



# Competition within Random Growth Models

Department of Mathematics and Statistics

Shane Turnbull

September 21, 2018

# Abstract

This thesis is concerned with introducing competition into random models. It can be observed that there are two natural mechanisms for the evolution of a random model; either by growth or by self interactions. What we do is look at two types of models and introduce competition within them. The first model, the voter model, is an example of a self interacting model and we introduce growth into it. The second model, the Hasting-Levitov model, is a random growth model and we introduce competition within the model.

In both cases we construct diffusion approximations to model these systems when the initial population is large for the first case and when the addition of incoming particles is small in the second. Once these diffusion processes have been constructed we then analyse the long term behaviour of them and find their asymptotic distribution, this is done by using the speed measure and scale function.

# Dedication

Throughout my time at university there have been many people who have made here such a fantastic experience. I have spent 8 years here at Lancaster and I am proud to say Lancaster is my home. I shall keep the list short so apologies for not naming many people!

Firstly, my thanks must go to my supervisor Amanda, her wisdom and guidance has helped me throughout my time here. Apologies for all the times I turned up unannounced at your door with a random query. I must also thank her for her optimism for when I clearly did not have any, she always knew what to say whenever I needed support.

I would also like to thank Tom, throughout my 8 years at university you have been someone who I knew would be there for me, through the highs and lows your support has always been appreciated. I know I have wasted much of your time discussing probability theory but I hope you enjoyed some of our conversations and am glad you have managed to retain some of it! I look forward to remaining friends even after Lancaster, I am sure we shall.

I also extend my thanks to my family and all my friends who offered their thoughts and encouragements while undertaking my PhD but for also offering me the distraction during the weekends and evenings when I much needed them.

Finally, I am very grateful for the Faculty of Science and Technology for supporting my studies. Without them and the fantastic environment the university as a whole

has provided me I would not have been able to do this. Thank you.

July 2018

I would also like to thank my examiners, Dmitry Korshunov and Tim Rogers, for both their time in reading my thesis and for their invaluable comments which helped me improve it further.

September 2018

# Declaration

I declare that the work in this thesis has been done by myself and has not been submitted elsewhere for the award of any other degree.

Shane Turnbull

# Contents

<b>1</b>	<b>Introduction</b>	<b>1</b>
<b>2</b>	<b>Random Models</b>	<b>3</b>
2.1	Random Growth Models . . . . .	3
2.1.1	Lattice Models . . . . .	4
2.1.2	Off Lattice Models . . . . .	9
2.1.3	Pólya Urn Model . . . . .	13
2.2	Model with Internal Behaviour . . . . .	14
2.2.1	Voter Model . . . . .	14
2.2.2	Moran Model . . . . .	15
2.2.3	Wright-Fisher Diffusion . . . . .	16
2.3	Random Growth Models with Competition . . . . .	17
2.3.1	Lotka-Volterra System . . . . .	17
2.3.2	Introducing randomness . . . . .	19
<b>3</b>	<b>Diffusion Theory</b>	<b>22</b>
3.1	Motivation . . . . .	22
3.2	Limit Theorem . . . . .	26
3.3	Analysis of Diffusion Processes . . . . .	33
<b>4</b>	<b>Addition of Growth to the Fully Connected Voter Model</b>	<b>43</b>

4.1	Introduction . . . . .	43
4.1.1	The fully connected voter model with the addition of growth . . . . .	44
4.1.2	Overview . . . . .	46
4.2	Diffusion Estimates . . . . .	47
4.3	Limit Distributions . . . . .	55
4.3.1	Deterministic . . . . .	56
4.3.2	Martingales and Diffusion Processes . . . . .	56
4.3.3	Limit Distribution for fixed $N$ . . . . .	58
4.4	Applications . . . . .	61
<b>5</b>	<b>Introducing Competition within HL(0)</b>	<b>66</b>
5.1	Introduction . . . . .	66
5.1.1	The Hastings-Levitov Model . . . . .	67
5.1.2	Harmonic Measure Flow . . . . .	68
5.1.3	Introducing Competition . . . . .	71
5.1.4	Outline . . . . .	73
5.2	Diffusion Estimates . . . . .	73
5.3	Limit Distributions . . . . .	78
5.4	Examples . . . . .	80

# Chapter 1

## Introduction

This thesis is split into four parts, each dealing with its own specific material. Chapters 2 and 3 serve as a literature review with Chapter 2 providing a grounding in where this area of research lies. We give a rough overview of two types of models; random growth models and self-interacting models. As the name suggests, random growth models are those that consist of a set which increases over time, given some probabilistic rule. We consider the Pólya urn model since it is simple to describe and also has an asymptotic distribution, something we are interested in with the models we consider. We also look at the Eden model and its continuous analogue, the Hastings-Levitov model, since part of our research focuses on this model.

Self interacting models are those that have states which change over time. We consider the voter model, which is a simplistic model for representing the votes of people. Again, this is another model which our research focuses on. We also include the Moran model and that of Wright-Fisher diffusion. This is because they are both used in population genetics where Wright-Fisher diffusion could be thought of as the continuous limit of the Moran model. We include these here as some of our stochastic differential equations that appear are similar to that of Wright-Fisher diffusion. We then end the section by considering a deterministic model which contains these two



types of behaviour.

Chapter 3 focuses on the tools required for us to construct and then analyse our models. We start by introducing the kernel of a pure jump process. This is a very useful object since it contains all the information of the process. With the kernel defined we then look at what a scaling limit is and state a theorem which ensures we have weak convergence of a sequence of stochastic processes to a diffusion process. Once we have done this, we then analyse the obtained diffusion process. We look at the speed measure and scale function, which are required for us to analyse our systems, along with how to classify the boundary behaviour. This is enough information to then find the asymptotic distribution.

The next two chapters, Chapter 4 and 5, contain our original work. We use the tools introduced in Chapter 3 to analyse the models and work out their long term behaviour.

Chapter 4 consists of the voter model, a self-interacting model, and we study the behaviour after the introduction of growth. This was originally studied in [Morris and Rogers, 2014], but we extend these results and reconfirm their results using the kernel of a process and Kurtz's theorem. We also give a full characterisation of the model. We show how the addition of growth has the ability to turn absorbing states into reflecting ones and then analyse the limiting behaviour.

In Chapter 5 we do the opposite. We consider a random growth model where the growth is constructed via a sequence of conformal maps, in this case the Hastings-Levitov model and introduce competition into it. We introduce a form of competition where we allow the growth to vary depending on the harmonic measure of the regions. We show it is necessary to allow the size and rate of the added particles to change to ensure that we do have coexistence between the competing regions.

# Chapter 2

## Random Models

The aim of this section is to give an overview of where our area of research lies. To do this we consider two types of random models. As the name suggests, a random model is intended to model something with a random element and is often influenced by the world around us. There are two natural mechanisms to think of when we consider this; growth and self-interactions. The first type of models that we consider are those that grow, such as the Eden model. The second type of models that we consider are those which evolve dynamically so that the states of them change over time; an examples of this would be the voter model. After we consider these two types of models, we consider a model that lies in the intersection of these areas; one which grows but also evolves dynamically. This is the area that we perform research in.

### 2.1 Random Growth Models

We shall consider two types of models in this section, those which grow on a lattice, called lattice models, and those which do not, called off lattice models. Lattice models are useful since the structure of the lattice allows us to grow clusters in an easy to explain way. The main disadvantage is that this does not represent many real life

examples. Off lattice models are a better representation for real world growth, but can be harder to model mathematically.

### 2.1.1 Lattice Models

In this section we look at three types of lattice models. In all cases, we are looking at the square lattice  $\mathbb{Z}^2$  but these can be expanded to others.

#### Eden Model

A contender as a model for cell growth, the Eden model is named after Murray Eden who studied it in [Eden, 1961] and is used to model the growth of a cluster where the growth is motivated by the splitting of cells. Consider the square lattice with a particle located at the origin. This has four unoccupied neighbours; we uniformly pick one of these neighbours at random and the cell divides to occupy this cell too. This cluster now has six neighbours and we repeat the process, letting the cluster grow, see Figure 2.1.

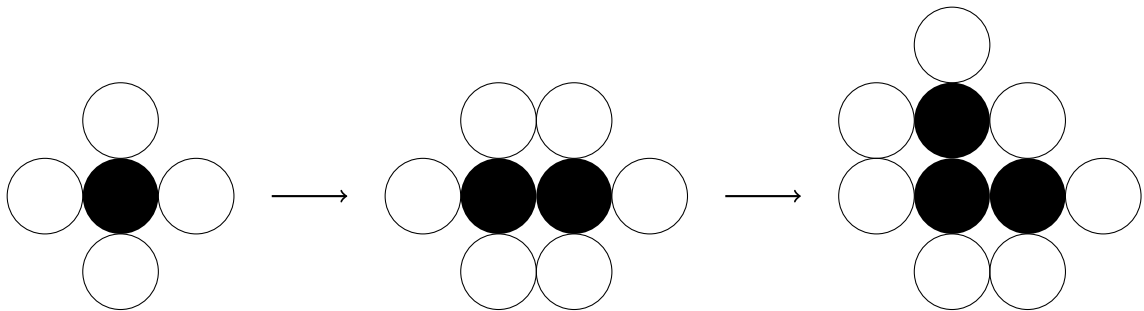


Figure 2.1: Possible first steps of the Eden model. The black cells indicate occupied sites and the white ones are the possible growth sites, which are picked with equal probability among them.

An interesting question to ask is, when we let the cluster grow for a long time, does a shape form? To be more precise, does there exist a scaling limit such that when

$t \rightarrow \infty$  we obtain a limit object? From computer simulations [Freche et al., 1985] the authors claim that the model does seem to exhibit anisotropic behaviour, and so is not isotropic, with large clusters forming a diamond shape, see Figure 2.2.

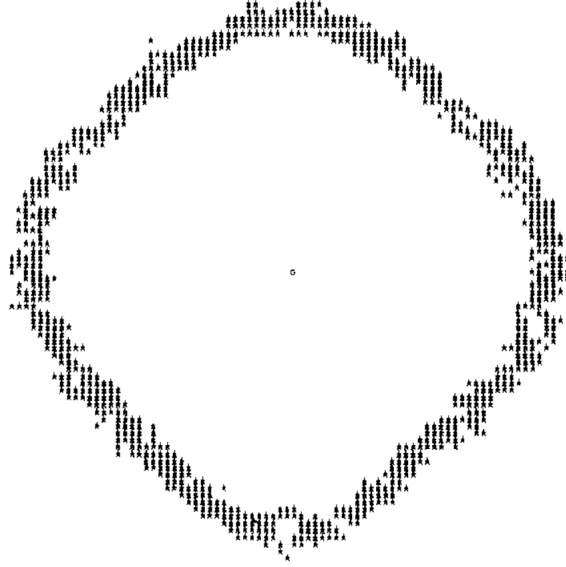


Figure 2.2: Simulation showing the anisotropy from the growing clusters. The middle has been removed to make the boundary clearer. This is Figure 2 from [Freche et al., 1985].

Though a simple question, it was very hard to prove and very little was known about the actual shape obtained in the limit for the first 10 years. Various attempts have been done and some results have been gained. One of the first rigorous results is listed below

**Theorem 2.1.1.** *Let  $A$  be the unit circle and  $A_t$  the Eden cluster at time  $t$ , then there exists a  $\lambda > 0$  such that for all  $\varepsilon > 0$  we have, with probability 1, a large enough  $t_0$  such that for all  $t > t_0$ ,*

$$(1 - \varepsilon)\lambda A \subset \bar{A}_t/t \subset (1 + \varepsilon)\lambda A,$$

where  $\bar{A}_t$  is known as the continuum versions of  $A_t$  formed by replacing each point in  $A_t$  by a unit square centered at the point.

This result was first shown to hold in probability in [Richardson, 1973] and then subsequently strengthened to the almost sure statement above in [Kesten, 1993]. The main approach to showing these results is to show that the Eden model can be linked to another model known as First-Passage percolation which we discuss in the next section and to then show that first passage percolation satisfies a property known as "sub-additivity". This was generalised by Kingman to produce the subadditive ergodic Theorem [Kingman, 1973]. For further details see Theorem 2.1 and the surrounding text in [Deijfen and Häggström, 2007].

Of course, this model can be generalised onto any graph. Consider a graph  $G = (V, E)$  comprising a set  $V$  of vertices and  $E$  of edges. Set  $C_0 = v_0$ , where  $v_0 \in V$ , to be the initial cluster. The model grows following the iteration

$$C_n = C_{n-1} \cup \{v_{n-1}\}$$

where  $v_{n-1}$  is a vertex which neighbours the cluster  $C_n$ , which is picked uniformly at random from all possible options.

## Percolation

As the name suggests, the following model is used to represent the idea of water seeping through a porous material. We shall restrict our study to the 2d case but this can be extended. Imagine an  $n \times n$  grid composed of  $1 \times 1$  squares, with probability  $p$  we let each square either be occupied or not. Some simulations for  $p = 0.2, 0.4, 0.6$  and  $0.8$  are shown in Figure 2.3.

This model is known as site percolation since it is the sites that are being occupied, a slight variant known as bond percolation exists where the edges of the lattice are considered closed or open. Site percolation can be viewed as a sub-model of the bond percolation since every bond model can be expressed in terms of a site model (on a different lattice) but the other way round does not hold. For further details see

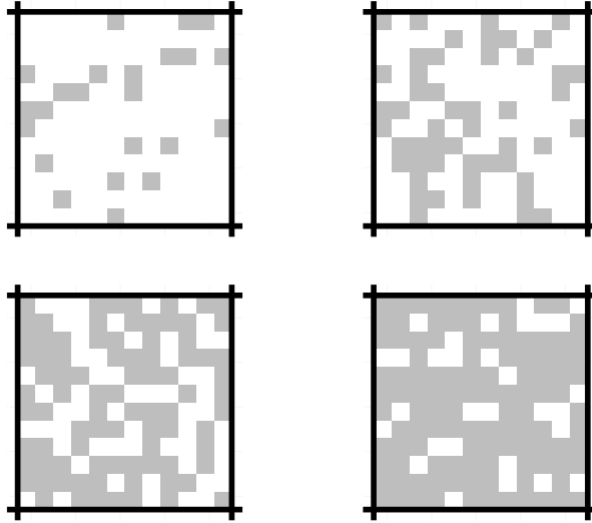


Figure 2.3: Simulations of the percolation model for  $p = 0.2, 0.4, 0.6$  and  $0.8$  listed from top left to bottom right. Filled in cells are coloured grey with empty ones in white.

[Grimmett, 1989]. A half way house model has been proposed called bridge percolation which links the two, see [Chayes and Schonmann, 2000].

The first question of interest is: is it possible for the water to seep through? That is, is there a path such that we can go from the top to the bottom? This question has been considered in the case where the lattice is infinite in which case the question is slightly reworded to; does there exist an infinitely large open cluster containing the origin? This can be shown to be a probability 0 or 1 event. This in-turn raises an observation; there must exist a critical probability, denoted by  $p_c$  such that for  $p < p_c$  we will find an infinitely large set and for  $p > p_c$  we can not. In our simulations we can see that in the case when  $p = 0.2$  and  $0.4$  we can find a path but in the other two cases there does not exist a path. This is something that has been well studied, it was shown in 1960 by Harris that for  $p = 1/2$  we can not find a path, see [Harris, 1960] but took another 20 years before Kesten was able to show that percolation does not occur for  $p > 1/2$  thus showing that  $p_c = 1/2$ , see [Kesten, 1980]. An overview of this is given in [Bollobás and Riordan, 2006].

As mentioned in the previous section, another model stemming from the percolation model exists and is called, *first passage percolation*, this was first introduced by Hammersley and Welsh [Hammersley and Welsh, 1965]. The idea is to introduce on each site a random variable which represents the amount of time it takes for the liquid to travel through the site, this is known as the passage time. If we denote the passage time on site  $s_i$  as  $t(s_i)$  then the passage time of a path  $r$  which travels through the site  $s_1, s_2, \dots, s_n$  is just given by the sum of the passage times, that is

$$T(r) = \sum_{i=1}^n t(s_i).$$

The *first passage time* between two points, say  $u$  and  $v$ , is denoted by  $T(u, v)$  and is defined to be the shortest time to travel between the two points, that is

$$T(u, v) = \inf\{T(r) : r \text{ is a path from } u \text{ to } v.\}.$$

The main object of study here is following

$$B(t) = \{v \in \mathbb{Z}^d : T(0, v) \leq t\}$$

the set of points which can be reached in time  $t$  from the origin. For further details concerning this and percolation in general please see [Kesten, 1987].

### **Diffusion Limited Aggregation**

We now consider another model that is similar to the Eden model, which was first studied by Witten and Sanders in [Witten and Sander, 1983]. Once again, consider the square lattice with an initial seed at  $(0, 0)$ . We start a random walk “at infinity” and once the random walk reaches an adjacent point to the cluster it sticks. We then repeat the process, this gives another growing cluster like the Eden model, but

instead of the addition sites being chosen uniformly over all boundary points, it is now selected with the harmonic measure of the boundary point, which is the probability of the point being hit by the random walk. This is different from the Eden model as in this model there is a possibility of holes being formed as they have no chance of being filled in, whereas in the Eden model this could not happen. This model is very hard to study and very little is known about it, a similar case holds for its continuous variant, off lattice diffusion limited aggregation.

### 2.1.2 Off Lattice Models

We now look at a type of growing model that takes place in continuous space. In the previous section, we mentioned a growth model where the growth site was picked based on the arrival of a simple random walk. The continuous analogue to this is called off site diffusion limited aggregation. Consider the unit disc and allow particles to perform a Brownian path until they hit the disc, at which point they join and increase the size of the cluster. This is very similar to the on site model, in the sense that we have an initial site and the cluster grows randomly but with the continuous analogue of the random walk. The probability of hitting a point is given by the harmonic measure; this is just the probability of a Brownian particle hitting the region. In the case of a circle, the harmonic measure is the uniform measure on the boundary. If the shape is a simply connected object, then we can find a conformal map between it and the circle. Since Brownian motion is invariant under conformal maps, we can then find the hitting probabilities from the circle. This fact will be of use later.

The problem with this model is that it is very hard to say anything about it, as it is not mathematically tractable. This is where the model proposed by Hastings and Levitov is considered, this was first studied in [Hastings and Levitov, 1998].



## Hastings-Levitov $HL(\alpha)$ Model

The model proposed is known as the Hastings-Levitov model and is one that we extend in Chapter 5. As such, we shall dedicate this section to giving a basic understanding of the original model. In Chapter 5, we go into more detail concerning the case when  $\alpha = 0$  as this is the one we shall work with. For a more thorough introduction, please see [Rohde and Zinsmeister, 2005].

Let  $K_0$  be the closed unit disc centered at the origin on the complex plane. We construct a sequence of compact sets  $(K_n)_{n \in \mathbb{N}}$ , which represents the cluster after the addition of  $n$  particles. The cluster after the addition of the first particle is given by  $K_1 = K_0 \cup P_0$  where  $P_0$  is the attachment of a particle at angle  $\theta = 0$ . In our case, the particle,  $P_0$ , will be a slit with length  $d_0$ , i.e.  $P_0 = (1, 1 + d_0]$ .

