Quantum Stochastic Flows on Universal Partial Isometry Matrix C*-Algebras

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Abstract

We give a simple algebraic construction of quantum stochastic flows on universal C*-algebras generated by partial isometry matrix relations. This is a large class of C*-algebras that subsumes the family of graph C*-algebras and, more generally, Cuntz-Krieger algebras. The construction expands on the main results of the 2015 paper by Belton and Wills which builds quantum stochastic flows from a stochastic flow generator defined on a dense *-subalgebra, subject to a growth condition. We characterise such quantum stochastic flows on the Cuntz algebras and give examples of when the growth condition is achieved.

As a specialisation of the above we then consider Lévy processes on universal C*-bialgebras generated by partial isometry matrices. Similarly to the C*-algebra scenario this class of C*-bialgebras is a large class which includes all universal compact quantum groups. The added structure of the C*-bialgebra removes the necessity of the growth condition required for quantum stochastic flows on C*-algebras. We construct a new family of universal C*-bialgebras which we call the deformed biunitary C*-bialgebras. The class of deformed biunitary C*-bialgebras includes the universal unitary compact quantum groups of Van Daele and Wang. We consider a sub-family of the deformed biunitary C*-bialgebras, the isometry C*-bialgebras and scrutinise the structure of Lévy processes on theses C*-bialgebras. Included in the isometry *-bialgebras is the Toeplitz algebra; we also examine this very closely. We investigate how Lévy processes on the Toeplitz algebra act on its commutative sub-C*-algebras.

To motivate the noncommutative setting we also consider classical Markov chains in terms of kernels. In so doing, we prove some characterisation results for bounded linear operators on some Banach spaces related to measurable and topological spaces in terms of kernels.

Acknowledgements

I owe gratitude to so many people for getting to this rather exceptional point in my life. For all of those who are not explicitly mentioned, this sentence is for you. Thank you.

To my wonderful partner Kelly, the love and support received was essential to keep me afloat in the latter and thoroughly exhausting days of this research. Every day I wake up and remember just how lucky I am to have you. I am eternally grateful and look forward to having my weekends and evenings back to spend them with you.

To my mother Patricia, without whom I may never have made it to university at all and followed my first instincts and became a travelling grifter. I think a doctorate was a better idea than that by a bit.

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To the examiners, whoever they may be, for (tentatively) taking the time to read my thesis.

To my siblings. To my aunts, uncles and cousins. To my friends from Cork. To my friends from Wexford. To all the band mates. To Tom Waits, Frank Zappa and Mike Patton. To music in general. To the hills of Bowland Forest to give my brain a reboot. To having a good cup of coffee to wake me up. To having a beer in the sun (where available). To eating a bowl of chili topped with cheese to take a break from marking. To following that bowl of chili with a bowl of Lancashire crisps, salt and vinegar to be exact (and another beer). To mathematics for just being plain interesting.

I feel like I did quite a bit of work myself so thank you reader for reading this patting of my own back. Good job me.

Declaration

All the work in this thesis is my own and has not been submitted for any other degree, either at Lancaster University or elsewhere.

Contents

Introduction

A Lévy process is a family of real random variables $(X_t)_{t\in\mathbb{R}_+}$ that satisfy the following conditions

(i) $X_0 = 0$ almost surely.

(ii) given $0 \leq t_1 < t_2 < \cdots < t_n$ then $X_{t_2} - X_{t_1}, X_{t_3} - X_{t_2}, \ldots, X_{t_n} - X_{t_{n-1}}$ are independent.

(iii) $X_t - X_s = X_{t-s}$ in distribution for all $0 \leq s \leq t$.

(iv) For all $\epsilon > 0$, $\mathbb{P}[|X_t| > \epsilon] \to 0$ as $t \to 0$.

where δ_0 is the at zero. These conditions tell us (i) the process has a predefined starting point, (ii) the increments are independent of one another, (iii) the increments are stationary and (iv) we have an element of continuity with respect to the "time" parameter. We can informally think of this as the movement of a particle on the real line that moves randomly in such a way that is invariant with respect to time and location. The time invariance is encoded in condition (iii) and the location invariance is a result of the use of additive inverses in (ii) and (iii). The prototypical examples of such processes are Brownian motion and the Poisson process in $\mathbb{R}$. In fact the Lévy-Khintchine formula tells us that all Lévy processes (often phrased as infinitely divisible distributions) satisfy the following

$$\mathbb{E}[e^{iX_t}] = \exp\left(ict - \frac{\sigma^2 t^2}{2} + \int \left(e^{itx} - 1 - \frac{itx}{1+x^2}\right)\mu(dx)\right)$$

where $c \in \mathbb{R}, \sigma^2 \geq 0$ and μ is a measure such that $\mu(\{0\}) = 0$ and $\int \frac{x^2}{1+x^2}\mu(dx) < \infty$ [Dur10, Theorem 3.8.2]. Informally, this tells us that all Lévy processes consist of combinations of continuous Brownian motion and jump type processes like Poisson processes for example. Applications of Lévy processes are ubiquitous finding their place in finance [App04] and many other areas of science and mathematics as documented in [App05, Page 3]. For more detailed information on classical Lévy processes see for example [Kal97, Chapter 13] [App05, Ber96].

Given two Borel probability measures μ and ν on $\mathbb{R}$ we can define a new measure called the convolution of measures by

$$(\mu * \nu)(A) := \int_{\mathbb{R}} 1_A(x+y)\mu(dx)\nu(dy)$$

for any measurable set A. Convolution gives a natural method of characterising independence of random variables. If we have two random variables X and Y then the associated measures or "laws" μ_X and μ_Y satisfy $\mu_X * \mu_Y = \mu_{X+Y}$ if and only if X and Y are independent. This gives us another way to think about Lévy processes. If we let $\mu_t = \mathbb{P} \circ X_t^{-1}$ then we can easily show that

(a) $\mu_0 = \delta_0$.

(b) $\mu_s * \mu_t = \mu_{s+t}$ for all $s, t \geq 0$.

(c) $\int_{\mathbb{R}} f(x)\mu_t(dx) \to f(0)$ as $t \to 0$ for all $f \in C_0(\mathbb{R})$.

In fact starting with such a family of measures we can construct a Lévy process by using the Kolmogorov construction theorem [Sat13, Theorem 10.4]. We will see that this approach allows for straightforward generalisation. Regardless of setting, if we have such a "C_0-semigroup" of probability measures (or objects that behave like probability measures) we can consider these to be the probability distributions of some Lévy process.

The notion of Lévy processes on the real line has been generalised in many ways. One of particular interest is the application to Lie groups as was introduced by Hunt in [Hun56]. This allowed many analogies from Lévy processes on the real line such as a version of the Lévy-Khintchine formula, appropriately termed Hunt's theorem [App05, Theorem 10.3]. For a modern introduction to Lévy processes on compact Lie groups see [App14, Chapter 5].

Furthermore, the notion of Lévy process on Lie groups has been generalised in the philosophy of noncommutative mathematics. This has been achieved by considering *-bialgebras, a not necessarily commutative algebra that carries with it a dual notion of multiplication and identity. The collection of *-bialgebras includes all Hopf *-algebras and it is through this lens we can see how this is a generalisation of the compact Lie group setting. It is well known that all compact groups have a canonical associated Hopf *-algebra in terms of what are called its representable functions [Tim08, Example 1.2.5].

This algebraic construction of noncommutative Lévy processes was introduced in [ASvW88] and Schürmann continued to develop an extensive theory as is documented in [Sch93]. Other good sources include [Fra06] [Mey93, Chapter VII] [FS16, Chapter 1]. The theory is built around the idea of convolution semigroups of states, that is families of linear functionals $(\phi_t)_{t\in\mathbb{R}_+}$ that are unital and positive, thought of as integration with respect to probability measures which satisfy

(a) $\phi_0 = \epsilon$.

(b) $\phi_s * \phi_t = \phi_{s+t}$ for all $s, t \geq 0$.

(c) $\phi_t(a) \to \epsilon(a)$ as $t \to 0$ for all a in the *-bialgebra.

where ϵ is the counit of the *-bialgebra and mimics the evaluation functional at the identity of a semigroup. This is analogous to the relationship between functionals and measures provided by the Markov-Riesz-Kakutani theorem [Rud87, Theorem 6.19]. One of the main appeals of this construction is its simplicity. It is easily shown that convolution semigroups of states can be characterised in terms of a single linear functional called the generating functional or equivalently in terms of a cohomological triple referred to as the Schürmann triple. Lévy processes on *-bialgebras have seen a lot of recent interest. A popular area of study is the existence of an analogue to the aforementioned Lévy-Khintchine decomposition or Hunt's formula and related questions about cohomology [FGT15, DFKS15, FKS16, DFKS17]. A recent preprint by Skalski and Viselter has characterised important operator algebraic properties such as the Haageruup property [DFSW16] and Kazhdan's property (T) [DSV17] for locally compact quantum groups in terms of Lévy processes [SV17]. See [KV00, KV03] for the original and full exposition on locally compact quantum groups.

To fully utilise the power of noncommutative mathematics it would be ideal to somehow extend the notions of *-bialgebraic Lévy processes to the level of C*-bialgebras. Using the celebrated Gelfand-Naimark theorem [Mur90, Theorem 2.1.10] we could then take an arbitrary commutative C*-bialgebra and construct concrete Lévy processes on a compact semigroup with identity element. Various approaches for defining Lévy processes on C*-bialgebras already exist. In the papers by Lindsay and Skalski [LS05, LS08, LS11, LS12] various ideas are proposed that exhaust the situation of Lévy process with bounded generating functionals. Cipriani, Franz and Kula provided a full characterisation for all compact quantum groups with arbitrary generating functionals in terms of associated Markov semigroups [CFK14, Theorem 3.4] with which they proceed to consider GNS and KMS symmetries of Lévy processes. In [LS08, Section 8] it can be seen that a theory for universal compact quantum groups with arbitrary generating functionals is developed. Das and Lindsay have also given a full description of Lévy processes on compact quantum groups which includes their representation as quantum stochastic flows on the reduced compact quantum group, and the implementation of these flows by unitary quantum stochastic cocycles on the GNS space with respect to the Haar state [DL].

In this thesis we give a method of extending the notion of arbitrary Lévy processes on *-bialgebras with partial isometry matrix generation properties to associated universal C*-bialgebras. The collection of *-bialgebras with these generation properties includes the polynomial Hopf *-algebra of compact quantum groups. Therefore, the collection extends the aforementioned method of [LS08, Section 8]. We also construct new examples of *-bialgebras with these generation properties that are not associated to compact quantum groups and characterise the Lévy processes on them in certain cases. This type of characterisation can be seen in [Fra06, Section 2.1] for the Glockner-von Waldenfels algebra, $SU_q(2)$ was completed by Schürmann and Skeide in [SS98] and more recently Franz, Kula and Skalski characterised the Lévy processes on the quantum permutation groups [FKS16, Section 8].

Using this construction of C*-bialgebraic Lévy processes the author and Hartung have considered some C*-bialgebras of this form and by canonical choice constructed analogues to the heat trace. Therefore analogues of dimension and volume can be calculated for "quantum manifolds" related to C*-algebras such as the Toeplitz algebra and $SU_q(2)$ [HH18]. Similar calculations by different methods can be found in [ALNJP15, LNJP16, Con04, HK05, HK10].

The method with which we extend *-bialgebraic Lévy processes to C*-bialgebra Lévy processes is a generalisation of the main result of [BW15]. This allows further generality in the sense that we can consider a larger class of noncommutative stochastic processes than Lévy processes on C*-bialgebras. We can instead consider quantum stochastic flows on C*-algebras with the same partial isometry matrix generation properties.

Quantum stochastic flows are *-homomorphic solutions to the Evans-Hudson quantum stochastic differential equation

$$dj_t = (j_t \otimes \mathrm{id}_{\mathcal{B}(\widehat{k})}) \circ \phi \; d\Lambda_t \quad \text{and} \quad j_0 = \mathrm{id} \tag{1}$$

where Λ is the quantum stochastic integral that is in some sense a combination of integration with respect to the processes of time, annihilation, creation and gauge familiar from quantum field theory and ϕ is called the quantum stochastic flow generator. Quantum stochastic calculus was introduced in [HP84] and is a generalisation of the L^2-theory of Ito stochastic calculus. For a modern exposition of these methods see [Lin05] and [Mey93, Chapters IV-VI]. Existence of *-homomorphic solutions to (1) implies the quantum stochastic flow generator satisfies certain structure relations. Conversely, in certain situations we can construct unital *-homomorphic solutions to (1) using a quantum stochastic flow generator ϕ that satisfies the same structure relations. Constructing these unital *-homomorphic solutions to equation (1) is the topic of [BW15, LW03, EH90]. This thesis contributes to the theory by showing the construction is possible for all universal C*-algebras with the aforementioned partial isometry matrix generation properties.

Layout of Thesis

Chapter 1 will discuss classical Markov processes and integration kernels. We begin with discrete Markov chains which introduce the most familiar form of kernels, stochastic matrices. A stochastic matrix is a matrix such that each row is a probability distribution. From there we extend the ideas from discrete state space by introducing measure theory, and most importantly measure kernels. We define general kernels and prove some characterisation theorems in terms of certain bounded linear operators (Theorems 1.2.31 and 1.2.37). In the course of these results we expand the main theorem of [MR17] and show that the unital C*-algebra of bounded measurable functions on certain measurable spaces is a commutative W*-algebra if and only if its σ-algebra is the power set (Corollary 1.2.14). Finally, we consider Markov chains on Polish state spaces, highlighting the connection to measures, kernels, functionals and operators (Corollaries 1.3.9, 1.3.11, 1.3.12,1.3.19 and 1.3.16).

Chapter 2 begins by following the exposition of quantum stochastic flows from [BW15]. We extend the main result of [BW15] and show that the quantum stochastic flows constructed on a dense *-subalgebra can be extended to the full C*-algebra given a partial isometry matrix generation property for the algebra and a growth condition for the quantum stochastic flow generator (Proposition 2.2.10). To conclude we construct some quantum stochastic flows on some non-commutative C*-algebras that satisfy these conditions. Further we characterise all such quantum stochastic flow generators on all Cuntz algebras in terms of the values given on the generators (Proposition 2.3.3).

Chapter 3 begins with an exposition of *-bialgebras, this follows the format of [Fra06] but with more examples to motivate the definitions as we go along. We introduce a new family of *-bialgebras which we call the deformed biunitary *-bialgebras (Proposition 3.1.5). These can be seen as an extension of the celebrated universal unitary and orthogonal compact quantum groups of Van Daele and Wang [VDW96], these take four not necessarily invertible matrices as parameters opposed to Van Daele and Wang's single invertible matrix. We prove some isomorphism results of these *-bialgebras in terms of the matrices chosen (Propositions 3.1.8 and 3.1.9).

We then proceed to give Schürmann's definition of Lévy processes on *-bialgebras and their various equivalent incarnations. We show the relationship between Lévy processes on *-bialgebras and the type of quantum stochastic flows introduced in Chapter 2 (Proposition 3.1.28). We show that similar to the classical setting we can characterise special types of Lévy processes, including Gaussian type processes and Poisson type processes, motivating the definitions as we go by considering commutative and cocommutative examples. We give a brief overview of the popular research topic of Lévy-Khintchine decompositions for *-bialgebras.

We then consider a purely noncommutative example. We take a specific class of the deformed biunitary *-bialgebra which we call the isometry *-bialgebras and characterise their Lévy processes (Theorem 3.2.2) and characterise when they are of a particular type, i.e. drift, Gaussian and Poisson (Propositions 3.2.8, 3.2.9 and 3.2.10). We conclude this chapter by adapting the main result from Chapter 2 and extending *-bialgebraic Lévy processes to universal C*-bialgebraic Lévy processes (Theorem 3.3.23). We also prove a limit theorem for the C*-bialgebras involved (Theorem 3.3.24). Finally we give an extensive overview of the Toeplitz algebra. We characterise the Lévy processes on this C*-bialgebra and give an explicit formulation for all possible generating functionals (Proposition 3.4.1) all while investigating the associated classical Lévy processes on the commutative sub-C*-bialgebras (Examples 3.4.11, 3.4.19, 3.4.20 and 3.4.21).

Notation and Conventions

The notation := is understood to mean "defined to be". For the positive integers we use $\mathbb{N} := \{1, 2, \dots\}$ and for the the non-negative integers we use $\mathbb{Z}_+ := \{0, 1, 2, \dots\}$. We use the notation $\mathbb{R}_+ := [0, \infty)$. Whenever we have a summation going the "wrong way" we assume it to be zero, i.e. given $k, l \in \mathbb{Z}$ such that $k < l$ then $\sum_{i=l}^{k} = 0$. Given normed vector spaces V and W we use $\mathcal{B}(V, W)$ to denote the bounded linear operators $V \to W$ if $V = W$ we abbreviate $\mathcal{B}(V, V) = \mathcal{B}(V)$. For H a Hilbert space we use the notations $\langle H| := \mathcal{B}(H, \mathbb{C})$ and $|H\rangle := \mathcal{B}(\mathbb{C}, H)$. For $h \in H$ then $\langle h| : H \to \mathbb{C}$ and $|h\rangle : \mathbb{C} \to H$ are the bounded linear operators such that $\langle h| (u) = \langle h, u\rangle$ and $|h\rangle (\lambda) = \lambda h$. Given a set E we denote the set of all subsets of E by 2^E. Given a vector space V we denote by V' the set of linear functionals on V. Given a normed vector space V we denote by V^* the set of bounded linear functionals on V. For a set A and $n \in \mathbb{N}$ we denote by $A^{\times n}$ the n-fold Cartesian product of A with itself, i.e. $A^{\times n} := A \times A \cdots \times A$.

Chapter 1

Classical Markov Processes and Kernels

We introduce Markov processes on a countable state space to provide some intuition. We generalise the methods from the finite case and in doing so we construct what we refer to as kernels. Under certain conditions these kernels act as analogues to the stochastic matrices that can be used to define Markov processes on finite state space. We then prove some characterisation theorems for kernels in terms of bounded linear operators. We conclude by applying kernels to characterisations of Markov and Feller processes.

1.1 Markov Chains with Discrete State Space

We begin by recalling some standard results and exposing some easy examples of Markov processes on a countable state space with discrete topology. This is similar in presentation to the first section of [FG06].

Definition 1.1.1. A **Markov chain** is a family of random variables $(X_n)_{n\in\mathbb{Z}_+}$ which take values in a countable set E, the **state space**, such that for all $n \in \mathbb{N}$ and $i_1, \ldots, i_n \in E$

$$\mathbb{P}(X_n = i_n | X_{n-1} = i_{n-1}, \ldots, X_0 = i_0) = \mathbb{P}(X_n = i_n | X_{n-1} = i_{n-1})$$

whenever $\mathbb{P}(X_{n-1} = i_{n-1}, \ldots, X_0 = i_0) > 0$. We only consider Markov chains that are time homogeneous, $\mathbb{P}(X_n = j | X_{n-1} = i) = \mathbb{P}(X_1 = j | X_0 = i)$ for all $n \in \mathbb{N}$ and $i, j \in E$.

Remark 1.1.2. We can define a family $(p^{(n)})_{n \in \mathbb{N}}$ by

$$p^{(n)} = (\mathbb{P}(X_n = j | X_{n-1} = i))_{i,j} = (p_{ij}^{(n)})_{i,j}$$

such that $p_{ij}^{(n)} \in [0, 1]$ and $\sum_k p_{ik}^{(n)} = 1$ for all $i, j \in E$. We call this the **transition probability family**. If the set E is finite, the family $(p_{ij})_{i,j}$ is called a **stochastic matrix**. The time homogeneous property in Definition 1.1.1 can thus be rephrased as the requirement that $p^{(n)} = p^{(1)}$ for all $n \in \mathbb{N}$.

We construct the following useful equivalent definition of Markov chains cf. [FG06, Propositon 1.2]. This definition is useful for comparison with some of the later constructions.

Proposition 1.1.3. *A stochastic process $(X_n)_{n \geq 0}$ on a countable state space E with initial distribution $\mathbb{P}(X_0 = i) = \lambda_i$ and transition probabilities*

$$\mathbb{P}(X_1 = j | X_0 = i) = p_{ij}$$

for all $i, j \in E$ is a Markov chain if and only if

$$\mathbb{P}(X_0 = i_0, \ldots, X_n = i_n) = \lambda_{i_0} p_{i_0 i_1} \ldots p_{i_{n-1} i_n}$$

for all $i_0, \ldots, i_n \in E$ and $n \in \mathbb{Z}_+$.

Proof. Assume $(X_n)_{n \geq 0}$ is a Markov chain. We prove this by induction on n. The case $n = 0$ is trivial; by definition, $\mathbb{P}(X_0 = i_0) = \lambda_{i_0}$. Let $r \in \mathbb{Z}_+, i_0, \ldots, i_r \in E$ and assume that $\mathbb{P}(X_0 = i_0, \ldots, X_{r-1} = i_{r-1}) = \lambda_{i_0} p_{i_0 i_1} \ldots p_{i_{r-2} i_{r-1}}$. By the definition of conditional probability and the Markov property,

$$
\begin{aligned}
&\mathbb{P}(X_0 = i_0, \dots, X_r = i_r) \\
&\quad = \mathbb{P}(X_0 = i_0, \dots, X_{r-1} = i_{r-1})\mathbb{P}(X_r = i_r | X_0 = i_0, \dots, X_{r-1} = i_{r-1}) \\
&\quad = \mathbb{P}(X_0 = i_0, \dots, X_{r-1} = i_{r-1})\mathbb{P}(X_r = i_r | X_{r-1} = i_{r-1}) \\
&\quad = (\lambda_{i_0} p_{i_0 i_1} \dots p_{i_{r-2} i_{r-1}})(p_{i_{r-1} i_r}).
\end{aligned}
$$

Assume that $\mathbb{P}(X_0 = i_0, \dots, X_n = i_n) = \lambda_{i_0} p_{i_0 i_1} \dots p_{i_{n-1} i_n}$ for all $n \geq 0$ and $i_0, \dots, i_n \in E$. Again by the definition of conditional probability

$$
\mathbb{P}(X_n = i_n | X_{n-1} = i_{n-1}, \dots, X_0 = i_0) = \frac{\mathbb{P}(X_0 = i_0, \dots, X_n = i_n)}{\mathbb{P}(X_0 = i_0, \dots, X_{n-1} = i_{n-1})} = p_{i_{n-1} i_n}
$$

assuming that $\mathbb{P}(X_0 = i_0, \dots, X_{n-1} = i_{n-1}) > 0$. Therefore

$$
\mathbb{P}(X_n = i_n | X_{n-1} = i_{n-1}, \dots, X_0 = i_0) = \mathbb{P}(X_n = i_n | X_{n-1} = i_{n-1})
$$

for all $n \in \mathbb{N}$ and $i_1, \dots i_n \in E$. Therefore, $(X_n)_{n \geq 0}$ is a Markov chain. $\square$

From the previous construction we can easily compute probabilities of specific events by an analogue of matrix calculation.

Corollary 1.1.4. *Let $(X_n)_{n \geq 0}$ be a Markov chain on a countable state space E with initial distribution $\lambda = (\lambda_i)_{i \in E}$ and transition probabilities $p = (p_{i,j})_{i,j \in E}$. For each $n \geq 0, i \in E$*

$$
\mathbb{P}(X_n = i) = (\lambda \cdot p^n)_i := \sum_{j_1, \dots, j_n \in E} \lambda_{j_1} p_{j_1, j_2} \dots p_{j_n, i}.
$$

Proof. This is a simple application of Proposition 1.1.3 as for all $i \in E$

$$
\mathbb{P}(X_n = i) = \sum_{j_1, \dots, j_n \in E} \mathbb{P}(X_0 = j_1, \dots, X_{n-1} = j_n, X_n = i). \quad \square
$$

When E is finite, Corollary 1.1.4 can be stated as follows: the distribution of the

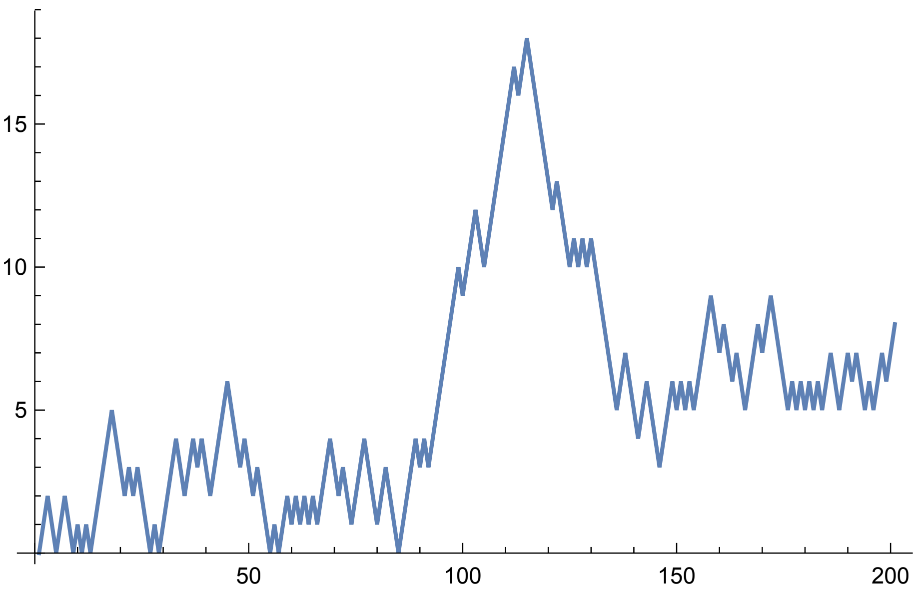


Figure 1.1: A sample path of Example 1.1.5 with 200 steps.

n-th step of the Markov chain is given by multiplication of the initial distribution (viewed as a row vector) with the n-th power of the transition probability matrix.

Example 1.1.5. Let $E = \mathbb{Z}_+$ and

$$\lambda_n = \delta_{0,n} := \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}, \quad p_{ij} = \begin{cases} 1 & i = 0 \text{ and } j = 1 \\ \frac{1}{2} & i > 0 \text{ and } |i - j| = 1 \\ 0 & \text{else} \end{cases}$$

defines a Markov chain on $\mathbb{Z}_+$. This is the simple random walk (to be introduced in Example 1.1.6) with a reflecting boundary at zero. For a sample path see Figure 1.1.

Random Walks

If our state space is a group G we get a family of Markov chains, $(g^{-1}X_n)_{n\geq 0}$, indexed by the elements of G. Furthermore, if

$$\mathbb{P}(X_n = j | X_{n-1} = i) = \mathbb{P}(X_n = g^{-1}j | X_{n-1} = g^{-1}i),$$

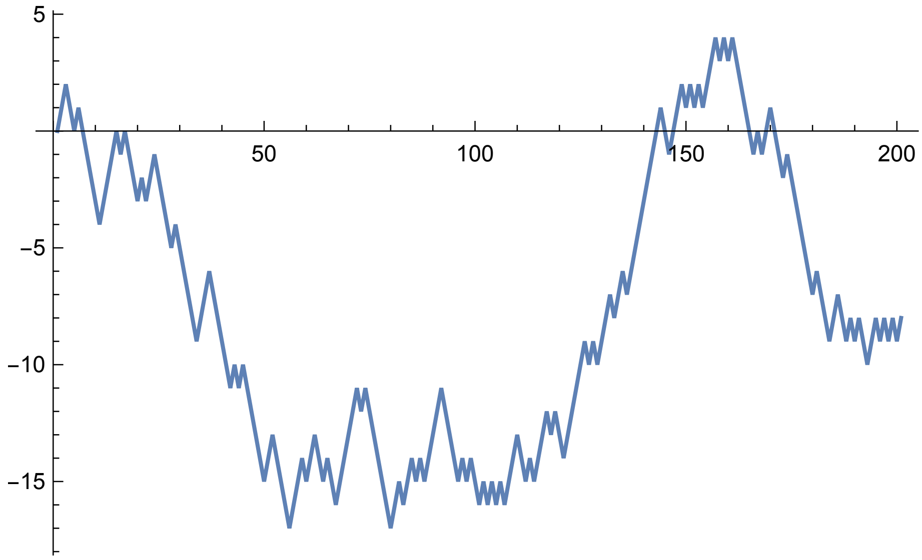


Figure 1.2: Sample path of Example 1.1.6 with 200 steps.

in other words $p_{ij} = p_{g^{-1}i,g^{-1}j}$, for all $g, i, j \in G$ we call $(X_n)_{n\geq 0}$ a **random walk on** E.

Example 1.1.6. The standard example of a random walk is the simple random walk on the group $(\mathbb{Z},+)$. This is given by initial distribution and transition probabilities

$$\lambda_n = \delta_{0,n} := \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}, \quad p_{ij} = \begin{cases} \frac{1}{2} & |i-j| = 1 \\ 0 & |i-j| \neq 1 \end{cases}.$$

This is a walk that starts at the point zero on the integers and at each step with equal probability will either move one unit in the negative or positive direction. For a sample path of this see Figure 1.2

1.2 Measure Theory and Kernels

What has preceded is a good toy example of how we want things to work. The next natural choice is to consider general state spaces. To achieve this we appeal to measure theory. We begin the section with the basics of measurable spaces and measures. This is in most cases quoted directly from [Rud87]. We follow this by introducing measure kernels and demonstrating their relationships to bounded linear operators on various function spaces. These correspondences are entirely new except for the result of Diestel and Uhl characterising the bounded linear operators on the bounded and measurable functions [DU77].

Measurable Spaces and Measures

Given a collection $\mathcal{F}$ of subsets of E, we denote $\sigma(\mathcal{F})$ as the **minimal σ-algebra that contains $\mathcal{F}$**. If (T, τ) is a topological space then $\sigma(\tau)$ is called the **Borel σ-algebra** of T. The Borel σ-algebra of $\mathbb{C}$ with the usual topology is denoted $\mathcal{B}$.

