# The Choice-theoretic Characterizations of Risk Changes and Risk Attitudes in Cumulative Prospect Theory: A Stochastic Dominance Approach


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# Declaration

I declare that no portion of the work referred to in the thesis has been submitted in support of an application for another degree or qualifications at this or any other university.

# Abstract

Stochastic dominance provides an effective tool to characterize individuals' risk attitudes in decision making under risk by comparing risky prospects. The emergence of the cumulative prospect theory (CPT), developed by Kahneman and Tversky (1979) and Tversky and Kahneman (1992), provides a prominent alternative to the expected utility theory. This thesis aims to provide a choice-theoretic characterization for risk changes and risk attitudes under CPT using a stochastic dominance approach.

This thesis identifies a set of stochastic dominance conditions to generalize the notions of increase in risk, strong risk aversion, downside risk and downside risk aversion to accommodate risk aversion, risk seeking and downside risk aversion preferences in the CPT paradigm. This study further investigates risk measures implied by risky choice behaviour of CPT decision makers. This thesis also extends the analyses to general reference point and inverse S-shaped value functions.

The stochastic dominance conditions identified in this thesis provide an approach for risk preference elicitation in the paradigm of CPT without prior knowledge on the shape of value functions or probability weighting functions, which complements to existing risk preference elicitation approaches. Consequently, the equivalence of risk measures and stochastic dominance conditions enables risk preference elicitation through pairwise comparisons of risky prospects. The implications of this work in experimental studies and optimal decision problems (e.g. portfolio choice) shed new light into the application of CPT.

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# List of Abbreviations

$\mathcal{V}_{CV}$ The class of increasing concave utility/value functions

$\mathcal{V}_{CX}$ The class of increasing convex e utility/value functions

$\mathcal{V}_{S}$ The class of increasing S-shaped value functions

$\mathcal{V}_{IS}$ The class of increasing inverse S-shaped value functions

$\mathcal{V}_{P}$ The class of increasing S-shaped value functions with positive third derivatives

$\mathcal{V}_{M}$ The class of inverse S-shaped value functions with positive third derivatives

$\mathcal{W}_{CV}$ The set of concave probability weighting functions

$\mathcal{W}_{CX}$ The set of increasing convex probability weighting functions

**PWF** Probability weighting functions

**FSD** First-degree stochastic dominance

**SSD** Second-degree stochastic dominance

**RSD** Risk seeking stochastic dominance

**TSD** Third-degree stochastic dominance

**PSD** Prospect stochastic dominance

**M-PSD** Mean-preserving stochastic dominance

**V-PSD** Prospect value preserving stochastic dominance

**MVP-DRI** Mean- and variance preserving downside risk increase

**TPSD** Third-degree prospect stochastic dominance

**W-TPSD** Weighted third-degree prospect stochastic dominance

**MU-W-TPSD** Mean and value preserving W-TPSD

**MUPT** Mean utility preserving transformation

**LPM** Lower partial moment

**UPM** Upper partial moment

**M-MUP-IDR** Markowitz mean utility preserving downside risk increase

**VaR** Value-at-risk

**CVaR** Conditional value-at-risk

**RVaR** Range value-at-risk

# Chapter 1

# Introduction

Expected utility theory (EUT) has reigned several decades as a dominant normative and descriptive model of decision-making under risk (Levy 1992). Extensive empirical studies have revealed that individuals' actual decisions may systematically violate the axioms of EUT. The emergence of prospect theory, proposed by Kahneman and Tversky (1979), and then further enhanced as *cumulative prospect theory* (CPT) by Tversky and Kahneman (1992), provides one of the most prominent alternatives to EUT as a descriptive model for decision making under risk, and has wide applications in operations management, economics and finance (see, e.g., Barberis et al. 2001, Kyle et al. 2006, He and Zhou 2011a, Schweitzer and Cachon 2000, Nagarajan and Shechter 2013, Long and Nasiry 2014).

In the EUT paradigm, the risk preference is fully represented by the curvature properties of a utility function with concavity (convexity) for risk averters (risk seekers). In the CPT paradigm, the risk preference is represented simultaneously by a reference-dependent value function and a pair of probability weighting functions with certain curvature properties.

In an effort to bridge EUT and CPT, a number of studies have strived to provide choice-theoretic characterizations for CPT by linking *stochastic dominance* conditions with the curvature properties of CPT value functions and probability weighting functions. The notion of stochastic dominance, defines partial ordering relationships between probability distributions (risky prospects), provides an effective way to reveal risk preference through pairwise comparisons of risky prospects. To capture risk seeking (in losses) and risk aversion (in gains), Levy and Wiener (1998) introduce the notion of prospect stochastic dominance. They show that when decision makers with S-shaped value functions impose a single rank-dependent cumulative weighting function on risky prospects, their preferences agree with prospect stochastic dominance if and only if the weighting function is S-shaped. Levy and Levy (2002) propose another stochastic dominance rule, called Markowitz stochastic dominance, for all inverse S-shaped value functions in an attempt to interpret the experimental result for mixed prospects that cannot be interpreted by CPT. To further account for the pair of nonlinear sign-dependent probability weighting functions under CPT, Baucells and Heukamp (2006) identify a stochastic dominance condition which, in addition to Levy and Wiener (1998), imposes additional restrictions on the pair of prospects under comparison and they show that the corresponding preference representation is a combination of S-shaped value functions and a special class of inverse S-shaped probability weighting functions.

More recently, Schmidt and Zank (2008) investigate the relationship between (strong) risk aversion and CPT. They show that strong risk aversion holds under CPT if and only if the utility function is concave in the domains of losses and gains, respectively, and the weighting functions for losses (gains) is concave (convex). This, however, does not capture the risk seeking preference observed in the experimental studies in CPT. This thesis aims to generalize the notion of increase in risk and strong risk aversion to accommodate both risk aversion and risk seeking preferences in the CPT paradigm.

Intuitively, the curvature characteristics of CPT provide a natural linkage between CPT and downside risk aversion, especially when the reference point is related to the return target and the downside risk is measured by expected losses. Empirical studies show a tradeoff exists between skewness (or downside risk) and variance (or overall risk): risk averters may be willing to accept a lower expected return, or a higher level of variance, if the distribution of the return is more skewed to the right (smaller downside risk) (see, e.g., Garrett and Sobel (1999), Harvey and Siddique 2000 and Chiu 2005), which underscores the importance of incorporating downside risk into the characterization of risk taking behaviour. To the best of my knowledge, however, there has so far been no attempt to provide a general choice-theoretic characterization of the tradeoff between overall risk and downside risk for CPT. This thesis aims to fill the gap.

The notion of risk plays a critical role in many problems in operations management and finance. A number of studies have attempted to define and quantify risk. Value-at-risk (VaR) and conditional value-at-risk (CVaR) are two risk measures that are commonly used by practitioners, especially by financial institutions (Levy 2016). Ogryczak and Ruszczynski (2002) first show that VaR is equivalent to first-degree stochastic dominance, and CVaR is equivalent to second-degree stochastic dominance. Ma and Wong (2010) use a stochastic dominance approach to provide a decision theoretic characterization for VaR and CVaR. Inspired by the linkage between risk measures and stochastic dominance conditions in the paradigm of EUT, this thesis investigates the risk measure that can be implied by decision makers with CPT risk preferences, and provides a behaviour foundation for various distortion risk measures in CPT.

In summary, the objective of this thesis is to provide a choice-theoretic foundation of risk change and risk attitudes in the paradigm of CPT. In particular, this thesis aims to address the following research questions:

- How should increase in risk and risk aversion be characterized in CPT?
- How should downside risk and downside risk aversion be characterized in CPT?
- What is the risk measure that can be implied by decision makers with CPT risk preferences?

These three research questions correspond to three elements in this thesis. Firstly, in order to address increases in risk and risk aversion, the thesis extends the notions of increase in risk and strong risk aversion to the CPT paradigm by identifying a set of SD rules and provide choice-theoretic characterizations for the preference towards these SD rules under CPT. To this end, this thesis first generalizes prospect stochastic dominance for S-shaped value functions by accounting for probability weighting functions. Inspired by Schmidt and Zank (2008), who consider the mean-preserving spread in CPT, this thesis introduces a new stochastic dominance condition which combines mean-preserving spread and mean-preserving contraction, to elicit the risk preferences of strong risk aversion in gains and strong risk seeking in losses. In addition, to characterize the intensity of risk aversion, this study generalizes the analysis of Diamond and Stiglitz (1974) by introducing a compensated adjustment stochastic dominance condition to capture the trade-off between mean and variance.

Secondly, to address increases in downside risk and downside risk aversion, this thesis first generalizes the notion of increase in downside risk from EUT to CPT, and defines the preference against such risk changes as prospect downside risk aversion. Then, the thesis establishes the stochastic dominance condition under which the degree of prospect downside risk aversion can be characterized by the prudence measure. The thesis also discusses some common statistical downside risk measures such as semi-variance, third-central moment, and strong skewness, and extends the analyses to inverse S-shaped value functions.

Finally, in order to address the risk measures implied by risky behaviour of CPT decision makers, this thesis establishes the connections among several distortion risk measures, stochastic dominance and choices made by decision makers with CPT risk preferences, to provide behaviour foundations for those risk measures in the paradigm of CPT. The thesis proposes range value-at-risk (RVaR), that are defined as the average level of VaR level across a range of loss/return probabilities, to characterize prospect stochastic dominance. The relationship between stochastic dominance rules and risk measures in turn provides the choice-theoretic foundation for CPT risk preferences, and captures the curvature properties of both value functions and probability weighting functions.

To further explore economic implications for the stochastic dominance rules and risk measures, the thesis perform comparative statics analysis with the respect to the stochastic dominance conditions and risk measures for CPT-value maximizing problems. The analyses can be further extended to preferences with general reference points and inverse S-shaped value functions.

The contribution of the thesis is fourfold:

- First, the thesis generalizes the notion of increases in risk and downside risk to CPT by identifying a set of stochastic dominance conditions to provide choice-theoretic characterizations for strong risk aversion and downside risk aversion, thereby enhancing the theoretical foundation of CPT;

- Second, this thesis provides a theoretical guideline for risk preference elicitation under CPT. In particular, the stochastic dominance conditions identified in this thesis can serve as a guideline for the design of experiments testing the joint hypotheses on the curvature properties of value functions and probability weighting functions;

- Third, the stochastic dominance conditions identified in this thesis shed new light into the applications of CPT. The thesis investigates the effect of changes in risk and risk attitudes on optimal solutions of expected utility maximization problems such as portfolio choice problems, thereby demonstrating the potential applications to more general optimal decision problems;

- Fourth, this thesis investigates robust risk measures in the paradigm of CPT and adds to the literature by establishing various logical connections among risk measures, stochastic dominance, and choices made by decision makers with CPT risk preferences.

The remainder of the thesis is organized as described below.

Chapter 2 provides a brief literature review and collects some mathematical notions and definitions that will be used in this thesis.

Chapter 3 focuses on increases in risk and strong risk aversion in CPT.

Chapter 4 addresses downside risk in CPT.

Chapter 5 investigates robust risk measures in CPT.

Chapter 6 concludes the thesis.

All technical proofs are provided in the appendices.

# Chapter 2

# Background and Related Work

This chapter aims to provide a brief literature review and to collect some mathematical notions that will be used in this thesis. Section 2.1 reviews the measurement of risk and downside risk. Section 2.2 collects some notions of risk preference/attitudes and comparative statics for risk aversion and downside risk aversion. Section 2.3 provides an overview of cumulative prospect theory. Section 2.4 revisits some notions of stochastic dominance.

## 2.1 On the Measurement of Risk

The notion of *risk* plays a critical role in many problems in operations management and finance. A number of studies have attempted to define and quantify risk. There are two major dimensions of risk identified in the literature: the probability of occurrence of loss and the amount of potential loss (Brachinger 2002). In his seminal work, Markowitz (1952a) proposes that variance can used as a proxy for overall risk in portfolios and that investors make tradeoffs between mean (expected return) and variance of portfolios. *Variance* or *standard deviation* are the most commonly used risk measures and the mean-variance rule is by far the most popular investment decision rule that has been widely adopted by both academics and practitioners (Levy 2016). The mean-variance rule provides a foundation for the development of the Capital Asset Pricing Model by Sharpe (1964) and Lintner (1965). It has long been criticized, however, on the basis that the variance is not consistent with investors' actual perception of risk since it allocates the weights of negative deviations (undesirable) and positive deviations (desirable) of returns evenly. A large body of empirical studies in finance, economics and psychology have argued that individuals view dispersions of returns in an asymmetric manner: losses weighs more heavily than gains (see, e.g., Markowitz 1959, Kahneman and Tversky 1979).

It has been argued that downside risk is a more appropriate risk measure because investors are more concerned about outcomes below the target return. Since then, various downside risk measures have been proposed. Markowitz (1959) proposes *semi-variance*, which takes into account only deviations below the mean or a selected critical value. Roy (1952) was the first to model downside risk formally, in what is known is known as 'safety first' technique. In his model, investors are assumed to minimize their risk and would prefer the principle of safety first. Similar to semi-variance, this measure only looks at the lower tail rather than the entire distribution of returns.

*Value-at-risk* (VaR) and *conditional value-at-risk* (CVaR) are two downside risk measures that are commonly used by practitioners, especially by financial institutions (Levy 2016). Jorion (2000) define the VaR of a portfolio as the worst loss over target horizon such that there is a low, pre-specified probability that the actual loss will be greater. VaR can be either calculated from the probability distribution of returns (gains) or the probability distribution of losses (i.e., the negative gains). Mathematically, for a portfolio investment over a period with return (gain) $X$, VaR at $100 \times \alpha\%$ risk tolerance level (or, equally speaking, $100 \times (1-\alpha)\%$ confidence level) is defined as the cutoff loss such that the probability of suffering a greater loss is less than $100 \times \alpha\%$ (i.e., the loss at the lower $100\alpha$th percentile of the distribution of $X$): $VaR_\alpha(X) = \sup\{a | P(-X > a) \leq \alpha\}$. For example, a portfolio having a 95% one-day VaR of \$100 million implies that there is only a 5% chance that the portfolio will lose more than \$100 million over the next day.

To regulate definitions of risk measures, Artzner et al. (1999) propose the *coherent risk measure*. A risk measure $\rho$ is coherent if the following four axioms hold.

(1) $\rho(X) \leq \rho(Y)$ if $X \leq Y$, (Monotonicity);

(2) $\rho(X + a) = \rho(X) - a$ for any constant $a$, (Translation invariance);

(3) $\rho(\lambda X) = \lambda \rho(X)$ for all $\lambda \geq 0$, (Positive homogeneity);

(4) $\rho(X + Y) \leq \rho(X) + \rho(Y)$, (Subadditivity).

Axioms (3) and (4) together imply that $\rho(X)$ is convex in $X$. That is, a coherent risk measure must be convex. It is known that VaR satisfies axioms (1) and (2) but fails to be subadditive for general loss distributions and is therefore not a convex risk measure, which makes it difficult to implement in portfolio optimization (see, e.g., Artzner et al. 1999, Pflug 2000).

The drawbacks of VaR have led many researchers to propose new risk measures that are coherent. CVaR is first proposed by Artzner et al. (1997) as an alternative risk measure, (also called *expected shortfall*, *average VaR*, *mean excess loss*, *expected tail loss* or *tail VaR*). Rockafellar and Uryasev (2000) demonstrate the use of CVaR in portfolio optimization and show that CVaR surpasses VaR computationally. Formally, the CVaR of the random payoff $X$ at the $(1-\alpha)100\%$ confidence level is defined as $CVaR_\alpha(X) = E[-X|-X \geq VaR_\alpha(X)]$, $\forall \alpha \in (0,1]$.

Rockafellar and Uryasev (2002) show that $CVaR_\alpha(X)$ can be defined as the minimum objective value of the convex optimization problem with $VaR_\alpha(X)$ being the minimizer: $CVaR_\alpha(X) := \inf \left\{ a + \frac{1}{\alpha} E[(-X-a)^+] : a \in \mathbb{R} \right\}$. That is, one can obtain $CVaR_\alpha(X)$ and $VaR_\alpha(X)$ simultaneously by solving the convex optimization problem in one dimension, which is appealing in computation. Moreover, Pflug (2000) and Rockafellar and Uryasev (2002) show that CVaR is a coherent risk measure.

On the other hand, Gneiting (2011) shows that CVaR suffers from backtesting and thus is not elicitable. Cont et al. (2010) introduce the notion of *qualitative robustness* for risk measurement procedure and show that practitioners should not only focus on the coherence of risk measures. According to Cont et al. (2010), practitioners should consider the necessity of the sub-additivity axiom proposed by Artzner et al. (1999) as it is in conflict with robustness for spectral risk measures. To overcome the non-robustness of CVaR, Cont et al. (2010) propose *range value-at-risk* (RVaR) as an alternative risk measure. RVaR can be seen as the average level of VaR level across a range of loss/return probabilities. RVaR is in general a non-convex risk measure. Embrechts et al. (2018) show that

RVaR satisfies a special form of sub-additivity and can be applied to risk sharing problems.

## 2.2 Risk Preference

It has been accepted that risk is different from risk preference. Brachinger (2002) states risk preference as "*the preferability of an alternative under conditions of risk and it is a matter of preference.*" Under EUT, the risk preference of a decision maker is fully characterized by the curvature properties of a utility function with concavity for risk averters and convexity for risk seekers.

In the EUT paradigm, risk aversion can be defined in both the weak sense and the strong sense. Weak risk aversion holds if the expected value of a prospect for certain is always preferred to the prospect itself, which is equivalent to the increasing concavity of the utility function of a decision maker. Hence weak risk aversion can be characterized by second-degree stochastic dominance (Pratt and Zeckhauser 1987). Strong risk aversion holds if a decision maker always dislikes increase in risk in the sense of mean-preserving spread (Rothschild and Stiglitz 1970). To characterize the intensity of risk aversion in the sense of the Arrow-Pratt measure of absolute risk aversion, defined as $R_u(x) = -u''(x)/u'(x)$, Diamond and Stiglitz (1974) define the mean utility preserving increase in risk as a compensated adjustment of mean-preserving increase in risk.

Since the 1980s, a number of studies have attempted to provide a choice-theoretic foundation for downside risk preference. Empirical studies show a tradeoff exists between skewness (or downside risk) and variance (or overall risk): risk averters may be willing to accept a lower expected return, or a higher level of variance, if the distribution of the return is more skewed to the right (smaller downside risk) (e.g., Garrett and Sobel (1999), Harvey and Siddique 2000 and Chiu 2005), which underscores the importance of incorporating downside risk into the characterization of risk taking behaviour.

In an empirical experiment, Mao (1970) shows that comparing two prospects, $(1, 3/4; 3, 1/4)$ and $(0, 1/4; 2, 3/4)$, decision makers prefer the former one, which cannot be interpreted by risk aversion. Note that the second prospect in fact represents a downside risk increase from the first one. Using this as evidence of the importance of downside risk preference, Menezes et al. (1980) identify downside-risk aversion with convexity of the marginal utility function ($u^{'''} > 0$) and show that a downside risk increase leads to a decrease of the expected utility under such a utility function. Menezes et al. (1980) introduce the mean and variance preserving downside risk increase to describe the change in risk that shifts probability to the lower tail (negative skewness) while preserving mean and variance. Meanwhile, it has been recognized by Leland (1968) and Sandmo (1970) that the positive third derivative of an utility function can also be used to characterize the precautionary saving motive in consumption theory, where the precautionary saving motive is a behaviour that uncertainty about future income will reduce current consumption and increase current saving. There is, therefore a natural linkage between precautionary saving motive and downside risk aversion.

To characterize the intensity of downside risk aversion, there are six different downside risk aversion measures proposed in the literature. The most well-known measure for downside risk aversion is prudence. Kimball (1990) introduces the notion of prudence, defined as $P_u(x) = -u'''(x)/u''(x)$, as a measure of downside risk aversion and show that this index is directly related to precautionary saving. Chiu (2005) comments that although it was known that there is a linkage between the concepts of prudence and downside-risk aversion, their precise relationship had not been clarified formally. To characterize their relationship, Chiu (2005) identifies a stochastic dominance condition under which an individual's choice between risky alternatives can be determined by his prudence measure, which provides a general choice-theoretic characterization of the tradeoff between overall risk and downside risk. Chiu (2005) shows that for risk averse decision makers, the greater the prudence, the greater the intensity of downside risk aversion.

Although the properties of prudence have many useful contexts in downside risk aversion, and it is recognized as a measure of downside risk aversion, both Keenan and Snow (2002) and Modica and Scarsini (2005) argue that the degree of local downside risk aversion should be measured by $u'''/u'$. Modica and Scarsini (2005) follow the analogue of Ross (1981) but use third derivatives and third-degree stochastic dominance between variables instead. They propose $u'''/u'$ by noting two major properties: (a) a decision maker should be ready to pay a higher amount to insure against a risk with higher skewness if he has a higher local measure than another decision maker; (b) they prove that the decreasing property of $u'''/u'$ is equivalent to the concept of local risk vulnerability (Gollier and Pratt 1996), where risk vulnerability refers to any unfair background risk that will increase risk aversion. They also note that, however, this measure could not translate a local comparison into a global comparison.

Crainich and Eeckhoudt (2008) also propose $u'''/u'$ as a measure for the intensity of downside risk aversion. They follow the approach of Pratt (1964) and establish that this measure, rather than contrasting with the prudence measure, complements it, since the properties of $u'''/u'$ are largely similar to the classical Arrow-Pratt absolute risk aversion. They show that a concave transformation function with a positive third derivative will increase the intensity of downside risk aversion in the sense of $u'''/u'$, but the same does not apply to the prudence measure. In particular, they show that such a transformation function (concave with a positive third derivative) will increase the concavity of $u$ and the convexity of $u'$, and hence increase the intensity of downside risk aversion in the sense of $u'''/u'$. While the prudence measure is directly related to precautionary saving (Kimball 1990), Crainich and Eeckhoudt (2008) show that $u'''/u'$ can be interpreted as "*a component of a price in a simple general equilibrium of savings*".

It is recognized that the positive third derivative of an utility function is a necessary, but not sufficient condition for decreasing absolute risk aversion. (Liu and Meyer 2012) propose decreasing absolute risk aversion as a measure for downside risk aversion. Keenan and Snow (2012) introduce the Schwarzian measure, defined as $S_u = -(R_u)' - \frac{1}{2}R_u^2$, where $R_u$ is the Arrow-Pratt absolute risk aversion as defined above, as an alternative measure for the intensity of downside risk aversion. They show that the Schwarzian measure can be used to characterize the general class of mean and variance preserving increase in downside risk. Huang and Stapleton (2014) introduce cautiousness as a new downside risk aversion measure which provides a linkage between cautiousness and skewness preference. They show that a decision maker with more cautiousness is more likely to buy options with the aim of pursuing a higher skewness of his portfolio, where cautiousness is defined as the rate of change in risk tolerance by Wilson (1968) in his theory of syndicates.

## 2.3 Cumulative Prospect Theory

Prospect theory, originated from Kahneman and Tversky (1979), has emerged as a prominent alternative to EUT. In their seminal paper, Kahneman and Tversky (1979) use experimental studies to show that the actual decisions of individuals under risk may systematically violate the axioms of expected utility theory. To describe the risk preferences observed in choice experiments better, they propose prospect theory under which utilities or values are defined over gains and losses related to a reference point rather than over final wealth (reference dependence, which was originally proposed by Markowitz 1952a), the value function is concave for gains and convex for losses (diminishing sensitivity) and steeper for losses than for gains (loss aversion) with a nonlinear transformation of the probability scale that overweights small probabilities and underweights moderate and high probabilities.

Prospect theory may violate first-degree stochastic dominance and it is not readily extended to prospects with a large number of outcomes (Tversky and Kahneman 1992). To overcome this shortcoming, Tversky and Kahneman (1992) develop *cumulative prospect theory* (CPT) by incorporating a pair of cumulative weighting functions (also called capacities) that transform cumulative, rather than individual probabilities of gains and losses into decision weights. CPT belongs to the class of rank-dependent utility models introduced by Quiggin (1981, 1982), which generalize expected utility theory by incorporating weighting functions that transforms cumulative probabilities into decision weights. CPT is a rank- and sign-dependent utility (RDU) model (Luce 1991; Luce and Fishburn 1991; Luce and Marley 2005) that incorporates reference dependence and sign dependence (a pair of probability weighting functions for gains and losses respectively). The key features of CPT include:

(1) individuals make decisions based on relative gains and losses against a certain reference point rather than total wealths (reference dependence);

(2) individuals are risk averse regarding gains and risk seeking regarding losses (S-shaped value function and diminishing sensitivity);

(3) individuals are more sensitive to losses than to gains (loss aversion); and

(4) individuals allocate decision weights to values of outcomes with a pair of weighting functions that transform the cumulative distribution of losses and the decumulative distribution of gains respectively, which generalizes the rank-dependent utility introduced by Quiggin (1982) to accommodate the sign and reference dependence of CPT.

Ever since then, successive efforts have been made to provide axiomatic characterizations for CPT (see, e.g., Wakker and Tversky 1993, Chateauneuf and Wakker 1999, Zank 2001, Wakker and Zank 2002, Schmidt 2003).

According to Tversky and Kahneman (1992), the risk preference under CPT is jointly represented by a value function and a pair of probability weighting functions. The non-decreasing S-shaped value function is defined over a monetary gain or loss (change in wealth with respect to a reference point $r$ and a pair of probability weighting functions $w^-(\cdot)$ and $w^+(\cdot)$ that transform the objective probabilities (cumulative distribution functions) to decision weights for losses and gains respectively, with boundary conditions $w^-(0) = w^+(0) = 0$ and $w^-(1) = w^+(1) = 1$. In addition, the gain weighting function equals the dual of the loss weighting function (i.e., $w^-(p) + w^+(1-p) = 1$) for all $p \in [0,1]$. A number of studies have attempted to characterize the shape of probability weighting functions using various methodologies (see, e.g., Prelec 1990, Tversky and Wakker 1995, Wu and Gonzalez 1996). A typical example of S-shaped value function and inverse S-shaped probability weighting function is provided by Tversky and Kahneman (1992):

$$v(x) = \begin{cases} x^\alpha, & \text{if } x \geq 0, \\ -\lambda(-x)^\beta, & \text{if } x < 0, \end{cases} \quad \text{and} \quad \begin{cases} w^+(p) = \frac{p^\gamma}{(p^\gamma+(1-p)^\gamma)^{1/\gamma}}, \\ w^-(p) = \frac{p^\delta}{(p^\delta+(1-p)^\delta)^{1/\delta}}, \end{cases}$$

with $0 < \alpha < 1$, $0 < \beta < 1$, $\lambda \geq 1$, and $\gamma, \delta \geq 0.28$ Wakker 2010.

In discrete cases, a prospect $F : (x_1, p_1; ...; x_n, p_n)$ with $a < x_1 < ... < x_{k-1} < x_k = 0 < x_{k+1} < ... < x_n < b$ is evaluated by $V_F = \sum_{i=0}^n \pi_i u(x_i)$, where the decision weights $\pi_i$ are defined as:

$$\pi_i = \begin{cases} w^+(p_i + \cdots + p_n) - w^+(p_{i+1} + \cdots + p_n) & \text{if } i \geq k, \\ w^-(p_1 + \cdots + p_i) - w^-(p_1 + \cdots + p_{i-1}) & \text{if } i < k. \end{cases}$$

Without loss of generality, the reference point is normalised to zero and $u(0) = 0$. Positive outcomes are defined as *gains* and negative outcomes are *losses*. When all the outcomes are positive, CPT coincides with rank-dependent expected utility

(Quiggin 1981; Quiggin 1982; Yaari 1987; Green and Jullien 1988; Wakker 1989; Quiggin and Wakker 1994).

## 2.4 Stochastic Dominance

*Stochastic dominance* (SD) provides a powerful alternative decision rule for risk and risk preferences within expected utility theory. The notion of SD describes partial ordering relationships between probability distributions (risky prospects) by pairwise comparisons, providing a way to divide the set of feasible risky prospects into efficient and inefficient sets (Levy 2016). The early contributions of Hadar and Russell (1969), Hanoch and Levy (1969), Rothschild and Stiglitz (1970) and Whitmore (1970) paved the way for employing the SD rule in decision making under risk. In what follows, we briefly review some notions of SD that will be used in this paper and the corresponding utility characterization.

The first-degree stochastic dominance (FSD) proposed by Hadar and Russell (1969) provides a choice-theoretic foundation for risk preferences with nondecreasing utility functions (for choice rationality). The notion $u' \geq 0$ represents the weakest assumption in respect to the risk preference, that the decision makers like more money to less. Formally, $F$ dominates $G$ by first-degree stochastic dominance, denoted by $F \succeq_{FSD} G$, if $F(x) \leq G(x)$ for all $x$.

The second-degree stochastic dominance (SSD) rules proposed by Hanoch and Levy (1969) and Rothschild and Stiglitz (1970) provides a choice-theoretic foundation for risk preferences with non-decreasing and concave functions (for risk aversion). Formally, $F$ dominates $G$ by second-degree stochastic dominance, denoted by $F \succeq_{SSD} G$ if $\int_{-\infty}^{x} F(z)dz \leq \int_{-\infty}^{x} G(z)dz$ for all $x$.

Levy and Wiener (1998) introduce the corresponding second-degree stochastic dominance condition for risk seekers to characterize risk preferences with non-decreasing and convex functions. Formally, $F$ dominates $G$ by inverse second degree stochastic dominance, denoted by $F \succeq_{RSD} G$, if $\int_{x}^{\infty} F(z)dz \leq \int_{x}^{\infty} G(z)dz$ for all $x$. The SD conditions of FSD, SSD are RSD are also called *stochastic increasing*, *increasing concave order* and *increasing convex order*, respectively, in the literature on stochastic ordering (Shaked and Shanthikumar 2007).

The third-degree stochastic dominance (TSD) is proposed by Whitmore (1970) for non-decreasing and concave utility functions with positive third derivatives, i.e., for decision makers who are risk averse and downside risk averse. Formally, $F$ dominates $G$ by third-degree stochastic dominance, denoted by $F \succeq_{TSD} G$, if (1) $\int_{-\infty}^{x} F(z)dz \leq \int_{-\infty}^{x} G(z)dz$ for all $x$; and (2) $\mu_F \geq \mu_G$. Note that the second condition of TSD specifically requires that the mean of $F$ is greater than that of $G$. Menezes et al. (1980) define that the risk change from $F$ to $G$ is an increase in downside risk if (i) $\int_{a}^{b}[G(z) - F(z)]dz = 0$; and (ii) $\int_{a}^{x}\int_{a}^{z}[G(t) - F(t)]dtdz \geq 0$ for all $x \in [a, b]$ and with equality holding at $x = b$. They show that an individual $u$ prefers $F$ to $G$ for all risk changes from $F$ to $G$ being increases in downside risk if and only if $u''' > 0$. These equalities imply equal mean and variance and that there is an unambiguous risk shift from the right to the left for the risk change from $F$ to $G$. The notion of mean- and variance-preserving increase in downside risk is a special case of TSD when $F$ and $G$ have the same mean and variance. Menezes et al. (1980) show that an increase in downside risk implies that $F$ is more skewed to the right than $G$, in the sense that the third central moment of

$F$ is greater than that of $G$. The SD rule implied by the increase in downside risk is also called *3-convex order* (Shaked and Shanthikumar 2007).

To capture the tradeoff between mean and variance, Diamond and Stiglitz (1974) introduce utility preserving increase in risk. In the same spirit, Chiu (2005) generalizes the SD conditions of increase in downside risk by introducing mean and utility preserving increase in downside risk. A risk change from $F$ to $G$ is mean and utility preserving increase in downside risk for a reference decision maker with an increasing and concave utility function $u$ if (i) $\int_a^b [G(z)-F(z)]dz = 0$ and (ii) $\int_a^x u''(z) \int_a^z [G(t) - F(t)]dtdz \leq 0$, $\forall x \in [a,b]$ with at least one strict inequality and equality holding at $x = b$. Note that in Chiu's definition, the accumulative advantage function of $F$ over $G$, defined as $\int_a^z [G(t) - F(t)]dt$, is weighted by a negative and decreasing function $u''$, which, together with the binding condition in (ii), implies that $G$ is more skewed to the left of $F$ while having a smaller variance. Chiu (2005) shows that, for risk-averse individuals, the greater the prudence measure, the greater the strength of downside risk aversion, which supports the role of prudence as a measure of downside risk aversion.

To capture risk seeking (in losses) and risk aversion (in gains), Levy and Wiener (1998) introduce the notion of *prospect stochastic dominance* (PSD): $F$ dominates $G$ by PSD if $\int_x^y [G(z) - F(z)]dz \geq 0$ for any $x \leq 0 \leq y$, with at least one strict inequality. Levy and Wiener (1998) show that when decision makers with S-shaped value functions impose a single rank-dependent cumulative weighting function on risky prospects their preferences agree with PSD if and only if the weighting function is S-shaped. Levy and Levy (2002) propose another SD rule called Markowitz stochastic dominance (MSD) for all inverse S-shaped value functions to interpret those experimental results for mixed prospects that cannot be interpreted by CPT.

To further account for the pair of nonlinear probability weighting functions under CPT, Baucells and Heukamp (2006) identify a set of SD conditions which, in addition to PSD, imposing additional restrictions on the pair of prospects under comparison, and show that the corresponding preference representation is a combination of S-shaped value functions and a special class of inverse S-shaped probability weighting functions. They also show that in the domain of losses, a convex (concave) value function is conjugate to a concave (convex) probability weighting function and, in the domain of gains, a concave (convex) value function is conjugate to a concave (convex) probability weighting function.

More recently, Schmidt and Zank (2008) investigate the relationship between (strong) risk aversion and CPT. They show that decision makers are averse to mean-preserving SSD if and only if the utility function is concave in the domains of losses and gains, respectively, and the weighting function for losses (gains) is concave (convex), which is partly in conflict with the curvatures of value functions and probability weighting functions described in Tversky and Kahneman (1992).

To account for higher order risk preferences, Wong and Chan (2008) further extend PSD and MSD to third-degree prospect stochastic dominance for all S-shaped value functions and third-degree Markowitz stochastic dominance for all inverse S-shaped value functions. Their analysis of utility representation, however, relies on the assumption that the second derivative of the utility function is zero at the reference point. This thesis does not impose this assumption since many commonly used CPT value functions are not necessarily differentiable at the reference points that are typically kinks.

# Chapter 3

# Risk Aversion and Risk Seeking in Cumulative Prospect Theory

## 3.1 Introduction

In this chapter, we aim to extend the notions of increase in risk and strong risk aversion to the CPT paradigm by identifying a set of SD rules and provide choice-theoretic characterizations for the preference towards these SD rules under CPT.

In the EUT paradigm, risk preference (e.g., risk aversion or risk seeking) is fully represented by the shape of the utility function (concavity for risk aversion and convexity for risk seeking). Risk aversion can be defined in both the weak sense and the strong sense. Weak risk aversion holds if the expected value of a prospect for certain is always preferred to the prospect itself, which is characterized by SSD and increasing concave utility functions (Pratt and Zeckhauser 1987). Strong risk aversion holds if a decision maker always dislikes an increase in risk in the sense of mean-preserving spread (Rothschild and Stiglitz 1970). To characterize the intensity of risk aversion in terms of Arrow-Pratt absolute risk aversion, Diamond and Stiglitz (1974) define the *mean utility preserving* increase in risk as a compensated adjustment of mean-preserving increase in risk.

In the CPT paradigm, the risk preference is represented simultaneously by the curvatures of a value function and a pair of PWFs. A number of studies have attempted to provide choice-theoretic characterizations for CPT by identifying some SD rules that can be represented by some curvature forms of CPT value functions and PWFs. To capture both risk seeking (in losses) and risk aversion (in gains), Levy and Wiener (1998) introduce the SD rule of prospect stochastic dominance (PSD) to generalize SSD and RSD in the CPT paradigm. They show that an individual with an S-shaped value function and a single rank-dependent PWF has a preference agreeing with the PSD order if and only if the PWF is S-shaped. Levy and Levy (2002) identify another SD rule, Markowitz stochastic dominance (MSD), for inverse S-shaped value functions to interpret the experimental result for mixed prospects that cannot be interpreted by CPT. Wakker 2003 argues that the experimental results of Levy and Levy (2002) can actually support prospect theory if PWFs are considered. Schmidt and Zank (2008) study the implication of strong risk aversion for CPT. They show that strong risk aversion holds under CPT if and only if the utility function is concave in the domains of losses and gains, respectively, and the weighting functions for losses (gains) is concave (convex), which is partly in conflict with curvatures of the value function and PWFs described in Tversky and Kahneman (1992). Their analysis generalizes Chew et al. (1987) who shows that strong risk aversion holds under RDU model with single PWF if and only if the utility function is concave and the PWF is convex. However, it does not capture the risk seeking preference observed in the experimental studies in CPT.