After the addition of the first particle, we allow particles to arrive with lengths  $d_n$  and at angle  $\theta_n$ . Though we talk about the slits being of length  $d_n$  we work with what is known as the capacity,  $c_n$ , of the particles. This can be expressed as a function of the diameter via

$$e^{c_n} = 1 + \frac{d_n^2}{4(1 + d_n)}. \quad (2.1)$$

Figure 2.4 shows the relation between the values  $c$  and  $d$ , note that as  $c \rightarrow 0$ ,  $d \asymp c^{1/2}$ . The addition of the first particle is given by the conformal map  $f : D_1 \rightarrow D_0$ . This can be constructed from a composition of conformal maps

$$f(z) = (m^{-1} \circ r \circ h \circ m)(z)$$

where

- $m$  maps the unit circle to the real line

$$m(z) = \frac{iz - i}{z + 1};$$

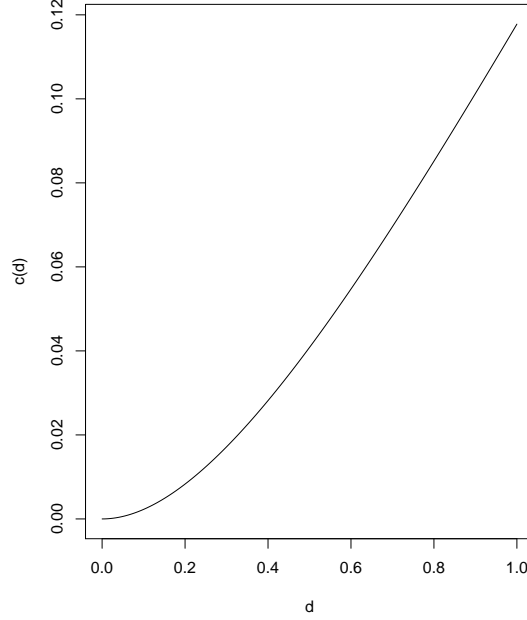


Figure 2.4: A plot comparing  $c$  and  $d$  with relation given in Equation 2.1.

- $r$  is a linear scaling

$$r(z) = \frac{z}{\sqrt{1 - \left(\frac{c^2}{2+c^2}\right)}};$$

- and  $h$  maps the upper half plane, minus a line  $\left\{z = si : s \in \left(0, \frac{1}{4} \left(\frac{c^2}{2+c^2}\right)^2\right)\right\}$  to the upper half plane

$$h(z) = \left(\frac{c^2}{2+c^2}\right)^2 + z^2.$$

Combining this, it is possible to obtain a closed form expression for the function  $f$  [Turner, 2006].

The function  $f$  maps the exterior of the cluster unioned with a point attached at angle  $\theta = 0$  back to the exterior of the initial cluster. This can be extended to map the attachment of any particle at angle  $\theta$  by  $e^{i\theta} f(e^{-i\theta} z)$ .

Set  $D_n = (\mathbb{C} \cup \{\infty\})/K_n$ . By the Riemann mapping theorem we know that there

exists a unique conformal map  $\Phi_n : D_0 \rightarrow D_n$ . The map  $\Phi_n$  is formed from a sequence of the maps  $f_n$ . This in turn will construct our sequence of growing clusters,  $(K_n)$ . With  $\theta_n$  representing the addition of a particle at that angle and  $c_n$  being a sequence of non-negative numbers we can generate a sequence of maps  $f_n$  and  $\Phi_n$

$$f_n(z) = e^{i\theta_n} f_{c_n}(e^{-i\theta_n} z) \quad \text{and} \quad \Phi_n = f_1 \circ \dots \circ f_n.$$

In our case, the angles  $\theta_n$  are distributed uniformly and  $c_n = c$  a constant. The Hastings-Levitov model differs from DLA since the conformal maps cause the slits to become deformed.

The model  $HL(\alpha)$  has the parameter  $\alpha \in [0, 2)$  which is related to the size of the attached particle  $d_n$ . The role of  $\alpha$  is given by

$$d_n = d |\Phi_{n-1}(e^{i\theta_n})|^{-\alpha/2}.$$

Recall that  $c_n$  and  $d_n$  are linked by Equation (2.1) and when  $\alpha = 0$  we obtain the constant case as mentioned.

*Remark 2.1.1.* When  $\alpha = 0$  the sequence of maps are independently identically distributed. This means that before we apply the conformal maps the added particles are the same size but they become deformed after we apply the maps. When  $\alpha = 2$ , the addition of particles is changed such that, after applying the conformal maps, the size of the particles being added is constant. The case  $\alpha = 2$  is of interest since it was thought that this could be a good model to represent DLA since in DLA the size of the added particles are all the same, after the application of the conformal maps.

From this set up, it is now possible to analyse the system. It is shown in [Norris and Turner, 2012] that the limiting shape is a circle. Questions such as the boundary behaviour and the anisotropic case have also been considered, see [Johansson Viklund et al., 2012] and [Silvestri, 2017] for example.

### 2.1.3 Pólya Urn Model

There are many different types of urn models and the rules that govern them vary, but they all follow the same set up. Suppose we start with an urn which contains  $r$  red balls and  $b$  black balls. We operate in discrete time and at each time step we pick a ball out of the urn, we then return that ball into the urn but also place an additional ball into the urn of the same colour. The first question of interest is, does such a system stabilise? That is, if we continue picking and adding balls a large number of times, will the proportion of black and red balls converge? The answer to this question is yes; this is easy to see as the proportion of balls is a martingale and since the proportion of balls is bounded between 0 and 1, we can conclude that it will converge as all bounded martingales do.

Once we know that the process converges, the next question of interest is, what does the process converge to? If we were to repeat the process again would we expect the process to converge to the same point or something else? In this case the proportion of balls will converge to a distribution.

We state a more general result, intuition may suggest that if we start with an equal number of black and red balls so that  $b = r$  then we would expect the proportion of balls to remain 50-50. However, this is not the case, in fact, the limit is given by the following theorem.

**Theorem 2.1.2.** *Suppose we have a Pólya urn model with  $r$  red balls and  $b$  black balls and at each time step we pick a ball from the urn at random. We then return the ball and add  $k$  balls of the same colour. Let  $R_n$  and  $B_n$  represent the number of red and black balls at time  $n$  and let  $P_n = \frac{R_n}{R_n+B_n}$  be the proportion of red balls at time  $n$ , then  $P_n \Rightarrow P_\infty$  (in distribution) where  $P_\infty \sim \text{Beta} \left( \frac{r}{k}, \frac{b}{k} \right)$*

*Proof.* See [Mahmoud, 2009], Theorem 3.2 for further details. □

This model is of interest as we will see that it is very similar to a special case of

the model we study in Chapter 4; the fully connected voter model when there are no internal mechanics and growth is governed by replication. The main difference is that the voter model is a continuous process whereas the Pólya urn model is a discrete time model. This still enables the study of the long term behaviour via the Pólya urn model as we shall discuss later.

As mentioned, there are many variants on the Pólya urn model. We have listed a well known result for a generalised case when there are two balls. This model could be extended to the case where there are a different number of colours or when balls are returned by following a distribution.

## 2.2 Model with Internal Behaviour

So far we have only considered models that over time grew, we now consider another type of model known as self-interacting models. As the name suggests, these are models which change states over time based on the current configuration of the system. We start this section by looking at the voter model and then consider two models well known in the field of population genetics, the Moran model and Wright-Fisher diffusion.

### 2.2.1 Voter Model

The voter model was first studied in [Clifford and Sudbury, 1973] and is a model for the way people vote. This model allows people to vote one of two ways and allows people to update their vote in continuous time based on the way their neighbours are planning on voting. We describe the standard voter model, which is a generalisation of the model we shall be working with.

Consider a connected graph  $G = (V, E)$ . At each vertex there is a person who can vote one of two ways. We let  $v \in V$  denote a person and  $\sigma_v$  denote the person's

vote with  $\sigma_v \in \{-1, 1\}$ . The set of edges,  $e \in E$ , represents connections between two people. If there is an edge linking them then they shall notify each other of how they plan on voting and this in turn influences their vote and whether or not they will change their vote.

In the standard model, each person changes their decision at rate 1. This can be simulated by selecting a person at random with the time given by an exponential 1 random variable. The probability of them changing their vote is proportional to the number of people who have the opposite view to them divided by the total number of neighbours. In Figure 2.5 we can see that the selected cell has three neighbours, all of which share the opposite view to it and so the probability of switching is 1.

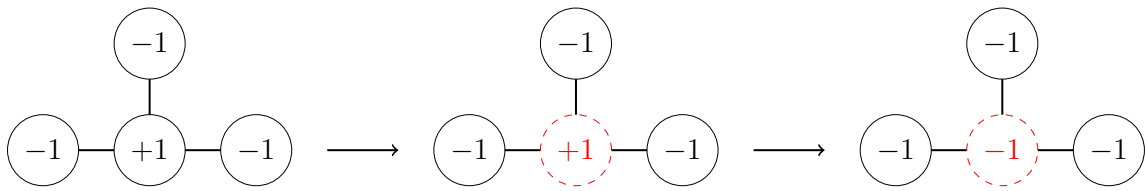


Figure 2.5: A person is picked at random, highlighted in dashed red and changes their vote with probability based on the proportion of votes it has from its neighbours. In this case, since all are opposite to it, they will change with probability 1.

It is clear that, given the rules of evolution, there are two invariant distributions, when all voters agree with each other, either  $\pm 1$ . It can be shown that these are in fact asymptotic distributions and, regardless of the initial configuration, will always converge to one of these.

### 2.2.2 Moran Model

The Moran model, which is named after Patrick Moran, is a model for the reproduction of genes. Suppose that initially we have  $N$  genes not necessarily all distinct. At each time step we select two different genes at random, one of which divides into two

and the other of which dies. This ensure the population remains constant and can be thought of as two parent cells generating an offspring which inherits one of the parents genes. In the long term it can be seen that one gene will eventually dominate and is an absorbing state, meaning once reached we remain there. If we label the genes  $1, 2, \dots, N$  and set  $X_n^N$  to be the number of type  $k \in \{1, 2, \dots, N\}$  genes at time  $n$ , then it can be seen that this is simply a Markov process with jumps  $\pm 1$  or  $0$ , as at each time step only one cell reproduces and one cell dies. The transition probabilities for  $X_n^N$  are given by

$$\begin{aligned}\mathbb{P}(X_{n+1}^N = i + 1 | X_n^N = i) &= \frac{i(N - i)}{N^2} \\ \mathbb{P}(X_{n+1}^N = i - 1 | X_n^N = i) &= \frac{i(N - i)}{N^2} \\ \mathbb{P}(X_{n+1}^N = i | X_n^N = i) &= \frac{(N - i)^2 + i^2}{N^2}.\end{aligned}$$

This clearly shows that there are two absorbing states for  $X^N$ , when  $X^N$  is equal to  $N$  or  $0$ . When  $X^N = N$  this represents dominance among the other genes which must all be equal to  $0$  which represents extinction. This model has been studied further by considering whether or not the specific genes dominate. This can be studied using knowledge of hitting times and is discussed in further detail in [Etheridge, 2011].

### 2.2.3 Wright-Fisher Diffusion

In the previous section we looked at the Moran model and looked at the process  $X_n^N$ , which represented the number of genes of type  $N$  and time  $n$ . If we scale this process by  $1/N$  we then obtain the proportion of total genes that are of type  $N$ . If we send  $N$  to infinity we obtain a continuous time Markov process known as Wright-Fisher Diffusion. Such a process satisfies the following stochastic differential equation

$$dX_t = \sqrt{X_t(1 - X_t)} dB_t.$$

Observe that when  $X_t = 0$  or  $X_t = 1$  the derivative is zero, which shows they are stationary states. It can further be shown that the process will always converge to 0 or 1. By symmetry, if the process starts at  $\frac{1}{2}$ , then it must reach each point with probability  $\frac{1}{2}$  thus making it an asymptotic distribution, see Chapter 3 and the scale measure for further details.

What is useful about Wright-Fisher diffusion, is that it carries many of the properties of the finite Moran model but allows us to use a different set of tools to study it. This is an example of a *scaling limit*, where we have changed the scale of the original process and looked at the limiting behaviour, which is a continuous time process.

This method is useful since continuous time processes are very well understood and have a large number of tools from Itô calculus to analyse them.

## 2.3 Random Growth Models with Competition

In this section we consider a model that lies at the intersection of the two previous areas. That is, a model that has some form of interaction but also where the population size changes. We shall look at the Lotka-Volterra model which is a simple deterministic model representing the population of two competing species.

### 2.3.1 Lotka-Volterra System

Consider the populations of two species of animals, one a predator, the other the prey. As time evolves the population of these two species will evolve but will depend on each other. As the number of predators increase the number of prey will decrease. However, once the population of prey has decreased sufficiently, there will not be enough food for the predator and this will cause a decrease in their population. This will then have a knock on effect and cause the population of the prey to increase. The system then repeats. This model was first studied independently by Alfred Lotka and



Vito Volterra, who were both looking at these types of interactions. For Lotka, he was concerned with a chemical reaction and Volterra was interested in applications in Biology, see [Goel et al., 1971] for more details.

## The System

Let  $x(t)$  and  $y(t)$  represent the number of prey and predators respectively, then the Lotka-Volterra equations are

$$\frac{dx_t}{dt} = ax_t - bx_t y_t \quad \text{and} \quad \frac{dy_t}{dt} = dx_t y_t - cy_t$$

where  $a, b, c, d$  are non-negative constants.

It can easily be seen that  $x_t = 0$  and  $y_t = 0$  are fixed points of the above equation, this would represent extinction of both parties. There is also another fixed point to the above pair of equations, this is given by  $x_t = c/d$  and  $y_t = a/b$  and would represent steady co-existence between the two species, the rate at which they die would be equal to the rate of birth and so we would see no change in the population.

Figure 2.6 shows a numerical solution to the differential system with certain parameters. As you can see, the system appears to be periodically fluctuating up and down as the other species does. Figure 2.7 highlights this periodic nature by plotting the two populations against each other. We can see that it does indeed look periodic and this fact is true for any values of  $a, b, c$  and  $d$  and any choice of starting points.

If we multiply and rearrange the two equations to separate the variables and then integrate, we obtain the following

$$-c \log(x_t) + dx_t = a \log(y_t) - by_t + D,$$

where  $D$  is the constant of integration. This is the equation plotted in Figure 2.7.

This is a very simple model since it is deterministic and so does not allow any

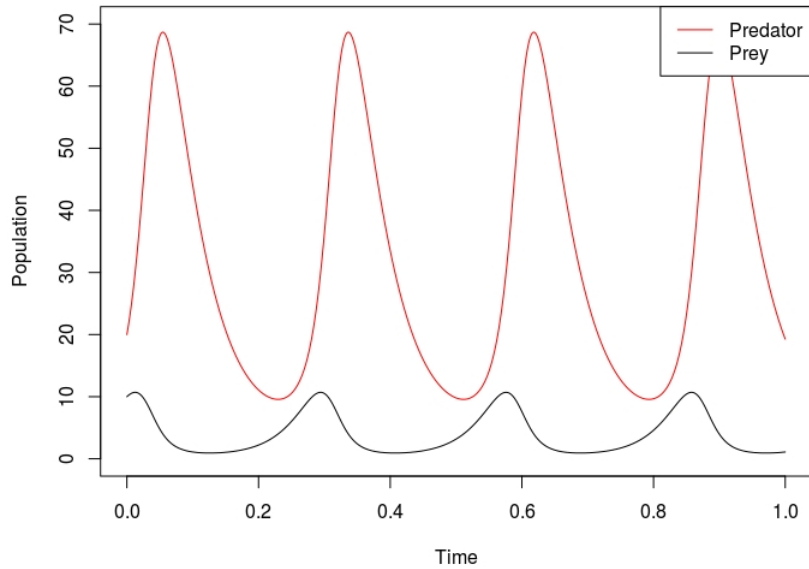


Figure 2.6: Plot of solution for  $a = 30, b = 1, c = 20$  and  $d = 5$ .

random fluctuations, which would be expected in real life. Another shortfall of this model is that for low values the behaviour seems unrealistic. For certain choices of the parameters, it is possible to have the population size incredibly small, for example, less than one. This is unrealistic since we know that a population size cannot be this small but also, even after reaching this small size, the population still recovers and can become large again.

Even though the model is simple, it still highlights the interest in trying to model population size and how competition can affect this.

### 2.3.2 Introducing randomness

The model we have described so far is a model with growth and competition which is something we are interested in but it does not contain any random elements. This means we know for certain how the model will evolve over time and does not take into account any random fluctuation that may appear such as from a harsh winter

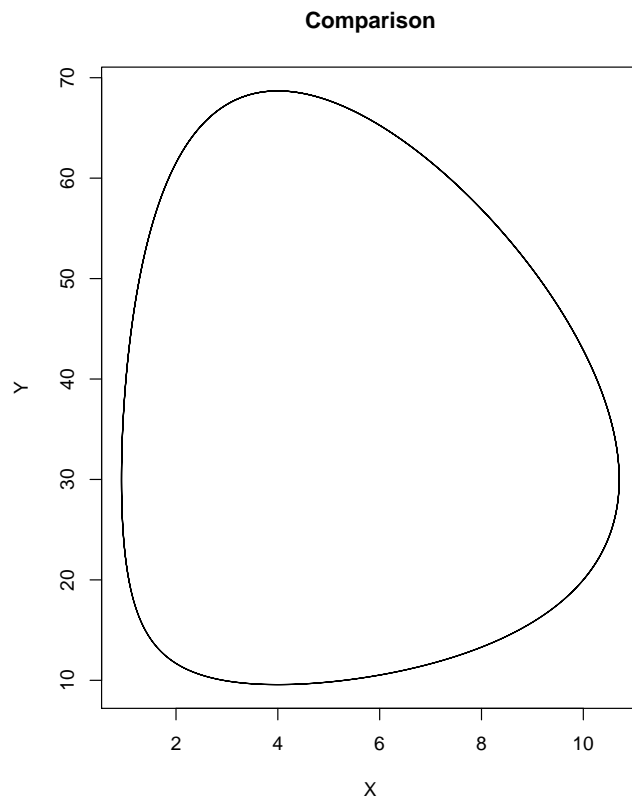


Figure 2.7: Limit cycle of  $X$ , the predator against  $Y$ , the prey.

etc. This is one reason why this model is not a good fit to the real world. One way to try and make the model more realistic is to allow randomness to occur, this is done by making the process a pure jump process with the rates being motivated by the differential equations above which describes the evolution of the system. Consider the following system comprising of predators  $X_t$  and prey  $Y_t$  which evolves like so

$$(x, y) \rightarrow \begin{cases} (x + 1, y) & \text{at rate } ax \\ (x - 1, y) & \text{at rate } bxy \\ (x, y + 1) & \text{at rate } dxy \\ (x, y - 1) & \text{at rate } cy \end{cases}$$

The first two lines relate the the number of prey in the system and are related to the birth and death of one respectively. The last two are the same for the predator. Again, this system can be seen that when both are zero we have a fixed point but now, because of the randomness we have, there cannot be any other fixed point. An issue with the system described above is that it does allow the prey to become extinct to which the predator will only evolve by growing, similar to the deterministic model. Other attempts to introduce randomness into the Lotka-Volterra system have been considered and for further reading please see [Dimentberg, 2002] and [Cai and Lin, 2004] for example.

# Chapter 3

## Diffusion Theory

In the previous section we had a look at a few models; some of which were discrete others of which were continuous. In this section, we shall look at how, from a continuous time Markov chain, we can scale the process to generate a sequence of stochastic processes such that they converge to a continuous time Markov process. The obtained continuous Markov process will be defined via a stochastic differential equation and we shall look at the tools required to understand such a process. We look at the speed measure and scale function along with the Feller boundary conditions. Once we have classified the boundary behaviour of our process we show how that is enough to understand the long term behaviour of the process.

### 3.1 Motivation

Let us have a look at a simple random process which takes jumps  $\pm 1$  both with rate 1 and starts at 0. Some simulations of this process are given after a different number of steps, shown in Figures 3.1 and 3.2. As you can see, the process after 1000 steps looks like a continuous process. This is because we have constrained the plot to fit inside the box. A question arises though, is it possible to scale the process such that when we tend the jump size to zero we obtain a continuous process? The answer to

this is yes, and there exists many theorems depending on the type of process you are considering and the limit process you wish to obtain. In this section, we shall look at a few that have been of use to us along with a few examples to help illustrate things.