We require the following standard σ-algebra construction.

$$\sigma(f) := \{f^{-1}(A) : A \in \mathcal{F}\}.$$

We denote the algebra of bounded and measurable complex valued functions $b\mathcal{E}$. The algebra $b\mathcal{E}$ is obviously a unital *-subalgebra of $\ell^\infty(E)$ and so is a commutative unital C*-algebra. See [DLS12, Definition 3.9 to Example 3.16] for an exposition of bounded Borel measurable functions as unital C*-algebras.

For standard results about complex measures on a measurable space see [Rud87, Chapter 6].

Definition 1.2.1. Let $(E, \mathcal{E})$ be a measurable space and μ be a complex measure. The **variation measure** is a given by

$$|\mu|(A) := \sup_{\pi \in P(A)} \sum_{B \in \pi} |\mu(B)| < \infty \tag{1.2.1}$$

for a set $A \in \mathcal{E}$ where $P(A)$ is the collection of finite measurable partitions of A.

Example 1.2.2. Given any locally compact group G there is a unique (up to positive multiple) left translation invariant Borel measure. That is a measure μ such $\mu(g \cdot A) = \mu(A)$ for all $g \in G$ and all Borel measurable sets A. This is referred to as the Haar measure [HR79, Chapter 4]. For $\mathbb{R}$ with addition as the group action the Haar measure is Lebesgue measure restricted to Borel measurable sets.

It is easily seen that the set of finite complex measures is a normed vector space with pointwise addition and norm given by (1.2.1). We denote the space of finite complex measures $M\mathcal{E}$.

Proposition 1.2.3. *For any complex measure μ on $(E, \mathcal{E})$, the function $|\mu| : \mathcal{E} \to \mathbb{R}_+$ is a measure. Furthermore, $M\mathcal{E}$ with the norm $\|\mu\| = |\mu|(E)$ is a Banach space.*

Proof. For the first statement see [Rud87, Theorem 6.2]. The normed vector space $M\mathcal{E}$ is complete by the Vitali-Hahn-Saks theorem [DS88]. □

If we have a locally compact Hausdorff group G the set of Borel measures becomes an algebra with a multiplication as follows [HR79, (19.8) Definition].

Definition 1.2.4. Let G be a locally compact Hausdorff group with Borel σ-algebra $\mathcal{G}$. If μ and ν are two finite complex measures then the measure $\mu * \nu : \mathcal{G} \to \mathbb{C}$ defined by

$$\mu * \nu(A) := \int_G 1_A(xy)\mu(dx)\nu(dy)$$

for each $A \in \mathcal{G}$ is called the **convolution measure of μ and ν.**

The convolution product encodes independence of random variables in the following way: Let X and Y be complex valued random variables on probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let μ_X and μ_Y be the associated probability measures on the complex numbers given by

$$\mu_X(A) = \mathbb{P}(X^{-1}(A)) \quad \text{and} \quad \mu_Y(A) = \mathbb{P}(Y^{-1}(A))$$

then we have $\mu_X * \mu_Y = \mu_{X+Y}$ if and only if

$$\mathbb{P}(X^{-1}(A))\mathbb{P}(Y^{-1}(A)) = \mathbb{P}(X^{-1}(A) \cap Y^{-1}(A))$$

for all Borel subsets of $\mathbb{C}$ [Kal97, Corollary 2.12].

We extend the main theorem of [MR17] and in doing so give a partial characterisation of the measurable spaces $(E, \mathcal{E})$ for which $b\mathcal{E}$ is a W*-algebra [Sak98, Sections 1.1 and 1.18]. This is used in Proposition 1.2.30 to show that the set of what we will term measure kernels is not equal to $\mathcal{B}(M\mathcal{E})$ in general.

Definition 1.2.5. Let $(E, \mathcal{E})$ be a measurable space. We call $A \in \mathcal{E}$ an **atom** if A is non-empty and given any non-empty $B \in \mathcal{E}$ such that $B \subseteq A$ then $B = A$.

We call $(E, \mathcal{E})$ an **atomic measurable space** if for all $x \in E$ there exists an atom $A \in \mathcal{E}$ such that $x \in A$.

Remark 1.2.6. For a similar notion of atomic see the comments before [Coh13, Theorem 8.6.7]. However, we assume that atoms are measurable.

Example 1.2.7. Any measurable space that has the property that each singleton set is measurable is clearly atomic. For example $\mathbb{R}^n$ with Borel measurable sets and $\mathbb{R}^n$ with Lebesgue measurable sets.

Example 1.2.8. Let $(E, \mathcal{E}) = (\mathbb{N}, \mathcal{N})$ where

$$\mathcal{N} = \{\emptyset, \{2n; n \in \mathbb{N}\}, \{2n+1; n \in \mathbb{N}\}, \mathbb{N}\}$$

is an atomic measurable space with atoms given by the set of the odd numbers and the set of the even numbers.

Proposition 1.2.9. *Let $(E, \mathcal{E})$ be an atomic measurable space then the atoms partition the set E.*

Proof. Let $x \in E$ such that x is in two distinct atoms A and B. Then $A \cap B \neq \emptyset$ which implies that $A \cap B = A$ and $A \cap B = B$ which implies that $A = B$ for contradiction. Therefore as $(E, \mathcal{E})$ is atomic for all $x \in E$ there is a unique atom A such that $x \in A$ and the union of the set of atoms is equal to E. □

Lemma 1.2.10. *Let $(E, \mathcal{E})$ be a measurable space, $A \in \mathcal{E}$ an atom and $f \in b\mathcal{E}$. Then f is constant on A.*

Proof. Let $z \in \mathrm{Ran}(f|_A)$. Then $A \cap f^{-1}(\{z\})$ is a non-empty measurable subset of A and so equals A. Thus $f(a) = z$ for all $a \in A$. $\square$

Definition 1.2.11. Let A be a unital Banach *-algebra. A unital *-homomorphism $\phi : A \to \mathbb{C}$ is called a **character**. We denote the collection of all characters of a unital Banach *-algebra A by $\Phi(A)$.

Corollary 1.2.12. *Let $(E, \mathcal{E})$ be a measurable space and $A \in \mathcal{E}$ an atom. Then $\delta_A : b\mathcal{E} \to \mathbb{C}$, given by $\delta_A(f) = f(x)$ for some $x \in A$, defines a character on $b\mathcal{E}$.*

For $(E, \mathcal{E})$ an atomic measurable space, let $\Delta_{\mathcal{E}} := \{\delta_A; A \text{ atoms}\}\} \subseteq (b\mathcal{E})^*$. Note that if $\mathcal{E} = 2^E$ then $b\mathcal{E} = \ell^\infty(E)$.

For an algebra we need the following set

$$U_n(A) := \left\{ (f_1, \dots f_n) \in A^n : \text{ there exists } (g_1, \dots, g_n) \in A^n; \sum_{j=1}^{n} g_j f_j = 1 \right\}$$

where A^n is the n-fold Cartesian product of A with itself.

What follows is an adaptation of [MR17, Theorem 2]. Only part (9) of the original theorem does not translate directly as we do not have a specific measure to be “almost everywhere” with respect to. Recall that a topological space is called **totally disconnected** if its only connected components are one point sets [Mun00, Page 152], **extremely disconnected** if the closure of every open set is open [Eng89, Page 368] and **hyperstonean** if it is compact, extremely disconnected and has “sufficiently many measures” [Tak02, Chapter 3, Section 1, Definition 1.14]).

Theorem 1.2.13. *Let $(E, \mathcal{E})$ be an atomic measurable space. Then the following assertions hold*

(i) $U_n(b\mathcal{E}) = \left\{ (f_1, \dots f_n) \in b\mathcal{E}^n; \inf_{x \in E} \sum_{j=1}^{n} |f_j(x)| > 0 \right\}$.

(ii) *Let $\tau_{\mathcal{E}}$ be the Gelfand topology on $\Phi(b\mathcal{E})$ then $(\Delta_{\mathcal{E}}, \tau_{\mathcal{E}})$ is a discrete topological space.*

(iii) *Under the discrete topology the atoms of* $(E, \mathcal{E})$ *can be continuously embedded into* $(\Phi(b\mathcal{E}), \tau_{\mathcal{E}})$.

(iv) *The set* $\Delta_{\mathcal{E}}$ *is dense in* $(\Phi(b\mathcal{E}), \tau_{\mathcal{E}})$.

(v) *Any point in* $\Delta_{\mathcal{E}}$ *is an isolated point in* $\Phi(b\mathcal{E})$ *and* $\Phi(b\mathcal{E}) \setminus \Delta_{\mathcal{E}}$ *is closed.*

(vi) $\Phi(\ell^\infty(E))$ *is homeomorphic to the Stone-Čech compactification of* E *with the discrete topology and is extremely disconnected. Therefore* $\Phi(\ell^\infty(E))$ *is totally disconnected.*

(vii) *If* $\{x\} \in \mathcal{E}$ *for all* $x \in E$ *and* $\mathcal{E} \neq 2^E$ *then* $\Phi(b\mathcal{E})$ *is totally disconnected but is not extremely disconnected.*

Sketch of proof. The proof of all these statement can be directly adapted from [MR17, Theorem 2]. All are relatively straightforward, except for (vii). To show $b\mathcal{E}$ is totally disconnected we show that for every $A \in \mathcal{E}$ there is a closed-open set in $\Phi(b\mathcal{E})$, and that these sets form a basis for the topology. Since $\Phi(b\mathcal{E})$ is compact and that topology has a basis of closed-open sets it is totally disconnected [MR17, Theorem A].

To show $\Phi(b\mathcal{E})$ is not extremely disconnected we consider A a non-measurable subset of E. The non-measurable set and its complement $E \setminus A$ are both open in $\Phi(b\mathcal{E})$ as a result of (iii). We assume the closure of these sets is disjoint for a contradiction. We use Urysohn's Lemma to find a continuous function f that is equal to one on $\overline{A}$ and zero on $\overline{E \setminus A}$. As a result $f^{-1}(\{1\}) = A$ which implies that A is measurable. □

Corollary 1.2.14. *Let* $(E, \mathcal{E})$ *be a measurable space such that* $\{x\} \in \mathcal{E}$ *for all* $x \in E$. *The unital C*-algebra* $b\mathcal{E}$ *is a W*-algebra if and only if* $\mathcal{E} = 2^E$.

Proof. First assume that $\mathcal{E} \neq 2^E$. From [Tak02, Chapter 3, Section 1, Theorem 1.18] the spectrum of a commutative W*-algebra must be hyperstonean. Theorem 1.2.13 (vii) tells us that the spectrum of $b\mathcal{E}$ is not extremely disconnected which is a condition for a topological space to be hyperstonean.

If $\mathcal{E} = 2^E$ then $b\mathcal{E} = \ell^\infty(E)$ which is a W*-algebra. $\square$

Kernels

We define a generalisation of transition probability matrices. Furthermore we characterise certain algebras of bounded linear operators in terms of these generalisations. These types of kernel have been mentioned in [Dur10, Chapter 6] in application to Markov chains, which we also discuss in the next section. Also the same objects appear in [Mey93, Appendix 2]

Definition 1.2.15. Let $(E, \mathcal{E})$ be a measurable space. A map $p : E \times \mathcal{E} \to \mathbb{C}$ is called a **kernel** if $p(\,\cdot\,, A) : E \to \mathbb{C}$ is measurable for all $A \in \mathcal{E}$ and $p(x, \,\cdot\,) : \mathcal{E} \to \mathbb{C}$ is a complex measure for each $x \in E$.

- A kernel is said to be of **finite variation** or **finite** if

$$\|p\| := \sup_{x \in E} \sup_{\pi \in P(E)} \sum_{A \in \pi} |p(x, A)| < \infty \tag{1.2.2}$$

  where $P(E)$ is the collection of all finite measurable partitions of E.

- A kernel is said to be **real** if $p(x, A) \in \mathbb{R}$ for all $x \in E$ and $A \in \mathcal{E}$.

- A kernel is said to be **positive** if $p(x, A) \geq 0$ for all $x \in E$ and $A \in \mathcal{E}$.

A positive kernel with $p(x, E) = 1$ for all $x \in E$ is called a **transition kernel**.

It is easily seen that the collection of finite kernels can be given the structure of a normed vector space with pointwise addition and norm given by (1.2.2). We denote the normed vector space of finite kernels $\ker(E, \mathcal{E})$.

Example 1.2.16. Let $E = \mathbb{R}^2$ with Borel σ-algebra then $p(x, A) := \mu_x(A)$ where μ_x is the uniform measure on the compact set $\{y \in \mathbb{C}; |y - x| = |x|\}$ defines a transition kernel.

For a measurable space $(E, \mathcal{E})$ a set function $\mu : \mathcal{E} \to \mathbb{C}$ is called a **finitely additive complex measure** if for any $n \in \mathbb{N}$ and disjoint collection $(A_i)_{i=1}^n \subseteq \mathcal{E}$

we have $\mu(\cup A_i) = \sum \mu(A_i)$. A finitely additive complex measure is said to be finite if

$$\sup_{\pi \in P(E)} \sum_{A \in \pi} |\mu(A)| < \infty$$

where $P(E)$ is the set of finite partitions of E. We denote the set of finite finitely additive complex measures by $M^{\mathrm{ba}}\mathcal{E}$. Note the following duality $(b\mathcal{E})^* \cong M^{\mathrm{ba}}\mathcal{E}$ [DS88, Theorem IV.5.1].

Definition 1.2.17. If we alter the Definition 1.2.15 so that $A \mapsto p(x, A)$ is a finitely additive set function and $x \mapsto p(x, A)$ is bounded and measurable we call p a **finitely additive kernel**. We denote the set of finite finitely additive kernels by $\ker^{\mathrm{ba}}(E, \mathcal{E})$.

Theorem 1.2.18 ([DU77, Theorem I.1.13])**.** *Let $(E, \mathcal{E})$ be a measurable space. We have the following isometric isomorphism of Banach algebras*

$$\ker^{\mathrm{ba}}(E, \mathcal{E}) \cong \mathcal{B}(b\mathcal{E}).$$

given by $p \mapsto T_p$ such that $T_p(f)(x) := p_x(f)$ where $p_x \in (b\mathcal{E})^*$ is the functional associated to $p(x, \,\cdot\,) \in M^{\mathrm{ba}}\mathcal{E}$.

Example 1.2.19. If $(E, \mathcal{E}) = (\{1, \dots, d\}, 2^{\{1,\dots,d\}})$ then

$$\ker^{\mathrm{ba}}(E, \mathcal{E}) \cong \ker(E, \mathcal{E}) \cong \mathcal{B}(\ell^\infty_d).$$

Corollary 1.2.20. *For all $p \in \ker(E, \mathcal{E})$ the mapping*

$$x \mapsto T_p(f)(x) := \int_E p(x, dy) f(y)$$

is bounded and measurable for all $f \in b\mathcal{E}$. In other words $T_p(b\mathcal{E}) \subseteq b\mathcal{E}$.

Proof. This follows from Theorem 1.2.18 as all countably additive kernels are automatically finitely additive. $\square$

Proposition 1.2.21. *For $p \in \ker(E, \mathcal{E})$ we have*

$$\|p\| = \|T_p\| := \sup_{\|f\|=1} \sup_{x \in E} |T_p(f)(x)|$$

i.e. the map $p \mapsto T_p$ is an isometry with respect to the operator norm on $\mathcal{B}(b\mathcal{E})$.

Proof. By a standard argument we have that $|T_p(f)(x)| \leq \|f\| |p_x|(E)$ where $p_x = p(x, \,\cdot\,)$ and $|p_x|$ is the total variation of the measure p_x, as in Proposition 1.2.3. Taking the supremum on both sides we get $\|T_p\| \leq \|p\|$.

Let π be a finite measurable partition of E, and let $x \in E$. Let $g_{\pi,x} : E \to \mathbb{C}$ denote the function

$$g_{\pi,x} := \sum_{A \in \pi_x} \frac{\overline{p(x, A)}}{|p(x, A)|} 1_A \quad \text{where} \quad \pi_x := \{A \in \pi; p(x, A) \neq 0\}.$$

This function is clearly measurable and satisfies $\sup_{y \in E} |g_{\pi,x}(y)| \leq 1$ for all $x \in E$. From the construction $\|T_p\| \geq |T_p(g_{\pi,x})(x)| = \sum_{A \in \pi} |p(x, A)|$. Taking the supremum on both sides gives $\|T_p\| \geq \|p\|$ and the result. □

Proposition 1.2.22. *Let $(E, \mathcal{E})$ be a measurable space. The convolution product on $\ker(E, \mathcal{E})$ given by $(p*q)(x, A) = \int_E p(x, dy) q(y, A)$ for all $p, q \in \ker(E, \mathcal{E})$ gives $\ker(E, \mathcal{E})$ the structure of a normed algebra with unit $(x, A) \mapsto 1_A(x) = \delta_x(A)$.*

Proof. Let $p, q \in \ker(E, \mathcal{E})$. Let $y \in E$ and observe that for $A = \cup_{i=1}^{\infty} A_i$ a mutually disjoint family of measurable sets $(A_i)_{i=1}^{\infty}$ we have that

$$q(y, A) = \sum_{i=1}^{\infty} q(y, A_i) < \infty.$$

As the order of union does not matter we have that this sum converges unconditionally which is equivalent to absolute convergence on $\mathbb{C}$. Also

$$|q(y, A)| = \left| \sum_{i=1}^{\infty} q(y, A_i) \right| \leq \sum_{i=1}^{\infty} |q(y, A_i)| \leq \sum_{i=1}^{\infty} |q(y, A_i)| + |q(y, A^c)| \leq \|q\|. \tag{1.2.3}$$

As $p_x := p(x, \cdot)$ is a complex measure with finite variation for each $x \in E$ and $q^A := q(\cdot, A)$ is a bounded and measurable function for all $A \in \mathcal{E}$, we have that q^A is integrable with respect to p_x and using the the polar form of complex measures [Rud87, Theorem 6.12] we get the following inequalities:

$$\begin{aligned}\left|\int_E p(x,dy)q(y,A)\right| &= \left|\int_E q^A(y)h_x(y)|p_x|(dy)\right| \\ &\leq \int_E |q^A(y)||p_x|(dy) \\ &\leq \int_E \sum_{i=1}^{\infty} |q^{A_i}(y)||p_x|(dy) \\ &\leq \|q\||p_x|(E) \\ &\leq \|p\|\|q\| < \infty\end{aligned} \tag{1.2.4}$$

where h_x is a bounded measurable function such that $|h_x(y)| = 1$ for all $y \in E$ and $\int f(y)p_x(dy) = \int f(y)h_x(y)|p_x|(dy)$ for all $f \in b\mathcal{E}$. Hence, by the Fubini-Tonelli theorem [Rud87, Theorem 8.8] we have that

$$\int_E p(x,dy)q(y,A) = \int_E \sum_{i=1}^{\infty} p(x,dy)q(y,A_i) = \sum_{i=1}^{\infty} \int_E p(x,dy)q(y,A_i).$$

for all $x \in E$ and all $A = \cup_{i=1}^{\infty} A_i$ for disjoint $A_i \in \mathcal{E}$.

If the mapping $x \mapsto (p * q)(x, A)$ is measurable for all $A \in \mathcal{E}$ then $p * q \in \ker(E, \mathcal{E})$. Fix $A \in \mathcal{E}$. If $q(y, A) = 1_C(y)$ for some $C \in \mathcal{E}$ then $x \mapsto (p * q)(x, A) = p(x, C)$ which is measurable by the definition of transition kernels. Hence, the mapping is measurable if $q(\cdot, A)$ is a simple measurable function. Finally, by the monotone convergence theorem the mapping $x \mapsto (p * q)(x, A)$ is measurable because any measurable function $q(\cdot, A)$ can be written as an increasing limit of simple functions.

For any finite measurable partition π of E we use calculations from (1.2.4) and

Fubini-Tonelli theorem

$$\begin{aligned}
\sum_{A\in\pi}\left|\int_E p(x,dy)q(y,A)\right| &\leq \sum_{A\in\pi}\int_E |q(y,A)||p_x|(dy)| \\
&= \int_E \sum_{A\in\pi} |q(y,A)||p_x|(dy)| \\
&\leq \sup_{y\in E}\sum_{A\in\pi} |q(y,A)||p_x|(E)| \\
&\leq \|p\|\|q\|
\end{aligned}$$

and taking the supremum on the left over partitions and $x \in E$ we get that $\|p * q\| \leq \|p\|\|q\|$. Therefore, $p * q \in \ker(E, \mathcal{E})$.

We check associativity. If $p_1, p_2, p_3 \in \ker(E, \mathcal{E}), x \in E$ and $A \in \mathcal{E}$ then using Fubini-Tonelli theorem we have

$$\begin{aligned}
(p_1 * (p_2 * p_3))(x,A) &= \int_E p_1(x,dy_1)(p_2 * p_3)(y_1,A) \\
&= \int_E p_1(x,dy_1)\left(\int_E p_2(y_1,dy_2)p_3(y_2,A)\right) \\
&= \int_E \left(\int_E p_1(x,dy_1)p_2(y_1,dy_2)\right) p_3(y_2,A) \\
&= ((p_1 * p_2) * p_3)(x,A).
\end{aligned}$$

Finally, the map $\iota : E \times \mathcal{E} \to \mathbb{R}$ given by $(x, A) \mapsto 1_A(x) = \delta_x(A)$ is measurable in the first argument and a probability measure with respect to second argument and therefore $\iota \in \ker(E, \mathcal{E})$. A simple calculation shows that $p * \iota = \iota * p = p$ for all $p \in \ker(E, \mathcal{E})$. □

Proposition 1.2.23. *Let p and q be kernels on E. Then $p*q$ is real if both p and q are real, positive if both p and q are positive and a transition kernel if both p and q are also transition kernels.*

Proof. This is easily seen from the definition of $p * q$. □

Note that we can embed $M\mathcal{E}$ into $\ker(E, \mathcal{E})$ by considering every measure $\mu \in M\mathcal{E}$ as a kernel $(x, A) \mapsto \mu(A)$.

Corollary 1.2.24. *A kernel p is a transition kernel if and only if $\lambda * p : \mathcal{E} \to \mathbb{C}$ defined by*

$$A \mapsto \int_E \lambda(dx)p(x, A)$$

is a probability measure for all probability measures λ.

Proof. Any probability measure can be thought of as a specific type of transition kernel $\lambda(A) = \lambda(x, A)$ for all $x \in E$. This gives the forward implication. Conversely letting $\lambda = \delta_x$ the Dirac point measures we get the converse implication. □

Proposition 1.2.25. *The set of transition kernels is convex.*

Proof. A convex combination of measurable functions is measurable and a convex combination of probability measures is a probability measure. □

Recall that the algebra with the same elements and reversed multiplication

$$a \cdot^{\mathrm{op}} b = ba$$

for all $a, b \in A$ the opposite algebra and denote it A^{op}.

Corollary 1.2.26. *The map* $\ker(E, \mathcal{E})^{\mathrm{op}} \to \mathcal{B}(M\mathcal{E})$ *given by $p \mapsto T_p$ where*

$$T_p(\mu) = \int \mu(dy)p(y, \ \cdot \ ) = \mu * p.$$

is an isometric Banach algebra homomorphism.

Proof. Using Proposition 1.2.22 $T_p \in \mathcal{B}(M\mathcal{E})$ with $\|T_p\| \leq \|p\|$. Also

$$\begin{aligned} \|T_p\| &= \sup_{|\mu|(E)=1} \left| \int \mu(dy)p(y, \ \cdot \ ) \right| (E) \\ &= \sup_{|\mu|(E)=1} \sup_{\pi} \sum_{A \in \pi} \left| \int \mu(dy)p(y, A) \right| \\ &\geq \sup_{x \in E} \sup_{\pi} \sum_{A \in \pi} |p(x, A)| \, . \end{aligned}$$

□

We characterise the bounded linear operators on the Banach space ℓ^1. For this and other similar realisations for sequence spaces see [TL86, Section IV.6].

Corollary 1.2.27. *Let $E = \mathbb{N}$ and $\mathcal{E} = 2^{\mathbb{N}}$ then we have the following isometric Banach algebra isomorphism*

$$\ker(E, \mathcal{E})^{\mathrm{op}} \cong \mathcal{B}(\ell^1).$$

Proof. Corollary 1.2.26 gives us the injective map $p \mapsto T_p$. For $(E, \mathcal{E}) = (\mathbb{N}, 2^{\mathbb{N}})$ and $T \in \mathcal{B}(M2^{\mathbb{N}}) = \mathcal{B}(\ell^1)$ we can easily check that $p(i,j) = e^i(T(e_j))$ for all i, j extends linearly to a kernel where $(e_j)_{j\in\mathbb{N}}$ and $(e^j)_{j\in\mathbb{N}}$ are the standard coordinate elements for ℓ^1 and ℓ^∞ respectively. $\square$

This tells us that we can consider all bounded linear operators on ℓ^1 uniquely as infinite matrices

$$\begin{pmatrix} p_{11} & p_{12} & \cdots \\ p_{21} & \ddots & \\ \vdots & & \end{pmatrix}$$

where $\sup_{i\in\mathbb{N}} \sum_{j=1}^{n} |p_{ij}| < \infty$.

Definition 1.2.28. We call an operator $T \in \mathcal{B}(b\mathcal{E})$ a **kernel operator** if $\mu \circ T \in M\mathcal{E}$ for all $\mu \in M\mathcal{E}$. We denote the set of all kernel operators by $\mathcal{B}(b\mathcal{E})_{M\mathcal{E}}$.

Proposition 1.2.29. *The set of kernel operators is a unital Banach algebra.*

Proof. It is easily checked that $\mathcal{B}(b\mathcal{E})_{M\mathcal{E}}$ is a normed vector space. Let T and S be kernel operators then $\mu \circ T \in M\mathcal{E}$ and $\mu \circ (TS) = (\mu \circ T) \circ S \in M\mathcal{E}$ for all $\mu \in \mathcal{E}$. Therefore TS is a kernel operator.

We have shown that $\mathcal{B}(b\mathcal{E})_{M\mathcal{E}}$ is a normed algebra which contains the identity operator. The mapping $T \mapsto \mu \circ T$ is a bounded linear map for all $\mu \in M\mathcal{E}$. Therefore, $\mathcal{B}(b\mathcal{E})_{M\mathcal{E}}$ is a closed subalgebra of $\mathcal{B}(b\mathcal{E})$. $\square$

Recall the following duality $(b\mathcal{E})^* \cong M^{\mathrm{ba}}\mathcal{E}$ [DS88, Theorem IV.5.1]. It is easily seen that $M^{\mathrm{ba}}\mathcal{E}$ contains the Banach space of countably additive bounded complex measures $M\mathcal{E}$.

Proposition 1.2.30. *The isometric embedding from Corollary 1.2.26 is proper for any measurable space $(E, \mathcal{E})$ such that $\{x\} \in \mathcal{E}$ for all $x \in E$ and $\mathcal{E} \neq 2^E$.*

Proof. Assume the converse, that the map $p \mapsto T_p$ defined in Corollary 1.2.26 is an isometric isomorphism.

There is a straightforward dual pairing of $M\mathcal{E}$ and $b\mathcal{E}$ given by integration. However, if the dual of $M\mathcal{E}$ was $b\mathcal{E}$ then $b\mathcal{E}$ would be a W*-algebra. Therefore there exists $\phi \in (M\mathcal{E})^*$ that cannot be identified with an element of $b\mathcal{E}$ by Corollary 1.2.14.

Let $\nu \in M\mathcal{E}$ and $\phi \in (M\mathcal{E})^*$ be as above and define $T := |\nu\rangle\, \phi : M\mathcal{E} \to M\mathcal{E}$ given by $T(\mu) = \phi(\mu)\nu$ our assumption is there is a corresponding kernel $p(x, A) = (1_A \circ T)(\delta_x)$ but $1_A \circ T = \nu(A)\phi$ which is not a bounded measurable function by construction and therefore $x \mapsto p(x, A)$ is not a bounded measurable function. □

Theorem 1.2.31. *There is an isometric isomorphism of Banach algebras*

$$\ker(E, \mathcal{E}) \cong \mathcal{B}(b\mathcal{E})_{M\mathcal{E}}$$

given by the map $p \mapsto T_p$.

Proof. By Corollary 1.2.20 for any $p \in \ker(E, \mathcal{E})$ there is a unique $T_p \in \mathcal{B}(b\mathcal{E})$ such that $(T_p(f))(x) = \int p(x, dy) f(y)$. For any μ in $M\mathcal{E}$ we have that $(\mu \circ T_p)(f) = \int \mu(dx) p(x, dy) f(y)$ for all $f \in b\mathcal{E}$ and therefore $\mu \circ T_p = \mu * p \in M\mathcal{E}$. Therefore $T_p \in \mathcal{B}(b\mathcal{E})_{M\mathcal{E}}$.

Let $T \in \mathcal{B}(b\mathcal{E})_{M\mathcal{E}}$, the mapping $(x, A) \mapsto (\delta_x \circ T)(1_A)$ defines an element of $\ker(E, \mathcal{E})$. For all $x \in E$ there exists a measure $p(x,\ \cdot\ )$ such that $(\delta_x \circ T)(1_A) = p(x, A)$ for all $A \in \mathcal{E}$ and for all $A \in \mathcal{E}$ the mapping $x \mapsto (\delta_x \circ T)(1_A) = T(1_A)(x) = p(x, A)$ is bounded and measurable. The statement of Proposition 1.2.21 tells us that $\|p\| = \|T\|$.