Baucells and Heukamp (2006) introduce a special PSD rule (defined as PW-SD) that imposes FSD conditions on both tails of the prospects under comparison. They identify an interesting conjugate curvature relationship: a concave (convex) value function and a convex (concave) PWF are conjugate in gains and a convex (concave) value function and a convex (concave) PWFs are conjugate. They show that PW-SD can be characterized by the set of S-shaped value functions and the set of inverse S-shaped PWFs with the inflation points corresponding to the tails where FSD rule applies, which implies that all decision makers with S-shaped value functions and inverse S-shaped PWFs with the specific inflation points agreeing with PW-SD. Using this as the theoretic foundation, they conduct choice experiments to test the curvatures of value functions and PWFs and conclude that their experimental results suggest that they can reject the inverse S-shaped value function advocated by Levy and Levy (2002) in favor of the S-shaped form. However, they do not show whether an individual exhibiting the preference towards the PW-SD order implies inverse S-shaped PWFs. It remains ambiguous whether the preference agreeing with PW-SD order must be represented by a S-shaped value function and a pair inverse S-shaped PWFs.

Inspired by aforementioned papers, we extend the notions of increase in risk and strong risk aversion to the CPT paradigm by identifying a set of SD rules and provide choice-theoretic characterizations for the preference towards these SD rules under CPT. To this end, we first generalizes PSD for $S$-shaped value functions by accounting for PWFs. Inspired by Rothschild and Stiglitz (1970), we introduce the SD rule of *mean-preserving* PSD (M-PSD), which combines a mean-preserving spread in gains and a mean-preserving contraction in losses, and define it as the increase in risk in the CPT paradigm. Then the notion of strong risk aversion can be generalized in the CPT paradigm as the aversion to M-PSD and we call it *prospect strong risk aversion*. We show that a decision maker exhibits prospect strong risk aversion if and only if the value function is $S$-shaped and both the gain- and loss-weighting functions are convex. The convexity of PWFs leads to greater weights on smaller gains (larger losses), which implies the probabilistic risk attitude of pessimism in gains and optimism in losses (Wakker 2010). We then generalize the analysis of Diamond and Stiglitz (1974) by introducing the *prospect value preserving* PSD (V-PSD) condition as a compensated adjustment of PSD to describe the tradeoff between mean and variance, which allows us to characterize the increase in risk aversion. We also show that our analysis can be further extended to preferences with general reference points and inverse $S$-shaped value functions.

We next discuss the implications of our results on risk preference elicitation in the CPT paradigm. Although it is impractical to elicit an individual's risk preference through pairwise comparisons for all possible risky prospect pairs, our results can provide a theoretic guideline for preference elicitation experimental studies. To show this, we first discuss the preference elicitation methods (e.g., certainty equivalence). Note that Bleichrodt et al. (2001) argue that the standard utility elicitation methods in the EUT paradigm may be biased. They propose using the CPT paradigm with the PWFs specified in Tversky and Kahneman (1992) to correct the biases and improve the prescriptive use of expected utility to derive optimal decisions. Inspired by their analysis, we discuss the implication of strong risk aversion on the certainty equivalence without specifying the PWFs. We then revisit the experimental studies of Kahneman and Tversky (1979) and Baucells and Heukamp (2006) by examining the risky prospect pairs using the SD rules identified in this chapter and provide alternative interpretations of their experimental results.

To further explore economic implications for optimal decision making, we perform comparative statics analysis with the respect to the SD conditions for CPT-value maximizing problems. Using portfolio choice problems as examples, we show that increases in risk and in risk aversion may lead to less riskier portfolio choices for a decision maker with CPT preference.

Our analysis can be further extended to preferences with general reference points and inverse S-shaped value functions. To accommodate inverse S-shaped PWFs that favor risk aversion in small probabilities of the lower tail of losses and risk seeking in small probabilities of the upper tail of gains, we extend our analysis by further generalizing the SD condition and the curvature form of value functions to allow concavity in the lower tail of losses and convexity in the upper tail of gains, which complements the effort of Baucells and Heukamp (2006).

Our contribution is threefold. Firstly, we generalize the notions of increase in risk and strong risk aversion in the CPT paradigm by introducing the M-PSD order and providing a choice-theoretic characterization for strong risk aversion in gains and strong risk seeking in losses, which complements the successive efforts of Levy and Wiener (1998), Schmidt and Zank (2008) and Baucells and Heukamp (2006).

Secondly, our analysis provides a theoretic guideline for risk preference elicitation under CPT, which complements the classic elicitation approaches. Thirdly, the comparative statics analysis with respect to the PSD and M-PSD orders for CPT-value maximizing decision problems (e.g., portfolio choice) sheds new light into the applications of CPT.

The remainder of the chapter is organized as follows. Section 3.2 provides choice-theoretic characterizations of risk aversion and risk seeking in CPT. Section 3.3 and Section 3.4 discuss implications for risk preference elicitation and optimal decision problems, respectively. Section 3.5 extends our analysis to account for general reference points, inverse S-shaped value functions, and inverse S-shaped PWFs, respectively. Section 3.6 concludes the paper. All technical proofs are in the appendix.

## 3.2 Choice-theoretic Characterizations in CPT

In decision making under risk, a *prospect* is described by a probability distribution. In particular, a prospect of a finite probability distribution can be described as $(x_1, p_1; ...; x_n, p_n)$ where $x_i$ is an outcome with probability $p_i$, $i = 1, ..., n$. Throughout this paper, the pair of risky prospects in comparison are denoted by $F$ and $G$. Let $\mu_F$ and $\mu_G$ be the corresponding means and $\bar{F}$ and $\bar{G}$ be the decumulative (or tail) distribution functions. Assume that $F$ and $G$ are continuous with a common support $[a, b], a < 0 < b$, and $F(x) \neq G(x)$ for at least one $x$. A risk change from $F$ to $G$ refers to the change from prospect $F$ to prospect $G$, denoted by $F \to G$.

Let $u$ be the value function with $u(0) = 0$. In this thesis, we use the terms "value function" and "utility function" interchangeably. Denote by $\mathcal{V}_S$ and $\mathcal{V}_M$ the sets of all increasing $S$-shaped prospect value functions that are strictly convex in the losses and strictly concave in the gains (i.e., $u'(x) > 0,\ u''(x) > 0$ for $x < 0, u''(x) < 0$ for $x > 0$), and all increasing inverse $S$-shaped prospect value functions that are strictly concave in the losses and strictly convex in the gains (i.e., $u'(x) > 0,\ u''(x) < 0$ for $x < 0,\ u''(x) > 0$ for $x > 0$).

According to Tversky and Kahneman (1992), the CPT preference can be represented by a nondecreasing $S$-shaped value function defined over a monetary gain or loss (change in wealth with respect to a reference point $r$ that is normalized to zero) and a pair of PWFs $w = (w^-, w^+)$ that transform the objective probabilities (cumulative distribution functions) to decision weights for losses and gains respectively. Assume that the PWFs are nondecreasing with $w^-(0) = w^+(0) = 0$, $w^-(1) = w^+(1) = 1$, and $w^-(p) + w^+(1-p) = 1$ for all $p \in [0,1]$. Then the PWFs can be viewed as transformation functions that transfer a prospect $F$ to a "weighted" prospect $F^w$ such that

$$F^w(x) = \begin{cases} w^-(F(x)) & \text{for } x \in [a, 0], \\ 1 - w^+(\bar{F}(x)) & \text{for } x \in (0, b]. \end{cases}$$

A typical example of value function and PWFs is provided by Tversky and Kahneman (1992):

$$u(x) = \begin{cases} x^\alpha, & \text{if } x \geq 0, \\ -\lambda(-x)^\beta, & \text{if } x < 0, \end{cases} \quad \text{and} \quad \begin{cases} w^+(p) = \frac{p^\gamma}{(p^\gamma + (1-p)^\gamma)^{1/\gamma}}, \\ w^-(p) = \frac{p^\delta}{(p^\delta + (1-p)^\delta)^{1/\delta}}, \end{cases} \tag{3.1}$$

with $0 < \alpha, \beta < 1$, $\lambda \geq 1$, and $\gamma, \delta \geq 0.28$ (Wakker 2010). Another well-known specification of PWFs was proposed by Goldstein and Einhorn (1987):

$$\begin{cases} w^+(p) = \frac{ap^\gamma}{ap^\gamma + (1-p)^\gamma}, \\ w^-(p) = \frac{bp^\delta}{bp^\delta + (1-p)^\delta}, \end{cases} \tag{3.2}$$

with $a \geq 0, b \geq 0, \delta \geq 0$ and $\gamma \geq 0$. Wakker (2010) uses the term of *probabilistic risk attitude* to refer to the effect of probability weighting on risk attitudes: greater (smaller) weights on unfavorable outcomes reflects pessimistic (optimistic) attitudes. He shows that when $\gamma$ or $\delta$ decreases the degree of likelihood insensitivity increases and hence the shape of PWF may change from S-shaped to inverse-S-shaped. In particular, when $\gamma = \delta = 1$ and $a, b < 1$, the PWFs are convex, which implies pessimism in gains (i.e., decision weights of smaller gains are higher) and optimism in losses (i.e., decision weights of smaller losses are higher).

Under EUT, the choice between prospects $F$ and $G$ for an individual $u$ is determined by the difference of the values of expected utility:

$$\Delta_u(F, G) = \int_a^b u(x)dF(x) - \int_a^b u(x)dG(x) = \int_a^b u'(x)[G(x) - F(x)]dx, \tag{3.3}$$

which can be viewed as the accumulative advantages of $F$ over $G$ throughout the support set $[a, b]$ weighted by the first derivative of the utility function.

The CPT value of prospect $F$ for a decision maker $u$ with PWFs $(w^-, w^+)$ is defined as

$$V_F = \int_a^0 u(x)d[w^-(F(x))] + \int_0^b u(x)d[1 - w^+(\bar{F}(x))].$$

Then the decision maker under CPT prefers $F$ to $G$ if $\Delta_u^w(F,G) \geq 0$, where

$$\begin{aligned}\Delta_u^w(F,G) = V_F - V_G &= \int_a^0 [w^-(G(x)) - w^-(F(x))]u'(x)dx \\ &+ \int_0^b [w^+(\bar{F}(x)) - w^+(\bar{G}(x))]u'(x)dx. \qquad (3.4)\end{aligned}$$

In discrete cases, a prospect $(x_1, p_1; ...; x_n, p_n)$ with $a < x_1 \cdots < x_{k-1} < x_k = 0 < x_{k+1} ... < x_n < b$ is evaluated by $\sum_{i=0}^n \pi_i u(x_i)$, where the decision weights $\pi_i$ are defined as

$$\pi_i = \begin{cases} w^+(p_i + \cdots + p_n) - w^+(p_{i+1} + \cdots + p_n) & \text{if } i \geq k, \\ w^-(p_1 + \cdots + p_i) - w^-(p_1 + \cdots + p_{i-1}) & \text{if } i < k. \end{cases}$$

Correspondingly, we have

$$\begin{aligned}\Delta_u^w(F,G) &= \sum_{i=1}^{k-1}[w^-(G(x_i)) - w^-(F(x_i))][u(x_{i+1}) - u(x_i)] \\ &+ \sum_{i=k}^{n}[w^+(\bar{F}(x_i)) - w^+(\bar{G}(x_i))][u(x_{i+1}) - u(x_i)].\end{aligned}$$

### 3.2.1 Increase in Risk under CPT

Levy and Wiener (1998) introduce the notion of prospect stochastic dominance (PSD) to characterize the preference represented by $S$-shaped value functions. Specifically, $F$ dominates $G$ according to PSD, written as $F \succeq_{PSD} G$, if $\int_x^y [G(t) - F(t)]dt \geq 0$ for all $x \leq 0 \leq y$, with at least one strict inequality. PSD implies

SSD in gains and RSD in losses. Integrating $\Delta_u(F,G)$ by parts yields

$$\begin{aligned}\Delta_u(F,G) &= u'(a)\int_a^0 [G(z)-F(z)]dz + \int_a^0 u''(x)\int_x^0 [G(z)-F(z)]dzdx \\ &+ u'(b)\int_0^b [G(z)-F(z)]dz - \int_0^b u''(y)\int_0^y [G(z)-F(z)]dzdy.\end{aligned}$$

Levy and Wiener (1998) show that $F \succeq_{PSD} G$ if and only if $\Delta_u(F,G) \geq 0$ for all $u \in \mathcal{V}_S$. Accounting for PWFs, the following lemma follows immediately.

**Lemma 1** (PSD and S-shaped Value Functions) *Consider a risk change from $F$ to $G$ and a pair of PWFs $(w^-, w^+)$. Then $F^w \succeq_{PSD} G^w$ if and only $\Delta_u^w(F,G) \geq 0$ for all $u \in \mathcal{V}_S$.*

Levy and Wiener (1998) also consider the preference with a reference-dependent value function and a single transformation function defined on the cumulative probabilities over the full support. They show that $F$ dominates $G$ by PSD if and only if every transformation function that is increasing, concave in gains and convex in losses (i.e., $S$-shaped) preserves this dominance, which implies that the transformation function overweights larger (smaller) cumulative probabilities in losses (gains). Note that in the CPT paradigm a pair of PWFs (i.e., a loss-weighting function and a gain-weighting function) are used to account for the rank- and sign-dependent structure of CPT. The next lemma further generalizes their analysis by accounting for a pair of PWFs.

**Lemma 2** (Convex PWFs) *Consider a risk change from $F$ to $G$. Then $F \succeq_{PSD} G$ if and only if $F^w \succeq_{PSD} G^w$ for all $(w^-, w^+) \in \mathcal{W}_{CX} \times \mathcal{W}_{CX}$, where $\mathcal{W}_{CX}$ is the set of increasing convex PWFs.*

Combining Lemma 1 and Lemma 2 yields the following choice-theoretic characterization of PSD.

**Proposition 1** (Characterizing PSD) *Consider a risk change from $F$ to $G$.*

*Then* $F \succeq_{PSD} G$ *if and only if* $\Delta_u^w(F,G) \geq 0$ *for all* $u \in \mathcal{V}_S$ *and all* $(w^-, w^+) \in \mathcal{W}_{CX} \times \mathcal{W}_{CX}$.

Proposition 1 shows that the PSD relationship between two prospects can be characterized by the choices of a class of individuals with $S$-shaped value functions and pairs of convex PWFs. Such a result generalizes the SSD and RSD representations for risk averters and risk seekers.

Our result is similar to Levy and Wiener (1998) in spirit except that they adopt the conventional rank-dependent weighting function with only one PWF defined on the cumulative probability. As a result, in the paradigm they adopted, the preference towards the PSD order is characterized jointly by the common preference of a class of $S$-shaped value functions and $S$-shaped PWFs, whereas in the CPT paradigm the curvatures of PWFs are both convex.

Baucells and Heukamp (2006) introduce a special PSD order, defined as PW-SD, which imposes FSD in two tails. More specifically, they define that a risk change from $F$ to $G$ is PW-SD if $F \succeq_{PSD} G$ and $G(x) \geq F(x)$ for $a \leq x \leq x_p^-$ and $x_p^+ \leq x < b$, where $x_p^- = \inf\{x \leq 0 : F(x) \geq p, G(x) \geq p\}$ and $x_p^+ = \sup\{x \geq 0 : F(x) \leq 1-p, G(x) \leq 1-p\}$ for some probability $p \in [0,1]$. They show that $F \succeq_{PW-SD} G$ if and only if $\Delta_u^w \geq 0$ for all $u \in \mathcal{V}_P$ and $(w^-, w^+)$ are inverse S-shaped with some inflection point $(p^-, p^+)$ (i.e., concave in $[0, p^-]$ and $[0, p^+]$ and convex in $[p^-, 1]$ and $[p^+, 1]$, respectively). It is notable that $F(x) \leq G(x)$ implies that $F^w(x) \leq G^w(x)$ and hence the concavity of PWFs in $[0, p^-]$ and $[0, p^+]$ are not necessary. It suffices to require the functional set of the PWFs to include PWFs that are linear in $[0, p^-]$ and $[0, p^+]$. Hence, it remains unclear whether an individual agreeing with the PW-SD order must have inverse-S-shaped PWFs in the CPT paradigm.

In the EUT paradigm, Rothschild and Stiglitz (1970) define *increase in risk* as *mean-preserving spread* (MPS) and define *strong risk aversion* as the aversion to MPS. More specifically, a risk change from $F$ to $G$ is MPS if $F \succeq_{SSD} G$ with $\mu_F = \mu_G$ (Levy 2016, p228). If $G$ is a MPS from $F$, then $F$ is a *mean-preserving contraction* (MPC) from $G$. Similarly, we can use the notion of *strong risk seeking* to describe the aversion to MPC. In the CPT paradigm, Schmidt and Zank (2008) show that strong risk aversion for both losses and gains implies a concave value function in both losses and gains with a convex gain-weighting function (overweighting larger tail probabilities) and concave loss-weighting function (overweighting smaller accumulative probabilities).

To capture both risk aversion and risk seeking observed in experimental studies that are well documented in the CPT literature, we introduce the notion of *mean-preserving prospect stochastic dominance* (M-PSD) to generalize the notion of *increase in risk* in the CPT paradigm.

**Definition 1 (**M-PSD) *Prospect $F$ dominates prospect $G$ in the sense of M-PSD, written as $F \succeq_{M-PSD} G$, if $\int_x^y [G(t) - F(t)]dt \geq 0$ for any $x \leq 0 \leq y$, with at least one strict inequality and with equality holding at $(x, y) = (a, b)$.*

Note that $F \succeq_{M-PSD} G$ implies $\int_a^0 [G(t) - F(t)]dt = \int_0^b [G(t) - F(t)]dt = 0$, i.e., the expected values of losses and the expected values of gains of $F$ and $G$ are the same. That is, M-PSD combines MPS in gains and MPC in loses. Hence the preference agreeing with M-PSD displays strong risk aversion in gains and strong risk seeking in losses, respectively, which captures the reflection effect observed in the experimental study of (Kahneman and Tversky 1979). For expositional convenience, we call such a preference *prospect strong risk aversion.* Given a value function $u$ and PWFs $(w^-, w^+)$, an individual displays *prospect strong risk aversion* holds if $\Delta_u^w(F, G) \geq 0$ for any risk change from $F$ to $G$ such that $F \succeq_{M-PSD} G$. The following proposition characterizes the curvatures of the value function and PWFs that represent prospect strong risk aversion.

**Proposition 2** (Representation of Prospect Strong Risk Aversion) *A decision maker with an increasing value function $u$ and a pair of increasing PWFs $(w^-, w^+)$ displays prospect strong risk aversion if and only $u \in \mathcal{V}_S$ and $(w^-, w^+) \in \mathcal{W}_{CX} \times \mathcal{W}_{CX}$.*

Proposition 2 shows that prospect strong risk aversion can be represented by $S$-shaped value functions and convex PWFs, which provides a choice-theoretic foundation for CPT. The convexity of PWFs distorts the probability distributions by overweighting smaller values of losses or gains and underweighting larger values of losses and gains, which reflects pessimism in gains and optimism in losses. The more convex the PWFs, the greater the degrees of distortions and more pessimistic (optimistic) in gains (losses).

As an empirical implication for experimental studies, if someone is asked to make choices between all pairs of prospects satisfying the M-PSD order and his choices agree with the M-PSD order, then his/her preference can be represented by a nondecreasing $S$-shaped value function and a pair nondecreasing convex PWFs. Clearly, when rank- and sign-dependence is irrelevant, the CPT preference reduces to expected utility. Hence it follows immediately from Proposition 2 that strong risk aversion under EUT holds if and only if the utility function is concave.

Note that the experimental studies find that individuals may overweight small probabilities and underweight larger probabilities (see, e.g., Tversky and Kahneman 1992, Wu and Gonzalez 1996), which implies that inverse-S-shaped PWFs may better represent the risk preference in CPT. Tversky and Kahneman (1992) summarize that the shapes of the value and the probability weighting functions imply risk averse and risk seeking preferences, respectively, for gains and for losses of moderate or high probability. But the concavity of PWFs in two tails favors risk seeking for small probabilities of gain and risk aversion for small probabilities of loss. It remains open what SD order can be used to characterize such complicated preference structure.

### 3.2.2 Increase in Risk Aversion in CPT

In the EUT paradigm, a decision maker is said to be more risk averse if he has greater value of Arrow-Pratt absolute risk aversion, defined as $R_u = -\frac{u''}{u'}$. Diamond and Stiglitz (1974) define the *mean utility preserving increase in risk* as compensated increase in risk to capture the tradeoff between mean and variance. Specifically, a risk change from $F$ to $G$ is a mean utility preserving increase in risk for a reference decision maker $u$ if $\Delta_u(F, G) = 0$ and $\int_a^x u'(z)(G(z) - F(z))dz \geq 0$ for all $x \in [a, b]$, with at least one strict inequality. They show that all risk averse decision makers $v$ with $R_v \geq R_u$ dislike such a mean utility preserving increase in risk.

In the CPT paradigm, the risk preference can be represented by a reference-dependent value function and a pair of PWFs $(w^-, w^+)$. More specifically, strong prospect risk aversion implies $S$-shaped value function and a pair of convex PWFs.

Hence the changes in risk attitudes may include the changes in both the value function and the PWFs.

First, for the value function, instead of considering the effect of unidirectional change of intensity of risk aversion on preferences, we consider bidirectional preference changes: increase in risk aversion (more risk averse) in gains and decrease in risk aversion (more risk seeking) in losses. Note that the Arrow-Pratt absolute risk aversion is positive (negative) for risk averters (risk seekers). In what follows, we use the absolute value of Arrow-Pratt absolute risk aversion, denoted by $|R_u(x)| = |-u^{''}(x)/u^{'}(x)|$, to measure the degree of prospect strong risk aversion. For two value functions $v, u \in \mathcal{V}_S$, $v$ is said to be more risk averse than $u$ in the CPT paradigm if it is more risk averse (seeking) than $u$ in gains (losses), i.e., $|R_v(x)| \geq |R_u(x)|$ for all $x \in (a, 0) \cup (0, b)$. Let $T_u$ be the transformation from $u$ to $v$ such that $v(x) = T_u(u(x))$, i.e., $T_u(\cdot) = v(u^{-1}(\cdot))$. Then $|R_v(x)| \geq |R_u(x)|$ for $x \in (a, 0) \cup (0, b)$ if and only if $T$ is concave in $(u(a), 0)$ and convex in $(0, u(b))$, i.e., $T_u \in \mathcal{V}_S$.

Second, the prospect strong risk aversion implies that increase in risk aversion can also be represented by the increase in the degrees of convexity of PWFs. Similar to value functions, we can also use the absolute risk aversion to measure the degree of convexity for PWFs. More specifically, for any $(w^-, w^+)$, let $|R_{w^-}(p)| = |-(w^-)^{''}(p)/(w^-)^{'}(p)|$ and $|R_{w^+}(p)| = |-(w^+)^{''}(p)/(w^+)^{'}(p)|$. Then the pair of PWFs $(\tilde{w}^-, \tilde{w}^+)$ are more convex than $(w^-, w^+)$ if and only if $|R_{\tilde{w}^-}(p)| \geq |R_{w^-}(p)|$ and $|R_{\tilde{w}^+}(p)| \geq |R_{w^+}(p)|$. Let $(T_w^-, T_w^+)$ be the nondecreasing transformations from $(w^-, w^+)$ to $(\tilde{w}^-, \tilde{w}^+)$. Then $|R_{\tilde{w}^-}(p)| \geq |R_{w^-}(p)|$ and $|R_{\tilde{w}^+}(p)| \geq |R_{w^+}(p)|$ if and only if $(T_w^-, T_w^+)$ are convex.

In the same spirit of *expected utility preserving increase in risk*, we define *(CPT) value preserving prospect stochastic dominance* (V-PSD) as a compensated increase in risk under CPT.

**Definition 2** (V-PSD) *Prospect $F$ dominates $G$ by V-PSD, denoted by $F \succeq_{V-PSD} G$, for a decision maker $u \in \mathcal{V}_S$ with a pair of PWFs $w = (w^-, w^+)$ if*

*(i)* $\int_a^0 u'(z)[w^-(G(z)) - w^-(F(z))]dz + \int_0^b u'(x)[w^+(\bar{F}(z)) - w^+(\bar{G}(z))]dz = 0,$

*(ii)* $\int_x^0 u'(z)[w^-(G(z)) - w^-(F(z))]dz \geq 0$, *and* $\int_0^y u'(x)[w^+(\bar{F}(z)) - w^+(\bar{G}(z))]dz \geq 0$ *for all $x \leq 0 \leq y$, with at least one strict inequality.*

Note that conditions (i) and (ii) of V-PSD imply $\int_a^0 u'(z)[w^-(G(z)) - w^-(F(z))]dz = \int_0^b u'(x)[w^+(\bar{F}(z)) - w^+(\bar{G}(z))]dz = 0$. Different from PSD, the accumulative advantage of $G$ over $F$ under PWFs is now weighted by $u'$. The equalities holding at $x = a$ and $y = b$ imply that the reference decision maker $u$ with PWFs $(w^-, w^+)$ is indifferent between $F$ and $G$. This definition reduces to mean preserving PSD when PWFs are linear and $u$ is risk neutral, i.e., $u$ is linear. The concavity of $u$ in gains and the convexity of $u$ is losses imply that less weights are imposed on larger values of gains and losses respectively. Hence, the expected value preserving condition implies that relative advantages of $F^w$ over $G^w$ are stretched horizontally towards the reference point, which in turn implies that $F^w$ has a smaller mean than that of $G^w$.

Let $F^{(u,w)}(t) = F^w(u^{-1}(t))$ and $G^{(u,w)}(t) = G^w(u^{-1}(t))$. Clearly, $F^{(u,w)}$ and $G^{(u,w)}$

can be viewed as two weighted prospects with support $[u(a), u(b)]$ and

$$\begin{aligned}\int_x^0 u'(z)[w^-(G(z)) - w^-(F(z))]dz &= \int_{u(x)}^0 [G^{(u,w)}(t) - F^{(u,w)}(t)]dt,\\ \int_0^y u'(x)[w^+(\bar{F}(z)) - w^+(\bar{G}(z))]dz &= \int_0^{u(y)} [G^{(u,w)}(t) - F^{(u,w)}(t)]dt,\end{aligned}$$

which implies that $F \succeq_{V-PSD} G$ is equivalent to $F^{(u,w)} \succeq_{M-PSD} G^{(u,w)}$. The next representation proposition characterizes the change in risk attitude with V-PSD.

**Proposition 3** (Increase in Risk Aversion) *Consider a reference decision maker with value function $u \in \mathcal{V}_S$ and a pair of PWFs $(w^-, w^+) \in \mathcal{W}_{CX} \times \mathcal{W}_{CX}$. Then, for any decision maker with value function $v \in \mathcal{V}_S$ and $\tilde{w} = (\tilde{w}^-, \tilde{w}^+)$, $\Delta_v^{\tilde{w}}(F, G) \geq 0$ for all risk changes from $F$ to $G$ such that $F \succeq_{V-PSD} G$ for the reference decision maker if and only if $|R_v(x)| \geq |R_u(x)|$ for all $x$ and $|R_{\tilde{w}^-}(p)| \geq |R_{w^-}(p)|$ and $|R_{\tilde{w}^+}(p)| \geq |R_{w^+}(p)|$ for all $0 \leq p \leq 1$.*

Proposition 3 shows that the preference towards the compensated increase in risk in CPT, i.e., V-PSD, can be represented by an $S$-shaped transformation for value function (i.e., more risk averse in gains and more risk seeking in losses in the Arrow-Pratt sense than the reference decision maker) and convex transformations for PWFs (more pessimism in gains and more optimism in losses).

## 3.3 Implications for Risk Preference Elicitation in CPT

The choice-theoretic characterizations have important implications on risk preference elicitation. In particular, Proposition 2 shows that the preference agreeing with M-PSD can be represented by an $S$-shaped value function and a pair of convex PWFs, and Proposition 3 shows that the preference towards V-PSD order for a reference decision maker can be represented by an $S$-shaped value function that is more concave in gains and more convex in losses and a pair of more convex PWFs, which provide a linkage between risk changes and risk preferences.

Our results provide a theoretical guideline in preference elicitation experiments. More specifically, one could design pairwise choice experiments according to these SD rules. Although it is impractical to request the subjects to make choices between all possible risky prospect pairs subject to an SD rule so as to fully elicit their risk preferences (if it is possible at all), the choice outcome can still partially reveal the decision maker's preference. For instance, for a given pair of prospects $F$ and $G$ satisfying $F \succeq_{M-PSD} G$, if a subject chooses $F$ ($G$), then according to Proposition 2 he/she is likely (unlikely) to have an $S$-shaped value function and a pair of convex PWFs at the same time.

In the rest of this section, we first discuss the classic risk preference elicitation approach in the literature, and then revisit the experimental studies of Kahneman and Tversky (1979) and Baucells and Heukamp (2006) by examining the risky prospect pairs in their studies using the SD conditions identified in this chapter and provide alternative interpretations of their experimental results.

### 3.3.1 The Certainty Equivalence (CE) Approach

In the literature, two most commonly used preference elicitation approaches are certainty equivalence (CE) and probability equivalence (PE) (Kachelmeier and Shehata 1992, Wakker and Deneffe 1996). Under the CE approach, a subject is asked to make pairwise comparisons of prospects. Each prospect pair consists of a certain prospect $(x, 1)$ and a risky prospect $(m, 1-p; M, p)$ with $m \leq x \leq M$. Given a probability $p$, the outcome $x$ of the certain prospect is varied until the subject is indifferent between the two prospects and the corresponding certain outcome $x$ is the CE of the risky prospect. Different from the CE approach, the PE approach fixes the certain prospect but varies the probability $p$ until the subject is indifferent between the two prospects.

In the EUT paradigm, these approaches can elicit the utility function exactly if an individual follows expected value maximization and if there were no measurement error (Armbruster and Delage 2015). Bleichrodt et al. (2001) attempt to apply the CE approach in the CPT paradigm.

More specifically, when both prospects are restricted to the domain of gains, i.e., $0 \leq m < x < M$, for any given $p$, then CE, denoted by $x^+_{CE}(p)$, satisfies

$$\begin{aligned} u(x) &= w^+(p)u(M) + (1 - w^+(p))u(m) \\ &= w^+(p)\left(u(M) - u(m)\right) + u(m). \end{aligned} \tag{3.5}$$

Since $u$ and $w^+$ are strictly increasing and $u(M) > u(m)$, for any given $p$, there exists a unique value of CE:

$$x^+_{CE}(p) = u^{-1}\left(w^+(p)\left(u(M) - u(m)\right) + u(m)\right).$$

Similarly, when both prospects are restricted to the domain of losses, i.e., $m <$

$x < M \leq 0$, for any given $p$, the CE, denoted by $x^-_{CE}(p)$, satisfies

$$\begin{aligned} u(x) &= w^-(1-p)u(m) + (1 - w^-(1-p))u(M) \\ &= u(M) - w^-(1-p)\left(u(M) - u(m)\right). \end{aligned} \tag{3.6}$$

Since $u$ and $w^-$ are strictly increasing, there exists a unique value of CE,

$$x^-_{CE}(p) = u^{-1}\left(u(M) - w^-(1-p)\left(u(M) - u(m)\right)\right).$$

Bleichrodt et al. (2001) assume that the PWFs are specified as that of (3.1) and that utility function is normalized as $u(m) = 0$ and $u(M) = 1$. Then for each given $p$, given $x^+_{CE}(p)$ or $x^-_{CE}(p)$ obtained from the choice experiment, one obtain $u(x^+_{CE}(p)) = w^+(p)$ or $u(x^-_{CE}(p)) = 1 - w^-(1-p)$ according to (3.5) and (3.6). Clearly, the utility function elicited from their approach depends on the specification of the PWFs. However, a full risk preference elicitation requires eliciting both the value function and the PWFs at the same time.

We next show the implication of prospect strong risk aversion on the curvature of CEs for both losses and gains in the CPT paradigm.

**Proposition 4** **(**Certainty Equivalence**)** *Consider a risky prospect* $(m, 1-p; M, p)$. *Under CPT, if prospect strong risk aversion holds, then* $x^+_{CE}(p)$ *is increasing and convex in p for* $M > m \geq 0$ *and* $x^-_{CE}(p)$ *is increasing and concave in p for* $m < M \leq 0$.

Proposition 4 shows that prospect strong risk aversion implies that the CE of a positive (negative) risky prospect $(m, 1-p; M, p)$ is convex (concave) in $p$, which can be used to examine whether a subject displays prospect strong risk aversion in the experimental study. If $x^+_{CE}(p)$ is increasing and convex in $p$ and $x^-_{CE}(p)$ is increasing and concave in $p$, then this subject may display prospect strong risk aversion; otherwise, prospect strong risk aversion does not hold.

### 3.3.2 The Experimental Study of Kahneman and Tversky (1979)

We now revisit the experimental studies conducted by Kahneman and Tversky (1979). Table 3.1, replicated from Kahneman and Tversky (1979), reports the experimental results of four pairs of choice problems in each of which the preference between positive prospects in the left-hand column and its mirror image of the preference between negative prospects in the right-hand column are tested respectively. In this table, $N$ denotes the number of participants in each experiment, the number in each $[\cdot]$ denotes the percentage of the choice of this prospect, and the gains and losses are measured in Israeli currency. They argue that these experimental results indicate a reflection effect such that the reflection of prospects around zero reverses the preference order: risk aversion in the positive domain is accompanied by risk seeking in the negative domain.

Table 3.1: The choice experiments in Kahneman and Tversky (1979)

| Task | Positive Prospects | | Task | Negative Prospects | |
|---|---|---|---|---|---|
| 3 | $(4,000,.80)$ | $(3,000)$ | 3' | $(-4,000,.80)$ | $(-3,000)$ |
| $(N=95)$ | [20] | [80]* | $(N=95)$ | [92]* | [8] |
| 4 | $(4,000,.20)$ | $(3,000,.25)$ | 4' | $(-4,000,.20)$ | $(-3,000,.25)$ |
| $(N=95)$ | [65]* | [35] | $(N=95)$ | [42] | [58]* |
| 7 | $(3,000,.90)$ | $(6,000,.45)$ | 7' | $(-3,000,.90)$ | $(-6,000,.45)$ |
| $(N=66)$ | [86]* | [14] | $(N=66)$ | [8] | [92]* |
| 8 | $(3,000,.002)$ | $(6,000,.001)$ | 8' | $(-3,000,.002)$ | $(-6,000,.001)$ |
| $(N=66)$ | [27] | [73]* | $(N=66)$ | [70] | [30]* |

We next examine the SD relationships between the pairs of prospects. For convenience, let $F_i$ and $G_i$ denote the first and second prospects in problem $i$. The results and their verifications are summarized in Table 3.2 and the appendix respectively. Note that the introduction of M-MSD and its choice-theoretic characterization are provided in Section 3.5.

Table 3.2: The SD relationships

| Task | Majority | Minority | Task | Majority | Minority |
|---|---|---|---|---|---|
| 3 | $G_3 \succeq_{V-PSD} F_3$ | $F_3 \succeq_{MSD} G_3$ | 3' | $F_{3'} \succeq_{V-PSD} G_{3'}$ | $G_{3'} \succeq_{MSD} F_{3'}$ |
| 4 | $F_4 \succeq_{MSD} G_4$ | $G_4 \succeq_{V-PSD} F_4$ | 4' | $G_{4'} \succeq_{MSD} F_{4'}$ | $F_{4'} \succeq_{V-PSD} G_{4'}$ |
| 7 | $F_7 \succeq_{M-PSD} G_7$ | $G_7 \succeq_{M-MSD} F_7$ | 7' | $G_{7'} \succeq_{M-PSD} F_{7'}$ | $F_{7'} \succeq_{M-MSD} G_{7'}$ |
| 8 | $G_8 \succeq_{M-MSD} F_8$ | $F_8 \succeq_{M-PSD} G_8$ | 8' | $F_{8'} \succeq_{M-MSD} G_{8'}$ | $G_{8'} \succeq_{M-PSD} F_{8'}$ |

In task 3, the majority of the subjects preferred the sure gain of 3000 to a risk of .80 to gain 4000, though the latter has a greater mean. We show that $G_3$ dominates $F_3$ according to V-PSD with respect to a reference decision maker with an $S$-shaped value function as specified in (3.1) with $\alpha = \beta = \log 0.8/\log 0.75$ and linear PWFs. Hence, the majority of the subjects may have a greater degree of risk aversion than reference one for gains. For the minority of the subjects, $F_3 \succeq_{MSD} G_3$, which implies that they may be risk seeking in gains.

In task 3', the preference order is reversed for losses of the same magnitudes such that $F_{3'} \succeq_{V-PSD} G_{3'}$ with respect to the same reference decision maker, which implies that most subjects may be more risk seeking than the reference one for losses, while $G_{3'} \succeq_{MSD} F_{3'}$ implies that the minority of the subjects may be risk averse in losses.

Similarly, for tasks 4 and 4', we can show that $F_4 \succeq_{MSD} G_4$, $G_{4'} \succeq_{MSD} F_{4'}$, and $G_4 \succeq_{VSD} F_4$, $F_{4'} \succeq_{VSD} G_{4'}$ with respect the same reference decision maker as that of task 3.

