient condition for the set M_A to be the largest proper two-sided ideal in A. Implication (*2* $\Rightarrow$ *1*) of the following lemma is essentially the proof of [19, Proposition 5.1].

Lemma 4.4.6. *Let A be a unital algebra. Then the following are equivalent:*

1. *M_A is closed under addition;*

2. for every $a \in A$ either $a \notin M_A$ or $1_A - a \notin M_A$.

Proof. We prove (*2* ⇒ *1*) by showing the contrapositive. Suppose that M_A is not closed under addition. That is, there exist $a, b \in M_A$ such that $a + b \notin M_A$. Consequently there exist $c, d \in A$ such that $1_A = c(a + b)d$. Let us define $p := (a + b)dc$. By Lemma 4.4.3 it follows that $cpad, cpbd \in M_A$. Also,

$$cpad + cpbd = cp(a + b)d = c(a + b)dc(a + b)d = 1_A 1_A = 1_A. \tag{4.20}$$

Since both $cpad \in M_A$ and $1_A - cpad = cpbd \in M_A$ hold, we have proved ¬*2*.

(*1* ⇒ *2*). Assume towards a contradiction that *1* holds but *2* does not. So M_A is closed under addition and there exists $a \in A$ such that $a \in M_A$ and $1_A - a \in M_A$. Then $1_A = (1_A - a) + a \in M_A$ which is a contradiction. □

Lemma 4.4.7. *Let X be a Banach space and let $P \in \mathcal{B}(X)$ be an idempotent. Then* Ran(P) *is finite-dimensional if and only if $P \in \mathcal{E}(X)$.*

Proof. If Ran(P) is finite-dimensional then $P \in \mathcal{F}(X) \subseteq \mathcal{E}(X)$. For the other direction suppose $P \in \mathcal{E}(X)$. Then for any $A \in \mathcal{B}(X)$ the operator $I_X + AP$ is Fredholm. Consequently, $I_X - P$ is a Fredholm operator so in particular Ker$(I_X - P)$ is finite-dimensional. This is of course equivalent to Ran(P) being finite-dimensional. □

The next proposition is certainly known, see the text following [20, Theorem 1.3].

Proposition 4.4.8. *Let X be an infinite-dimensional Banach space with the following property:*

(P) For every $T \in M_{\mathcal{B}(X)}$ and every complemented subspace Y of X, such that $Y \simeq X$, there exists a complemented subspace Z of X with $Z \subseteq Y$ such that $Z \simeq X$ and $T|_Z \in \mathcal{E}(Z, X)$.

Then $M_{\mathcal{B}(X)}$ is the largest proper closed two-sided ideal in $\mathcal{B}(X)$.

Proof. Suppose X has property *(P)*. Let $T \in \mathcal{B}(X)$ be arbitrary, in view of Lemma 4.4.6 we need to show that $T \notin M_{\mathcal{B}(X)}$ or $I_X - T \notin M_{\mathcal{B}(X)}$. Assume towards a contradiction that both $T \in M_{\mathcal{B}(X)}$ and $I_X - T \in M_{\mathcal{B}(X)}$ hold. Then applying *(P)* to the operator $T \in M_{\mathcal{B}(X)}$ and closed subspace $Y := X$, it follows that there exists an idempotent $Q' \in \mathcal{B}(X)$ such that $\operatorname{Ran}(Q') \simeq X$ and $TQ' \in \mathcal{E}(X)$. Now applying *(P)* to the operator $I_X - T \in M_{\mathcal{B}(X)}$ and closed subspace $Y := \operatorname{Ran}(Q')$, there exists an idempotent $Q \in \mathcal{B}(X)$ such that $\operatorname{Ran}(Q) \subseteq \operatorname{Ran}(Q')$ and $\operatorname{Ran}(Q) \simeq X$ and $(I_X - T)Q \in \mathcal{E}(X)$. In particular $Q'Q = Q$ and thus $TQ = TQ'Q \in \mathcal{E}(X)$. Consequently $Q = (I_X - T)Q + TQ \in \mathcal{E}(X)$, which in view of Lemma 4.4.7 is equivalent to Q having finite-dimensional range. This contradicts $\operatorname{Ran}(Q) \simeq X$. □

Lemma 4.4.9. *Let A be a unital Dedekind-finite algebra. Then $M_A = A \setminus \operatorname{inv}(A)$.*