We shall restrict our study to that of *pure jump Markov processes*. A pure jump process is one whose movements are given purely by jumps and a Markov process is one whose movement is only dependant on its current position and does not require knowledge of the past. Though this may sound restrictive, this still incorporates a large class of processes such as Poisson processes which are used in queueing theory and the study of birth-death processes.

**Definition 3.1.1.** We say that a process  $X_t$  is a pure jump process if we can express the evolution of the process in terms of jumps and its current location only. That is, we can write

$$X \rightarrow \begin{cases} X + J_1(X) & \text{at rate} & R_1(X) \\ X + J_2(X) & \text{at rate} & R_2(X) \\ & \vdots & \\ X + J_n(X) & \text{at rate} & R_n(X) \end{cases}$$

where  $n$  is finite,  $J_i$  are the jumps of the process and  $R_i$  are the rates for the jumps  $J_i$  respectively.

Before we consider these limit theorems though, we need to introduce one thing that is central in the study of all these processes and this is known as the *kernel*. This gives us another identical way of describing a pure jump process.

**Definition 3.1.2.** We say that the process  $X_n$  is a pure jump Markov process with kernel,  $K$ , if it is a pure jump process and for all  $n \in \mathbb{N}$ ,

$$\mathbb{P}(J_n \in dt, \Delta X_{J_n} \in dy | J_n > t, X_{J_{n-1}} = x) = K(x, dy)dt$$

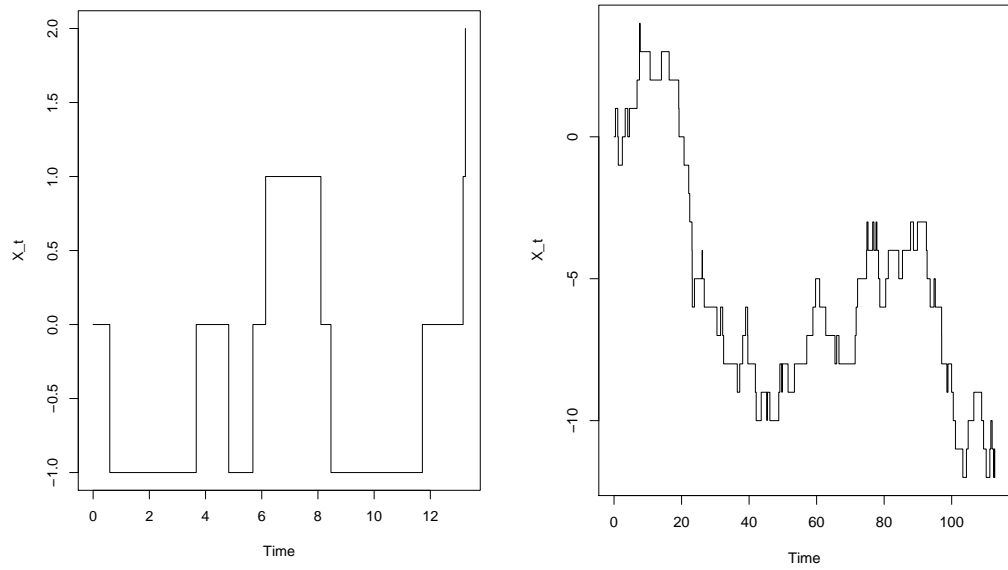


Figure 3.1: Sample paths of a birth death process after 10 and 100 steps.

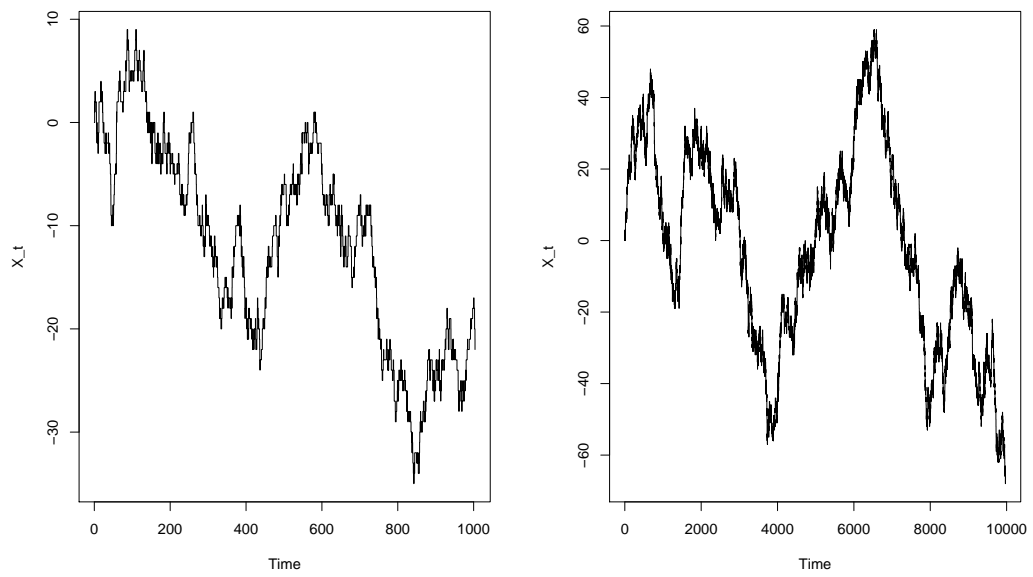


Figure 3.2: Sample paths of a birth death process after 1000 and 10000 steps.

where  $J_1, J_2, \dots$  are the jump times of the process.

The kernel is useful since it contains all the information we require about the behaviour of the process. It informs us of the probability of jumping and the direction of the jumps.

We say that the two definitions are identical in the sense that we can derive one from the other since they both inform us of the jumps and the rates of the process which is all we need. We end this section with a couple of illustrative examples.

**Example 3.1.1.** Consider the process  $X_t$  discussed so far with jumps given by the following,

$$x \longrightarrow \begin{cases} x + 1 & \text{at rate } 1 \\ x - 1 & \text{at rate } 1. \end{cases}$$

Such a process would have the following kernel associated with it,

$$K(x, dy) = \delta_1 + \delta_{-1}$$

where  $\delta_x$  represents the Dirac point measure which is defined such that for a subset  $A \subset \mathbb{R}$ ,  $\delta_x(A) = 1$  if  $x \in A$  and 0 otherwise.

We can also extend the process such that it takes jumps of  $\pm 1$  but this time with rates given by  $\lambda$  and  $\mu$  respectively. This process is known as a birth-death process. If  $\mu = 0$ , then the process is known as a pure birth process. Such a process would have kernel given by

$$K(x, dy) = \lambda\delta_1 + \mu\delta_{-1}.$$

We now turn our attention into how we can use the kernel to show that the process, after an appropriate scaling, converges.



## 3.2 Limit Theorem

In this section, we look at a theorem which states sufficient conditions to ensure that the processes introduced in the previous section converges to a continuous process. The result that we give is found in [Ethier and Kurtz, 1986] Chapter 7, though we have altered it such that it includes the kernel directly. Similar results exist for other types of diffusion processes, though we do not discuss them in much detail here. A similar result for Markov chains which converge to a non-deterministic process is given in [Gikhman and Skorokhod, 1996].

### Inhomogeneous Markov Processes

The result we use is concerned with space inhomogeneous Markov processes, that is, those processes whose movements are determined only by its current location. The first theorem in this section will state what are the required conditions such that a process converges weakly but before we state the theorem we need to introduce a few concepts.

We firstly define the Skorokhod metric, this is a metric on the space  $D[0, \infty)$ . Let  $\Lambda$  be the space of strictly increasing continuous functions mapping  $[0, \infty)$  to itself and let  $\|f\|$  be the supremum norm, that is

$$\|f\| = \sup_{x>0} |f(x)|,$$

then the Skorokhod metric,  $\sigma(f, g)$ , for  $f, g \in D[0, \infty)$  is given by

$$\sigma(f, g) = \inf_{\lambda \in \Lambda} \max \{ \lambda - I, f - g \circ \lambda \}$$

where  $I$  is the identity function on  $[0, \infty)$ .

Let  $(X_t^N)_{N \in \mathbb{N}}$  and  $X_t$  all be stochastic processes in  $D[0, \infty)$ . We shall define  $\mu^N$

(respectively  $\mu$ ) corresponding to  $X^N$  (respectively  $X$ ) to be the measure defined to be

$$\mu(A) = \mathbb{P}(X_t \in A) \quad (\text{respectively} \quad \mu^N(A) = \mathbb{P}(X_t^N \in A))$$

where  $A$  is an element of the  $\sigma$ -Algebra of Borel subsets of  $D[0, \infty)$ .

**Definition 3.2.1.** We say that a sequence of measures  $(\mu^N)_{N \in \mathbb{N}}$  converges weakly to  $\mu$  if for all continuous bounded functionals  $f$  on  $D[0, \infty)$  with the Skorokhod metric we have

$$\lim_{N \rightarrow \infty} \int f(x) \mu^N(dx) = \int f(x) \mu(dx).$$

If this is the case, we shall write  $\mu^N \Rightarrow \mu$  (or  $X^N \Rightarrow X$ ). Further details concerning the notion of weak convergence and the Skorokhod metric can be found in [Billingsley, 1968], Chapter 1 Section 5 and Chapter 3 Section 14 respectively.

Also the notion of Brownian motion will be of use throughout the rest of this thesis and so we include it here.

**Definition 3.2.2.** A stochastic process  $B_t$  which has the following properties is known as Brownian motion:

- For any  $s < t < u < v$  the increment  $B_t - B_s$  is independent of  $B_v - B_u$ .
- For any  $s, t > 0$ ,  $B_{s+t} - B_s$  is normally distribution with mean 0 and variance  $t$ .
- The process  $B_t$  has almost surely continuous sample paths.

When  $B_0 = 0$  then we shall refer to it as the *Standard Brownian Motion*.

The following theorem is what shall be key in our study of Markov processes to obtain a scaling limit. We shall firstly state it in the way it shall be used but this is not how it is stated in [Ethier and Kurtz, 1986]. After stating the theorem we shall explain how this is obtained from their theorem.

**Theorem 3.2.1.** Let  $a = (a_{i,j})$  be a Lipschitz continuous, symmetric, non-negative definite  $d \times d$  matrix valued function on  $\mathbb{R}^d$  and let  $b : \mathbb{R}^d \rightarrow \mathbb{R}^d$  be Lipschitz continuous. Let  $K^N(x, dy)$  be the kernel associated with the process  $X^N$ , which takes values on some subset  $I \subseteq \mathbb{R}^d$  and define

$$b^N(x) = \int_{\mathbb{R}^d} y K^N(x, dy) \quad \text{and} \quad a^N(x) = \int_{\mathbb{R}^d} yy^T K^N(x, dy).$$

Suppose that,

$$\sup_{x \in I} |a^N(x) - a(x)| \rightarrow 0 \quad \text{and} \quad \sup_{x \in I} |b^N(x) - b(x)| \rightarrow 0$$

and that

$$\sup_{t > 0} |X_t^N - X_{t-}^N| \rightarrow 0$$

as  $N \rightarrow \infty$ . If  $X_0^N = X_0$ , then, setting  $\sigma^2(x) = a(x)$ ,  $X_t^N \rightarrow X_t$  weakly in  $D[0, \infty)$  where  $X_t$  is a solution to the stochastic differential equation given by

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t.$$

where  $B_t$  is the standard one dimensional Brownian motion.

We now state the original Theorem and Corollary as stated in their book. Though their terminology and notation is slightly different we will explain how it relates to the theorem we use above.

**Theorem 3.2.2** (Theorem 4.1 [Ethier and Kurtz, 1986]). Let  $a = (a_{i,j})$  be a continuous, symmetric, nonnegative definite,  $d \times d$  matrix valued function on  $\mathbb{R}^d$  and let  $b : \mathbb{R}^d \rightarrow \mathbb{R}^d$  be continuous. Let

$$A = \{(f, \mathcal{G}f) : f \in C_c^\infty(\mathbb{R}^d)\}$$

where

$$\mathcal{G}f = \frac{1}{2} \sum a_{ij} \partial_i \partial_j f + \sum b_i \partial_i f$$

and suppose that the  $C_{\mathbb{R}^d}[0, \infty)$  martingale problem for  $A$  is well-posed. For  $n = 1, 2, \dots$ , let  $X^n$  and  $B^n$  be processes with sample paths in  $D_{\mathbb{R}^d}[0, \infty)$ , and let  $A_n = (A_{i,j}^n)$  be a symmetric  $d \times d$  matrix-valued process such that  $A_{i,j}^n$  has sample paths in  $D_{\mathbb{R}}[0, \infty)$  and  $A^n(t) - A^n(s)$  is nonnegative definite for  $0 \leq s < t$ . Set  $\mathcal{F}_t^n = \sigma(X^n(s), B^n(s), A^n(s) : s \leq t)$ .

Let  $\tau_r^n = \inf \{t : |X^n(t)| \geq r \text{ or } X^n(t-) \geq t\}$  and suppose that

$$M^n \equiv X^n - B^n$$

and

$$M_i^n M_j^n - A_{i,j}^n, \quad i, j = 1, 2, \dots, d,$$

are  $\mathcal{F}_t^n$ -local martingales, and that for each  $r > 0, T > 0$ , and  $i, j = 1, 2, \dots, d$

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[ \sup_{t \leq T \wedge t_r^n} |X^n(t) - X^n(t-)|^2 \right] = 0 \quad (3.1)$$

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[ \sup_{t \leq T \wedge t_r^n} |B^n(t) - B^n(t-)|^2 \right] = 0 \quad (3.2)$$

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[ \sup_{t \leq T \wedge t_r^n} |A_{i,j}^n(t) - A_{i,j}^n(t-)|^2 \right] = 0 \quad (3.3)$$

$$\sup_{t \leq T \wedge t_r^n} \left| B_i^n(t) - \int_0^t b_i(X^n(s)) ds \right| \rightarrow 0 \quad \text{in probability} \quad (3.4)$$

and

$$\sup_{t \leq T \wedge t_r^n} \left| A_{i,j}^n(t) - \int_0^t a_{i,j}(X^n(s)) ds \right| \rightarrow 0 \quad \text{in probability.} \quad (3.5)$$

Suppose that  $PX_n(0)^{-1} \Rightarrow \nu \in \mathcal{P}(\mathbb{R}^d)$ . Then  $(X^n)$  converges in distribution to the solution of the martingale problem for  $(A, \nu)$ .

In essence the above theorem states that if for a given sequence of processes  $(X^n)$  you can find functions  $a$  and  $b$  such that they satisfy the above conditions, then the sequence  $(X^n)$  converges weakly to the process  $X$  which satisfies the stochastic differential equation given by

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t.$$

This is precisely what is means to be a solution to the martingale problem. The problem arises with how to find the functions  $a$  and  $b$ . This is where the corollary is of use.

**Corollary 3.2.1** (Corollary 4.2 [Ethier and Kurtz, 1986]). *Let  $a, b$  and  $A$  be as in the previous theorem and suppose the martingale problem for  $(A, \nu)$  has a unique solution for each  $\nu \in \mathcal{P}(\mathbb{R}^d)$ . Let  $\mu^n(x, \Gamma), n = 1, 2, \dots$ , be the transition function on  $\mathbb{R}^d$ , and set*

$$b^n(x) = n \int_{|y-x| \leq 1} (y-x) \mu^n(x, dy)$$

and

$$a^n(x) = n \int_{|y-x| \leq 1} (y-x)(y-x)^T \mu^n(x, dy).$$

Suppose for each  $r > 0$  and  $\varepsilon > 0$ .

$$\sup_{|x| \leq r} |a^n(x) - a(x)| \rightarrow 0,$$

$$\sup_{|x| \leq r} |b^n(x) - b(x)| \rightarrow 0$$

and

$$\sup_{|x| \leq r} n \mu^n(x, \{y : |y-x| \geq \varepsilon\}) \rightarrow 0. \tag{3.6}$$

Let  $Y^n$  be a Markov chain with transition function  $\mu^n(x, \Gamma)$  and define  $X^n(t) = Y^n([nt])$ . If  $PY_n(0)^{-1} \Rightarrow \nu$ , then  $(X^n)$  converges in distribution to the solution of

the martingale problem for  $(A, \nu)$ .

The above corollary states that from the kernel we can construct functions  $a^n$  and  $b^n$  and if these function converges to  $a$  and  $b$  then subject to some other conditions we have weak convergence. The reason we do not go into much further detail concerning the other conditions is because they are automatically satisfied for us in all our cases. The condition given by equation (3.6) is a requirement that the jump sizes of the process  $X^n$  decrease as we increase  $n$ . The other condition given by  $PY_n(0)^{-1} \Rightarrow \nu$  is stating that the initial distributions must converge which in our case holds since they are all equal. The only other difference between this corollary and our theorem is that we use  $K^n = n\mu^n$

*Proof of Theorem 3.2.1.* See Theorem 7.4.1 and Corollary 7.4.2 in [Ethier and Kurtz, 1986]. □

In the previous theorem, the functions  $b(x)$  and  $a(x)$  can be thought of as the infinitesimal drift and diffusivity of the process  $X_t$  respectively. This is why we require the functions  $b^N(x)$  and  $a^N(x)$  to converge. We also require the jump sizes to tend to zero which is clearly necessary. Our final requirement is that the starting points of all the processes coincide, this can be relaxed such that the starting points follow a distribution and that this sequence of distributions must converge to the starting distribution of  $X_t$ . We do not include it here since it will not be of need to us. We shall now give a few examples where we can use the above theorem.

**Example 3.2.1.** Consider the birth death process given in Example 3.1.1. If we construct the process  $X_t^N := \frac{1}{\sqrt{N}}X_{Nt}$  then this would have kernel

$$K^N(x, dy) = N\delta_{\frac{-1}{\sqrt{N}}} + N\delta_{\frac{1}{\sqrt{N}}}$$

which yield the functions

$$b^N(x) = 0 = b(x) \quad \text{and} \quad a^N(x) = 1 = a(x)$$

hence, by tending  $N \rightarrow \infty$  we see that the jump sizes tend to zero. Assuming that  $X_0^N = X_0$ , we can conclude, by Theorem 3.2.1, that the process  $X_t^N$  converges to the process  $X_t$  which satisfies the stochastic differential equation

$$dX_t = 0dt + 1dB_t,$$

which in other words means that the process  $X_t = B_t$ , a Brownian path.

**Example 3.2.2.** Consider again the birth-death process given in Example 3.1.1 but with  $\lambda(x) = x$  and  $\mu = 0$ . Such a process is known as a pure birth process; since  $\mu = 0$ , the process can only increase. This is an example of an inhomogeneous space process since the  $\lambda$  depends on the location of the process  $X_t$ . The process is still a Markov chain as we only require knowledge of its current position. Consider the process  $X_t^N = \frac{1}{N}X_{Nt}$ , this has the kernel

$$K^N(x, dy) = Nx\delta_{\frac{1}{N}}$$

which yields the equations

$$a^N(x) = x = a(x) \quad \text{and} \quad b^N(x) = 0 = b(x).$$

Again, the jump sizes decrease to zero as we increase  $N$  and we make the assumption that  $X_0 = X_0^N$  for all  $N$ . Combining this, by Theorem 3.2.1, we obtain the ordinary differential equation given by

$$dX_t = X_t dt$$

which can be solved to give  $X_t = X_0 e^t$ .

One question that the above theorem does not answer is: though we know that we have convergence of the process, what is the speed at which it converges? This can be useful at times since this helps us obtain the error when approximating for finite  $N$  or when running simulations. In the case where the limit is deterministic, as in Example 3.2.2 we can apply what is known as the fluid limit which shows we have exponential convergence to the deterministic process, for further details see [Darling and Norris, 2008].

