Singular scaling limits in a planar random growth model

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(joint work with Alan Sola, Fredrik Viklund)

Planar random growth processes occur widely in the physical world. Examples include diffusion-limited aggregation (DLA) for mineral deposition and the Eden model for biological cell growth. One of the curious features of these models is that although the models are constructed in an isotropic way, scaling limits appear to be anisotropic. In this talk, we construct a family of models in which randomly growing clusters can be represented as compositions of conformal mappings. We are able to show rigorously that for certain parameter choices, the scaling limits are anisotropic and we obtain shape theorems in this case. This contrasts with earlier work on related growth models in which the scaling limits are shown to be growing disks [5, 2].

Clusters of particles can be represented using compositions of conformal mappings as follows. Let $\mathbf{c} > 0$, and let $f_{\mathbf{c}}$ denote the unique conformal map

$$f_{\mathbf{c}} \colon \Delta = \{z \in \mathbb{C} \colon |z| > 1\} \to D_1 = \Delta \setminus (1, 1 + d]$$

having $f_{\mathbf{c}}(z) = e^{\mathbf{c}}z + \mathcal{O}(1)$ at infinity, and sending the exterior disk Δ to the complement of the closed unit disk with a slit of length $d = d(\mathbf{c})$ attached at the point 1. The capacity increment $\mathbf{c}$ and the length d of the slit satisfy

$$e^{\mathbf{c}} = 1 + \frac{d^2}{4(1+d)}; \tag{1}$$

in particular, $d \asymp \mathbf{c}^{1/2}$ as $\mathbf{c} \to 0$. In terms of aggregation, the closed unit disk can be viewed as a seed, while the slit represents an attached particle. Typically, we think of the particle as being small compared to the seed.

A general two-parameter framework to model random or deterministic aggregation, based on conformal maps, is given by the following construction. Pick a sequence $\{\theta_k\}_{n=1}^{\infty}$ in $[-\pi, \pi)$, and let $\{d_k\}_{k=1}^{\infty}$, or, equivalently, $\{c_k\}_{k=1}^{\infty}$, be a sequence of non-negative numbers connected via (1). From the two numerical sequences $\{\theta_k\}$ and $\{c_k\}$, we obtain a sequence $\{f_k\}_{k=1}^{\infty}$ of rotated and rescaled conformal maps, referred to as building blocks, via

$$f_k(z) = e^{i\theta_k} f_{c_k}(e^{-i\theta_k} z).$$

Finally, we set

$$\Phi_n(z) = f_1 \circ \cdots \circ f_n(z), \quad n = 1, 2, \ldots. \tag{2}$$

Each Φ_n is itself a conformal map sending the exterior disk onto the complement of a compact set $K_n \subset \mathbb{C}$, that is,

$$\Phi_n \colon \Delta \to \mathbb{C} \setminus K_n.$$

The sets $\{K_n\}_{n=1}^{\infty}$ are called clusters. They satisfy $K_{n-1} \subset K_n$, and model a growing two-dimensional aggregate formed of n particles.

We introduce the Aggregate Loewner evolution model, abbreviated ALE(α, η), with parameters $\alpha \in \mathbb{R}$ and $\eta \in \mathbb{R}$. In ALE(α, η), conformal maps Φ_n are defined as in (2) as follows. Let θ_1 be uniform in $[0, 2\pi)$, and, for $k = 2, 3, \ldots$, let

$$\theta_k \propto \frac{|\Phi'_{k-1}(e^{\boldsymbol{\sigma}+i\theta})|^{-\eta}\mathrm{d}\theta}{\int_{\mathbb{T}} |\Phi'_{k-1}(e^{\boldsymbol{\sigma}+it})|^{\eta}\mathrm{d}t}.$$

Here, $\boldsymbol{\sigma} > 0$ is a regularization parameter, which ensures that the angle distributions are well defined even though Φ'_{k-1} has zeros and singularities on $\mathbb{T}$. The parameter $\boldsymbol{\sigma}$ is allowed to depend on the basic capacity parameter $\mathbf{c}$.

Next, we define a sequence of capacity increments for $k = 2, 3, \ldots$ by taking

$$c_k = \frac{\mathbf{c}}{|\Phi'_{k-1}(e^{\tilde{\boldsymbol{\sigma}}+i\theta_k})|^{\alpha/2}}$$

where $\tilde{\boldsymbol{\sigma}}$ is another regularization parameter. The ALE$(\alpha, 0)$ model (with $\tilde{\boldsymbol{\sigma}} = 0$) is the same model as the Hastings-Levitov HL(α) model [1], and in particular ALE$(0, 0)$ coincides with the HL(0) model studied in depth in [5]. When $\alpha = 2$, the model coincides with the dielectric breakdown model (DBM) of Mathiesen and Jensen [4]. When $\alpha = 0$, the growth process is reminiscent of the Quantum Loewner Evolution (QLE) of Miller and Sheffield [3] but without quantum gravity, that is, with $\gamma = 0$, and with SLE curves replaced by straight slits.