Proof. We have already seen that $M_A \subseteq A \setminus \operatorname{inv}(A)$ holds for any unital algebra A. Now suppose A is DF. Let $a \notin M_A$, there exist $b, c \in A$ such that $1_A = bac$. Now let $p := cba$ and $q := acb$. It is immediate from the definitions that $p, q \in A$ are idempotents such that $p, q \sim 1_A$. Since A is DF, it follows that $p = 1_A = q$ equivalently $cba = 1_A = acb$. Thus a is invertible with inverse $cb \in A$. This proves the containment $A \setminus \operatorname{inv}(A) \subseteq M_A$, as required. □

Lemma 4.4.10. *Let A be a unital Banach algebra with stable rank one. Then $M_A = \partial(\operatorname{inv}(A))$.*

Proof. It follows from Lemma 4.2.1 that A is DF, thus by Lemma 4.4.9 we have $M_A = A \setminus \operatorname{inv}(A)$. Since A has stable rank one, we obtain $M_A = \overline{\operatorname{inv}(A)} \setminus \operatorname{inv}(A) = \partial(\operatorname{inv}(A))$, as claimed. □

A unital algebra A is called *local* if A has a unique maximal left ideal. The following theorem is standard, we refer the reader to [44, Theorem 19.1]:

Theorem 4.4.11. *For a unital algebra A the following are equivalent:*

1. *A is local;*

2. *A has a unique maximal right ideal;*

3. *$A/\operatorname{rad}(A)$ is a division algebra;*

4. *$A\setminus\operatorname{inv}(A)$ is a two-sided ideal in A.*

Corollary 4.4.12. *A local semisimple algebra is a division algebra, thus simple.*

Proof. Let A be local and semisimple. Then $A/\operatorname{rad}(A)$ is a division algebra, thus by semisimplicity, A is a division algebra, thus A is simple. □

Remark 4.4.13. Note that for a non-zero Banach space X it is immediate that $\mathcal{B}(X)$ is local if and only if $\dim(X) = 1$. Indeed, this follows from the previous corollary and the fact that $M_n(\mathbb{C})$ is not a division algebra, it is simple and thus semisimple, whenever $n \geq 2$.

The following corollary is implicitly contained in the proof of Theorem 4.4.11, but since we will use it later we add the proof here for the sake of completeness.

Corollary 4.4.14. *Let A be a local algebra. Then* $\operatorname{rad}(A) = A\setminus\operatorname{inv}(A)$.

Proof. Let M be the unique maximal left ideal in A, then $M = \operatorname{rad}(A)$. Clearly $M \subseteq A\setminus\operatorname{inv}(A)$. But $A\setminus\operatorname{inv}(A)$ is a two-sided ideal by Theorem 4.4.11, consequently it is a left ideal, thus $A\setminus\operatorname{inv}(A) \subseteq M$ must hold. So $\operatorname{rad}(A) = M = A\setminus\operatorname{inv}(A)$, as required. □

Lemma 4.4.15. *Let A be a local algebra. Then the only idempotents in A are 1_A and 0_A.*

Proof. Assume towards a contradiction that $p \in A\setminus\{1_A, 0_A\}$ is an idempotent. Then p and $q := 1_A - p$ are orthogonal idempotents in A. We observe that Ap and Aq are proper left ideals in A. Since A is local, both Ap and Aq must be contained in the unique maximal left ideal, say M, of A. But clearly $1_A = p+(1_A-p) \in Ap+Aq \subseteq M+M \subseteq M$, which is impossible, since M is a proper ideal. $\square$

Corollary 4.4.16. *Every local algebra is Dedekind-finite.*

Proof. Let A be a local algebra and suppose $p \in A$ is an idempotent such that $p \sim 1_A$. By Lemma 4.4.15, either $p = 1_A$ or $p = 0_A$. The latter is clearly impossible. $\square$

We show now that under the assumption that M_A is closed under addition the converse is also true.