Let $p \mapsto T_p$ be as above, then a simple calculation shows that

$$\delta_x(T_p(T_q(1_A))) = p * q(x, A) = \delta_x(T_{p*q}(1_A))$$

for all $x \in E$ and $A \in \mathcal{E}$. $\square$

Corollary 1.2.32. *The unital normed algebra* $\ker(E, \mathcal{E})$ *is complete.*

Definition 1.2.33. We call an operator $T \in \mathcal{B}(b\mathcal{E})_{M\mathcal{E}}$ a **transition operator** if $T(f)$ is positive for all positive $f \in b\mathcal{E}$ and $T(1) = 1 \in b\mathcal{E}$.

Proposition 1.2.34. *Transition operators correspond to transition kernels.*

Proof. This is easily from the correspondence. Given T a transition operator we have $A \mapsto \delta_x \circ T(1_A) \geq 0$ for all $A \in \mathcal{E}$ and $\delta_x \circ T(1_E) = 1$ for all $x \in E$. Therefore $A \mapsto \delta_x \circ T(1_A)$ is a probability measure. Similarly $T(1_A) \in b\mathcal{E}$ for all $A \in \mathcal{E}$. Hence, $x \mapsto \delta_x \circ T(1_A) = T(1_A)(x)$ is a bounded measurable function for all $A \in \mathcal{E}$. $\square$

Feller Kernels

We introduce Feller kernels. These are the kernels that respect the topological properties of topological spaces with Borel σ-algebra. For E a locally compact Hausdorff space with Borel σ-algebra recall that a finite measure μ is called **regular** if $\mu(E) < \infty$

$$\mu(A) = \sup\{\mu(K); K \subseteq A, K \text{ is compact}\} \text{ and}$$
$$\mu(A) = \inf\{\mu(B); B \supseteq A, B \text{ is open}\}.$$

A complex measure μ is regular if $|\mu|$ is regular [Rud87, Definition 2.15].

Definition 1.2.35. Let E be a locally compact Hausdorff space with Borel σ-algebra $\mathcal{E}$. A kernel p is said to be a **Feller kernel** if $T_p(C_0(E)) \subseteq C_0(E)$, where

$$T_p(f)(x) := \int_E p(x, dy) f(y)$$

for all $f \in b\mathcal{E}$ and $x \in E$ and $p(x, \cdot)$ is a regular Borel measure for all $x \in E$. A kernel p is a **Feller transition kernel** if $p(x, \cdot)$ is a probability measure for all $x \in E$. We denote the set of finite Feller kernels by $\ker_F(E, \mathcal{E})$.

Example 1.2.36. Given a locally compact Hausdorff space with Borel σ-algebra the kernel $(x, A) \mapsto 1_A(x) = \delta_x(A)$ is Feller.

Note that from the definitions we have that

$$\ker_F(E, \mathcal{E}) \subseteq \ker(E, \mathcal{E}) \subseteq \ker^{\mathrm{ba}}(E, \mathcal{E}).$$

We denote the set of regular complex Borel measures by $M(E) \cong C_0(E)^*$.

Theorem 1.2.37. *Let E be a locally compact metrisable topological space with Borel σ-algebra $\mathcal{E}$. We have the following isometric isomorphism of unital Banach algebras*

$$\ker_F(E, \mathcal{E}) \cong \mathcal{B}(C_0(E))$$

given by $p \mapsto T_p$.

Proof. For any $p \in \ker_F(E, \mathcal{E})$ let T_p be as in Definition 1.2.35. It is clear that T_p is linear and

$$\|T_p\| = \sup_{\|f\|=1} \|T_p(f)\| = \sup_{\|f\|=1} \sup_{x \in E} \left| \int_E p(x, dy) f(y) \right| = \sup_{x \in E} |p_x|(E) = \|p\| < \infty \tag{1.2.5}$$

for all $f \in C_0(E)$. The penultimate equality in equation (1.2.5) is concluded from the Markov-Riesz-Kakutani theorem which says that if $p_x := p(x, \cdot)$ is a measure then the total variation of p_x is given by the norm of the associated functional [Rud87, Theorem 6.19]. Hence, T_p is bounded with $\|T_p\| = \|p\|$.

Let $T \in \mathcal{B}(C_0(E))$ and for any $x \in E$ consider the evaluation map $\mathrm{ev}_x \in C_0(E)^* \cong M(E)$ such that $\mathrm{ev}_x(f) = f(x)$ for all $f \in C_0(E)$. The composition $\mathrm{ev}_x \circ T$ is also in $C_0(E)^*$ which implies by the Riesz-Markov-Kakutani theorem that there exists a unique measure $p(x, \cdot)$ on E such that $(\mathrm{ev}_x \circ T)(f) = \int_E f(y) p(x, dy)$

for all $f \in C_0(E)$. By the same calculation as in (1.2.5) we get that $\|p\| = \|T\| < \infty$.

For A a non-empty open subset of E let $x \in E$ and d be a compatible metric then $f_n(x) = \min\{1, nd(x, E \setminus A)\}$ where $d(x, E \setminus A) = \inf_{y \in E \setminus A} d(x, y)$ defines a continuous function for each n and the sequence $f_n(x)$ increases up to the limit $1_A(x)$ as $n \to \infty$. Then by the monotone convergence theorem we get that the mapping $x \mapsto p(x, A)$ is measurable for all open $A \in \mathcal{E}$. Therefore the mapping $x \mapsto p(x, A)$ is measurable for all closed $A \in \mathcal{E}$. The closed sets form a π-system that includes E and applying the monotone class theorem [Dur10, Theorem 6.1.3] we can confirm that the mapping $x \mapsto p(x, A)$ is measurable for all $A \in \mathcal{E}$. Finally, as $T \in \mathcal{B}(C_0(E))$ the mapping $Tf \in C_0(E)$ for any $f \in C_0(E)$ and therefore $p \in \ker_F(E, \mathcal{E})$. Hence, the mapping $p \mapsto T_p$ is a bijection.

From (1.2.5) we get that $\|T_p\| = \|p\|$. It is clear that $T_p + T_q = T_{p+q}$ and also

$$\begin{aligned} ((T_p(T_q(f))))(x) &= \left(T_p\left(\int_E q(\,\cdot\,, dy_1) f(y_1)\right)\right)(x) \\ &= \int_E \int_E p(x, dy_2) q(y_2, dy_1) f(y_1) \\ &= \int_E f(y_1)(p * q)(x, dy_1) = T_{p*q}(f)(x) \end{aligned}$$

for all $p, q \in \ker_F(E, \mathcal{E})$, $f \in C_0(E)$ and $x \in E$. Hence $p \mapsto T_p$ is an isometric isomorphism. $\square$

Corollary 1.2.38. *The unital normed algebra* $\ker_F(E, \mathcal{E})$ *is complete.*

Definition 1.2.39. Let E be a locally compact metrisable topological space with Borel σ-algebra $\mathcal{E}$. We call an operator $T \in \mathcal{B}(C_0(E))$ a **Feller transition operator** if $T(f)$ is positive for all positive $f \in C_0(E)$ and $\delta_x \circ T \in M(E)$ is a probability measure for all $x \in E$.

Proposition 1.2.40. *Let E be a compact metrisable space. An operator $T \in \mathcal{B}(C(E))$ is a Feller transition operator if and only if $T(f)$ is positive for all positive $f \in C(E)$ and $T(1) = 1 \in C(E)$.*

Proof. First if T is a Feller transition kernel then $\delta_x \circ T(1) = 1$ for all $x \in E$ as $\delta_x \circ T$ is a probability measure for all $x \in E$.

Conversely if $T(1) = 1$ then using the positivity condition $A \mapsto \delta_x \circ T(1_A)$ defines a positive measure for all $x \in E$ such that $\delta_x \circ T(1_E) = 1$ for all $x \in E$. $\square$

Proposition 1.2.41. *Feller transition operators correspond to Feller transition kernels.*

Proof. This is easily seen from the correspondence in Theorem 1.2.37. $\square$

Example 1.2.42. Let $E = \mathbb{N}$ and $\mathcal{E} = 2^{\mathbb{N}}$ then $C_0(\mathbb{N}) = c_0$, $b\mathcal{E} = \ell^\infty$ and $(c_0^*)^* \cong (\ell^1)^* \cong \ell^\infty$ in this setting we have the following Banach algebra isomorphisms

$$\ker_F(E,\mathcal{E}) \cong \mathcal{B}(c_0), \quad \ker(E,\mathcal{E})^{\mathrm{op}} \cong \mathcal{B}(\ell^1), \quad \ker^{\mathrm{ba}}(E,\mathcal{E}) \cong \mathcal{B}(\ell^\infty).$$

We give a simple characterisation of bounded linear operators on c_0. For this and further similar results see [TL86, Chapter IV.6] and [Ban87, Chapter V.7]

Proposition 1.2.43. *Let $E = \mathbb{N}$, $\mathcal{E} = 2^{\mathbb{N}}$ and $p : E \times \mathcal{E} \to \mathbb{C}$ then $p \in \ker_F(E,\mathcal{E})$ if and only if $\lim_{i\to\infty} p_{ij} = 0$ for all $j \in \mathbb{N}$ and $\sup_i \sum_{j=1}^\infty |p_{ij}| < \infty$.*

Proof. Let $p_{ij} \to 0$ as $i \to \infty$ and assume $0 \neq \|p\| = \sup_i \sum_{j=1}^\infty |p_{ij}|$. Let $a \in c_0$ and $\epsilon > 0$ by definition there exists $M \in \mathbb{N}$ such that $|a_n| < \epsilon/2\|p\|$ for all $n \geq M$. Also there exists $N_j \in \mathbb{N}$ for $j = 1, \dots, M$ such that $|p_{nj}| < \epsilon/2M\|a\|$ for all $n \geq N_j$. Let $N = \max\{N_j; j = 1, \dots M\}$ then

$$\begin{aligned}
\left|\sum_{j=1}^\infty p_{nj}a_j\right| &\leq \sum_{j=1}^M |p_{nj}||a_j| + \sum_{j=M+1}^\infty |p_{nj}||a_j| \\
&< \sum_{j=1}^M \frac{\epsilon}{2M\|a\|}\|a\| + \|p\|\frac{\epsilon}{2\|p\|} \\
&= \epsilon
\end{aligned}$$

for all $n \geq N$.

Conversely let $p \in \ker_F(E, \mathcal{E})$ then by definition $\|p\| = \sup_i \sum_{j=1}^{\infty} |p_{ij}| < \infty$. Also by definition $\sum_{j=1}^{\infty} p_{ij}(e_k)_j = p_{ik} \to 0$ as $i \to \infty$ for all $k \in \mathbb{N}$ where $e_k \in c_0$ such that $(e_k)_j = \delta_{j,k}$. $\square$

Similarly to Corollary 1.2.27 we can consider bounded linear operators on c_0 as infinite matrices.

Corollary 1.2.44. *There is a one-to-one correspondence between bounded linear operators on c_0 and families $(p_{ij})_{i,j \in \mathbb{N}}$ such that $\lim_{i\to\infty} p_{ij} = 0$ for all $j \in \mathbb{N}$ and $\sup_i \sum_{j=1}^{\infty} |p_{ij}| < \infty$.*

1.3 Markov Chains with Kernels

We use kernels and their associated bounded linear operators to characterise Markov chains on some measurable spaces similarly to [Dur10, Chapter 6].

The following definition is needed to construct kernels as conditional expectations and to use Kolmogorov's Extension Theorem.

Definition 1.3.1. A measurable space $(E, \mathcal{E})$ is said to be a **standard Borel space** if E is a Polish space (i.e. separable and completely metrisable) and $\mathcal{E}$ is its Borel σ-algebra [Kal97, Page 7] [Dur10, Page 45].

Remark 1.3.2. As the σ-algebra is implied in this context we often only refer to E when extra topological adjectives are necessary, i.e. "For E a locally compact Polish space…".

Be reminded that for every E-valued random variable X we have an associated σ-algebra

$$\sigma(X) = \{X^{-1}(A); A \in \mathcal{E}\}.$$

Lemma 1.3.3 ([Kal97, Proposition 5.3])**.** *Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a standard Borel space $(E, \mathcal{E})$, and E-valued random variables X and Y, there exists* *a transition kernel p such that*

$$p(Y,A) = \mathbb{E}[1_A(X)|\sigma(Y)]$$

$\mathbb{P}$*-almost surely for all* $A \in \mathcal{E}$*. The transition kernel* p *is unique* $\mathbb{P} \circ Y^{-1}$ *almost surely.*

Corollary 1.3.4. *Given a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$*, a standard Borel space* $(E, \mathcal{E})$*, and* E*-valued random variables* X *and* Y*, any transition kernel* p *as above satisfies*

$$\mathbb{E}[f(X)|\sigma(Y)] = \int_E p(X, dy) f(y)$$

for any $f \in b\mathcal{E}$*.*

Definition 1.3.5. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $(E, \mathcal{E})$ a measurable space. A family of E-valued random variables $(X_n)_{n\geq 0}$ such that

$$\mathbb{E}[f(X_n)|\sigma(X_{n-1}, \ldots, X_0)] = \mathbb{E}[f(X_n)|\sigma(X_{n-1})]$$

for all $n \geq 1$ and all bounded and measurable functions $f : E \to \mathbb{R}$ is called a **Markov chain**. Markov chains are assumed to be time homogeneous, that is, the transition kernel $p^{(n)} : E \times \mathcal{E} \to [0,1]$ associated to the random variables X_n and X_{n-1} as in Lemma 1.3.3 satisfies the identity $p^{(n)} = p^{(1)}$ $\mathbb{P}$ almost surely for all $n \geq 1$. If the kernel p is Feller we call the process a **Feller chain**.

The following is an argument employing the monotone class theorem [Dur10, Theorem 6.1.3] to extend a map from the set of cylindrical sets of a Cartesian product of measurable spaces to the sigma algebra of the same Cartesian product.

Lemma 1.3.6. *Let* $(E, \mathcal{E})$ *be a measurable space with probability measure* λ *and transition kernel* p*. Then*

$$\underset{E^{n+1}}{\int \cdots \int} 1_S(x_0, \ldots, x_n) \lambda(dx_0) p(x_0, dx_1) \cdots p(x_{n-1}, dx_n)$$

is well defined for all $S \in \sigma(\mathcal{E}^{n+1})$ where $\mathcal{E}^{n+1}$ denotes the Cartesian product of $\mathcal{E}$ with itself $n+1$ times.

Proof. Let $\mathcal{H}$ denote the set of all functions $f : E^{n+1} \to \mathbb{R}$ such that

$$\underset{E^{n+1}}{\int \cdots \int} f(x_0, \ldots, x_n)\lambda(dx_0)p(x_0, dx_1)\cdots p(x_{n-1}, dx_n)$$

is well defined. Note that $\mathcal{E}^{n+1}$ is a π-system and includes E^{n+1}.

It is easily seen for all $A \in \mathcal{E}^{n+1}$ that $1_A \in \mathcal{H}$. By standard properties of integrals and bounded measurable functions we can see that if $f, g \in \mathcal{H}$ then $f + g \in \mathcal{H}$ and $cf \in \mathcal{H}$ for all $c \in \mathbb{R}$.

Finally, by the monotone convergence theorem if $(f_n)_{n\geq 1} \subseteq \mathcal{H}$ is an increasing sequence of functions with bounded limit f then $f \in \mathcal{H}$.

Therefore, by the monotone class theorem $\mathcal{H}$ contains the set of the functions which are bounded and measurable with respect to the sigma algebra $\sigma(\mathcal{E}^{n+1})$ which in turn contains all functions 1_S where $S \in \sigma(\mathcal{E}^{n+1})$. $\square$

The following two propositions are continuous generalisation of Proposition 1.1.3. This can be seen by replacing kernels with stochastic matrices and integrals with summations.

Proposition 1.3.7 ([Dur10, Theorem 6.1.1])**.** *Let $(E, \mathcal{E})$ be a standard Borel space, let $\lambda : \mathcal{E} \to [0, 1]$ be a probability measure and $p : E \times \mathcal{E} \to [0, 1]$ a transition kernel. There exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and an E-valued stochastic process $(X_n)_{n\geq 0}$ such that, for all $n \geq 0$ and $A_0, \ldots, A_n \in \mathcal{E}$,*

$$\mathbb{P}(X_0 \in A_0, \ldots, X_n \in A_n) = \int_{A_0} \lambda(dx_0) \int_{A_1} p(x_0, dx_1) \cdots \int_{A_n} p(x_{n-1}, dx_n).$$

Moreover, this stochastic process satisfies the Markov condition

$$\mathbb{E}[f(X_n)|\sigma(X_{n-1}, \ldots, X_0)] = \mathbb{E}[f(X_n)|\sigma(X_{n-1})]$$

almost surely for any $n \geq 0$ and $f \in b\mathcal{E}$.

Proof. Let $P_n : \sigma(\mathcal{E}^{(n+1)}) \to \mathbb{R}$ be given by the map

$$S \mapsto \underset{E^{n+1}}{\int \cdots \int} 1_S(x_0, \ldots, x_n)\lambda(dx_0)p(x_0, dx_1) \cdots p(x_{n-1}, dx_n),$$

which is well defined by Lemma 1.3.6. Note that if $S = (A_0, \ldots, A_n) \in \mathcal{E}^{n+1}$ then

$$P_n(A_0, \ldots, A_n) = \int_{A_0} \lambda(dx_0) \int_{A_1} p(x_0, dx_1) \cdots \int_{A_n} p(x_{n-1}, dx_n).$$

From its definition it is clear that P_n is positive and countably additive for all $n \geq 0$. For all $n \geq 0$

$$\begin{aligned} P_n(E, \ldots, E) &= \int_E \lambda(dx_0) \cdots \int_E p(x_{n-2}, dx_{n-1}) \int_E p(x_{n-1}, dx_n) \\ &= \int_E \lambda(dx_0) \cdots \int_E p(x_{n-2}, dx_{n-1}) \left(p(x_{n-1}, E)\right) \\ &= \int_E \lambda(dx_0) \cdots \int_E p(x_{n-2}, dx_{n-1})(1) \\ &= \cdots = \int_E \lambda(dx_0) = \lambda(E) = 1. \end{aligned}$$

Hence, P_n is a probability measure on $(E^{n+1}, \sigma(\mathcal{E}^{(n+1)}))$. It is easily seen by a similar calculation that

$$P_n(A_0, \ldots, A_n) = P_m(A_0, \ldots, A_n, E, \ldots, E)$$

for all $A_0, \ldots, A_n \in \mathcal{E}$ and $n \leq m$. Therefore by Kolmogorov's consistency theorem [Par67, Chapter 7, Theorem 5.1] there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and an E-valued stochastic process $(X_n)_{n\geq 0}$ such that for all $n \geq 0$ and $A_0, \ldots, A_n \in \mathcal{E}$

$$\mathbb{P}(X_0 \in A_0, \ldots, X_n \in A_n) = \int_{A_0} \lambda(dx_0) \int_{A_1} p(x_0, dx_1) \cdots \int_{A_n} p(x_{n-1}, dx_n). \tag{1.3.1}$$

It remains to show that the Markov property holds. Let $F = \{X_0 \in A_0, \ldots, X_{n-1} \in$

$A_{n-1}\} \in \mathcal{F}_{n-1} := \sigma(X_0, \ldots, X_{n-1})$ and $B \in \mathcal{E}$. We have that

$$\begin{aligned}\mathbb{E}[1_F 1_{X_n \in B}] &= \mathbb{P}(F \cap \{X_n \in B\}) \\ &= \int_{A_0} \lambda(dx_0) \int_{A_1} p(x_0, dx_1) \cdots \int_{A_{n-1}} p(x_{n-2}, dx_{n-1}) p(x_{n-1}, B).\end{aligned}$$

We first claim that

$$\int_{A_0} \lambda(dx_0) \int_{A_1} p(x_0, dx_1) \cdots \int_{A_{n-1}} p(x_{n-2}, dx_{n-1}) p(x_{n-1}, B) = \mathbb{E}[1_F p(X_{n-1}, B)]. \tag{1.3.2}$$

By assumption $p^{\mathcal{B}}(x) = p(x, B)$ is a bounded and measurable function. If p^B is an indicator function, then equation (1.3.2) holds by (1.3.1). Thus by linearity equation (1.3.2) holds if p^B is a simple function. Finally, by monotone convergence theorem equation (1.3.2) holds if p^B is any bounded and measurable function.

The statement is true for all cylindrical sets $(A_0, \ldots, A_{n-1}) \in \mathcal{E}^n$, so by a similar monotone class argument as in Lemma 1.3.6 we get the result for all $F \in \mathcal{F}_{n-1}$. Therefore, $p(X_{n-1}, B) = \mathbb{E}[1_{X_n \in B} | \mathcal{F}_{n-1}]$ $\mathbb{P}$-almost everywhere for all $n \geq 1$.

We have thus shown that $\mathbb{E}[1_{X_n \in B} | \mathcal{F}_{n-1}]$ is $\sigma(X_{n-1})$ measurable. Hence, by the tower property for conditional expectations with $\sigma(X_{n-1}) \subseteq \mathcal{F}_{n-1}$

$$\begin{aligned}\mathbb{E}[1_{X_n \in B} | \sigma(X_{n-1})] &= \mathbb{E}[\mathbb{E}[1_{X_n \in B} | \mathcal{F}_{n-1}] | \sigma(X_{n-1})] \\ &= \mathbb{E}[p(X_{n-1}, B) | \sigma(X_{n-1})] = p(X_{n-1}, B)\end{aligned}$$

Therefore $p(X_{n-1}, B) = \mathbb{E}[1_{X_n \in B} | \sigma(X_{n-1})]$ $\mathbb{P}$-almost everywhere for all $n \geq 1$. Hence $(X_n)_{n \geq 0}$ is a Markov chain. □

We have shown that given an initial distribution and a transition kernel we can construct a Markov chain. We can now prove the converse.

Proposition 1.3.8 ([Dur10, Theorem 6.1.2]). *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $(E, \mathcal{E})$ a standard Borel space and $(X_n)_{n \geq 0}$ an E-valued Markov chain. Then there exists a probability measure λ on E and transition kernel $p : E \times \mathcal{E} \to [0, 1]$ such* *that*

$$\mathbb{P}(X_n \in A_n, \ldots, X_0 \in A_0) = \int_{A_0} \lambda(dx_0) \int_{A_1} p(x_0, dx_1) \cdots \int_{A_n} p(x_{n-1}, dx_n) \quad (1.3.3)$$

for all $A_0, \ldots, A_n \in \mathcal{E}$ *and* $n \geq 0$.

Proof. Denote the initial distribution $\lambda(A) := \mathbb{P}(X_0 \in A)$ and transition kernel $p := p^{(1)}$ as defined in Definition 1.3.5. Using properties of conditional expectation and integration we get

$$\begin{aligned}
\mathbb{E}[f_0(X_0) \ldots f_n(X_n)] &= \mathbb{E}[\mathbb{E}[f_0(X_0) \ldots f_n(X_n) | \sigma(X_{n-1}, \ldots, X_0)]] \\
&= \mathbb{E}[f_0(X_0) \ldots f_{n-1}(X_{n-1}) \mathbb{E}[f_n(X_n) | \sigma(X_{n-1}, \ldots, X_0)]] \\
&= \mathbb{E}[f_0(X_0) \ldots f_{n-1}(X_{n-1}) \mathbb{E}[f_n(X_n) | \sigma(X_{n-1})]] \\
&= \mathbb{E}\left[f_0(X_0) \ldots f_{n-1}(X_{n-1}) \int p(X_{n-1}, dy) f_n(y)\right]
\end{aligned}$$

The integral $\int p(X_{n-1}, dy) f_n(y)$ is a bounded measurable function. Therefore continuing the process iteratively we get

$$\begin{aligned}
\mathbb{E}[f_0(X_0) \ldots f_n(X_n)] &= \mathbb{E}\left[f_0(X_0) \int p(X_0, dx_1) f_1(x_1) \ldots \int p(x_{n-1}, dx_n) f_n(x_n)\right] \\
&= \int \lambda(dx_0) f_0(x_0) \int p(x_0, dx_1) f_1(x_1) \ldots \int p(x_{n-1}, dx_n) f_n(x_n).
\end{aligned}$$

The result is thus obtained by letting $f_i = 1_{A_i}$ for $A_i \in \mathcal{E}$. $\square$

Corollary 1.3.9. *Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space,* $(E, \mathcal{E})$ *a standard Borel space and* $(X_n)_{n \geq 0}$ *be an* E*-valued Markov chain with initial distribution* $\lambda : \mathcal{E} \to [0, 1]$ *and transition kernel* $p : E \times \mathcal{E} \to [0, 1]$*. For each* $n \geq 0$ *and* $A \in \mathcal{E}$

$$\mathbb{P}(X_n \in A) = (\lambda * p^{*n})(A) = \int_E \lambda(dx_0) \int_E p(x_0, dx_1) \cdots \int_E p(x_{n-1}, dx_n) 1_A(x_n).$$

Example 1.3.10. Using the kernel from Example 1.2.16 and letting $\mu = \delta_1$ we get a highly unstable process which takes very small steps when close to zero and very large steps away from zero. See Figure 1.3 for a sample path.

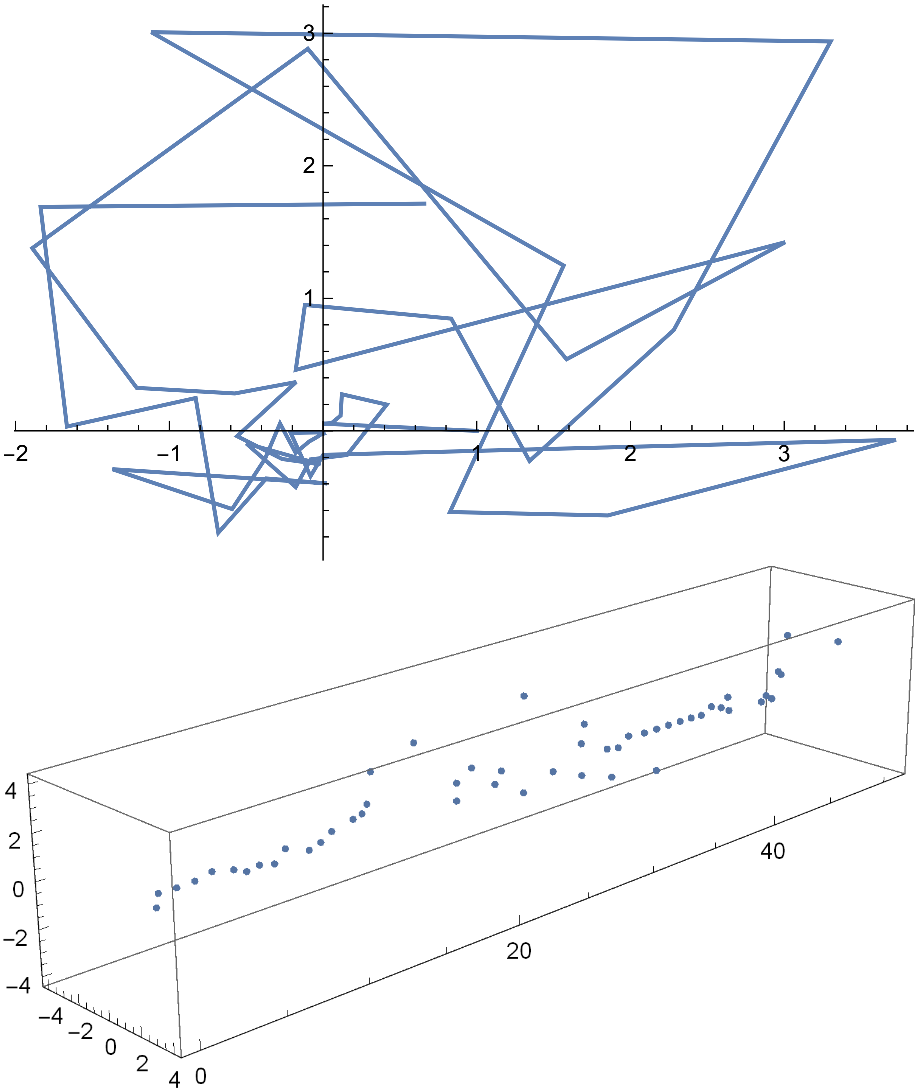


Figure 1.3: Sample path of the Markov chain in Example 1.3.10 starting at $x = 1$ with 50 steps.

Directly from Corollary 1.3.9 and our correspondence between kernel and kernel operators given by Theorem 1.2.31 we can deduce a relationship between Markov chains and transition operators.

Corollary 1.3.11. *Let $(E, \mathcal{E})$ be a standard Borel space.*

(i) *For $(X_n)_{n\geq 0}$ an E-valued Markov chain there exists a probability measure μ on $(E, \mathcal{E})$ and a transition operator $T \in \mathcal{B}(b\mathcal{E})_{M\mathcal{E}}$ such that*

$$\mathbb{E}(f(X_n)) = \mu \circ T^n(f)$$

for all $f \in b\mathcal{E}$ and $n \in \mathbb{Z}_+$.

(ii) *Given a probability measure μ on $(E, \mathcal{E})$ and a transition operator $T \in \mathcal{B}(b\mathcal{E})_{M\mathcal{E}}$ there exists a E-valued Markov chain $(X_n)_{n\geq 0}$ such that*

$$\mathbb{E}(f(X_n)) = \mu \circ T^n(f)$$

for all $f \in b\mathcal{E}$ and $n \in \mathbb{Z}_+$.

Similarly to above using Theorem 1.2.37 we have a relationship between Feller chains and Feller transition operators. Note that finite Borel measures on Polish spaces are automatically regular [Par67, Chapter 2, Theorem 3.2].