In tasks 7-8', all the prospect pairs have equal means and we show that either M-PSD or M-MSD holds for each pair. For instance, in task 7, the majority of subjects prefer $F_7$ to $G_7$. Their preferences may agree with the M-PSD rule (i.e., prospect strong risk aversion) and hence can be represented by an $S$-shaped value function and a pair of convex PWFs. The preferences of the minority of subjects may agree with the M-MSD rule and the curvatures of value function and PWFs are inverse $S$-shaped and concave respectively.

**Remark 1** *Note that the choice experiments in tasks 3 and 4 were conducted for the same group of subjects. Kahneman and Tversky (1979) argue that the experimental results contradict EUT: a decision maker u prefers $G_3$ to $F_3$ if and only if* $0.8u(4000) < u(3000)$ *and prefers $F_4$ to $G_4$ if and only if* $0.2u(4000) > 0.25u(3000)$ *(i.e.,* $0.8u(4000) > u(3000)$*). In a CPT paradigm with $u \in \mathcal{V}_S$ and PWFs $(w^-, w^+)$, the results of the two choice experiments imply* $(1-w^+(1-0.8))u(4000) < u(3000)$ *and* $(1-w^+(1-0.2))u(4000) > (1-w^+(1-0.25))u(3000)$*, respectively, which implies that*

$$1 - w^+(0.2) < \frac{u(3000)}{u(4000)} < \frac{1 - w^+(0.8)}{1 - w^+(0.75)}.$$

*One can verify that the inequalities hold when the PWFs are specified as that of (3.1) with* $\gamma = 0.3$.

### 3.3.3 The Experimental Study of Baucells and Heukamp (2006)

Baucells and Heukamp (2006) conduct an experiment study to test different features of CPT, namely S-shaped value functions, inverse-S-shaped PWFs and loss aversion. The setups and results of their experiments are summarized in and Table 3.4. We refer to Baucells and Heukamp (2006) for their definitions of SW-SD, WL-SD, S*W-SD, P*W-SD and PWL-SD rules.

Table 3.3: Experiments of Baucells and Heukamp (2006)

| Task | Prospect $F$ | Prospect $G$ |
|---|---|---|
| I<br>$N = 277$ | $(0, .1; 1000, .4; 2000, .4; 3000, .1)$<br>[74] | $(0, .5; 3000, .5)$<br>[26] |
| II<br>$N = 273$ | $(-3000, .5; 0, .5)$<br>[65] | $(-3000, .1; -2000, .4; -1000, .4; 0, .1)$<br>[35] |
| III<br>$N = 276$ | $(-6000, 1/3; 3000, 1/2; 4500, 1/6)$<br>[76] | $(-6000, 1/6; -3000, 1/3; 4500, 1/2)$<br>[24] |
| IV<br>$N = 273$ | $(-6000, .26; 3000, .72; 4500, .02)$<br>[81] | $(-6000, .02; -3000, .48; 4500, 0.5)$<br>[19] |
| V<br>$N = 271$ | $(-6000, .25; 3000, .75)$<br>[37] | $(-3000, .5; 4500, .5)$<br>[63] |
| VI<br>$N = 216$ | $(-1000, .1; 0, .8; 1000, .1)$<br>[43] | $(-1000, .5; 1000, .5)$<br>[57] |
| VII<br>$N = 208$ | $(-3000, .2; 0, .6; 3000, .2)$<br>[61] | $(-3000, .5; 3000, .5)$<br>[39] |
| VIII<br>$N = 208$ | $(-3000, .2; -1000, .3; 1000, .3; 3000, .2)$<br>[64] | $(-3000, .5; 3000, .5)$<br>[36] |
| IX<br>$N = 209$ | $(-500, .1; 0, .4; 1500, .4; 2000, .1)$<br>[46] | $(-500, .3; 500, .2; 1000, .2; 2000, .3)$<br>[64] |
| X<br>$N = 216$ | $(-2000, .3; 0, .6; 1000, .1)$<br>[70] | $(-2000, .1; -1000, .6; 1000, .3)$<br>[30] |
| XI<br>$N = 209$ | $(-5000, .15; -3000, .3; 0, .2; 3000, .2; 5000, .15)$<br>[46] | $(-5000, .35; -1000, .3; 5000, .35)$<br>[64] |
| XII<br>$N = 177$ | $(-1500, .2; 1500, .6; 4500, .2)$<br>[77] | $(-1500, .5; 4500, .5)$<br>[23] |

More specifically, tasks I-III test the shape of the value function, assuming that the PWFs are inverse $S$-shaped. In fact, the assumption of inverse $S$-shaped PWFs implies that the PWFs must be linear for different cumulative probabilities of the pair of prospects. Hence, essentially, prospect $F$ dominates prospect $G$ according to SSD, RSD and PSD in Tasks I, II and III, respectively. They therefore argue that the value function cannot be convex in gains and concave in losses (under the aforementioned assumption on PWFs) and the $S$-shaped value function remains as a specification that is consistent with the preferences of the majority of the subjects. Similarly, Tasks IV and V test the shape of PWFs, assuming the value function is $S$-shaped (which, again, implies that the PWFs must be linear when the cumulative probabilities under comparison are different); Tasks VI to VIII test loss aversion, assuming the value function is $S$-shaped and the PWFs are inverse $S$-shaped; and Tasks IX to XII test whether the preferences of the subjects are consistent with the predictions of CPT, assuming all empirical specifications of CPT, namely, $S$-shaped value functions, inverse-shaped PWFs and loss aversion, hold. Again, in all these experiments, the prospects under comparisons and the assumption of inverse-shaped PWFs imply that the PWFs must be linear when the underlying cumulative probabilities are different. Their result, however, is not sufficient to elicit preferences. For example, for Task I, rejecting convexity of the value function for gains does not imply that the value function has to be concave in gains, although preferences under concave value functions agree with the dominance relationship between two prospects.

Observing that the prospects in each pair have an equal mean, which motivates us to examine the relationships of these prospect pairs using the notions introduced in this chapter. It turns out that all pairs of prospects in their experiments follow the M-PSD order if the reference points are chosen properly (see Table 3.4). According to Proposition 1, a subject agreeing with the M-PSD order displays the prospect strong risk aversion which can be represented by an $S$-shaped value function and a pair of convex PWFs. Hence, in their experiments, the preferences of those subjects who consistently made choices that are in line with the corresponding M-PSD orders may be described as prospect strong risk aversion. Hence, to further elicit their curvature forms of value functions and PWFs using the classic approaches (i.e., the CE approach), one could focus on the classes of $S$-shaped value functions and convex PWFs.

Table 3.4: SD Rules of Baucells and Heukamp (2006) and M-PSD/M-MSD Rules

| Task | SD Orders (B&H 2006) | Alternative SD Orders |
|---|---|---|
| I | $F \succeq_{SW-SD} G$, $G \succeq_{S^*W-SD} F$ | $F \succeq_{M-PSD} G$, $G \succeq_{M-MSD} F$ $(r=0)$ |
| II | $F \succeq_{S^*W-SD} G$, $F \succeq_{SW-SD} G$ | $F \succeq_{M-PSD} G$, $G \succeq_{M-MSD} F$ $(r=0)$ |
| III | $F \succeq_{PW-SD} G$, $G \succeq_{P^*W-SD} F$ | $F \succeq_{M-PSD} G$, $G \succeq_{M-MSD} F$ $(r=0)$ |
| IV | $F \succeq_{PW-SD} G$, $G \succeq_{P^*W-SD} F$ | $F \succeq_{M-PSD} G$, $G \succeq_{M-MSD} F$ $(r=0)$ |
| V | $F \succeq_{PW-SD} G$, $G \succeq_{P^*W-SD} F$ | $F \succeq_{M-PSD} G$, $G \succeq_{M-MSD} F$ $(r=0)$ |
| VI | $F \succeq_{WL-SD} G$ | $F \succeq_{M-PSD} G$, $G \succeq_{M-MSD} F$ $(r=-1000)$ |
| VII | $F \succeq_{WL-SD} G$ | $F \succeq_{M-PSD} G$, $G \succeq_{M-MSD} F$ $(r=-3000)$ |
| VIII | $F \succeq_{WL-SD} G$ | $F \succeq_{M-PSD} G$, $G \succeq_{M-MSD} F$ $(r=-3000)$ |
| IX | $F \succeq_{PWL-SD} G$ | $F \succeq_{M-PSD} G$, $G \succeq_{M-MSD} F$ $(r=-500)$ |
| X | $F \succeq_{PWL-SD} G$ | $F \succeq_{M-PSD} G$, $G \succeq_{M-MSD} F$ $(r=-500)$ |
| XI | $F \succeq_{PWL-SD} G$ | $F \succeq_{M-PSD} G$, $G \succeq_{M-MSD} F$ $(r=-5000)$ |
| XII | $F \succeq_{PWL-SD} G$ | $F \succeq_{M-PSD} G$, $G \succeq_{M-MSD} F$ $(r=-1500)$ |

We next provide an example in which a decision maker with an S-shaped value function and a pair of convex PWFs prefers $F$ to $G$ such that $F \succeq_{PW-SD} G$.

**Example 1** *Consider two prospects* $F : (-2000, 0.4; 1500, 0.5; 2500, 0.1)$ *and* $G : (-2500, 0.1; -1500, 0.5; 2000, 0.4)$. *Note that* $x_c^- = -2000$ *and* $x_c^+ = 2000$ *for* $c = 0.1$, *and* $G(x) \geq F(x)$ *for all* $x < x_c^-$, *and* $G(x) \geq F(x)$ *for all* $x > x_c^+$. *We next verify that* $F \succeq_{PSD} G$:

$$\int_x^0 [G(z) - F(z)]dz = \begin{cases} -0.2x \geq 0 & \text{for } x \in [-1500, 0], \\ 300 - 0.3(x + 1500) \geq 0 & \text{for } x \in [-2000, -1500), \\ 150 + 0.1(x + 2000) \geq 0 & \text{for } x \in [-2500, -2000), \end{cases}$$

*and*

$$\int_0^y [G(z) - F(z)]dz = \begin{cases} 0.2x \geq 0 & \text{for } x \in [0, 1500], \\ 300 - 0.3(x - 1500) \geq 0 & \text{for } x \in (1500, 2000], \\ 150 + 0.1(x - 2000) \geq 0 & \text{for } x \in (2000, 2500]. \end{cases}$$

*Then* $F \succeq_{PW-SD} G$. *Consider a decision maker with an S-shaped value function*

*$v$ specified as (3.1) with $\alpha = \beta = 0.5$, and a pair of convex PWFs specified as follows:*

$$w^-(p) = \frac{1.5p}{1.5p + (1-p)}, \text{ and } w^+(p) = \frac{1.5p}{1.5p + (1-p)}.$$

*We can show that the decision maker prefers $F$ to $G$:*

$$\begin{aligned} \Delta_v^w(F,G) &= \left(-0.5\sqrt{2000} + 0.55\sqrt{1500} + 0.14\sqrt{2500}\right) \\ &- \left(0.14\sqrt{2000} - 0.55\sqrt{1500} + 0.5\sqrt{2000}\right) > 0. \end{aligned}$$

## 3.4 Implications for Optimal Decision Problems in CPT

To explore the economic implications of increased risk under CPT, i.e., M-PSD, on optimal decisions, we follow Diamond and Stiglitz (1974) to consider a utility function $U(x, \theta)$ which depends on a return value of $x$ and a control variable $\theta \in [0, 1]$, and a pair of PWFs $(w^-, w^+)$ which satisfy the regularity properties of CPT. Assume that $U(x, \theta)$ is continuously differentiable in $\theta$.

Given a prospect $F$, the decision maker aims to solve the following optimization problem:

$$\max_{\theta \in [0,1]} \psi_F(\theta) \triangleq \int_a^0 U(x,\theta) dw^-(F(x)) + \int_0^b U(x,\theta) d[1 - w^+(\bar{F}(x))]. \quad (3.7)$$

Let $\arg\max_{\theta \in [0,1]} \psi_F(\theta)$ be the set of maximizers for $\psi_F(\theta)$ for any given $\gamma$ and prospect $F$. We are interested to relate the optimal solution to increases in risk in the sense of PSD or M-PSD.

The next theorem identifies sufficient conditions of $U(x, \theta)$ under which an increase in risk in the sense of PSD or M-PSD leads to smaller optimal solutions.

**Proposition 5** (Monotone Comparative Statics) *Consider two prospects $F$ and $G$ and assume that $U_\theta(\cdot,\theta) \in \mathcal{V}_S$ with a pair of PWFs $w = (w^-, w^+)$. Then $\arg\max_{\theta\in[0,1]} \psi_F(\theta) \geq \arg\max_{\theta\in[0,1]} \psi_G(\theta)$ if (i) a risk change from $F^w$ to $G^w$ is PSD, or (ii) a risk change from $F$ to $G$ is M-PSD and the PWFs $(w^+, w^-) \in \mathcal{W}_{CX} \times \mathcal{W}_{CX}$.*

It is notable that to obtain the monotonicity result, it suffices to show that $\psi_F(\theta) - \psi_G(\theta)$ is a decreasing function of $\theta$, or, equivalently, the objective function is submodular in the control variable $\theta$ and the level of risk in the sense of M-PSD (see, e.g., Topkis 1998).

We next apply this result to a portfolio choice problem.

**Example 2** (Portfolio Choice) *Consider a portfolio choice problem in a market consisting of one risky asset (e.g., stock) and one risk-free asset (e.g., bond) with an investment planning horizon from date 0 to date $T$. The return rate of the risk-free asset over the investment period is deterministic, denoted by $\tau$. We follow the literature and assume that the reference point of the investor is equal to the risk-free return rate $\tau$ (see, e.g., Barberis and Xiong 2009, He and Zhou 2011a, Li and Yang 2013, and Meng and Weng 2017). The return rate of the risky asset is random, denoted by $R$. Let $\theta \in [0,1]$ be the percentage of the wealth invested in the stock and the reminder in the risk free asset. Then the final normalized wealth is $\Pi(\theta) = (1-\theta)\tau + \theta R$. For convenience, let $X = R - \tau$ and we normalize the initial wealth level to one. The stock total excess return $R - \tau$ is a random variable with distribution function $F$. Then we have $\Pi(X,\theta) = \tau + \theta X$. Let $\theta \in [0,1]$ be the percentage of the wealth invested in the stock and the remainder in the risk-free asset. Then the relative gain or loss is $\theta X$. Suppose the distribution function of $X$ is $F$. Then the portfolio choice problem under CPT preference for $u \in \mathcal{V}_S$ is, therefore,*

$$\max_{\theta\in[0,1]} E[u(\theta X)]. \tag{3.8}$$

*Consider the portfolio choice problem (4.4) and the $S$-shaped value function spec-*

*ified in (3.1). We have*

$$U(x,\theta) = u(\theta x) = \begin{cases} \theta^\alpha x^\alpha, & \text{if } x > 0, \\ -\lambda\theta^\beta(-x)^\beta, & \text{if } x < 0. \end{cases}$$

*where* $0 < \alpha, \beta < 1$, $\lambda > 0$. *Then we have*

$$U_{\theta x} = \begin{cases} \alpha^2\theta^{\alpha-1}x^{\alpha-1}, & \text{if } x > 0, \\ \lambda\beta^2\theta^{\beta-1}(-x)^{\beta-1}, & \text{if } x < 0, \end{cases} \quad \text{and} \quad U_{\theta xx} = \begin{cases} -\alpha^2(1-\alpha)\theta^{\alpha-1}x^{\alpha-2}, & \text{if } x > 0, \\ \lambda\beta^2(1-\beta)\theta^{\beta-1}(-x)^{\beta-2}, & \text{if } x < 0. \end{cases}$$

*Clearly,* $U$ *is thrice differentiable with respect to* $\theta$ *and* $x$, $U_{\theta x} > 0$ *for all* $x$, $U_{\theta xx} > 0$ *for* $x < 0$ *and* $U_{\theta xx} < 0$ *for* $x > 0$, *i.e.,* $U_\theta \in \mathcal{V}_S$. *According to Proposition 5, under condition (1) or (2), an increase in risk in the sense of PSD or M-PSD leads to smaller values of optimal portfolio decision* $\theta$ *or less riskier portfolio choices.*

## 3.5 Extensions

### 3.5.1 Reference Points

In the preceding analysis, the reference point is normalized to zero. We now consider general reference points for PSD with and without PWFs.

For two prospects $F$ and $G$, let $\phi(x) = \int_a^x [G(z) - F(z)]dz$. Clearly, according to the definition of PSD, $F \succeq_{PSD} G$ with respect to a reference point $r$ if $\int_x^y [G(z) - F(z)]dz \geq 0$, or, equivalently, $\phi(y) \geq \phi(x)$ for all $x \leq r \leq y$, with at least one strict inequality. The definition of PSD implies that the function $\phi(x)$ crosses over the horizontal line at $\phi(r)$ at most once from below.

Figure 3.1 provides an illustrative graph for the range of possible reference points. Suppose that there exists a reference point $r_0$ such that $F \succeq_{PSD} G$ with respect to $r_0$. Let $\underline{r} = \inf\{x \leq r_0 : \phi(x') \leq \phi(x) \leq \phi(x'')\ \forall x' \in [a, x],\ \forall x'' \in [x, r_0]\}$ and

$\bar{r} = \sup\{x \geq r_0 : \phi(x') \leq \phi(x) \leq \phi(x'')\ \forall x' \in [r_0, x],\ \forall x'' \in [x, b]\}$. Suppose that $\underline{r}$ and $\bar{r}$ are well-defined. Then $F \succeq_{PSD} G$ with respect to any $r \in [\underline{r}, \bar{r}]$. Such an observation provides a way to identify reference points at which the risk change from $F$ to $G$ is PSD. The similar analysis applies to full CPT preference with PWFs.

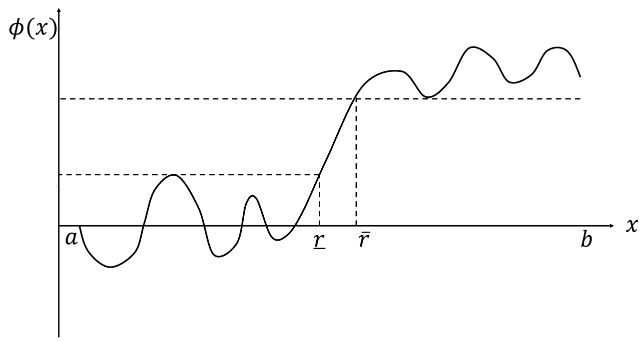


Figure 3.1: An Example of General PSD

The following proposition shows that PSD is implied by RSD under some conditions if the reference point is chosen properly.

**Proposition 6** (SSD AND PSD) *Consider two prospects $F$ and $G$ and a pair of PWFs $(w^-(\cdot), w^+(\cdot))$. Suppose $G$ crosses $F$ from below exactly once at $x_0$ and $F \succeq_{RSD} G$. Then*

*(i) there exists $\underline{r} \in [x_0, b]$ such that $F \succeq_{PSD} G$ for all $r \in [\underline{r}, b]$; and furthermore*

*(ii) if the PWFs are convex, there exists $\underline{r}' \leq \underline{r}$ such that $F^w \succeq_{PSD} G^w$ for all $r \in [\underline{r}', b]$.*

### 3.5.2 Markowitz Stochastic Dominance and Inverse S-Shaped Value Functions

Markowitz (1952) proposes that decisions are based on *change* in wealth and individuals are risk averse for losses and risk seeking for gains as long as the possible outcomes are not very extreme, i.e., value functions are inverse $S$-shaped (concave in the losses and convex in the gains). For extreme outcomes, individuals are risk seeking for losses and risk averse for gains. Levy and Levy (2002) introduce the concept of Markowitz Stochastic Dominance (MSD) for inverse $S$-shaped functions. More specifically, $F$ dominates $G$ by MSD if $\int_a^x [G(z) - F(z)]dx \geq 0$ for all $x \leq 0$ and $\int_x^b [G(z) - F(z)]dz \geq 0$ for all $x \geq 0$ with at least one strict inequality. They show that $F$ dominates $G$ by MSD if and only if all $u \in \mathcal{V}_M$ prefer $F$ to $G$. In what follows, we show that the choice-theoretic characterizations for CPT can be readily extended to that with inverse $S$-shaped value functions.

We first extend the definitions of M-PSD and V-PSD for inverse $S$-shaped value functions. Similar to M-PSD, to characterize the preferences with strong risk aversion (strong risk seeking) in losses (gains), we can define the SD condition of mean-preserving MSD as follows.

**Definition 3 (Mean Preserving MSD)** *Prospect $F$ dominates prospect $G$ by* mean preserving MSD*, denoted by $F \succeq_{M-MSD} G$, if (i) $\int_a^b [G(z) - F(z)]dz = 0$ and (ii) $\int_a^x [G(z) - F(z)]dz \geq 0$ for all $x \in [a, 0)$ and $\int_y^b [G(z) - F(z)]dz \geq 0$ for all $y \in (0, b]$, with at least one strict inequality.*

The following lemma provides a linkage between M-PSD and M-MSD.

**Lemma 3** (M-PSD AND M-MSD) *Consider two prospects $F$ and $G$. Then $F \succeq_{M-PSD} G$ if and only if $G \succeq_{M-MSD} F$.*

Note that in general $F \succeq_{PSD} G$ does not necessarily imply $G \succeq_{MSD} F$ and vice versa. Lemma 3 shows that under the mean preserving condition, $F \succeq_{PSD} G$ is equivalent to $G \succeq_{MSD} F$.

We can also provide a choice-theoretic characterization for M-MSD when considering PWFs as that in the CPT paradigm. The following theorem shows that, different from M-PSD, the preference towards M-MSD can be characterized by inverse $S$-shaped value functions and concave PWFs.

**Proposition 7** (Preference Representation of M-MSD) *A decision maker with an increasing value function $u$ and a pair of increasing PWFs $(w^-, w^+)$ displays a preference that agrees with M-MSD orders if and only if $u \in \mathcal{V}_M$ and PWFs $(w^-, w^+) \in \mathcal{W}_{CV} \times \mathcal{W}_{CV}$, where $\mathcal{W}_{CV}$ is the set of concave PWF pairs.*

Similar to the choice-theoretic characterization for $S$-shaped value functions in Proposition 3, the following representation proposition characterizes the change in risk attitude.

**Proposition 8** (Changes in Risk Attitudes) *Consider a reference decision maker with value function $u \in \mathcal{V}_M$ and a pair of PWFs $w = (w^-, w^+) \in \mathcal{W}_{CV} \times \mathcal{W}_{CV}$. Then, for any decision maker with value function $v \in \mathcal{V}_M$ and $\tilde{w} = (\tilde{w}^-, \tilde{w}^+)$, $\Delta_v^{\tilde{w}} \geq 0$ for all risk changes from $F$ to $G$ such that $F^{(u,w)} \succeq_{M-MSD} G^{(u,w)}$ for the reference decision maker if and only if $|R_v(x)| \geq |R_u(x)|$ for all $x$ and $|R_{\tilde{w}^-}(p)| \geq |R_{w^-}(p)|$ and $|R_{\tilde{w}^+}(p)| \geq |R_{w^+}(p)|$ for all $0 \leq p \leq 1$.*

Proposition 8 shows that the changes in risk attitudes (e.g., more risk seeking in gains and more risk averse in losses) can be characterized by the preferences towards M-MSD.

### 3.5.3 Accounting for Inverse S-shaped PWFs

Note that based on their experimental results Tversky and Kahneman (1992) and Wu and Gonzalez (1996) argue that decision makers are risk averse and risk seeking, respectively, for gains and losses of moderate or high (cumulative) probabilities, and that the PWFs are concave in (and hence overweight) small (cumulative) probabilities of the lower tail of losses and the upper tail of gains, which favors risk seeking for extreme gains and risk aversion for extreme losses. Although the S-shaped value function and convex PWFs can characterize the risk aversion in gains and risk seeking in losses, they are not able to accommodate the risk seeking and risk averse preferences for the extreme gains and losses, respectively. Our preceding analysis implies that concavity of value functions and convexity of PWFs are paired (or conjugate) for risk aversion in gains and that convexity of value functions and convexity of PWFs are paired for risk seeking in losses.

To accommodate the risk preferences in both tails, we next generalize the PSD and M-PSD orders. To this end, let $\mathcal{V}_P^{c,d}$ be the class of value functions that are increasing concave in $(a, c)$ and $(0, d)$, and increasing convex in $(c, 0)$ and $(d, b)$, respectively, where $c$ and $d$ are the inflection points satisfying $a \leq c \leq 0 \leq d \leq b$. Denote by $\mathcal{W}_{IS}^p$ the class of inverse S-shaped PWFs that are increasing concave in $(0, p)$ and increasing convex in $(p, 1)$ for some inflation point $p \in (0, 1)$.

We first introduce the definition of *generalized prospect stochastic dominance* (GPSD).

**Definition 4 (GPSD)** *Prospect $F$ dominates $G$ by GPSD, denoted by $F \succeq_{GPSD} G$, if for some inflection points $c \in [a, 0]$ and $d \in [0, b]$ we have (i) $\int_a^x [G(z) - F(z)]dz \geq 0$ for all $x \in [a, c)$, (ii)$\int_x^y [G(z) - F(z)]dz \geq 0$ for all $x \in [c, 0], y \in [0, d]$, and (iii)$\int_y^b [G(z) - F(z)]dz \geq 0$ for all $y \in (d, b]$.*

The stochastic change defined by GPSD can be seen as a combination of PSD with SSD in the lower tail and RSD in the upper tail. It is different from the PW-SD of Baucells and Heukamp (2006) that requires FSD in both tails. In the following analysis, we will show that GPSD can be used to accommodate the risk aversion and risk seeking preferences in the lower tail of losses and the upper tail of gains, respectively. To this end, we first generalize Lemma 2 to link GPSD and inverse S-shaped PWFs.

**Lemma 4** *Prospect $F$ dominates $G$ by GPSD with inflection points $c \in [a, 0]$ and $d \in [0, b]$ if and only if $F^w$ dominates $G^w$ by GPSD for all PWFs $(w^-, w^+) \in \mathcal{W}_{IS}^{p_c} \times \mathcal{W}_{IS}^{p_d}$, where $F(c) = G(c) = p_c$ and $\bar{F}(d) = \bar{G}(d) = p_d$.*

The following proposition generalizes Proposition 1 to characterize a risk change of GPSD with the class of value functions, $\mathcal{V}_P^{c,d}$, and the class of PWFs, $\mathcal{W}_{IS}^{p_c}$ and $\mathcal{W}_{IS}^{p_c}$, respectively.

**Proposition 9** (GPSD) *Prospect $F$ dominates $G$ by GPSD, i.e., $F \succeq_{GPSD} G$, with inflection points $c \in [a, 0]$ and $d \in [0, b]$ if and only if $\Delta_u^w(F, G) \geq 0$ for all $u \in \mathcal{V}_P^{c,d}$ and $(w^-, w^+) \in \mathcal{W}_{IS}^{p_c} \times \mathcal{W}_{IS}^{p_d}$.*

We next define *mean-preserving generalized prospect stochastic dominance* (M-GPSD).

**Definition 5 (M-GPSD)** *Prospect $F$ dominates $G$ by M-GPSD, denoted by $F \succeq_{M-GPSD} G$, if $\int_a^b [G(z) - F(z)]dz = 0$ and $F \succeq_{GPSD} G$ with inflection points $c \in [a, 0]$ and $d \in [0, b]$.*

Note that the mean preserving condition and the inequalities conditions of GPSD imply that

$$\begin{aligned}
\int_a^c [G(z) - F(z)]dz &= \int_c^0 [G(z) - F(z)]dz \\
&= \int_0^d [G(z) - F(z)]dz \\
&= \int_d^b [G(z) - F(z)]dz = 0. \qquad (3.9)
\end{aligned}$$

We next present the preference representation proposition for M-GPSD.

**Proposition 10** (Preference Representation of M-GPSD) *For any decision maker with an increasing value function $u$ and a pair of increasing PWFs $w = (w^-, w^+)$, $\Delta_u^w(F, G) \geq 0$ for any risk change from $F$ to $G$ such that $F \succeq_{M-GPSD} G$ with inflection points $c \in [a, 0]$ and $d \in [0, b]$ and $F(c) = G(c) = p_c$, $\bar{F}(d) = \bar{G}(d) = p_d$ if and only if $u \in \mathcal{V}_P^{c,d}$ and $(w^-, w^+) \in \mathcal{W}_{IS}^{p_c} \times \mathcal{W}_{IS}^{p_d}$.*

Proposition 10 shows that the preference towards M-GPSD order can be represented by the value functions in $\mathcal{V}_P^{c,d}$ and the PWFs in $\mathcal{W}_{IS}^{p_c} \times \mathcal{W}_{IS}^{p_d}$. Such a curvature representation allows us to accommodate the inverse S-shaped PWFs that favors risk aversion in the lower tail of losses and risk seeking in the upper tail of gains, and to this end we have to generalize the S-shaped curvature form of the value function to allow it to be concave in the corresponding lower tail of losses and convex in the corresponding upper tail of gains.

## 3.6 Conclusion

This chapter provides choice-theoretic characterizations for increases in risk and in risk aversion in the paradigm of CPT. We first generalize the notions of increase in risk (mean-preserving spread) and strong risk aversion introduced by Rothschild and Stiglitz (1970) to the CPT paradigm by introducing the SD rule of mean-preserving prospect stochastic dominance (M-PSD). M-PSD consists of MPC in losses and MPS in gains. Hence, we define the preference agreeing with the M-PSD order as strong prospect risk aversion. We show that strong prospect risk aversion can be represented by an S-shaped value function and a pair of convex PWFs. To further characterize the increase in risk aversion in the CPT paradigm, we extend the SD rule of mean utility preserving increase in risk introduced by Diamond and Stiglitz (1974) to the CPT paradigm by introducing the SD rule of CPT-value preserving increase in risk (V-PSD). V-PSD serves as a compensated adjustment to M-PSD to address the tradeoff between means and variances in gains and domain of losses respectively. We show that the preference towards V-PSD can be used to characterize the increase in the absolute value of the absolute risk aversion (more risk averse/seeking in gains/gains) and the increase of convexity of PWFs. We further extend our analysis to account for general reference points, inverse S-shaped value functions and inverse S-shaped PWFs, respectively.

We also explore the implications of our choice-theoretic characterizations on risk preference elicitation in the CPT paradigm. We first discuss classic preference elicitation approaches (e.g., certainty equivalence) and then revisit the experimental studies of Kahneman and Tversky (1979) and Baucells and Heukamp (2006). We examine the prospect pairs in their studies using the SD rules introduced in our paper and provide alternative interpretations of the experimental results.

We also show that these SD rules can also provide economic implications for optimal decision problems in the CPT paradigm. Using portfolio choice as an example, we show that for a CPT-value maximizing investor will invest less in risky assets in the presence of increases in risk in the sense of M-PSD or PSD.

# Chapter 4

# Downside Risk in Cumulative Prospect Theory

## 4.1 Introduction

This chapter aims to generalize the choice-theoretic characterizations of downside risk aversion and increase in downside risk aversion from the expected utility theory (EUT) paradigm to the cumulative prospect theory (CPT) paradigm.

In modern portfolio theory that was built upon von Neuman-Morgenstern EUT, investment decisions focus on the tradeoffs between risks and returns of portfolios. The Markowitz (1952a) mean-variance decision rule, using variance as a proxy for the overall riskiness of a portfolio, is by far the most popular investment decision rule that has been widely adopted by both academics and practitioners (Levy (2016)). However, it has long been criticized that variance is not consistent with investors' actual perception of risk as it allocates weights on (undesirable) negative deviations and (desirable) positive deviations of returns from their means evenly. It has been argued that downside risk is a more appropriate risk measure because investors are more concerned about loss below the target return (Markowitz 1959). Various downside risk measures have been proposed; see, e.g., semi-variance (Markowitz 1959), lower partial moment (Fishburn 1977), value-at-risk (VaR) and conditional value-at-risk (CVaR) (Rockafellar and Uryasev 2002). Empirical studies also show that individuals may be willing to accept a lower expected return or a higher level of variance if the distribution of the return is more skewed to the right (smaller downside risk) (e.g., Harvey and Siddique 2000 and Garrett and Sobel 1999).

*Stochastic dominance* (SD) provides an alternative decision rule based on partial orders of stochastic changes between distributions. The first-degree stochastic dominance (FSD) and second-degree stochastic dominance (SSD) were proposed by Hadar and Russell (1969), Hanoch and Levy (1969) and Rothschild and Stiglitz (1970) to characterize the preferences of individuals with nondecreasing utility functions (choice rationality) and nondecreasing concave utility functions (for risk aversion), respectively. The third-degree stochastic dominance (TSD) was defined by Whitmore (1970) for nondecreasing and concave utility functions with positive third derivatives.

Menezes et al. (1980) introduce the increase in downside risk to describe the risk change that shifts probability to the lower tail (negative skewness) while preserving mean and variance, and use the positive third derivative of the VNM utility function to indicate downside risk aversion. Note that this type of utility functions can also be used to interpret the precautionary saving motive (to save for future uncertainty) in consumption theory (Leland 1968). Generalizing the

Arrow-Pratt measure of absolute risk aversion (defined as $R_u = -u''/u'$), Kimball (1990) introduces the notion of prudence measure (defined as $P_u = -u'''/u''$) to characterize the strength of precautionary saving motive. Following Diamond and Stiglitz (1974), Chiu (2005) identifies the SD condition of mean utility preserving increase in downside risk to address the tradeoff between risk and downside risk, using prudence to measure the intensity of downside risk aversion. This chapter aims to generalize the choice-theoretic characterizations of downside risk aversion and increase in downside risk aversion from EUT to the CPT.

To bridge EUT and CPT, a few studies employ SD approach to provide a choice-theoretic foundation for CPT by addressing the key curvature properties of CPT. Levy and Wiener (1998) generalize the second-degree stochastic dominance condition to the prospect stochastic dominance (PSD) condition for all S-shaped value functions and S-shaped probability weighting functions. Levy and Levy (2002) introduce the Markowitz stochastic dominance (MSD) condition for all inverse S-shaped value functions. Baucells and Heukamp (2006) further extend the PSD condition to accommodate inverse S-shaped probability weighting functions and loss aversion. To the best of our knowledge, there has so far no attempt to provide a general choice-theoretic characterization of the tradeoff between overall risk and downside risk for CPT. This chapter aims to fill this gap.

More specifically, we first identify generalised mean and variance preserving increase in downside risk defined by Menezes et al. (1980), which allows us to characterize downside risk aversion in the CPT paradigm. We then identify another SD condition, defined as *mean utility preserving third degree cumulative prospect stochastic dominance*, to describe the tradeoff between overall risk and downside risk under CPT, and characterize the intensity of downside risk aversion using absolute value of prudence measure. Note that to show the necessity of the increase in downside risk conditions for the characterizations of downside risk aversion and intensity of downside risk aversion we need to construct value functions with desired curvature properties of CPT, which appears to be challenging.

We also address several commonly used downside risk measures: lower partial moment and upper partial moment, third central moment and strong skewness. We also extend our analysis to alternative downside risk increases under which the intensity of downside risk aversion can be characterized using decreasing absolute risk aversion. Finally, we extend the choice-theoretic characterization of downside risk aversion to inverse S-shaped value functions.

The remainder of the chapter is organized as follows. Section 4.2 briefly reviews the concepts of SD and downside risk aversion in EUT. Section 4.3 provides the choice-theoretic characterization for downside risk aversion under CPT. Section 4.4 provides an application in portfolio choice problem. Section 4.5 presents several extensions. Section 4.6 concludes the chapter with a few remarks. All technical proofs are provided in the appendix.

## 4.2 Preliminaries: Downside Risk Aversion in EUT

In decision making under risk, a *prospect* is described by a probability distribution. In particular, the prospect of a finite probability distribution can be described as $(x_1, p_1; ...; x_n, p_n)$ where $x_i$ is an outcome with probability $p_i$, $i = 1, ..., n$.

Throughout this chapter, we will compare a pair of risky prospects, represented by (cumulative) distributions $F$ and $G$ with a common support $[a, b]$ for $a < 0 < b$, with $\bar{F}$ and $\bar{G}$ being the corresponding decumulative (or tail) distribution functions. Assume that $F$ and $G$ are continuous and $F(x) \neq G(x)$ at least for some $x \in [a, b]$. The means, standard deviations of $F$ and $G$ are represented by $\mu_F,\ \mu_G,\ \sigma_F,\ \sigma_G$ respectively. A risk change from $F$ to $G$ refers to the change from distribution $F$ to distribution $G$.