### 3.3 Analysis of Diffusion Processes

In this section, we analyse the behaviour of the stochastic differential equations that we can obtain by applying the theorem in the previous section. The main question that we are interested in is: what is the long term behaviour of the process as  $t \rightarrow \infty$ ? The process may settle down and converge to a point, it may converge to a set of points with certain probabilities or it may not settle down at all. If the process does not settle down, is it possible to find a distribution for  $X_\infty$ ? To answer these questions, it turns out that it is sufficient to know about the boundary behaviour along with something called the speed measure. We shall look at the speed measure and see how this is of use to us along with the scale function, this is a function which converts suitable diffusion processes into martingales, which reduces the complexity. Once this is done we shall use the speed measure to classify the boundary behaviour into one of three types, either absorbing, reflecting, or inaccessible and we will then combine this knowledge to understand the long term behaviour.



## Scale function

Firstly, suppose we have a stochastic process  $X_t$  which satisfies the following stochastic differential equation,

$$dX_t = \mu(X_t)dt + \sigma(X_t)dB_t. \quad (3.7)$$

Such a process is known as a diffusion process and the functions  $\mu$  and  $\sigma$  are known as the drift and diffusivity respectively. As mentioned earlier, if we could reduce the problem such that the above process, or some invertible function of the above process, was a martingale, then we could analyse the martingales behaviour and that in turn would tell us the behaviour of our original process.

To do this, we use Itô's formula which allows us to calculate functions of stochastic processes given in the form above. We state the theorem below

**Theorem 3.3.1** (Itô's Formula). *Let  $X_t$  be given in (3.7) and let  $f$  be a twice differentiable continuous function, then*

$$df(X_t) = \left( \mu(X_t)f'(X_t) + \frac{1}{2}\sigma^2(X_t)f''(X_t) \right) dt + \sigma(X_t)f'(X_t)dB_t.$$

*Proof.* See Theorem 15.19 in [Kallenberg, 2002]. □

This is of use since if we have a process of the form (3.7) and we suppose that we have a function  $f$  which satisfies the assumed conditions, we can apply  $f$  to the process and determine it's stochastic differential equation by applying Itô's theorem to see what the resulting process would be. If we now suppose that  $f$  satisfies

$$\mu(x)f'(x) + \frac{1}{2}\sigma^2(x)f''(x) = 0$$

then after Itô's formula we would obtain a process which only has a diffusive part and so must be a martingale. Such a function is called the scale function and we denote

it by  $\rho$ . It can be shown that  $\rho$  must satisfy

$$\rho(x) = \int_0^x \exp\left(-2 \int_m^y \frac{\mu(u)}{\sigma^2(u)} du\right) dy$$

where  $m$  is a point in the interior of the interval the process takes, provided that  $\mu/\sigma^2$  is integrable. This is an assumption we shall need to make throughout since if this does not hold we would not be able to transform the process.

After applying the scale function  $\rho$  we obtain the process  $Y_t = \rho(X_t)$  which is of the form

$$dY_t = \tilde{\sigma}(Y_t)dB_t$$

and so is a martingale.

**Definition 3.3.1.** Any process whose scale function is linear is said to be in *natural scale*.

If the process is not in natural scale, then after applying  $\rho$  it will be. This can be seen since any martingale is in natural scale as  $\mu(x) \equiv 0$ . This reduces the study of the diffusion process into that of a martingale. We now turn our attention to them. For us to analyse martingales, we need to look at something called the speed measure.

### Speed Measure

As we have shown with the use of the scale function, we can reduce any suitable diffusion process into that of a martingale. This is useful since all martingales are time changes of Brownian motion. This time change is given by the speed measure,  $m$ . Combining this we obtain that

$$\rho(X_t) = B_{m(t)}.$$

For a process of the form

$$dX_t = \sigma(X_t)dB_t$$

the speed measure is given by

$$m(dx) = \frac{1}{\sigma^2(x)}dx.$$

In general, for a process of the form (3.7), the speed measure is given by

$$m(dx) = \frac{1}{\sigma^2(x)} \exp\left(2 \int_0^x \frac{\mu(y)}{\sigma^2(y)} dy\right) dx.$$

The speed measure is of use since for any martingale, knowledge of the boundary behaviour is given explicitly in terms of the speed measure. This in turn will allow us to classify the long term behaviour of the process.

## Boundary Behaviour

We now turn our attention to the boundary points of a process. Suppose we have a process  $X_t$  which takes values on the interval  $[\theta, \lambda]$ , which may be infinite. We wish to understand what these boundary points are. There are three types of boundary points; *inaccessible*, *absorbing* and *reflecting*.

**Definition 3.3.2.** For a stochastic process  $X_t$  we say that an endpoint  $\theta$  is *inaccessible* if the probability of hitting it from any internal point is 0.

*Remark 3.3.1.* If the boundary point is  $\pm\infty$  then it is inaccessible.

**Definition 3.3.3.** For a stochastic process  $X_t$  we say that an endpoint  $\theta$  is *absorbing* if the event  $\{X_s = \theta\} \Rightarrow \{X_u = \theta\}$  for all  $u \geq s$ .

*Remark 3.3.2.* Though we do not consider it, if  $Z_\theta$  has positive Lebesgue measure but not equal to the whole real line, then we say that the process  $X_t$  is *sticky* or *elastic*.

If the processes is sticky, it spends some positive amount of time at the end points before leaving. See [Feller, 1952] for his original work.

**Definition 3.3.4.** For a stochastic process  $X_t$  we say that an endpoint  $\theta$  is *reflecting* if the set  $Z_\theta = \{t \geq 0 : X_t = \theta\}$  is such that  $\text{int}(Z_\theta) = \emptyset$ .

Informally, an end point is inaccessible if we can not reach it, an end point is accessible and reflecting if once reached the particle is ejected out immediately and the endpoint is absorbing if the process remains in the end point indefinitely.

If the endpoint is accessible then it must be either absorbing or reflecting, if it is not accessible then it must be inaccessible. The following theorem helps us classify the end points.

**Theorem 3.3.2** (Boundary behaviour, Feller). *Let  $m$  be the speed measure of a regular diffusion on a natural scale in some interval  $I = [\theta, \lambda]$ , and fix any  $u \in \text{int}(I)$ . Then,*

(i)  $\lambda$  is accessible iff it is finite with  $\int_u^\lambda (\lambda - x)m(dx) < \infty$

(ii)  $\lambda$  is accessible and reflecting iff it is finite with  $m(u, \lambda) < \infty$ .

*Proof.* See Theorem 20.12 of [Kallenberg, 2002]. □

Let us look at some examples to see how this can be of use.

**Example 3.3.1.** Consider the process  $X_t$  such that  $X_0 = 0$  and the process satisfies the stochastic differential equation given by

$$dX_t = \sqrt{1 - X_t^2} dB_t.$$

Clearly such a process must take values on the interval  $[-1, 1]$  and so we wish to understand the boundary points  $\pm 1$ . Firstly we calculate the speed measure, in this case we have

$$m(dx) = \frac{dx}{1 - x^2}.$$

It can be seen that

$$\int_u^1 (1-x)m(dx) = \int_u^1 \frac{dx}{1+x} < \infty$$

but

$$m(u, 1] = \int_u^1 \frac{dx}{1-x^2} = \infty$$

and so, from Theorem 3.3.2 the boundary point 1 is accessible but not reflecting and so must be an absorbing boundary point. Similar calculations can be done on  $-1$  to show that it too is an absorbing point.

**Example 3.3.2.** Now consider a slight variant on the one above, consider the process  $X_t$  which satisfies the equation

$$dX_t = (1 - X_t^2)dB_t.$$

Again,  $\pm 1$  are boundary points but this time we have

$$\begin{aligned} \int_u^1 (1-x)m(dx) &= \int_u^1 \frac{dx}{(1-x)(1+x)^2} \\ &\geq \int_u^1 \frac{dx}{4(1-x)} = \infty \end{aligned}$$

and so in this case the boundary point 1 is not accessible, similarly for  $-1$ . This is because, as the process approaches the boundary points, the rate of change decreases faster than the process can move and so does not reach the boundary point.

This raises the question of how quickly does the diffusive term need to go to zero to ensure the end point in the above cases are accessible.

*Proposition 3.3.1.* Consider the stochastic process  $X_t$  which satisfies the stochastic differential equation

$$dX_t = (1 - X_t^2)^\alpha dB_t$$

for  $\alpha > 0$ . Then the boundary points are accessible for  $\alpha \leq 1/2$  and inaccessible for  $\alpha > 1/2$ .

*Proof.* Since the process is in natural scale it is clear to see that the speed measure is  $m(dx) = (1 - x^2)^{-2\alpha} dx$ . Applying Theorem 3.3.2 (i) we see that the  $\int_0^1 (1 - x)m(dx)$  is finite when  $\alpha \leq 1/2$  and infinite otherwise. A similar argument holds for the lower boundary point.  $\square$

**Example 3.3.3.** Consider another variant which takes values on  $[-1, 1]$ . Suppose we have a process  $X_t$  which satisfies

$$dX_t = -\frac{1}{2}X_t dt + \sqrt{1 - X_t^2} dB_t.$$

This is not in natural scale and so we cannot classify the end points directly. We must first calculate the scale function which in this case is

$$\rho(x) = \sin^{-1}(x).$$

Considering the process  $Y_t = \sin^{-1}(X_t)$  and applying Itô's formula yields,

$$dY_t = dB_t \Rightarrow Y_t = B_t.$$

Observe that this shows that the process  $X_t = \sin(B_t)$ . Since the original process  $X_t$  takes value on the interval  $[-1, 1]$  and we are now considering the process  $Y_t$ , this must takes values on  $[-\pi, \pi]$ . We firstly classify these boundary points and then this in turn will classify those of  $X_t$ . Since  $Y_t = B_t$  it can easily be verified that  $m(dx) = dx$  the standard Lebesgue measure. This means that for the boundary point  $\pi$  we have

$$m(u, \pi] = \pi - u < \infty$$

and so the point  $\pi$  is an accessible reflecting point. A similar argument shows that  $-\pi$  is an accessible reflecting point. Since both these points are reflecting, when we look at  $X_t$  we can conclude that they too must be reflecting points.

We now know how to classify the end points for a wide variety of processes, but how can we use this knowledge to work out the long time behaviour of such processes? The next theorem is what shall be of use. Beforehand, we need the following definitions.

**Definition 3.3.5.** A stochastic process  $X_t$  taking values on an interval  $A$  is said to be *recurrent* if for any  $x, y \in A$

$$\mathbb{P}_x(T_y < \infty) = 1$$

where  $\mathbb{P}_x$  is the measure associated with the process  $X_t$  with  $X_0 = x$  and  $T_y = \inf\{t > 0 : X_t = y\}$ .

A recurrent process can be further split into being *null recurrent* or *positive recurrent*.

**Definition 3.3.6.** A stochastic process  $X_t$  taking values in  $A$  which is recurrent is called positive recurrent if for all  $x, y \in A$ ,

$$\mathbb{E}_x(T_y) < \infty.$$

If this is not the case then the process is called null recurrent.

**Definition 3.3.7.** A diffusion  $X_t$  with speed measure  $m(dx)$  on an interval  $I$  shall be called *m-ergodic* if it is recurrent and the limit distribution of  $X_t$  has density proportional to  $m$  for all  $X_0 \in I$ .

We now let  $(, [$  and  $[[$  denote the boundary point being inaccessible, absorbing or reflecting respectively (e.g.  $[[0, 1]$  means that the point 0 is a reflecting point and 1 is an absorbing point).

**Theorem 3.3.3** (Feller, Maruyama and Tanaka, Theorem 20.15 [Kallenberg, 2002]).

*For any regular diffusion on a natural scale and with speed measure  $m$ , the ergodic behaviour is the following, depending on the initial position  $x$  and the nature of the boundaries,*

1.  $(-\infty, \infty)$ :  *$m$ -ergodic if  $m$  is bounded, otherwise null-recurrent;*
2.  $(0, \infty)$ : *converges to 0 a.s.;*
3.  $[0, \infty)$ : *absorbed at 0 a.s.;*
4.  $[[0, \infty)$ :  *$m$ -ergodic if  $m$  is bounded, otherwise null-recurrent;*
5.  $(0, 1)$ : *converges to 0 or 1 with probabilities  $x$  and  $1 - x$ , respectively;*
6.  $[0, 1)$ : *absorbed at 0 or converges to 1 with probabilities  $x$  and  $1 - x$ , respectively;*
7.  $[0, 1]$ : *absorbed at 0 or 1 with probabilities  $x$  and  $1 - x$ , respectively;*
8.  $[[0, 1)$ : *converges to 1 a.s.;*
9.  $[[0, 1]$ : *absorbed at 1 a.s.;*
10.  $[[0, 1]]$ :  *$m$ -ergodic.*

As such, for us to understand the long term behaviour of the process, we first use the function  $\rho$  to transform the process into a martingale so that it is in natural scale. We then calculate the speed measure of the process, this will allow us to work out the behaviour of the boundary points. Once we have done that we can use the above theorem, after applying an affine transformation, to work out what the limiting behaviour of the transformed process is and then, using the fact that  $\rho$  is strictly increasing (and so invertible), we can work out the limiting distribution of the original process. We end this section with a few examples.



**Example 3.3.4** (Example 3.3.1 Cont.). In Example 3.3.1 we showed that  $\pm 1$  are both absorbing points for the process  $X_t$  satisfying  $dX_t = \sqrt{1 - X_t^2} dB_t$ . As such, we are in case 7 of Theorem 3.3.3, and so the probabilities are directly proportional to the distance to the boundaries. Since our process takes values on  $[-1, 1]$  we need a linear map to map it to the interval  $[0, 1]$ . In this case we have  $f(x) = \frac{1}{2}(1 + x)$  and so if our process starts at  $x_0 \in [-1, 1]$  we obtain that the distribution  $X_\infty$  is given by

$$X_\infty = \begin{cases} +1 & \text{with probability } \frac{1}{2}(1 + x_0) \\ -1 & \text{with probability } \frac{1}{2}(1 - x_0). \end{cases}$$

**Example 3.3.5** (Example 3.3.2 Cont.). Example 3.3.2 showed that  $\pm 1$  are inaccessible for the process  $X_t$  satisfying  $dX_t = (1 - X_t^2)dB_t$ . As such, we are in case 5 of Theorem 3.3.3, and so the probabilities are directly proportional to the distance to the boundaries. Though they are not absorbed at the endpoints, they still converge to these points and so we obtain the same distribution as the previous example.

**Example 3.3.6** (Example 3.3.3 Cont.). For the process given as a solution to the stochastic differential equation

$$dX_t = -\frac{1}{2}X_t dt + \sqrt{1 - X_t^2} dB_t$$

we know that we have reflecting boundary points at  $\pm 1$ , this puts us in case 10 of the theorem. This means our process is ergodic and the distribution at infinity is given by the arcsine distribution. This can be seen since the process,  $X_t$  above has scale function  $\rho(x) = \sin^{-1}(x)$  giving  $Y_t = \rho(X_t) = B_t$  with the boundary points for the process  $Y_t$  being reflecting. The distribution is given by the uniform distribution on the interval  $[-\pi, \pi]$ , mapping this back through gives us the arcsine distribution.

# Chapter 4

## Addition of Growth to the Fully Connected Voter Model

### 4.1 Introduction

Discrete random dynamical systems typically evolve according to one of two mechanisms: growth, in which the addition of individuals or particles are governed by some rule; or internal dynamics, in which particles within the system interact with each other. Examples of pure growth dynamical systems include the Eden model, 2d percolation and the Pólya urn model. See [Lindgren, 1963] for the original work by Eden concerning his model, [Grimmett, 1989] for details on percolation and [Johnson and Kotz, 1977] concerning the Pólya urn model. Examples of models with purely internal dynamics include the voter model, which is studied in [Clifford and Sudbury, 1973] or the Metropolis-Hastings algorithm for sampling from the Ising model (originally published in [Ising, 1925]). In this chapter, we consider a model that incorporates both of these mechanisms; a model that has growth and internal dynamics. The model we look at is the fully-connected voter model with growth, which was first introduced in [Morris and Rogers, 2014] by Morris and Rogers. In their paper, Morris and Rogers

consider one type of internal behaviour and two different types of growth. We extend these results in a rigorous manner. We show how to approximate the proportion of votes for a large initial population with a diffusion process. We also calculate the limit distributions and show how the behaviour of states can change by increasing the rate from constant growth to exponential growth. The case where the initial population size is small is also discussed and we comment on how our limit distribution results still hold for all except two cases. In the cases where it does not, we show how in one case we obtain a system that is similar to the Pólya urn model and so the limit distribution can be explicitly calculated.

#### 4.1.1 The fully connected voter model with the addition of growth

Following the set up given by [Morris and Rogers, 2014], we firstly introduce the basics of the voter model. Consider a graph  $G = (V, E)$  comprising a set  $V$  of vertices and  $E$  of edges. On each vertex  $v \in V$  there exists a discrete random variable, which we denote as  $\sigma_v$  where  $\sigma_v \in \{-1, 1\}$ . The point  $v$  represents a voter and  $\sigma_v$  is the vote of a person on the site. The way that each person may vote is influenced by their neighbours. We shall consider the mean-field case where everyone is connected to one another. Since we are not concerned with the individual behaviour of each person, but the average of them, we shall look at the proportion of votes, which we denote by  $x$ . This is a function based upon the configurations of the votes and is defined to be

$$x(\sigma) = \frac{1}{|V|} \sum_{v \in V} \sigma_v.$$

This means  $x \in [-1, 1]$  with the boundary points being achieved only when the voters all align with the same party, either  $-1$  or  $+1$ , respectively. We denote  $N = |V|$  to be the number of people initially in our system. If we let  $f_u(x)$  be the rate at which a

person changes their vote from  $+1$  to  $-1$  and  $f_d(x)$  be the rate of switching the other way. We can express such a system in the following way.

**Definition 4.1.1.** The *fully connected voter model* is a process  $X_t^N$  which has jumps given by

$$x \longrightarrow \begin{cases} x - \frac{2}{N} & \text{at rate } f_u(x)N(1+x)/2 \\ x + \frac{2}{N} & \text{at rate } f_d(x)N(1-x)/2. \end{cases}$$

Though originally intended for use as a model for votes, we shall use the terminology of Morris and Rogers [Morris and Rogers, 2014], in which it is formulated as a model for magnetism. Instead of calling each element  $v$  a voter we shall refer to them as a particle and the value  $\sigma_v$  shall be the spin of that particle. The values  $+1$  and  $-1$  represent whether the spin is up or down.

We now introduce growth into the system. Again, the initial size of our system shall be  $N$ , but now  $S_t$  is defined to be the scale of the system at time  $t$ , so that  $NS_t$  is the number of particles in the system at time  $t$ . Particles are introduced with the rate of addition being proportional to the size of the system to some power, which we denote by  $\alpha$  with  $\alpha \in [0, 1]$ . This condition can be relaxed further without changing the work so that it is only asymptotically proportional, but we do not provide the details here. Particles are introduced into the system at rate  $g_u(x)$  being positive and at rate  $g_d(x)$  being negative. The combined process  $Z_t^N = (X_t^N, S_t^N)$  is a pure jump process, which is defined in Section 4.2 [Ethier and Kurtz, 1986].

**Definition 4.1.2.** The *growing voter model with rate  $\alpha$*  is a process  $(X_t^N, S_t^N)$  with the initial condition that  $(X_0^N, S_0^N) = (x_0, 1)$  where  $x_0 \in [-1, 1]$ , which has jumps

given by the following:

$$(x, s) \longrightarrow \begin{cases} \left(x - \frac{2}{sN}, s\right) & \text{at rate } f_u(x)sN(1+x)/2 \\ \left(x + \frac{2}{sN}, s\right) & \text{at rate } f_d(x)sN(1-x)/2 \\ \left(x + \frac{1-x}{sN+1}, s + \frac{1}{N}\right) & \text{at rate } \lambda s^\alpha N^\alpha g_u(x) \\ \left(x - \frac{1+x}{sN+1}, s + \frac{1}{N}\right) & \text{at rate } \lambda s^\alpha N^\alpha g_d(x), \end{cases}$$

where  $\lambda$  is a constant of proportionality,  $0 \leq \alpha \leq 1$  and  $f_u(x), f_v(x), g_u(x)$  and  $g_v(x)$  are as defined.