Proposition 4.4.17. *Let A be a unital algebra such that M_A is closed under addition. Then the following are equivalent:*

1. *A is local,*

2. *A is Dedekind-finite.*

Proof. Suppose M_A is closed under addition and A is Dedekind-finite. Then by Lemma 4.4.9 it follows that $A \backslash \operatorname{inv}(A)$ is an ideal, therefore by Theorem 4.4.11, A is local. $\square$

Theorem 4.4.18. *Let A be a complex unital semisimple Banach algebra. If A is Dedekind-finite and M_A is closed under addition then $A \simeq \mathbb{C}$.*

Proof. By the Proposition 4.4.17, A is local. Equivalently, by Theorem 4.4.11, $A/\operatorname{rad}(A)$ is a division algebra. Thus by the Gel'fand–Mazur Theorem, $A/\operatorname{rad}(A) \simeq \mathbb{C}$. Since A is semisimple, this amounts to $A \simeq \mathbb{C}$, as required. $\square$

Remark 4.4.19. Let us recall (see [63, Theorem 5.3.2]) that for $X := \ell_q \oplus \ell_p$, where $1 \leq p < q < \infty$, the set $M_{\mathcal{B}(X)}$ is not closed under addition. More precisely, $\mathcal{B}(X)$ has exactly two maximal two-sided ideals, $\overline{\mathscr{G}}_{\ell_p}(X)$ and $\overline{\mathscr{G}}_{\ell_q}(X)$ which are the sets of operators which approximately factor through ℓ_p and ℓ_q, respectively.

In light of the previous remark the following question naturally arises: If X is an infinite-dimensional Banach space such that $M_{\mathcal{B}(X)}$ is not closed under addition, does it necessarily follow that $\mathcal{B}(X)$ contains at least two distinct maximal two-sided ideals? In the following we observe that the answer to this question is negative.

Corollary 4.4.20. *Let X be an infinite-dimensional Banach space such that $\mathcal{B}(X)$ is Dedekind-finite. Then $M_{\mathcal{B}(X)}$ is not closed under addition.*

Proof. Since $\mathcal{B}(X)$ is semisimple, by Theorem 4.4.18 it follows that $M_{\mathcal{B}(X)}$ cannot be closed under addition. □

Lemma 4.4.21. *If X is an infinite-dimensional Banach space, then every maximal two-sided ideal of $\mathcal{B}(X)$ contains $\mathcal{E}(X)$. In particular, if $\mathcal{E}(X)$ is a maximal two-sided ideal in $\mathcal{B}(X)$ then it is the unique maximal two-sided ideal.*

Proof. Let $M \trianglelefteq \mathcal{B}(X)$ be a maximal two-sided ideal. Then $M/\mathcal{A}(X)$ is a maximal two-sided ideal in $\mathcal{B}(X)/\mathcal{A}(X)$. We recall that in a unital algebra the Jacobson radical is contained in any maximal two-sided ideal (see [44, Exercise 4.8]), so in particular $\mathrm{rad}(\mathcal{B}(X)/\mathcal{A}(X)) \subseteq M/\mathcal{A}(X)$. By Kleinecke's Thereom 3.3.5 this is equivalent to $\mathcal{E}(X) \subseteq M$. The second part of the claim follows trivially. □

Proposition 4.4.22. *Let X be a hereditarily indecomposable Banach space. Then $M_{\mathcal{B}(X)}$ is not closed under addition and $\mathcal{S}(X)$ is the unique maximal ideal in $\mathcal{B}(X)$.*

Proof. We recall that by [28, Theorem 18] it follows that $\mathcal{S}(X)$ is a maximal two-sided ideal in $\mathcal{B}(X)$. Since X is infinite-dimensional, $\mathcal{E}(X)$ must be a proper closed two-sided ideal, thus from $\mathcal{S}(X) \subseteq \mathcal{E}(X)$ in fact $\mathcal{S}(X) = \mathcal{E}(X)$ follows. Thus by Lemma 4.4.21, $\mathcal{S}(X)$ is the unique maximal two-sided ideal in $\mathcal{B}(X)$. Since X is HI, it follows from Lemma 4.1.2 that $\mathcal{B}(X)$ is DF, hence by Corollary 4.4.20 the set $M_{\mathcal{B}(X)}$ cannot be closed under addition. □