Corollary 1.3.12. *Let E be a locally compact Polish space.*

(i) *For $(X_n)_{n\geq 0}$ an E-valued Feller chain there exists a Borel probability measure μ and a Feller transition operator $T \in \mathcal{B}(C_0(E))$ such that*

$$\mathbb{E}(f(X_n)) = \mu \circ T^n(f)$$

for all $f \in C_0(E)$ and $n \geq 0$.

(ii) *Given a Borel probability measure μ and a Feller transition operator $T \in \mathcal{B}(C_0(E))$ there exists a E-valued Feller chain $(X_n)_{n\geq 0}$ such that*

$$\mathbb{E}(f(X_n)) = \mu \circ T^n(f)$$

for all $f \in C_0(E)$ and $n \geq 0$.

Random Walks

If G is a locally compact group with Borel σ-algebra $\mathcal{G}$ we denote the shift maps $s_g : b\mathcal{G} \to b\mathcal{G}$ given by $s_g(f)(x) = f(g^{-1}x)$ for each $g \in G$. Note that since multiplication is continuous we also have that $s_g(C_0(G)) \subseteq C_0(G)$ for each $g \in G$.

Definition 1.3.13. Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a Polish group G we call a G-valued Markov chain (X_n) such that

$$s_{g^{-1}}(\mathbb{E}[s_g(1_A(X_n))|\sigma(X_{n-1})]) = \mathbb{E}[1_A(X_n)|\sigma(X_{n-1})]$$

for all $g \in G$ and all bounded and $A \in \mathcal{G}$ a **random walk** on G. In other words $p(X_{n-1}, A) = p(g^{-1}X_{n-1}, g^{-1} \cdot A)$, for all $g \in G, A \in \mathcal{G}, n \geq 1$ $\mathbb{P}$-almost surely where $g \cdot A := \{ga; a \in A\}$. A kernel with this property is called **translation invariant**.

Proposition 1.3.14. *Let G be a Polish group. A kernel operator $T \in \mathcal{B}(b\mathcal{G})_{M\mathcal{G}}$ satisfies $s_g \circ T = T \circ s_g$ for all $g \in G$ if and only if its associated kernel satisfies $p(x, A) = p(g^{-1}x, g^{-1} \cdot A)$ for all $x, g \in G$ and $A \in \mathcal{G}$.*

Proof. Let $g, x \in G$ and $f \in b\mathcal{G}$ then

$$\begin{aligned} s_g \circ T(f)(x) = T \circ s_g(f)(x) &\iff s_g \circ T \circ s_{g^{-1}}(f)(x) = T(f)(x) \\ &\iff \int p(g^{-1}x, dy) f(gy) = \int p(x, dy) f(y) \\ &\iff \int p(g^{-1}x, g^{-1}dy) f(y) = \int p(x, dy) f(y). \quad \square \end{aligned}$$

Corollary 1.3.15. *Let G be a Polish group.*

(i) *For $(X_n)_{n\geq 0}$ a G-valued random walk there exists a probability measure $\mu \in M\mathcal{G}$ and a transition operator $T \in \mathcal{B}(b\mathcal{G})_{M\mathcal{G}}$ such that*

$$\mathbb{E}(f(X_n)) = \mu \circ T^n(f) \text{ and } s_g \circ T = T \circ s_g$$

for all $f \in b\mathcal{G}, g \in G$ and $n \geq 0$.

(ii) *Given a probability measure $\mu \in M\mathcal{G}$ and a transition operator $T \in \mathcal{B}(b\mathcal{G})_{M\mathcal{G}}$ such that $s_g \circ T = T \circ s_g$ for all $g \in G$ there exists a G-valued random walk $(X_n)_{n\geq 0}$ such that*

$$\mathbb{E}(f(X_n)) = \mu \circ T^n(f)$$

for all $f \in b\mathcal{G}$ and $n \geq 0$.

Corollary 1.3.16. *Let G be a locally compact Polish group and $(X_n)_{n\geq 0}$ be a G-valued random walk with initial distribution $\lambda : \mathcal{G} \to [0,1]$ and transition kernel $p : G \times \mathcal{G} \to [0,1]$. For each $n \geq 0, A \in \mathcal{G}$*

$$\mathbb{P}(X_n \in A) = (\lambda * p_e^{*n})(A) := \int_{G^{n+1}} 1_A(x_0 x_1 \dots x_n)\lambda(dx_0)p_e(dx_1)\cdots p_e(dx_n)$$

where $p_e : \mathcal{G} \to [0,1]$ is the probability measure $p(e,\cdot)$.

Example 1.3.17. Let $G = \mathbb{R}^2$ and p_e be the uniform measure on the compact set $\{(x,y); x^2 + y^2 = 1\}$ i.e. the circle with centre zero and radius one and $\mu = \delta_0$. These two measures can be used to define a random walk on $\mathbb{R}^2$. A sample path is given in Figure 1.4.

If G is a compact Hausdorff group we can realise it as a compact quantum group, similar ideas are expanded upon in Chapter 3. There exist unital *-homomorphisms $\Delta : C(G) \to C(G) \otimes C(G) \cong C(G \times G)$ called the comultiplication and $\epsilon : C(G) \to \mathbb{C}$ called the counit which satisfy the following equations

$$(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta \quad \text{ and } \quad (\epsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id} = (\mathrm{id} \otimes \epsilon) \circ \Delta$$

which are referred to as coassociativity and the counital property respectively. These unital *-homomorphism are given explicitly by $\Delta(f)(x,y) = f(xy)$ and $\epsilon(f) = f(e)$ for all $f \in C(G)$ and $x, y \in G$ where $e \in G$ is the identity element of G. Given two linear functionals $\phi, \omega \in C(G)^*$ we can define the convolution functional by $\phi * \omega = (\phi \otimes \omega) \circ \Delta$. For a quick introduction to this topic see [FKS16, Section 2].

We can consider translation invariant operators in the language of quantum groups. For a more "quantum" or noncommutative exposition of similar ideas see [CFK14, Section 3] and [LS11, Theorem 2.4].

Note that a space that is compact and metrisable is Polish [Mun00, Theorem

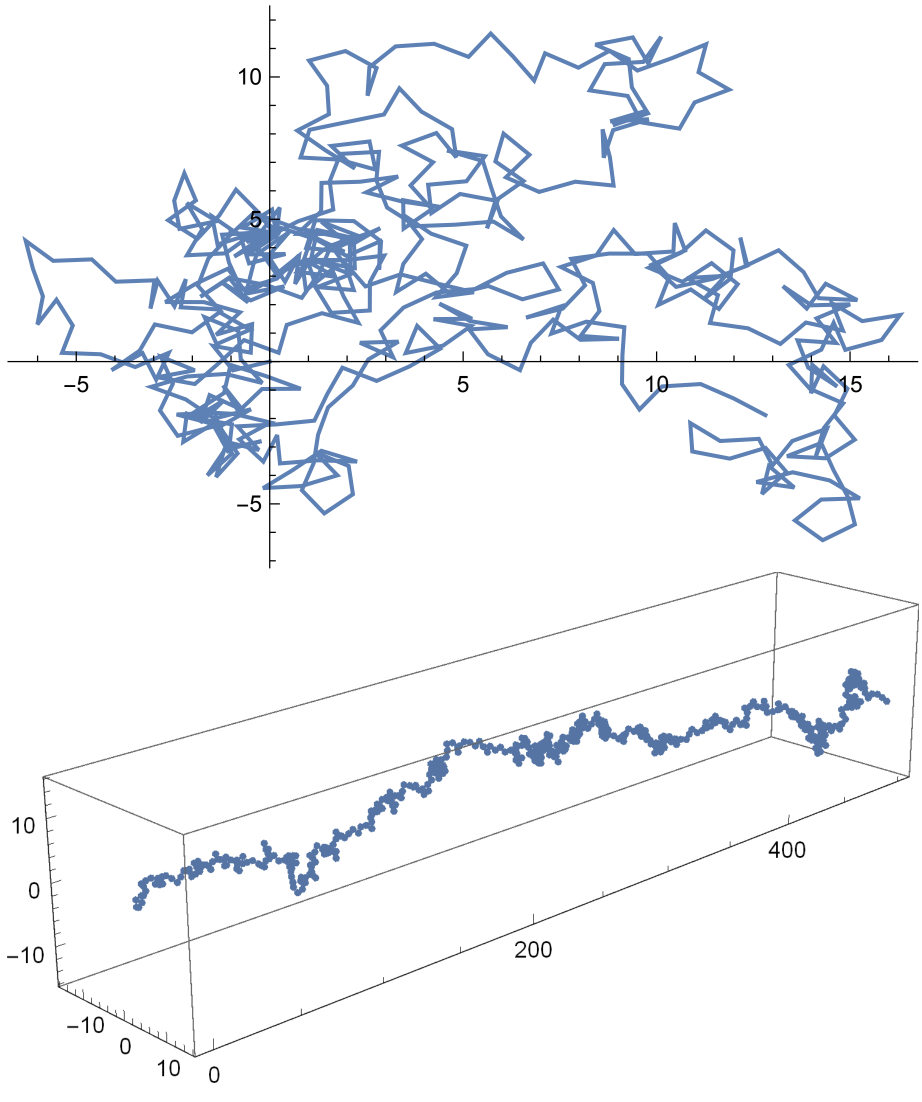


Figure 1.4: Sample path of Example 1.3.17 starting at $x = 0$ with 500 steps.

45.1].

Proposition 1.3.18. *Let G be a compact metrisable group and $T \in \mathcal{B}(C(G))$. The following are equivalent:*

(i) $s_g \circ T = T \circ s_g$ *for all* $g \in G$*;*

(ii) $T = (\mathrm{id} \otimes \phi) \circ \Delta$ *for some state* $\phi \in C(G)^*$*;*

(iii) $(\mathrm{id} \otimes T) \circ \Delta = \Delta \circ T$.

Proof. (i) $\implies$ (ii): The associated kernel to T satisfies $p(g^{-1}x, g^{-1}\cdot A) = p(x, A)$ for all $x, g \in G$ and $A \in \mathcal{G}$. Hence, $\int p(x, dy)f(y) = \int p(e, dy)f(xy)$ for all $f \in C(G)$. Therefore, $T(f)(x) = (\mathrm{id} \otimes \phi)(\Delta(f))(x)$ for all $x \in G$ and $f \in C(G)$ where ϕ is the functional given by $\phi(f) = \int f(x)p(e, dx)$ for all $f \in C(G)$.

(ii) $\implies$ (iii): This follows by coassociativity

$$\begin{aligned}
\Delta \circ T &= (\Delta \otimes \phi) \circ \Delta \\
&= (\mathrm{id} \otimes \mathrm{id} \otimes \phi) \circ (\Delta \otimes \mathrm{id}) \circ \Delta \\
&= (\mathrm{id} \otimes \mathrm{id} \otimes \phi) \circ (\mathrm{id} \otimes \Delta) \circ \Delta \\
&= (\mathrm{id} \otimes T) \circ \Delta.
\end{aligned}$$

(iii) $\implies$ (i): This follow from direct calculation

$$\begin{aligned}
s_g \circ T(f)(y) &= T(f)(g^{-1}y) \\
&= \Delta(T(f))(g^{-1}, y) \\
&= ((\mathrm{id} \otimes T) \circ \Delta)(f)(g^{-1}, y) \\
&= T(f(g^{-1} \cdot\ ))(y) \\
&= T \circ s_g(f)(y)
\end{aligned}$$

for all $g, y \in G$ and $f \in C(G)$ $\square$

Corollary 1.3.19. *Let $(E, \mathcal{E})$ be a compact metrisable group.*

(i) *For* $(X_n)_{n\geq 0}$ *a* G*-valued random walk then there exists states* $\phi, \omega \in C(G)^*$ *such that*

$$\mathbb{E}(f(X_n)) = \phi * \omega^{*n}(f)$$

for all $f \in C(G)$ *and* $n \geq 0$*.*

(ii) *Given states* $\phi, \omega \in C(G)^*$ *there exists a* G*-valued random walk* $(X_n)_{n\geq 0}$ *such that*

$$\mathbb{E}(f(X_n)) = \phi * \omega^{*n}(f)$$

for all $f \in C(G)$ *and* $n \geq 0$*.*

Proof. (i): Corollary 1.3.15 gives the existence of a probability measure μ and a translation invariant operator $T = (\mathrm{id} \otimes \omega) \circ \Delta$. As finite Borel measures on Polish spaces are regular, let $\phi \in C(G)$ be the associated functional to μ then

$$\phi \circ T = \phi \circ (\mathrm{id} \otimes \omega) \circ \Delta = (\phi \otimes \omega) \circ \Delta = \phi * \omega.$$

The proof that $\phi \circ T^n = \phi * \omega^{*n}$ follows similarly.

(ii): This follows from Propositions 1.3.18 and 1.3.7. □

What follows is a simple proof using quantum group methods that transition kernels associated to random walks are in fact Feller kernels.

Proposition 1.3.20. *If* G *be a compact metrisable group and* $T \in \mathcal{B}(b\mathcal{G})_{M\mathcal{G}}$ *such that* $s_g \circ T = T \circ s_g$ *for all* $g \in G$ *then* $T(C(G)) \subseteq C(G)$*.*

Proof. The associated kernel to T satisfies $p(gx, g \cdot A) = p(x, A)$ for all $x, g \in G$ and $A \in \mathcal{G}$ therefore $\int p(x, dy) f(y) = \int p(e, dy) f(xy)$ for all $f \in b\mathcal{G}$. As all finite Borel measures on Polish spaces are regular if we let ϕ be the associated state to $p(e, \cdot) \in M(E)$ then we can see that $T_{C(G)} = (\mathrm{id} \otimes \phi) \circ \Delta \in \mathcal{B}(C(G))$. □

Chapter 2

Quantum Stochastic Flows on C*-Algebras

We follow the exposition of quantum stochastic flows as in [BW15]. This gives a quick introduction for everything we need in terms of quantum stochastic analysis, for more detail see [Lin05]. As soon as we have developed enough background material we extend the main result of [BW15]. This provides *-homomorphic solutions to the Evans-Hudson quantum stochastic differential equation on C*-algebras that are generated by partial isometry matrices. The construction requires that the quantum stochastic flow generators satisfy a growth restriction. The method employed to find these solutions is by iterating quantum stochastic integrals of our quantum stochastic flow generator realising the method of Picard iteration cf. [LW03, Section 2].

For this chapter we set A to be a unital C*-algebra with A_0 a dense unital *-subalgebra of A. We let k be a finite dimensional Hilbert space referred to as the noise space and let $\widehat{k} = \mathbb{C} \oplus k$ with $\widehat{x} = (1, x)$ for any $x \in k$. All the results extend easily for a general Hilbert space k but with greater attention to detail needed for relevant tensor products. Given a map $\delta : A_0 \to A_0 \otimes |k\rangle$ we define an associated map $\delta^\dagger : A_0 \to A_0 \otimes \langle k|$ such that $\delta^\dagger(a) = \delta(a^*)^*$.

To illustrate this consider a simple tensor example. Let $\delta_i : A_0 \to A_0$ be a collection of linear maps then $\delta(a) = \delta_i(a) \otimes |e_i\rangle$ for any basis (e_i) of k has associated map $\delta^\dagger(a) = f(a^*)^* \otimes \langle e_i|$ for all $a \in A_0$. Where appropriate we abbreviate the trivial ampliation $b \otimes \mathrm{id}_{\mathbb{C}} = b$ for example $\delta(a)b := \delta(a)(b \otimes \mathrm{id}_{\mathbb{C}})$ and $b\delta^\dagger(a) := (b \otimes \mathrm{id}_{\mathbb{C}})\delta^\dagger(a)$ for all $a, b \in A_0$.

2.1 Quantum Stochastic Flow Generators

We describe quantum stochastic flow generators in a purely algebraic way. We motivate the definitions and results by use of three examples on the commutative C*-algebra $C[-1, 1]$. We prove the existence of an associated unital *-subalgebra that captures the elements where the quantum stochastic flow generator can be exponentiated.

Definition 2.1.1. A **quantum stochastic flow generator** is a map $\phi : A_0 \to A_0 \otimes \mathcal{B}(\widehat{k})$ given by

$$\phi = \begin{pmatrix} \tau & \delta^\dagger \\ \delta & \pi - \iota \end{pmatrix}$$

where $\pi : A_0 \to A_0 \otimes \mathcal{B}(k)$ is a unital *-homomorphism, $\iota : A_0 \to A_0 \otimes \mathcal{B}(k)$ is the ampliation $a \mapsto a \otimes I_k$, $\delta : A_0 \to A_0 \otimes |k\rangle$ is a π derivation i.e.

$$\delta(ab) = \delta(a)b + \pi(a)\delta(b) \tag{2.1.1}$$

for all $a, b \in A_0$ and $\tau : A_0 \to A_0$ is a *-linear map such that

$$\tau(ab) = \tau(a)b + a\tau(b) + \delta^\dagger(a)\delta(b). \tag{2.1.2}$$

We call the triple (π, δ, τ) the **components** of a flow generator.

Example 2.1.2. Let $\pi : A_0 \to A_0 \otimes \mathcal{B}(k)$ and let $w \in k$ then the maps given by

$$\delta(a) = (\pi(a) - \iota(a))(1_{A_0} \otimes |w\rangle) \text{ and } \tau = (1_{A_0} \otimes \langle w|)(\pi(a) - \iota(a))(1_{A_0} \otimes |w\rangle)$$

for all $a \in A_0$ make the components (π, δ, τ) of a flow generator. This is easily seen, if we let $\pi' = \pi - \iota$ then

$$\pi'(ab) = \pi'(a)\iota(b) + \pi(a)\pi'(b) = \pi'(a)\iota(b) + \iota(a)\pi'(b) + \pi'(a)\pi'(b)$$

for all $a, b \in A_0$.

Let $P_k := \{0\} \oplus I_k \in \mathcal{B}(\widehat{k})$ and $\tilde{P_k} := 1_{A_0} \otimes P_k \in A_0 \otimes \mathcal{B}(\widehat{k})$ These maps are the orthogonal projection from $\widehat{k} \to k$ and its ampliation respectively. The ampliation is also a projection in the abstract C*-algebraic sense, i.e. $\tilde{P_k} = \tilde{P_k}^2 = \tilde{P_k}^*$.

Note that if A_1 and A_2 are Banach spaces then for any $T \in \mathcal{B}(A_1 \oplus A_2)$ there exists unique $T_{i,j} \in \mathcal{B}(A_j, A_i)$ for $1 \leq i, j \leq 2$ such that

$$T = \begin{pmatrix} T_{1,1} & T_{1,2} \\ T_{2,1} & T_{2,2} \end{pmatrix}.$$

Using this decomposition, given a map $\phi : A_0 \to A_0 \otimes \mathcal{B}(\mathbb{C} \oplus k)$ we get the **component maps** $\phi_{1,1} : A_0 \to A_0, \phi_{1,2} : A_0 \otimes \langle k| , \phi(1,2) : A_0 \to A_0 \otimes |k\rangle$ and $\phi_{2,2} : A_0 \to A_0 \otimes \mathcal{B}(k)$ such that

$$\phi(a) = \begin{pmatrix} \phi_{1,1}(a) & \phi_{1,2}(a) \\ \phi_{2,1}(a) & \phi_{2,2}(a) \end{pmatrix}$$

for all $a \in A_0$.

Note the following useful identities

$$a \otimes \mathrm{id}_{\widehat{k}} = \begin{pmatrix} a & 0 \\ 0 & \iota(a) \end{pmatrix} \qquad \text{and} \qquad \tilde{P_k}\phi(a) = \begin{pmatrix} 0 & 0 \\ \delta(a) & \pi(a) - \iota(a) \end{pmatrix}$$

for all $a \in A_0$.

Proposition 2.1.3 ([BW15, Lemma 2.2])**.** *There is a one to one correspondence between quantum stochastic flow generators and *-linear maps* $\phi : A_0 \to A_0 \otimes \mathcal{B}(\widehat{k})$

such that $\phi(1) = 0$ *and*

$$\phi(ab) = \phi(a)(b \otimes \mathrm{id}_{\widehat{k}}) + (a \otimes \mathrm{id}_{\widehat{k}})\phi(b) + \phi(a)\tilde{P}_k\phi(b). \tag{2.1.3}$$

We introduce some straightforward examples of quantum stochastic flow generators on a commutative unital C*-algebra. As we progress through the chapter we highlight definitions and results in terms of these examples.

Example 2.1.4. Let $A = C([-1,1])$. This is the universal C*-algebra generated by a single self adjoint element x such that $\|x\| \leq 1$, see Example 3.3.5. Let A_0 be the unital *-subalgebra generated by x, the polynomial subalgebra. By convention we let $x^n = 0$ for all $n < 0$.

Let $k = \mathbb{C}$ then some example of quantum stochastic flow generators $\phi^i : A_0 \to A_0 \otimes \mathcal{B}(\widehat{k}) \cong A_0 \otimes M_2(\mathbb{C})$ with components $(\pi^i, \delta^i, \tau^i)$ for $i = 1, 2, 3$ are

$$\phi^1(x^n) = x^{n-1} \otimes \begin{pmatrix} n & 0 \\ 0 & 0 \end{pmatrix} \qquad \begin{aligned} \pi^1(x^n) &= x^n \\ \delta^1(x^n) &= 0 \\ \tau^1(x^n) &= nx^{n-1}, \end{aligned}$$

$$\phi^2(x^n) = x^{n-1} \otimes \begin{pmatrix} 0 & n \\ n & 0 \end{pmatrix} + x^{n-2} \otimes \begin{pmatrix} \frac{n(n-1)}{2} & 0 \\ 0 & 0 \end{pmatrix} \qquad \begin{aligned} \pi^2(x^n) &= x^n \\ \delta^2(x^n) &= nx^{n-1} \\ \tau^2(x^n) &= \frac{n(n-1)}{2}x^{n-2}, \end{aligned}$$

$$\phi^3(x^n) = x^{2n} \otimes \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} - x^n \otimes \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \qquad \begin{aligned} \pi^3(x^n) &= x^{2n} \\ \delta^3(x^n) &= x^{2n} - x^n \\ \tau^3(x^n) &= x^{2n} - x^n \end{aligned}$$

for $n \in \mathbb{Z}_+$. The quantum stochastic flow generators ϕ^1 and ϕ^2 are easily constructed by considering the derivative type properties in (2.1.1) and (2.1.2) of Definition 2.1.1. The quantum stochastic flow generator ϕ^3 is of the form in Example 2.1.2.

Definition 2.1.5. Given a quantum stochastic flow generator $\phi : A_0 \to A_0 \otimes \mathcal{B}(\widehat{k})$

we define the iterates of ϕ by $\phi_n : A_0 \to A_0 \otimes \mathcal{B}(\widehat{k})^{\otimes n}$ such that

$$\phi_0 = \mathrm{id}_{A_0}, \quad \phi_{n+1} = (\phi_n \otimes \mathrm{id}_{\widehat{k}}) \circ \phi$$

for all $n \in \mathbb{Z}_+$.

Example 2.1.6. Let $A = C([-1,1])$ and let A_0 be the polynomial subalgebra of A. By convention we let $x^n = 0$ for all $n < 0$.

Consider the quantum stochastic flow generators from Example 2.1.4 then the iterates are

$$\begin{aligned}
\phi_m^1(x^n) &= x^{n-m} \bigotimes_{k=0}^{m-1} \begin{pmatrix} n-k & 0 \\ 0 & 0 \end{pmatrix} = \frac{n!}{(n-m)!} x^{(n-m)} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}^{\otimes m} \\
\phi_m^2(x^n) &= \sum_{k=0}^{m} \sum_{a_1+\cdots+a_m=k} \frac{n!}{(n-m-k)!} \frac{x^{(n-m-k)}}{2^k} \otimes D_m(a_1, \ldots, a_m) \\
\phi_m^3(x^n) &= \sum_{k=0}^{m} \binom{m}{k} (-1)^k x^{n 2^{m-k}} \otimes \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}^{\otimes m}
\end{aligned}$$

for all $n \in \mathbb{Z}_+$ where $D_m : \{0,1\}^m \to M_2(\mathbb{C})^{\otimes m}$ where $D_m(a_1, \ldots, a_m) = D_{a_1} \otimes \cdots \otimes D_{a_m}$ and

$$D_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad D_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Proposition 2.1.7 ([BW15, Definition 2.7])**.** *Let $\zeta, \eta \in \widehat{k}$ and*

$$\phi_\zeta^\eta := (\mathrm{id}_{A_0} \otimes \langle \zeta |) \phi(\,\cdot\,)(\mathrm{id}_{A_0} \otimes |\eta\rangle) : A_0 \to A_0$$

then

$$(\mathrm{id}_{A_0} \otimes \langle \zeta_1 | \otimes \cdots \otimes \langle \zeta_n |) \phi_n(a)(\mathrm{id}_{A_0} \otimes |\eta_1\rangle \otimes \cdots \otimes |\eta_n\rangle) = \phi_{\zeta_1}^{\eta_1} \circ \ldots \phi_{\zeta_n}^{\eta_n}(a)$$

for all $a \in A_0$ and $\zeta_1, \eta_1, \ldots, \zeta_n, \eta_n \in \widehat{k}$.

In general for $\omega \in \mathcal{B}(\widehat{k})^*$ we use the notation $\phi_\omega := (\mathrm{id}_{A_0} \otimes \omega) \circ \phi$.

Example 2.1.8. Let $A = C([-1,1])$ and let A_0 be the polynomial subalgebra. By convention we let $x^n = 0$ for all $n < 0$.

Let us consider the quantum stochastic flow generators from Example 2.1.4 and $\omega = \langle (1,1)| \cdot |(1,1)\rangle \in M_2(\mathbb{C})^*$ then we have that $(\phi^i_\omega)^m = (\mathrm{id}_{A_0} \otimes \omega \otimes \cdots \otimes \omega) \circ \phi^i_m$ and

$$\begin{aligned}
(\phi^1_\omega)^m(x^n) &= x^{n-m} \prod_{k=0}^{m} (n-k) = \frac{n!}{(n-m)!} x^{(n-m)} \\
(\phi^2_\omega)^m(x^n) &= \sum_{j=0}^{m} \frac{n!}{(n-m-j)!} \frac{x^{(n-m-j)}}{2^j} \left(\binom{m}{j} 2^j \right) \\
&= \sum_{j=0}^{m} \frac{n!m!}{(n-m-j)!j!(m-j)!} x^{(n-m-j)} \\
(\phi^3_\omega)^m(x^n) &= \left(\sum_{j=0}^{m} \binom{m}{j} (-1)^j x^{2^{m-j}} \right) 4^m
\end{aligned}$$

Using a combinatorial argument we can make a generalised product rule of the type (2.1.3) for all ϕ_n and $n \in \mathbb{Z}_+$ [BW15, Theorem 2.10] originally from [LW03, Section 2.4]. Using the extended product rule we get the following result that gives us a natural *-subalgebra of A_0. This subalgebra can be thought of as the elements that allow exponentiation with respect to a quantum stochastic integral.

Proposition 2.1.9 ([BW15, Corollary 2.12])**.** *For a flow generator $\phi : A_0 \to A_0 \otimes \mathcal{B}(\widehat{k})$ let*

$$A_\phi := \{x \in A_0; \|\phi_n(x)\| \leq C_x M_x^n \text{ for some } C_x, M_x > 0 \text{ and all } n \in \mathbb{Z}_+\}.$$

*Then A_ϕ is a unital *-subalgebra of A_0, which is equal to A_0 if A_ϕ contains the generators of A_0.*

Example 2.1.10. Let $A = C([-1,1])$ and let A_0 be the polynomial subalgebra. By convention we let $x^n = 0$ for all $n < 0$.

Let us consider the quantum stochastic flow generators from Example 2.1.4

then

$$\|\phi_m^1(x)\| \leq 1, \quad \|\phi_m^2(x)\| \leq 1, \quad \|\phi_m^3(x)\| \leq 2(4^m)$$

for all $m \in \mathbb{Z}_+$. Hence by Proposition 2.1.9 for each of these quantum stochastic flow generators ϕ^i we have $A_0 = A_{\phi^i}$.

We make a minor generalisation of [BW15, Lemma 2.14] allowing for all functionals in place of functionals of the form $\langle\zeta| \cdot |\eta\rangle$ for $\zeta, \eta \in \widehat{k}$.

Lemma 2.1.11. *Let $\phi : A_0 \to A_0 \otimes \mathcal{B}(\widehat{k})$ be a flow generator. For all $\omega \in \mathcal{B}(\widehat{k})^*$ we have $\phi_\omega(A_\phi) \subseteq A_\phi$ and the series*

$$\exp(z\phi_\omega) := \sum_{n=0}^{\infty} \frac{z^n \phi_\omega^n}{n!}$$

is strongly absolutely convergent on A_ϕ for all $z \in \mathbb{C}$.

Proof. Let $a \in A_\phi$ and $\omega \in \mathcal{B}(\widehat{k})^*$ then $\phi_n(\phi_\omega(a)) = (\mathrm{id}_{A_0} \otimes \omega)\phi_{n+1}(a)$ and

$$\|\phi_n(\phi_\omega(a))\| \leq \|\omega\| C_a M_a^{n+1}$$

and $\phi_\omega(a) \in A_\phi$.

Moreover, for all $a \in A_\phi$ and $\omega_1 \ldots \omega_n \in \mathcal{B}(\widehat{k})^*$ we have that

$$\|\phi_{\omega_1} \circ \ldots \circ \phi_{\omega_n}(a)\| \leq \|\omega_1\| \ldots \|\omega_n\| C_a M_a^n \tag{2.1.4}$$

and strong absolute convergence of $\exp(z\phi_\omega)$ follows. □

Example 2.1.12. Let $A = C([-1, 1])$ and let A_0 be the polynomial subalgebra. By convention we let $x^n = 0$ for all $n < 0$.