Denote by $\mathcal{V}_{CV}, \mathcal{V}_{CX},\ \mathcal{V}_P$ and $\mathcal{V}_M$ the classes of all nondecreasing strictly concave utility functions (denoted by $u$) with strictly positive third derivatives (i.e., $u'(x) > 0,\ u^{''}(x) < 0,\ u^{'''}(x) > 0$), all nondecreasing strictly convex utility functions with strictly positive third derivatives (i.e., $u'(x) > 0,\ u^{''}(x) > 0,\ u^{'''}(x) > 0$), all nondecreasing $S$-shaped prospect value functions that are strictly convex in the losses and strictly concave in the gains with positive third derivatives (i.e., $u'(x) > 0,\ u^{''}(x) > 0$ for $x < 0, u^{''}(x) < 0$ for $x \geq 0,\ u^{'''}(x) > 0$), and all nondecreasing inverse-$S$-shaped prospect value functions that are strictly concave in the losses and stictly convex in the gains with positive third derivatives (i.e., $u'(x) > 0,\ u^{''}(x) < 0$ for $x < 0,\ u^{''}(x) > 0$ for $x \geq 0,\ u^{'''}(x) > 0$ ), where all the inequalities hold almost everywhere. Without loss of generality, the reference point or *status quo* is normalized to zero. To accommodate CPT, we will use in the rest of the chapter the terms "value function" and "utility function" interchangeably. The choice between prospects $F$ and $G$ for an individual $u$ is determined by the difference of the expected prospect values

$$\Delta_u(F, G) = \int_a^b u(x)dF(x) - \int_a^b u(x)dG(x) = \int_a^b u'(x)[G(x) - F(x)]dx,$$

which can be viewed intuitively as the accumulative advantages of $F$ over $G$ throughout the support set $[a, b]$ weighted by the first derive of the utility function. SD approach can also be used to describe risk changes and preferences towards alternative risky prospects. Specifically, prospect $F$ stochastically dominates prospect $G$ by FSD, SSD, and TSD if and only if $F(x) \leq G(x)$ for all $x$ (FSD), $\int_a^x [G(t) - F(t)]dt \geq 0$ for all $x$ (SSD), and $\int_a^x \int_a^z [G(t) - F(t)]dtdz \geq 0$ for all $x$ and $\mu_F \geq \mu_G$ (TSD). The corresponding utility functions satisfy: $u'(x) \geq 0$ (FSD), $u'(x) \geq 0$ and $u^{''}(x) \leq 0$ (SSD) and $u'(x) \geq 0, u^{''}(x) \leq 0$ and $u^{'''}(x) \geq 0$ (TSD) for all $x$, respectively Levy (2016). These curvature properties have intuitive economic interpretations: the positive first derivative implies that the decision maker prefers more payoff to less payoff, which stems from the monotonicity axiom, the positive second derivative implies that decision makers prefer less uncertainty or risk (risk aversion), and the positive third derivative implies the preference toward positive skewness or less downside risk Levy (2016).

Intuitively, an individual is downside risk averse if he is decreasingly risk averse ($R_u' \leq 0$), which implies that $u^{'''} \geq 0$. Menezes et al. (1980) were the first to provide the choice-theoretic characterization for downside risk aversion by formally introducing the stochastic change concept of *mean- and variance-preserving downside risk increase* (MV-DRI).

**Definition 6** (MV-DRI) *A risk change from $F$ to $G$ is MV-DRI if (i) $\int_a^b [G(z) - F(z)]dz = 0$, and (ii) $\int_a^x \int_a^z [G(t) - F(t)]dtdz \geq 0$ for all $x \in [a, b]$, with equality holding at $x = b$, and strict inequality holding for some $x \in (a, b)$.*

Condition (i) of MV-DRI implies that the two risky alternatives have the same mean. The inequality of condition (ii) implies that there is an unambiguous risk shift from the right to the left for the risk change from $F$ to $G$. The equality of condition (ii) implies that the magnitudes of risk increases and decreases are equal and therefore the variances are equal. Menezes et al. (1980) show that MV-DRI implies that $F$ is more skewed to the right than $G$ in the sense that the third central moment of $F$ is greater than that of $G$. Moreover, they show that a downside risk change can be expressed in terms of a set of mean-preserving-spread and mean-preserving-contraction probability transformation functions. They show that an individual $u$ prefers $F$ to $G$ for all risk changes from $F$ to $G$ being increases in downside risk if and only if $u'''(x) > 0$. The SD rule of MV-DRI is also called *3-convex order* (Shaked and Shanthikumar 2007).

**Remark 2** (MVP-DRI AND TSD) *Both MV-DRI and TSD describe downside risk changes. MV-DRI differs from TSD (in characterizing the downside risk aversion) in that (1) the utility function does not have to be nondecreasing, and (2) the utility function does not have to be concave, which together drive the mean- and variance-preserving properties. In other words, if we restrict the class of preferences with nondecreasing utility functions, the corresponding downside risk changes may no longer preserve the mean and variance, which can be readily verified by extending the analysis of Menezes et al. (1980).*

To capture the tradeoff between mean and variance, Diamond and Stiglitz (1974) introduce utility preserving increase in risk. In the same spirit, Chiu (2005) generalizes the SD conditions of increase in downside risk by introducing mean utility preserving increase in downside risk. A risk change from $F$ to $G$ is mean utility preserving increase in downside risk for a reference decision maker $u \in \mathcal{V}_{CV}$ if (i) $\int_a^b [G(z) - F(z)]dz = 0$ and (ii) $\int_a^x u''(z) \int_a^z [G(t) - F(t)]dtdz \leq 0, \forall x \in [a, b]$ with at least one strict inequality and equality holding at $x = b$.

The accumulative advantage function of $F$ over $G$, $\int_a^z [G(t) - F(t)]dt$, is weighted by a negative and decreasing function $u''$, which, together with the binding condition of (ii), implies that $G$ is more skewed to the left of $F$ while having a smaller variance. Chiu (2005) shows that for risk-averse individuals, the greater the prudence measure, the greater the strength of downside risk aversion, which supports the role of prudence to measure the strength of downside risk aversion.

## 4.3 Downside Risk Aversion in CPT

According to Tversky and Kahneman (1992), the CPT preference can be described with a nondecreasing S-shaped value function defined over a monetary gain or loss (change in wealth with respect to a reference point $r$ that is normalized to zero in this chapter) and a pair of probability weighting functions (PWFs) $w^-(\cdot)$ and $w^+(\cdot)$ that transform the objective probabilities (cumulative distribution functions) to decision weights for losses and gains respectively, with boundary conditions $w^-(0) = w^+(0) = 0$ and $w^-(1) = w^+(1) = 1$. In addition, the gain weighting function equals the dual of the loss weighting function (i.e., $w^-(p) + w^+(1-p) = 1$) for all $p \in [0,1]$. A typical example of S-shaped value function and PWFs are provided by Tversky and Kahneman (1992):

$$v(x) = \begin{cases} x^\alpha, & \text{if } x \geq 0, \\ -\lambda(-x)^\beta, & \text{if } x < 0, \end{cases} \quad \text{and} \quad \begin{cases} w^+(p) = \frac{p^\gamma}{(p^\gamma + (1-p)^\gamma)^{1/\gamma}}, \\ w^-(p) = \frac{p^\delta}{(p^\delta + (1-p)^\delta)^{1/\delta}}, \end{cases} \tag{4.1}$$

with $0 < \alpha < 1$, $0 < \beta < 1$, $\lambda \geq 1$, and $\gamma, \delta \geq 0.28$.

Under CPT, an individual $u \in \mathcal{V}_P$ prefers $F$ to $G$ if and only if $\Delta_u^w(F, G) \geq 0$, where

$$\Delta_u^w(F,G) = \int_a^0 [w^-(G(x) - w^-(F(x)]u'(x)dx + \int_0^b [w^+(\bar{F}(x)) - w^+(\bar{G}(x))]u'(x)dx. \quad (4.2)$$

### 4.3.1 Increase in Downside Risk under CPT

To describe the SD rule with which decision makers with S-shaped value functions agree, Levy and Wiener (1998) introduce the notion of *prospect stochastic dominance* (PSD): $F$ dominates $G$ in the sense of PSD if $\int_x^y [G(t) - F(t)]dt \geq 0$ for any $x \leq 0 \leq y$. When PWFs are linear, i.e., $w^+(p) = w^-(p) = p$, they show that $\Delta_u^w(F,G) \geq 0$ if $F$ dominates $G$ by PSD. To characterize downside risk aversion under CPT, we first extend PSD to its third-degree counterpart, called *third degree prospect stochastic dominance* (TPSD).

**Definition 7** (TPSD) *Prospect $F$ dominates prospect $G$ by TPSD, denoted by $F \succeq_{TPSD} G$, if*

*(i)* $\int_a^0 [G(z) - F(z)]dz \geq 0$ *and* $\int_0^b [G(z) - F(z)]dz \geq 0$, *and*

*(ii)* $\int_x^0 \int_z^0 [G(t) - F(t)]dtdz \geq 0$ *for all* $x \in [a, 0]$, *and* $\int_0^y \int_0^z [G(t) - F(t)]dtdz \geq 0$ *for all* $y \in [0, b]$, *with at least one strict inequality.*

Condition (i) of TPSD implies that the expected loss and gain of prospect $F$ are greater than that of prospect $G$, which is driven by the monotonicity of value functions. Recall that Menezes et al. (1980) only require positive third derivatives for the utility functions of downside risk averters while leaving the signs of first and second derivatives arbitrary, which leads to the mean and variance preserving property of the SD condition of increase in downside risk. The following proposition provides an utility representation of TPSD.

**Proposition 11** *Prospect $F$ dominates $G$ by TPSD if and only if $\Delta_u(F,G) \geq 0$ for all $u \in \mathcal{V}_P$.*

Proposition 11 shows that the risk change described by TPSD is indeed an increase in downside risk for decision makers with $S$-shaped value functions that have positive third derivatives. Note that Wong and Chan (2008) introduce a similar condition, also called TPSD, that only contains condition (ii) of Definition 7. However, their analysis on utility representation relies on the assumption that the second derivative of the utility function is zero at the reference point, i.e., $u''(0) = 0$. We do not impose this assumption as many commonly used CPT value functions, including the specification of (4.1), are not necessarily differentiable at the reference points that are typically kinks.

To account for PWFs, we now extend TPSD to *weighted third-degree prospect stochastic dominance* (W-TPSD).

**Definition 8** (W-TPSD) *Given a pair of PWFs $(w^-(\cdot), w^+(\cdot))$, $F$ dominates $G$ by W-TPSD, denoted by $F \succeq_{W-TPSD} G$, if*

*(i)* $\int_a^0 [w^-(G(z)) - w^-(F(z))]dz \geq 0$ *and* $\int_0^b [w^+(\bar{F}(z)) - w^+(\bar{G}(z))]dz \geq 0$, *and*

*(ii)* $\int_x^0 \int_z^0 [w^-(G(t)) - w^-(F(t))]dtdz \geq 0$ *for all* $x \in [a,0]$, *and* $\int_0^y \int_0^z [w^+(\bar{F}(t)) - w^+(\bar{G}(t))]dtdz \geq 0$ *for all* $y \in [0,b]$, *with at least one strict inequality.*

Clearly, when PWFs are linear, i.e., $w^+(p) = w^-(p) = p$, W-TPSD reduces to TPSD. Using the weighted prospects $F^w$ and $G^w$, $F \succeq_{W-TPSD} G$ is equivalent to $F^w \succeq_{TPSD} G^w$. The following lemma follows immediately from Proposition 11.

**Lemma 5** (Downside Risk Aversion and W-TPSD) *Consider a risk change from $F$ to $G$ and a pair of PWFs $(w^-(\cdot), w^+(\cdot))$. Then $\Delta_u^w(F,G) \geq 0$ for all $u \in \mathcal{V}_P$ if and only if $F \succeq_{W-TPSD} G$.*

Lemma 5 shows that the risk change described by W-TPSD is indeed an increase in downside risk for all downside risk averters with S-shaped value functions and a common pair of PWFs.

### 4.3.2 Increase in Downside Risk Aversion under CPT

Diamond and Stiglitz (1974) define the *mean utility preserving increase in risk* as compensated increase in risk to characterize the intensity of risk aversion. In the same spirit, we now define *mean and value preserving W-TPSD* (MU-W-TPSD) as a compensated adjustment of W-TPSD to characterize the intensity of downside risk aversion under CPT using prudence measure.

**Definition 9 (MU-W-TPSD)** *Consider a reference decision maker* $u \in \mathcal{V}_P$ *with PWFs* $(w^-(\cdot), w^+(\cdot))$*. The risk change from* $F$ *to* $G$ *is an MU-W-TPSD if:*
*(i)* $\int_a^0 [w^-(G(x)) - w^-(F(x))]dx + \int_0^b [w^+(\bar{F}(x)) - w^+(\bar{G}(x))]dx = 0$*, and*
*(ii)* $\int_x^0 u''(t) \int_t^0 [w^-(G(z)) - w^-(F(z))]dzdt \geq 0$ *for* $x \leq 0$*, and* $\int_0^y u''(t) \int_0^t [w^+(\bar{F}(z)) - w^+(\bar{G}(z))]dzdt \leq 0$ *for* $y \geq 0$*, with at least one strict inequality and with equalities at* $x = a$*,* $y = b$*.*

Condition (i) of MU-W-TPSD implies that the weighted mean of $F$ and $G$ are the same under the pair of PWFs. Note that condition (ii) of the above definition describes the downside risk changes based on lower tails of losses domain and upper tails of gains domain respectively. MU-W-TPSD reduces to W-TPSD when $u''$ is constant. Condition (i) and the equalities holding at $x = a$ and $y = b$ in condition (ii) imply that the reference decision maker $u$ with PWFs $(w^-, w^+)$ is indifferent between $F$ and $G$.

Rothschild and Stiglitz (1970) define a mean preserving spread (MPS) as a function that transforms weights from the center of a distribution to its tails without changing the mean. The converse transformation, which transforms weights from the tails of a distribution to its center without changing mean, is a mean-preserving contraction (MPC). Similarly, Menezes et al. (1980) introduce a mean and variance preserving transformation by combining pairs of MPS and MPC transformations. We further generalize their analyses by introducing a mean utility preserving transformation (MUPT) for any $u \in \mathcal{V}_P$ such that the risk change driven by MUPT is an MU-W-TPSD for $u$.

**Definition 10 (MUPT)** *For any $u \in \mathcal{V}_P$ and prospect $F$ with probability or density function $f$ and nonzero probabilities in both losses and gains domains, a transformation function $t(x)$ is an MUPT for $F$ and $u$ if (i) $t(x) = s(x) + c(x)$ where $s(x)$ and $c(x)$ are MPS and MPC transformation functions respectively (i.e., $f(x) + t(x) \geq 0$ almost everywhere in $[a, b]$, $\int_a^b s(x)dx = \int_a^b c(x)dx = 0$, and $\int_a^b xs(x)dx = \int_a^b xc(x)dx = 0$), and (ii) $\int_x^0 u''(z) \int_z^0 T(\xi)d\xi dy \geq 0$ for $x \in [a, 0]$ and $\int_0^y u''(z) \int_0^z T(\xi)d\xi dz \geq 0$ for $y \in [0, b]$ with strict inequality for some $x \in (a, b)$ and equalities holding at $x = a$ and $y = b$), where $T(x) = \int_a^x t(\xi)d\xi$.*

**Lemma 6** (Construction of MU-W-TPSD) *Consider a reference decision maker $u \in \mathcal{V}_P$ and PWFs $(w^-(\cdot), w^+(\cdot))$. For any given distribution $F$, there always exist a pair of MUPT functions $t_1,\ t_2$ such that the risk change is MU-W-TPSD from $F$ for $u$. In particular, $w^-(G(x)) = w^-(F(x)) + \int_a^x t_1(\xi)d\xi$ for $x \leq 0$ and $w^+(G(x)) = w^+(\bar{F}(x)) - \int_0^x t_2(\xi)d\xi$ for $x \geq 0$.*

Lemma 6 implies that for any given reference decision maker $u \in \mathcal{V}_P$ and a prospect $F$, the risk change of MU-W-TPSD always exists. To gain the idea of the construction approach, we briefly sketch the proof. First, following Menezes et al.'s approach, we construct a MPS transformation function $s(x)$, which introduces a greater variance without changing the mean, and a MPC transformation function $c(x)$, which reduces the variance without changing the mean and introduces negative skewness. We can then observe that the cumulative risk changes of the MPS-MPC combination $t(x)$, denoted by $T(x) = \int_0^x t(\xi)d\xi$, changes signs at most twice, first from positive to negative and then from negative to positive. We then vary the MPC transformation function $c(x)$ until the expected utility of $u$ is preserved. Figure 4.1 provides a graphical illustrative example for the MUPT, which is used in the proof of Lemma 6 to construct MU-W-TPSD.

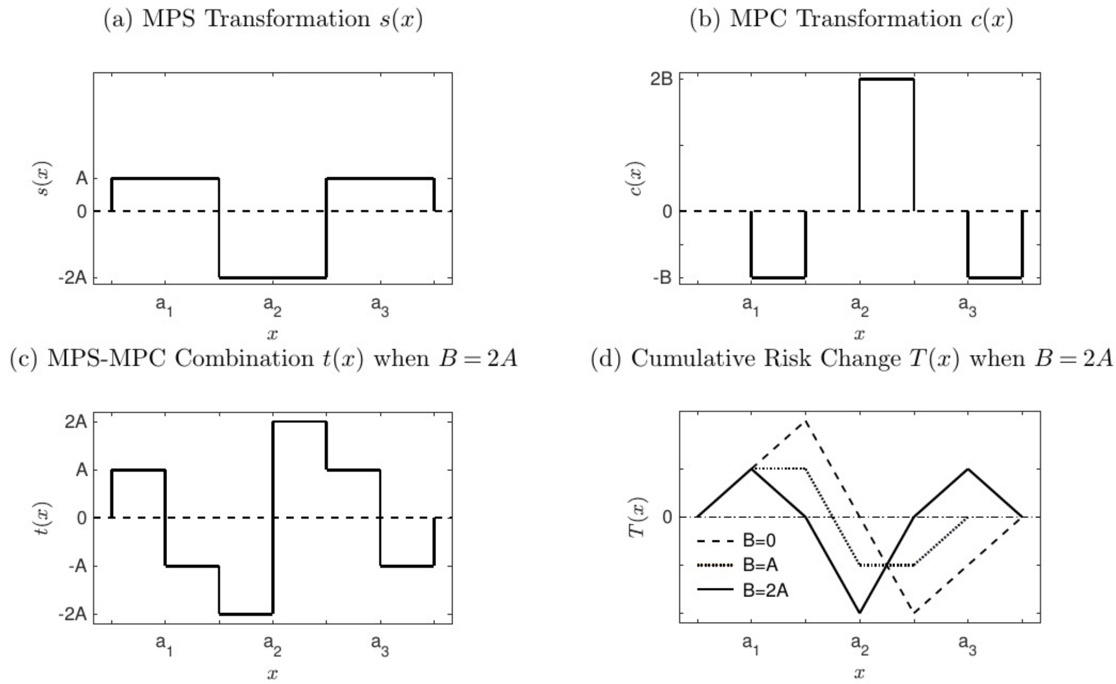


Figure 4.1: A Graphic Illustration of MUPT

We next characterize the intensity of downside risk aversion by showing that under the SD condition of MU-W-TPSD an individual's choice between two risky prospects can be determined by the prudence measure.

**Proposition 12** (Increase in Downside Risk Aversion under CPT) *Consider a reference decision maker $u \in \mathcal{V}_P$ with PWFs $(w^-(\cdot), w^+(\cdot))$, who is indifferent between $F$ and $G$ , i.e., $\Delta_u^w(F,G) = 0$. For any $v \in \mathcal{V}_P$, $\Delta_v^w(F,G) \geq 0$ for all risky prospect pairs $F$ and $G$, such that the risk change from $F$ to $G$ is a MU-W-TPSD for $u$, if and only if $|P_v(x)| \geq |P_u(x)|$ for any $x$.*

Proposition 12 extends the analysis of Chiu (2005) for risk averters to that for downside risk averters under CPT. First, notice that in the special case when $b = 0$, the relationship between prudence and downside risk aversion for risk-seekers is opposite to that for risk averters. It indicates that $v$ prefers $F$ to $G$, when $G$ is mean utility preserving increase in downside risk from $F$ for $u$, if and only if $v$ has smaller prudence (rather than greater prudence when $r = a$) than $u$. The relationship between prudence and downside risk aversion under CPT can then be established: if we increase prudence in the gains domain or decrease prudence in the losses domain, the intensity of downside risk aversion will be increased. That is, for S-shaped value functions, the absolute value of the prudence can be used to measure the intensity of downside risk aversion.

We next provide a remarks on the relationship between prudence measure and loss aversion.

**Remark 3** (Index of Loss Aversion and Prudence Measure) *To describe CPT, Tversky and Kahneman (1992) argue that the principle of loss aversion implies that the value function is steeper for losses than gains, i.e.,* $u'(-x) > u'(x)$ *for* $x > 0$*. They use* $\frac{-u(-1)}{u(1)}$ *to index the degree of loss aversion. Köbberling and Wakker (2005) propose an alternative index of loss aversion, defined as* $\lambda = \frac{u'(0-)}{u'(0+)}$*, where* $u'(0-)$ *and* $u'(0+)$ *are the left and right derivatives of* $v$ *at zero. In the power-utility specification (4.1), when* $\alpha = \beta$*, the two definitions coincide, i.e.,* $\lambda = \frac{u'(0-)}{u'(0+)} = \frac{-u(-1)}{u(1)}$*. To explicitly model the index of loss aversion, Köbberling and Wakker (2005) propose a more general form of prospect value function* $u(x) = \begin{cases} \varphi(x) & \text{if } x \geq 0 \\ \lambda\varphi(x) & \text{if } x < 0 \end{cases}$*, where* $\varphi$ *is concave for* $x \geq 0$ *and convex for* $x < 0$*, and* $\lambda$ *is the index of loss aversion. Clearly, under the aforementioned specifications, the prudence measure* $P_u$ *does not depend on the index of loss aversion* $\lambda$ *and hence changing the index of loss aversion does not change the prudence measure. Therefore, if a risk change from* $F$ *to* $G$ *is a MU-W-TPSD for some* $u$ *of the above form with* $\varphi'''(x) > 0$ *for all* $x$ *and some* $\lambda$*, then changing the index of loss aversion (*$\lambda$*) does not change the preference.*

# 4.4 Applications

## 4.4.1 Monotonicity of Optimal Decisions

To explore the economic implications of increased downside risk in the sense of W-TPSD and MU-W-TPSD, we follow a similar setting of Diamond and Stiglitz (1974) to consider a decision maker with a utility function $U(x, \theta)$, which depends on a return value $x$ and a control variable $\theta \in [0, 1]$, and a pair of PWFs $(w^-(\cdot), w^+(\cdot))$. Given a prospect $F$, the decision maker aims to solve the following optimization problem:

$$\max_{\theta \in [0,1]} \psi_F(\theta) \triangleq \int_a^0 U(x, \theta) dw^-(F(x)) + \int_0^b U(x, \theta) d[1 - w^+(\bar{F}(x))]. \qquad (4.3)$$

Let $\arg\max_{\theta \in [0,1]} \psi_F(\theta)$ be the set of maximizers for $\psi_F(\theta)$ for any given $\gamma$ and prospect $F$. We are interested to relate the optimal solution to increases in risk in the sense of W-TPSD and MU-W-TPSD.

The next theorem identifies sufficient conditions of $U(x, \theta)$ under which an increase in risk in the sense of W-TPSD or MU-W-TPSD for $u$ leads to smaller optimal solutions.

**Theorem 1** (Monotonicity in Increase in Risk) *Consider two prospects $F$ and $G$ and assume that $U_\theta(\cdot, \theta) \in \mathcal{V}_P$ with a pair of PWFs $(w^-(\cdot), w^+(\cdot))$. Then* $\arg\max_{\theta \in [0,1]} \psi_F(\theta) \geq \arg\max_{\theta \in [0,1]} \psi_G(\theta)$ *if (i) the risk change from $F$ to $G$ is W-TPSD; or (ii) the risk change from $F$ to $G$ is MU-W-TPSD for some $u \in \mathcal{V}_P$, and $\left|\frac{U_{\theta xxx}}{U_{\theta xx}}\right| \geq |P_u(x)|$ for all $x \in [a, b]$ and $\theta \in [0, 1]$.*

It is notable that to perform the comparative statics with respect to an increase in risk it suffices to show that $\psi_F(\theta) - \psi_G(\theta)$ is increasing in $\theta$, or, equivalently, the objective function is submodular in the control variable $\theta$ and the level of risk in the W-TPSD or MU-W-TPSD sense (see, e.g., Topkis 1998 ).

We next apply Theorem 1 to a portfolio choice problem.

**Example 3** (Portfolio Choice) *Consider a portfolio choice problem in a market consisting of one risky asset (e.g., stock) and one risk-free asset (e.g., bond) with an investment planning horizon from date* $0$ *to date* $T$*. The return rate of the risk-free asset over the investment period is deterministic, denoted by* $r$*. We follow He and Zhou (2011b) and assume that the reference point of the investor is equal to the risk-free return rate* $r$*. Let the return rate of the risky asset is random, denoted by* $R$*. Let* $\theta \in [0,1]$ *be the percentage of the wealth invested in the stock and the reminder in the risk free asset. Then the final normalized wealth is* $\Pi(\theta) = (1-\theta)r + \theta R$*. For convenience, let* $X = R - r$ *and we normalized the initial wealth level to one. The stock total excess return* $R-r$ *is a random variable with CDF* $F$*. Then we have* $\Pi(X,\theta) = r + \theta X$*. Let* $\theta \in [0,1]$ *be the percentage of the wealth invested in the stock and the remainder in the risk-free asset. Then the relative gain or loss is* $\theta X$*. Suppose the distribution function of* $X$ *is* $F$*. Then the portfolio choice problem under CPT preference for* $v \in \mathcal{V}_P$ *is, therefore,*

$$\max_{\theta \in [0,1]} E[v(\theta X)]. \tag{4.4}$$

*Assume* $v$ *is specified as in equation (4.1). Thus,*

$$U(x,\theta) = v(\theta x) = \begin{cases} \theta^\alpha x^\alpha, & \text{if } x > 0, \\ -\lambda \theta^\beta (-x)^\beta, & \text{if } x < 0. \end{cases}$$

*Then we have*

$$\begin{aligned} U_{\theta x} &= \begin{cases} \alpha^2 \theta^{\alpha-1} x^{\alpha-1}, \text{if } x > 0, \\ \lambda \beta^2 \theta^{\beta-1} (-x)^{\beta-1}, \text{if } x < 0, \end{cases} \\ U_{\theta xx} &= \begin{cases} -\alpha^2 (1-\alpha) \theta^{\alpha-1} x^{\alpha-2}, \text{if } x > 0, \\ \lambda \beta^2 (1-\beta) \theta^{\beta-1} (-x)^{\beta-2}, \text{if } x < 0, \end{cases} \\ U_{\theta xxx} &= \begin{cases} \alpha^2 (1-\alpha)(2-\alpha) \theta^{\alpha-1} x^{\alpha-3}, \text{if } x > 0, \\ \lambda \beta^2 (1-\beta)(2-\beta) \theta^{\beta-1} (-x)^{\beta-3}, \text{if } x < 0. \end{cases} \end{aligned}$$

*Clearly,* $U_{\theta x} > 0$ *and* $U_{\theta xxx} > 0$ *for all* $x \neq 0$*, and* $U_{\theta xx} > 0$ *for* $x < 0$*,* $U_{\theta xx} < 0$ *for*

*$x > 0$, which verifies that $U_\theta \in \mathcal{V}_P$. Treating $U_\theta$ as an utility function, calculating the absolute value of prudence measure, we have*

$$\left|\frac{U_{\theta xxx}}{U_{\theta xx}}\right| = \begin{cases} \frac{2-\alpha}{x}, & \textit{if } x > 0, \\ -\frac{2-\beta}{x}, & \textit{if } x < 0. \end{cases}$$

*According to Theorem 1, an increase in risk in the sense W-TPSD leads to smaller values of optimal portfolio decision $\theta$ or less riskier portfolio choices.*

*Now consider a reference decision maker $u$, which is also of the form (4.1), with parameters $\alpha_0$, $\beta_0$ and $\lambda_0$, such that $\alpha \leq \alpha_0$ and $\beta \leq \beta_0$. Then its absolute value of prudence measure is*

$$|P_u(x)| = \begin{cases} \frac{2-\alpha_0}{x}, & \textit{if } x > 0, \\ -\frac{2-\beta_0}{x}, & \textit{if } x < 0. \end{cases}$$

*Clearly, $\left|\frac{U_{\theta xxx}}{U_{\theta xx}}\right| \geq |P_u(x)|$ for all $x \neq 0$ and $\theta$, i.e., the 'individual' $U_\theta$ is more downside risk averse than $u$. Applying Theorem 1 again, an increase in risk in the sense of MU-W-TPSD for $u$ induces $v$ to choose a less riskier portfolio choice.*

## 4.5 Extensions and Further Discussions

### 4.5.1 Alternative Downside Risk Measures

We now associate the concept of MU-W-TPSD to the familiar notions of *below target semivariance* (Markowitz 1959), *lower partial moment* (LPM, Fishburn 1977), *third central moment* and *strong skewness* that are commonly used as measures of downside risk.

First, the second-degree LPM, which is identical to the *below target semivariance*, measures the moments below certain return target (reference point) as a proxy of downside risk measure in the distribution, and the greater the LPM the greater the negative skewness or downside risk. Correspondingly, *upper partial moment* (UPM), or, equivalently, *above target semivariance*, measures upside potential above the reference point (Farinelli and Tibiletti 2008). With PWFs $(w^-(\cdot), w^+(\cdot))$, the second-degree LPMs/UPMs for $F$ can be defined as

$$LPM^w(F) = \int_a^0 x^2 d[w^-(F(x))], \text{ and } UPM(F)^w = \int_0^b x^2 d[-w^+(\bar{F}(z))].$$

We next show that a MU-W-TPSD can be viewed as a combination of LPM and UPM.

**Lemma 7 (**MU-W-TPSD AND LPM/UPM) *Consider a reference decision maker $u \in \mathcal{V}_P$ with PWFs $(w^-(\cdot), w^+(\cdot))$. A MU-W-TPSD from $F$ to $G$ for $u$ implies an increase in the second-degree LPM and a decrease in the second-degree UPM.*

Combining Lemma 7 and Proposition 12, we know that a downside risk averter, who has a greater prudence measure than the reference decision maker $u$ for whom the risk change from $F$ to $G$ is a MU-W-TPSD, prefers a smaller second-degree LPM defined on the losses domain and a greater second-degree UPM defined on the gains domain.

Another common downside risk measure is *third central moment*. Menezes et al. (1980) show that increase in downside risk from $F$ to $G$ implies that $F$ is more skewed to the right than $G$ in the sense that the *third central moment* of $F$ is greater than that of $G$. For prospect $F$, the third central moment with PWFs $(w^-(\cdot), w^+(\cdot))$ can be defined as

$$m_F^w = \int_a^0 (x - \mu_F^w)^3 d\left[w^-(F(x))\right] + \int_0^b (x - \mu_F^w)^3 d\left[-w^+(\bar{F}(x))\right],$$

where $\mu_F^w = \int_a^0 x d[w^-(F(x))] + \int_0^b x d[-w^+(\bar{F}(x))]$. The following lemma identifies a condition under which a MU-W-TPSD from $F$ to $G$ implies a decrease in third central moment.

**Lemma 8** (MU-W-TPSD and Third Central Moment) *Consider a decision maker $u \in \mathcal{V}_P$ with PWFs $(w^-(\cdot), w^+(\cdot))$, and two prospects $F$ and $G$. Suppose that $G$ crosses over $F$ exactly twice, first from above and then from below, in both losses and gains domains, and $\mu^w_F = \mu^w_G$ is between the average values of crossing points in losses and gains domains respectively. Then a MU-W-TPSD for $u \in \mathcal{V}_P$ implies that $m^w_F > m^w_G$.*

The condition of having two crossing points in both losses and gains domains, with $G$ crossing over $F$ first from above and then from below, implies that $F$ is more skewed to the right than $G$ in both domains. Intuitively, the mean being between the mid-points of two crossing points in losses and gains domains is consistent with its role of being the "central" point. Lemma 8 then shows that these two conditions together with the MU-W-TPSD conditions imply that the risk change from $F$ to $G$ leads to a decrease of third central moment.

The last relevant concept is *strong skewness.* van Zwet (1964) argues that a distribution $F$ is more skewed to the right than $G$ if the transformation $G^{-1}(F(x))$ is concave, which is defined by Oja (1983) as *strong skewness comparability.* The concavity of the distribution transformation $G^{-1}(F(x))$ implies that $F$ crosses $G$ at most twice, and that if they cross exactly twice $G$ must first cross $F$ from above (Oja 1983). Chiu (2005) shows that the strong right skewness increase condition plus the mean utility preserving with variance decrease from $F$ to $G$ for a risk averse reference decision maker $u$ imply that the increase in skewness is just sufficient to compensate $u$ for the increase of overall riskiness and that such a risk change is an MUP-IDR.

The following lemma identifies a set of sufficient conditions, which ensure the MU-W-TPSD conditions, using below/above target semi-variance measures (or LPM/UPM) and a pair of modified strong skewness conditions defined on losses and gains domains respectively under CPT.

**Lemma 9** (Strong Skewness and MU-W-TPSD) *A risk change from $F$ to $G$ for some $u \in \mathcal{V}_P$ is MU-W-TPSD when the following conditions are satisfied.*

*(1)* $\int_a^0 [w^-(G(z)) - w^-(F(z))]dz = \int_0^b [w^+(\bar{F}(z)) - w^+(\bar{G}(z))]dz = 0,$

*(2)* $\int_a^0 u''(x) \int_x^0 [w^-(G(z)) - w^-(F(z))]dzdx = \int_0^b u''(x) \int_0^x [w^+(\bar{F}(z)) - w^+(\bar{G}(z))]dzdx = 0$*, and*

*(3)* $G^{-1}(F(x)) - x$ *is quasi-concave in losses and gains domains respectively.*

Lemma 9 specifies three conditions which ensure that the risk change from $F$ to $G$ is a MU-W-TPSD. In particular, condition (1) implies that $F$ and $G$ have the same weighted means in the losses and gains domains respectively, i.e., $\int_a^0 [w^-(G(x)) - w^-(F(x))]dx = \int_0^b [w^+(\bar{F}(x)) - w^+(\bar{G}(x))] = 0$, which, together, imply that the overall weighted means are equal. Condition (2), in addition to (1), implies that $\Delta_u^w(F, G) = 0$. Moreover, as we show in the proof, conditions (1) and (2) imply that there are at least two crossing points in the losses and gains domains respectively. Condition (3) generalizes the strong skewness conditions introduced by van Zwet (1964), who assumes that $G^{-1}(F(\cdot))$ is concave, for S-shaped value functions, which implies that $G$ crosses $F$ at most twice from above in both losses and gains domains and hence $w(G)$ is strongly more skewed to the left than $w(F)$. Combining all three conditions, we know that $G$ crosses over $F$ exactly twice, first from above, in both losses and gains domains, which leads to the conditions of MU-W-TPSD.

### 4.5.2 Markowitz Downside Risk Increase

Markowitz (1952b) proposes that decisions are based on *change* in wealth and individuals are risk averse for losses and risk seeking for gains as long as the possible outcomes are not very extreme, i.e., value functions are inverse S-shaped (concave in the losses and convex in the gains). For extreme outcomes, individuals are risk seeking for losses and risk averse for gains. Levy and Levy (2002) extend the work of Markowitz (1952b) to propose (second-degree) Markowitz Stochastic Dominance (MSD) for all inverse S-shaped functions.Wong and Chan (2008) further generalize second-degree MSD to the third-degree MSD, or TMSD, conditions: (i) $\int_a^b [G(z) - F(z)]dz \geq 0$, and (ii) $\int_a^x \int_a^z [G(t) - F(t)]dtdz \geq 0$ for all $x \in [a, 0]$, and $\int_y^b \int_z^b [G(t) - F(t)]dtdz \geq 0$ for all $y \in [0, b]$. It is notable that Wong and Chan (2008) implicitly restrict that the second derivatives of the inverse S-shaped value functions are equal to zero at the reference point (zero). We next provide alternative SD conditions for increases in downside risk and downside risk aversion without requiring the second derivatives of the inverse S-shaped value functions to be zero at the reference point. Following Levy and Levy (2002), we omit PWFs.

**Proposition 13** *Consider two prospects $F$ and $G$. $\Delta_u(F, G) \geq 0$ for all $u \in \mathcal{V}_M$ if and only if*

*(i)* $\int_a^b [G(z) - F(z)]dz \geq 0$*;*

*(ii)* $\int_x^0 \int_z^0 [G(t) - F(t)]dtdz \geq 0$ *for all* $x \in [a, 0]$*, and* $\int_0^y \int_0^z [G(t) - F(t)]dtdz \geq 0$ *for all* $y \in [0, b]$*, with at least one strict inequality and with equalities holding at* $x = a$ *and* $y = b$*.*

Note that the inequalities in condition (ii) are binding at the lower bound of the losses domain and the upper bound of the gains domain respectively, which is driven by the fact that the second derivative at the reference point is not necessarily zero.

Similar to MU-W-TPSD, we define the concept of Markowitz mean utility preserving downside risk increase (M-MUP-IDR) to characterize increase in downside risk aversion.

**Definition 11** (M-MUP-IDR) *A risk change from $F$ to $G$ is an M-MUP-IDR for some $u \in \mathcal{V}_M$ if*

*(i)* $\int_a^0 [G(z) - F(z)]dz = \int_0^b [G(z) - F(z)]dz = 0;$

*(ii)* $\int_x^0 u''(t) \int_t^0 [G(\xi) - F(\xi)]d\xi dt \leq 0 \; \forall x \in [a, 0)$ *and* $\int_0^x u''(t) \int_0^t [G(\xi) - F(\xi)d\xi dt \geq 0 \; \forall x \in (0, b]$*, with at least one strict inequality and equalities holding at* $y = a$ *and* $x = b$.