Chapter 5

List of Symbols and Index

T^*	the adjoint of $T \in \mathcal{B}(X)$, where X is a Banach space, *7*
$\aleph_0$	the cardinality of the natural numbers, *1*
α^+	the ordinal successor of the ordinal α, *96*
$\mathcal{A}$	the closed operator ideal of approximable operators, *10*
$\mathcal{B}(X, Y)$	the Banach space of bounded linear operators between Banach spaces X and Y, *7*
$\mathcal{B}(X)$	the Banach algebra $\mathcal{B}(X, X)$, *7*
$\mathcal{B}_1(A, B)$	$= \{S \in \mathcal{B}(A, B) : S(1_A) = 1_B\}$, where A, B are unital Banach algebras, *26*
$B_r(y; X)$	the open ball of radius r around the point y in a normed space X, *5*
$\overline{B}_r(y; X)$	the closed ball of radius r around the point y in a normed space X, *5*
$\mathrm{Bil}(X, X; Y)$	the Banach space of bounded bilinear maps from $X \times X$ to Y, *63*
$\mathbb{C}$	the field of complex numbers, *1*
c_0	the Banach space of complex sequences on $\mathbb{N}$ with 0 limit, *13*
$c_0(\lambda)$	the Banach space of complex transfinite sequences on an infinite cardinal λ with 0 limit, *87*
$C(K; X)$	the Banach space of X valued continuous functions on a compact space K, *100*

$C(K)$	$= C(K;\mathbb{C})$, *99*
$C_0(L)$	the Banach space of continuous functions vanishing at infinity on a locally compact space L, *99*
$\mathrm{Cal}(\mathcal{H})$	the Calkin algebra of a separable Hilbert space $\mathcal{H}$, *72*
$\lvert S\rvert$	the cardinality of a set S, *1*
$\overline{S}$	the closure of a subset S of a topological space, *2*
$\delta(T)$	the coboundary operator evaluated against $T \in \mathrm{Bil}(A, A; X)$, *64*
$\hat{X}$	the complexification of a real Banach space [algebra] X, *121*
$2^{\aleph_0}$	continuum, *1*
$\mathbb{D}$	the open unit disc in the complex plane, *120*
$\mathrm{def}(A)$	the multiplicative defect of a Banach algebra A, *19*
$\partial(S)$	the topological boundary of a subset S of a topological space, *115*
δ_α	the Dirac-measure centred at α, *102*
$\mathrm{dens}(X)$	the density character of a Banach space X, *73*
$\mathrm{dist}(T)$	$= \inf\{\lVert T - S\rVert : S \in \mathrm{Mult}(A, B)\}$, *19*
$\text{d-}\lim\limits_{i\to\mathcal{U}} x_i$	$= \varphi^{-1}(\text{w*-}\lim\limits_{i\to\mathcal{U}} \varphi(x_i))$, where $(x_i)_{i\in I}$ is a net in a dual Banach algebra with predual (B_*, φ) and $\mathcal{U}$ is an ultrafilter on I, *23*
X^*	the dual of a normed space X, *6*
X^{**}	the bidual of a normed space X, *6*
$\langle x, f\rangle$	the duality pairing, *6*
$\mathcal{E}$	the closed operator ideal of inessential operators, *10*
$\exp(A)$	$= \{\exp(a) : a \in A\}$, where A is complex unital Banach algebra, *119*
$\mathcal{F}$	the operator ideal of finite-rank operators, *10*
$\overline{\mathscr{G}}_Y(X)$	the closed, two-sided ideal of operators on X factoring through Y approximately, *88*
$\mathcal{H}^2(A, X)$	the second bounded Hochschild cohomology group of A with coefficients in X, *65*