Let us consider the quantum stochastic flow generators from Example 2.1.4

and $\omega = \langle (1,1)| \cdot |(1,1)\rangle \in M_2(\mathbb{C})^*$ then

$$\begin{aligned}
\exp(z\phi^1_\omega)(x^n) &= \sum_{k=0}^{\infty} \frac{z^k}{k!} \frac{n!}{(n-k)!} x^{(n-k)} = (z1+x)^n \\
\exp(z\phi^2_\omega)(x^n) &= \sum_{k=0}^{\infty} \frac{z^k}{k!} \sum_{j=0}^{k} \frac{n!k!}{(n-k-j)!j!(k-j)!} x^{(n-k-j)} \\
&= \sum_{j=0}^{\lceil n/2 \rceil} \frac{z^{-j}}{j!} \sum_{k=j}^{n-j} \frac{n!}{(n-k-j)!(k-j)!} z^{k+j} x^{(n-k-j)} \\
\exp(z\phi^3_\omega)(x^n) &= \sum_{k=0}^{\infty} \frac{z^k}{k!} \left(\sum_{j=0}^{k} \binom{k}{j} (-1)^j x^{2^{k-j}} \right) 4^k
\end{aligned}$$

for all $n \in \mathbb{Z}_+$ and $z \in \mathbb{C}$.

2.2 Quantum Stochastic Flows

In this section we take the quantum stochastic flow generators of the previous section and construct quantum stochastic flows. The method uses quantum stochastic integration iteratively. We prove that if the quantum stochastic flow is well behaved on the generators of a *-algebra it can be extended to the associated unital C*-algebra. We also prove that a large class of *-algebras, namely those that are generated by partial isometry matrices, automatically have this property.

Let k be a Hilbert space and $\mathcal{F} \cong \Gamma(L^2(\mathbb{R}_+; k))$ be the symmetric Fock space [Lin05, Section 1.5]. Note that the factorisation

$$\mathcal{F} \cong \mathcal{F}_{[0,a)} \otimes \mathcal{F}_{[a,b)} \otimes \mathcal{F}_{[b,\infty)}$$

for all $0 < a < b$ gives a natural embedding of $\mathcal{B}(\mathcal{F}_{[a,b)})$ into $\mathcal{B}(\mathcal{F})$ given by $T \mapsto I_{\mathcal{F}_{[0,a)}} \otimes T \otimes I_{\mathcal{F}_{[b,\infty)}}$.

For $f \in L^2(\mathbb{R}_+; k)$ we define the the exponential vectors

$$e(f) := (1, f, \frac{f \otimes f}{\sqrt{2}}, \ldots, \frac{f^{\otimes n}}{\sqrt{n!}}, \ldots) \in \mathcal{F}$$

we define the linear span of all exponential vectors

$$\mathcal{E} := \operatorname{Lin}\{e(f); f \in L^2(\mathbb{R}_+; k)\} \subseteq \mathcal{F}.$$

Let H be a Hilbert space. We follow the standard convention of omitting the tensor product notation for elements of $H \otimes \mathcal{E}$. For example $u \otimes e(f) = ue(f) \in H \otimes \mathcal{E}$ for $u \in H$ and $f \in L^2(\mathbb{R}_+; k)$. For any $f \in L^2(\mathbb{R}_+; k)$ we use the notation $f_{t)} = f 1_{[0,t)}$ and $f_{[t} = f 1_{[t,\infty)}$.

For this section $A \subseteq \mathcal{B}(H)$ is a unital C*-algebra and A_0 is a unital *-subalgebra.

Definition 2.2.1. A family of linear operators $(T_t)_{t\geq 0}$ on $H \otimes \mathcal{F}$ with domains including $H \otimes \mathcal{E}$ is called **adapted** if

$$\langle ue(f), T_t(ve(g))\rangle = \left\langle ue(f_{t)}), T_t(ve(g_{t)}))\right\rangle e^{\left\langle f_{[t}, g_{[t}\right\rangle}$$

for all $u, v \in H$ and $f, g \in L^2(\mathbb{R}_+; k)$ and $t \geq 0$.

The main construction needed for this chapter is the quantum stochastic integral, Λ. This extends the L^2 theory of classical stochastic integration of Itô and consists of an amalgamation of four integral operators, annihilation, creation, gauge and time. The following theorems only tell us the main properties of these integrals and how we can iterate them. For a full exposition of this theory see [Lin05, Section 3].

Theorem 2.2.2 ([BW15, Theorem 3.3])**.** *For all $n \in \mathbb{Z}_+$ and $T \in \mathcal{B}(H \otimes \widehat{k}^{\otimes n})$ there exists a family $\Lambda^n(T) = (\Lambda^n_t(T))_{t\geq 0}$ of linear operators on $H \otimes \mathcal{F}$ with domains including $H \otimes \mathcal{E}$, that is adapted and such that*

$$\langle ue(f), \Lambda^n_t(T)(ve(g))\rangle = e^{\langle f,g\rangle} \int_{D_n(t)} \left\langle u \otimes \widehat{f}(\underline{t}), T(v \otimes \widehat{g}(\underline{t}))\right\rangle d\underline{t}$$

where $D_n(t) := \{(t_1, \ldots, t_n); 0 \leq t_1 \leq \cdots \leq t_n \leq t\} \subseteq \mathbb{R}^n_+$ and $\widehat{f}(\underline{t}) := \widehat{f}(t_1) \otimes \cdots \otimes \widehat{f}(t_n) \in \widehat{k}^{\otimes n}$ for $\underline{t} \in D_n(t)$ and $\Lambda^0_t(T) = T \otimes \mathrm{id}_{\mathcal{F}}$ for all $t \geq 0$.

If $f \in L^2(\mathbb{R}_+; k)$ then

$$\|\Lambda_t^n(T)ue(f)\| \leq \frac{K_{f,t}^n}{\sqrt{n!}}\|T\|\|ue(f)\|$$

where $K_{f,t} := \sqrt{(2+4\|f\|^2)(t+\|f\|^2)}$ and the map

$$\mathbb{R}_+ \to \mathcal{B}(H; H \otimes \mathcal{F}); \quad t \mapsto \Lambda_t^n(T)(\mathrm{id}_H \otimes |e(f)\rangle)$$

is norm continuous.

Example 2.2.3. Consider the quantum stochastic flow generators from Example 2.1.4 where x is the generator of $A = C[-1,1] \subseteq \mathcal{B}(L^2[-1,1])$ then we have the following

$$\Lambda_t^m(\phi_m^1(x))_{ve(g)}^{ue(f)} = \begin{cases} e^{\langle f,g\rangle} \langle u, xv\rangle & m = 0 \\ e^{\langle f,g\rangle} \langle u, v\rangle \, t & m = 1 \\ 0 & \text{else.} \end{cases}$$

$$\Lambda_t^m(\phi_m^2(x))_{ve(g)}^{ue(f)} = \begin{cases} e^{\langle f,g\rangle} \langle u, xv\rangle & m = 0 \\ e^{\langle f,g\rangle} \langle u, v\rangle \int_0^t (g(s) + \overline{f(s)})ds & m = 1 \\ 0 & \text{else.} \end{cases}$$

$$\Lambda_t^m(\phi_m^3(x))_{ve(g)}^{ue(f)} = e^{\langle f,g\rangle} \left\langle u, \left(\sum_{k=0}^{m} \binom{m}{k} (-1)^k x^{2^{m-k}}\right) v \right\rangle I_m(f,g)$$

where

$$I_m(f,g) = \int_{t_{m-1}}^{t} \cdots \int_0^{t_2} \prod_{i=1}^{m} \overline{(1+f(t_i))}(1+g(t_i))dt_1 dt_2 \ldots dt_m$$

for all $m \in \mathbb{Z}_+$.

Given our quantum stochastic integral and its iterates we can use a Picard iteration type method to solve quantum stochastic differential equations of the form

$$dj_t = (j_t \otimes \mathrm{id}_{\mathcal{B}(\widehat{k})}) \circ \phi \, d\Lambda_t \quad \text{and} \quad j_0 = \mathrm{id}_{\mathcal{F}} \,.$$

Theorem 2.2.4 ([BW15, Theorem 3.5]). *Let $\phi : A_0 \to A_0 \otimes \mathcal{B}(\widehat{k})$ be a flow generator. If $x \in A_\phi$ then the series*

$$j_t(x) := \sum_{n=0}^{\infty} \Lambda_t^n(\phi_n(x))$$

is strongly absolutely convergent on $H \otimes \mathcal{E}$ for all $t \geq 0$, uniformly so on compact subsets of $\mathbb{R}_+$. The map

$$\mathbb{R}_+ \to \mathcal{B}(H; H \otimes \mathcal{F}); \quad t \mapsto j_t(x)(\mathrm{id}_H \otimes |e(f)\rangle)$$

is norm continuous for all $f \in L^2(\mathbb{R}_+; k)$. The family $(j_t(x))_{t\geq 0}$ is adapted and

$$\langle ue(f), j_t(x)ve(g)\rangle = \langle ue(f), xve(g)\rangle + \int_0^t \left\langle ue(f), j_s\left(\phi_{\widehat{g}(s)}^{\widehat{f}(s)}(x)\right) ve(g)\right\rangle ds \quad (2.2.1)$$

for all $u, v \in H, f, g \in L^2(\mathbb{R}_+; k), x \in A_\phi$ and $t \geq 0$. Furthermore,

$$(\mathrm{id}_H \otimes \langle e(f)|)j_t(x)(\mathrm{id}_H \otimes |e(g)\rangle) \in A$$

for all $x \in A_\phi, f, g \in L^2(\mathbb{R}_+; k)$ and $t \geq 0$.

Given a quantum stochastic flow generator ϕ the associated $(j_t)_{t\geq 0}$ is referred to as the associated **quantum stochastic flow**.

Example 2.2.5. Consider the quantum stochastic flow generators from Example 2.1.4 where x is the generator of A then we have the following

$$\begin{aligned}
j_t^1(x)_{ve(g)}^{ue(f)} &= e^{\langle f,g\rangle}(\langle u, xv\rangle + \langle u, v\rangle\, t) \\
j_t^2(x)_{ve(g)}^{ue(f)} &= e^{\langle f,g\rangle}\langle u, xv\rangle + e^{\langle f,g\rangle}\langle u, v\rangle \int_0^t (g(s) + \overline{f(s)})ds \\
j_t^3(x)_{ve(g)}^{ue(f)} &= e^{\langle f,g\rangle} \sum_{m=0}^{\infty} \int_{D_m(\underline{t})} \overline{(1 + f(\underline{t}))}(1 + g(\underline{t}))d\underline{t} \sum_{k=0}^{m} \binom{m}{k} (-1)^k \left\langle u, x^{2^{m-k}} v\right\rangle
\end{aligned}$$

where

$$\int_{D_m(\underline{t})} \overline{(1+f(\underline{t}))}(1+g(\underline{t}))d\underline{t} = \int_{t_{m-1}}^{t} \cdots \int_0^{t_2} \prod_{i=1}^{m} \overline{(1+f(t_i))}(1+g(t_i))dt_1 dt_2 \ldots dt_m$$

for all $m \in \mathbb{Z}_+$. These are the solutions of the quantum stochastic differential equations given by

$$dj_t^1(x) = j_t^1(x) \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} d\Lambda_t, \qquad j_0^1 = \mathrm{id}\,.$$

$$dj_t^2(x) = j_t^2(x) \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} d\Lambda_t, \qquad j_0^2 = \mathrm{id}\,.$$

$$dj_t^3(x) = (j_t^3(x^2) - j_t^3(x)) \otimes \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} d\Lambda_t, \qquad j_0^3 = \mathrm{id}\,.$$

where x is the generator of $C[-1,1]$. Hence, we can deduce that

$$j_t^1(x^n) = x^n \otimes \Gamma\left(n1_{[0,t)}, 0, I, 0\right)$$
$$j_t^2(x^n) = x^n \otimes \Gamma\left(\frac{n(n-1)}{2}1_{[0,t)}, n1_{[0,t)}, I, n1_{[0,t)}\right)$$

for each $n \in \mathbb{N}$ and $t \geq 0$ where Γ is the exponential operator [Lin05, Page 209]. The quantum stochastic flow j_t^3 is less straightforward and is omitted in such generality for this reason. The quantum stochastic flows j^1 and j^2 correspond to a drift process and a Gaussian process respectively.

The maps j_t are unital and linear. We also have the following weak *-homomorphic property. This is pivotal in a number of proofs later on in the chapter.

Proposition 2.2.6 ([BW15, Lemma 3.7]). *Let $\phi : A_0 \to A_0 \otimes \mathcal{B}(\widehat{k})$ be a flow generator and let j_t be as in Theorem 2.2.4 for all $t \geq 0$. If $x, y \in A_\phi$ then*

$$\langle j_t(x)ue(f), j_t(y)ve(g)\rangle = \langle ue(f), j_t(x^*y)ve(g)\rangle$$

for all $u, v \in H$ and $f, g \in L^2(\mathbb{R}_+; k)$. In particular if $x \in A_\phi$ then $j_t(x^*) \subseteq j_t(x)^*$.

Proposition 2.2.7 ([BW15, Lemma 3.8])**.** *If A_ϕ is dense in A then there is at most one family of *-homomorphisms $(j_t)_{t\geq 0}$ from $A \to \mathcal{B}(H \otimes \mathcal{F})$ that satisfies* (2.2.1).

We focus on universal C*-algebras. Paraphrased from the criteria in [Bla06, Section II.8.3], to construct a universal C*-algebra from a set of generators and relations we require:

(i) there exists a unital *-homomorphism from the unital *-algebra with the same generators and relations to $\mathcal{B}(H)$ for some Hilbert space H; there exists a realisation of these generators and relations as bounded operators on some Hilbert space;

(ii) the quantity $\sup_\pi \|\pi(x)\|$ is finite for all generators x where the supremum is taken over the unital *-homomorphisms form part (i).

If the preceding criteria hold then the universal C*-algebra is given by the completion of the *-algebra A_0 with the given generators and relations with respect to the norm

$$\|a\|_u := \sup\{\|\pi(a)\|; \pi : A_0 \to \mathcal{B}(H) \text{ is unital *-homomorphism}\}.$$

Such a C*-algebra A is universal in the sense that if we have another C*-algebra B with elements that satisfy the same relations then there exists a unital *-homomorphism $\pi : A \to B$. In the next chapter we discuss universal C*-algebras in greater detail giving many examples and non-examples.

The following tells us that if our quantum stochastic flow $(j_t)_{t\geq 0}$ is bounded operator valued on the generators of the algebra it is bounded operator valued everywhere and as a result of universality has an extension to the universal C*-algebra.

Lemma 2.2.8. *Let A be the universal unital C*-algebra generated by $(s_i)_{i\in I}$ and a certain set of relations. Let A_0 be the unital *-algebra generated by (s_i) and the same relations. If ϕ is a flow generator such that $A_\phi = A_0$ and $j_t(s_i)$ is bounded for each i and $t \geq 0$ then, for all $t \geq 0$ there exists a unital *-homomorphism $\bar{j}_t : A \to \mathcal{B}(H \otimes \mathcal{F})$.*

Proof. First note that if $x \in A_\phi$ such that $j_t(x)$ is bounded then from Proposition 2.2.6 we have that $j_t(x^*) \subseteq j_t(x)^*$ and thus $j_t(x) \subseteq j_t(x^*)^*$ as A_ϕ is closed under involution. This implies that $j_t(x^*)^*$ is bounded and therefore $j_t(x^*)$ is too.

By the assumption we have that $j_t(s)$ and $j_t(s^*)$ are bounded for all generators s. If s_1 and s_2 are generators then

$$|\langle \psi, j_t(s_1 s_2)\theta\rangle| = |\langle j_t(s_1^*)\theta, j_t(s_2)\psi\rangle| \leq \|j_t(s_1^*)\theta\| \|j_t(s_2)\psi\|$$

for all $\theta, \psi \in H \otimes \mathcal{E}$ and $t \geq 0$. Taking the supremum over θ and ψ with norm equal to one we get that $\|j_t(s_1 s_2)\| \leq \|j_t(s_1^*)\| \|j_t(s_2)\|$.

Using the same argument inductively we see that $j_t(x)$ is bounded for all $x \in A_0$. Moreover $j_t(s_i)$ satisfy the same relations as the generators, so by the universality of A we have a unital *-homomorphism $\pi_t : A \to \mathcal{B}(H \otimes \mathcal{F})$ such that $\pi_t(s_i) = j_t(s_i)$ for all $i \in I$ and $\pi_t := \bar{j}_t$. $\square$

The following result preserves positivity of the diagonal elements of partial isometry matrices on C*-algebras at the algebraic level. For a unital *-algebra A_0 we use the notation $M_n(A_0)$ to denote the algebra of $n \times n$ matrices with components from A_0. Let $a = (a_{ij}) \in M_n(A_0)$ and $b = (b_{ij}) \in M_n(A_0)$ then

$$a \cdot b = \left(\sum_{k=1}^{n} a_{ik} b_{kj} \right), \quad (a_{ij})^* = (a_{ji}^*) \quad \text{and } 1_{M_n(A_0)} = (1_{A_0}\delta_{i,j})$$

Lemma 2.2.9. *Let A_0 be a unital *-algebra and $a \in M_n(A_0)$ such that $a^2 = a = a^*$. The diagonal elements of a are positive such that $a_{ii} = \sum_l a_{il}^* a_{il}$ for each i.*

Proof. This follows directly from the fact that $a = a^2 = a^* a$. $\square$

From Lemma 2.2.8 we know that we can extend our flows to the full C*-algebra if the flow is bounded operator valued on the generators. This is the case for a large class of C*-algebras. That is, the C*-algebras that are generated by the aforementioned partial isometry matrices.

Proposition 2.2.10. *Let $v \in M_n(A_0)$ be a partial isometry in $M_n(A)$ and j_t as in Theorem 2.2.4. Then $j_t(v_{ij}) \in B(H \otimes \mathcal{F})$ for all i and j.*

Proof. As v is a partial isometry we have that $I - v^*v$ is an orthogonal projection with elements in A_0. Hence the diagonal elements $(I - v^*v)_{ii} = 1 - \sum_k v_{ki}^* v_{ki}$ are of the form $\sum_k a_{ik}^* a_{ik}$ for some $a_{ik} \in A_0$ by Lemma 2.2.9. Then as a result of this and Proposition 2.2.6 we get

$$\begin{aligned} 0 &\leq \sum_{k=1}^{n} \langle j_t(a_{ik})\theta, j_t(a_{ik})\theta \rangle \\ &= \langle \theta, j_t((1 - v^*v)_{i,i})\theta \rangle \\ &= \left\langle \theta, j_t(1 - \sum_k v_{ki}^* v_{ki})\theta \right\rangle \\ &= \|\theta\|^2 - \sum_k \|j_t(v_{ki})\theta\|^2 \end{aligned}$$

for all i and $\theta \in H \otimes \mathcal{E}$. Therefore by extension $j_t(v_{ij}) \in B(H \otimes \mathcal{F})$ and is contractive for all i and j. □

Corollary 2.2.11. *Let A be the C*-algebra universally generated by the elements of a family of partial isometry matrices $(p_k \in M_{n_k}(A))_k$ and A_0 the unital *-subalgebra generated by the same elements. If ϕ is a flow generator such that $A_\phi = A_0$ then there exists a family of unital *-homomorphisms $\overline{j}_t : A \to B(H \otimes \mathcal{F})$.*

Remark 2.2.12. The previous result covers a very large class of examples. Including C*-algebras universally generated by (and combinations of)

- unitary matrices (for example compact quantum groups);
- partial isometries (for example graph C*-algebras);

- $(s_i)_{i\in I}$ and J be some collection of finite subsets of some set I such that $\cup J = I$ and for each $p \in J$ at least one of $\sum_{i\in p} s_i^* s_i = 1$ or $\sum_{i\in p} s_i s_i^* = 1$ holds (for example Cuntz algebras and Jones "Pythagorean" C*-algebra used in the talk [Jon].). To see this consider $(s_i)_{i=1}^n$ such that $\sum_{i=1}^n s_i^* s_i = 1$ then

$$u := \begin{pmatrix} s_1 & 0 & \dots & 0 \\ s_2 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ s_n & 0 & \dots & 0 \end{pmatrix}$$

is a partial isometry.

2.3 Noncommutative Quantum Stochastic Flow Examples

The examples of quantum stochastic flows and quantum stochastic flow generators we have already encountered have been on the commutative C*-algebra $C[-1,1]$. In this section we provide examples of quantum stochastic flow generators on a noncommutative C*-algebra. The preceding results of this chapter then ensures the existence of corresponding flows when growth conditions are adhered to. Throughout we make use of the following useful result [FKS16, Lemma 5.8] adapted to quantum stochastic flows generators.

Lemma 2.3.1. *Let A_0 be a unital *-algebra generated by a collection of elements, $a_1, \dots, a_n$ and let (π, δ, τ) be the components of a quantum stochastic flow on A_0. Let B_0 be the quotient of A_0 by the two-sided *-ideal generated by the polynomial relations $r_1(a_1, a_1^*, \dots, a_n, a_n^*) = 0, \dots, r_k(a_1, a_1^*, \dots, a_n, a_n^*) = 0$ and their adjoints.*

If π and δ vanish on $r_1, r_1^*, \dots, r_k, r_k^*$, then π is a representation of B_0 and δ is a π-derivation on B_0. If, moreover, τ vanishes on $r_1, r_1^*, \dots, r_k, r_k^*$, then (π, δ, τ) is the components of a quantum stochastic flow generator on B_0.

Cuntz Algebras

We give a characterisation for all such quantum stochastic flows on the Cuntz algebras $\mathcal{O}^n$.

Definition 2.3.2. Let $n \in \mathbb{N}$. Let $\mathcal{O}^n$ denote the **Cuntz algebras**, that is, the universal C*-algebra generated by $(s_i)_{i=1}^n$ such that

$$s_i^* s_j = \delta_{i,j} 1 \quad \text{and} \quad \sum_{k=1}^n s_k s_k^* = 1.$$

Let $\mathcal{O}_0^n$ denote the dense *-subalgebra with the same generators and relations.

Proposition 2.3.3. *Let k be a Hilbert space, $\pi_i \in \mathcal{O}_0^n \otimes \mathcal{B}(k)$ for $i \in \{1, \dots, n\}$ with $\pi_i^* \pi_j = \delta_{i,j} 1_{\mathcal{O}_0^n \otimes \mathcal{B}(k)}$ and $\sum_k \pi_k \pi_k^* = 1_{\mathcal{O}_0^n \otimes \mathcal{B}(k)}$, $d_i \in \mathcal{O}_0^n \otimes |k\rangle$ for $i \in \{1, \dots, n\}$ and $T_{ij} \in \mathcal{O}_0^n$ such that $T_{ij}^* = T_{ji}$ for $i, j \in \{1, \dots, n\}$. Then there exists a unique quantum stochastic flow generator with component maps that satisfy*

$$\pi(s_j) = \pi_j, \quad \delta(s_j) = d_j \quad \text{and} \quad \tau(s_j - s_k) = 2iT_{jk}.$$

Moreover, all quantum stochastic flow generators on $\mathcal{O}^n$ arise this way.

Proof. Let ϕ be a flow generator. The existence of the relevant elements is trivial.

Conversely let π_i be as above, then the assignment $\pi(s_i) = \pi_i$ extends to a unital *-homomorphism by universality.

Setting $\delta(s_i) = d_i$, $\delta(s_j^*) = -\sum_{k=1}^n \pi_j^* d_k s_k^*$ and $\delta(ab) = \pi(a)\delta(b) + \delta(a)b$ makes a well defined map $\delta : \mathcal{O}_0^n \to \mathcal{O}_0^n \otimes |k\rangle$ because

$$\delta(s_i^* s_j) = \pi_i^* d_j + \left(-\sum_{j=1}^n \pi_i^* d_k s_k^* \right) s_j = 0 = \delta(\delta_{i,j} 1)$$

and

$$\sum_{k=1}^n \delta(s_k s_k^*) = \sum_{k=1}^n \left[\pi_k \left(-\sum_{j=1}^n \pi_k^* d_j s_j^* \right) + d_k s_k^* \right] = -\sum_{j=1}^n d_j s_j^* + \sum_{k=1}^n d_k s_k^* = 0 = \delta(1).$$

We similarly construct τ. Let

$$\tau(s_j) = \sum_{k=1}^{n} \left(i s_k T_{kj} - \frac{s_k}{2} d_k^* d_j \right) = \tau(s_j^*)^*$$

and $\tau(ab) = a\tau(b) + \tau(a)b + \delta^\dagger(a)\delta(b)$ then τ is well defined because

$$\begin{aligned}
\tau(s_j^* s_k) &= \tau(s_j^*) s_k + s_j^* \tau(s_k) + \delta^\dagger(s_j^*)\delta(s_k) \\
&= \left(\sum_{l=1}^{n} \left(i s_l T_{lj} - \frac{s_l}{2} d_l^* d_j \right) \right)^* s_k + s_j^* \left(\sum_{l=1}^{n} \left(i s_l T_{lk} - \frac{s_l}{2} d_l^* d_k \right) \right) + d_j^* d_k \\
&= \sum_{l=1}^{n} \left(-i T_{jl} s_l^* s_k - d_j^* d_l \frac{s_l^*}{2} \right) s_k + s_j^* \left(\sum_{l=1}^{n} \left(i s_l T_{lk} - \frac{s_l}{2} d_l^* d_k \right) \right) + d_j^* d_k \\
&= -i T_{jk} - \frac{d_j^* d_k}{2} + i T_{jk} - \frac{d_j^* d_k}{2} + d_j^* d_k \\
&= 0 \\
&= \tau(\delta_{i,j} 1)
\end{aligned}$$

and

$$\begin{aligned}
\sum_{k=1}^{n} \tau(s_k s_k^*) &= \sum_{k=1}^{n} \left(\tau(s_k) s_k^* + s_k \tau(s_k^*) + \delta^\dagger(s_k)\delta(s_k^*) \right) \\
&= \sum_{k=1}^{n} \left[\left(\sum_{l=1}^{n} \left(i s_l T_{lk} - \frac{s_l}{2} d_l^* d_k \right) \right) s_k^* \right. \\
&\qquad \left. + s_k \left(\sum_{l=1}^{n} \left(i s_l T_{lk} - \frac{s_l}{2} d_l^* d_k \right) \right)^* + \delta^\dagger(s_k)\delta(s_k^*) \right] \\
&= \sum_{k=1}^{n} \left[\sum_{l=1}^{n} \left(i s_l T_{lk} s_k^* - \frac{s_l d_l^* d_k s_k^*}{2} - i s_k T_{kl} s_l^* - \frac{s_k d_k^* d_l s_l^*}{2} \right) \right. \\
&\qquad \left. + \left(- \sum_{l=1}^{n} \pi_k^* d_l s_l^* \right)^* \left(- \sum_{l=1}^{n} \pi_k^* d_l s_l^* \right) \right] \\
&= \sum_{k,l=1}^{n} \left[-\frac{s_l d_l^* d_k s_k^*}{2} - \frac{s_k d_k^* d_l s_l^*}{2} \right] + \sum_{k,l_1,l_2=1}^{n} s_{l_1} d_{l_1}^* \pi_k \pi_k^* d_{l_2} s_{l_2}^* \\
&= 0 \\
&= \tau(1). \qquad \square
\end{aligned}$$

Corollary 2.3.4. *Let k be a Hilbert space, $h_i \in k$, $\pi_i = s_i \otimes I_k$, $d_i = s_i \otimes |h_i\rangle$ and $T_{ij} = 0$ for all $i, j \in \{1, \dots n\}$ as in Proposition 2.3.3 then the associated flow generator ϕ satisfies $A_\phi = A_0$.*

Proof. The iterated flow generator is of the form

$$\phi_n(s_i) = s_i \otimes \begin{pmatrix} -\frac{\|h_i\|^2}{2} & -\langle h_i| \\ |h_i\rangle & 0 \end{pmatrix}^{\otimes n}$$

and the result follows by Proposition 2.1.9. $\square$

Example 2.3.5. let $\mathcal{O}^n \subseteq \mathcal{B}(H)$ for some Hilbert space H and consider the quantum stochastic flow generators from Corollary 2.3.4. Then

$$j_t(s_i)^{ue(f)}_{ve(g)} = \langle u, s_i(v)\rangle\, e^{\langle f,g\rangle} \sum_{m=0}^{\infty} \int_{D_m(\underline{t})} \left(-\frac{\|h_i\|^2}{2} + \langle f(\underline{t}), h_i\rangle - \langle h_i, g(\underline{t})\rangle \right) d\underline{t}$$

for all $t \in \mathbb{R}_+, u, v \in H, f, g \in L^2(\mathbb{R}_+; k)$ and $i = 1, 2, \dots n$. These are the solutions of the quantum stochastic differential equations given by

$$dj_t(s_i) = j_t(s_i) \otimes \begin{pmatrix} -\frac{\|h_i\|}{2} & -\langle h_i| \\ |h_i\rangle & 0 \end{pmatrix} d\Lambda_t, \qquad j_0 = \mathrm{id}\,.$$

where s_i are the generators of $\mathcal{O}^n$. Hence, we can deduce that

$$j_t(s_i) = s_i \otimes W(h_i 1_{[0,t)})$$

for each i and $t \geq 0$ where W is the Fock-Weyl operator [Lin05, Page 209]. Using the properties of W as listed in this reference we see that W is an isometry and

$$j_t(s_i^*) = \left(s_i \otimes W\left(h_i 1_{[0,t)}\right)\right)^* = s_i^* \otimes W\left(-h_i 1_{[0,t)}\right)$$

for each i and $t \geq 0$. Therefore,

$$j_t(s_i^n s_i^{*m}) = s_i \otimes W\left((n-m)h_i 1_{[0,t)}\right)$$

for each i $n \in \mathbb{N}_+$ and $t \in \mathbb{R}_+$. Using that $j_t : \mathcal{O}^n \to \mathcal{B}(H \otimes \mathcal{F})$ is a unital *-homomorphism we can calculate $j_t(a)$ for all $a \in \mathcal{O}^n$.