It is readily to generalize the analysis of Proposition 12 to provide the choice-theoretic characterization of increase in downside risk aversion for individuals with inverse S-shaped value functions in $\mathcal{V}_M$.

**Proposition 14** *Consider a reference decision maker with $u \in \mathcal{V}_M$.*

*(a) For any $v \in \mathcal{V}_M$, $\int_a^b v(x)dF(x) \geq \int_a^b v(x)dG(x)$ for all pairs of prospects $F$ and $G$, such that the risk change from $F$ to $G$ is an M-MUP-DRI for $u$, if and only if $|P_v(x)| \geq |P_u(x)|$ for all $x$.*

*(b) Suppose $u$ is indifferent between $F$ and $G$. $\int_a^b v(x)dF(x) \geq \int_a^b v(x)dG(x)$ for all $v \in \mathcal{V}_M$ with $|P_v(x)| \geq |P_u(x)|$ for all $x$ if and only if the risk change from $F$ to $G$ is an M-MUP-IDR for $u$.*

Proposition 14 shows that the absolute value of the prudence can also be used to characterize the intensity of downside risk aversion for decision makers with inverse S-shaped value functions.

## 4.6 Conclusion

This chapter provides choice-theoretic characterizations for the increases in downside risk and downside risk aversion in CPT. We identify SD conditions of increases in downside risk to characterize downside risk aversion and intensity of downside risk aversion under CPT. In particular, our definition of MU-W-TPSD generalizes the downside risk increase introduced by Menezes et al. (1980) and Chiu (2005) by accommodating the S-shaped curvature of value functions and probability weighting functions in CPT. We also extend the analysis to inverse S-shaped value functions.

There are a few interesting future research directions along this line. First, this chapter characterizes the intensity of downside risk aversion using prudence measure. We can extend our analysis to other alternative downside risk aversion measures introduced in the literature, such as Liu and Meyer (2012) for decreasing absolute risk aversion (DARA), Modica and Scarsini (2005) and Crainich and Eeckhoudt (2008) for the ratio of the third and first derivatives, Keenan and Snow (2002) for Schwarzian derivatives, and Huang and Stapleton (2014) for cautiousness measure. Second, the downside risk increase conditions are in the class of third-degree SD conditions, which can be further generalized to higher degree SD conditions. Last but not least, one can conduct experimental studies to test how individuals make the tradeoff between risk and downside risk.

# Chapter 5

# Robust Risk Measures for Cumulative Prospect Theory

## 5.1 Introduction

The measurement of risk plays a central role in managing risk. A number of studies have attempted to quantify risk by identifying appropriate risk measures. In his seminal work, Markowitz (1952a) proposes that variance can be used as a proxy for the overall riskiness of a portfolio and risk averse investors make tradeoffs between expected returns and the resulting variances of portfolios. The Markowitz mean-variance (MV) rule is by far the most popular investment decision rule that has been widely adopted by both academics and practitioners (Levy 2016). It has long been criticized that variance is not consistent with investors' actual perception of risk since it allocates weights on negative (undesirable) deviations and positive (desirable) deviations of returns evenly. Markowitz himself also recognized that downside risk is a more appropriate risk measure because investors are more concerned about loss below the target return (Markowitz 1959).

Since then various downside risk measures have been proposed. *Value-at-risk* (VaR) and *conditional value-at-risk* (CVaR) are two commonly used downside risk measures by practitioners and financial institutions (Levy 2016).

Jorion (2007) defines the VaR of a portfolio as *the worst loss over target horizon such that there is a low, pre-specified probability that the actual loss will be greater*. It represents the maximum potential loss expected on a portfolio at a given confidence level over a given time period. It has been widely used by financial institutions, portfolio managers and regulators to measure risks and determine the amount of adequate economic capital to sustain potential losses (i.e., capital adequacy) (Jorion 2007). However, VaR is often criticized as it is not a coherent risk measure (see, e.g. Artzner et al. 1999, Acerbi 2002). CVaR is first proposed by Artzner et al. (1997) as an alternative risk measure. Rockafellar and Uryasev (2000) demonstrate the use of CVaR in portfolio optimization and show that CVaR surpasses VaR computationally.

On the other hand, Gneiting (2011) shows that CVaR suffers from backtesting and thus is not elicitable. Cont et al. (2010) introduce the notion of qualitative robustness for risk measurement procedure and argue that practitioners should not only focus on the coherence of risk measures, and propose to consider again the necessity of the subbaitivity axiom proposed by Artzner et al. (1999) as it is in conflict with robustness for spectral risk measures. To overcome the non-robust property of CVaR, Cont et al. (2010) propose *range value-at-risk* (RVaR) as an alternative risk measure. RVaR can be seen as the average level of VaR level across a range of loss/return probabilities. The notion of RVaR is in general a non-convex risk measure. Embrechts et al. (2018) show that RVaR satisfies a special form of sub-additivity and can be applied to risk sharing problems.

The notion *stochastic dominance* (SD) describes partial ordering relationships between probability distributions (risky prospects) by pairwise comparisons, providing a way to divide the set of feasible risky prospects into efficient and inefficient sets (Levy 2016). Commonly used SD conditions include first-degree stochastic dominance (FSD) and second-degree stochastic dominance (SSD) which were proposed by Hadar and Russell (1969), Hanoch and Levy (1969) and Rothschild and Stiglitz (1970) to characterize the preferences of decision makers who prefer more to less in a stochastic sense and who are risk averse, respectively.

Although the decision criterion of SD seems very different from the decision criterion of using risk measures, there are logical connections among SD and risk measures. Ogryczak and Ruszczynski (2002) first show that VaR is equivalent to FSD, and CVaR is equivalent to SSD. Ma and Wong (2010) use a stochastic dominance approach to provide a decision theoretic characterization for VaR and CVaR. They show that the risk preference of rational and risk averse decision makers can be represented by VaR and CVaR at all levels of risk tolerance, respectively.

The emergence of the cumulative prospect theory (CPT), developed by Kahneman and Tversky (1979) and Tversky and Kahneman (1992), provides a prominent alternative to EUT. In the EUT paradigm, the risk preference is fully represented by the curvature properties of utility functions with concavity (convexity) for risk averters (risk seekers). In the CPT paradigm, the risk preference is typically represented simultaneously by a reference-dependent value function and a pair of probability weighting functions (PWFs) with certain curvature properties.

Inspired the linkage among VaR, CVaR and SD in the paradigm of EUT, this chapter investigates the corresponding connections in the paradigm of CPT. In this chapter, we establish the connections among VaR, CVaR, RVaR, SD and choices made by decision makers with CPT risk preferences, to provide behaviour foundations for those risk measures in the paradigm of CPT. More specifically, we first establish the connections among RVaR, SD and choices made by decision makers with S-shaped value functions. We then extend the analysis to the paradigm of CPT by incorporating PWFs. We thereby provide linkages between risk preferences and risk measures.

To demonstrate the economic implications of increases in RVaR, we identify sufficient conditions under which the optimal solutions to an expected utility maximizing problem are monotone in the risk increases in the sense of RVaR. Using portfolio choice problems as examples, we show that increases in risk in the sense of RVaR may lead to less riskier portfolio choices for a decision maker with CPT preference. Finally, we show that our analyses can be further extended to risk preferences with inverse S-shaped value functions.

The remainder of this chapter is organized as follows. Section 5.2 reviews and recalls the risk measures that includes VaR, CVaR and RVaR. Section 5.3 provides a theoretic foundation of various risk measures in the paradigm of CPT. Section 5.4 presents implications for optimal decision problems. Section 5.5 extends the analysis to reverse S-shaped value functions. Section 5.6 concludes the chapter. All technical proofs are provided in the appendix.

## 5.2 Preliminaries

Throughout this chapter, we focus on quantifying the risks associated with two random variables $X$ and $Y$ that represent the potential returns of two portfolios over a certain period with $F$ and $G$ being the prospects or cumulative distribution functions. For simplicity, we assume that $F$ and $G$ are differentiable with the inverse functions $F^{-1}$ and $G^{-1}$. Denote by $\mathcal{U}_I$, $\mathcal{U}_{CV}, \mathcal{U}_{CX}$, $\mathcal{V}_S$ and $\mathcal{V}_{IS}$ the classes of all nondecreasing function, nondecreasing concave utility functions (i.e., $u' \geq 0, u^{''} \leq 0$), all nondecreasing convex utility functions (i.e., $u' \geq 0, u^{''} \geq 0$), all nondecreasing $S$-shaped value functions with $u^{''}(x) \geq 0$ for $x < 0$, $u^{''}(x) \leq 0$ for $x > 0$ and a reference point 0, and all nondecreasing inverse-$S$-shaped value functions with $u^{''}(x) \leq 0$ for $x < 0$, $u^{''}(x) \geq 0$ for $x > 0$ and a reference point 0. Without loss of generality, we assume that $u(0) = 0$.

The choice between prospects $F$ and $G$ for an individual with utility function $u$ is determined by the difference of the expected prospect values

$$\Delta_u(F, G) = \int_a^b u(x)dF(x) - \int_a^b u(x)dG(x) = \int_a^b u'(x)[G(x) - F(x)]dx,$$

which can be viewed intuitively as the accumulative advantages of $F$ over $G$ throughout the support set $[a, b]$ weighted by the first derivative of the utility function.

### 5.2.1 Value-at-Risk and Conditional Value-at-Risk

A *risk measure* is a real-valued function $\rho(\cdot)$ defined on a space of measurable functions. VaR can be either calculated from the probability distribution of returns (gains) or the probability distribution of losses (i.e., the negative gains) (Hull 2015). In this chapter, for convenience, we use the former approach. Mathematically, for a portfolio investment over a period with return (gain) $X$, VaR at $100 \times \alpha\%$ risk tolerance level (or, equally speaking, $100 \times (1-\alpha)\%$ confidence level) is defined as the cutoff loss such that the probability of suffering a greater loss is less than $100 \times \alpha\%$ (i.e., the loss at the lower $100\alpha$th percentile of the distribution of $X$):

$$VaR_\alpha(X) = \sup\{a|Pr(-X > a) \leq \alpha\}.$$

For example, a portfolio having a 95% one-day VaR of $100 million implies that there is only a 5% chance that the portfolio will lose more than $100 million over the next day. In what follows, for convenience, we will use $VaR_\alpha(X)$ and $VaR_\alpha^F$ interchangeably. We follow the literature and make the convention that the argument of VaR counts losses positive and profits negative.

Note that VaR is the maximum potential loss expected on a portfolio at a given confidence level over a given time period. The VaR distinguishes normal and abnormal market risks at certain confidence level over the overall risk: the potential losses below the VaR are treated as normal market risks and the losses above the VaR are treated as abnormal market risks or catastrophic losses. According to the capital adequacy and regulatory principles for financial institute risk

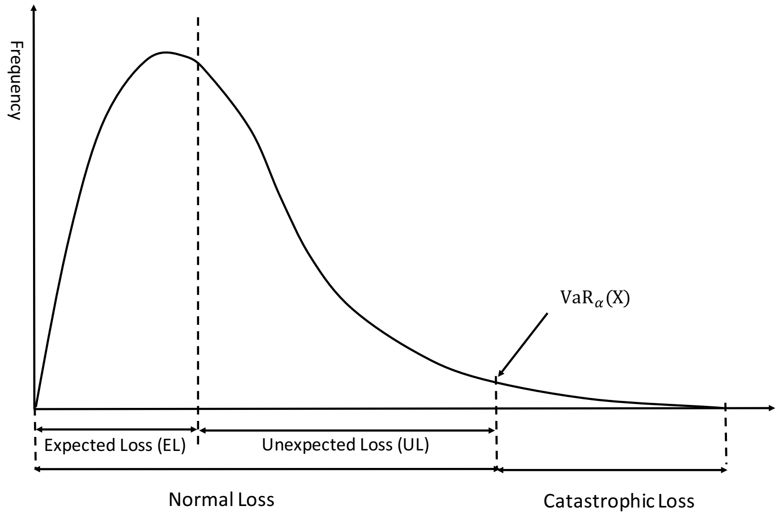


Figure 5.1: Loss distribution

management (see, e.g., The Basel II Capital Accord), a regulatory capital is reserved to cover the *unexpected loss* (UL) which is equal to the difference between VaR and the expected value of overall potential losses (i.e., *expected loss*, EL) and the catastrophic losses can be protected by insurance or hedged by financial derivatives (see, e.g., Chernobai et al. 2008, Moosa 2008).

To regulate definitions of risk measure, Artzner et al. (1999) propose *coherent risk measure*.

**Definition 12 (Coherent risk measure)** *A risk measure $\rho$ is coherent if the following four axioms hold.*

*(1)* $\rho(X) \leq \rho(Y)$ *if* $X \leq Y$, (Monotonicity);

*(2)* $\rho(X + a) = \rho(X) - a$ *for any constant* $a$, (Translation invariance);

*(3)* $\rho(\lambda X) = \lambda\rho(X)$ *for all* $\lambda \geq 0$, (Positive homogeneity);

*(4)* $\rho(X + Y) \leq \rho(X) + \rho(Y)$, (Subadditivity).

Axioms (3) and (4) together imply that $\rho(X)$ is convex in $X$. That is, a coherent risk measure must be convex. It is known that VaR satisfies axioms (1) and (2) but fails to be subadditive for general loss distributions and is therefore not a convex risk measure, which makes such a risk measure difficult to implement in portfolio optimization (see, e.g., Artzner et al. 1999, Pflug 2000).

The drawbacks of VaR have led many researchers to propose new risk measures that are coherent. Artzner et al. (1997) introduce an alternative risk measure known as CVaR (also called *expected shortfall* (ES), *average VaR* (AVaR), *mean excess loss*, *expected tail loss* (ETL) or *tail VaR*). CVaR is the conditional expectation of loss given that the loss is beyond the VaR level, the maximum potential loss expected, at a given confidence level over a given period. In other words, CVaR is the mean of abnormal or catastrophic loss. Formally, the CVaR of the random payoff $X$ at the $(1-\alpha)100\%$ confidence level is defined as

$$CVaR_\alpha(X) = E[-X| - X \geq VaR_\alpha(X)], \ \forall \alpha \in (0,1]. \tag{5.1}$$

It is readily to show that CVaR can be represented in the following forms:

$$\begin{aligned} CVaR_\alpha(X) &= \frac{1}{\alpha}\int_{-\infty}^{-VaR_\alpha(X)} (-x)dF(x), \\ &= \frac{1}{\alpha}\int_0^\alpha VaR_\gamma(X)d\gamma, \\ &= VaR_\alpha(X) + \frac{1}{\alpha}\int_{-\infty}^{-VaR_\alpha(X)} F(x)dx, \\ &= VaR_\alpha(X) + \frac{1}{\alpha}E[(-X - VaR_\alpha(X))^+]. \end{aligned}$$

Rockafellar and Uryasev (2002) show that $CVaR_\alpha(X)$ can be defined as minimum objective value of the following convex optimization problem:

$$CVaR_\alpha(X) := \inf \left\{ a + \frac{1}{\alpha} E[(-X-a)^+] : a \in \mathbb{R} \right\}, \tag{5.2}$$

with $VaR_\alpha(X)$ being the minimizer. That is, one can obtain simultaneously $CVaR_\alpha(X)$ and $VaR_\alpha(X)$ by solving the above convex optimization problem in one dimension, which is appealing in computation. Moreover, Pflug (2000) and Rockafellar and Uryasev (2002) show that CVaR is a coherent risk measure.

### 5.2.2 Range Value-at-Risk

Cont et al. (2010) introduce an alternative risk measure, called *range value-at-risk* (RVaR). RVaR can be seen as the average VaR over a range of return.

**Definition 13** (Range Value at Risk) *For any $0 < \alpha < 1-\beta < 1$, the RVaR at the $(1-\beta-\alpha)100\%$ confidence level is*

$$RVaR_{[\alpha,1-\beta]}(X) = \frac{1}{1-\beta-\alpha} \int_\alpha^{1-\beta} VaR_\gamma(X) d\gamma.$$

It is notable that RVaR comprises CVaR as special cases: $RVaR_{[0,1-\beta]}(X) = CVaR_{1-\beta}(X)$. The risk measure RVaR is in general a non convex measure, but similar to VaR, it satisfies some other properties like monotonicity, translation invariance and positive homogeneity.

**Proposition 15** *For any $0 < \alpha < F(0) < 1-\beta < 1$, , RVaR has the following representations:*

*(i)* $RVaR_{[\alpha,1-\beta]}(X) = E[-X| - VaR_\alpha \leq X \leq -VaR_{1-\beta}]$;

*(ii)* $RVaR_{[\alpha,1-\beta]}(X) = \frac{1}{1-\beta-\alpha}\big(\beta CVaR_{1-\beta}(X) - \alpha CVaR_\alpha(X)\big)$;

*(iii)* $RVaR_{[\alpha,1-\beta]}(X) = -\frac{1}{1-\beta-\alpha}\int_\alpha^{1-\beta} F^{-1}(\gamma) d\gamma$.

All VaR, CVaR and RVaR belong to the family of distortion risk measures.

# 5.3 Risk Measures in CPT

## 5.3.1 Accounting for Value Function

Levy and Wiener (1998) introduce the notion of prospect stochastic dominance (PSD) to generalize the concepts of SSD and RSD and show that the preferences towards PSD can be represented by S-shaped value functions in the EUT paradigm. Specifically, $F$ dominates $G$ in the sense of PSD, denoted by $F \succeq_{PSD} G$, if $\int_x^y [G(t) - F(t)]dt \geq 0$ for all $x \leq 0 \leq y$, with at least one strict inequality, which implies SSD in gains and RSD in losses.

Levy and Wiener (1998) also provide a characterization of PSD using quantiles. In particular, $F \succeq_{PSD} G$ if

$$\int_p^q [F^{-1}(t) - G^{-1}(t)]dt \geq 0 \text{ for all } 0 \leq p \leq F(0) \leq G(0) \leq q \leq 1. \tag{5.3}$$

The quantile rule of PSD indicates that $F(0) \leq G(0)$, which can be deduced from the fact that for PSD rules, the area enclosed by $F$ and $G$, denoted by $G - F$, is always greater than zero around the reference point.

The following proposition establishes the connection between PSD and RVaR as a measure of risk, and characterizes decision makers with S-shaped value function.

**Proposition 16** *Consider two prospects $F$ and $G$ with $F(0) \leq G(0)$. The following are equivalent* *(i)* $F \succeq_{PSD} G$;

*(ii)* $RVaR_{[\alpha,1-\beta]}(X) \leq RVaR_{[\alpha,1-\beta]}(Y)$
*for all* $0 \leq \alpha \leq F(0) \leq G(0) \leq 1-\beta \leq 1$;

*(iii)* $\Delta_u(F,G) \geq 0$ *for all* $u \in \mathcal{V}_S$.

Levy and Wiener (1998) establishes the equivalence of (i) and (iii). Proposition 16 presents the equivalence relationship between PSD and RVaR, and shows that decision makers with S-shaped value functions prefer smaller RVaR at all levels of risk tolerance.

**Remark 4** *Note that when* $1-\beta \geq F(0)$, *we have* $VaR_{1-\beta}(X) \leq 0$.

The following lemma reveals the connections among VaR, CVaR, stochastic dominance and curvature structures of utility functions (preferences), providing behavior foundations for VaR and CVaR, respectively (see, e.g., Pflug 2000, Ogryczak and Ruszczynski 2002 and Ma and Wong 2010).

**Lemma 10 (VaR, CVaR and SD)** *Let* $F$ *and* $G$ *be defined as above.*

*(i)* $F \succeq_{FSD} G \Longleftrightarrow VaR_\alpha(X) \leq VaR_\alpha(Y)$ *for all* $\alpha \in (0,1] \Longleftrightarrow \Delta_u(F,G) \geq 0$ *for all* $u \in \mathcal{U}_I$.

*(ii)* $F \succeq_{SSD} G \Longleftrightarrow CVaR_\alpha(X) \leq CVaR_\alpha(Y)$ *for all* $\alpha \in (0,1] \Longleftrightarrow \Delta_u(F,G) \geq 0$ *for all* $u \in \mathcal{U}_{CV}$.

The equivalence of VaR and FSD shows that a decision maker who is rational (prefers more to less) prefers $F$ to $G$ if $F$ has smaller VaR than $G$ at all levels of risk tolerance. The equivalence of CVaR and SSD shows that a decision makers who is risk averse prefers $F$ to $G$ if $F$ has smaller CVaR than $G$ at all levels of risk tolerance.

**Remark 5** *If* $CVaR_\alpha(X) \leq CVaR_\alpha(Y)$ *for only one* $\alpha$ *level, the risk preference* *for a decision maker is still unknown.*

### 5.3.2 Accounting for Probability Weighting Function

According to Tversky and Kahneman (1992), the CPT preference can be represented by a nondecreasing S-shaped value function defined over a monetary gain or loss (change in wealth with respect to a reference point 0 that is normalized to zero) and a pair of PWFs $w^-(\cdot)$ and $w^+(\cdot)$ that transform the objective probabilities (cumulative distribution functions) to decision weights for losses and gains respectively. Without loss of generality, we assume that the PWFs are nondecreasing with boundary conditions $w^-(0) = w^+(0) = 0$ and $w^-(1) = w^+(1) = 1$. In addition, the gain weighting function equals the dual of the loss weighting function (i.e., $w^-(p) + w^+(1-p) = 1$) for all $p \in [0, 1]$. Then $w = (w^-, w^+)$ can be viewed as a transformation function that transfers a prospect $F$ to a "weighted" prospect $F^w$ such that

$$F^w(x) = \begin{cases} w^-(F(x)) & \text{for } x \in [a, 0], \\ 1 - w^+(\bar{F}(x)) & \text{for } x \in (0, b]. \end{cases}$$

A typical example of value function and PWFs are provided by Tversky and Kahneman (1992):

$$u(x) = \begin{cases} x^\alpha, & \text{if } x \geq 0, \\ -\lambda(-x)^\beta, & \text{if } x < 0, \end{cases} \quad \text{and} \quad \begin{cases} w^+(p) = \frac{p^\gamma}{(p^\gamma + (1-p)^\gamma)^{1/\gamma}}, \\ w^-(p) = \frac{p^\delta}{(p^\delta + (1-p)^\delta)^{1/\delta}}, \end{cases} \tag{5.4}$$

with $0 < \alpha < 1$, $0 < \beta < 1$, $\lambda \geq 1$, and $\gamma, \delta \geq 0.28$. For convenience, denote by $\mathcal{W}_I$, $\mathcal{W}_{CX}$, $\mathcal{W}_{CV}$, $\mathcal{W}_{IS}$ the sets of increasing, increasing convex, increasing concave, and increasing inverse S-shaped probability weighting function pairs respectively

The CPT value of prospect $F$ for a decision maker $u$ with PWFs $(w^-(\cdot), w^+(\cdot))$ is defined as

$$V_F = \int_a^0 u(x)d[w^-(F(x))] + \int_0^b u(x)d[1 - w^+(\bar{F}(x))].$$

Then the decision maker under CPT prefers $F$ to $G$ if and only if $\Delta_u^w(F, G) \geq 0$, where

$$\begin{aligned} & \Delta_u^w(F, G) = V_F - V_G \\ = & \int_a^0 [w^-(G(x) - w^-(F(x)]u'(x)dx + \int_0^b [w^+(\bar{F}(x)) - w^+(\bar{G}(x))]u'(x)dx. \end{aligned}$$

The next lemma provides a linkage between PWFs and RVaR.

**Lemma 11** (RVaR and PWFs) *Let $F$ and $G$ be defined as above with $F(0) \leq G(0)$. The following are equivalent*

*(i)* $RVaR_{[\alpha,1-\beta]}(X) \leq RVaR_{[\alpha,1-\beta]}(Y)$ *for all* $0 \leq \alpha \leq F(0) \leq G(0) \leq 1 - \beta \leq 1$*;*

*(ii)* $F^w \succeq_{PSD} G^w$ *for all* $(w^-, w^+) \in \mathcal{W}_{CX} \times \mathcal{W}_{CX}$.

Combining Proposition 16 and Lemma 11, we have the following choice characterization for RVaR.

**Proposition 17** *Let $F$ and $G$ be defined as above with $F(0) \leq G(0)$. Then $RVaR_{[\alpha,1-\beta]}(X) \leq RVaR_{[\alpha,1-\beta]}(Y)$ for all $0 \leq \alpha \leq F(0) \leq G(0) \leq 1 - \beta \leq 1$ if and only if $\Delta_u^w(F, G) \geq 0$ for all $u \in \mathcal{V}_S$ and $(w^-, w^+) \in \mathcal{W}_{CX} \times \mathcal{W}_{CX}$.*

Recall that in Chapter 3, we have defined M-PSD to characterize strong risk seeking in losses and strong risk aversion in gains. More specifically, $F$ dominates $G$ according to M-PSD, denoted by $F \succeq_{M-PSD} G$ if $F \succeq_{PSD} G$ and $\int_a^b [G(z) - F(z)]dz = 0$. Note that $F \succeq_{M-PSD} G$ implies $\int_a^0 [G(t) - F(t)]dt = \int_0^b [G(t) - F(t)]dt = 0$, and $RVaR_{[0,1]}(X) = RVaR_{[0,1]}(Y)$, i.e., the expected values of losses and the expected values of gains of $F$ and $G$ are the same. That is, M-PSD combines MPS in gains and MPC in loses. Hence the preference agreeing with M-PSD displays strong risk aversion in gains and strong risk seeking in losses, respectively, which captures the reflection effect observed in the experimental study of (Kahneman and Tversky 1979). The following proposition characterizes the curvatures of the value function and PWFs that can be represented by preference towards RVaR.

**Proposition 18** *Consider a decision maker with an increasing value function $u$ and a pair of increasing PWFs $(w^-, w^+)$. Then $\Delta_u^w(F, G) \geq 0$ for all prospect pairs $F$ and $G$ such that $RVaR_{[\alpha,1-\beta]}(X) \leq RVaR_{[\alpha,1-\beta]}(Y)$ for all $0 \leq \alpha \leq F(0) \leq G(0) \leq 1 - \beta \leq 1$ with $RVaR_{[0,1]}(X) = RVaR_{[0,1]}(Y)$ if and only if $u \in \mathcal{V}_S$ and $(w^-, w^+) \in \mathcal{W}_{CX} \times \mathcal{W}_{CX}$.*

Proposition 18 applies the equivalence of RVaR and M-PSD when $F$ and $G$ have the same mean. It shows that a decision maker exhibits strong risk aversion in gains and strong risk seeking in losses respectively will prefer smaller RVaR for all $\alpha$ and $\beta$ level. Such preferences can be characterized using S-shaped value functions and convex PWFs. The convexity of PWFs distorts the probability distributions by overweighting smaller values of losses or gains and underweighting larger values of losses and gains, which reflects pessimism in gains and optimism in losses. The more convex the PWFs, the greater the degrees of distortions and more pessimistic (optimistic) in gains (losses).

Similarly, the following lemma characterizes the curvatures of the value function and PWFs that can be represented by preference towards VaR and CVaR.

**Lemma 12** (VaR, CVaR and PWFs) *Suppose $F(0) = G(0) = 1$. Then*

*(i) $VaR_\alpha(X) \leq VaR_\alpha(Y)$ for all $\alpha \in (0,1)$ if and only if $F^w \succeq_{FSD} G^w$ for all $w^- \in \mathcal{W}_I$.*

*(ii) $CVaR_\alpha(X) \leq CVaR_\alpha(Y)$ for all $\alpha \in (0,1)$ if and only if $F^w \succeq_{SSD} G^w$ for all $w^- \in \mathcal{W}_{CV}$.*

**Proposition 19** *Suppose $F(0) = G(0) = 1$. Then*

*(i) $VaR_\alpha(X) \leq VaR_\alpha(Y)$ for all $\alpha \in (0,1)$ if and only if $\Delta_u^w(F,G) \geq 0$ for all $u \in \mathcal{U}_I$ and $w^- \in \mathcal{W}_I$.*

*(ii) $CVaR_\alpha(X) \leq CVaR_\alpha(Y)$ for all $\alpha \in (0,1)$ if and only if $\Delta_u^w(F,G) \geq 0$ for all $u \in \mathcal{U}_{CV}$ and $w^- \in \mathcal{W}_{CV}$.*

## 5.4 Implications for Optimal Decision Problems

To explore the economic implications of increased risk in the sense of RVaR under CPT on optimal decisions, we follow a similar setting of Diamond and Stiglitz (1974) to consider a utility function $U(x,\theta)$ which depends on a return value of $x$ and a control variable $\theta \in [0,1]$, and a pair of PWFs $(w^-(\cdot), w^+(\cdot))$ which satisfy the regularity properties of CPT. Assume that $U(x,\theta)$ is continuously differentiable in $\theta$.

Given a prospect $F$, the decision maker aims to solve the following optimization problem:

$$\max_{\theta\in[0,1]} \psi_F(\theta) \triangleq \int_a^0 U(x,\theta)dw^-(F(x)) + \int_0^b U(x,\theta)d[1 - w^+(\bar{F}(x))]. \tag{5.5}$$

Let $\arg\max_{\theta\in[0,1]} \psi_F(\theta)$ be the set of maximizers for $\psi_F(\theta)$ for any given $\gamma$ and prospect $F$. We are interested to relate the optimal solution to increases in RVaR.

The next theorem identifies sufficient conditions of $U(x,\theta)$ under which an increase in RVaR leads to smaller optimal solutions.

**Proposition 20** (Monotone Comparative Statics) *Consider two prospects $F$ and $G$ such that $F(0) \le G(0)$ and assume that $U_\theta(\cdot,\theta) \in \mathcal{V}_S$ with a pair of PWFs $(w^-, w^+)$. Then* $\arg\max_{\theta\in[0,1]} \psi_F(\theta) \ge \arg\max_{\theta\in[0,1]} \psi_G(\theta)$ *if*

*(1) $RVaR_{[\alpha,1-\beta]}(X) \le RVaR_{[\alpha,1-\beta]}(Y)$ for all $0 \le \alpha \le F(0) \le G(0) \le 1-\beta \le 1$, or*

*(2) $RVaR_{[\alpha,1-\beta]}(X) \le RVaR_{[\alpha,1-\beta]}(Y)$ for all $0 \le \alpha \le F(0) \le G(0) \le 1-\beta \le 1$ and the PWFs $(w^-, w^+) \in \mathcal{W}_{CX} \times \mathcal{W}_{CX}$.*

It is notable that to obtain the monotonicity result, it suffices to show that $\psi_F(\theta) - \psi_G(\theta)$ a decreasing function of $\theta$, or, equivalently, the objective function is submodular in the control variable $\theta$ and the level of risk in the sense of RVaR (see, e.g., Topkis 1998).

We next apply this result to a portfolio choice problem.

**Example 4** (Portfolio Choice) *Consider a portfolio choice problem in a market consisting of one risky asset (e.g., stock) and one risk-free asset (e.g., bond) with an investment planning horizon from date $0$ to date $T$. The return rate of the risk-free asset over the investment period is deterministic, denoted by $\tau$. We follow the literature and assume that the reference point of the investor is equal to the risk-free return rate $\tau$ (see, e.g., Barberis and Xiong 2009, He and Zhou 2011a, Li and Yang 2013, and Meng and Weng 2017). The return rate of the risky asset is random, denoted by $R$. Let $\theta \in [0,1]$ be the percentage of the wealth invested in the stock and the reminder in the risk free asset. Then the final normalized wealth is $\Pi(\theta) = (1-\theta)\tau + \theta R$. For convenience, let $X = R - \tau$ and we normalize the initial wealth level to one. The stock total excess return $R-\tau$ is a random variable with distribution function $F$. Then we have $\Pi(X,\theta) = \tau + \theta X$. Let $\theta \in [0,1]$ be the percentage of the wealth invested in the stock and the remainder in the risk-free asset. Then the relative gain or loss is $\theta X$. Suppose the distribution function of $X$ is $F$. Then the portfolio choice problem under CPT* *preference for* $u \in \mathcal{V}_S$ *is, therefore,*

$$\max_{\theta \in [0,1]} E[u(\theta X)]. \tag{5.6}$$

*Consider the portfolio choice problem (5.6) and the S-shaped value function specified in (5.4). We have*

$$U(x,\theta) = u(\theta x) = \begin{cases} \theta^\alpha x^\alpha, & \text{if } x > 0, \\ -\lambda \theta^\beta (-x)^\beta, & \text{if } x < 0. \end{cases}$$

*where* $0 < \alpha, \beta < 1$, $\lambda > 0$. *Then we have*

$$\begin{aligned} U_{\theta x} &= \begin{cases} \alpha^2 \theta^{\alpha-1} x^{\alpha-1}, & \text{if } x > 0, \\ \lambda \beta^2 \theta^{\beta-1} (-x)^{\beta-1}, & \text{if } x < 0, \end{cases} \quad \text{and} \\ U_{\theta xx} &= \begin{cases} -\alpha^2 (1-\alpha) \theta^{\alpha-1} x^{\alpha-2}, & \text{if } x > 0, \\ \lambda \beta^2 (1-\beta) \theta^{\beta-1} (-x)^{\beta-2}, & \text{if } x < 0. \end{cases} \end{aligned}$$

*Clearly,* $U$ *is thrice differentiable with respect to* $\theta$ *and* $x$, $U_{\theta x} > 0$ *for all* $x$, $U_{\theta xx} > 0$ *for* $x < 0$ *and* $U_{\theta xx} < 0$ *for* $x > 0$, *i.e.,* $U_\theta \in \mathcal{V}_S$. *According to Proposition 20, under the condition (1) or (2), an increase in risk in the sense of RVaR leads to smaller values of optimal portfolio decision* $\theta$ *or less riskier portfolio choices.*

**Remark 6** (PSD and Certainty Equivalent) *For all decision maker* $u \in \mathcal{V}_P$, *the certainty equivalence for a prospect* $F$, *denoted by* $CE(F)$, *satisfies* $u(CE(F)) = E_F u(x)$, *and* $CE(F) = E_F u(x) - \pi_F$, *where* $\pi_F$ *is risk premium. When* $F \succeq_{M-PSD} G$, $CE(F) \geq CE(G)$. *When* $F \succeq_{PSD} G$, *the relationship between* $\pi_F$ *and* $\pi_G$ *remains unclear unless* $E_F u(x) = E_G u(x)$.

## 5.5 Extensions

Markowitz (1952) proposes that decisions are based on *change* in wealth and individuals are risk averse for losses and risk seeking for gains as long as the possible outcomes are not very extreme, i.e., value functions are inverse S-shaped (concave in the losses and convex in the gains). For extreme outcomes, individuals are risk seeking for losses and risk averse for gains. Levy and Levy (2002) introduce the concept of Markowitz Stochastic Dominance (MSD) for inverse S-shaped functions. More specifically, $F$ dominates $G$ by MSD if $\int_a^x [G(z) - F(z)]dx \geq 0$ for all $x \leq 0$ and $\int_x^b [G(z) - F(z)]dz \geq 0$ for all $x \geq 0$ with at least one strict inequality. They show that $F$ dominates $G$ by MSD if and only if all $u \in \mathcal{V}_{IS}$ prefer $F$ to $G$. In what follows, we show that the choice-theoretic characterizations for CPT can be readily extended to that with inverse S-shaped value functions.

**Proposition 21** (CVaR and MSD) *Let $F$, $G$ and $\alpha$ be defined as above. The following are equivalent*

*(i)* $F \succeq_{MSD} G$;

*(ii)* $CVaR_\alpha(X) \leq CVaR_\alpha(Y)$ *for all* $\alpha \in (0, \min(F(0), G(0)))$, *and* $\mu_F - (1-\beta)CVaR_{1-\beta}(X) \leq \mu_G - (1-\beta)CVaR_{1-\beta}(Y)$ *for all* $\beta \in (0, \max(F(0), H(0)))$;

*(iii)* $\Delta_u(F, G) \geq 0$ *for all* $u \in \mathcal{V}_{IS}$.

Similar to M-PSD, to characterize the preferences with strong risk aversion (strong risk seeking) in losses (gains), we can define the SD condition of mean-preserving MSD as follows.

**Definition 14 (Mean Preserving MSD)** *Prospect $F$ dominates prospect $G$ by* mean preserving MSD, *denoted by* $F \succeq_{M-MSD} G$, *if*

*(i)* $\int_a^b [G(z) - F(z)]dz = 0$;

*(ii)* $\int_a^x [G(z) - F(z)]dz \geq 0$ *for all* $x \in [a, 0)$ *and* $\int_y^b [G(z) - F(z)]dz \geq 0$ *for all* $y \in (0, b]$, *with at least one strict inequality.*

The following lemma provides a linkage between M-PSD and M-MSD.

**Lemma 13** (M-PSD and M-MSD) *Consider two prospects $F$ and $G$. Then* $F \succeq_{M-PSD} G$ *if and only if* $G \succeq_{M-MSD} F$.

Note that in general $F \succeq_{PSD} G$ does not necessarily imply $G \succeq_{MSD} F$ and vice versa. Lemma 13 shows that under the mean preserving condition, $F \succeq_{PSD} G$ is equivalent to $G \succeq_{MSD} F$.