Clusters that are formed by successively composing slit maps come with a natural notion of ancestry for their constituent particles. We say that a particle j has parent 0 if it attaches directly to the unit disk. We say that the particle j has parent k if the j^{th} particle is directly attached to the k^{th} particle. In the ALE$(0, \eta)$ model, each successive particle chooses its attachment point on the cluster according to the relative density of harmonic measure (as seen from infinity) raised to the power η. As the highest concentration of harmonic measure occurs at the tips of slits, intuitively one would expect that for sufficiently large values of η each particle is likely to attach near the tip of the previous particle. We show that this indeed happens, and we identify the values of η for which the above event occurs with high probability in the limit as $\mathbf{c} \to 0$. Figure 1 displays ALE$(0, \eta)$ clusters for different values of η.

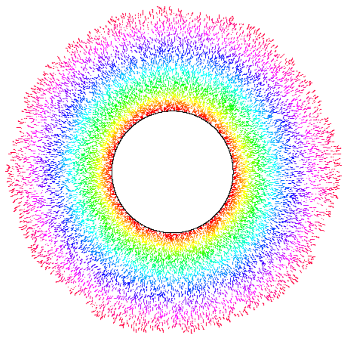
Define the event

$$\Omega_N = \{\text{Particle } j \text{ has parent } j-1 \text{ for all } j = 1, \ldots, N\}$$

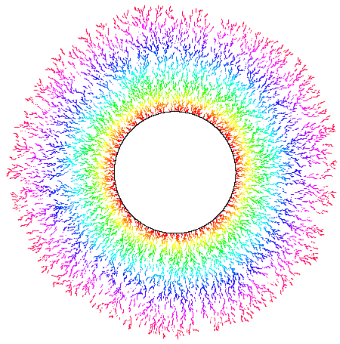
in which there is a simple ancestral line whereby each particle is attached to the previous particle. Set $n(t) = \lfloor t\mathbf{c}^{-1} \rfloor$. Our main result states that if $N = n(T)$ for some fixed $T > 0$, then

$$\lim_{\mathbf{c}\to 0} \mathbb{P}(\Omega_N) = 1 \quad \text{if } \eta > 1$$
$$\limsup_{\mathbf{c}\to 0} \mathbb{P}(\Omega_N) < 1 \quad \text{if } \eta = 1,$$

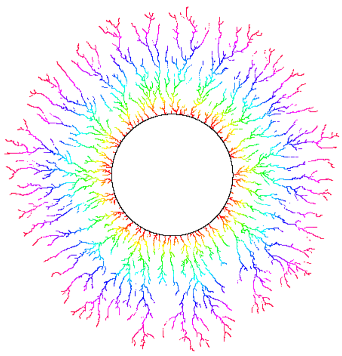
provided $\boldsymbol{\sigma}$ is small enough. When $\eta > 1$, we characterise how fast one must let $\boldsymbol{\sigma} \to 0$ as $\mathbf{c} \to 0$ in order for $\mathbb{P}(\Omega_N) \to 1$. We show that when this happens, the cluster $K_{n(t)}$ converges in the Hausdorff topology to a slit of capacity t at position $e^{i\theta_1}$.



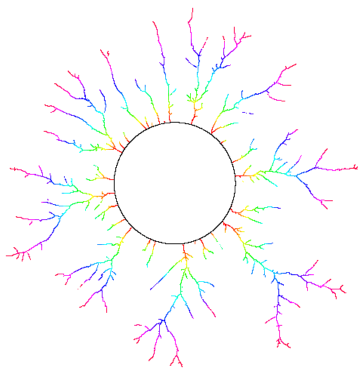
(a) ALE(-1.0)



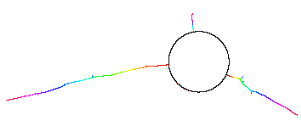
(b) ALE(0.0)



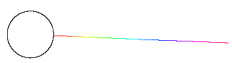
(c) ALE(1.0)



(d) ALE(1.5)



(e) ALE(2.0)



(f) ALE(4.0)

FIGURE 1. ALE *clusters with* $\mathbf{c} = 10^{-4}$, $\boldsymbol{\sigma} = \mathbf{c}^2$, *and* $n = 10,000$.

References

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