If $\langle h_i, h_j \rangle \in \mathbb{R}$ for all i, j then there is the following straightforward presentation for the dense unital *-subalgebra spanned by elements of the form $s_{i_1} \ldots s_{i_\alpha} s_{j_\beta}^* \ldots s_{j_1}^*$

$$j_t(s_{i_1} \ldots s_{i_\alpha} s_{j_\beta}^* \ldots s_{j_1}^*) = s_{i_1} \ldots s_{i_\alpha} s_{j_\beta}^* \ldots s_{j_1}^* \otimes W\left(h_{i,j} 1_{[0,t)}\right)$$

for all $\alpha, \beta \in \mathbb{N}$ and $i \in \{1, \ldots n\}^\alpha$ and $j \in \{1, \ldots n\}^\beta$ where

$$h_{i,j} = h_{i_1} + \cdots + h_{i_\alpha} - h_{j_\beta} - \cdots - h_{j_1}.$$

Chapter 3

Lévy Processes on C*-Bialgebras

We specialise the previous chapter to the setting of C*-bialgebras. This includes a detailed exposition of the theory of Lévy processes on *-bialgebras in the style of Schürmann [Sch93, Fra06]. We introduce the deformed biunitary *-bialgebras. These are a generalisation of the universal unitary compact quantum groups of Van Daele and Wang [VDW96]. We provide the standard results of Lévy processes on *-bialgebras following the example of [Fra06]. We proceed to characterise Lévy processes on certain *-bialgebras, including specifically a subclass of the deformed biunitary *-bialgebras referred to as the isometry *-bialgebras. From there we introduce C*-bialgebras. We discuss under what conditions the deformed biunitary *-bialgebras have a universal C*-completion.

Using similar methods to Chapter 2 we show that C*-bialgebras that are generated by partial isometry matrices have the property that Lévy processes are uniquely determined by the *-bialgebraic formulation of Lévy processes. We use this to prove a limit theorem on C*-bialgebras that are generated by partial isometry matrices. We conclude with an in depth investigation into the Lévy processes on the Toeplitz algebra. Using the newly developed C*-bialgebraic setting of Lévy processes to construct concrete Lévy processes on the spectra of commutative sub-C*-bialgebras contained in the Toeplitz algebra.

3.1 Levy Processes on *-Bialgebras

We begin by defining Lévy processes in the purely algebraic setting. Contained is the introduction of a new class of *-bialgebras, the deformed biunitary *-bialgebras. We prove some similarity results for this class. We describe some of the equivalent formulations of Lévy processes on *-bialgebras and introduce the three standard types of Lévy processes on *-bialgebras: drift, Gaussian and Poisson. We also briefly discuss the Lévy-Khintchine decomposition.

*-Bialgebras

Definition 3.1.1. A ***-bialgebra** is a unital *-algebra A with $\Delta : A \to A \otimes A$ and $\epsilon : A \to \mathbb{C}$ unital *-homomorphisms such that

$$(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta \quad \text{and} \quad (\epsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id} = (\mathrm{id} \otimes \epsilon) \circ \Delta.$$

The map Δ is called the **coproduct** or **comultiplication** and ϵ is called the **counit**.

If further there exists a unital antihomomorphism $S : A \to A$ such that

$$m \circ (S \otimes \mathrm{id}) \circ \Delta = \epsilon(\,\cdot\,)1 = m \circ (\mathrm{id} \otimes S) \circ \Delta$$

where $m : A \otimes A \to A$ is the multiplication of the algebra, we call A a **Hopf *-algebra**. The map S is called the **antipode** or the **coinverse**.

Given two *-bialgebras A_1 and A_2, a map $\phi : A_1 \to A_2$ is called a ***-bialgebra morphism** if ϕ is a unital *-homomorphism of algebras such that $\epsilon_{A_2} \circ \phi = \epsilon_{A_1}$ and $(\phi \otimes \phi) \circ \Delta_{A_1} = \Delta_{A_2} \circ \phi$.

Let $\Sigma : A \otimes A \to A \otimes A$ denote the flip map that acts on elementary tensors so that $\Sigma(a \otimes b) = b \otimes a$ for all $a, b \in A$. If $m : A \otimes A \to A$ is the multiplication of the algebra we can define the opposite multiplication as $m^{\mathrm{op}} = m \circ \Sigma$ and the co-opposite comultiplication $\Delta^{\mathrm{cop}} := \Sigma \circ \Delta$.

The *-bialgebra A with the opposite multiplication is denoted A^{op}, with the co-opposite multiplication A^{cop} and with both the opposite multiplication and co-opposite comultiplication $A^{\mathrm{op,cop}}$.

If $m^{\mathrm{op}} = m$ the multiplication is called **commutative** and if $\Delta^{\mathrm{cop}} = \Delta$ we say that the comultiplication is **cocommutative**.

The following example is used repeatedly to show how the definitions relating to Lévy processes on *-bialgebras compare with classical Lévy processes on the real line.

Example 3.1.2. Let $\mathbb{C}[x]$ be the algebra of polynomials of some real variable with complex coefficients, that is the complex algebra with one self-adjoint generator x. This can be given the structure of a *-bialgebra with comultiplication and counit given by the unital *-representation extensions of

$$\Delta(x) = 1 \otimes x + x \otimes 1 \quad \text{and} \quad \epsilon(x) = 0.$$

Let $\mathbb{C}[x, y]$ denote the algebra of polynomials of two commuting real variables with complex coefficients. It is not difficult to see that $\mathbb{C}[x, y] \cong \mathbb{C}[x] \otimes \mathbb{C}[x]$ using the linear extension of the map $x^n y^m \mapsto x^n \otimes x^m$.

Using this algebra isomorphism we can see that the coalgebraic structure of $\mathbb{C}[x]$ mimics the group structure of $\mathbb{R}$ in that given $p \in \mathbb{C}[x]$ we have that $\Delta(p)(s, t) = p(s + t)$ for all $s, t \in \mathbb{R}$ and $\epsilon(p) = p(0)$.

In fact, $\mathbb{C}[x]$ is a Hopf *-algebra with antipode given by $S(x) = -x$.

Further motivating examples are derived from monoids, that is semigroups with an identity element.

Example 3.1.3. let S be a finite semigroup with identity e and let $\mathbb{C}^S$ denote the algebra of complex functions $f : S \to \mathbb{C}$ with pointwise multiplication and addition and involution given by complex conjugation. Let $(\delta_s)_{s\in S}$ be the standard basis on $\mathbb{C}^S$ such that

$$\delta_s(t) = \begin{cases} 1 & s = t \\ 0 & \text{else} \end{cases}$$

for all $s, t \in S$ then $\mathbb{C}^S$ can be given the structure a *-bialgebra with comultiplication and counit given by the linear extension of

$$\Delta(\delta_s) = \sum_{gh=s} \delta_g \otimes \delta_h \quad \text{ and } \quad \epsilon(\delta_s) = \delta_s(e)$$

for all $s \in S$. Note that $\delta_s \otimes \delta_t \mapsto \delta_{(s,t)}$ defines an algebra isomorphism $\mathbb{C}^S \otimes \mathbb{C}^S \cong \mathbb{C}^{S\times S}$.

Using this algebra isomorphism we see that $\Delta(f)(x,y) = f(xy)$ for all $f \in \mathbb{C}^S$ and $x, y \in S$. Similarly the counit is given in terms of the identity of the semigroup $\epsilon(f) = f(e)$ for all $f \in \mathbb{C}^S$.

Example 3.1.4. Let S be an involutive semigroup with identity element, that is there exists a map $\dagger : S \to S$ such that $\dagger(\dagger(s)) = s$ and $\dagger(st) = \dagger(t) \cdot \dagger(s)$. We call this an involution and denote $\dagger(s)$ by $s^\dagger$. If S is a group we take the involution to be the inverse.

The semigroup algebra $\mathbb{C}S$ given by formal finite linear combinations of elements of S is a *-bialgebra. The involution is given by the antilinear extension of the semigroup involution and the comultiplication and counit are given respectively by linear extension of

$$\Delta(s) = s \otimes s \quad \text{ and } \quad \epsilon(s) = 1$$

for all $s \in S$.

The preceding examples are commutative and cocommutative respectively. This is expected because in the general framework of noncommutative mathematics, commutativity of some sort usually implies a relationship with some form of "classical object". Hence, we can think of *-bialgebras as similar to the algebras associated to monoids.

Good examples that are neither commutative nor cocommutative include the Glockner-von Waldenfels algebras introduced in [GvW89], the polynomial algebra of the compact quantum group $SU_q(2)$ introduced in [Wor87] and the Kac-Paljutkin eight dimensional finite quantum group introduced in [KP66]. We proceed to construct a class of *-bialgebras that includes the Glockner-von Waldenfels algebras, $SU_q(2)$ and the polynomial algebras of the universal unitary compact quantum groups introduced by Van Daele and Wang [VDW96].

Notation. Let A be a *-algebra and $M_d(A)$ be the set of $d \times d$ matrices with entries from A. This is itself an algebra given the usual matrix multiplication and pointwise addition.

If $u = (u_{ij})_{i,j=1}^d \in M_d(A)$ then we have the following involutions

$$u^* := (u_{ji}^*)_{i,j=1}^d, \quad u^t := (u_{ji})_{i,j=1}^d \quad \text{and} \quad \overline{u} := (u_{ij}^*)_{i,j=1}^d.$$

Note that for non commutative algebras we do not necessarily have that $(ab)^t = b^t a^t$ for all $a, b \in M_d(A)$.

Also note that $u = \sum_{i,j=1}^d e_{ij} \otimes u_{ij} \in M_d(\mathbb{C}) \otimes A$ where e_{ij} are the standard basis elements of the matrix algebra $M_d(\mathbb{C})$. We denote the natural ampliations of $M_d(\mathbb{C}) \otimes A$ into $M_d(\mathbb{C}) \otimes A \otimes A$ using the standard leg notation:

$$u_{[12]} := \sum_{i,j=1}^d e_{ij} \otimes u_{ij} \otimes 1 \quad \text{and} \quad u_{[13]} := \sum_{i,j=1}^d e_{ij} \otimes 1 \otimes u_{ij}.$$

We can embed $M_d(\mathbb{C})$ into $M_d(A)$ for any unital algebra A using the embedding

$$M_d(\mathbb{C}) \ni (q_{ij})_{i,j=1}^d \mapsto (q_{ij} 1_A)_{i,j=1}^d \in M_d(A).$$

Proposition 3.1.5. *For any $d \in \mathbb{N}$ and $Q_1, Q_2, Q_3, Q_4 \in M_d(\mathbb{C})$ the unital *-algebra $A^d(Q_1, Q_2, Q_3, Q_4)_0$ generated by elements u_{ij} for $i, j \in \{1, \ldots, d\}$ with* *relations*

$$u^*Q_1u = Q_1, \quad uQ_2u^* = Q_2, \quad \overline{u}Q_3^tu^t = Q_3^t, \quad u^tQ_4^t\overline{u} = Q_4^t$$

*can be given the structure of *-bialgebra. The comultiplication and counit take the following values on the generators respectively:*

$$\Delta(u_{ij}) = \sum_{k=1}^{d} u_{ik} \otimes u_{kj}, \qquad \textit{and} \qquad \epsilon(u_{ij}) = \delta_{i,j} \qquad \textit{for all } i, j \in \{1, \ldots, d\}.$$

Proof. Denote $A := A^d(Q_1, Q_2, Q_3, Q_4)_0$. If $v = (v_{ij})_{i,j=1}^d := (\sum_{k=1}^d u_{ik} \otimes u_{kj})_{i,j=1}^d = u_{[12]}u_{[13]}$ and if $I = (\delta_{i,j})_{i,j=1}^d$ satisfy the relations of the algebra then then there exist unital *-homomorphisms $\Delta : A \to A \otimes A$ and $\epsilon : A \to \mathbb{C}$ as above by universality.

The identity matrix trivially satisfies the relations of the algebra as $I = I^* = I^t = \overline{I}$. Hence $\epsilon(u_{ij}) = \delta(u_{ij})$ defines a character on A.

For the existence of the comultiplication we need the following easily proved identities:

$$\begin{aligned} (u_{[12]}u_{[13]})^* &= u_{[13]}^*u_{[12]}^*, \\ (u_{[12]}u_{[13]})^t &= u_{[13]}^tu_{[12]}^t \text{ and} \\ \overline{(u_{[12]}u_{[13]})} &= \overline{u_{[12]}}\ \overline{u_{[13]}}. \end{aligned} \tag{3.1.1}$$

The calculations that v satisfies the relations of the algebra are as follows:

$$\begin{aligned} v^*Q_1v &= u_{[13]}^*u_{[12]}^*Q_1u_{[12]}u_{[13]} & vQ_2v^* &= u_{[12]}u_{[13]}Q_2u_{[13]}^*u_{[12]}^* \\ &= u_{[13]}^*Q_1u_{[13]} & &= u_{[12]}Q_2u_{[12]}^* \\ &= Q_1 & &= Q_2 \\ \overline{v}Q_3^tv^t &= \overline{u_{[12]}}\ \overline{u_{[13]}}Q_3^tu_{[13]}^tu_{[12]}^t & v^tQ_4^t\overline{v} &= u_{[13]}^tu_{[12]}^tQ_4^t\overline{u_{[12]}}\ \overline{u_{[13]}} \\ &= \overline{u_{[12]}}Q_3^tu_{[12]}^t & &= u_{[13]}^tQ_4^t\overline{u_{[13]}} \\ &= Q_3^t & &= Q_4^t. \end{aligned}$$

The coassociativity property of the coproduct is apparent on generating elements as it mirrors matrix multiplication and is easily shown by direct calculation. Therefore, coassociativity holds on all elements by using induction on word length and the unital *-homomorphic property of Δ. The counital property of ϵ holds similarly. $\square$

We call these the **deformed biunitary *-bialgebras**. They are a generalisation of polynomial algebras of universal free unitary quantum groups, as introduced by Van Daele and Wang [VDW96]. We often abbreviate the notation $A^d\left(Q_1, Q_2, Q_3, Q_4\right)_0$ to $A^d(\underline{Q})_0$

Example 3.1.6. Let $d \in \mathbb{N}$ and $Q \in M_d(\mathbb{C})$ be invertible. Then the *-bialgebra $A_u^d(Q)_0 := A^d\left(I, I, Q^t, (Q^t)^{-1}\right)_0$ has an antipode defined by the anti-homomorphic extension of

$$S(u_{ij}) = u_{ji}^*, \quad \text{ and } S(u_{ij}^*) = (Qu^tQ^{-1})_{i,j}$$

for all i, j. This is the polynomial algebra of the universal unitary compact quantum groups.

Example 3.1.7. Let $d \in \mathbb{N}$. We call $A^d\left(I, 0, 0, 0\right)_0$ the $d \times d$ **Isometry *-bialgebra** and denote it $\mathcal{I}(d)_0$. It is easily seen to be non commutative for all d as in all cases $u_{11}^* u_{11} \neq u_{11} u_{11}^*$. For $d \geq 2$ these *-bialgebras are non-cocommutative

$$\Delta(u_{11}) = \sum_{k=1}^{d} u_{1k} \otimes u_{k1} \neq \sum_{k=1}^{d} u_{k1} \otimes u_{1k} = \Sigma \circ \Delta(u_{11}).$$

We can discuss similarities between the deformed biunitary *-bialgebras. It is not difficult to see that $A^d\left(Q_1, Q_2, Q_3, Q_4\right)_0 = A^d\left(z_1 Q_1, z_2 Q_2, z_3 Q_3, z_4 Q_4\right)_0$ for any $z_1, z_2, z_3, z_4 \in \mathbb{C} \setminus \{0\}$.

Proposition 3.1.8. *We have the following isomorphisms of *-bialgebras*

$$\begin{aligned} A^d\left(Q_1, Q_2, Q_3, Q_4\right)_0 &\cong A^d\left(Q_4^t, Q_3^t, Q_2^t, Q_1^t\right)_0 \\ &\cong A^d\left(Q_2, Q_1, Q_4, Q_3\right)_0^{\mathrm{cop}} \\ &\cong A^d\left(Q_3^t, Q_4^t, Q_1^t, Q_2^t\right)_0^{\mathrm{cop}} \end{aligned}$$

such that $u \mapsto \overline{v}$, $u \mapsto v^*$ and $u \mapsto v^t$, respectively, where u and v are the matrices of generators of the respective algebras.

Proof. First we need to check if $\phi(u_{ij}) = v_{ij}^*$ extends to a unital *-homomorphism of algebras $A^d\left(Q_1, Q_2, Q_3, Q_4\right)_0 \to A^d\left(Q_4^t, Q_3^t, Q_2^t, Q_1^t\right)_0$. Noting that $\phi(u) = \overline{v}$ a sample calculation follows:

$$\phi(u)^* Q_1 \phi(u) = \overline{v}^* Q_1 \overline{v} = v^t Q_1 \overline{v} = Q_1.$$

The other three similar calculations verify that ϕ extends to a *-algebra homomorphism. It is clearly invertible as it is an involution.

For all $1 \leq i, j \leq d$

$$\epsilon_2 \circ \phi(u_{ij}) = \epsilon_2(v_{ij}^*) = \delta_{i,j} = \epsilon_1(u_{ij})$$

and therefore $\epsilon_2 \circ \phi = \epsilon_1$ everywhere on the algebra by extension.

Similarly, for all $1 \leq i, j \leq d$

$$\Delta_2 \circ \phi(u_{ij}) = \Delta_2(v_{ij}^*) = \sum_{k=1}^{d} v_{ik}^* \otimes v_{kj}^* = (\phi \otimes \phi) \circ \Delta_1(u_{ij}).$$

A similar argument to the counit proves the identity for all elements of the algebra.

The other two isomorphisms follow similarly, the "co-opposite" arises as a result of the fact that $(u_{[12]} u_{[13]})^* = u_{[13]}^* u_{[12]}^*$ and $(u_{[12]} u_{[13]})^t = u_{[13]}^t u_{[12]}^t$. $\square$

It is known for the universal unitary compact quantum groups that for all unitary matrices $U \in M_d(\mathbb{C})$ that $A_u^d(Q) \cong A_u^d(U^* Q U)$ [VDW96, 2.1 Lemma].

Proposition 3.1.9. *There is an isomorphism of *-bialgebras*

$$A^d\left(Q_1, Q_2, Q_3, Q_4\right)_0 \cong A^d\left(UQ_1U^*, UQ_2U^*, UQ_3U^*, UQ_4U^*\right)_0$$

for any unitary $U \in M_d(\mathbb{C})$ given by $u \mapsto U^*vU$ where u and v are the matrices of generators of the respective algebras.

Proof. First we need to check if $\phi(u_{ij}) = (U^*vU)_{ij}$ can be extended to a unital *-homomorphism of algebras. Two sample calculations follow:

$$\phi(u)^*Q_1\phi(u) = (U^*vU)^*Q_1U^*vU = U^*(v^*UQ_1U^*v)U = U^*(UQ_1U^*)U = Q_1$$

and

$$\begin{aligned}
\phi(u)^tQ_4^t\overline{\phi(u)} &= (U^*vU)^tQ_4^t\overline{(U^*vU)} \\
&= U^tv^t\overline{U}Q_4^tU^t\overline{v}\overline{U} \\
&= U^t(v^t(UQ_4U^*)^t\overline{v})\overline{U} \\
&= U^t(UQ_4U^*)^t\overline{U} \\
&= (U^*UQ_4U^*U)^t \\
&= Q_4^t.
\end{aligned}$$

Two more similar calculations verify that ϕ defines a unital *-homomorphism of algebras.

It is trivial that $\epsilon_2 \circ \phi = \epsilon_1$ on the generators of the algebra. The composition of unital *-homomorphisms is a unital *-homomorphism so the the counit identity holds everywhere.

The comultiplicative property $\Delta_2 \circ \phi = (\phi \otimes \phi) \circ \Delta_1$ holds on generators because $U^*v_{[12]}v_{[13]}U = (U^*v_{[12]}U)(U^*v_{[13]}U)$ and then a similar argument as above proves the identity in general. $\square$

Example 3.1.10 (Cocommutative examples)**.** For $d = 1$ the matrix multiplication in the relations of $A^1(\underline{Q})_0$ is scalar multiplication which reduces the relations to whether or not the single generator u satisfies $u^*u = 1$ or $uu^* = 1$.

For $d = 1$ we also see that the comultiplication is cocommutative. Therefore we should expect classical semigroups to arise.

Let $F := \langle p, q;\ p^\dagger = q\rangle$ be the free involutive semigroup with identity with one generator, $B := \langle p, q;\ pq = e,\ p^\dagger = q\rangle$ be the bicyclic semigroup and $\mathbb{Z}$ be the group of integers. Recall that $\mathbb{C}S$ denotes the semigroup algebra (Example 3.1.4). Then

(i) $A^d\,(0,0,0,0)_0 \cong \mathbb{C}F$.

(ii) If at least one of λ_1 and λ_2 is non-zero then

$$A^1\,(\lambda_1, 0, \lambda_2, 0)_0 \cong A^1\,(0, \lambda_1, 0, \lambda_2)_0 \cong \mathbb{C}B.$$

(iii) If at least one of λ_1 and λ_2 is non-zero and at least one of μ_1 and μ_2 is non-zero then $A^1\,(\lambda_1, \mu_1, \lambda_2, \mu_2)_0 \cong \mathbb{C}\mathbb{Z}$.

Lévy Processes

We introduce Lévy processes on *-bialgebras with three equivalent formulations: convolution semigroups of states, generating functionals and Schürmann triples. This follows the exposition in [Fra06, Section 1]. To begin we need the following standard result.

Proposition 3.1.11. *Given a *-bialgebra A the dual A' is a unital algebra with multiplication given by $\phi_1 * \phi_2 = (\phi_1 \otimes \phi_2) \circ \Delta$ and unit given by ϵ.*

Proof. It is clear that $* : A' \times A' \to A'$ is bilinear. Associativity follows from coassociativity of the comultiplication. If $\phi_1, \phi_2, \phi_3 \in A'$ then

$$\begin{aligned}(\phi_1 * \phi_2) * \phi_3 &= ((\phi_1 \otimes \phi_2) \circ \Delta \otimes \phi_3) \circ \Delta \\ &= (\phi_1 \otimes \phi_2 \otimes \phi_3) \circ (\Delta \otimes \mathrm{id}) \circ \Delta \\ &= (\phi_1 \otimes \phi_2 \otimes \phi_3) \circ (\mathrm{id} \otimes \Delta) \circ \Delta \\ &= (\phi_1 \otimes (\phi_2 \otimes \phi_3) \circ \Delta) \circ \Delta \\ &= \phi_1 * (\phi_2 * \phi_3).\end{aligned}$$

The counit is the unit for this multiplication as

$$\phi * \epsilon = \phi \circ (\mathrm{id} \otimes \epsilon) \circ \Delta = \phi \circ \mathrm{id} = \phi \circ (\epsilon \otimes \mathrm{id}) \circ \Delta = \epsilon * \phi$$

for all $\phi \in A'$ □

We refer to this multiplication as the **convolution**.

Definition 3.1.12. A linear functional $\phi \in A'$ is called

(i) **positive** if $\phi(a^*a) \geq 0$ for all $a \in A$;

(ii) **normalised** if $\phi(1) = 1$.

A linear functional that is positive and normalised is called a **state**.

Note that the convolution of two states is a state [Mey93, Page 198].

For a good notion of probability we use states. It is well understood that linear functionals are an analogue of integrating with respect to complex measures. States are the natural analogue of integrating with respect to a probability measure.

Example 3.1.13. Let $\mathbb{C}[x]$ be as in Example 3.1.2 and let $f : \mathbb{R} \to \mathbb{R}$ be a Lebesgue integrable function such that $\int_{\mathbb{R}} x^n f(x)dx$ exists for all $n \in \mathbb{Z}_+$. Then $\phi(x^n) = \int_{\mathbb{R}} x^n f(x)dx$ defines a real functional on $\mathbb{C}[x]$.

If f is positive except on a Lebesgue null subset of $\mathbb{R}$ then ϕ is a positive functional. If $\int_{\mathbb{R}} f(x)dx = 1$ then ϕ is a normalised functional.

Thus, if f is positive except on a Lebesgue null subset of $\mathbb{R}$ and $\int_R f(x)dx = 1$ then ϕ is a state. In this case f is the probability density function for a distribution with all finite moments.

Example 3.1.14. Let S be a finite semigroup with identity and consider $\mathbb{C}^S$ as in Example 3.1.3. Given $\mu : S \to \mathbb{C}$, setting $\phi(\delta_s) = \mu(s)$ for all $s \in S$ and define a linear functional on $\mathbb{C}^S$ by extending linearly.

If $\mu(s) \in \mathbb{R}$ for all $s \in S$ then ϕ is real. If $\mu(s) \geq 0$ for all $s \in S$ then ϕ is positive. If $\sum_{s\in S} \mu(s) = 1$ then ϕ is normalised.

Thus if both conditions hold then ϕ is a state. In this case μ is a probability measure on S.

We introduce our first equivalent definition of a Lévy process. The definition that follows is a weak one, in the sense that we are defining Lévy processes in terms of their distributions.

Definition 3.1.15. A **convolution semigroup of states** is a family of states on A $(\phi_t)_{t\in\mathbb{R}_+}$ such that the following hold

(i) $\phi_{t+s} = \phi_t * \phi_s$ for all $s, t \in \mathbb{R}_+$;

(ii) $\phi_0 = \epsilon$;

(iii) $\lim_{t\to 0} \phi_t(a) = \epsilon(a)$ for all $a \in A$.

Property (i) encodes a form of time and "space" invariance. The fact there is only one index on the family indicates the "stationary increments" property. Property (ii) tells us the Lévy process starts at the identity. Property (iii) is the weak continuity condition expected of a Lévy processes.

Example 3.1.16. Let $\mathbb{C}[x]$ be as in Example 3.1.2. Given $t \geq 0$ let $f_t : \mathbb{R} \to \mathbb{R}$ be the probability density function for the normal distribution with mean zero and variance t $f_t(x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}$ then ϕ_t defined as in Example 3.1.13 defines a convolution semigroup of states. In fact this is the convolution semigroup of states associated to the standard Brownian motion.

Example 3.1.17. Let $\mathbb{C}B$ be the bicyclic semigroup algebra introduced in Example 3.1.10. Setting $\phi_t(q^n p^m) = e^{-\frac{(n-m)^2 t}{2}}$ for all $n, m \in \mathbb{Z}_+$ and extending by linearity defines a convolution semigroup of states. In the final section of this thesis it shown that this is the convolution semigroup of states associated to Brownian motion on the circle.

Whenever a semigroup that is indexed by $\mathbb{R}_+$ arises the natural question to ask is: "What is the generator?". In this algebraic setting the generator always exists and is of a particularly nice form.

Definition 3.1.18. A functional $L \in A'$ is called a **generating functional** if

$$L(1) = 0, \quad L(a^*) = \overline{L(a)} \quad \text{and} \quad L((a - \epsilon(a)1)^*(a - \epsilon(a)1)) \geq 0$$

for all $a \in A$.

Example 3.1.19. Let $\mathbb{C}[x]$ be as in Example 3.1.2. The functional $L(x^n) = \delta_{n,2}$ defines a generating functional. In other words $L = \frac{1}{2}\frac{\partial^2}{\partial x^2}|_{x=0}$, half times the one dimensional Laplacian operator. Therefore this generating functional corresponds again to the standard Brownian motion.

Example 3.1.20. Let $\mathbb{C}B$ be the bicyclic semigroup algebra introduced in Example 3.1.10. The functional $L(q^n p^m) = \frac{(n-m)^2}{2}$ defines a generating functional.

For any vector space D we denote the set of linear operators by $\mathcal{L}(D)$. For D a pre-Hilbert space we denote the set of **adjointable linear operators** by $\mathcal{L}^\dagger(D)$. That is, the set of all linear operators on D that have an adjoint defined everywhere

$$\mathcal{L}^\dagger(D) := \left\{ T \in \mathcal{L}(D); \begin{array}{l} \text{there exists } T^* \in \mathcal{L}(D) \text{ with } \langle u, Tv\rangle = \langle T^*u, v\rangle \text{ for} \\ \text{all } u, v \in D \end{array} \right\}.$$

Definition 3.1.21. A **Schürmann triple** (ρ, η, L) on a unital *-algebra A with character ϵ consists of a unital *-homomorphism $\rho : A \to \mathcal{L}^\dagger(D)$ for some pre-

Hilbert space D, a $\rho - \epsilon$ cocycle $\eta : A \to D$, i.e. a linear map such that

$$\eta(ab) = \rho(a)\eta(b) + \eta(a)\epsilon(b)$$

and a *-linear functional $L : A \to \mathbb{C}$ such that

$$L(ab) = L(a)\epsilon(b) + \epsilon(a)L(b) + \langle \eta(a^*), \eta(b) \rangle .$$

If the map $\eta : A \to D$ is surjective we call (ρ, η, L) a **surjective Schürmann triple**.

Example 3.1.22. Let $\mathbb{C}[x]$ be as in Example 3.1.2. The triple (ρ, η, L) with associated Hilbert space $\mathbb{C}$ and $\rho(x^n) = \delta_{n,0}$, $\eta(x^n) = \delta_{n,1}$ and $L(x^n) = \delta_{n,2}$ defines a surjective Schürmann triple.