Analogous to the analysis in the CPT paradigm, we show that the preference towards smaller RVaR is characterized by S-shaped value functions and convex PWFs (Proposition 18), we can also provide a choice-theoretic characterization for when considering the PWFs as that in the CPT paradigm.

**Proposition 22** (Curvature Representation for CVaR) *Consider an increasing value function $u$ and a pair of increasing PWFs $(w^-(\cdot), w^+(\cdot))$. Then $\Delta_u^w(F,G) \geq 0$ for all pairs of $F$ and $G$ such that $CVaR_\alpha(X) \leq CVaR_\alpha(Y)$ for all $\alpha \in (0, \min(F(0), G(0)))$, $CVaR_{1-\beta}(X) \geq CVaR_{1-\beta}(Y)$ for all $\beta \in (0, \max(F(0), H(0)))$ and $\mu_F = \mu_G$ if and only if $u \in \mathcal{V}_{IS}$ and PWFs $(w^-, w^+) \in \mathcal{W}_{CV} \times \mathcal{W}_{CV}$.*

Proposition 22 characterizes strong risk aversion in losses and strong risk seeking in gains. It is a direct application of Proposition 7.

## 5.6 Conclusion

This chapter provides behaviour foundations for VaR, CVaR and RVaR as risk measures in the paradigm of CPT. We demonstrate various logical linkages among risk measures, stochastic dominance and choices made by decision makers with CPT risk preferences. We show that the SD rule of mean-preserving prospect stochastic dominance and mean-preserving Markowitz stochastic dominance are compatible with RVaR and CVaR, respectively. The equivalence of these risk measures and stochastic dominance conditions enables the elicitation of risk preference through pairwise comparisons of risky prospects.

We also show that RVaR as a risk measure has economic implications for optimal decision problems in the CPT paradigm. Using portfolio choice as an example, we show that for a CPT value maximizing investor will invest less in risky assets in the presence of increases in risk in the sense of RVaR.

# Chapter 6

# Conclusion

This thesis investigates risk changes and risk attitudes in the framework of non-expected utility theories. A set of stochastic dominance conditions are generalised to provide characterizations for changes in risk and risk attitudes in CPT.

Specifically, Chapter 2 provides choice-theoretic characterizations for increases in risk and in risk aversion in the paradigm of CPT. We first generalize the notions of increase in risk (mean-preserving spread) and strong risk aversion introduced by Rothschild and Stiglitz (1970) to the CPT paradigm by introducing the SD rule of mean-preserving prospect stochastic dominance (M-PSD). We show that strong prospect risk aversion can be represented by an S-shaped value function and a pair of convex PWFs. To further characterize the increase in risk aversion in the CPT paradigm, we extend the SD rule of mean utility preserving increase in risk introduced by Diamond and Stiglitz (1974) to the CPT paradigm by introducing the SD rule of CPT-value preserving increase in risk (V-PSD) to address the tradeoff between means and variances in gains and domain of losses respectively. We further extend our analysis to account for general reference points, inverse S-shaped value functions and inverse S-shaped PWFs, respectively. We also explore the implications of our choice-theoretic characterizations on risk preference elicitation in the CPT paradigm.

Chapter 3 provides choice-theoretic characterizations for the increases in downside risk and downside risk aversion in CPT. We identify another set of SD conditions of increases in downside risk to characterize downside risk aversion and intensity of downside risk aversion under CPT using prudence. We also extend the analysis to inverse S-shaped value functions.

Chapter 4 provides behaviour foundations for VaR, CVaR and RVaR as risk measures in the paradigm of CPT. We demonstrate various logical linkages among risk measures, stochastic dominance and choices made by decision makers with CPT risk preferences. We show that the SD rule of mean-preserving prospect stochastic dominance and mean-preserving Markowitz stochastic dominance are compatible with RVaR and CVaR, respectively. The equivalence of these risk measures and stochastic dominance conditions enables the elicitation of risk preference through pairwise comparisons of risky prospects. We also show that RVaR as a risk measure has economic implications for optimal decision problems in the CPT paradigm. Using portfolio choice as an example, we show that for a CPT value maximizing investor will invest less in risky assets in the presence of increases in risk in the sense of RVaR.

A set of stochastic dominance conditions are generalised to provide characterizations for changes in risk and risk attitudes in CPT. The SD conditions identified in the study provides an approach for risk preference elicitation in the paradigm of CPT without prior knowledge of the value function and probability weighting functions. This work further demonstrates that the increase in risk aversion in the CPT paradigm can be characterized using increase in the absolute value of the absolute risk aversion (more risk averse in gains and more risk seeking in losses), and the increase of convexity of PWFs. In addition, the increase in downside risk aversion in the CPT paradigm can be characterized using increase in the the absolute value of prudence (more downside risk averse in gains and losses).

In addition, this thesis explores the implications of the choice-theoretic characterizations in risk preference elicitation in the CPT paradigm. The research also demonstrates that these stochastic dominance rules and risk measures have economic implications for optimal decision problems in the paradigm of CPT, which sheds new light into the applications of CPT.

The current research can be further developed in two interesting directions. Firstly, an important role of SD is to guide the design of choice-based experimental studies for preference elicitation. It is also notable that our approach can be integrated with the classic elicitation approaches such as certainty equivalence or probability equivalence approach in the experimental studies. The experiments in Kahneman and Tversky (1979) are either positive or negative prospects and they do not always satisfy the SD conditions that are characterized by CPT preferences. It is important to redesign pairwise choice experiments with mixed prospects that satisfy the SD conditions in the CPT paradigm in order to elicit the CPT preferences. As an empirical implication for experimental studies, if someone is asked to make choices between all pairs of prospects satisfying the M-PSD order and his choices agree with the M-PSD order, then his/her preference can be represented by a nondecreasing $S$-shaped value function and a pair nondecreasing convex PWFs. Clearly, when rank- and sign-dependence is irrelevant, the CPT preference reduces to expected utility. However, it is impossible to ask an individual to make choices between all pairs of prospects satisfying the desired stochastic dominance order, it remains unclear how many such pairwise comparisons are sufficient for preference elicitation.

Secondly, this thesis characterizes the intensity of downside risk aversion using prudence measure. The analyses can be extended to other alternative downside risk aversion measures introduced in the literature, such as Liu and Meyer (2012) for decreasing absolute risk aversion (DARA), Modica and Scarsini (2005) and Crainich and Eeckhoudt (2008) for the ratio of the third and first derivatives, Keenan and Snow (2002) for Schwarzian derivatives, and Huang and Stapleton (2014) for cautiousness measure.

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# Appendix for Chapter 3

## Proof of Statements

### Proof of Lemma 2

We first prove the "only if" part. For convenience, we define the following slop functions

$$\rho_-(x) = \frac{w^-(G(x)) - w^-(F(x))}{G(x) - F(x)}, \text{ and } \rho_+(x) = \frac{w^+(\bar{F}(x)) - w^+(\bar{G}(x))}{\bar{F}(x) - \bar{G}(x)}.$$

We argue that the convexity of $w^-(\cdot)$ and $w^+(\cdot)$ implies that $\rho_-(x)$ is increasing in $x$ and $\rho_+(x)$ is decreasing in $x$. To show this, for any $x < x' \leq 0$, we have

$$\frac{w^-(G(x)) - w^-(F(x))}{G(x) - F(x)} \leq \frac{w^-(G(x)) - w^-(F(x'))}{G(x) - F(x')} \leq \frac{w^-(G(x')) - w^-(F(x'))}{G(x') - F(x')},$$

where both inequalities are due to the convexity of $w^-$ and the monotonicity of $F$, $G$ and $w^-$. Similarly, for any $0 \leq x < x'$, we have

$$\frac{w^+(\bar{F}(x')) - w^+(\bar{G}(x'))}{\bar{F}(x') - \bar{G}(x')} \leq \frac{w^+(\bar{F}(x')) - w^+(\bar{G}(x))}{\bar{F}(x') - \bar{G}(x)} \leq \frac{w^+(\bar{F}(x)) - w^+(\bar{G}(x))}{\bar{F}(x) - \bar{G}(x)},$$

where both inequalities are also due to the convexity of $w^+$ and the monotonicity of $\bar{F}$, $\bar{G}$ and $w^+$.

Then for $x \in [a, 0]$, we have

$$\begin{aligned}
& \int_x^0 \left[w^-\left(G(z)\right) - w^-\left(F(z)\right)\right] dz \\
= & \int_x^0 \rho_-(z)[G(z) - F(z)]dz \\
= & -\int_x^0 \rho_-(z)d\left(\int_z^0 [G(t) - F(t)]dt\right) \\
= & \rho_-(x)\int_x^0 [G(t) - F(t)]dt + \int_x^0 \rho_-^{'}(z)\int_z^0 [G(t) - F(t)]dtdz \geq 0,
\end{aligned}$$

where the inequality follows from the fact that $\rho_-(x) \geq 0$, $\rho_-^{'}(x) \geq 0$, and $\int_x^0 [G(z) - F(z)]dz \geq 0$ for all $x \leq 0$.

Similarly, for $y \in [0, b]$, we have

$$\begin{aligned}
& \int_0^y \left[w^+\left(\bar{F}(z)\right) - w^+\left(\bar{G}(z)\right)\right] dz \\
= & \int_0^y \rho_+(z)\left[\bar{F}(z) - \bar{G}(z)\right] dz \\
= & \int_0^y \rho_+(z)d\left(\int_0^z \left[\bar{F}(t) - \bar{G}(t)\right] dt\right) \\
= & \rho_+(y)\int_0^y [G(t) - F(t)]dt - \int_0^y \rho_+^{'}(z)\int_0^z [G(t) - F(t)]dtdz \geq 0,
\end{aligned}$$

where the inequality follows from the fact that $\rho_+(x) \geq 0$, $\rho_+^{'}(x) \leq 0$, and $\int_0^y [G(z) - F(z)]dz \geq 0$ for all $y \geq 0$. Combining the two cases, we know that $F \succeq_{PSD} G$ implies $F^w \succeq_{PSD} G^w$ when $w^-$ and $w^+$ are convex.

For the "if" part, note that among the class of convex PWFs, there is a pair of PWFs that almost does not change the distribution, e.g., linear PWFs, under which $F^w \succeq_{PSD} G^w$ implies $F \succeq_{PSD} G$.

## Proof of Proposition 1

We first prove the "if" part. It follows from the "if" part of Lemma 1 that for any given pair of PWFs $(w^-, w^+) \in \mathcal{W}_{CX}$, $\Delta_u^w(F, G) \geq 0$ for all $u \in \mathcal{V}_S$ implies that the risk change from $F^w$ to $G^w$ is PSD. It then follows from the "only if" part of Lemma 2 that $F^w \succeq_{PSD} G^w$ for all $(w^-, w^+) \in \mathcal{W}_{CX}$ implies that $F \succeq_{PSD} G$

The "only if" part follows immediately from the "only if" part of Lemma 2 and "if" part of Lemma 1.

## Proof of Proposition 2

For the "if" part, it follows from Lemma 2 that $F \succeq_{M-PSD} G$ implies $F \succeq_{W-PSD} G$ when $w^-(\cdot)$ and $w^+(\cdot)$ are convex. Then we immediately have $\Delta_u^w(F, G) \geq 0$ for all $u \in \mathcal{V}_S$ by Lemma 1.

For the "only if" part, we first consider the domain of losses by showing that $\Delta_u^w(F, G) \geq 0$ for all risk changes from $F$ to $G$ such that $F \succeq_{M-PSD} G$ implies that $w^-(\cdot)$ is convex and value function is convex in the domain of losses. To this end, consider a prospect $F := (x_1, p_1; ...; x_i, p_i; x_{i+1}, p_{i+1}; ...; x_n, p_n)$ where $x_1 \leq ... \leq x_n \leq 0$, and, in particular, for some $i$, $x_{i+1} = x_i + \gamma$ for any $\gamma \geq 0$ and $p_i = p_{i+1} = p$ for any $p \in [0, 1/2]$. Let $G$ be the prospect such that

$$G := (x_1, p_1; ...; x_i + \delta, p; x_{i+1} - \delta, p; ...; x_n, p_n),$$

where $0 < \delta \leq \gamma/2$ so that $x_{i-1} \leq x_i + \delta \leq x_{i+1} - \delta \leq x_{i+2}$. Then the risk change from $F$ to $G$ is a mean-preserving contraction, which also implies that

$F \succeq_{M-PSD} G$. Since $\Delta_u^w(F,G) \geq 0$ for such risk changes,

$$\Delta_u^w(F,G) \;=\; \pi_{i+1}u(x_{i+1}) + \pi_i u(x_i) - \pi_{i+1}u\left(x_{i+1}-\delta\right) - \pi_i u\left(x_i+\delta\right) \geq 0.$$

Since $u$ is increasing, we have $u\left(x_i+\delta\right) \geq u(x_i)$, and $u(x_{i+1}) \geq u\left(x_{i+1}-\delta\right)$, which implies that

$$\pi_{i+1}\left[u(x_{i+1}) - u(x_{i+1}-\delta)\right] \geq \pi_i\left[u(x_i+\delta) - u(x_i)\right].$$

Consequently, we have

$$\frac{\pi_i}{\pi_{i+1}} \leq \frac{u(x_{i+1}) - u(x_{i+1}-\delta)}{u(x_i+\delta) - u(x_i)}. \tag{1}$$

Since $\delta$ can be arbitrarily small, taking limit with respect to $\delta$ on both sides of (1) yields

$$\frac{\pi_i}{\pi_{i+1}} \leq \lim_{\delta\to 0}\frac{u(x_{i+1}) - u(x_{i+1}-\delta)}{u(x_i+\delta) - u(x_i)} = \lim_{\delta\to 0}\frac{[u(x_{i+1}) - u(x_{i+1}-\delta)]/\delta}{[u(x_i+\delta) - u(x_i)]/\delta} = \frac{u'(x_{i+1})}{u'(x_i)}.$$

Since the value of $\gamma$ can also be arbitrarily small, further taking limit with respect to $\gamma$ yields

$$\frac{\pi_i}{\pi_{i+1}} \leq \lim_{\gamma\to 0}\frac{u'(x_{i+1})}{u'(x_i)} = 1,$$

which implies that

$$\pi_i = w^-\left(\sum_{j=1}^{i-1} p_j + p\right) - w^-\left(\sum_{j=1}^{i-1} p_j\right) \leq \pi_{i+1} = w^-\left(\sum_{j=1}^{i-1} p_j + 2p\right) - w^-\left(\sum_{j=1}^{i-1} p_j + p\right),$$

for all $p \in [0, 1/2]$. This implies the convexity of $w^-$.

We next show that the value function is convex in losses. Since $p$ can be arbitrarily small, taking limit with respect to $p$ on both sides of (1) leads to

$$\lim_{p\to 0}\frac{\pi_i}{\pi_{i+1}} = 1 \leq \frac{u(x_{i+1}) - u(x_{i+1}-\delta)}{u(x_i+\delta) - u(x_i)},$$

for all $\gamma > 0$. Since $\delta$ can be arbitrarily small, we know that $u$ must be convex in the domain of losses.

We now address the domain of gains. Consider a prospect

$$F := (x_1, p_1; ...; x_i, p_i; x_{i+1}, p_{i+1}; ...; x_n, p_n)$$

where where $0 \leq x_1 \leq ... \leq x_n$, and, in particular, for some $i$, $x_{i+1} = x_i + \gamma$ for any $\gamma \geq 0$ and $p_i = p_{i+1} = p$ for any $p \in [0, 1/2]$. Let $G$ be a prospect such that

$$G := (x_1, p_1; ...; x_i - \delta, p; x_{i+1} + \delta, p; ...; x_n, p_n),$$

where $0 < \delta \leq \min(x_i - x_{i-1}, x_{i+2} - x_{i+1})$ so that $x_{i-1} \leq x_i - \delta \leq x_{i+1} + \delta \leq x_{i+2}$. Clearly, the risk change from $F$ to $G$ is a mean-preserving spread, which also implies that $F \succeq_{M-PSD} G$. Since $\Delta_u^w(F, G) \geq 0$ for all such risk changes, we have

$$\Delta_u^w(F,G) \quad = \quad \pi_{i+1}u(x_{i+1}) + \pi_i u(x_i) - \pi_{i+1}u\left(x_{i+1}+\delta\right) - \pi_i u\left(x_i - \delta\right) \geq 0.$$

Since $u$ is increasing, we have $u\left(x_{i+1}+\delta\right) \geq u(x_{i+1})$, and $u(x_i) \geq u\left(x_i - \delta\right)$, which implies that

$$\pi_{i+1}\left[u(x_{i+1}+\delta) - u(x_{i+1})\right] \leq \pi_i\left[u(x_i) - u(x_i-\delta)\right],$$

for all such $\delta$. Consequently, we have

$$\frac{\pi_i}{\pi_{i+1}} \geq \frac{u(x_{i+1}+\delta) - u(x_{i+1})}{u(x_i) - u(x_i-\delta)}. \tag{2}$$

Similar to the analysis for the domain of losses, taking limits with respect to $\delta$ and $\gamma$ on both side of (2) yields

$$\frac{\pi_i}{\pi_{i+1}} \geq 1,$$

which implies that

$$w^+\left(2p + \sum_{j=i+2}^{n} p_j\right) - w^+\left(p + \sum_{j=i+2}^{n} p_j\right) \geq w^+\left(p + \sum_{j=i+2}^{n} p_j\right) - w^+\left(\sum_{j=i+2}^{n} p_j\right),$$

for any $p \in [0, 1/2]$. This implies the convexity of $w^+$.

We next show that the value function is concave in gains. Since $p$ can be arbitrarily small, taking limit with respect to $p$ on both side of (2) yields

$$\lim_{p\to 0} \frac{\pi_i}{\pi_{i+1}} = 1 \geq \frac{u(x_{i+1}+\delta) - u(x_{i+1})}{u(x_i) - u(x_i - \delta)},$$

for all $\gamma > 0$. Since $\delta$ can be arbitrarily small, $u$ must be concave in gains.

## Proof of Proposition 3

Note that $F \succeq_{V-PSD} G$ is equivalent to $F^{(u,w)} \succeq_{M-PSD} G^{(u,w)}$. Then, treating $T_u$ as a value function, by Proposition 2 we know that

$$\Delta_v^{\tilde{w}}(F, G) = \Delta_{T_u}^{T_w}(F^{(u,w)}, G^{(u,w)}) \geq 0$$

for all $F^{(u,w)} \to G^{(u,w)}$ such that $F^{(u,w)} \succeq_{M-PSD} G^{(u,w)}$ if and only if $T_u \in \mathcal{V}_S$ and $(T_w^+, T_w^-) \in \mathcal{W}_{CX}$. Note that $|R_v(x)| \geq |R_u(x)|$ for any $x$ if and only if $T_u \in \mathcal{V}_S$, and that $|R_{\tilde{w}^-}(x)| \geq |R_{w^-}(x)|$ and $|R_{\tilde{w}^+}(x)| \geq |R_{w^+}(x)|$ if and only if $(T_w^-, T_w^+)$ are convex. The desired result holds.

## Proof of Proposition 4

According to Proposition 2, an individual displays prospect strong risk aversion if and only if the value function $u$ is $S$-shaped and the PWFs $(w^-, w^+)$ are convex. Since $u$ is strictly increasing, we have $u(M) > u(m)$ for all $M > m$. Then $w^+(p)\,(u(M) - u(m)) + u(m)$ is strictly increasing and convex in $p$. As $u$ is strictly increasing and concave in gains, it is easy to verify that $(u^{-1})'(y) = \frac{1}{u'(u^{-1}(y))} > 0$ and $(u^{-1})''(y) = \frac{-u''(u^{-1}(y))}{(u'(u^{-1}(y)))^3} > 0$, i.e., $u^{-1}$ is strictly increasing and convex. It is known that the composition of two increasing convex functions remains increasing and convex. Hence, $x^+_{CE}(p)$ is increasing and convex in $p$.

Similarly, in the domain of losses, the monotonicity and convexity of $u$ implies that $u^{-1}$ is strictly increasing and concave. As $w^-$ is strictly increasing and convex, $u(M) - w^-(1-p)\,(u(M) - u(m))$ is strictly increasing and concave in $p$. It is known that the composition of two increasing concave functions remains increasing and concave. Hence, $x^-_{CE}(p)$ is increasing and concave in $p$.

## Proof of Proposition 5

Note that $\frac{\partial}{\partial\theta}\psi_F(\theta) = \int_a^0 U_\theta(x,\theta)dw^-(F(x)) + \int_0^b U_\theta(x,\theta)d[1 - w^+(\bar{F}(x))]$. Since $U_\theta(\cdot,\theta) \in \mathcal{V}_S$, we can treat $\frac{\partial}{\partial\theta}\psi_F(\theta)$ as if it were a CPT value of prospect $F$ for an individual with the value function $U_\theta$ and PWFs $(w^-, w^+)$. Then according to Lemma 1 and Proposition 2, under condition (1) or (2), we have $\frac{\partial}{\partial\theta}\psi_F(\theta) \geq \frac{\partial}{\partial\theta}\psi_G(\theta)$, which implies that $\psi_F(\theta) - \psi_G(\theta)$ is increasing in $\theta$. In other words, using a shift factor $\gamma$ to index the level of risk in the sense of PSD or M-PSD, the objective function is submodular in $\theta$ and the risk level. It follows from

Proposition 2.8.2 in Topkis (1998) that $\arg\max_{\theta\in[0,1]} \psi_F(\theta) \geq \arg\max_{\theta\in[0,1]} \psi_G(\theta)$.

## Proof of Proposition 6

Note that $G$ crosses $F$ exactly once at $x_0$ and $F \succeq_{RSD} G$, which implies that $\int_{x_0}^b [G(z) - F(z)]dz > 0$ and $\int_a^x [G(z) - F(z)]dz \leq 0$ for $x \leq x_0$ with a strict inequality at $x = x_0$. Since $F \succeq_{RSD} G$ implies that $\int_a^b [G(z) - F(z)]dz \geq 0$, the function $\int_a^x [G(z) - F(z)]dz$ must cross over zero from below exactly once in $(x_0, b]$. Let $\underline{r} = \inf\{x \in (x_0, b] : \int_a^x [G(z) - F(z)]dz \geq 0\}$. Pick any $r \in [\underline{r}, b]$. Clearly, $\int_r^y [G(z) - F(z)]dz \geq 0$ for all $y \geq r$ and $\int_x^r [G(z) - F(z)]dz \geq 0$ for all $x \in [x_0, r]$ since $G(x) \geq F(x)$ for all $x \geq x_0$. We also have $\int_x^r [G(z) - F(z)]dz \geq \int_a^r [G(z) - F(z)]dz \geq 0$ for all $x \in [a, x_0)$ where the first inequality is by the fact that $G(x) \leq F(x)$ for all $x \leq x_0$ and the second by the definition of $\underline{r}$ and $r \geq \underline{r}$. Then, by the definition of PSD, we know that $F$ dominates $G$ by PSD with respect to any $r \in [\underline{r}, b]$.

We now prove assertion (ii). Note that $w^-(G(x))$ and $w^-(F(x))$ have the same crossing point as that of $G$ and $F$. Hence $w^-(G(x))$ crosses $w^-(F(x))$ exactly once at $x_0$, which implies $\int_a^{x_0} [w^-(G(z)) - w^-(F(z))]dz \leq 0$ for $x \leq x_0$ with strict inequality at $x = x_0$. We have

$$
\begin{aligned}
& \int_a^{\underline{r}} [w^-(G(z)) - w^-(F(z))]dz \\
= \; & \int_a^{\underline{r}} \rho_-(z)[G(z) - F(z)]dz \\
= \; & \left[\rho_-(x)\int_a^x [G(z) - F(z)]dz\right]_{x=a}^{x=\underline{r}} - \int_a^{\underline{r}} \rho_-^{'}(x)\int_a^x [G(z) - F(z)]dzdx \\
= \; & \rho_-(\underline{r})\int_a^{\underline{r}} [G(z) - F(z)]dz - \int_a^{\underline{r}} \rho_-^{'}(x)\int_a^x [G(z) - F(z)]dzdx \geq 0,
\end{aligned}
$$

where $\rho_-(x) = \frac{w^-(G(x))-w^-(F(x))}{G(x)-F(x)}$ and satisfying $\rho_-(x) > 0$ and $\rho_-^{'}(x) \geq 0$ for all $x$ by the proof of Lemma 2, and the inequality is due to the facts that $\rho_-(x) > 0$ and $\rho_-^{'}(x) \geq 0$, $\int_a^{\underline{r}}[G(z) - F(z)]dz \geq 0$ and $\int_a^x[G(z) - F(z)]dz \leq 0$ for $x \in [a, \underline{r}]$. Hence the function $\int_a^x[w^-(G(z)) - w^-(F(z))]dz$ must cross over zero from below exactly once in $[x, \underline{r}]$. Let $\underline{r}^{'} = \inf\{x \in (x_0, \underline{r}] : \int_a^x[w^-(G(z)) - w^-(F(z))]dz \geq 0\}$. Pick any $r \in [\underline{r}^{'}, b]$. Clearly, $\int_r^y[w^+(\bar{F}(z)) - w^+(\bar{G}(z))]dz \geq 0$ for all $y \in [\underline{r}^{'}, b]$ and $\int_x^r[w^-(G(z)) - w^-(F(z))]dz \geq 0$ for all $x \in [x_0, r]$ since $w^-(\cdot)$ and $w^+(\cdot)$ are increasing and $G(x) \geq F(x)$ for all $x \in [x_0, b]$. We also have $\int_x^r[w^-(G(z)) - w^-(F(z))]dz \geq \int_a^r[w^-(G(z)) - w^-(F(z))]dz \geq 0$ for all $x \in [a, x_0]$, where the first inequality is by the fact that $w^-(G(x)) \leq w^-(F(x))$ for all $x \leq x_0$, and the second inequality is by the definition of $\underline{r}^{'}$ and $r \geq \underline{r}^{'}$. Then by definition of PSD, we know that $F^w \succeq_{PSD} G^w$ with respect to any $r \in [\underline{r}^{'}, b]$.

## Proof of Lemma 3

According to the definition of M-PSD, $F \succeq_{M-PSD} G$ implies $\int_a^0[G(z)-F(z)]dz = \int_0^b[G(z) - F(z)] = 0$, $\int_x^0[G(z) - F(z)]dz \geq 0$ for all $x \in [a, 0]$, and $\int_0^y[G(z) - F(z)]dz \geq 0$ for all $y \in [0, b]$. Then we have

$$\int_a^x [G(z)-F(z)]dz = \int_a^0 [G(z)-F(z)]dz - \int_x^0 [G(z)-F(z)]dz = -\int_x^0 [G(z)-F(z)]dz \leq 0$$

for all $x \in [a, 0]$,

$$\begin{aligned}\int_y^b [G(z) - F(z)]dz &= \int_0^b [G(z) - F(z)]dz - \int_0^y [G(z) - F(z)]dz \\ &= -\int_0^y [G(z) - F(z)]dz \leq 0\end{aligned}$$

for all $y \in [0,b]$. which implies that $G \succeq_{M-MSD} F$ by the definition of M-MSD. Conversely, following the same logic, we can show that $G \succeq_{M-MSD} F$ implies $F \succeq_{M-PSD} G$.

## Proof of Proposition 7

We first the "if" part. Taking integration by parts for $\Delta_u^w(F,G)$ yields

$$\begin{aligned}\Delta_u^w(F,G) &= u^{'}(0)\int_a^0 \left[w^-\left(G(x)\right) - w^-\left(F(x)\right)\right] dx \\ &\quad -\int_a^0 u^{''}(x)\int_a^x \left[w^-\left(G(z)\right) - w^-\left(F(z)\right)\right] dzdx \\ &\quad +u^{'}(0)\int_0^b \left[w^+\left(\bar{F}(y)\right) - w^+\left(\bar{G}(y)\right)\right] dy \\ &\quad +\int_0^b u^{''}(y)\int_y^b \left[w^+\left(\bar{F}(z)\right) - w^+\left(\bar{G}(z)\right)\right] dzdy.\end{aligned}$$

Note that $u'(0) > 0$, $u''(x) < 0$ for $x < 0$ and $u''(x) > 0$ for $x > 0$. It suffices to show $\int_a^x \left[w^-\left(G(z)\right) - w^-\left(F(z)\right)\right] dz \geq 0$ for all $x \in (a,0]$ and

$$\int_y^b \left[w^+\left(\bar{F}(z)\right) - w^+\left(\bar{G}(z)\right)\right] dz \geq 0$$

for all $y \in [0,b)$.

Recall that in the proof of Lemma 2, we defined the following slop functions

$$\rho_-(x) = \frac{w^-(G(x)) - w^-(F(x))}{G(x) - F(x)} \text{ and } \rho_+(x) = \frac{w^+(\bar{F}(x)) - w^+(\bar{G}(x))}{\bar{F}(x) - \bar{G}(x)}.$$

We argue that the concavity of $w^-(\cdot)$ and $w^+(\cdot)$ implies that $\rho_-(x)$ is decreasing in $x$ and $\rho_+(x)$ is increasing in $x$. To show this, for any $x < x' \leq 0$, we have

$$\frac{w^-(G(x)) - w^-(F(x))}{G(x) - F(x)} \geq \frac{w^-(G(x)) - w^-(F(x'))}{G(x) - F(x')} \geq \frac{w^-(G(x')) - w^-(F(x'))}{G(x') - F(x')},$$

where both inequalities are due to the concavity of $w^-$ and the monotonicity of $G$, $F$ and $w^-$. Similarly, for any $0 \leq x < x'$ we have

$$\frac{w^+(\bar{F}(x')) - w^+(\bar{G}(x'))}{\bar{F}(x') - \bar{G}(x')} \geq \frac{w^+(\bar{F}(x')) - w^+(\bar{G}(x))}{\bar{F}(x') - \bar{G}(x)} \geq \frac{w^+(\bar{F}(x)) - w^+(\bar{G}(x))}{\bar{F}(x) - \bar{G}(x)},$$

where both inequalities are also due to the concavity of $w^+$ and the monotonicity of $\bar{G}$, $\bar{F}$ and $w^+$.

Then for $x \in (a, 0]$, we have

$$\begin{aligned}
& \int_a^x \left[w^-\left(G(z)\right) - w^-\left(F(z)\right)\right] dz \\
= \;& \int_a^x \rho_-(z)[G(z) - F(z)]dz \\
= \;& \int_a^x \rho_-(z)d\left(\int_a^z [G(t) - F(t)]dt\right) \\
= \;& \rho_-(x)\int_a^x [G(t) - F(t)]dt - \int_a^x \rho_-^{'}(z)\int_a^z [G(t) - F(t)]dtdz \geq 0,
\end{aligned}$$

where the inequality is due to $\rho_-(x) \geq 0$, $\rho_-^{'}(x) \leq 0$, and $\int_a^x [G(z) - F(z)]dz \geq 0$ for all $x \leq 0$.

Similarly, for $y \in [0, b)$, we have

$$\begin{aligned}
& \int_y^b \left[w^+\left(\bar{F}(z)\right) - w^+\left(\bar{G}(z)\right)\right] dz \\
= \;& \int_y^b \rho_+(z)\left[\bar{F}(z) - \bar{G}(z)\right] dz \\
= \;& -\int_y^b \rho_+(z)d\left(\int_z^b \left[\bar{F}(t) - \bar{G}(t)\right] dt\right) \\
= \;& \rho_+(y)\int_y^b [G(t) - F(t)]dt + \int_y^b \rho_+^{'}(z)\int_z^b [G(t) - F(t)]dtdz \geq 0,
\end{aligned}$$

where the inequality is due to $\rho_+(x) \geq 0$, $\rho_+^{'}(x) \geq 0$, and $\int_y^b [G(z) - F(z)]dz \geq 0$

for all $y \geq 0$. Combining the two cases, we know that $\Delta_u^w(F, G) \geq 0$.

For the "only if" part, we can prove with the same technique as that in the proof of Proposition 2 where a mean-preserving contraction in the domain of losses and a mean-preserving spread in the domain of gains were constructed. In particular, using a mean-preserving spread in the domain of losses and a mean-preserving contraction in the domain of gains respectively, we can show that $\Delta_u^w(F, G) \geq 0$ for any M-MSD risk change from $F$ to $G$ implies value function is inverse $S$-shaped and PWFs $w^-$ and $w^+$ are both concave. The detailed arguments are omitted for brevity.

## Proof of Proposition 8

Similar to the proof of Proposition 3, we can treat $T_u$ as a value function, according to Proposition 7 we know that $\Delta_v^{\tilde{w}}(F, G) = \Delta_{T_u}^{T_w}(F^{(u,w)}, G^{(u,w)}) \geq 0$ for all $F^{(u,w)} \to G^{(u,w)}$ such that $F^{(u,w)} \succeq_{M-MSD} G^{(u,w)}$ if and only if $T_u \in \mathcal{V}_M$ and $(T_w^+, T_w^-) \in \mathcal{W}_{CV}$. Note that $|R_v(x)| \geq |R_u(x)|$ for any $x$ if and only if $T_u \in \mathcal{V}_M$, and that $|R_{\tilde{w}^-}(x)| \geq |R_{w^-}(x)|$ and $|R_{\tilde{w}^+}(x)| \geq |R_{w^+}(x)|$ if and only if $(T_w^-, T_w^+)$ are concave. The desired result holds.

## Proof of Lemma 4

For the "only if" part, given that $(w^-, w^+) \in \mathcal{W}_{IS}^{p_c} \times \mathcal{W}_{IS}^{p_d}$ and $F \succeq_{GPSD} G$, it follows from the proof of Lemma 2 that the slop function $\rho_-(x) = \frac{w^-(G(x)) - w^-(F(x))}{G(x) - F(x)}$ is increasing in $x$ for $x \in [a, c]$ and that $\rho_+(y) = \frac{w^+(\bar{F}(y)) - w^+(\bar{G}(y))}{\bar{F}(y) - \bar{G}(y)}$ is decreasing in $y$ for $y \in [0, d]$ as $w^-$ and $w^+$ are convex in $[p_c, 1]$ and $[p_d, 1]$, respectively, for

$F(c) = G(c) = p_c$ and $\bar{F}(d) = \bar{G}(d) = p_d$. In a similar fashion, we can show that $\rho_-(x)$ is decreasing in $x$ for $x \in [0, c]$ and $\rho_+(x)$ is increasing in $x$ for $x \in [d, b]$, which is due to the concavity of $w^-$ and $w^+$ in $[0, p_c]$ and $[0, p_d]$, respectively. Then the inequalities of GPSD follow immediately. For the "if" part, note that among the class of PWFs $\mathcal{W}_{IS}^{p_c} \times \mathcal{W}_{IS}^{p_d}$, there is a pair of PWFs that almost does not change the distribution, including linear PWFs under which $F^w \succeq_{GPSD} G^w$ implies $F \succeq_{GPSD} G$.

## Proof of Proposition 9

We first prove the "only if" part. Integrating $\Delta_u^w(F,G)$ by parts yields

$$\begin{aligned}
\Delta_u^w(F,G) &= \int_a^c u'(z)[w^-(G(z)) - w^-(F(z))]dz + \int_c^0 u'(z)[w^-(G(z)) - w^-(F(z))]dz \\
&\quad + \int_0^d u'(z)[w^+(\bar{F}(z)) - w^+(\bar{G}(z))]dz + \int_d^b u'(z)[w^+(\bar{F}(z)) - w^+(\bar{G}(z))]dz \\
&= u'(c)\int_a^c u'(z)[w^-(G(z)) - w^-(F(z))]dz \\
&\quad - \int_a^c u''(x)\int_a^x [w^-(G(z)) - w^-(F(z))]dzdx \\
&\quad + u'(c)\int_c^0 [w^-(G(z)) - w^-(F(z))]dz \\
&\quad + \int_c^0 u''(x)\int_x^0 [w^-(G(z)) - w^-(F(z))]dzdx \\
&\quad + u'(d)\int_0^d [w^+(\bar{F}(z)) - w^+(\bar{G}(z))]dz \\
&\quad - \int_0^d u''(x)\int_0^x [w^+(\bar{F}(z)) - w^+(\bar{G}(z))]dzdx \\
&\quad + u'(d)\int_d^b [w^+(\bar{F}(z)) - w^+(\bar{G}(z))]dz \\
&\quad + \int_d^b u''(x)\int_x^0 [w^+(\bar{F}(z)) - w^+(\bar{G}(z))]dzdx \\
&\geq 0
\end{aligned}$$

where the second equality follows from integration by parts, the inequality follows from the necessity of Lemma 4 that $F \succeq_{GPSD} G$ implies $F^w \succeq_{GPSD} G^w$ for $(w^-, w^+) \in \mathcal{W}_{IS}^{p_c} \times \mathcal{W}_{IS}^{p_d}$, and $u'(x) > 0$ for all $x$, $u''(x) < 0$ for $x \in (a,c)$, and $x \in (0,d)$, $u''(x) > 0$ for $x \in (c,0)$ and $x \in (d,b)$.