id_A	the identity map on an algebra A, *7*
$\mathbf{1}_S$	the indicator function of a set S, *89*
$\mathrm{int}(S)$	the interior of a subset S of a topological space, *117*
$[\alpha, \beta)$	$= \{\gamma : \alpha \leq \gamma < \beta\}$, where α, β are ordinals, *97*
$[\alpha, \beta]$	$= \{\gamma : \alpha \leq \gamma \leq \beta\}$, where α, β are ordinals, *97*
$\mathrm{inv}(A)$	the set of invertible elements in an algebra A, *8*
$\iota_\Delta(a)$	$= \Delta \otimes a$, where $a \in A$ and $\Delta \in A \hat{\otimes}_\pi A$, *21*
$X \simeq Y$	isomorphism of the Banach spaces [algebras] X and Y, *8*
$\mathcal{J}_p$	the p^{th} James space, *77*
κ_X	the canonical embedding from X to X^{**}, *6*
$\mathcal{K}$	the closed operator ideal of compact operators, *10*
K_b	the basis constant corresponding to a Schauder basis, *12*
K_u	the unconditional basis constant corresponding to an unconditional basis, *13*
K_{sub}	the subsymmetric basis constant corresponding to a subsymmetric basis, *14*
$\mathrm{Ker}(T)$	the kernel of the bounded linear operator T, *6*
$\ell_1(\mathbb{N}_0)$	the Banach algebra of complex polynomials completed in the $\|\cdot\|_1$-norm, *77*
ℓ_p	the Banach space of p-summable sequences on $\mathbb{N}$ for $p \in [1, \infty)$, the Banach space of bounded functions on $\mathbb{N}$ for $p = \infty$, *13*
$\ell_p(\lambda)$	the Banach space of p-summable transfinite sequences on an infinite cardinal λ for $p \in [1, \infty)$, *100*
$\ell_\infty^c(\lambda)$	the Banach space of countably supported, bounded functions on an infinite cardinal λ, *87*
$L_p(X, \mu)$	the Banach space of p-integrable functions on a measure space (X, μ) for $p \in [0, \infty)$, and the Banach space of essentially bounded functions on (X, μ) for $p = \infty$, *89*

$L_p[0,1]$	$= L_p([0,1],\lambda)$, where λ is the Lebesgue measure on $[0,1]$, *13*
$\lambda_a^{\mathrm{mod}}(x)$	$= ax$, where $a \in A$ and $x \in X$, *21*
$\lim\limits_{i\to\mathcal{U}} x_i$	the limit of $(x_i)_{i\in I}$ along some ultrafilter $\mathcal{U}$ on I, *3*
$\mathrm{linv}(A)$	the set of left-invertible elements in a unital algebra A, *128*
M_A	the set of elements a in a unital algebra A through which the identity 1_A does not factor, *128*
$M_n(\mathbb{C})$	the algebra of (2×2) complex valued matrices, *71*
$\mathcal{M}_{LW}$	the Loy–Willis ideal, *103*
$\mathrm{Mult}(A,B)$	the closed set of multiplicative bounded linear maps between Banach algebras A and B, *19*
$\mathcal{N}^2(A,X)$	the linear space of 2-coboundaries of A with coefficients in X, *64*
$\mathbb{N}$	the set of natural numbers, excluding 0, *1*
$\mathbb{N}_0$	the set of natural numbers and 0, *1*
ω	the first non-zero limit ordinal, *96*
ω_1	the first uncountable ordinal, *96*
$p \sim q$	the idempotents p,q are equivalent, *4*
$p \perp q$	the idempotents p,q are orthogonal, *73*
$p \leq q$	$pq = p = qp$, where p,q are idempotents, *74*
$\pi_A(a\otimes b)$	$= ab$, where $a,b \in A$, *21*
$\mathcal{P}(S)$	the power set of a set S, *1*
$\mathcal{P}_\infty(S)$	the set of infinite subsets of a set S, *2*
$\mathrm{rad}(A)$	the Jacobson radical of a unital algebra A, *75*
$\mathrm{Ran}(f)$	the range of a function f, *7*
$\mathbb{R}$	the field of real numbers, *1*
$\rho_a^{\mathrm{mod}}(x)$	$= xa$, where $a \in A$ and $x \in X$, *21*
$T\vert_W$	the restriction of T to W, *7*
$T\vert^H$	the restriction of T onto H, where $\mathrm{Ran}(T) \subseteq H$, *7*