### 4.1.2 Overview

Such a model is of interest as the addition of growth can substantially alter the behaviour of the process. For example, it was observed in [Morris and Rogers, 2014] that it can turn what was once an absorbing state into a repelling state. They showed how a change in the growth rate from  $\alpha = 0$  to  $0 < \alpha \leq 1$  can break ergodicity of the process.

In Section 2, we construct diffusion processes to approximate our system for large  $N$  (after an appropriate time change). We need to consider different cases depending on the functions  $f_u, f_d, g_u$  and  $g_d$ . It is shown that for any non-deterministic behaviour to be seen, we must have it that the internal behaviour is set so that  $f_u(x) = \frac{1}{2}(1+x)$  and  $f_v(x) = \frac{1}{2}(1-x)$ , so that the system is self-stabilising. Where possible, we use Kurtz's Theorem (Theorem 3.2.1) to show that as  $N \rightarrow \infty$  we can find a time change such that we obtain a diffusion process for the combined process  $Z_t = (X_t, S_t)$ , where  $X_t$  and  $S_t$  have the property of being decoupled, i.e. they can be written as a differential equation in terms of themselves only. In the cases where we can not apply Kurtz's Theorem, we show that there exists no such time change for us to see any non-trivial behaviour.

Section 3 then looks at the limit distribution of the processes obtained in Section 2. We look at the speed measure and scale functions of a process and show how these, along with knowledge of the boundary points, can be used to determine the limit distribution of the process. To simplify arguments, we restrict ourselves to the case where  $f_u, f_v, g_u$  and  $g_v$  are linear, however standard arguments can be applied to the general case. We end this section by considering the case where  $N$  is fixed and show that when the limiting distribution for the asymptotic case is not dependent on the initial condition, then the limiting distribution is the same for the case where  $N \rightarrow \infty$ .

Finally, in Section 4 we end with some illustrative examples, which show how the behaviour does change, and we compare our results to some simulations.

## 4.2 Diffusion Estimates

Our main tool in finding diffusion approximations is Kurtz's Theorem which we stated in Chapter 3, from this we can construct diffusion processes.

We will firstly apply Kurtz's theorem to our process  $(X_t^N, S_t^N)$  with the initial conditions that  $(X_0^N, S_0^N) = (x_0, 1)$ .

**Example 4.2.1** (No Growth). Consider the fully connected case but without growth as defined in Definition 4.1.1. We shall apply Theorem 3.2.1 to find its natural limit as  $N \rightarrow \infty$ . This process has the kernel given by

$$K^N(x, dy) = \frac{1}{2}f_u(x)N(1+x)\delta_{-2/N} + \frac{1}{2}f_d(x)N(1+x)\delta_{2/N}$$

where we have used the notation that  $\delta_x$  is the Dirac point measure already defined. This yields

$$b^N(x) = -f_u(x)(1+x) + f_d(x)(1-x) = b(x)$$

and

$$a^N(x) = \frac{2}{N} [f_u(x)(1+x) + f_d(x)(1-x)] \rightarrow 0 = a(x).$$

Hence, given any Lipschitz functions  $f_u(x)$  and  $f_d(x)$  the process  $X_t^N$  converges weakly to the process  $X_t$ , which is given by the ordinary differential equation

$$dX_t = [-f_u(X_t)(1+X_t) + f_d(X_t)(1-X_t)] dt \quad \text{with} \quad X_0 = x_0. \quad (4.1)$$

If we set  $f_u(x) = \frac{1-x}{1+x} f_d(x)$  then (4.1) is equal to zero and so we do not see any interesting behaviour. To see any non-trivial limiting behaviour, it is necessary to apply a time change. Increasing all the rates by  $N^\gamma$  (for some  $\gamma$  to be determined) we obtain the following functions

$$b^N(x) = [-f_u(x)(1+x) + f_d(x)(1-x)] N^\gamma = 0 = b(x)$$

and

$$a^N(x) = 2N^{\gamma-1} [f_u(x)(1+x) + f_d(x)(1-x)].$$

Setting  $\gamma = 1$ , we obtain convergence of  $a^N$  to  $a$  given by

$$a(x) = 2 [f_u(x)(1+x) + f_d(x)(1-x)].$$

By Theorem 3.2.1, we can deduce that in this case  $X_{\tau(t)}^N$ , with  $\frac{d\tau}{dt} = N$  (i.e.  $\tau(t) = Nt$ ), converges weakly to  $X_t$ , which is a solution to the stochastic differential equation

$$dX_t = \sqrt{f_u(x)(1+x) + f_d(x)(1-x)} dB_t \quad \text{with} \quad X_0 = x_0.$$

Combining we get the following:

Suppose  $\tau(t) = N^\gamma t$  and consider the limit of  $X_{\tau(t)}^N$  as  $N \rightarrow \infty$ .

a) If  $f_d(x) \neq \frac{1+x}{1-x} f_u(x)$  then, for a non-trivial limit to exist,  $\gamma = 0$  and the limit

process  $X_t$  satisfies the following ordinary differential equation

$$dX_t = [f_d(X_t)(1 - X_t) - f_u(X_t)(1 + X_t)] dt \quad \text{with} \quad X_0 = x_0.$$

b) If  $f_d(x) \equiv \frac{1+x}{1-x} f_u(x)$  then, for a non-trivial limit to exist,  $\gamma = 1$  and the limit process  $X_t$  satisfies the following stochastic differential equation

$$dX_t = \sqrt{2(1 + X_t)f_u(X_t)} dB_t \quad \text{with} \quad X_0 = x_0.$$

**Example 4.2.2** (Pure Growth). Consider the process  $Z_t^N = (X_t^N, S_t^N)$  which we introduced in Definition 4.1.2. The scale process is independent from the magnetisation and so we can obtain a one dimensional process  $S_t^N$  satisfying  $S_0^N = 1$  and with jumps given by

$$s \longrightarrow s + \frac{1}{N} \quad \text{at rate} \quad s^\alpha N^\alpha$$

where  $\alpha \in \mathbb{R}$ . This is a simple birth process with kernel given by

$$K^N(s, dy) = s^\alpha N^\alpha \delta_{1/N}$$

which allows us to compute

$$b^N(s) = s^\alpha N^{\alpha-1} \quad \text{and} \quad a^N(s) = s^\alpha N^{\alpha-2}.$$

By Theorem 3.2.1, the process  $S_{\tau(t)}^N$ , where  $\frac{d\tau}{dt} = N^{1-\alpha}$ , converges to the process  $S_t$  which satisfies the following differential equation

$$dS_t = S_t^\alpha dt.$$



This has solution

$$S_t = \begin{cases} [(1 - \alpha)t + 1]^{\frac{1}{1-\alpha}} & \text{when } \alpha < 1 \text{ for } t \geq 0 \\ \exp(t) & \text{when } \alpha = 1 \text{ for } t \geq 0 \\ [(1 - \alpha)t + 1]^{\frac{1}{1-\alpha}} & \text{when } \alpha > 1 \text{ for } 0 \leq t < t^* \end{cases} \quad (4.2)$$

where  $t^* = 1/(\alpha - 1)$  is the explosion time of the process after which the process is infinite.

Example 4.2.1 is informative as it shows us that it is necessary to consider different cases depending on whether the functions satisfy certain equations (namely  $f_u(x) = \frac{1-x}{1+x}f_d(x)$  in this example). Example 4.2.1 also shows us that we may need to speed up the system to see any interesting behaviour which is of use when considering the growing system.

We now turn to our system of interest, the fully connected voter model with growth included. The limit will vary depending on the functions  $f_u, f_d, g_u$  and  $g_d$  and, depending on what these functions are, determine the value of  $\gamma$  required to rescale the rates to obtain a non-zero limit such that the processes are independent of one another. In the theorem we use the notation  $f(x) \equiv g(x)$  which means  $f(x) = g(x)$  for all  $x$ . We state our results in the following theorem.

**Theorem 4.2.1.** *For all  $0 \leq \alpha \leq 1$ , and Lipschitz functions  $f_u, f_d, g_u$  and  $g_d$ , with  $g_u$  and  $g_d$  non-negative either:*

- (1) *There exists a  $\gamma = \gamma(\alpha)$  such that by setting  $\frac{d\tau}{dt} = (S_t^N N)^\gamma$ , the process  $Z_{\tau(t)}^N = (X_{\tau(t)}^N, S_{\tau(t)}^N)$  converges weakly in  $D[0, \infty)$  to  $Z_t = (X_t, S_t)$  where  $X_t$  and  $S_t$  are decoupled, with  $X_t$  satisfying a non-trivial differential equation.*
- (2) *There does not exist a time change for which the limit process is non-trivial.*

*We list all the possible cases below. In all cases  $(X_0, S_0) = (x_0, 1)$  where  $x_0 \in [-1, 1]$ .*

(a) If  $f_d(x) \not\equiv \frac{1+x}{1-x}f_u(x)$  then  $\gamma = 0$ ,  $S_t = 1$  for all  $t$  and

(i) for  $0 \leq \alpha < 1$ ,

$$dX_t = [f_d(X_t)(1 - X_t) - f_u(X_t)(1 + X_t)] dt \quad (4.3)$$

(ii) for  $\alpha = 1$ ,

$$\begin{aligned} dX_t = & [(1 - X_t)f_d(X_t) - (1 + X_t)f_u(X_t) \\ & + g_u(X_t)(1 - X_t) - g_d(X_t)(1 + X_t)] dt. \end{aligned} \quad (4.4)$$

(b) If  $f_d(x) \equiv \frac{1+x}{1-x}f_u(x)$  and  $g_u(x) \not\equiv \frac{1+x}{1-x}g_d(x)$  then  $\gamma = 1 - \alpha$ ,  $S_t = e^t$  and

(i) for  $\alpha = 0$ ,

$$dX_t = [g_u(X_t)(1 - X_t) - g_d(X_t)(1 + X_t)] dt + \sqrt{4(1 + X_t)f_u(X_t)} dB_t \quad (4.5)$$

(ii) for  $0 < \alpha \leq 1$ ,

$$dX_t = [g_u(X_t)(1 - X_t) - g_d(X_t)(1 + X_t)] dt. \quad (4.6)$$

(c) If  $f_d(x) \equiv \frac{1+x}{1-x}f_u(x)$  with  $f_u(x) \not\equiv 0$  and  $g_u(x) \equiv \frac{1+x}{1-x}g_d(x)$  then

(i) for  $\alpha = 0$ ,  $\gamma = 1 - \alpha$ ,  $S_t = e^t$  and

$$dX_t = \sqrt{4(1 + X_t)f_u(X_t)} dB_t \quad (4.7)$$

(ii) for  $0 < \alpha \leq 1$ , there is no non-trivial limit as

$$\lim_{N \rightarrow \infty} \sup_{t \geq 0} |X_t^N - x_0| = 0. \quad (4.8)$$

(d) If  $f_u(x) \equiv f_d(x) \equiv 0$  and  $g_u(x) \equiv \frac{1+x}{1-x}g_d(x)$  then there is no non-trivial limit as

$$\lim_{N \rightarrow \infty} \sup_{t \geq 0} |X_t^N - x_0| = 0. \quad (4.9)$$

*Proof.* The kernel of the process  $Z_{\tau(t)}^N$  with  $\frac{d\tau}{dt} = (S_t N)^\gamma$  is given by

$$\begin{aligned} K^N(x, s, dy) &= f_u(x)(sN)^{\gamma+1} \frac{1+x}{2} \delta\left(-\frac{2}{sN}, 0\right) + f_d(x)(sN)^{\gamma+1} \frac{1-x}{2} \delta\left(\frac{2}{sN}, 0\right) \\ &\quad + (sN)^{\alpha+\gamma} g_u(x) \delta\left(\frac{1-x}{sN+1}, \frac{1}{N}\right) + (sN)^{\alpha+\gamma} g_d(x) \delta\left(-\frac{1+x}{sN+1}, \frac{1}{N}\right) \end{aligned}$$

where  $\delta(x) = \delta_x$  is the Dirac point measure already defined. This gives the following drift and diffusive functions

$$b^N(x, s) = b_1^N(x, s)(sN)^\gamma + b_2^N(x, s)(sN)^{\alpha+\gamma-1} \quad (4.10)$$

$$a^N(x, s) = \lambda a_1^N(x, s)(sN)^{\gamma-1} + \lambda a_2^N(x, s)(sN)^{\alpha+\gamma-2} \quad (4.11)$$

where

$$b_1^N(x, s) = [f_d(x)(1-x) - f_u(x)(1+x)](1, 0) \quad (4.12)$$

$$b_2^N(x, s) = \left( g_u(x) \frac{1-x}{1+1/(sN)} - g_d(x) \frac{1+x}{1+1/(sN)}, s \right) \quad (4.13)$$

$$a_1^N(x, s) = [2f_u(x)(1+x) + 2f_d(x)(1-x)] \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad (4.14)$$

$$a_2^N(x, s) = \begin{bmatrix} g_u(x) \frac{(1-x)^2}{(1+1/(sN))^2} + g_d(x) \frac{(1+x)^2}{(1+1/(sN))^2} & s \frac{g_u(x)(1-x) - g_d(x)(1+x)}{1+1/(sN)} \\ s \frac{g_u(x)(1-x) - g_d(x)(1+x)}{1+1/(sN)} & s^2 \end{bmatrix}. \quad (4.15)$$

We now consider  $a^N$  and  $b^N$  as  $N \rightarrow \infty$ . For convergence they both must be finite, this is our first constraint and so the leading power of  $N$  in each function must be no greater than 0, with at least one of the leading powers being equal to 0 for one of the

two functions. Our second constraint is that we want the limit to be such that the limit processes have components decoupled.

Observe that for  $\alpha \in [0, 1]$ , equations (4.12)-(4.15) are the coefficients from equations (4.10) and (4.11) in descending power of  $N$  and, as a result, show the required  $\gamma$  depending on the choice of functions  $f_u, f_d, g_u$  and  $g_d$ . We then obtain the following cases

1. If  $b_1^N(x, s) \not\equiv 0$  for all  $0 \leq \alpha \leq 1$  then  $\gamma = 0$
2. If  $b_1^N(x, s) \equiv 0$  and  $b_2^N(x, s) \not\equiv 0$  then  $\gamma = 1 - \alpha$
3. If  $b_1^N(x, s) \equiv 0$  and  $b_2^N(x, s) \equiv 0$  but  $a_1^N(x, s) \not\equiv 0$  then for  $\alpha = 0, \gamma = 1$
4. If we are in any other case then no  $\gamma$  exists to give a non-trivial limit.

The conditions on the function  $f_u, f_v, g_u$  and  $g_v$  that must hold in order that the above functions be identically zero correspond to the different cases in Theorem 4.2.1.

Observe that the jump sizes of  $(X^N, S^N)$  are no greater than  $1/N$  and that for cases (a), (b) and (c)(i) there exists functions  $a(x)$  and  $b(x)$  such that  $b^N(x) \rightarrow b(x)$  and  $a^N(x) \rightarrow a(x)$  as  $N \rightarrow \infty$ . Thus, by Theorem 3.2.1 we deduce that the process  $(X_t^N, S_t^N)$  converges weakly to the limit processes  $(X_t, S_t)$ .

An issue arises for cases (c)(ii) and (d) as in this change of time the scale function explodes instantaneously in the limit and so we are not able to apply Theorem 3.2.1 to deduce a limit. However, we show that regardless of the scaling used, the process shall be identically zero in the limit. We employ the use of the exponential martingale inequality (see [Darling and Norris, 2008], Theorem 4.2) for case (c)(ii) and Bernstein's inequality (see [Freedman, 1975], Proposition 2.1) for case (d). In both cases we can only use these inequalities if the process is a martingale. This is true for the processes given in case (c)(ii) and (d) since if we calculate  $b^N(x, s)$  we will see that this is identically zero, this means that there is no drift for each  $N$ , and so we expect the process to remain where it is, exactly what a martingale is.

Starting with case (c)(ii), our aim is to firstly use Doob's inequality from [Darling and Norris, 2008] to show that

$$\mathbb{P}\left(\sup_t |X_t^N| > \frac{1}{N^{1/3}}\right) \leq 4N^{2/3} \sup_t \mathbb{E}[|X_t^N|^2] \leq \frac{4}{N^{1/3}}.$$

Setting  $A_T$  to be the event that  $\{\sup_{t < T} |S_t^N - e^t| < \frac{1}{2}e^t\}$ , observe that we have the following,

$$\begin{aligned} \mathbb{E}\left(|X_t^N|^2\right) &= \mathbb{E}\left((X_t^N)^2 \middle| A_T\right) \mathbb{P}(A_T) + \mathbb{E}\left((X_t^N)^2 \middle| A_T^c\right) \mathbb{P}(A_T^c) \\ &\leq \mathbb{E}\left((X_t^N)^2 \middle| A_T\right) + \mathbb{P}(A_T^c). \end{aligned}$$

where we have used the fact that  $|X_t| \leq 1$  for all  $t$  for the second line. It can be shown, with  $\gamma = 1 - \alpha$  so that  $S_t^N \rightarrow e^t$ , that

$$\begin{aligned} \mathbb{E}\left(|X_t^N|^2 \middle| A_t\right) &\leq \mathbb{E}\left[\frac{f_u(X_t^N)(1 + X_t^N) + f_d(X_t^N)(1 - X_t^N)}{2S_t^N N} \middle| A_T\right] \\ &\leq \frac{K}{N}, \end{aligned}$$

where we have obtained the first line from the kernel of the process and used the fact that  $S_t^N > 1$  and set  $K = 2 \sup_x (f_u(x) \vee f_d(x))$  in the second line. Note that the final line is independent of  $t$ . Using Itô's formula with  $f(x, t) = e^{-t}x$  it is possible to show that the process  $M_t^N = e^{-t}S_t^N - 1$  is a martingale and so by use of Doob's inequality we get that

$$\mathbb{P}\left(\sup_{t < T} |S_t^N - e^t| \geq \frac{1}{2}e^t\right) = \mathbb{P}\left(\sup_{t < T} |M_t^N| \geq \frac{1}{2}\right) \leq 16\mathbb{E}\left[(M_T^N)^2\right].$$

Furthermore, it can be shown that, with  $K^N(s, dy) = Ns\delta_y(1/N)$ , the kernel of the

process  $S_t^N$  that

$$\begin{aligned}
\mathbb{E} \left[ (M_T^N)^2 \right] &\leq \int_0^T \int_{\mathbb{R}} \mathbb{E} \left[ (e^{-u}y)^2 K^N(S^N, dy) du \right] \\
&= \int_0^T \frac{e^{-2u}}{N} \mathbb{E} (S_{u-}^N) du \\
&= \int_0^T \frac{e^{-u}}{N} du = \frac{1 - e^{-T}}{N} \\
&\leq \frac{1}{N}
\end{aligned}$$

which is also independent of  $T$ . Combining all this together we deduce (4.2). Case (d) is simpler since the only way this process can evolve is by the addition of particle. Since we are only interested in the asymptotic distribution we can consider the jump process  $Y_n^N = X_{J_n}^N$  where  $J_n$  are the jump times of the process since the limiting behaviour of this process will be the same as the original process. We then use Bernstein's inequality with a suitable constant  $C$  and with  $\mathbb{E} |X_n^N - X_{n-1}^N|^2 \leq \frac{4}{(N+n)^2}$  to obtain the same conclusion. An alternate proof is given later on.  $\square$

**Remarks** It is interesting that while  $\alpha$  plays an important role in determining what limit process is obtained, the limit processes do not feature  $\alpha$ .