For the "if" part, since Lemma 4 has shown that $F \succeq_{GPSD} G$ if $F^w \succeq_{GPSD} G^w$ for all $(w^-, w^+) \in \mathcal{W}_{IS}^{p_c} \times \mathcal{W}_{IS}^{p_d}$, it suffices to show that $F \succeq_{GPSD} G$ if $\Delta_u(F,G) \geq 0$

for all $u \in \mathcal{V}_P^{c,d}$. In fact, using the construction method as that for Lemma 1 in Levy and Wiener (1998), we can prove by contradiction in the intervals $[a, c)$, $[c, 0)$, $(0, d]$ and $(d, b]$, respectively, to verify that the conditions (i), (ii) and (iii) of the definition of GPSD hold. The details are omitted for brevity.

### Proof of Proposition 10

The "if" part is the same as Proposition 9. For the "only if" part, following the same approach as that in the proof of Proposition 2, one can readily show that $v \in \mathcal{V}_P^{c,d}$ and $(w^-, w^+) \in \mathcal{W}_{IS}^{p_c} \times \mathcal{W}_{IS}^{p_d}$ by constructing mean-preserving spreads in $[a, c)$ and $(0, d]$, and mean-preserving contraction in $[c, 0)$ and $(d, b]$, respectively. The details are omitted for brevity.

## Verifying SD Relationships of Prospects in Experimental Studies

### The Experimental Study of Kahneman and Tversky (1979)

**Tasks** 3-3′

We first verify $G_3 \succeq_{V-PSD} F_3$ and $F_{3'} \succeq_{V-PSD} G_{3'}$ for a reference decision maker with the value function $u$ as specified in (3.1) with $\alpha = \beta = \log 0.8/\log 0.75$ and linear PWFs (i.e., $w^-(p) = p$, $w^+(p) = p$).

For task 3, we first verify the expected utility preserving condition of V-PSD:

$$\Delta_u^w(F, G) \;=\; 0.8(4000)^{\frac{\log 0.8}{\log 0.75}} - (3000)^{\frac{\log 0.8}{\log 0.75}} = 0.$$

To verify condition (ii) of Definition 2, we have

$$\int_0^x u'(z)[w^+(\bar{G}(z)) - w^+(\bar{F}(z))]dz = \begin{cases} 0.2u(x) \geq 0 & \text{for } x \in [0, 3000], \\ u(3000) - 0.8u(x) \geq 0 & \text{for } x \in (3000, 4000]. \end{cases}$$

Then the risk change from $G_3$ to $F_3$ is indeed V-PSD for the above reference decision maker $u$.

Similarly, for task 3', we have

$$\Delta_u^w(F, G) \;=\; -0.8\lambda(-4000)^{\frac{\log 0.8}{\log 0.75}} + \lambda(-3000)^{\frac{\log 0.8}{\log 0.75}} = 0, \quad \text{and}$$

$$\begin{aligned} &\int_x^0 u'(x)[w^-(G(z)) - w^-(F(z))]dz \\ = &\begin{cases} -0.2\lambda u(x) & \text{for } x \in [-3000, 0], \\ -\lambda u(-3000) - 0.8\lambda u(x) \geq 0 & \text{for } x \in [-4000, -3000), \end{cases} \end{aligned}$$

which implies that $F_{3'} \succeq_{V-PSD} G_{3'}$.

We next show that $F_3 \succeq_{MSD} G_3$ for task 3 and $G_{3'} \succeq_{MSD} F_{3'}$ for task 3, respectively. We have

$$\int_x^{4000} [G_3(z) - F_3(z)]dz \;=\; \begin{cases} 800 - 0.2(3000 - x) \geq 0 & \text{for } x \in [0, 3000], \\ 0.8(4000 - x) \geq 0 & \text{for } x \in (3000, 4000], \end{cases} \quad \text{and}$$

$$\int_{-4000}^x [G_{3'}(z) - F_{3'}(z)]dz \;=\; \begin{cases} 800 - 0.2(x + 3000) \geq 0 & \text{for } x \in (-3000, 0], \\ 0.8(x + 4000) \geq 0 & \text{for } x \in [-4000, -3000], \end{cases}$$

which verifies that $F_3 \succeq_{MSD} G_3$ and $G_{3'} \succeq_{MSD} F_{3'}$.

**Tasks** 4-4$'$

With respect to the same reference decision maker $u$ as in Task 3, we can show that $G_4 \succeq_{V-PSD} F_4$ and $F_{4'} \succeq_{V-PSD} G_{4'}$. For task 4, we have

$$\Delta_u^w(F,G) = 0.2(4000)^{\frac{\log 0.8}{\log 0.75}} - 0.25(3000)^{\frac{\log 0.8}{\log 0.75}} = 0, \text{ and}$$
$$\int_0^x u'(x)[w^+(\bar{G}(z)) - w^+(\bar{F}(z))]dz$$
$$= \begin{cases} 0.05u(x) \geq 0 & \text{for } x \in [0, 3000], \\ 0.25u(3000) - 0.2u(x) \geq 0 & \text{for } x \in (3000, 4000]. \end{cases}$$

For task 4', we have

$$\Delta_u^w(F,G) = -0.2\lambda(-4000)^{\frac{\log 0.8}{\log 0.75}} + 0.25\lambda(-3000)^{\frac{\log 0.8}{\log 0.75}} = 0, \text{ and}$$
$$\int_x^0 u'(x)[w^-(G(z)) - w^-(F(z))]dz$$
$$= \begin{cases} -0.05\lambda u(x) \geq 0 & \text{for } x \in [-3000, 0], \\ -1/4\lambda u(-3000) - 0.2\lambda u(x) & \text{for } x \in [-4000, -3000]. \end{cases}$$

We next prove that $F_4 \succeq_{MSD} G_4$, and $G_{4'} \succeq_{MSD} F_{4'}$.

For task 4, we have $\mu_{F_4} = 800$ and $\mu_{G_4} = 750$, and

$$\int_x^{4000} [G_4(z) - F_4(z)]dz = \begin{cases} 200 - 0.05(3000 - x) > 0 & \text{for } x \in [0, 3000), \\ 0.2(4000 - x) > 0 & \text{for } x \in [3000, 4000]. \end{cases}$$

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For task $4'$, we have $\mu_{F_{4'}} = -800$ and $\mu_{G_{4'}} = -750$, and

$$\int_{-4000}^{x} [F_{4'}(z) - G_{4'}(z)]dz = \begin{cases} 200 - 0.05(x + 3000) > 0 & \text{for } x \in [-3000, 0], \\ 0.2(x + 4000) > 0 & \text{for } x \in [-4000, -3000), \end{cases}$$

which verify that $F_4 \succeq_{MSD} G_4$, and $G_{4'} \succeq_{MSD} F_{4'}$.

**Tasks** 7-$7'$

For task 7, we have $\mu_{F_7} = \mu_{G_7} = 270$, and

$$\int_{0}^{x} [G_7(z) - F_7(z)]dz = \begin{cases} 0.45x > 0 & \text{for } x \in (0, 3000), \\ 1350 - 0.45(x - 3000) > 0 & \text{for } x \in [3000, 6000], \end{cases}$$

which implies that $F_7 \succeq_{M-PSD} G_7$ and hence $G_7 \succeq_{M-MSD} F_7$.

Similarly, for task 7', we have $\mu_{F_{7'}} = \mu_{G_{7'}} = -270$, and

$$\int_{x}^{0} [F_{7'}(z) - G_{7'}(z)]dz = \begin{cases} 0.45(-x) > 0 & \text{for } x \in (-3000, 0], \\ 1350 - 0.45(-3000 - x) \geq 0 & \text{for } x \in [-6000, -3000], \end{cases}$$

which implies that $G_{7'} \succeq_{M-PSD} F_{7'}$ and hence $F_{7'} \succeq_{M-MSD} G_{7'}$.

**Task** 8-$8'$

For task 8, we have $\mu_{F_8} = \mu_{G_8} = 6$, and

$$\int_{0}^{x} [G_8(z) - F_8(z)]dz = \begin{cases} 0.001x > 0 & \text{for } x \in (0, 3000), \\ 3 - 0.001(x - 3000) \geq 0 & \text{for } x \in [3000, 6000], \end{cases}$$

which implies $F_8 \succeq_{M-PSD} G_8$, or equivalently, $G_8 \succeq_{M-MSD} F_8$.

For task $8'$, we have $\mu_{F_{8'}} = \mu_{G_{8'}} = -6$, and

$$\int_x^0 [F_{8'}(z) - G_{8'}(z)]dz = \begin{cases} 0.001(-x) > 0 & \text{for } x \in [-3000, 0], \\ 3 - 0.001(-3000 - x) > 0 & \text{for } x \in [-6000, -3000), \end{cases}$$

which implies $G_{8'} \succeq_{M-PSD} F_{8'}$, or equivalently, $F_{8'} \succeq_{M-MSD} G_{8'}$.

## The Experimental Study of Baucells and Heukamp (2006)

### Task I

We have $\mu_F = \mu_G = 1500$ and

$$\int_0^x [G(z) - F(z)]dz \quad = \quad \begin{cases} 0.4x \geq 0 & \text{for } x \in [0, 1000], \\ 400 \geq 0 & \text{for } x \in (1000, 2000], \\ 400 - 0.4(x - 2000) \geq 0 & \text{for } x \in (2000, 3000], \end{cases}$$

which, by Definition 1, imply that $F \succeq_{M-PSD} G$ and hence $G \succeq_{M-MSD} F$.

### Task II

We have $\mu_F = \mu_G = -1500$ and

$$\int_x^0 [G(z) - F(z)]dz \quad = \quad \begin{cases} -0.4x \geq 0 & \text{for } x \in [-1000, 0], \\ 400 \geq 0 & \text{for } x \in [-2000, -1000), \\ 400 - 0.4(-x + 2000) \geq 0 & \text{for } x \in [-3000, -2000), \end{cases}$$

which, by Definition 1, imply that $F \succeq_{M-PSD} G$ and hence $G \succeq_{M-MSD} F$.

**Task III**

We have $\mu_F = \mu_G = 250$ and

$$\int_x^0 [G(z) - F(z)]dz = \begin{cases} -1/6x \geq 0 & \text{for } x \in [-3000, 0], \\ 500 - 1/6(-x + 3000) \geq 0 & \text{for } x \in [-6000, -3000), \end{cases}$$

$$\int_0^y [G(z) - F(z)]dz = \begin{cases} 1/6x \geq 0 & \text{for } y \in [0, 3000], \\ 500 - 1/3(x - 3000) \geq 0 & \text{for } y \in (3000, 4500], \end{cases}$$

which imply that $F \succeq_{M-PSD} G$ and hence $G \succeq_{M-MSD} F$.

**Task IV**

We have $\mu_F = \mu_G = 1500$ and

$$\int_x^0 [G(z) - F(z)]dz = \begin{cases} -0.24x \geq 0 & \text{for } x \in [-3000, 0], \\ 720 - 0.24(-x + 3000) \geq 0 & \text{for } x \in [-6000, -3000), \end{cases}$$

$$\int_0^y [G(z) - F(z)]dz = \begin{cases} 0.24x \geq 0 & \text{for } y \in [0, 3000], \\ 720 - 0.48(x - 3000) \geq 0 & \text{for } y \in (3000, 4500], \end{cases}$$

which verify that $F \succeq_{M-PSD} G$ and hence $G \succeq_{M-MSD} F$.

**Task V**

We have $\mu_F = \mu_G = 750$ and

$$\int_x^0 [G(z) - F(z)]dz = \begin{cases} -0.25x \geq 0 & \text{for } x \in [-3000, 0], \\ 750 - 0.25(-x + 3000) \geq 0 & \text{for } x \in [-6000, -3000), \end{cases}$$

$$\int_0^y [G(z) - F(z)]dz = \begin{cases} 0.25x \geq 0 & \text{for } y \in [0, 3000], \\ 750 - 0.5(x - 3000) \geq 0 & \text{for } y \in (3000, 4500], \end{cases}$$

which imply that $F \succeq_{M-PSD} G$ and hence $G \succeq_{M-MSD} F$.

**Task VI**

Let $r = -1000$. We have $\mu_F = \mu_G = 0$, and

$$\int_{-1000}^x [G(z) - F(z)]dz = \begin{cases} 0.4(1000 + x) \geq 0 & \text{for } x \in [-1000, 0], \\ 4000 - 0.4x \geq 0 & \text{for } x \in (0, 1000], \end{cases}$$

which verify that $F \succeq_{M-PSD} G$ and hence $G \succeq_{M-MSD} F$ with respect to $r = -1000$.

**Task VII**

Let $r = -3000$. We have $\mu_F = \mu_G = 0$ and

$$\int_{-3000}^x [G(z) - F(z)]dz = \begin{cases} 0.3(x + 3000) \geq 0 & \text{for } x \in [-3000, 0], \\ 900 - 0.3x \geq 0 & \text{for } x \in (0, 3000], \end{cases}$$

which verify that $F \succeq_{M-PSD} G$ and hence $G \succeq_{M-MSD} F$ with respect to $r = -3000$.

**Task VIII**

Let $r = -3000$. We have $\mu_F = \mu_G = 0$ and

$$\int_{-3000}^{x} [G(z) - F(z)]dz = \begin{cases} 0.3(x + 3000) \geq 0 & \text{for } x \in [-3000, -1000], \\ 600 & \text{for } x \in (-1000, 1000], \\ 600 - 0.3(x - 1000) \geq 0 & \text{for } x \in (1000, 3000], \end{cases}$$

which verify that $F \succeq_{M-PSD} G$ and hence $G \succeq_{M-MSD} F$ with respect to $r = -3000$.

**Task IX**

Let $r = -500$. We have $\mu_F = \mu_G = 750$ and

$$\int_{-500}^{x} [F(z) - G(z)]dz = \begin{cases} 100 + 0.2x \geq 0 & \text{for } x \in [-500, 0], \\ 100 - 0.2x \geq 0 & \text{for } x \in (0, 500], \\ 0 & \text{for } x \in (500, 1000], \\ 0.2(x - 1000) \geq 0 & \text{for } x \in (1000, 1500], \\ 100 - 0.2(x - 1500) \geq 0 & \text{for } x \in (1500, 2000]. \end{cases}$$

which verify that $F \succeq_{M-PSD} G$ and hence $G \succeq_{M-MSD} F$ with respect to $r = -500$.

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**Task X**

Let $r = -500$. It is easy to verify that $\mu_F = \mu_G = -500$, and

$$\int_x^{-500} [G(z) - F(z)]dz = \begin{cases} -0.4(x+500) \geq 0 & \text{for } x \in [-1000, -500], \\ 200 + 0.2(x+1000) \geq 0 & \text{for } x \in [-2000, -1000), \end{cases}$$

$$\int_{-500}^{y} [G(z) - F(z)]dz = \begin{cases} 0.4(x+500) \geq 0 & \text{for } y \in [-500, 0], \\ 200 - 0.2x \geq 0 & \text{for } y \in (0, 1000], \end{cases}$$

which verify that $F \succeq_{M-PSD} G$ and hence $G \succeq_{M-MSD} F$ with respect to $r = -500$.

**Task XI**

Let $r = -5000$. We have $\mu_F = \mu_G = -300$, and

$$\int_{-5000}^{x} [G(z) - F(z)]dz = \begin{cases} 0.2(x+5000) \geq 0 & \text{for } x \in [-5000, -3000], \\ 400 - 0.1(x+3000) \geq 0 & \text{for } x \in (-3000, -1000], \\ 200 + 0.2(x+1000) & \text{for } x \in (-1000, 0], \\ 400 & \text{for } x \in (0, 3000], \\ 400 - 0.2(x-3000) \geq 0 & \text{for } x \in (3000, 5000], \end{cases}$$

which verify that $F \succeq_{M-PSD} G$ and hence $G \succeq_{M-MSD} F$ with respect to $r = -5000$.

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**Task XII**

Let $r = -1500$. It is easy to verify that $\mu_F = \mu_G = -1500$, and

$$\int_{-1500}^{x} [G(z) - F(z)]dz \;\; = \;\; \begin{cases} 0.3(x + 1500) \geq 0 & \text{for } x \in [-1500, 1500], \\ 900 - 0.3(x - 1500) \geq 0 & \text{for } x \in (1500, 4500], \end{cases}$$

which verify that $F \succeq_{M-PSD} G$ and hence $G \succeq_{M-MSD} F$ with respect to $r = -1500$.

# Appendix For Chapter 4

## Proof of Proposition 11

**Proof of the "if" part.** Integrating $\Delta_u(F,G)$ twice yields

$$\begin{aligned}
\Delta_u(F,G) &= u^{'}(a)\int_a^0 [G(z)-F(z)]dz + \int_a^0 u^{''}(x)\int_x^0 [G(z)-F(z)]dzdx \\
&\quad + u^{'}(b)\int_0^b [G(z)-F(z)]dz - \int_0^b u^{''}(y)\int_0^y [G(z)-F(z)]dzdy \\
&= u^{'}(a)\int_a^0 [G(z)-F(z)]dz + u^{'}(b)\int_0^b [G(z)-F(z)]dz \\
&\quad + u^{''}(a)\int_a^0\int_z^0 [G(t)-F(t)]dtdz \\
&\quad + \int_a^0 u^{'''}(x)\int_x^0\int_z^0 [G(t)-F(t)]dtdzdx \\
&\quad - u^{''}(b)\int_0^b\int_0^z [G(t)-F(t)]dtdz \\
&\quad + \int_0^b u^{'''}(y)\int_0^y\int_0^z [G(t)-F(t)]dtdzdy \geq 0
\end{aligned}$$

where the inequality follows immediately from conditions (i) and (ii) of TPSD.

---

**Proof of the "only if" part.** We first show that $\Delta_u(F,G) \geq 0$ for all $u \in \mathcal{V}_P$ implies condition (i) of TPSD. Consider a value function

$$u_1(x) = \begin{cases} \frac{\epsilon(e^{\alpha x}-1)}{\alpha} + x & \text{for } x < 0 \\ \frac{\epsilon(1-e^{-\beta x})}{\beta} & \text{for } x > 0 \end{cases},$$

with $0 < \alpha < 1,\ 0 < \beta < 1,$ and $\epsilon > 0$. Clearly, $u_1 \in \mathcal{V}_P$ with

$$u_1^{'}(x) = \begin{cases} \epsilon e^{\alpha x} + 1, & \text{for } x < 0, \\ \epsilon e^{-\beta x}, & \text{for } x > 0. \end{cases}$$

Taking the limit with respect to $\epsilon$ for $\Delta_{u_1}$ for losses and gains respectively,

$$\begin{aligned} \lim_{\epsilon \to 0} \Delta_{u_1}^{w} &= \lim_{\epsilon \to 0} \left\{ \int_a^0 u'(t)[G(t)-F(t)]dt + \int_0^b u'(t)[G(t)-F(t)]dt \right\} \\ &= \int_a^0 [G(t)-F(t)]dt \end{aligned}$$

Therefore $\Delta_u(F,G) \geq 0$ implies $\int_a^0 [G(t)-F(t)]dt \geq 0$. Similarly, considering $u_2(x) = -u_1(-x)$, we can show that $\int_0^b [G(t)-F(t)]dt \geq 0$.

We next show by contradiction that $\Delta_u(F,G) \geq 0$ for all $u \in \mathcal{V}_P$ implies condition(ii). In the losses domain, suppose by contradiction that there exists $y^* \in [a,0)$ such that $\int_a^{y^*} \int_a^z [G(t)-F(t)]dtdz < 0$. Then there must exist an interval $[y_1, y_2] \subsetneq [a,0)$ such that $\int_a^y \int_a^z [G(t)-F(t)]dtdz < 0$ for all $y \in [y_1, y_2]$.

Now consider the following value function

$$u_3(x) = \begin{cases} \int_a^x \int_a^z \int_a^t \epsilon d\xi dt dz - C, & \text{if } x \in [a, y_1), \\ \int_{y_1}^x \int_{y_1}^z \int_{y_1}^t 1 d\xi dt dz + A(x) - C, & \text{if } x \in [y_1, y_2], \\ \int_{y_2}^x \int_{y_2}^z \int_{y_2}^t \epsilon d\xi dt dz + B(x) - C, & \text{if } x \in (y_2, 0), \\ \frac{10\epsilon}{9} x^{1/3}, & \text{if } x \in (0, b), \end{cases}$$

where $0 < \epsilon < 1$, $A(x) = \epsilon \left( \int_{y_1}^x \int_{y_1}^z \int_a^{y_1} d\xi dt dz + \int_a^x \int_a^{y_1} \int_a^{y_1} d\xi dt dz \right)$, $B(x) = \int_{y_2}^x \int_{y_2}^z \left[ \int_{y_1}^{y_2} d\xi + \epsilon \int_a^{y_1} d\xi \right] dt dz + \int_{y_2}^x \left[ \int_{y_1}^{y_2} \int_{y_1}^t d\xi dt + A'(y_2) \right] dz$ and $C = B(0) + \int_{y_2}^0 \int_{y_2}^z \int_{y_2}^t \epsilon d\xi dt dz$. Clearly, $u \in \mathcal{V}_P$, $A^{'''}(x) = 0$, $B^{'''}(x) = 0$, and

$$u_3^{'''}(x) = \begin{cases} \epsilon, & \text{if } x \in [a, y_1), \\ 1, & \text{if } x \in [y_1, y_2], \\ \epsilon, & \text{if } x \in (y_2, 0], \\ \epsilon x^{-\frac{8}{3}}, & \text{if } x \in (0, b). \end{cases}$$

Taking the limit with respect to $\epsilon$ and using the facts that $\lim_{\epsilon \to 0} u''(a) = 0$, and $\lim_{\epsilon \to 0} u''(b) = 0$, we have $\lim_{\epsilon \to 0} u'(a) = \lim_{\epsilon \to 0} u'(b) = 0$, and

$$\lim_{\epsilon \to 0} \Delta_{u_3}^w = \int_{y_0 - \delta}^{y_0 + \delta} u'''(y) \int_a^y \int_a^z [G(t) - F(t)] dt dz dy < 0$$

as $u'''(y) > 0$ and $\int_a^y \int_a^z [w^-(G(t)) - w^-(F(t))] dt dz < 0$ for $y \in [y_1, y_2]$. This contradicts $\Delta_u(F, G) \geq 0$. Similarly, in the gains domain, assume by contradiction that there exists $x^* \in [0, b]$ such that $\int_{x^*}^b \int_z^b [G(t) - F(t)] dt dz < 0$, then there exists an interval $[x_1, x_2] \subsetneq [0, b]$, such that $\int_x^b \int_z^b [G(t) - F(t)] dt dz < 0$ for all $x \in [x_1, x_2]$. Then construct $u_4(x) = -u_3(-x)$ and the desired results follow.

## Proof of Lemma 6

Denote the density function for $w^-(F)$ and $w^+(\bar{F})$ by $f^-(x)$ and $f^+(x)$ respectively, and $f^-(x) > 0$, $f^+(x) > 0$ almost everywhere in $[a, b]$. It suffices to consider only the losses domain as the transformation in the gains domain can be constructed similarly. We first construct an MPS transformation function for $f^-(x)$:

$$s(x) = \begin{cases} A & \text{for } x \in [a_1 - \delta, a_1 + \delta), \\ -2A & \text{for } x \in [a_2 - \delta, a_2 + \delta], \\ A & \text{for } x \in (a_3 - \delta, a_3 + \delta], \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

where $a \leq a_1 - \delta$, $a_1 + \delta = a_2 - \delta$, $a_2 + \delta = a_3 - \delta$, $a_3 + \delta \leq 0$, $\delta > 0$, and $0 < A \leq \frac{1}{2} f^-(x)$ for all $x \in [a_1 - \delta, a_3 + \delta]$. It is easy to verify that $s$ satisfies (1)$f^-(x) + s(x) \geq 0$ for all $x$; (2) $\int_a^0 s(x)dx = 0$; and (3)$\int_a^0 xs(x)dx = 0$, which implies that $s(x)$ is an MPS transformation function for $f^-(x)$. Clearly, $s(x)$ is symmetric with respect to $a_2$ and the cumulative change of the MPS transformation $s$ from 0 to $x$ can be expressed as

$$S(x) = \int_0^x s(\xi)d\xi = \begin{cases} A(x - a_1 + \delta) & \text{if } a_1 - \delta \leq x \leq a_1 + \delta, \\ 2A(a_2 - x) & \text{if } a_1 + \delta \leq x \leq a_2 + \delta, \\ A(x - a_3 - \delta) & \text{if } a_2 + \delta \leq x \leq a_3 + \delta, \\ 0 & \text{otherwise,} \end{cases}$$

which changes sign only once from positive to negative at $a_2$ with the positive area (to the left hand side of $a_2$) and the negative area (to the right hand side of

$a_2$) being equal. We can then further verify $\int_x^0 S(\xi)d\xi \leq 0$ with strict inequality for $x \in (a_1 - \delta, a_3 + \delta)$ and $\int_a^0 S(\xi)d\xi = \int_{a_1-\delta}^{a_3+\delta} \int_0^x s(\xi)d\xi dx = 0$.

We next construct an MPC transformation function for $f(x) + s(x)$:

$$c(x) = \begin{cases} -B & \text{for } x \in [a_1, a_1 + \delta], \\ 2B & \text{for } x \in [a_2, a_2 + \delta], \\ -B & \text{for } x \in [a_3, a_3 + \delta], \\ 0 & \text{otherwise}, \end{cases} \tag{4}$$

where $0 \leq B \leq 2A$ for all $x \in [a_1, a_1+\delta] \cup [a_3, a_3+\delta]$. It is easy to verify that $c(x)$ satisfies (1) $f^-(x) + s(x) + c(x) \geq 0$; (2)$\int_a^0 c(x)dx = 0$; and (3)$\int_a^0 xc(x)dx = 0$, which implies that $c(x)$ is an MPC transformation function for $f^-(x)+s(x)$. The corresponding cumulative change of the MPC transformation $c$ from 0 to $x$ can be expressed as

$$C(x) = \int_0^x c(\xi)d\xi = \begin{cases} -B(x - a_1) & \text{if } x \in [a_1, a_1 + \delta], \\ -B\delta & \text{if } x \in [a_1 + \delta, a_2], \\ -B\delta + 2B(x - a_2) & \text{if } x \in [a_2, a_2 + \delta], \\ B\delta & \text{if } x \in [a_2 + \delta, a_3], \\ B\delta - B(x - a_3) & \text{if } x \in [a_3, a_3 + \delta], \\ 0 & \text{otherwise.} \end{cases}$$

The cumulative changes of the combination of the subsequent MPS and MPC transformations $t(x) = s(x) + c(x)$ from 0 to $x$ can be expressed as

$$T(x) = \int_0^x t(z)dz = \begin{cases} A(x - a_1 + \delta) & \text{if } x \in [a_1 - \delta, a_1], \\ A\delta - (B - A)(x - a_1) & \text{if } x \in [a_1, a_1 + \delta], \\ (2A - B)\delta - 2A(x - a_1 - \delta) & \text{if } x \in [a_1 + \delta, a_2], \\ 2(B - A)(x - a_2) - B\delta & \text{if } x \in [a_2, a_2 + \delta], \\ A(x - a_2 - \delta) - (2A - B)\delta & \text{if } x \in [a_2 + \delta, a_3], \\ (B - A)(\delta + a_3 - x) & \text{if } x \in [a_3, a_3 + \delta], \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $T(a) = T(0) = 0$, $\int_a^0 T(x)dx = 0$ and $T(x)$ changes signs at most twice when $B \in [0, 2A]$.

In particular, when $B = A$, we have

$$T(x) = \begin{cases} A(x - a_1 + \delta) & \text{if } x \in [a_1 - \delta, a_1], \\ A\delta & \text{if } x \in [a_1, a_1 + \delta], \\ A\delta - 2A(x - a_1 - \delta) & \text{if } x \in [a_1 + \delta, a_2], \\ -A\delta & \text{if } x \in [a_2, a_2 + \delta], \\ A(x - a_2 - \delta) - A\delta & \text{if } x \in [a_2 + \delta, a_3], \\ 0 & \text{otherwise,} \end{cases}$$

which changes signs only once from positive to negative at the mid-point of the interval $[a_1 + \delta, a_2]$ and the sizes of the positive area and the negative area are equal, i.e., $\int_a^0 T(x)dx = 0$. Clearly, $\int_x^0 T(\xi)d\xi < 0$ if and only if $x \in (a_1 - \delta, a_3)$. Since $u''(x) > 0$ and $u'''(x) > 0$ for $x < 0$, we have $\int_a^0 u''(z) \int_z^0 T(\xi)d\xi dz < 0$.

When $A < B \leq 2A$, we can observe that $T(x)$ changes signs exactly twice, first from positive to negative in $[a_2-\delta, a_2]$, and then from negative to positive in $[a_3-\delta, a_3]$. In particular, when $B=2A$, $T(x)$ changes signs at $a_2-\delta$ and $a_2+\delta$ respectively, and $T(x)$ is symmetric with respect to $a_2$. We can also observe that $\int_x^0 T(\xi)d\xi \geq 0$ for $x \in [a_2, a_3+\delta]$ with the equality holding at $a_2$, i.e., $\int_{a_2}^0 T(\xi)d\xi = 0$, and $\int_x^0 T(\xi)d\xi \leq 0$ for $x \in [a_1-\delta, a_2]$ with the equality holding at $a_1-\delta$, i.e., $\int_{a_1-\delta}^0 T(\xi)d\xi = 0$. We can further observe that $\int_x^0 T(\xi)d\xi \geq 0$ changes sign exactly once at $a_2$ from negative to positive, with equal sizes of the areas contained by $T(x)$ symmetrically at both sides of $a_2$, which implies that $\int_x^0 T(\xi)d\xi > 0$ for $x \in (a_2, a_3+\delta)$ and $\int_x^0 T(\xi)d\xi < 0$ for $x \in (a_1-\delta, a_2)$. We have

$$\begin{aligned}\int_a^0 u^{''}(z)\int_z^0 T(\xi)d\xi dz &= \int_a^{a_2} u^{''}(z)\int_z^0 T(\xi)d\xi dz + \int_{a_2}^0 u^{''}(z)\int_z^0 T(\xi)d\xi dz \\ &> u^{''}(a_2)\left[\int_a^{a_2}\int_z^0 T(\xi)d\xi dz + \int_{a_2}^0\int_z^0 T(\xi)d\xi dz\right] = 0,\end{aligned}$$

where the inequality is due to the fact that $u^{''}(x) > 0$ and $u^{'''}(x) > 0$ for $x < 0$.

Combining the above two cases (i.e., $B=A$ and $B=2A$) and by the continuity of the term $\int_a^0 u^{''}(z)\int_z^0 T(\xi)d\xi dz$ in terms of $B$, there must exist a value $B_0 \in (A, 2A)$ such that the equality $\int_a^0 u^{''}(z)\int_z^0 T(\xi)d\xi dz = 0$ holds when $B = B_0$. As $T(x)$ changes signs exactly twice when $B \in (A, 2A)$, first from positive to negative and then from negative to positive, and $\int_x^0 T(\xi)d\xi$ changes sign at most once from negative to positive for $x \in [a_1-\delta, a_3+\delta]$ and it must be strictly positive for $x \in (a_3, a_3+\delta)$. The equality $\int_a^0 u^{''}(z)\int_z^0 T(\xi)d\xi dz = 0$ and the fact that $u''(x) > 0$ for $x < 0$, implies that $\int_x^0 T(\xi)d\xi$ must change sign exactly once; if, otherwise, there is no sign change, we must have $\int_x^0 T(\xi)d\xi \geq 0$ with at least one strict inequality for $x \in (a_1-\delta, a_3+\delta)$ and hence $\int_a^0 u^{''}(z)\int_z^0 T(\xi)d\xi dz > 0$, which violates the equality. Let $x_0$ be the sign-changing point. Since $u^{''}(x) > 0$ for $x < 0$, we know that $\int_x^0 u''(z)\int_z^0 T(\xi)d\xi dz > 0$ for $x \in (x_0, a_3+\delta)$ and that $\int_x^0 u''(z)\int_z^0 T(\xi)d\xi dz$ will change sign at most once in $(a_1-\delta, x_0)$. We argue that $\int_x^0 u''(z)\int_z^0 T(\xi)d\xi dz$ is always positive. If, otherwise, $\int_x^0 u''(z)\int_z^0 T(\xi)d\xi dz$ changes sign at some point $x_0' \in (a_1-\delta, x_0)$, we must have $\int_a^0 u''(z)\int_z^0 T(\xi)d\xi dz < \int_{x_0'}^0 u''(z)\int_z^0 T(\xi)d\xi dz = 0$, which yields a contradiction. This proves that $\int_x^0 u''(z)\int_z^0 T(\xi)d\xi dz \geq 0$ for all $x \leq 0$ with equality holding at $x = a$, which implies that the transformation function $T(\cdot)$ with $B = B_0$ is indeed an MUPT for $F$ and $u$ .

## Proof of Proposition 12

We first prove the "if" part. Applying integration by parts yields

$$\begin{aligned}
\Delta_v^w(F,G) &= v'(a)\int_a^0 [w^-(G(z)) - w^-(F(z))]dz \\
&\quad + \int_a^0 v''(z)\int_z^0 [w^-(G(t)) - w^-(F(t))]dtdz \\
&\quad + v'(b)\int_0^b [w^+(\bar{F}(z)) - w^+(\bar{G}(z))]dz \\
&\quad - \int_0^b v''(z)\int_0^z [w^+(\bar{F}(t)) - w^+(\bar{G}(t))]dtdz \\
&= \int_a^0 v''(z)\int_z^0 [w^-(G(z)) - w^-(F(z))]dtdz \\
&\quad - \int_0^b v''(z)\int_0^z [w^+(\bar{F}(t)) - w^+(\bar{G}(t))]dtdz \qquad (5)
\end{aligned}$$

where the second equality is due to condition (i) in the definition of MU-W-TPSD. It suffices to show that the two terms of the right hand side of the equality (5) are both nonnegative. For the first term, we have

$$
\begin{aligned}
& \int_a^0 v''(z) \int_z^0 [w^-(G(t)) - w^-(F(t))]dtdz \\
= & \int_a^0 \frac{v''(z)}{u''(z)} u''(z) \int_z^0 [w^-(G(t)) - w^-(F(t))]dtdz \\
& - \int_a^0 u''(z) \frac{v''(a)}{u''(a)} \int_z^0 [w^-(G(t)) - w^-(F(t))]dtdz \\
= & \int_a^0 u''(z) \left( \frac{v''(z)}{u''(z)} - \frac{v''(a)}{u''(a)} \right) \int_z^0 [w^-(G(t)) - w^-(F(t))]dtdz \\
= & \int_a^0 u''(z) \left[ \int_a^z \left( \frac{v''(x)}{u''(x)} \right)' dx \right] \int_z^0 [w^-(G(t)) - w^-(F(t))]dtdz \\
= & \int_a^0 \left( \frac{v''(x)}{u''(x)} \right)' \int_x^0 u''(z) \int_z^0 [w^-(G(t)) - w^-(F(t))]dtdzdx \\
= & \int_a^0 \frac{v''(x)}{u''(x)} [P_u(x) - P_v(x)] \int_x^0 u''(z) \int_z^0 [w^-(G(t)) - w^-(F(t))]dtdzdx \quad (6) \\
\geq & \; 0
\end{aligned}
$$

where the first equality is due to condition (ii) that $\int_a^0 u''(z) \int_z^0 [w^-(G(t)) - w^-(F(t))]dtdz = 0$, the fourth equality is obtained by changing the order of integrals, and the inequality is due to Condition (ii) of Definition 9 ($\int_x^0 u''(z) \int_z^0 [w^-(G(t)) - w^-(F(t))]dtdz \geq 0$) and the assumption $|P_v(x)| \geq |P_u(x)|$, or, equivalently, $P_v(x) \leq P_u(x)$ (since the prudence measure is negative in the losses domain).

---

Similarly, for the second term in Equation (5), we have

$$
\begin{aligned}
& -\int_0^b v''(z)\int_0^z [w^+(\bar{F}(t)) - w^+(\bar{G}(t))]dtdz \\
= \ & \frac{v''(b)}{u''(b)}\int_0^b u''(z)\int_0^z [w^+(\bar{F}(t)) - w^+(\bar{G}(t))]dtdz \\
& -\int_0^b \frac{v''(z)}{u''(z)}u''(z)\int_0^z [w^+(\bar{F}(t)) - w^+(\bar{G}(t))]dtdz \\
= \ & \int_0^b u''(z)\left(\frac{v''(b)}{u''(b)} - \frac{v''(z)}{u''(z)}\right)\int_0^z [w^+(\bar{F}(t)) - w^+(\bar{G}(t))]dtdz \\
= \ & \int_0^b u''(z)\left[\int_z^b \left(\frac{v''(y)}{u''(y)}\right)' dy\right]\int_0^z [w^+(\bar{F}(t)) - w^+(\bar{G}(t))]dtdz \\
= \ & \int_0^b \left(\frac{v''(y)}{u''(y)}\right)'\int_0^y u''(z)\int_0^z [w^+(\bar{F}(t)) - w^+(\bar{G}(t))]dtdzdy \\
= \ & \int_0^b \frac{v''(y)}{u''(y)}[P_u(y) - P_v(y)]\int_0^y u''(z)\int_0^z [w^+(\bar{F}(t)) - w^+(\bar{G}(t))]dtdzdy \quad (7) \\
\geq \ & 0
\end{aligned}
$$

where the inequality is due to Condition (ii) of Definition 9 ($\int_0^y u''(z)\int_0^z [w^+(\bar{F}(t)) - w^+(\bar{G}(t))]dtdz \leq 0$) and the assumption $|P_v(y)| \geq |P_u(y)|$, i.e., $P_v(y) > P_u(y)$ (since the prudence measure is positive in the gains domain). This completes the proof of the "if" part for both statements.