$\mathrm{rinv}(A)$	the set of right-invertible elements in a unital algebra A, *128*
$\mathcal{S}$	the closed operator ideal of strictly singular operators, *10*
$\mathbf{S}$	Schlumprecht's arbitrarily distortable Banach space, *72*
$\mathcal{S}_Y(X)$	the set of Y-singular operators on a Banach space X, *83*
$A \ltimes X$	the semi-direct product of A with A, *63*
$\sigma_\Delta(a)$	$= a \otimes \Delta$, where $a \in A$ and $\Delta \in A\hat{\otimes}_\pi A$, *21*
$\sigma(X, X^*)$	the weak topology on a Banach space X, *6*
$\sigma(X^*, X)$	the weak* topology on a Banach space X, *6*
$\bigoplus_{i=1}^{n} X_i$	the direct sum of Banach spaces $X_1, \ldots, X_n$, *8*
$\left(\bigoplus_{n\in\mathbb{N}} X_n\right)_Y$	the Y-sum of Banach spaces X_n, where $Y = \ell_1$ or $Y = c_0$, *73*
$\mathbb{T}$	the unit circle in the complex plane, *119*
$\mathcal{T}$	the Toeplitz algebra, *119*
$\mathrm{supp}(f)$	the support of a function f, *1*
$X\hat{\otimes}_\epsilon Y$	the injective tensor product of Banach spaces X and Y, *18*
$T \otimes_\epsilon S$	the injective tensor product operator of operators T and S, *18*
$X\hat{\otimes}_\pi Y$	the projective tensor product of Banach spaces X and Y, *17*
$T \otimes_\pi S$	the projective tensor product operator of operators T and S, *17*
$A^\sharp$	the (forced) unitisation of a Banach algebra A, *9*
$\mathcal{V}$	the Volterra algebra, *69*
$T^\vee(a, b)$	$= T(ab) - T(a)T(b)$, where $T \in \mathcal{B}(A, B)$ and $a, b \in A$, *26*
$\mathcal{W}$	the closed operator ideal of weakly compact operators, *10*
$\mathrm{w}^*\text{-}\lim_{i\to\mathcal{U}} f_i$	the weak* limit of the net $(f_i)_{i\in I}$ where $\mathcal{U}$ is an ultrafilter on I, *22*
$\mathcal{X}$	the closed operator ideal of operators with separable range, *10*
Y^X	the set of functions from a set X to a set Y, *1*
Y_X	$= \{F \in C([0, \omega_1]; X) :\ F(\omega_1) = 0\}$, *100*
$\mathcal{Z}^2(A, X)$	the Banach space of 2-cocycles of A with coefficients in X, *64*

Index

Y-singular, 83
δ-multiplicative, 60
ℓ_1-sum, 73
c_0-sum, 73
2-coboundary, 64
2-cocycle, 64

almost disjoint family, 2
almost orthogonal, idempotent, 15
Alspach–Benyamini character, 103
amenable Banach algebra, 25
AMNM property, 20
AMNM-bootstrap property, 39
approximable operator, 10
approximation property, 11

Banach left A-module, 20
basic sequence, 11
basis, 11
bounded appoximate identity, 25
bounded approximate diagonal, 25
bounded approximation property, 11
bounded below, 7

character, 8
club subset, 98
compact operator, 10
complementably homogenous, 83
complementably minimal, 83
complemented, 7
complexification, 121
continuum, 1
contractive approximate identity, 25
coordinate functional, 11
coordinate projection, 11

Dedekind-finite, 113
Dedekind-infinite, 113
density character, 73
DF, 113
direct sum, finite, 8
dual Banach algebra, 21

Eidelheit's Theorem, 71
equivalent, basic sequence, 13

equivalent, idempotent, 4

factors through, 128

filter, 2

finite-rank operator, 10

fixed ultrafilter, 3

Fréchet filter, 3

Fredholm operator, 10

free ultrafilter, 3

Fundamental Isomorphism Theorem, 6

hereditarily indecomposable, 9

HI, 9

idempotent, 4

indecomposable, 8

inessential operator, 10

injective tensor product, 18

isomorphic to its square, 8

Jacobson radical, 75

Kleinecke's Theorem, 81

Koszmider space, 125

limit ordinal, 96

local, 131

Loy–Willis ideal, 103

metric approximation property, 11

minimal idempotent, 74

monotone basis, 12

multiplication operator, 125

multiplicative defect, 19

mutually orthogonal, idempotent, 73

net, 2

normalised basis, 11

operator ideal, 9

order filter, 4

order topology, 97

ordinal, 96

pre-AMNM property, 41

primary, 124

prime, 123

projection, 74

projective tensor product, 17

properly infinite, 113

Schauder basis, 11

second bounded Hochschild cohomology group, 65

Semadeni space, 99

semi-direct product, 63

semisimple, 75

SHAI property, 72

shrinking basis, 14

simple, 75

stable rank one, 114
strictly singular operator, 10
subsymmetric basis, 13
successor, 96
system, 2

Toeplitz-algebra, 119
topological zero divisor, 115
transitive set, 96

ultrafilter, 3
ultralimit, 3
unconditional basis, 12
unitisation, 9

weak multiplication, 125
weak topology, 6
weak* topology, 6
weakly compact operator, 10

References

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