### 4.3 Limit Distributions

In Section 4.2 we showed that if the internal behaviour, type of addition and the rate at which particles are added are known, we can find diffusion processes to approximate the system when  $N \rightarrow \infty$ . In this section we study the long time behaviour of these processes, that is we find the distribution of  $X_\infty = \lim_{t \rightarrow \infty} X_t$  which exists in all cases. We first consider when we obtain deterministic limits, then martingales and then diffusion processes.

### 4.3.1 Deterministic

We obtained five cases where the limiting process from Theorem 4.2.1 was given by an ordinary differential equation. These were cases (a)(i), (a)(ii), (b)(ii), (c)(ii) and (d). Cases (c)(ii) and (d) are trivial with  $X_\infty = X_0$ . In the remaining cases, to simplify the presentation, we make the assumption that the functions  $f_u, f_d, g_u$  and  $g_d$  are linear; our results can be extended to the non-linear case.

In all cases we can write  $dX_t = \mu(X_t)dt$  for some suitable function  $\mu$  with the property that  $\mu$  is Lipschitz and  $\mu(1) > 0$  and  $\mu(-1) < 0$  (or vice versa). By Picard's existence Theorem we know that a unique solution must exist for any given initial condition (see pages 463-468 [Blazy et al., 2013] for more details). Piecing this together we can conclude that there must exist a point  $x \in [-1, 1]$  such that  $\mu(x) = 0$ . Since we have assumed that our functions are linear we know there can be only one root. This point must be the limit point for any other solution with a different starting point since on both sides the derivative is towards the fixed point and so it is an attractive fixed point.

### 4.3.2 Martingales and Diffusion Processes

We now use our knowledge of the speed measure and scale function to work out the long term behaviour of the processes we obtained.

**Theorem 4.3.1.** *Suppose that the functions  $f_u, f_d, g_u$  and  $g_d$  are all linear. Then, if  $X_t$  is the limit process obtained from the process  $X_\tau^N$  in Theorem 4.2.1,  $X_t$  always converges in distribution to  $X_\infty$  as  $t \rightarrow \infty$  with the distribution of  $X_\infty$  depending on  $f_u, f_d, g_u$  and  $g_d$  in the following ways:*

(a) ***Deterministic***

- (i) *If  $f_d(x) \neq \frac{1+x}{1-x}f_u(x)$  then the limit distribution is a point mass at the unique fixed point of (4.3) or (4.4) for  $0 \leq \alpha < 1$  and  $\alpha = 1$  respectively.*

(ii) If  $f_d(x) \equiv \frac{1+x}{1-x}f_u(x)$ ,  $g_u(x) \equiv \frac{1+x}{1-x}g_d(x)$  and  $0 < \alpha \leq 1$  then the limit distribution is a point mass at the unique fixed point of (4.6).

(iii) If  $f_d(x) \equiv \frac{1+x}{1-x}f_u(x)$ ,  $f_u(x) \not\equiv 0$ ,  $g_u(x) \not\equiv \frac{1+x}{1-x}g_d(x)$  and  $0 < \alpha \leq 1$  then  $X_\infty = x_0$ .

(iv) If  $f_d(x) \equiv f_u(x) \equiv 0$ ,  $g_u(x) \not\equiv \frac{1+x}{1-x}g_d(x)$  and  $0 < \alpha \leq 1$  then  $X_\infty = x_0$ .

(b) **Point-masses**

(i) If  $f_d(x) \equiv \frac{1+x}{1-x}f_u(x)$ ,  $f_u(x) \not\equiv 0$ ,  $g_u(x) \equiv \frac{1+x}{1-x}g_d(x)$  and  $\alpha = 0$  then

$$X_\infty = \begin{cases} +1 & \text{w.p. } \frac{1}{2}(1+x_0) \\ -1 & \text{w.p. } \frac{1}{2}(1-x_0) \end{cases}$$

(c) **Ergodic**

(i) If  $f_d(x) \equiv \frac{1+x}{1-x}f_u(x)$ ,  $f_u(x) \not\equiv 0$  and  $g_u(x) \not\equiv \frac{1+x}{1-x}g_d(x)$ , then  $\alpha = 0$  with distribution given by

$$m(dx) = \frac{1}{4(1-x)f_u(x)} \exp\left(2 \int_0^x \frac{g_u(y)(1-y) - g_d(y)(1+y)}{4(1-y)f_u(y)} dy\right).$$

*Proof.* In cases (a)(i) and (a)(ii) the limit process is deterministic and it is well known that the process will converge to the fixed point of its equation. Case (a)(iii) and (a)(iv) are also obvious since the limit process is the constant function. These are the only deterministic cases.

Case (b)(i) obtains a martingale in the limit, this is already in natural scale and so we do not need to consider the scale function. Since we have assumed that the functions are all linear we know that  $\pm 1$  are the only roots to the stochastic differential equation. The end points are clearly not repelling so are either absorbing or inaccessible, either way, by Theorem 3.3.3 we know that the process converges to the



end points with probability proportional to the starting points. Hence we obtain the only point mass case.

The final case has conditions which yield a stochastic differential equation with drift and diffusive terms. Since we are not in natural scale, we are required to calculate the scale function, in this case we get

$$\rho(z) = \int_0^z \exp \left( -2 \int_0^y \frac{g_u(x)(1-x) - g_d(x)(1+x)}{4(1+x)f_u(x)} dx \right) dy.$$

It can be shown that  $\rho(\pm 1) < \infty$  and so we need to determine the boundary behaviour. Using the definition of  $m(dx)$  given, it can be shown that  $m(0, 1] < \infty$  and  $m[-1, 0) < \infty$  and so by Theorem 3.3.2 we can deduce they are both reflecting points and so by Theorem 3.3.3 we conclude the process is  $m$ -ergodic.  $\square$

### 4.3.3 Limit Distribution for fixed $N$

Throughout we have only considered the situation when  $N \rightarrow \infty$ . We now look at the process  $X_t^N$  when  $N$  is fixed and find the distribution of  $X_\infty^N$ . Again, we make the assumption that the functions are linear since the matter is more complicated when the functions are not linear as there could be multiple roots to the differential equation obtained. If this is the case, then the starting point does play a part since in the finite case the process could ‘jump’ over these stationary points.

The reason we discuss the finite  $N$  case is to compare our results to that obtained in [Morris and Rogers, 2014] which motivated our study. In their paper, they consider for fixed  $N$  the limit as they tend  $t \rightarrow \infty$  and then tend  $N \rightarrow \infty$  whereas in ours, we tend  $N \rightarrow \infty$  and then  $t \rightarrow \infty$ . It is interesting to compare the two as naively you may expect the two results to coincide but this is not necessarily true. A simple

counter example would be the double sequence given by  $a_{n,t} = n/(t + n)$  since

$$\lim_{t \rightarrow \infty} \lim_{n \rightarrow \infty} a_{n,t} = 1 \quad \text{but} \quad \lim_{n \rightarrow \infty} \lim_{t \rightarrow \infty} a_{n,t} = 0.$$

However, we shall argue that in most cases the limits do coincide. We do not go into full details since this is not what is of interest to us but we shall give a strong argument as to why this must be the case. In this section we shall make the assumption the  $N$  is even, this is simply so that we can start with  $X_0^N = 0$  as we have done throughout and it simplifies the mathematics since the process in this case is symmetric about 0. The case where  $N$  is odd can be calculate in a similar way albeit slightly more complicated. Another assumption that we shall make is that for the case when the process converges to a fixed point, we shall assume that fixed point is rational. The reasoning behind this is that our process takes values on the rationals and so this assumption means that for the process the fixed point is accessible. When the limit point is not rational it will need to be argued that the values  $X^N$  take on the interval  $[-1, 1]$  becomes dense as  $t \rightarrow \infty$  which can easily be seen since  $S_t^N \rightarrow \infty$ .

Another motivation for considering the case when  $N$  is fixed it to see how it relates to our results when  $N \rightarrow \infty$ , to see if they coincide. We shall argue that in all cases, except those of case (c)(ii) and (d), the asymptotic distributions are identical, that is  $X_\infty^N = X_\infty$ .

The case of (c)(ii) and (d) can very easily be seen to be different since in the finite case  $X_\infty^N$  is non-degenerate. The process is still a bounded martingale and so will converge but this distribution is no longer a point mass at the starting point. We give an illustrative example by considering the functions in case (d) but with fixed  $N < \infty$ . In this case the system is identical to that of the Pólya urn model which has been extensively studied (see [Mahmoud, 2009] for example). If we impose the initial condition that there are  $m$  up particles and  $n$  down particles then the distribution of

$X_\infty$  is given by

$$\mathbb{P}(X_\infty = x) = \frac{\Gamma(m+n)}{2^{m+n-1}\Gamma(m)\Gamma(n)}(1+x)^{m-1}(1-x)^{n-1}.$$

**Remark** This gives an alternate proof for case (d) when  $N \rightarrow \infty$ . By having  $m = N(X_0 + 1)/2$  and  $n = N(X_0 - 1)/2$  and tending  $N \rightarrow \infty$  we obtain the limit distribution in the infinite case, which is a point mass at  $X_0$ .

The above reasoning also explains why our results for case (c)(ii) differ from that obtained by Roger and Morris, in their paper, since  $N$  is fixed the process is able to move more than ours and so has positive probability of reaching  $\pm 1$  causing point masses to accrue there unlike ours.

Arguing that in the other cases the limits are identical is slightly more complicated. Firstly, since  $N$  is finite the process  $X^N$  is a Markov chain, this will be of use since the process only takes discrete jumps. We shall explain the situation for case (c)(i) which can be modified for all the other cases except case (b)(i) which we discuss separately.

In case (c)(i) we have some knowledge of how  $X_t^N$  and  $S_t^N$  can behave. Firstly, it should be noted that  $X_t^N$  is a Markov chain with the points  $+1$  and  $-1$  being accessible absorbing points. This means that, letting  $A$  represent that state-space of the process  $X^N$  we have that whenever the process is in  $\text{Int}(A)$  it has positive probability of reaching  $+1$  or  $-1$  and once reaching one of these two states it remains there. Furthermore, the process is symmetric about 0 so if  $X_0^N = 0$  we can conclude that the absorbing points must be reached with equal probability. That is,

$$X_\infty^N = \begin{cases} +1 & w.p & \frac{1}{2} \\ -1 & w.p & \frac{1}{2}. \end{cases}$$

A similar argument can be used for all other cases remaining except for case (b)(i). Since again from any state the process has positive probability of reaching the ab-

sorbing point as it is an accessible point. Since it is an absorbing point it remains there once reached and so again we conclude that  $X_\infty^N = x^*$  where  $x^*$  denotes the fixed point to the system. The reason the point is accessible is this point is rational and so can be written in the form  $x^* = p/q$  where  $p, q \in \mathbb{N}, q \neq 0$ . If we take it that  $p, q$  do not share any factors we can write  $x^* = (2p)/(2q)$  which is useful since the process  $X^N$  is of this form infinitely often since  $S^N \rightarrow \infty$  and so is even infinitely often.

Case (b)(i) requires a different argument since the process is ergodic. We shall use this fact to help us. We know the following two facts: 1) The process  $X_t$  converges to  $X_\infty$  as  $t \rightarrow \infty$  regardless of the initial value of  $X_0$  and 2) That  $X^N$  converges to  $X$  as  $N \rightarrow \infty$ . As such, if we can show that  $X_t^N$  converges to  $X$  as  $t \rightarrow \infty$  then we are done since  $X$  converges to  $X_\infty$ . This holds since as  $t$  increases  $S_t^N \rightarrow \infty$  meaning that  $\Delta X_t^N \rightarrow 0$  thus that the process converges to a continuous one, further more by looking at the drift and diffusive terms we see that this converges to that of  $X$ .

## 4.4 Applications

In this section we shall focus on some specific examples that are of particular interest and highlight how the addition of growth may change the behaviour of our process. In these examples we take  $f_u(x) = \frac{1}{2}(1 - x)$  and  $f_d(x) = \frac{1}{2}(1 + x)$  so that the system is self-stabilising.

**Example 4.4.1** (Growth by Replication). Consider a non-growing system with the flips given by the functions  $f_u(x) = \frac{1}{2}(1 - x)$  and  $f_d(x) = \frac{1}{2}(1 + x)$ . From Example 1 in Section 2 we can see that the limit process would follow a path given by

$$dX_t = \sqrt{1 - X_t^2} dB_t.$$

Since there is no drift term this is a martingale and so is in natural scale. The end points are absorbing points and with our process starting at  $X_0 = 0$ , both points are

equidistant and so the limit distribution,  $X_\infty$ , of our process is given by equal point masses at the points  $\pm 1$  (see Figure 4.1).

We now add growth to see how this changes the behaviour of the system. Set  $g_u(x) = \frac{1}{2}(1+x)$  so that the addition of particles is dependent on the magnetisation of the system. When  $\alpha = 0$ , Theorem 4.2.1 gives the limit process  $X_t$  as identical to that of the non-growing case and so the limit distribution  $X_\infty$  is identical (see Figure 4.2).

If we increase the rate of addition of the particles further so that  $0 < \alpha \leq 1$  then the addition of the particles now overwhelms the system and the internal mechanics do not have time to play a part. This causes the process to remain in the original state so that  $X_t = X_0$  as seen in Figure 4.4. This recovers the results obtain in [Morris and Rogers, 2014].

**Example 4.4.2** (Growth from Unmagnetised Reservoir). In the last example the growth could be considered as the replication of the system, in this case we shall introduce particles from an unmagnetised reservoir so that the addition of particles has an equal chance of being up and down. This is done by setting  $g_u(x) = g_d(x) = \frac{1}{2}$ . We already know what would happen in the non-growing case from the previous example. Consider  $\alpha = 0$ . In this case we would obtain the limit given by case (b)(i), i.e.

$$dX_t = -X_t dt + \sqrt{2(1-X_t^2)} dB_t.$$

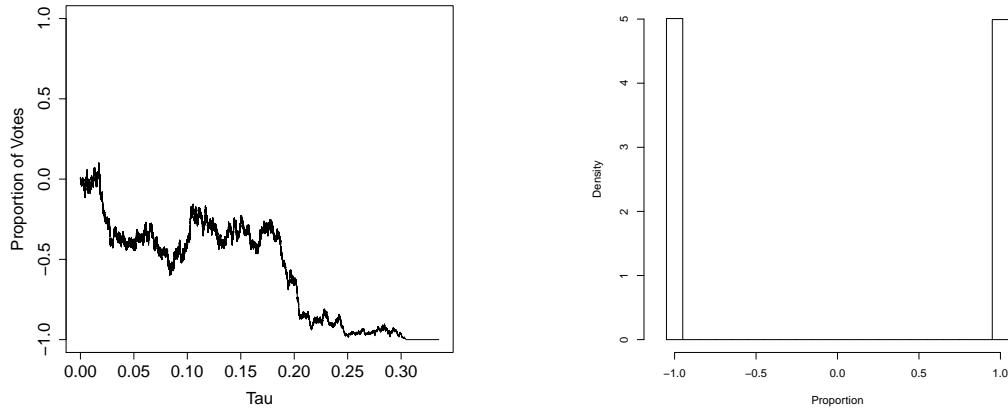
We can see straight away that the drift and diffusivity terms do not share any roots and so we can conclude that the process is ergodic. To see what the distribution is we look at the speed measure and scale function which in this case are

$$m(dx) = \frac{1}{2\sqrt{1-x^2}} dx \quad \text{and} \quad \rho(x) = \sin^{-1}(x).$$

Since  $\rho(x)$  is strictly increasing on  $[-1, 1]$  we deduce that  $X_\infty$  has the distribution

given by  $m(x)$  which is an arcsine distribution. Simulations are given in Figure 4.3.

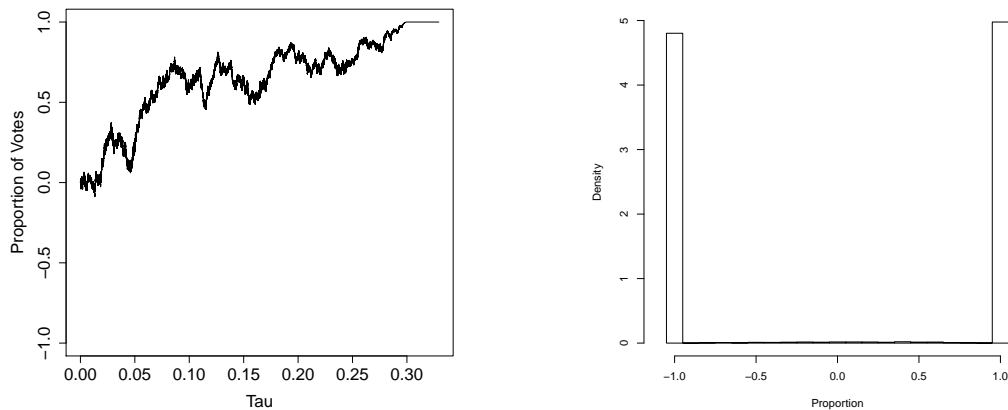
Increasing the rate of addition further, so that  $\alpha > 0$ , the process becomes deterministic given by the ordinary differential equation  $dX_t = -X_t dt$ . This has solution  $X_t = x_0 e^{-t}$  which tends to zero as  $t$  increases. This case is of interest as there is a change in the behaviour when there is no growth,  $\alpha = 0$  and  $\alpha \in (0, 1]$  (Figure 4.5).



(a) Simulated sample path .

(b) The distribution of 10000 sample paths.

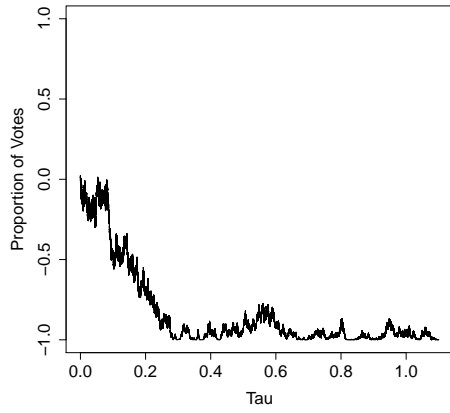
Figure 4.1: Sample path and distribution of the no growth process (see Definition 1) with  $N = 100$  and the process stopped at  $\tau = 1$ .



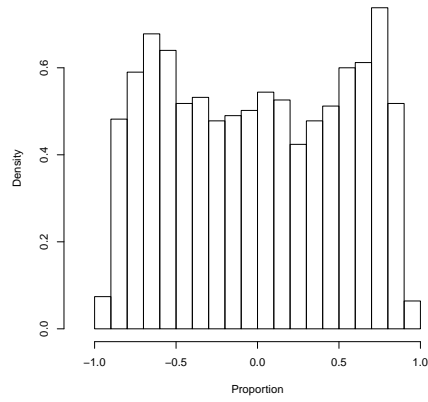
(a) Simulated sample path.

(b) The distribution of 10000 sample paths.

Figure 4.2: Sample path and distribution in Example 3 (Constant Replication Growth) with  $N = 100$  and the process stopped at  $\tau = 1$ .

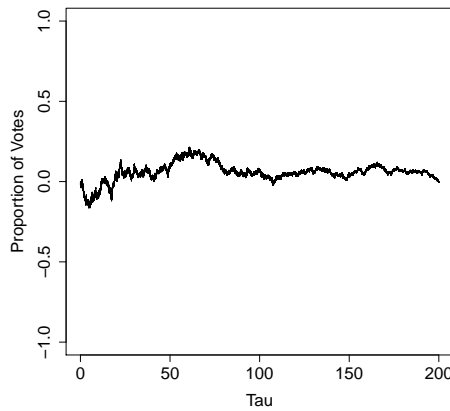


(a) Simulated sample path.

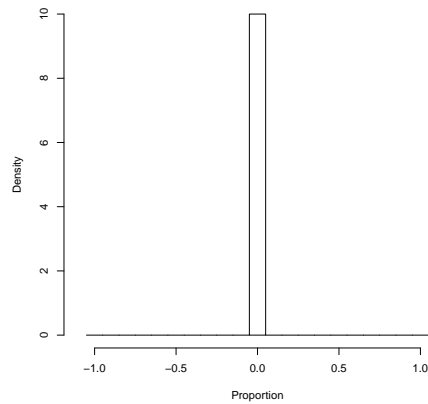


(b) The distribution of 10000 sample paths.