Example 3.1.23. Let $\mathbb{C}B$ be the bicyclic semigroup algebra introduced in Example 3.1.10. The triple (ρ, η, L) with associated Hilbert space $\mathbb{C}$ and $\rho(q^n p^m) = 1$, $\eta(q^n p^m) = n - m$ and $L(q^n p^m) = \frac{(n-m)^2}{2}$ defines a surjective Schürmann triple.

A pair of surjective Schürmann triples (ρ, η, L) and (ρ', η', L') are said to be **unitarily equivalent** if there exists a bijective isometric linear map $U : D \to D'$ such that

$$\rho'(\,\cdot\,)U = U\rho(\,\cdot\,), \quad \eta' = U\eta, \quad \text{and} \quad L' = L.$$

In many cases the representation of the Schürmann triple automatically take values in the bounded operators on the pre-Hilbert space. In this case it is equivalent to consider the Hilbert space $H = \overline{D}$ the completion of the pre-Hilbert space and the unital *-representation as $\rho : A \to \mathcal{B}(H)$. In this setting the U in the unitary equivalence above extends to a unitary operator $U : \overline{D} \to \overline{D'}$.

We provide a very broad condition for when this automatic boundedness holds.

Proposition 3.1.24. *Let A be the *-algebra generated by $(s_i)_{i \in I}$ and a set of polynomial relations $r_1, r_1^*, \ldots r_k, r_k^*$. If $\pi : A \to \mathcal{L}(D)$ is a unital *-homomorphism* *and $\pi(s_i)$ is bounded for each i then π extends to a unital *-homomorphism $\pi : A \to \mathcal{B}(\overline{D})$.*

Proof. This follows the same method as the proof to Lemma 2.2.8. $\square$

Proposition 3.1.25. *Let $u \in M_n(A)$ such that $(u^*u)^2 = u^*u$ and let $\pi : A \to \mathcal{L}(D)$ be a unital *-homomorphism. Then $\pi(u_{ij}) \in \mathcal{B}(D)$ for all i and j.*

Proof. This follows the same method of proof as Proposition 2.2.10. $\square$

The definitions and examples we have encountered in this section, in some cases, are clearly related. The following statements prove that the definitions are in fact equivalent, which in turn shows that the examples in many cases are duplicates of one another.

Lemma 3.1.26 ([Fra06, Lemma 1.6])**.** *Let A be a *-bialgebra.*

(i) *Given a functional L the series*

$$\exp_*(L)(a) := \sum_{n=0} \frac{1}{n!} L^{*n}(a) = \epsilon(a) + L(a) + \frac{1}{2}(L * L)(a) + \dots$$

converges for all $a \in A$.

(ii) *Let $(\phi_t)_{t\geq 0}$ be a convolution semigroup of states. The limit*

$$\lim_{t\to 0} \frac{\phi_t(a) - \epsilon(a)}{t}$$

exists for all $a \in A$.

Proof. In both cases we consider the convolution operators associated to functionals on *-bialgebras. These are given by $\phi \mapsto (\mathrm{id} \otimes \phi) \circ \Delta$. The Fundamental Theorem of Coalgebras tells us that every element of a coalgebra is contained in some finite dimensional subcoalgebra [DNR01, Theorem 1.4.7].

This reduces the lemma to the level of linear operators on finite dimensional vector spaces. The statement of (i) becomes the existence of matrix exponentials and (ii) is the existence of a generator for C_0-semigroup on a finite dimensional Banach space [RS80, Section X.8]. $\square$

Proposition 3.1.27 ([Fra06, Theorem 1.9]). *Let A be a *-bialgebra. There is a one to one correspondence between the following:*

(i) *convolution semigroups of states on A;*

(ii) *generating functionals on A;*

(iii) *surjective Schürmann triples (modulo unitary equivalence) on A.*

Proof. Let $(\phi_t)_{t\geq 0}$ be a convolution semigroup of states. The following defines a generating functional

$$L(a) = \lim_{t\to 0} \frac{\phi_t(a) - \epsilon(a)}{t}$$

for all $a \in A$. This is easily checked: $L(1) = 0$ because $\phi_t(1) = \epsilon(1) = 1$, L is real because both of ϕ_t and ϵ are real and L is conditionally positive because ϕ_t is positive for all $t \in \mathbb{R}_+$.

Let L be a generating functional. We can construct a Schürmann triple using a GNS type construction.

Consider the sesquilinear form on A given by

$$\langle a, b\rangle := L((a - \epsilon(a)1)^*(b - \epsilon(b)1)).$$

Given the null space

$$\mathcal{N}_L := \{a \in A; \langle a, a\rangle = 0\}$$

$D := A/\mathcal{N}_L$ is a pre-Hilbert space. Denote the quotient map $\eta : A \to D$ and $\rho : A \to \mathcal{L}^\dagger(D)$ by the action of left multiplication of A on D, in other words

$$\rho(a)\eta(b - \epsilon(b)1) = \eta(a(b - \epsilon(b)1)).$$

This is well defined because $\mathcal{N}_L$ is an ideal by the Cauchy-Schwarz inequality. These three maps make (ρ, η, L) a Schürmann triple.

Given a Schürmann triple (ρ, η, L) the solution to the quantum stochastic convolution differential equation

$$dj_t = j_s * \begin{pmatrix} L & \langle \eta | \\ |\eta\rangle & \rho - \epsilon \end{pmatrix} \circ d\Lambda_s$$

with initial condition $j_0 = \epsilon 1$ for all $s \in \mathbb{R}_+$ has a solution. Moreover, $\phi_t := \langle \Omega, j_t(\,\cdot\,)\Omega\rangle$ defines a convolution semigroup of states with generator L. This is a technical argument originally proved by Schürmann [Sch93, Theorem 2.5.3]. For a more straightforward and modern approach see [LS05, Section7]. $\square$

Note that given a non-surjective Schürmann triple we can make an associated surjective Schürmann triple by replacing the associated pre-Hilbert space with the image of the cocycle η.

We prove the connection between quantum stochastic flows (Definition 2.1.1) and Lévy processes on C*-bialgebras. For a comprehensive overview of all such algebraic quantum stochastic processes see [LS05].

Proposition 3.1.28. *For A a *-bialgebra there is an injection from the set of Schürmann triples on A to the set of flow generators on A.*

Proof. The mapping from Schürmann triples to the structure maps of flow generators given by

$$(\rho, \eta, L) \mapsto ((\mathrm{id} \otimes \rho) \circ \Delta, (\mathrm{id} \otimes \eta) \circ \Delta, (\mathrm{id} \otimes L) \circ \Delta)$$

is easily seen to be injective by use of the map $\epsilon \otimes \mathrm{id}$. It is a routine calculation that these are indeed the structure maps of a flow generator. $\square$

Types of Lévy Processes

We introduce the three main types of Lévy processes. The definition are originally due to Schürmann as found in [Sch93]. The presentation of this section is similar to [FS16, Section 1.5.3]. We include proofs and examples as we progress. To begin the classification of Lévy process types we need the following decreasing family of ideals

$$K^n = \mathrm{Lin}\{a_1 \dots a_n; a_1, \dots, a_n \in \ker(\epsilon)\}.$$

Proposition 3.1.29 ([FS16, Proposition 1.5.12]). *Let A be a *-bialgebra, L be a generating functional on A with associated surjective Schürmann triple (ρ, η, L) on A and convolution semigroup of states $(\phi_t)_{t\in\mathbb{R}_+}$ on A. The following are equivalent:*

(i) $\eta = 0$;

(ii) $L|_{K^2} = 0$;

(iii) *L is an ϵ-derivation i.e. $L(ab) = L(a)\epsilon(b) + \epsilon(a)L(b)$ for all $a, b \in A$;*

(iv) *The states ϕ_t are homomorphisms for all $t \in \mathbb{R}_+$.*

Proof. (i) $\implies$ (iii) $\implies$ (ii) $\implies$ (i): If $\eta = 0$ then $L(ab) = \epsilon(a)L(b) + L(a)\epsilon(b)$ which easily implies $L|_{K^2} = 0$. If $L|_{K^2} = 0$ then the GNS type construction gives that the associated pre-Hilbert is $\{0\}$ and therefore $\eta = 0$.

(iii) $\implies$ (iv): Note that if L is a ϵ-derivation then by an induction argument we can show

$$L^{*n}(ab) = \sum_{k=0}^{n} \binom{n}{k} L^{*k}(a)L^{*(n-k)}(b).$$

Therefore, by direct calculation

$$\begin{aligned}
\phi_t(ab) &= \sum_{n=0} \frac{t^n}{n!} \sum_{k=0}^{n} \binom{n}{k} L^{*k}(a)L^{*(n-k)}(b) \\
&= \sum_{n=0} \sum_{k=0}^{n} \frac{t^k t^{n-k}}{k!(n-k)!} L^{*k}(a)L^{*(n-k)}(b) \\
&= \left(\sum_{n=0} \frac{t^n}{n!} L^{*n}(a)\right) \left(\sum_{n=0} \frac{t^n}{n!} L^{*n}(b)\right) = \phi_t(a)\phi_t(b).
\end{aligned}$$

(iv) $\implies$ (iii): This is straightforward as

$$\begin{aligned}
L(ab) &= \lim_{t\to 0} \frac{\phi_t(ab) - \epsilon(ab)}{t} \\
&= \lim_{t\to 0} \frac{\phi_t(a)\phi_t(b) - \phi_t(a)\epsilon(b) + \phi_t(a)\epsilon(b) - \epsilon(a)\epsilon(b)}{t} \\
&= \lim_{t\to 0} \frac{\phi_t(a)(\phi_t(b) - \epsilon(b)) + (\phi_t(a) - \epsilon(a))\epsilon(b)}{t} \\
&= \epsilon(a)L(b) + L(a)\epsilon(b). \qquad \square
\end{aligned}$$

A Lévy process in the form of Proposition 3.1.29 is called a **drift**. To illustrate this analogy consider property (iv). Integration with respect to a probability measure is a multiplicative operation as a functional on a suitable algebra of functions if and only if the probability measure is Dirac point mass. That is, a sure probability measure of a specific point. This is why the drift processes are considered the trivial or "predictable" Lévy processes.

The GNS type construction for a generating functional of a drift makes a Schürmann triple given by $(\rho, \eta, L) = (0, 0, L)$ because the associated pre-Hilbert space is given by $D = \{0\}$.

Example 3.1.30. Let $\mathbb{C}[x]$ be as in Example 3.1.2 and let $\phi_t(x^n) = t^n$ for all $t \in \mathbb{R}$. Clearly ϕ_t is a unital *-homomorphism for all $t \in \mathbb{R}$. Also

$$\phi_t * \phi_s(x^n) = \sum_{k=0}^{n} \binom{n}{k} \phi_t(x^k)\phi_s(x^{n-k}) = \sum_{k=0}^{n} \binom{n}{k} t^k s^{n-k} = (t+s)^n = \phi_{t+s}(x^n)$$

for all $t, s \in \mathbb{R}_+$ and $n \in \mathbb{Z}_+$. Weak continuity and $\phi_0(x^n) = \delta_{n,0}$ are trivial from the definition. Therefore, $(\phi_t)_{t\in\mathbb{R}_+}$ defines a drift process.

It is not difficult to see that ϕ_t corresponds to the Dirac point mass at $t \in \mathbb{R}$. for all $t \in \mathbb{R}$.

The generating functional associated to this semigroup of states is $L(x^n) = \delta_{n,1}$, in other words the first derivative with respect to t evaluated at $t = 0$.

Example 3.1.31. Let $\mathbb{C}B$ be the bicyclic semigroup algebra introduced in Example 3.1.10 and let $\phi_t(q^n p^m) = e^{i(n-m)t}$. It is a lot easier to see that this is a convolution semigroup of states. It is less obvious that it is multiplicative, it is still a straightforward calculation.

The associated generating functional is given by $L(q^n p^m) = n - m$, again, the first derivative of $\phi_t(q^n p^m)$ with respect to t evaluated at $t = 0$.

We see that this corresponds to a Dirac point mass on the circle group at the point e^{it}.

Proposition 3.1.32 ([FS16, Proposition 1.5.13])**.** *Let A be a *-bialgebra, L be a generating functional on A with associated Schürmann triple (ρ, η, L) on A. The following are equivalent:*

(i) $L|_{K^3} = 0$*;*

(ii) $L(b^*b) = 0$ *for all* $b \in K^2$;

(iii) $L(abc) = L(ab)\epsilon(c)+L(ac)\epsilon(b)+L(bc)\epsilon(a)-L(a)\epsilon(bc)-L(b)\epsilon(ac)-L(c)\epsilon(ab)$ *for all* $a, b, c \in A$*;*

(iv) $\eta|_{K^2} = 0$*;*

(v) $\eta(ab) = \epsilon(a)\eta(b) + \eta(a)\epsilon(b)$ *for all* $a, b \in A$*;*

(vi) $\rho|_K = 0$*;*

(vii) $\rho = \epsilon 1$*;*

Proof. (i) $\Longrightarrow$ (iii): Rearrange $L((a - \epsilon(a)1)(b - \epsilon(b)1)(c - \epsilon(c))) = 0$.

(iii) $\Longrightarrow$ (i): Trivial.

(i) $\Longrightarrow$ (ii): Trivial.

(ii) $\Longrightarrow$ (iv): For all $b \in K^2$ we have that $\|\eta(b)\|^2 = L(b^*b) = 0$.

(iv) $\Longrightarrow$ (i): Let $a, b, c \in K$ then

$$L(abc) = L(a)\epsilon(bc) + \epsilon(a)L(bc) + \langle \eta(a^*), \eta(bc) \rangle = 0.$$

(iv) $\Longleftrightarrow$ (v): This is similar to (i) $\Longleftrightarrow$ (iii).

(iv) $\implies$ (vi): For all $a, b \in A$ we have that

$$0 = \eta((a - \epsilon(a)1)(b - \epsilon(b)1)) = \rho(a - \epsilon(a)1)\eta(b)$$

as D is defined as the image of η we have that $\rho|_K = 0$

(vi) $\iff$ (vii): This is similar to (i) $\iff$ (iii).

(vii) $\implies$ (v): By definition of Schürmann triple. □

A Lévy process with any of the properties listed in Proposition 3.1.32 is called **Gaussian** or **quadratic**. This definition of quadratic is a generalisation inspired by the definition of Gaussian Lévy processes on compact Lie groups taking Hunt's formula in to consideration [Sch93, Section 5]. For more on Gaussian Lévy processes on compact Lie groups see [App14, Section 5.6].

Example 3.1.33. Let $\mathbb{C}[x]$ be as in Example 3.1.2. The convolution semigroup of states in Example 3.1.16 is a quadratic Lévy process. The generating functional is given by the generating functional in Example 3.1.19.

This generating functional satisfies property (i) of Proposition 3.1.32 as K^3 for $\mathbb{C}[x]$ is the set of polynomials whose lowest order monomial term is three.

The associated Schürmann triple is given in Example 3.1.22.

Example 3.1.34. Let $\mathbb{C}B$ be the bicyclic semigroup algebra introduced in Example 3.1.10.The convolution semigroup of states in Example 3.1.17 is a quadratic Lévy process. The generating functional is given by the generating functional in Example 3.1.20.

The associated Schürmann triple is given in Example 3.1.23. We prove the relationships between these objects later. It is clear from the definition of the Schürmann triple that this is a quadratic Lévy process.

Proposition 3.1.35 ([FS16, Proposition 1.5.14])**.** *Let A be a *-bialgebra and L be a generating functional on A.*

(i) *There exists a state ϕ and $\lambda > 0$ such that*

$$L(a) = \lambda(\phi(a) - \epsilon(a))$$

for all $a \in A$.

(ii) *There exists a Schürmann triple (ρ, η, L) that contains L such that η is trivial i.e.*

$$\eta(a) = (\rho(a) - \epsilon(a)1)w$$

for all $a \in A$ and some $w \in D \setminus \{0\}$.

Proof. (i) $\Longrightarrow$ (ii): Construct the GNS-space (D, ρ, w) for ϕ. Check that (ρ, η, L) with η as defined makes a Schürmann triple.

(ii) $\Longrightarrow$ (i): Let $\phi = \langle w, \rho(\,\cdot\,)w\rangle / \|w\|^2$ and $\lambda = \|w\|^2$. $\square$

A Lévy process with either of the properties in Proposition 3.1.35 is called **Poisson**. Note that in the previous proposition, the GNS construction in (i) $\Longrightarrow$ (ii) may not give a surjective Schürmann triple. As always, this can be alleviated by considering the image of η instead of the pre-Hilbert space associated to the state ϕ. Note that the characterisation of Lévy processes is in terms of surjective Schürmann triples

Example 3.1.36. Let $\mathbb{C}[x]$ be as in Example 3.1.2. The generating functional given by $L(x^n) = 1 - \delta_{n,0}$ defines a Poisson Lévy process on $\mathbb{C}[x]$. It corresponds to the compound Poisson process on the real line which jumps in the positive direction plus one with rate one.

Example 3.1.37. Let $\mathbb{C}B$ be the bicyclic semigroup algebra introduced in Example 3.1.10. The generating functional $L(q^n p^m) = e^{(n-m)i} - 1$ describes a Poisson process on $\mathbb{C}B$.

Lévy-Khintchine Decomposition

Similar to the classical setting we can consider an analogue to the Lévy-Khintchine decomposition, in which we decompose generating functionals into a maximal Gaussian part and a remaining purely non-Gaussian part. In general the Lévy-Khintchine decomposition does not exist for all generating functionals. It is a matter of current research as to which *-bialgebras have the property that all generating functionals have such a decomposition. For a comprehensive overview of this topic see [FGT15]. We give a brief introduction to the topic here.

Definition 3.1.38. A **cocycle pair** (ρ, η) on a *-bialgebra A consists of unital *-representation $\rho : A \to \mathcal{L}(D)$ for some pre-Hilbert space D and a cocycle $\eta : A \to D$ i.e.

$$\eta(ab) = \rho(a)\eta(b) + \eta(a)\epsilon(b)$$

for all $a, b \in A$.

A cocycle pair (ρ, η) is called **surjective** if η is surjective.

A cocycle pair (ρ, η) is called **Gaussian** if $\rho = \epsilon$.

Let (ρ, η) be a surjective cocycle pair with pre-Hilbert space D. Let $H = \overline{D}$ and consider the closed subspaces of H:

$$\begin{aligned} H_G &= \{(\rho(a) - \epsilon(a))v; a \in A, v \in D\}^{\perp} \\ H_R &= \overline{\text{Lin}\{(\rho(a) - \epsilon(a))v; a \in A, v \in D\}} \end{aligned}$$

with projections P_G and P_R respectively. We define unital *-representations $\rho_G : A \to D_G$ and $\rho_R : A \to D_R$ where $D_G = P_G(D)$ and $D_R = P_R(D)$ such that

$$\rho_G(a)P_G(\eta(b)) = \epsilon(a)P_G(\eta(b)) \quad \text{ and } \rho_R(a)P_R(\eta(b)) = \rho(a)\eta(b) - \epsilon(a)P_G(\eta(b))$$

for all $a, b \in A$. If $\eta_G := P_G \circ \eta$ and $\eta_R := P_R \circ \eta$ then (ρ_G, η_G) and (ρ_R, η_R) are cocycle pairs with pre-Hilbert spaces D_G and D_R respectively.

Definition 3.1.39. A Schürmann triple (ρ, η, L) has the **Lévy-Khintchine decomposition property** if there exists generating functionals L_G and L_R such that (ρ_G, η_G, L_G) and (ρ_R, η_R, L_R) are Schürmann triples and $L = L_G + L_R$.

The method of determining the existence of Lévy-Khintchine decompositions of Schürmann triples has generally taken the form of determining which *-bialgebras have various properties that imply all the Schürmann triples have such a decomposition.

Definition 3.1.40. A *-bialgebra A has the **Lévy-Khintchine property (LK)** if for all Schürmann triples on A have a Lévy-Khintchine decomposition.

A *-bialgebra A has the **Gaussian complete (GC)** property if given any Gaussian cocycle pair (ρ, η) on A there exists a generating functional L such that (ρ, η, L) is a Schürmann triple on A.

A *-bialgebra A has the **all cocycle complete (AC)** property if given any cocycle pair (ρ, η) on A there exists a generating functional L such that (ρ, η, L) is a Schürmann triple on A.

It is well known that none of these properties are equivalent and that there exists *-bialgebras that do not have the property (LK). For an overview of counterexamples and further useful and similar properties see [FGT15, Section 4]. In the Schürmann texts [Sch93, Section 5] and [Sch90] (LK), (GC) and (AC) are referred to as (C),(C') and (D) respectively.

Proposition 3.1.41. *For any *-bialgebra*

$$\text{(AC)} \implies \text{(GC)} \implies \text{(LK)}.$$

Proof. It is trivial that (AC) $\implies$ (GC).

If a *-bialgebra has (GC) then given a Schürmann triple (ρ, η, L) we can construct the associated cocycle pairs (ρ_G, η_G) and (ρ_R, η_R). As a result there exists a a generating functional L_G such that (ρ_G, η_G, L_G) is a Schürmann triple. A straightforward calculation shows that if we let $L_R := L - L_G$ then

$$L_R(ab) = L_R(a)\epsilon(b) + \epsilon(a)L_R(b) + \langle \eta_R(a^*), \eta_R(b)\rangle$$

for all $a, b \in A$ and (ρ_R, η_R, L_R) is a Schürmann triple. □

3.2 Classifications of Lévy Processes

We take some examples of *-bialgebras and give straightforward classifications of the possible Lévy processes on them and as a result which (LK) type properties each has. We specifically focus on the isometry *-bialgebras.

We make use of the following useful result which tells us that if the maps in our Schürmann triple are well defined with respect to the relations on our algebra they are well defined everywhere.

Lemma 3.2.1 ([FKS16, Lemma 5.8]). *Let A be a *-algebra generated by a collection of elements, $a_1, \dots, a_n$, let ϵ be a character on A, and let (ρ, η, L) be a Schürmann triple on A. Let B_0 be the quotient of A by the two-sided ideal generated by the polynomial relations $r_1(a_1, a_1^*, \dots, a_n, a_n^*) = 0, \dots, r_k(a_1, a_1^*, \dots, a_n, a_n^*) = 0$ and their adjoints.*

If ρ, ϵ and η vanish on $r_1, r_1^*, \dots, r_k, r_k^*$, then ρ is a representation of B_0 and η is a $\rho-\epsilon$-cocycle on B_0. If, moreover, L vanishes on $r_1, r_1^*, \dots, r_k, r_k^*$, then (ρ, η, L) is a Schürmann triple on B_0.

Lévy Processes on Isometry *-Bialgebras

In [Fra06, Section 2.1] a characterisation of Lévy processes on the unitary *-bialgebras which correspond to the deformed biunitary *-bialgebras $A^d\left(I, I, 0, 0\right)_0$ is given. A similar characterisation holds for the isometry *-bialgebras that is

$$\mathcal{I}(d)_0 := A^d\left(I, 0, 0, 0\right)_0 \cong \left\langle (u_{ij})_{1\le i,j\le d}; \sum_{k=1}^{d} u_{ki}^* u_{kj} = \delta_{i,j} 1 \right\rangle.$$

This section is devoted to this characterisation and related results including the characterisation of Lévy processes on the bicyclic semigroup algebra.

The isometry algebras fit into the framework of Propositions 3.1.24 and 3.1.25 so we can use Hilbert spaces instead of pre-Hilbert spaces for our Schürmann triples.

Theorem 3.2.2. *Let H be a Hilbert space, $V \in M_d(\mathcal{B}(H))$ be an isometry, $A \in M_d(H)$ and $\lambda \in M_d(\mathbb{C})$ be Hermitian. Then there exists a unique Schürmann triple on $\mathcal{I}(d)_0$ such that*

$$\rho(u_{ij}) = P_i V P_j^*, \quad \eta(u_{ij}) = a_{ij}, \text{ and } L(u_{ij} - u_{ji}^*) = 2i\lambda_{ij}.$$

For all i, j where $P_i : H \otimes \mathbb{C}^d \mapsto H \otimes \mathbb{C}e_i$, is the projection into the i-th copy of H.

Furthermore, every Schürmann triple on $\mathcal{I}(d)_0$ arises this way.

Proof. If we have a Schürmann triple on $\mathcal{I}(d)_0$ we can easily see the existence of (V, A, λ) as above.

Let (V, A, λ) be as above. We can define maps on the free algebra with generators (u_{ij}) by the values the linear maps take on the generators:

$$\rho(u_{ij}) = V_{ij}, \quad \eta(u_{ij}) = A_{ij} \text{ and } L(u_{ij}) = i\lambda_{ij} - \frac{1}{2}\sum_{k=1} \langle A_{ki}, A_{kj}\rangle$$

and applying the necessary product and involutive rules:

$$\begin{aligned} \rho(a^*) &= \rho(a)^* & &\text{and} & \rho(ab) &= \rho(a)\rho(b), \\ \eta(u_{ji}^*) &= -\sum_{k=1}^{d} V_{ki}^* A_{kj} & &\text{and} & \eta(ab) &= \rho(a)\eta(b) + \eta(a)\epsilon(b), \\ L(a^*) &= \overline{L(a)} & &\text{and} & L(ab) &= \epsilon(a)L(b) + \langle \eta(a^*), \eta(b)\rangle + L(a)\epsilon(b). \end{aligned}$$

These product rules act associatively i.e. $\eta(a(bc)) = \eta((ab)c)$. Therefore, these maps are well defined on the free algebra. The maps are constructed to be a Schürmann triple on the free algebra. It is now only a matter to check that the maps vanish on the relations of the algebra by Lemma 3.2.1 to get a Schürmann triple on $\mathcal{I}(d)_0$.

The V_{ij} satisfy the relations of the algebra, therefore ρ is a unital *-homomorphism from $\mathcal{I}(d)_0$ to $\mathcal{B}(H)$. Fix i and j

$$\begin{aligned}\eta\left(\sum_k (u_{ki}^* u_{kj} - \delta_{ij,1})\right) &= \sum_k (\rho(u_{ki}^*)\eta(u_{kj}) + \eta(u_{ki}^*)\epsilon(u_{kj})) - 0 \\ &= \sum_k \left(V_{ki}^* A_{kj} - \sum_l V_{li}^* A_{lk}\delta_{k,j}\right) \\ &= 0\end{aligned}$$

Note that $L(u_{ij} + u_{ji}^*) = -\sum_k \langle A_{ki}, A_{kj}\rangle$

$$\begin{aligned}L\left(\sum_k (u_{ki}^* u_{kj} - \delta_{i,j})\right) &= \sum_k \left(L(u_{ki}^*)\epsilon(u_{kj}) + \epsilon(u_{ki}^*)L(u_{kj}) + \langle \eta(u_{ki}), \eta(u_{kj})\rangle\right) - 0 \\ &= L(u_{ji}^* + u_{ij}) + \sum_k \langle A_{ki}, A_{kj}\rangle \\ &= 0\end{aligned}$$

for all i, j. Therefore, (ρ, η, L) defines a Schürmann triple on $\mathcal{I}(d)_0$. $\square$

Contained within the preceding proof is the fact the isometry *-bialgebras have the property (AC) and therefore (LK).

Definition 3.2.3. We call (V, A, λ) from Theorem 3.2.2 a **Lévy process triple** on $\mathcal{I}(d)_0$.

Definition 3.2.4. Given a Schürmann triple (ρ, η, L) we refer to $\overline{\mathrm{Ran}}(\eta)$ as the **associated Hilbert space**.

Proposition 3.2.5. *Given a surjective Schürmann triple (ρ, η, L) on $\mathcal{I}(d)_0$ with Lévy process triple (V, A, λ) the associated Hilbert space is equal to the closure of*

$$K := \mathrm{Lin}\{V_{i_k,j_k}^{\theta_k} \ldots V_{i_1,j_1}^{\theta_1} A_{i_0,j_0}^{\theta_0}; k \in \mathbb{Z}_+, \theta_l \in \{1, *\}, i_l, j_l \in \{1, \ldots, d\}, l \in \{0, \ldots, k\}\}.$$

Proof. It is clear to see that $\eta(u_{ij}) = A_{ij}$ and $\eta(u_{ji}^*) = -\sum_k V_{ki}^* A_{kj}$ are elements of K. Using the cocycle property we see that

$$\eta\left(u_{ij}^{\theta_1} u_{kl}^{\theta_0}\right) = V_{ij}^{\theta_1}\eta\left(u_{kl}^{\theta_0}\right) + \eta\left(u_{ij}^{\theta_1}\right) = V_{ij}^{\theta_1} A_{kl}^{\theta_0} + A_{ij}^{\theta_1} \in K$$

for $i, j, k, l \in \{1, \dots d\}$ and $\theta_0, \theta_1 \in \{1, *\}$. Continuing inductively on word length and using linearity of η we find that $\mathrm{Ran}(\eta) = K$. □

Example 3.2.6. If $d = 1$ the associated Hilbert space is given by $K = \overline{\mathrm{Lin}}\{V^k A; k \in \mathbb{Z}\}$ where $V^{-k} := V^{*k}$ for all $k \in \mathbb{N}$. For example if $V = I$ then $K \cong \mathbb{C}$ for any choice of A.

Proposition 3.2.7. *Let (V_1, A_1, λ_1) and (V_2, A_2, λ_2) be Lévy process triples on $\mathcal{I}(d)_0$ with associated Hilbert spaces K_1 and K_2 respectively. The existence of a unitary operator $U : K_2 \to K_1$ such that $(V_1)_{ij}|_{K_1} = U(V_2)_{ij}|_{K_2}U^*$, $(A_1)_{ij} = U(A_2)_{ij}$ and $\lambda_1 = \lambda_2$ is an equivalence relation between Lévy process triples on $\mathcal{I}(d)_0$.*

If such a unitary operator exists the associated generating functionals are identical.