We now prove the "only if" by contradictions for losses and gains domains subsequently. In the losses domain, suppose for contradiction that there exists $x^- \in [a, 0)$ such that $P_v(x^-) > P_u(x^-)$. Since both $u$ and $v$ are continuous in $\mathcal{V}_P$, there must exist an interval $[x_1^-, x_2^-] \subsetneq [a, 0)$ such that $P_v(x) > P_u(x)$ $\forall x \in [x_1^-, x_2^-]$. We can construct a risky prospect pair $F$ and $G$ such that the risk change from $F$ to $G$ is a MU-W-TPSD for $u$ (see Lemma 6), and satisfies that $\int_x^0 u''(z)\int_z^0 [w^-(G(t)) - w^-(F(t))]dtdz > 0$ for $x \in [x_1^-, x_2^-]$ and

$w^-(G(x)) - w^-(F(x)) = 0$ for $x \in [a, x_1^-) \cup (x_2^-, b]$. Then, by equation (6), we have

$$\Delta_v^w(F,G) = \int_{x_1^-}^{x_2^-} \frac{v''(x)}{u''(x)}[P_u(x) - P_v(x)] \int_x^0 u''(z) \int_z^0 [w^-(G(t)) - w^-(F(t))]dtdzdx < 0$$

where the inequality is due to the fact that $\frac{v''(x)}{u''(x)}[P_u(x) - P_v(x)] < 0$ for $x \in [x_1^-, x_2^-]$. This contradicts the assumption $\Delta_v^w \geq 0$ for all MU-W-TPSD risk changes for $u$.

Similarly, suppose that in the gains domain there exists $y_+ \in (0, b]$ such that $P_v(y^+) < P_u(y^+)$. Then, there must exist an interval $[y_1^+, y_2^+] \subsetneq (0, b]$ such that $P_v(y) < P_u(y)$ for $\forall y \in [y_1^+, y_2^+]$. We can construct a risky prospect pair $F$ and $G$ such that the risk change from $F$ to $G$ is a MU-W-TPSD for $u$, but $\int_0^y u''(z) \int_0^z [w^+(\bar{F}(t)) - w^+(\bar{G}(t))]dtdz < 0$ for all $x \in [y_1^+, y_2^+]$ and $w^+(\bar{F}(y)) - w^+(\bar{G}(y)) = 0$ for $y \in [a, y_1^+) \cup (y_2^+, b]$. Then by equation (7), we have

$$\Delta_v^w = \int_{y_1^+}^{y_2^+} \frac{v''(y)}{u''(y)}[P_u(y) - P_v(y)] \int_0^y u''(z) \int_0^z [w^+(\bar{F}(t)) - w^+(\bar{G}(t))]dtdzdx < 0$$

where the inequality is due to the fact that $\frac{v''(y)}{u''(y)}[P_u(y) - P_v(y)] > 0$ as $P_v(y) < P_u(y)$ for $x \in [y_1^+, y_2^+]$. This contradicts the assumption $\Delta_v^w \geq 0$ for all MU-W-TPSD risk changes for $u$. This completes the proof for the "only if" part.

## Proof of Theorem 1

Note that under prospect $F$ we have

$$\frac{\partial}{\partial \theta}\psi_F(\theta) = \int_a^0 U_\theta(x,\theta)dw^-(F(x)) + \int_0^b U_\theta(x,\theta)d[1 - w^+(\bar{F}(x))].$$

Since $U_\theta(\cdot,\theta) \in \mathcal{V}_P$, we can treat $\frac{\partial}{\partial\theta}\psi_F(\theta)$ as a CPT value of prospect $F$ for an individual with value function $U_\theta$ and PWFs $(w^-(\cdot), w^+(\cdot))$. Then according to Theorems 5 or Proposition 12, we have $\frac{\partial}{\partial\theta}\psi_F(\theta) \geq \frac{\partial}{\partial\theta}\psi_G(\theta)$, which implies that $\psi_F(\theta) - \psi_G(\theta)$ is increasing in $\theta$. In other words, using a shift factor $\gamma$ to index the level of risk in the sense of W-TPSD or MU-W-TPSD, the objective function is submodular in $\theta$ and the risk level. It follows Theorem 2.8.2 in Topkis (1998) that $\arg\max_{\theta\in[0,1]} \psi_F(\theta) \geq \arg\max_{\theta\in[0,1]} \psi_G(\theta)$.

## Proof of Lemma 7

We first prove $UPM(G) - UPM(F) = 2\int_0^b \int_0^x [w^+(\bar{F}(z)) - w^+(\bar{G}(z))]dz \leq 0$ in the gains domain. For convenience, let $\varphi(x) = -u''(x)$. Note that the definition of MU-W-TPSD with linear PWFs requires $\int_0^x u''(t)\int_0^t [w^+(\bar{F}(\xi)) - w^+(\bar{G}(\xi))]d\xi dt < 0$ for some $x \in (0,b)$, we rule out the case where $\int_0^x [w^+(\bar{F}(t)) - w^+(\bar{G}(t))] = 0$ for all $x \in (0,b]$, in other words, $\int_0^x [w^+(\bar{F}(z)) - w^+(\bar{G}(z))]dz \neq 0$ for some $x \in (0,b]$. Then $\int_0^x [w^+(\bar{F}(z)) - w^+(\bar{G}(z))]dz$ crosses over zero in $(0,b)$ at least once. If, otherwise, $\int_0^x [w^+(\bar{F}(z)) - w^+(\bar{G}(z))]dz > 0$ or $\int_0^x [w^+(\bar{F}(z)) - w^+(\bar{G}(z))]dz < 0$ for all $x \in (0,b)$, we must have $\int_0^b \varphi(t)\int_0^t [w^+(\bar{F}(\xi)) - w^+(\bar{G}(\xi))]d\xi dt > 0$ or $\int_0^b \varphi(t)\int_0^t [w^+(\bar{F}(\xi)) - w^+(\bar{G}(\xi))]d\xi dt < 0$, which leads to a contraction. Since $\varphi(x) > 0$ and $\varphi'(x) = -u'''(x) < 0$, the inequality

$$\int_0^x \varphi(t)\int_0^t [w^+(\bar{F}(\xi)) - w^+(\bar{G}(\xi))]d\xi dt \geq 0$$

implies that there exists a value $x' \in (0,b)$ such that $\int_0^x [w^+(\bar{F}(z)) - w^+(\bar{G}(z))]dz \geq 0$ for all $x \in (0,x')$. Similarly, the equality $\int_0^b \varphi(x)\int_x^b [w^+(\bar{F}(\xi)) - w^+(\bar{G}(\xi))]d\xi dx = -\int_0^b \varphi(x)\int_0^x [w^+(\bar{F}(\xi)) - w^+(\bar{G}(\xi))]d\xi dx \leq 0$ implies that there exists a value

$x'' \in (0, b)$ such that $\int_x^b [w^+(\bar{F}(z)) - w^+(\bar{G}(z))]dz \leq 0$ for all $x \in (x'', b)$.

We then show $\int_0^b x^2 d[w^+(\bar{F}(z)) - w^+(\bar{G}(z))] = 2\int_0^b \int_0^x [w^+(\bar{F}(z)) - w^+(\bar{G}(z))]dz \leq 0$ by contradiction. Suppose that $\int_0^b \int_0^t [w^+(\bar{F}(\xi)) - w^+(\bar{G}(\xi))]d\xi dt > 0$. Without loss of generality, we assume that there are $n$ separated closed intervals $A_1 = [x_1, y_1], ..., A_n = [x_n, y_n = b]$ in $(0, b]$, where $x_i < x_j$ for $i < j$, $\int_0^t [w^+(\bar{F}(\xi)) - w^+(\bar{G}(\xi))]d\xi \leq 0$ for $t \in A_i$ and $\int_0^t [w^+(\bar{F}(\xi)) - w^+(\bar{G}(\xi))]d\xi > 0$ for $t \in (y_1, x_2) \cup ... \cup (y_{n-1}, x_n)$. Let $A = A_1 \cup ... \cup A_n$, and $\bar{A} = [0, b]/A$ (the complement of $A$ in $[0, b]$). Since $\int_0^b \int_0^t [w^+(\bar{F}(\xi)) - w^+(\bar{G}(\xi))]d\xi dt = \int_{A\cup\bar{A}} \int_0^t [w^+(\bar{F}(\xi)) - w^+(\bar{G}(\xi))]d\xi dt > 0$, we can then identify a sequence of $n$ separable intervals in $\bar{A}$ in a backward fashion from $b$ to 0, namely, $\bar{A}_1, ..., \bar{A}_n$. Each $\bar{A}_i$ contains the greatest possible values that can be paired with the corresponding interval in $A$ according to the same order such that $\int_{\bar{A}_i \cup A_i} \int_0^t [w^+(\bar{F}(\xi)) - w^+(\bar{G}(\xi))]d\xi dt = 0$ (i.e., the accumulative disadvantages of $F$ over $G$ in each interval $A_i$ is cancelled out by that in the its paired interval $\bar{A}_i$). More specifically, for each interval $A_i, i = n, ..., 1$, we can correspondingly identify a point $x'_i < \min(x_i, x'_{i+1})$ to define its paired interval subsequently $\bar{A}_i = (x'_i, \min(x_i, x'_{i+1})) \cap \bar{A}$ (with $x'_{n+1} = x_n$) such that $\int_{\bar{A}_i \cup A_i} \int_0^t [w^+(\bar{F}(\xi)) - w^+(\bar{G}(\xi))]d\xi dt = 0$. Note that $\int_0^b \int_0^t [w^+(\bar{F}(\xi)) - w^+(\bar{G}(\xi))]d\xi dt > 0$ implies $0 < x'_1 < x_1$. Since $\phi(x)$ strictly decreases in $x$, for each pair $\bar{A}_i \cup A_i$ we must have

$$\begin{aligned}
& \int_{\bar{A}_i \cup A_i} \varphi(t) \int_0^t [w^+(\bar{F}(\xi)) - w^+(\bar{G}(\xi))]d\xi dt \\
= & \int_{\bar{A}_i} \varphi(t) \int_0^t [w^+(\bar{F}(\xi)) - w^+(\bar{G}(\xi))]d\xi dt + \int_{A_i} \varphi(t) \int_0^t [w^+(\bar{F}(\xi)) - w^+(\bar{G}(\xi))]d\xi dt \\
> & \varphi(x_i) \left[ \int_{\bar{A}_i} \int_0^t [w^+(\bar{F}(\xi)) - w^+(\bar{G}(\xi))]d\xi dt + \int_{A_i} \int_0^t [w^+(\bar{F}(\xi)) - w^+(\bar{G}(\xi))]d\xi dt \right] \\
= & 0.
\end{aligned}$$

Then

$$\begin{aligned}
& \int_0^b \varphi(t) \int_0^t [w^+(\bar{F}(\xi)) - w^+(\bar{G}(\xi))] d\xi dt \\
\geq \; & \int_0^{x_1'} \varphi(t) \int_0^t [w^+(\bar{F}(\xi)) - w^+(\bar{G}(\xi))] d\xi dt \\
= \; & \int_{\bar{A} \cup A} \varphi(t) \int_0^t [w^+(\bar{F}(\xi)) - w^+(\bar{G}(\xi))] d\xi dt > 0,
\end{aligned}$$

which leads to a contradiction to the condition

$$\int_0^b \varphi(x) \int_0^x [w^+(\bar{F}(z)) - w^+(\bar{G}(z))] dz dx = 0.$$

We therefore must have $\int_0^b \int_0^x [w^+(\bar{F}(z)) - w^+(\bar{G}(z))] dz dx \leq 0$. In a similar fashion, we can show $\int_a^0 \int_y^0 [w^-(G(x)) - w^-(F(x))] dz dy \leq 0$.

## Proof of Lemma 8

Let $\mu_{F^w} = \mu_{G^w} = \mu$. Integrating by parts, we have

$$\begin{aligned}
m_{F^w} - m_{G^w} \; = \; & \int_a^0 (x-\mu)^3 d\left[w^-(F(x)) - w^-(G(x))\right] \\
& + \int_0^b (x-\mu)^3 d\left[-w^+(\bar{F}(x)) - (-w^+(\bar{G}(x)))\right] \\
= \; & \int_a^0 3(x-\mu)^2 \left[w^-(G(x)) - w^-(F(x))\right] dx \\
& + \int_0^b 3(x-\mu)^2 \left[w^+(\bar{F}(x)) - w^+(\bar{G}(x))\right]] dx \\
= \; & 3\left(\int_a^b (x^2 - 2\mu x)[G(x) - F(x)] dx + \mu^2 \int_a^b [G(x) - F(x)] dx\right) \\
= \; & 3\int_a^0 (x^2 - 2\mu x)[w^-(G(x)) - w^-(F(x))] dx \\
& + 3\int_0^b (x^2 - 2\mu x)[w^+(\bar{F}(x)) - w^+(\bar{G}(x)) dx,
\end{aligned}$$

where the fourth equation is due to the equal mean. It suffices to show that

$$\int_a^0 (x^2 - 2\mu x)[w^-(G(x)) - w^-(F(x))]dx \geq 0, \text{ and}$$

$$\int_0^b (x^2 - 2\mu x)[w^+(\bar{F}(x)) - w^+(\bar{G}(x))]dx \geq 0.$$

Applying integration by parts, we have

$$\begin{aligned}
& \int_a^0 (x^2 - 2\mu x)[w^-(G(x)) - w^-(F(x))]dx \\
= & \int_a^0 [(x - x_1)(x - x_2) + (x_1 + x_2 - 2\mu)x][w^-(G(x)) - w^-(F(x))]dx \\
& -(x_1 x_2) \int_a^0 [w^-(G(x)) - w^-(F(x))]dx \\
= & \int_a^0 (x - x_1)(x - x_2)[w^-(G(x)) - w^-(F(x))]dx \\
& + \int_a^0 (x_1 + x_2 - 2\mu)x[w^-(G(x)) - w^-(F(x))]dx \\
= & \int_a^0 (x - x_1)(x - x_2)[w^-(G(x)) - w^-(F(x))]dx \\
& +2\left[\frac{x_1 + x_2}{2} - \mu\right] \int_a^0 \int_x^0 [w^-(G(z)) - w^-(F(z))]dzdx \geq 0,
\end{aligned}$$

where the second equality is due to the fact that $F$ and $G$ have the same expected losses. The inequality is due to the conditions regarding the crossing points in the losses domain and the fact that $\int_a^0 \int_x^0 [w^-(G(z)) - w^-(F(z))]dzdx \leq 0$ by Lemma

7. Similarly, we have

$$
\begin{aligned}
& \int_0^b (x^2-2\mu x)[w^+(\bar{F}(x))-w^+(\bar{G}(x))]dx \\
= & \int_0^b [(x-x_3)(x-x_4)+(x_3+x_4-2\mu)x][w^+(\bar{F}(x))-w^+(\bar{G}(x))]dx \\
= & \int_0^b (x-x_3)(x-x_4)[w^+(\bar{F}(x))-w^+(\bar{G}(x))]dx \\
& -2\left[\frac{x_3+x_4}{2}-\mu\right]\int_0^b\int_0^y [w^+(\bar{F}(z))-w^+(\bar{G}(z))]dzdy \geq 0,
\end{aligned}
$$

where the inequality is due to the conditions regarding the crossing points in the gains domain and the fact that $\int_0^b\int_y^b [w^+(\bar{F}(z))-w^+(\bar{G}(z))]dzdy \leq 0$ by Lemma 7. The desired result holds.

## Proof of Lemma 9

Using equation (5), $\Delta_u^w(F,G)=0$ follows immediately from condition (1) and (2). It suffices to prove the two inequalities of condition (ii) of Definition 9. In what follows, we only prove the inequality, $\int_0^x\int_0^z [w^+(\bar{F}(t))-w^+(\bar{G}(t))]dtdz \leq 0$, for the gains domain. The other inequality for the losses domain can be proved using the same arguments and its proof is therefore omitted.

Note that the quasi-concavity of $G^{-1}(F(x))-x$ implies that it crosses over zero in the gains domain at most twice, first from below, which implies that $G(x)$ crosses over $F(x)$ in the gains domain at most twice, first from above. If $F(x)$ and $G(x)$ do not cross in the gains domain, then $w^+(\bar{G}(z)) > w^+(\bar{F}(z))$ or $w^+(\bar{G}(z)) < w^+(\bar{F}(z))$ for all $x \in (0,b]$, then $\int_0^b [w^+(\bar{F}(z))-w^+(\bar{G}(z))]dz \neq 0$, which leads to a contradiction. If $G(x)$ crosses $F(x)$ exactly once, from above, then we have

$\int_0^x [w^+(\bar{F}(z)) - w^+(\bar{G}(z))]dz \geq 0$ for all $x \in (0, b]$, with strict inequality for some $x \in (0, b)$ as $\int_0^b [w^+(\bar{F}(z)) - w^+(\bar{G}(z))]dz = 0$. Since $u''(x) < 0$ for all $x \in (0, b]$, we have $\int_0^b u''(x) \int_0^x [w^+(\bar{F}(z)) - w^+(\bar{G}(z))]dzdx < 0$, which contradicts condition (2). Hence, $G(x)$ crosses over $F(x)$ exactly twice in $(0, b)$, first from above and then from below. Then we have $\int_0^x [w^+(\bar{F}(z)) - w^+(\bar{G}(z))]dz$ crosses over zero only once in $(0, b)$ from above as $\int_0^b [w^+(\bar{F}(x)) - w^+(\bar{G}(x))]dx = 0$.

We now show $\int_0^x u''(t) \int_0^t [w^+(\bar{F}(\xi)) - w^+(\bar{G}(\xi))]d\xi dt \leq 0$. If, otherwise, we have $\int_0^x u''(t) \int_0^t [w^+(\bar{F}(\xi)) - w^+(\bar{G}(\xi))]d\xi dt$ changes sign at most once from negative to positive at some point $x_0' \in (0, b)$, since $\int_0^x [w^+(\bar{F}(t)) - w^+(\bar{G}(t))]dt$ changes sign only once from positive to negative and $u''(x) < 0$ for $x \in (0, b]$, then $\int_0^b u''(t) \int_0^t [w^+(\bar{F}(\xi)) - w^+(\bar{G}(\xi))]d\xi dt > \int_0^{x_0'} u''(t) \int_0^t [w^+(\bar{F}(\xi)) - w^+(\bar{G}(\xi))]d\xi dt = 0$, which yields a contradiction. This proves that

$$\int_0^x u''(t) \int_0^t [w^+(\bar{F}(\xi)) - w^+(\bar{G}(\xi))]d\xi dt \geq 0$$

for all $x \geq 0$ with equality holding at $x = b$. The inequality becomes strict inequality at some $x$ if $F(x) \neq G(x)$ for some $x \in (0, b)$ .

## Proof of Theorem 14

The "if" part of the proof follows the same arguments of the proof of Proposition 11 , where concave and convex domains are treated separately, and hence it is omitted. For the "only if" part, we need to exchange $\rho(x, \epsilon)$ and $\lambda(x, \epsilon)$ in Lemma 6, which makes sure that $v \in \mathcal{V}_M$ and $|P_v(x)| \geq |P_u(x)|$.

## Proof of Proposition 13

The "if" part of the proof follows the same arguments of the proof in Proposition 11, therefore we only present the proof for "only if" part. Following the similar arguments for the "only if" part of Proposition 11, we can show the inequalities. It suffices to prove the equality conditions, $\int_a^0 \int_z^0 [G(\xi) - F(\xi)]d\xi dz = 0$ and $\int_0^b \int_0^x [G(\xi) - F(\xi)]d\xi dz = 0$. Suppose by contradiction that $\int_0^b \int_0^z [G(\xi) - F(\xi)]d\xi dz > 0$. Consider

$$u_5(x) = \begin{cases} \int_a^x [z + \frac{e^{-\epsilon z}-1}{\epsilon}]dz - \int_a^0 [z + \frac{e^{-\epsilon z}-1}{\epsilon}]dz, & \text{if } x < 0, \\ \int_0^x \frac{e^{\epsilon(z-b)}}{\epsilon} dz, & \text{if } x > 0. \end{cases}$$

It is easy to verify that

$$u_5^{'}(x) = \begin{cases} x + \frac{e^{-\epsilon x}-1}{\epsilon}, & \text{if } x < 0, \\ \frac{e^{\epsilon(x-b)}}{\epsilon}, & \text{if } x > 0, \end{cases}$$

$$u_5^{''}(x) = \begin{cases} 1 - e^{-\epsilon x}, & \text{if } x < 0, \\ e^{\epsilon(x-b)}, & \text{if } x > 0, \end{cases} \text{ and}$$

$$u_5^{'''}(x) = \begin{cases} \epsilon e^{-\epsilon x}, & \text{if } x < 0, \\ \epsilon e^{\epsilon(x-b)}, & \text{if } x > 0. \end{cases}$$

Clearly $u \in \mathcal{V}_M$. Taking the limit with respect to $\epsilon$, we have $\lim_{\epsilon\to 0} u_5'(a) = \lim_{\epsilon\to 0} u_5'(b) = \lim_{\epsilon\to 0} u_5''(a) = \lim_{\epsilon\to 0} u_5'''(x) = 0$, $\lim_{\epsilon\to 0} u_5''(b) = 1$, and $\lim_{\epsilon\to 0} \Delta_{u_5} = -\int_0^b \int_0^z [G(\xi) - F(\xi)]d\xi dz < 0$, contradicting the assumption that $\Delta_{u_5} \geq 0$.

Similarly, we can show that $\int_a^0 \int_z^a [G(\xi) - F(\xi)] d\xi dz = 0$ by considering the following value function

$$u_6(x) = \begin{cases} \int_a^x \frac{e^{\epsilon(a-x)}}{\epsilon} dz - \int_a^0 \frac{e^{\epsilon(a-x)}}{\epsilon} dz, & \text{if } x < 0, \\ \int_0^x [\frac{e^{\epsilon z}+1}{\epsilon} - z] dz, & \text{if } x > 0. \end{cases}$$

This completes the proof.

# Appendix for Chapter 5

## Proof of Proposition 15

We first show assertion (i) and (ii). We have

$$
\begin{aligned}
& E[-X| - VaR_\alpha \leq X \leq -VaR_{1-\beta}] \\
= \; & \frac{1}{1-\beta-\alpha}\big((1-\beta)E[-X|X \leq -VaR_{1-\beta}(X)] - \alpha E[-X|X \leq -VaR_\alpha(X)]\big) \\
= \; & \frac{1}{1-\beta-\alpha}\left((1-\beta)CVaR_{1-\beta}(X) - \alpha CVaR_\alpha(X)\right) \\
= \; & \frac{1}{1-\beta-\alpha}\left(\int_0^{1-\beta} VaR_\gamma(X)d\gamma - \int_0^\alpha VaR_\gamma(X)d\gamma\right) \\
= \; & \frac{1}{1-\beta-\alpha} VaR_\gamma(X)d\gamma.
\end{aligned}
$$

where the second and third equalities follow from the definition of CVaR. Assertion (iii) follows immediately from the fact that $VaR_\gamma(X) = -F^{-1}(\gamma)$.

## Proof of Proposition 16

Note that (i) $\Leftrightarrow$ (iii) is documented in Levy and Wiener (1998). We only need to show (i)$\Leftrightarrow$ (ii). We have

$$
\begin{aligned}
& RVaR_{[\alpha,1-\beta]}(X) - RVaR_{[\alpha,1-\beta]}(Y) \\
= & \frac{1}{1-\beta-\alpha}\int_{\alpha}^{1-\beta} VaR_{\gamma}(X)d\gamma - \frac{1}{1-\beta-\alpha}\int_{\alpha}^{1-\beta} VaR_{\gamma}(Y)d\gamma \\
= & -\frac{1}{1-\beta-\alpha}\int_{\alpha}^{1-\beta} F^{-1}(\gamma)d\gamma + \frac{1}{1-\beta-\alpha}\int_{\alpha}^{1-\beta} G^{-1}(\gamma)d\gamma \\
= & -\frac{1}{1-\beta-\alpha}\int_{\alpha}^{1-\beta} [F^{-1}(\gamma) - G^{-1}(\gamma)]d\gamma
\end{aligned}
$$

where the second equality follows from the quantile representations of RVaR in Proposition 15, assertion (iii). Then

$$
\begin{aligned}
& RVaR_{[\alpha,1-\beta]}(X) \leq RVaR_{[\alpha,1-\beta]}(Y) \text{ for all } 0 \leq \alpha \leq F(0) \leq G(0) \leq 1-\beta \leq 1, \\
\Leftrightarrow\ & -\frac{1}{1-\beta-\alpha}\int_{\alpha}^{1-\beta} [F^{-1}(\gamma) - G^{-1}(\gamma)]d\gamma \leq 0 \text{ for all } 0 \leq \alpha \leq F(0) \leq G(0) \leq 1-\beta \leq 1, \\
\Leftrightarrow\ & \int_{\alpha}^{1-\beta} [F^{-1}(\gamma) - G^{-1}(\gamma)]d\gamma \geq 0, \text{ for all } 0 \leq \alpha \leq F(0) \leq G(0) \leq 1-\beta \leq 1, \\
\Leftrightarrow\ & F \succeq_{PSD} G.
\end{aligned}
$$

This completes the proof.

### Proof of Lemma 11

It follows from Proposition 16 that $RVaR_{[\alpha,1-\beta]}(X) \leq RVaR_{[\alpha,1-\beta]}(Y)$ for all $0 \leq \alpha \leq F(0) \leq G(0) \leq 1-\beta \leq 1$ if and only if $F \succeq_{PSD} G$. So we only need to show $F \succeq_{PSD} G$ if and only if $F^w \succeq_{PSD} G^w$ for all $(w^-, w^+) \in \mathcal{W}_{CX} \times \mathcal{W}_{CX}$.

We first prove the "only if" part. For convenience, we define the following slop functions

$$\rho_-(x) = \frac{w^-(G(x)) - w^-(F(x))}{G(x) - F(x)}, \text{ and } \rho_+(x) = \frac{w^+(\bar{F}(x)) - w^+(\bar{G}(x))}{\bar{F}(x) - \bar{G}(x)}.$$

We argue that the convexity of $w^-(\cdot)$ and $w^+(\cdot)$ implies that $\rho_-(x)$ is increasing in $x$ and $\rho_+(x)$ is decreasing in $x$. To show this, for any $x < x' \leq 0$, we have

$$\frac{w^-(G(x)) - w^-(F(x))}{G(x) - F(x)} \leq \frac{w^-(G(x)) - w^-(F(x'))}{G(x) - F(x')} \leq \frac{w^-(G(x')) - w^-(F(x'))}{G(x') - F(x')},$$

where both inequalities are due to the convexity of $w^-$ and the monotonicity of $F$, $G$ and $w^-$. Similarly, for any $0 \leq x < x'$, we have

$$\frac{w^+(\bar{F}(x')) - w^+(\bar{G}(x'))}{\bar{F}(x') - \bar{G}(x')} \leq \frac{w^+(\bar{F}(x')) - w^+(\bar{G}(x))}{\bar{F}(x') - \bar{G}(x)} \leq \frac{w^+(\bar{F}(x)) - w^+(\bar{G}(x))}{\bar{F}(x) - \bar{G}(x)},$$

where both inequalities are also due to the convexity of $w^+$ and the monotonicity of $\bar{F}$, $\bar{G}$ and $w^+$.

Then for $x \in [a, 0]$, we have

$$\begin{aligned}
& \int_x^0 \left[w^-\left(G(z)\right) - w^-\left(F(z)\right)\right] dz \\
= & \int_x^0 \rho_-(z)[G(z) - F(z)]dz \\
= & -\int_x^0 \rho_-(z) d\left(\int_z^0 [G(t) - F(t)]dt\right) \\
= & \rho_-(x)\int_x^0 [G(t) - F(t)]dt + \int_x^0 \rho_-^{'}(z)\int_z^0 [G(t) - F(t)]dtdz \geq 0,
\end{aligned}$$

where the inequality follows from the fact that $\rho_-(x) \geq 0$, $\rho_-^{'}(x) \geq 0$, and $\int_x^0 [G(z) - F(z)]dz \geq 0$ for all $x \leq 0$.

Similarly, for $y \in [0,b]$, we have

$$\begin{aligned}
& \int_0^y \left[w^+\left(\bar{F}(z)\right) - w^+\left(\bar{G}(z)\right)\right] dz \\
= & \int_0^y \rho_+(z)\left[\bar{F}(z) - \bar{G}(z)\right] dz \\
= & \int_0^y \rho_+(z) d\left(\int_0^z \left[\bar{F}(t) - \bar{G}(t)\right] dt\right) \\
= & \rho_+(y)\int_0^y [G(t) - F(t)]dt - \int_0^y \rho_+^{'}(z)\int_0^z [G(t) - F(t)]dt dz \geq 0,
\end{aligned}$$

where the inequality follows from the fact that $\rho_+(x) \geq 0$, $\rho_+^{'}(x) \leq 0$, and $\int_0^y [G(z) - F(z)]dz \geq 0$ for all $y \geq 0$. Combining the two cases, we know that $F \succeq_{PSD} G$ implies $F^w \succeq_{PSD} G^w$ when $w^-$ and $w^+$ are convex.

For the "if" part, note that among the class of convex PWFs, there is a pair of PWFs that almost does not change the distribution, e.g., linear PWFs, under which $F^w \succeq_{PSD} G^w$ implies $F \succeq_{PSD} G$.

## Proof of Proposition 18

It follows from Proposition 16 that $RVaR_{[\alpha,1-\beta]}(X) \leq RVaR_{[\alpha,1-\beta]}(Y)$ for all $0 \leq \alpha \leq F(0) \leq G(0) \leq 1-\beta \leq 1$ implies $F \succeq_{PSD} G$. Since $RVaR_{[0,1]}(X) = RVaR_{[0,1]}(Y)$, it further implies that $F \succeq_{M-PSD} G$. The result then follows immediately from Proposition 2.

## Proof of Lemma 12

Assertion (i) follows immediately from the fact that $F \succeq_{FSD} G$ if and only if $F^w \succeq_{FSD} G^w$ for all $w^-$ that is increasing (Levy and Wiener 1998), and assertion

(i) in Lemma 10.

For assertion (ii), it follows from Lemma 10 that $CVaR_\alpha(X) \le CVaR_\alpha(Y)$ for all $\alpha \in (0,1)$ is equivalent to $F \succeq_{PSD} G$. So we only need to show $F \succeq_{SSD} G$ if and only if $F^w \succeq_{SSD} G^w$ for all $w^- \in \mathcal{W}_I$. To see this, recall that we have defined the slop function:

$$\rho_-(x) = \frac{w^-(G(x)) - w^-(F(x))}{G(x) - F(x)}.$$

The concavity of $w^-(\cdot)$ implies that $\rho_-(x)$ is decreasing in $x$. Suppose $F \succeq_{SSD} G$, then for $x \in [a,0]$, we have

$$\begin{aligned}
& \int_a^x \left[w^-\left(G(z)\right) - w^-\left(F(z)\right)\right] dz \\
= & \int_a^x \rho_-(z)[G(z) - F(z)]dz \\
= & \int_a^x \rho_-(z) d\left(\int_a^z [G(t) - F(t)]dt\right) \\
= & \ \rho_-(x)\int_a^x [G(t) - F(t)]dt - \int_a^x \rho_-^{'}(z)\int_a^z [G(t) - F(t)]dtdz \ge 0,
\end{aligned}$$

where the inequality is due to $\rho_-(x) \ge 0$, $\rho_-^{'}(x) \le 0$, and $\int_a^x [G(z) - F(z)]dz \ge 0$ for all $x \in [a,0]$ as $F \succeq_{SSD} G$. Hence we have $F^w \succeq_{SSD} G^w$.

For the "only if" part, note that among the class of convex PWFs, there is a pair of PWFs that almost does not change the distribution, e.g., linear PWFs, under which $F^w \succeq_{SSD} G^w$ implies $F \succeq_{SSD} G$.

## Proof of Proposition 19

We first prove the "if" part of assertion (i). It follows from assertion (i) of Lemma 10 that for any given pair of PWFs $w^- \in \mathcal{W}_I$, $\Delta_u^w(F,G) \ge 0$ for all

$u \in \mathcal{V}_I$ implies the risk change from $F^w$ to $G^w$ is PSD. It then follows from the "only if" part of Lemma 12 that $F^w \succeq_{FSD} G^w$ for all $w^- \in \mathcal{W}_I$ implies that $VaR_\alpha(X) \leq VaR_\alpha(Y)$ for all $\alpha \in (0,1)$.

The "only if" part of (i) follows immediately from Lemma 10 and "if" part of assertion (i) in Lemma 12.

We now prove assertion the "if" part of assertion (ii). It follows from assertion (ii) Lemma 10 that for any given pair of PWFs $w^- \in \mathcal{W}_{CV}$, $\Delta_u^w(F,G) \geq 0$ for all $u \in \mathcal{V}_{CV}$ implies the risk change from $F^w$ to $G^w$ is SSD. It then follows from the "only if" part of Lemma 12 that $F^w \succeq_{SSD} G^w$ for all $w^- \in \mathcal{W}_{CV}$ implies that $CVaR_\alpha(X) \leq CVaR_\alpha(Y)$ for all $\alpha \in (0,1)$.

The "only if" part follows immediately from Lemma 10 and "if" part of assertion (i) in Lemma 12.

## Proof of Proposition 20

Note that $\frac{\partial}{\partial\theta}\psi_F(\theta) = \int_a^0 U_\theta(x,\theta)dw^-(F(x)) + \int_0^b U_\theta(x,\theta)d[1-w^+(\bar{F}(x))]$. Since $U_\theta(\cdot,\theta) \in \mathcal{V}_S$, we can treat $\frac{\partial}{\partial\theta}\psi_F(\theta)$ as if it were a CPT value of prospect $F$ for an individual with the value function $U_\theta$ and PWFs $(w^-, w^+)$. Then according to Proposition 16 and 18, under condition (i) or (ii), we have $\frac{\partial}{\partial\theta}\psi_F(\theta) \geq \frac{\partial}{\partial\theta}\psi_G(\theta)$, which implies that $\psi_F(\theta) - \psi_G(\theta)$ is increasing in $\theta$. In other words, using a shift factor $\gamma$ to index the level of risk in the sense RVaR, the objective function is submodular in $\theta$ and the risk level. It follows from Proposition 2.8.2 in Topkis (1998) that $\arg\max_{\theta\in[0,1]} \psi_F(\theta) \geq \arg\max_{\theta\in[0,1]} \psi_G(\theta)$.

## Proof of Proposition 21

Note that in the domain of losses, $\int_a^x [G(z) - F(z)]dz \geq 0$ for all $x \leq 0$ implies and is implied by $CVaR_\alpha(X) \leq CVaR_\alpha(Y)$ for all $\alpha \in (0, \min(F(0), G(0)))$ due to Lemma 10. We only need to show that in the domain of gains, $\mu_F - (1-\beta)CVaR_{1-\beta}(X) \leq \mu_G - CVaR_{1-\beta}(Y)$ for all $\beta \in (0, \max(F(0), G(0)))$ implies and is implied by $\int_x^b [G(z) - F(z)]dz \geq 0$ for all $x \geq 0$. We then have

$$
\begin{aligned}
\int_x^b [G(z) - F(z)]dz &= \int_{1-\beta}^1 [F^{-1}(t) - G^{-1}(t)]dt \\
&= \int_{1-\beta}^1 F^{-1}(t)dt - \int_{1-\beta}^1 G^{-1}(t)dt \\
&= \mu_F - \int_{1-\beta}^1 F^{-1}(t)dt - \mu_G + \int_{1-\beta}^1 G^{-1}(t)dt \\
&= \mu_F - (1-\beta)CVaR_{1-\beta}(X) - (\mu_G - (1-\beta)CVaR_{1-\beta}(Y))
\end{aligned}
$$

where the first equality follows from the quantile representations of MSD, and the fourth equality follows from the quantile representation of CVaR. Then

$$
\begin{aligned}
& \mu_F - (1-\beta)CVaR_{1-\beta}(X) \leq \mu_G - CVaR_{1-\beta}(Y) \text{ for all } \beta \in (0, \max(F(0), G(0))) \\
\Leftrightarrow \quad & \int_x^b [G(z) - F(z)]dz \geq 0 \text{ for all } x \geq 0.
\end{aligned}
$$

This completes the proof.

## Proof of Proposition 22

Since $\mu_F = \mu_G$, it follows from Proposition 21 that $CVaR_\alpha(X) \leq CVaR_\alpha(Y)$ for all $\alpha \in (0, \min(F(0), G(0)))$ and $CVaR_{1-\beta}(X) \geq CVaR_{1-\beta}(Y)$ implies $F \succeq_{M-PSD} G$. The result then follows immediately from Proposition 7.