Figure 4.3: Sample path and distribution in Example 3 (Constant Unmagnetised Growth) with  $N = 100$  and the process stopped at  $\tau = 1$ .

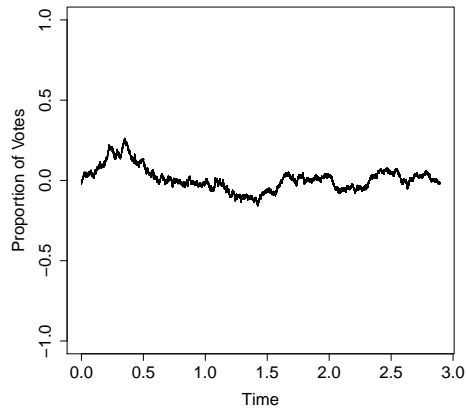


(a) Simulated sample path.

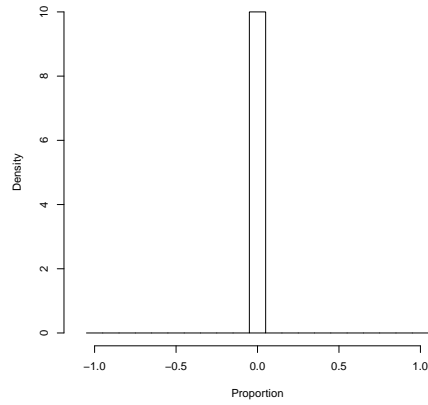


(b) The distribution of 10000 sample paths.

Figure 4.4: Sample path and distribution in Example 3 (Accelerated Replication Growth) with  $N = 50000$ ,  $\alpha = 0.75$  and the process stopped at  $\tau = 1$ .



(a) Simulated sample path.



(b) The distribution of 10000 sample paths.

Figure 4.5: Sample path and distribution in Example 4 (Accelerated Unmagnetised Growth) with  $N = 50000$ ,  $\alpha = 0.75$  and the process stopped at  $\tau = 1$ .



# Chapter 5

## Introducing Competition within HL(0)

### 5.1 Introduction

We consider a planar random growth model devised by Hastings and Levitov [Hastings and Levitov, 1998]. This is a model which uses conformal mapping theory to model a growing cluster formed by the aggregation of particles. The model has a parameter  $\alpha \in [0, 2]$ , however in this paper we only consider the case when  $\alpha = 0$ . Much research has been conducted regarding this model, see [Johansson Viklund et al., 2012] and [Silvestri, 2017] for example. We introduce competition into the model by considering two competing areas and having incoming particles join the team that they land in. By considering the harmonic measure of the areas, we show that by changing the size and rate of the addition of particles, based on the harmonic measure, allows us to construct cases where at any given time there is a positive probability that each area has positive measure.

### 5.1.1 The Hastings-Levitov Model

Before we can introduce competition into our model of interest, we must firstly define the model itself. The model of interest, known as the Hasting-Levitov model, is a growing sequence of clusters formed by a sequence of conformal maps which represent the addition of a particle to the cluster. Let  $c > 0$ , we consider a slit of length  $d = d(c)$  and attach it to the unit circle,  $\mathbb{T}$  at the point 1. The value  $c$  is known as the capacity of the particle and is another way to consider the ‘size’ of the particles being added. It is easier to describe the length of a slit and so we introduce  $d$  but it is mathematically more convenient to work with the capacity and so we include the relationship between them here. For further work concerning the capacity please see the introduction of [Johansson Viklund et al., 2012]. The relationship between  $c$  and  $d$  is given by

$$e^c = 1 + \frac{d^2}{4(1+d)}.$$

Observe that  $d \asymp c^{1/2}$  as  $c \rightarrow 0$ . The attachment of such a particle can be given uniquely by the conformal map

$$f^c : \Delta := \{z \in \mathbb{C} : |z| > 1\} \cup \{\infty\} \rightarrow D_1 := \Delta \setminus (1, 1+d]$$

with  $f^c(z) = e^c z + \mathcal{O}(1)$  at infinity. From this we can construct a model to represent random aggregation which is constructed from a sequence of conformal maps derived from the function  $f^c$ . Let  $(\theta_n)_{n \in \mathbb{N}}$  be a sequence of  $\text{Uniform}[-\pi, \pi)$  random variables and  $t_1, t_2, \dots$  be the arrival times of the particles which arrive at rate 1. From this we obtain a sequence of maps  $(f_n)_{n \in \mathbb{N}}$  from  $f^c$  by

$$f_n^c(z) = e^{i\theta_n} f^c(e^{-i\theta_n} z),$$

which is a rotation and rescaling on the original map. From this sequence of maps set

$$\Phi_t(z) = f_1^c \circ f_2^c \circ \dots \circ f_n^c(z), \quad t_n \leq t < t_{n+1}.$$

Since  $\Phi_t$  is a composition of conformal maps, it too is one, and maps the exterior disc to the complement of a compact set, which we denote by  $K_t$ . In other words,

$$\Phi_t : \Delta \rightarrow \mathbb{C} \setminus K_t.$$

The sets  $(K_t)_{t \in \mathbb{R}}$  are called clusters which satisfy  $K_s \subseteq K_t$  for  $s \leq t$ . The set  $K_t$  represents the growing cluster after the addition of  $n$  particles for  $t_n \leq t < t_{n+1}$ . Throughout the rest of this paper we shall omit the  $c$  to simplify notation when needed.

### 5.1.2 Harmonic Measure Flow

Our motivation behind the Hastings-Levitov model stems from the aim of modelling growing clusters formed by the aggregation of particles. Such a model is known as diffusion limited aggregation (DLA). The DLA model is constructed by having an initial cluster and adding particles. The addition of each particle is from a Brownian path which starts at infinity and is added to the cluster at the point it hits the unit disc. This model is very hard to analyse mathematically and this is why we study the Hastings-Levitov model, it is similar to DLA except that the conformal maps distort the size and shape of the added particles.

For the unit disc, the probability of the Brownian particle hitting the boundary is given by a uniform distribution. This is an example of the harmonic measure.

**Definition 5.1.1.** Let  $D \subset \mathbb{R}^2$  and let  $\partial D$  represent the boundary of  $D$ . Then, for

any  $A \subseteq \partial D$  and  $x \in D$  the harmonic measure of the set  $A$  is defined to be

$$\mu_D^x(A) = \mathbb{P}(B_\tau \in A | B_0 = x)$$

where  $\tau = \inf_{t>0} \{B_t \in \partial D\}$ .

The harmonic measure of a subset of the boundary is the probability a Brownian path, starting from within the set, exits through that subset. In our case, the set is  $(\mathbb{C} \cup \{\infty\}) \setminus K_t$  with the starting point being  $\infty$ .

Calculating the harmonic measure of such a set directly is complicated, since the set  $K_t$  can be quite intricate. What is of use though, is that the set  $K_t$  is simply connected and the harmonic measure of a simply connected set can be given by the harmonic measure of the unit disc and the mapping between the set  $K_t$  and the unit disc. This is possible since Brownian motion is invariant under conformal maps.

Since the harmonic measure is a one dimensional object taking values in  $\mathbb{R}^+$ , we can turn the maps into a one dimensional object too. If we define  $\gamma_n = (\log f_n^{-1}) / 2\pi$ , we can consider the movement of a point after the addition of  $n$  particles. The sequence  $(Z_t)_{t \in \mathbb{R}}$  formed by the composition of map,  $Z_t(x) = \gamma_n \circ \gamma_{n-1} \circ \dots \circ \gamma_1(x)$ , for  $t_n \leq t < t_{n+1}$ , is known as the harmonic measure flow. The individual functions  $\gamma_n$  represent the change in harmonic measure.

**Example 5.1.1.** If we denote  $A_0 = [0, \pi]$ , the the evolution of the harmonic measure as the process evolves would be given by

$$\mu_{K_t}^\infty(A_t) = Z_t(\pi) - Z_t(0).$$

In a similar way that  $f_n$  was constructed from  $f$  by rotating for the addition of a particle at angle  $\theta_n$  The function  $\gamma_n(x) = \gamma_n^c(x)$  can be expressed explicitly from the

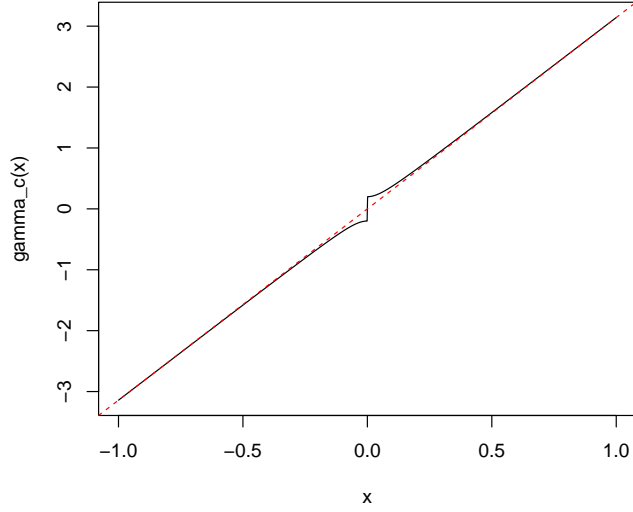


Figure 5.1: Plot of the function  $\gamma^c(x)$  with  $c = 0.01$  along with the identity function (dashed).

function  $\gamma^c$  where

$$\gamma^c(x) = 2\text{sgn}(x) \tan^{-1} \sqrt{e^c \tan^2(\pi x/2) + e^c - 1}, \quad x \in (-1, 1] \setminus \{0\}$$

with  $t_1, t_2, \dots$  being the arrival time of the particles, see Figure 5.1 for a plot.

The relation between  $\gamma^c$  and  $\gamma_n^c$  is given by

$$\gamma_n^c(x) = \theta_n + \gamma^c(x - \theta_n).$$

The process  $Z_t$ , known as the harmonic measure flow, can be constructed iteratively from the functions  $\gamma_n^c$ . If we define  $Z_0 = 0$  then we can define  $Z_t$  by

$$Z_t = \theta_n + \gamma_n^c(Z_s - \theta_n), \quad Z_0 = z_0 \in (1, 1] \quad (5.1)$$

where  $t_{n-1} \leq s < t_n \leq t < t_{n+1}$ . In the limit, after scaling the space, as  $c \rightarrow 0$  this gives Brownian motion.

This can further be extended by considering an infinite number of starting positions simultaneously. If we consider an infinite sequence,  $Z_t = (Z_t^1, Z_t^2, \dots)$ , with  $Z_0^1 = z_0^1, Z_0^2 = z_0^2, \dots$  with each  $z_0^n$  unique, then in the limit as  $c \rightarrow 0$ , the process  $Z_t$ , after an appropriate scaling, converges to the Brownian web. This, along with the previous result was shown in [Turner, 2006].

### 5.1.3 Introducing Competition

So far we have introduced the original growth model. In this section we add some form of competition. To do this, we split the unit disc into two regions and colour the upper half red and the lower half blue. We then run the model as normal but upon hitting the disc the particle attaches itself and becomes the colour of the region it attaches to. That is, if it attaches to the blue region it becomes blue, similarly for the red area.

What we consider is the harmonic measure of these two regions. The measure of a section being zero means that the probability of a Brownian path hitting it starting at infinity is zero and so the cluster must be surrounded by the other region, which would have measure 1. Our aim is to ensure that, at any time, the probability of each region having positive harmonic measure has a positive probability.

To understand this model it is sufficient to look at the harmonic measure and so we are only required to look at the boundary points between the two, marked as the black dots in Figure 5.2.

If we consider the two regions and do not alter the way that particles are added we would be in the same case as in the previous section and so would obtain a process that performs Brownian motion. This in turn would mean that eventually one of the areas harmonic measure will be zero. As such, for us to ensure some form of competition, we allow the size of the incoming particles and rate of growth to be dependant on the harmonic measure of the red area.

If we set  $X_0^+ = 0$  and  $X_0^- = \pi$  to be the initial boundary points between the red and blue surfaces, we can then look at the evolution of these points after the addition of each particle as  $c \rightarrow 0$ . The evolution of these points is given precisely by (5.1). As previously mentioned, in the original case, these points would behave like Brownian motions and so with probability one these two points will meet. This would mean the measure of one region is zero and the other is one and so one area will always surround the other and there is no competition.

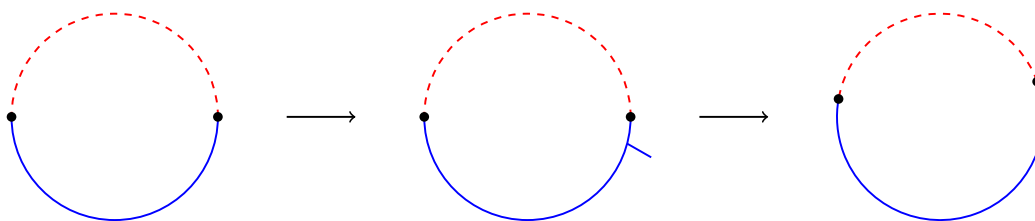


Figure 5.2: An illustrative example showing the initial system, a particle joining, then applying the conformal mapping which absorbs the particle. This causes the boundary between the dashed red curve and solid blue curve to change.

Our aim is to ensure the survival of both areas. As mentioned, we allow the size of the incoming particle and the rate of addition to be functions of the harmonic measure of the red area. To introduce competition, we shall change the rate such that it is a function of the harmonic measure of the red area and change the size of the particles in the red and blue region, again as a function of the harmonic measure of the red area. To do this, we introduce functions  $r(x, c)$ ,  $s^+(x, c)$  and  $s^-(x, c)$ . These are defined such that the rate of addition is given by  $r(x, c)$  and the size of particles in the red and blue region is given by  $s^+(x, c)$  and  $s^-(x, c)$  respectively. To generalise the process to take into account the rate change and to allow the particle size to vary as a function of the harmonic measure, we redefine  $\gamma(u)$  to be

$$\gamma_{cs(x,c)}(u) = 2\text{sgn}(u) \tan^{-1} \sqrt{e^{cs(x,c)} \tan^2(\pi u/2) + e^{cs(x,c)} - 1}.$$

### 5.1.4 Outline

In this section we show that it is possible to ensure coexistence of the two regions by choosing the rate and size of the particle to be appropriate functions of the harmonic measure of the red region. In Section 5.2 we introduce the kernel of a process and apply Kurtz's Theorem to find a scaling limit and list all possible cases, that is we find a limiting process  $X_t$  from the process  $X_t^c$  after an appropriate scaling. Section 5.3 then turns our attention to the asymptotic distribution,  $X_\infty$ , of the process  $X_t$  obtained in Section 5.2. We use the speed measure and scale function to analyse the boundary behaviour which in turn will give us our distributions. Finally we end with a few illustrative examples in Section 5.4.

## 5.2 Diffusion Estimates

To analyse the model, we use the framework given by Kurtz to obtain diffusion processes as  $c \rightarrow 0$ . We will make use of the material covered in Chapter 3.

**Example 5.2.1.** The kernel for the process associated with the harmonic measure flow of a single point is given by

$$K^c(x, dy) = \frac{1}{2} \int_{-1}^1 \delta_y(\gamma_c(x - \theta) - (x - \theta)) d\theta dy. \quad (5.2)$$

where  $\delta_y(x)$  is the dirac delta for  $y$ . It was shown in [Turner, 2006] that if  $X_t^c$  is the process associated with the above kernel then  $X_{\tau(t)}^c$  converges weakly in  $D[0, \infty)$  to  $B_t$ , Brownian motion where  $\tau(t) = c^{-3/2}t$ .

Since we are only interested in the harmonic measure of the red region, we can rotate the cluster object and this will not affect the harmonic measure. This is of use since we can attach a particle which will cause the boundary points to move and then rotate the cluster such that the left most boundary point is  $\pi$ . This makes it easier to



construct the kernel and yields

$$K^c(x, dy) = \frac{1}{2} \int_{-1}^1 \delta_y (\tilde{\gamma}_c(1 - \theta) - \tilde{\gamma}_c(1 - \theta - x)) d\theta dy, \quad (5.3)$$

with  $\tilde{\gamma}_c(x) = \gamma_c(x) - x$

**Example 5.2.2.** Consider the process  $X_t^c$  with kernel given by (5.3). Using Theorem 3.2.1 we get weak convergence of  $X_t^c$  to  $X_t$  as  $c \rightarrow 0$ . This diffusion process satisfies the stochastic differential equation

$$dX_t = \mathbb{I}_{(-1,1)}(X_t) dB_t.$$

*Remark 5.2.1.* In the above example we saw that if the process reaches the end points 1 or -1 the process then stops. To avoid the need for indicator functions while writing stochastic differential equations, whenever we write  $t$  we mean  $t \wedge \tau$  where  $\tau = \min\{t > 0 : X_t \in \{-1, 1\}\}$ .

Extending the previous kernel to take into account the change in rate and size of the incoming particles, as functions of the harmonic measure of the red region, we obtain,

$$\begin{aligned} K^c(x, dy) = & \frac{1}{2} r(x, c) \int_{1-x}^1 \delta_y (\tilde{\gamma}_{cs^+(x,c)}(1 - \theta) - \tilde{\gamma}_{cs^+(x,c)}(1 - \theta - x)) d\theta dy \\ & + \frac{1}{2} r(x, c) \int_{-1}^{1-x} \delta_y (\tilde{\gamma}_{cs^-(x,c)}(1 - \theta) - \tilde{\gamma}_{cs^-(x,c)}(1 - \theta - x)) d\theta dy. \end{aligned} \quad (5.4)$$

*Remark 5.2.2.* Setting  $r(x, c) = 1$  retrieves the original case considered.

**Theorem 5.2.1.** Consider the process  $X_t^c$  with kernel given by (5.4) with  $X_0^c = x_0$  for some  $x_0 \in [0, 2]$ , then  $X_t^c$  converges weakly to the process  $X_t$  in  $D[0, \infty)$  with  $X_0 = x_0$ . The limit is determined by the function  $r(x, c)$ ,  $s^+(x, c)$  and  $s^-(x, c)$ . We list the possible cases below:

(a) If there exists a function  $\mu(x)$  such that

$$\sup_x |r(x, c)s^+(x, c) - s^-(x, c)c \log(c) - \mu(x)| \rightarrow 0$$

and

$$\sup_x |c^{-3/2}r(x, c)| \rightarrow 0$$

as  $c \rightarrow 0$ , then the limit process satisfies

$$dX_t = \mu(X_t)dt \tag{5.5}$$

(b) If there exists functions  $\mu(x)$  and  $\sigma(x)$  such that

$$\sup_x |r(x, c)s^+(x, c) - s^-(x, c)c \log(c) - \mu(x)| \rightarrow 0$$

and

$$\sup_x |c^{-3/2}r(x, c) - \sigma^2(x)| \rightarrow 0$$

as  $c \rightarrow 0$ , then the limit process satisfies

$$dX_t = \mu(X_t)dt + \sigma(X_t)dB_t.$$

(c) If there exists a function  $\sigma(x)$  such that

$$\sup_x |s^+(x, c) - s^-(x, c)|c \log(c) \rightarrow 0$$

and

$$\sup_x \left| \frac{32}{3}c^{-3/2}r(x, c) - \sigma^2(x) \right| \rightarrow 0$$

as  $c \rightarrow 0$ , then the limit process satisfies

$$dX_t = \sigma(X_t)dB_t.$$

*Proof.* We use Theorem 3.2.1 to prove weak convergence. To do this, we are required to calculate the functions  $a^c$  and  $b^c$  to ensure they converge to functions  $a$  and  $b$  as  $c \rightarrow 0$ , after a suitable scaling to be determined. Observe that

$$\begin{aligned} b^c(x) &= \frac{1}{2}r(x, c) \int_0^2 \int_{1-x}^1 \delta_y (\tilde{\gamma}_{cs^+(x,c)}(1-\theta) - \tilde{\gamma}_{cs^+(x,c)}(1-\theta-x)) d\theta dy \\ &\quad + \frac{1}{2}r(x, c) \int_0^2 \int_{-1}^{1-x} \delta_y (\tilde{\gamma}_{cs^-(x,c)}(1-\theta) - \tilde{\gamma}_{cs^-(x,c)}(1-\theta-x)) d\theta dy \\ &= \frac{1}{2}r(x, c) \int_{1-x}^1 (\tilde{\gamma}_{cs^+(x,c)}(1-\theta) - \tilde{\gamma}_{cs^+(x,c)}(1-\theta-x)) d\theta \\ &\quad + \frac{1}{2}r(x, c) \int_{-1}^{1-x} (\tilde{\gamma}_{cs^-(x,c)}(1-\theta) - \tilde{\gamma}_{cs^-(x,c)}(1-\theta-x)) d\theta \\ &= r(x, c)\mathbb{I}_{[0,1]}(x) \int_0^x (\tilde{\gamma}_{cs^+(c)}(u) - \tilde{\gamma}_{cs^-(c)}(u)) du \\ &\quad + r(x, c)\mathbb{I}_{[1,2]}(x) \int_0^{2-x} (\tilde{\gamma}_{cs^+(c)}(u) - \tilde{\gamma}_{cs^-(c)}(u)) du. \end{aligned}$$

We used Fubini's Theorem in the first line and then a change of variables and the periodicity of  $\gamma$  in the final line. With a similar argument it can be shown that

$$\begin{aligned} a^c(x) &= r(x, c)\mathbb{I}_{[0,1]}(x) \left[ \int_0^x (\tilde{\gamma}_{cs^+(x,c)}^2(u) - \tilde{\gamma}_{cs^-(x,c)}^2(u)) du - 2 \int_0^1 \tilde{\gamma}_{cs^-(x,c)}^2(u) du \right] \\ &\quad - r(x, c)\mathbb{I}_{[0,1]}(x) \int_0^x (\tilde{\gamma}_{cs^-(x,c)}(u)\tilde{\gamma}_{cs^-(x,c)}(u+x) - \tilde{\gamma}_{cs^+(x,c)}(u)\tilde{\gamma}_{cs^+(x,c)}(u+x)) du \\ &\quad + 2r(x, c)\mathbb{I}_{[0,1]}(x) \int_0^1 \tilde{\gamma}_{cs^-(x,c)}(u)\tilde{\gamma}_{cs^-(x,c)}(u+x) du \\ &\quad + r(x, c)\mathbb{I}_{[1,2]}(x) \left[ \int_0^{2-x} (\tilde{\gamma}_{cs^-(x,c)}^2(u) - \tilde{\gamma}_{cs^+(x,c)}^2(u)) du + 2 \int_0^1 \tilde{\gamma}_{cs^+(x,c)}^2(u) du \right] \\ &\quad - r(x, c)\mathbb{I}_{[1,2]}(x) \int_0^x (\tilde{\gamma}_{cs^+(x,c)}(u)\tilde{\gamma}_{cs^+(x,c)}(u+x) - \tilde{\gamma}_{cs^-(x,c)}(u)\tilde{\gamma}_{cs^-(x,c)}(u+x)) du \\ &\quad - 2r(x, c)\mathbb{I}_{[1,2]}(x) \int_0^1 \tilde{\gamma}_{cs^+(x,c)}(u)\tilde{\gamma}_{cs^+(x,c)}(u+x) du. \end{aligned}$$

As such, we need to obtain approximations for  $\tilde{\gamma}_{cs(x,c)}$  to calculate the above integrals as  $c \rightarrow 0$ . To obtain these approximations we shall approximate  $\gamma$  for when  $x \leq c$  and  $x > c$ . By suitable Taylor expansions, it can be shown that

$$\sup_{u \leq c} \left| \gamma_{cs(x,c)}(u) - \sqrt{u^2 + 4cs(x,c)} \right| \rightarrow 0$$

and

$$\sup_{u > c} \left| \gamma_{cs(x,c)}(u) - \left[ u + \frac{cs(x,c)}{\tan(u/2)} \right] \right| \rightarrow 0.$$

as  $c \rightarrow 0$ . Hence we deduce that

$$\sup_c \left| c^{-3/2} a^c(x) - \frac{16}{3} r(x,c) \right| \rightarrow 0$$

and

$$\begin{aligned} b^c(x) &= cr(x,c)(s^+(x,c) - s^-(x,c)) (1 + 2 \log \sin(x/2) + \log 16) \\ &\quad + cr(x,c)s^-(x,c) \log(4s^-(x,c)) - cr(x,c)s^+(x,c) \log(4s^+(x,c)) \\ &\quad - r(x,c)(s^+(x,c) - s^-(x,c))c \log c + o(c). \end{aligned}$$

For us to use Kurtz's theorem, we must check that both function  $a^c$  and  $b^c$  converge. To ensure that the limit is not trivial, we must ensure that the limit for at least one of them is non-zero. These are precisely the conditions we state in the theorem. If we are outside these cases then either the limiting process is trivial or we do not obtain convergence in at least one of the processes  $a^c$  or  $b^c$ .

Finally, we observe that the other condition also holds since  $X_0^c = X_0$  for all  $c$  and the sizes of the jumps decrease as  $c \rightarrow 0$ .  $\square$

## 5.3 Limit Distributions

In this section we aim to analyse the long term behaviour of the processes obtained in the previous section. To do this, we make use of the speed function,  $\rho(x)$ , and scale measure,  $m(dx)$ .

**Theorem 5.3.1.** *Let  $X_t$  be the limit process obtained from  $X_t^c$  in Theorem 5.2.1.*

*Then the limit distribution  $X_\infty$  of  $X_t$  depends solely on  $r(x, c)$ ,  $s^+(x, c)$  and  $s^-(x, c)$ .*

*We list the possible cases below in terms of  $\rho(x)$  where*

$$\rho(x) = \int_1^x \exp\left(-2 \int_c^y \frac{\mu(u)}{\sigma^2(u)} du\right) dy, \quad c \in (0, 2).$$

- **Deterministic** *If  $s^+(x, c) \neq s^-(x, c)$  then the limit is a fixed point to (5.5) which we denote by  $x^*$  if  $0 < x^* < 2$  else it is 0 if  $x^* \leq 0$  or 2 if  $x^* \geq 2$ .*

- **Point mass**

- *If  $\rho(\pm 1) < \infty$  then the limit is given by*

$$X_\infty = \begin{cases} 2 & \text{w.p. } x_0/2 \\ 0 & \text{w.p. } (2 - x_0)/2. \end{cases}$$

- *If  $\rho(-1) = \infty$  and  $\rho(1) < \infty$  then  $X_\infty = 1$ .*

- *If  $\rho(1) = \infty$  and  $\rho(-1) < \infty$  then  $X_\infty = -1$ .*

- **Null Recurrent** *If  $\rho(\pm 1) = \infty$  and  $m(-1, 1) = \infty$ .*

- **Ergodic** *If  $\rho(\pm 1) = \infty$  and  $m(-1, 1) < \infty$  then the process is ergodic and the distribution is given by*

$$m(dx) = \frac{3}{64r(x, c)} \exp\left(\frac{3}{16} \int_1^x h^+(y) - h^-(y) dy\right) dx.$$

*Proof.* We do not discuss the deterministic case further as the results are well known. In the case where the function is a martingale, the process is already in natural scale. Since the end points can not be repelling they are either absorbing or inaccessible, in either case the process converges to the boundary points proportional to the distance away thus obtaining our first point mass result.

The case where we obtained a diffusion process with both drift and diffusive parts is more complicated since the behaviour will depend on the choice of rate and size functions. In all cases, the process is not in natural scale. If we apply  $\rho$  to the process  $X_t$  and then classify the boundary behaviour of  $Y_t = \rho(X_t)$  we can then identify the behaviour of  $Y_t$  from the above theorem. Since  $\rho$  is invertible this in turn allows us to understand  $X_t$ . The cases that then need to be considered are exactly those listed in the theorem, mainly whether or not the function  $\rho(x)$  maps one or both of the boundary points to  $\infty$  or to a finite point.  $\square$

As such, there exist cases where there is coexistence between the two regions. To construct a case where there is coexistence we need to change the rate of the system as we approach the boundary and increase the size of the particles that land in the dying region and decrease the size of those in the larger region.

What is of interest is that the way we change the size of the particles is very subtle and is not as large as seen in the previous chapter. In the previous chapter where we considered the voter model and introduced competition we saw that a linear function of the system generated different behaviour, in this case, if we were to apply a linear function, we would obtain a deterministic process as this would be too strong an influence.

## 5.4 Examples

We now consider a few specific cases to see how the theory can be applied to turn the naturally absorbing points into inaccessible points and produce an ergodic process.

### Inaccessible case

Set  $r(x, c) = c^{-3/2}x^2(2 - x)^2$  and set  $s^+(x, c) \equiv s^-(x, c) \equiv 1$  we obtain in the limit a process with the stochastic differential equation

$$dX_t = X_t(2 - X_t)dB_t.$$

Such a process can easily be verified as a martingale and so already is in natural scale. Furthermore, by calculating the speed measure, it can be shown that the end points are inaccessible. As such, we have managed to turn absorbing endpoints into inaccessible points by changing the rate of the process. If we look at the long term behaviour though, it can be shown that in the limit, we have  $X_\infty$  distributed as

$$X_\infty = \begin{cases} 0 & \text{w.p. } x_0/2 \\ 2 & \text{w.p. } (2 - x_0)/2 \end{cases}$$

which is identical in the limit to the original case.

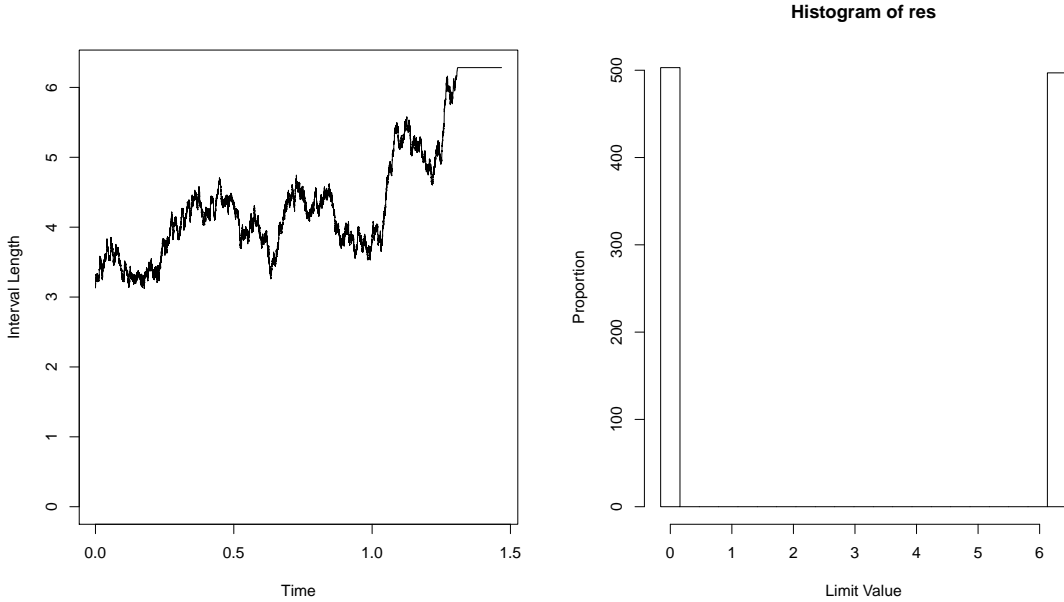


Figure 5.3: Sample path and distribution in Case 1 (Inaccessible) with  $c = 0.001$  and the process stopped once time jumps exceeded 2. Observe how we still converge to one of the end points.

### An Ergodic Process

Set  $r(x, c) = c^{-3/2} \frac{3}{64} x(2 - x)$  and

$$s^+(x, c) = 1 - \frac{3(1 - x)}{4r(x, c)c \log c} \quad \text{and} \quad s^-(x, c) = 1 - \frac{3x}{4r(x, c)c \log c}.$$

This puts us in case (c) of Theorem 5.2.1 and so we obtain in the limit the process  $X_t$  which satisfies the stochastic differential equation

$$dX_t = \frac{3}{2}(1 - X_t)dt + \sqrt{X_t(2 - X_t)} dB_t.$$

This gives a stochastic process which is not in natural scale. After calculating the scale function  $\rho$ , we can see that we map the process  $X_t$  to  $Y_t = \rho(X_t)$  which takes values on the interval  $(-\infty, \infty)$ , and so we are in case 1 of Theorem 5.2.1. Furthermore, this process is ergodic and so mapping back to  $X_t$  must be ergodic too. The process  $X_t$



can be shown to be ergodic with distribution given by the Lebesgue measure on the interval  $[-1, 1]$ .

# Bibliography

- [Billingsley, 1968] Billingsley, P. (1968). *Convergence of Probability Measures*. John Wiley & Sons, Inc.
- [Blazy et al., 2013] Blazy, S., Paulin-Mohring, C., and Pichardie, D., editors (2013). *Interactive Theorem Proving*. Springer.
- [Bollobás and Riordan, 2006] Bollobás, B. and Riordan, O. M. (2006). A short proof of the harris-kesten theorem. *Bull. London Math. Soc.*
- [Cai and Lin, 2004] Cai, G. Q. and Lin, Y. K. (2004). Stochastic analysis of the lotka-volterra model for ecosystems. *Phys. Rev. E*, 70:041910.
- [Chayes and Schonmann, 2000] Chayes, L. and Schonmann, R. H. (2000). Mixed percolation as a bridge between site and bond percolation. *Ann. Applied Probability*.
- [Clifford and Sudbury, 1973] Clifford, P. and Sudbury, A. (1973). A model for spatial conflict. *Biometrika*, 60:581–588.
- [Darling and Norris, 2008] Darling, R. W. R. and Norris, J. R. (2008). Differential equation approximations for Markov chains. *Probab. Surv.*, 5:37–79.
- [Deijfen and Häggström, 2007] Deijfen, M. and Häggström, O. (2007). The two type richardson model with unbounded initial configurations. *Ann. Applied Probability*.
- [Dimentberg, 2002] Dimentberg, M. F. (2002). Lotka-volterra system in a random environment. *Phys. Rev. E*, 65:036204.

- [Eden, 1961] Eden, M. (1961). A two-dimensional growth process. In *Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability, Volume 4: Contributions to Biology and Problems of Medicine*, pages 223–239, Berkeley, Calif. University of California Press.
- [Etheridge, 2011] Etheridge, A. (2011). *Some mathematical models from population genetics*, volume 2012 of *Lecture Notes in Mathematics*. Springer, Heidelberg. Lectures from the 39th Probability Summer School held in Saint-Flour, 2009, École d’Été de Probabilités de Saint-Flour. [Saint-Flour Probability Summer School].
- [Ethier and Kurtz, 1986] Ethier, S. N. and Kurtz, T. G. (1986). *Markov processes: Characterization and convergence*. Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics. John Wiley & Sons, Inc., New York.
- [Feller, 1952] Feller, W. (1952). The parabolic differential equations and the associated semi-groups of transformations. *Ann. of Math. (2)*, 55:468–519.
- [Freche et al., 1985] Freche, P., Stauffer, D., and Stanley, H. E. (1985). Surface structure and anisotropy of eden clusters. *Journal of Physics A: Mathematical and General*, 18(18):L1163.
- [Freedman, 1975] Freedman, D. A. (1975). On tail probabilities for martingales. *Ann. Probability*, 3:100–118.
- [Gikhman and Skorokhod, 1996] Gikhman, I. I. and Skorokhod, A. V. (1996). *Introduction to the theory of random processes*. Dover Publications, Inc., Mineola, NY. Translated from the 1965 Russian original, Reprint of the 1969 English translation, With a preface by Warren M. Hirsch.

- [Goel et al., 1971] Goel, N. S., Maitra, S. C., and Montroll, E. W. (1971). On the Volterra and other nonlinear models of interacting populations. *Rev. Modern Phys.*, 43:231–276.
- [Grimmett, 1989] Grimmett, G. (1989). *Percolation*. Springer Science+Business Media, New York.
- [Hammersley and Welsh, 1965] Hammersley, J. M. and Welsh, D. J. A. (1965). *First-Passage Percolation, Subadditive Processes, Stochastic Networks, and Generalized Renewal Theory*. Springer, Berlin, Heidelberg.
- [Harris, 1960] Harris, T. E. (1960). A lower bound for the critical probability in a certain percolation process. *Proc. Cambridge Phil. Soc.*
- [Hastings and Levitov, 1998] Hastings, M. B. and Levitov, L. S. (1998). Laplacian growth as one-dimensional turbulence. *Physica D*.
- [Ising, 1925] Ising, E. (1925). Beitrag zur theorie des ferromagnetismus. *Zeitschrift für Physik*.
- [Johansson Viklund et al., 2012] Johansson Viklund, F., Sola, A., and Turner, A. (2012). Scaling limits of anisotropic Hastings-Levitov clusters. *Ann. Inst. Henri Poincaré Probab. Stat.*, 48(1):235–257.
- [Johnson and Kotz, 1977] Johnson, N. L. and Kotz, S. (1977). *Urn models and their application*. John Wiley & Sons, New York-London-Sydney. An approach to modern discrete probability theory, Wiley Series in Probability and Mathematical Statistics.
- [Kallenberg, 2002] Kallenberg, O. (2002). *Foundations of modern probability*. Probability and its Applications (New York). Springer-Verlag, New York, second edition.
- [Kesten, 1980] Kesten, H. (1980). The critical probability of bond percolation on the square lattice equals  $1/2$ . *Comm. Math. Phys.*

- [Kesten, 1987] Kesten, H. (1987). Percolation theory and first-passage percolation. *Ann. Probability*.
- [Kesten, 1993] Kesten, H. (1993). On the speed of convergence in first-passage percolation. *Ann. Applied Probability*.
- [Kingman, 1973] Kingman, J. F. C. (1973). Subadditive ergodic theory. *Ann. Probability*.
- [Lindgren, 1963] Lindgren, B. W. (1963). Recent Publications and Presentations: Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability. Volume IV: Biology and Problems of Health. *Amer. Math. Monthly*, 70(9):1027.
- [Mahmoud, 2009] Mahmoud, H. M. (2009). *Pólya Urn Models*. CRC Press, Boca Raton, FL.
- [Morris and Rogers, 2014] Morris, R. G. and Rogers, T. (2014). Growth-induced breaking and unbreaking of ergodicity in fully-connected spin systems. *J. Phys. A*, 47(34):342003, 9.
- [Norris and Turner, 2012] Norris, J. and Turner, A. (2012). Hastings-Levitov aggregation in the small-particle limit. *Comm. Math. Phys.*, 316(3):809–841.
- [Richardson, 1973] Richardson, D. (1973). Random growth in a tessellation. *Proc. Cambridge Phil. Soc.*
- [Rohde and Zinsmeister, 2005] Rohde, S. and Zinsmeister, M. (2005). Some remarks on laplacian growth. *Topology and its Applications*, 152(1):26 – 43. Proceedings of the Dynamical Systems Conference.
- [Silvestri, 2017] Silvestri, V. (2017). Fluctuation results for Hastings-Levitov planar growth. *Probab. Theory Related Fields*, 167(1-2):417–460.

[Turner, 2006] Turner, A. (2006). *Scaling Limits of Stochastic Processes*. PhD thesis, University of Cambridge.

[Witten and Sander, 1983] Witten, T. A. and Sander, L. M. (1983). Diffusion-limited aggregation. *Phys. Rev. B* (3), 27(9):5686–5697.