Proof. The first claim is straightforward, clearly every Lévy process triple on $\mathcal{I}(d)_0$ is related to itself this way by the existence of the identity operator. If T_1 and T_2 are Lévy process triples on $\mathcal{I}(d)_0$ related by the unitary U then T_2 is related to T_1 by the unitary U^*. The transitivity relation follows from the fact that products of unitary operators are themselves unitary.

For the second claim it is enough to check that $L_1(u_{ij} \pm u_{ij}^*) = L_2(u_{ij} \pm u_{ij}^*)$:

$$\begin{aligned} L_1(u_{ij} - u_{ij}^*) &= 2i(\lambda_1)_{ij} \\ &= 2i(\lambda_2)_{ij} \\ &= L_2(u_{ij} - u_{ij}^*) \end{aligned} \qquad \begin{aligned} L_1(u_{ij} + u_{ij}^*) &= -\sum_k \langle (A_1)_{ki}, (A_1)_{kj} \rangle \\ &= -\sum_k \langle U(A_2)_{ki}, U(A_2)_{kj} \rangle \\ &= -\sum_k \langle (A_2)_{ki}, (A_2)_{kj} \rangle \\ &= L_2(u_{ij} + u_{ij}^*). \end{aligned}$$

□

What follows is the classification of certain types of Lévy processes on the isometry *-bialgebras as previously discussed.

Proposition 3.2.8. *The triple (V, A, λ) corresponds to a drift Lévy process if and only if the associated Hilbert space is trivial i.e. $H = \{0\}$.*

Proof. If $H = \{0\}$ then $\eta = 0$ and $V = \mathrm{id}_{\{0\}} = 0$ and therefore $L(ab) = \epsilon(a)L(b) + L(a)\epsilon(b)$. Conversely if L is an ϵ-derivation then $L((a - \epsilon(a)1)^*(a - \epsilon(a)1)) = 0$ for all $a \in A$ and the associated Hilbert space is therefore trivial. □

Proposition 3.2.9. *The triple (V, A, λ) corresponds to a Gaussian Lévy process if and only if V is the identity operator.*

Proof. This is straightforward:

$$(V_{ij}) = (\rho(u_{ij})I_H) = I_{M_d(\mathcal{B}(H))} \iff \rho(u_{ij}) = \delta_{i,j}I_H = \epsilon(u_{ij})I_H. \quad \square$$

Proposition 3.2.10. *The triple (V, A, λ) corresponds to a Poisson Lévy process if and only if*

$$A_{ij} = V_{ij}w - \delta_{i,j}w \text{ and } \lambda_{ij} = \frac{\langle w, \left(V_{ij} - V_{ji}^*\right) w\rangle}{2i}$$

for some $w \in H$.

Proof. If $L(a) = \tau(\phi(a) - \epsilon(a))$ for some state ϕ and $\tau > 0$ then by a GNS construction we get a unital *-representation ρ to some Hilbert space H and some $w \in H$ such that $\phi(a) = \langle w, \rho(a)w\rangle$. As ρ is a unital *-homomorphism we get $\rho(u_{ij}) = V_{ij}$ is in the necessary form. Finally simple calculations show that $\eta(u_{ij}) = (V_{ij} - \delta_{i,j}1)w$ defines a cocycle that completes the Schürmann triple.

Let $A_{ij} = V_{ij}w - \delta_{i,j}w$ for some $w \in H$. Clearly $\eta(u_{ij}) = (\rho(u_{ij}) - \epsilon(u_{ij})w$ and

$$\eta(u_{ij}^*) = -\sum_k V_{kj}^*\eta(u_{ki}) = -\sum_k V_{kj}^*(V_{ki}w - \delta_{ki}w) = V_{ij}^*w - \delta_{i,j}w.$$

If we assume $\eta(a) = (\rho(a) - \epsilon(a))w$ and $\eta(b) = (\rho(b) - \epsilon(b))w$ we can use the product rule on η to see that

$$\eta(ab) = \rho(a)(\rho(b) - \epsilon(b)1)w + (\rho(a) - \epsilon(a)1)w\epsilon(b) = (\rho(ab) - \epsilon(ab)1)w.$$

From here we can use induction on word length and use linearity to show that $\eta(a) = (\rho(a) - \epsilon(a))w$ for all $a \in \mathcal{I}(d)_0$. Fix i and j then

$$\begin{aligned} L(u_{ij}) &= \frac{\left\langle w, (V_{ij} - V_{ji}^*)w \right\rangle - \sum_k \left\langle V_{ki}w - \delta_{ki}w, V_{kj}w - \delta_{kj}w \right\rangle}{2} \\ &= \frac{\left\langle w, (V_{ij} - V_{ji}^*)w \right\rangle - \left\langle w, \delta_{i,j}w - V_{ji}^*w - V_{ij}w + \delta_{i,j}w \right\rangle}{2} \\ &= \langle w, V_{ij}w \rangle - \delta_{i,j}\|w\|^2. \end{aligned}$$

In other words $L(u_{ij}) = \|w\|^2(\phi(u_{ij}) - \epsilon(u_{ij}))$ where ϕ is the state given by $\phi(a) = \langle w, \rho(a)w \rangle / \|w\|^2$. Similarly to before let $a, b \in A$ such that $L(a) = \|w\|^2(\phi(a) - \epsilon(a))$ and $L(b) = \|w\|^2(\phi(b) - \epsilon(b))$. Note that

$$\begin{aligned} \langle \eta(a^*), \eta(b) \rangle &= \langle w, (\rho(a) - \epsilon(a)I)(\rho(b) - \epsilon(b)I)w \rangle \\ &= \langle w, \rho(ab)w \rangle - \epsilon(a) \langle w, \rho(b)w \rangle - \langle w, \rho(a)w \rangle \epsilon(b) + \epsilon(ab)\|w\|^2 \\ &= \|w\|^2(\phi(ab) - \epsilon(a)\phi(b) - \phi(a)\epsilon(b) + \epsilon(ab)). \end{aligned}$$

Therefore

$$\begin{aligned} L(ab) &= L(a)\epsilon(b) + \epsilon(a)L(b) + \langle \eta(a^*), \eta(b) \rangle \\ &= \|w\|^2(\epsilon(a)\phi(b) - \epsilon(ab) + \phi(a)\epsilon(b) - \epsilon(ab)) + \langle \eta(a^*), \eta(b) \rangle \\ &= \|w\|^2(\phi(ab) - \epsilon(ab)). \end{aligned}$$

The proof then follows once more by induction on word length and linearity. $\square$

Gaussian Processes on the Universal Rotation Algebra

We consider the universal rotation *-bialgebra and characterise the Gaussian Lévy processes on it.

Example 3.2.11. Let $A := \langle U, V, Z \text{ unitary }; UV = ZVU, UZ = ZU, VZ = ZV\rangle$ where $T \in A$ is unitary if $T^*T = TT^* = 1$. This is referred to as the polynomial algebra of the universal rotation algebra. This algebra has a basis of the form $(U^nV^mZ^p)_{n,m,p\in\mathbb{Z}}$ where $T^{-k} = T^{*k}$ for $k > 0$ for T unitary.

Therefore, we can give A the structure of a *-bialgebra by extending the maps

$$\Delta(U) = U \otimes U, \qquad \Delta(V) = V \otimes V, \qquad \Delta(Z) = Z \otimes Z$$

and $\epsilon(U) = \epsilon(V) = \epsilon(Z) = 1$.

We apply Lemma 3.2.1 to characterise the Gaussian Lévy processes on this *-bialgebra.

Proposition 3.2.12. *Let H be a Hilbert space, $\eta_U, \eta_V \in H$ and $\lambda_U, \lambda_V \in \mathbb{R}$ then there exists a unique Gaussian Schürmann triple on A such that*

$$\eta(U) = \eta_U \qquad \eta(V) = \eta_V$$
$$L(U - U^*) = 2i\lambda_U \qquad L(V - V^*) = 2i\lambda_V.$$

Furthermore, every Gaussian Schürmann triple on A arises this way.

Proof. Let (ρ, η, L) be a Gaussian Schürmann triple on A. It is clear that the relevant elements exist.

Let $\eta_U, \eta_V \in H$ and $\lambda_U, \lambda_V \in \mathbb{R}$. Let $\rho(U^nV^mZ^p) = \mathrm{id}_H$ for all $n, m, p \in \mathbb{Z}$, this is trivially a unital *-homomorphism. Let $\eta(U) = \eta_U$ and $\eta(U^*) = -\eta_U$, similarly for V and let $\eta(Z) = \eta(Z^*) = 0$. Then using the product rule we see that

$$\eta(U^*U) = \eta(UU^*) = \eta(V^*V) = \eta(VV^*) = \eta(Z^*Z) = \eta(ZZ^*) = 0 = \eta(1)$$

and

$$\eta(UV) = \eta(U) + \eta(V) = \eta(Z) + \eta(V) + \eta(U) = \eta(ZVU).$$

The commutative relations $\eta(UZ) = \eta(ZU)$ and $\eta(VZ) = \eta(ZV)$ are trivial in the Gaussian case.

Similarly let

$$L(U) = i\lambda_U - \frac{1}{2}\|\eta_U\|^2, \quad L(V) = i\lambda_V - \frac{1}{2}\|\eta_V\|^2 \quad \text{and} \quad L(Z) = \langle \eta_V, \eta_U \rangle - \langle \eta_U, \eta_V \rangle$$

Then using the product rule we see that

$$L(U^*U) = L(U^*) + L(U) + \|\eta_U\| = 0 \quad \text{and} \quad L(V^*V) = L(V^*) + L(V) + \|\eta_V\| = 0$$

and

$$L(Z^*Z) = L(Z) + L(Z^*) = 0$$

and similarly for the coisometry relations. The commutativity relations are satisfied as follows

$$\begin{aligned} L(UV) &= L(U) + L(V) - \langle \eta_U, \eta_V \rangle \\ &= L(U) + L(V) + \langle \eta_V, \eta_U \rangle - \langle \eta_U, \eta_V \rangle - \langle \eta_V, \eta_U \rangle \\ &= L(U) + L(V) + L(Z) + \langle \eta(V^*), \eta(U) \rangle + \langle \eta(Z^*), \eta(V) \rangle + \langle \eta(Z^*), \eta(U) \rangle \\ &= L(U) + L(V) + L(Z) + \langle \eta(Z^*), \eta(VU) \rangle \\ &= L(ZVU). \end{aligned}$$

The remaining commutativity relations are straightforward. □

Contained within the preceding proof is the fact that the polynomial algebra of the universal rotation algebra has the property (GC) and therefore (LK).

Corollary 3.2.13. *All Gaussian generating functionals on the polynomial algebra* *of the universal rotation algebra are of the form*

$$L(U^nV^mZ^p) = i(n\lambda_U+m\lambda_V+p(2\operatorname{im}\langle\eta_V,\eta_U\rangle))-\frac{1}{2}(n^2\|\eta_U\|+2nm\langle\eta_U,\eta_V\rangle+m^2\|\eta_V\|^2)$$

where H is a Hilbert space with $\eta_U, \eta_V \in H$ and $\lambda_U, \lambda_V \in \mathbb{R}$.

Proof. First, let us show that

$$L(U^n) = in\lambda_U - \frac{1}{2}n^2\|\eta_U\|$$

for all $n \in \mathbb{N}$ by induction. The base case is trivial. Assume the statement is true for some $n = r$ then using the product rule of L

$$\begin{aligned} L(U^{r+1}) &= L(U) + L(U^r) + \langle\eta(U^*), \eta(U^r)\rangle \\ &= i\lambda_U - \frac{1}{2}\|\eta_U\| + ir\lambda_U - \frac{1}{2}r^2\|\eta_U\| + \langle-\eta_U, r\eta_U\rangle \\ &= i(r+1)\lambda_U - \frac{1}{2}(r+1)^2\|\eta_U\|. \end{aligned}$$

Using the *-linear property of L we then get $L(U^{-n}) = -in\lambda_U - \frac{1}{2}n^2\|\eta_U\|$ for all $n \in \mathbb{N}$. Therefore, $L(U^n) = in\lambda_U - \frac{1}{2}n^2\|\eta_U\|$ for all $n \in \mathbb{Z}$. Similarly we can show $L(V^n) = in\lambda_V - \frac{1}{2}n^2\|\eta_V\|$ and $L(Z^n) = 2in\operatorname{im}\langle\eta_V, \eta_U\rangle$. The result then follows by a final application of the product rule on L. □

It is easy to see that if $\langle\eta_U, \eta_V\rangle \in \mathbb{R}$ then the presentation is simpler still:

$$L(U^nV^mZ^p) = i(n\lambda_U + m\lambda_V) - \frac{1}{2}\langle n\eta_U + m\eta_V, n\eta_U + m\eta_V\rangle.$$

Lévy Processes on the One Generator Free Inverse Monoid

Example 3.2.14. Consider the unital *-algebra with generator p such that $pp^*p = p$. We refer to this as the semigroup algebra of the free inverse semigroup with one generator and identity. This has the *-bialgebra structure as given in Example 3.1.4.

Proposition 3.2.15. *Let H be a Hilbert space, $V \in \mathcal{B}(H)$, $h_1, h_2 \in H$ such that $Vh_2 = -VV^*h_1$ and $\lambda \in \mathbb{R}$. There exists a unique Schürmann triple (ρ, η, L) such that*

$$\rho(p) = V, \quad \eta(p) = h_1, \quad \eta(p^*) = h_2 \quad \text{and} \quad L(p - p^*) = 2i\lambda.$$

Furthermore every Schürmann triple arises this way.

Proof. Let (ρ, η, L) be Schürmann triple. The existence of appropriate V and λ be as above is trivial. Let $h_1 = \eta(p)$ and $h_2 = \eta(p^*)$ then

$$0 = \eta(pp^*p) - \eta(p) = (VV^*h_1 + Vh_2 + h_1) - h_1 \implies Vh_2 = -VV^*h_1.$$

Let V, h_1, h_2 and λ as above. The mapping $\rho(p) = V$ extends to a unital *-homomorphism. Using the product rule we have that

$$\eta(pp^*p) = VV^*h_1 + Vh_2 + h_1 = h_1 = \eta(p)$$

and by Lemma 3.2.1 η is a cocycle on A_0.

Let $L(p) = i\lambda - \frac{\|h_1\|^2 + \|h_2\|^2 - \|VV^*h_1\|^2}{2}$ then note that $Vh_2 = -VV^*h_1$ and $(VV^*)^2 = VV^*$ and $\langle Vh_2, h_1 \rangle = -\langle VV^*h_1, h_1 \rangle = -\|VV^*h_1\|^2$.

$$\begin{aligned}
L(pp^*p) &= L(p) + L(p^*) + L(p) + \langle h_2, h_2 \rangle + \langle h_1, h_1 \rangle + \langle h_2, V^*h_1 \rangle \\
&= i\lambda - \frac{\|h_1\|^2 + \|h_2\|^2 - \|VV^*h_1\|^2}{2} + (-i\lambda - \frac{\|h_1\|^2 + \|h_2\|^2 - \|VV^*h_1\|^2}{2}) \\
&\quad + i\lambda - \frac{\|h_1\|^2 + \|h_2\|^2 - \|VV^*h_1\|^2}{2} + \|h_2\|^2 + \|h_1\|^2 + \langle h_2, V^*h_1 \rangle \\
&= i\lambda - \frac{\|h_1\|^2 + \|h_2\|^2 - 3\|VV^*h_1\|^2 + 2\langle Vh_2, h_1 \rangle}{2} \\
&= i\lambda - \frac{\|h_1\|^2 + \|h_2\|^2 - \|VV^*h_1\|^2}{2} \\
&= L(p).
\end{aligned}$$

Therefore, L is well defined and completes the Schürmann triple. $\square$

A set of examples for the previous theorem is given by letting h_1 be arbitrary and $h_2 = -V^*h_1$. Contained within the preceding proof is the fact the semigroup of the free inverse semigroup with one generator and identity has the property (AC) and therefore (LK).

3.3 Lévy Processes on C*-Bialgebras

In this section we construct C*-bialgebras using the universal C*-completion method. We discuss what is required for this method to be successfully employed and investigate under what conditions the deformed biunitary C*-bialgebras meet these requirements. We prove that Fock space Lévy processes on a C*-bialgebra generated by partial isometry matrices are in one-to-one correspondence with the *-bialgebra Lévy processes on the dense underlying sub-*-bialgebra. We then use this to prove a limit theorem for C*-bialgebras generated by partial isometry matrices.

C*-Bialgebras

We begin by defining the C*-algebraic counterpart to a *-bialgebra and present some examples. For the remainder of the thesis the symbol $\otimes$ denotes the C*-algebraic spatial tensor product [WO93, Appendix T.5].

Definition 3.3.1. A **C*-bialgebra** is a unital C*-algebra A with unital C*-homomorphisms $\Delta : A \to A \otimes A$ and $\epsilon : A \to \mathbb{C}$ such that

$$(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta \quad \text{and} \quad (\epsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id} = (\mathrm{id} \otimes \epsilon) \circ \Delta.$$

The map Δ is called the **coproduct** or **comultiplication** and ϵ is called the **counit**.

Given two C*-bialgebras A_1 and A_2, a map $\phi : A_1 \to A_2$ is called a **C*-bialgebra morphism** if ϕ is a unital C*-homomorphism of algebras such that $\epsilon_{A_2} \circ \phi = \epsilon_{A_1}$ and $(\phi \otimes \phi) \circ \Delta_{A_1} = \Delta_{A_2} \circ \phi$.

Example 3.3.2. Let S be a compact topological semigroup with identity e. Compactness of S implies that $C(S \times S) \cong C(S) \otimes C(S)$. The set $C(S)$ is well known to be a unital C*-algebra and can be given the structure of a C*-bialgebra where $\Delta(f)(x, y) = f(xy)$ and $\epsilon(f) = f(e)$ for all $f \in C(S)$ and $x, y \in S$.

For the previous example we can consider $S = [-1, 1]$ with the usual topology and multiplication which has unit $1 \in [-1, 1]$.

Unlike in the algebraic setting, for C*-algebras it is not enough to consider a set of algebraic generators and relations. The definition of a C*-algebra includes a norm and as a result if we are to describe a C*-algebra by generators there must be some sort of implicit norm bound on those generators.

One way to do this is to take a set of generators in $\mathcal{B}(H)$ for some Hilbert space H and close with respect to the operator norm on $\mathcal{B}(H)$.

Similarly to Chapter 2 we focus on universal C*-algebras. Let us remind ourselves of the criteria for the existence of universal C*-algebra taken from [Bla06, Section II.8.3]. To construct a universal unital C*-algebra from a set of generators and relations we require:

(i) there exists a unital *-homomorphism from the unital *-algebra with the same generators and relations to $\mathcal{B}(H)$ for some Hilbert space H; there exists a realisation of these generators and relations as bounded operators on some Hilbert space;

(ii) the quantity $\sup_\pi \|\pi(x)\|$ is finite for all generators x where the supremum is taken over the unital *-homomorphisms form part (i).

If the preceding criteria hold then the universal C*-algebra is given by the completion of the *-algebra A_0 with the given generators and relations with respect to the norm

$$\|a\|_u := \sup\{\|\pi(a)\|; \pi : A_0 \to \mathcal{B}(H) \text{ is unital *-homomorphism}\}.$$

The appeal of the universal C*-algebra construction is the universal property. Remember that such a C*-algebra A is universal in the sense that if we have another C*-algebra B with elements that satisfy the same relations then there exists a unital *-homomorphism $\pi : A \to B$.

Non-Example 3.3.3. Let a be a generator with relation $a^*a = -1$. This can not be realised as an operator on a Hilbert space. If we assume there was a unital *-homomorphism π to $\mathcal{B}(H)$ for some Hilbert space then note $\|\pi(a)\| = \|-1\| = 1$ and as a result $\pi(a) \neq 0$. Hence, there exists $h \in H \setminus \{0\}$ such that $\pi(a)h \neq 0$. Therefore

$$0 > -\langle h, h\rangle = \langle h, \pi(a^*a)h\rangle = \langle \pi(a)h, \pi(a)h\rangle > 0.$$

Non-Example 3.3.4. Let $\mathbb{C}[x]$ be the algebra generated by one self adjoint element as in Example 3.1.2. For any real number θ we have a unital *-homomorphism $\pi_\theta(x) = \theta \in \mathcal{B}(\mathbb{C}) \cong \mathbb{C}$. Clearly $\sup_\theta(\|\pi_\theta(x)\|)$ is not finite. Therefore, the lack of upper bound stops this algebra from having a universal C*-algebraic counterpart. This is clear from the fact that $\|\pi_\theta(x)\| = |\theta|$ for all $\theta \in \mathbb{R}$ and therefore the universal C*-norm of the generator can not be finite.

Example 3.3.5. If we take the previous non-example and add the additional norm relation that $\|x\| \leq \eta$ for some $\eta \geq 0$ we again have realisations on $\mathcal{B}(\mathbb{C})$ for $\theta \in [-\eta, \eta]$ and explicit upper bound for the generator. Using the functional calculus on the normal element x we get that universal C*-algebra is $C[-\eta, \eta]$.

If we added the relation that $x \geq 0$ then we would similarly have the associated universal C*-algebra is given by $C[0, \eta]$.

Example 3.3.6. Let u be a generator with the relation $u^*u = 1 = uu^*$. This is an example of a generator with an implicit norm bound for some H. Let H be a Hilbert space and let π be a unital *-homomorphism from the algebra generated by u to $\mathcal{B}(H)$. Clearly $\pi(u)$ is a unitary element in a C*-algebra, therefore the norm of $\pi(u)$ is equal to one regardless of choice of representation π.

Again using the functional calculus the universal C*-algebra generated by a single unitary element is given by $C(\mathbb{T})$ where $\mathbb{T} := \{z \in \mathbb{C}; |z| = 1\}$.

Example 3.3.7. Let v be a generator with the relation $v^*v = 1$. Similarly to the previous example this has the implicit norm bound of one. The universal C*-algebra generated by a single isometry is referred to as the **Toeplitz algebra** and is denoted by $\mathcal{T}$.

This C*-algebra is isomorphic to the C*-algebra generated by $R \in \mathcal{B}(\ell_2)$ such that

$$R(x_0, x_1, \dots) = (0, x_0, x_1, \dots).$$

Example 3.3.8. Let p be a generator with the relation $pp^*p = p$. Again any Hilbert space realisation of this algebra maps this generator to a partial isometry and has implicit universal norm one. This C*-algebra has been investigated in [BN12] for example and is shown to have similar but, unsurprisingly, more complicated properties than the Toeplitz algebra.

Example 3.3.9. Let $d \in \mathbb{N}$ and $(p_{ij})_{1 \le i,j \le d}$ be a family of generators with d^2 relations

$$\sum_{k_1,k_2} p_{ik_1} p^*_{k_2k_1} p_{k_2j} = p_{ij}.$$

This is equivalent to a matrix relation $pp^*p = p$, from this we can see that any realisation of these generators on a Hilbert space H can be placed in the components of a partial isometry operator on H^d. In other words if $\pi(p_{ij}) = V_{ij} \in \mathcal{B}(H)$ then

$$V = \begin{pmatrix} V_{11} & \dots & V_{1d} \\ \vdots & \ddots & \vdots \\ V_{d1} & \dots & V_{dd} \end{pmatrix} \in M_d(\mathcal{B}(H)) \cong \mathcal{B}(H^d)$$

is such that $VV^*V = V$. Therefore, $\|V\| = 1$ and using operator space results we get

$$\begin{aligned} \|V_{ij}\| &= \|(0, \dots 0, 1, 0, \dots, 0)^t V (0, \dots 0, 1, 0, \dots, 0)\| \\ &\le \|(0, \dots 0, 1, 0, \dots, 0)^t\| \|V\| \|(0, \dots 0, 1, 0, \dots, 0)\| \\ &= 1. \end{aligned}$$

Hence the algebra generated as above can be completed to a universal C*-algebra.

One of the useful properties of the universal C*-completion in this context is that if the *-algebra we are completing to a universal C*-algebra is a *-bialgebra then the universal property allows the unital *-homomorphisms Δ and ϵ to extend from the *-algebra to the C*-algebra which makes the completion a C*-bialgebra.

Deformed Biunitary C*-Bialgebras

We investigate for what choice of $\underline{Q}$ the deformed biunitary *-bialgebra $A^d(\underline{Q})_0$ can be completed to a universal C*-bialgebra.

Example 3.3.10. The isometry *-bialgebra $\mathcal{I}(d)_0$ from Theorem 3.2.2 can be completed to a universal C*-bialgebra. This is similar to Example 3.3.9. We denote the universal C*-completion of $\mathcal{I}(d)_0$ by $\mathcal{I}(d)$ and call it the **isometry C*-bialgebra** .

Non-Example 3.3.11. Consider the deformed biunitary $A^2\left(\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}, 0, 0, 0\right)_0$ with generating matrix $u = (u_{ij})_{1\leq i,j\leq 2}$. For any $x \in \mathbb{C}$ the map

$$\pi_x(u) = \begin{pmatrix}1 & 0\\ x & x\end{pmatrix}$$

on the generators has a unital *-homomorphic extension such that $\|\pi_x(u_{22})\| = |x|$. Therefore, no upper bound on the norm of the generator u_{22} exists and thus a universal C*-completion is not possible.

Non-Example 3.3.12. Consider the deformed biunitary $A^2\left(\begin{pmatrix}1 & 0\\ 0 & -t\end{pmatrix}, 0, 0, 0\right)_0$ for $t > 0$ with generating matrix $u = (u_{ij})_{1\leq i,j\leq 2}$. For any $x > 1$ the map

$$\pi_x(u) = \begin{pmatrix}x & \sqrt{t(x^2-1)}\\ \sqrt{\frac{x^2-1}{t}} & x\end{pmatrix}$$

on the generators has a unital *-homomorphic extension such that $\|\pi_x(u_{11})\| = x$. Therefore, no upper bound on the norm of the generator u_{11} exists and thus a universal C*-completion is not possible.

Example 3.3.13. Consider the deformed biunitary $A^2\left(\begin{pmatrix}1 & 0\\ 0 & i\end{pmatrix}, 0, 0, 0\right)_0$ for $t > 0$ with generating matrix $u = (u_{ij})_{1\le i,j\le 2}$. Then in the relations we get the equations

$$u_{11}^* u_{11} + i u_{12}^* u_{12} = 1 \quad \text{and } u_{21}^* u_{21} + i u_{22}^* u_{22} = i.$$

By taking the involution on both sides we get

$$u_{11}^* u_{11} - i u_{12}^* u_{12} = 1 \quad \text{and } u_{21}^* u_{21} - i u_{22}^* u_{22} = -i.$$

Therefore $u_{12}^* u_{12} = u_{21}^* u_{21} = 0$ and $u_{11}^* u_{11} = u_{22}^* u_{22} = 1$. Hence, $\|u_{11}\| = \|u_{22}\| = 1$ and $\|u_{12}\| = \|u_{21}\| = 0$.

Example 3.3.14. The deformed biunitary $A^2\left(\begin{pmatrix}1 & \lambda\\ 0 & 1\end{pmatrix}, 0, 0, 0\right)_0$ with generating matrix $u = (u_{ij})_{1\le i,j\le 2}$ for any $|\lambda| < 2$ has the universal C*-completion property. Two of the relations included are

$$u_{11}^* u_{11} + \lambda u_{11}^* u_{12} + u_{12}^* u_{12} = 1 \quad \text{and} \quad u_{21}^* u_{21} + \lambda u_{21}^* u_{22} + u_{22}^* u_{22} = 1.$$

The first of these equations implies that $\lambda u_{11}^* u_{12} = \overline{\lambda} u_{12}^* u_{11}$ because everything else in the equation is self adjoint. Therefore, completing the square we get that

$$0 \le \left(u_{11} + \frac{\lambda}{|\lambda|} u_{12}\right)^* \left(u_{11} + \frac{\lambda}{|\lambda|} u_{12}\right) = 1 + \frac{\lambda}{|\lambda|}(2 - |\lambda|) u_{11}^* u_{12}.$$

Which implies that $-\lambda u_{11}^* u_{12} \le \frac{|\lambda|}{2-|\lambda|} 1$. Therefore,

$$u_{11}^* u_{11} + u_{12}^* u_{12} = 1 - \lambda u_{11}^* u_{12} \le 1 + \frac{|\lambda|}{2 - |\lambda|} 1$$

and

$$u_{11}^*u_{11} \leq 1 + \frac{|\lambda|}{2-|\lambda|}1 \quad \text{and} \quad u_{12}^*u_{12} \leq 1 + \frac{|\lambda|}{2-|\lambda|}1.$$

We can conclude similarly for u_{21} and u_{22}.

Non-Example 3.3.15. Consider the deformed biunitary $A^2\left(\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, 0, 0, 0\right)_0$ with generating matrix $u = (u_{ij})_{1\leq i,j\leq 2}$. For any $x \in \mathbb{R}$ the map

$$\pi_x(u) = \begin{pmatrix} x & x-1 \\ 1-x & 2-x \end{pmatrix}$$

on the generators has a unital *-homomorphic extension such that $\|\pi_x(u_{11})\| = |x|$. Therefore, no upper bound on the norm of the generator u_{11} exists and thus a universal C*-completion is not possible.

We can use Specht's theorem on unitary equivalence of matrices [DJ07, Theorem 2.1, Theorem 2.4 and following comments] to see that for $A^2\left(\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}, 0, 0, 0\right)_0$ a universal C*-completion is not possible for all $|\lambda| = 2$. If we let

$$Q_\lambda = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$$

then it is easily checked that $\mathrm{Tr}(Q_\lambda) = \mathrm{Tr}(Q_\lambda^2) = 2$ and $\mathrm{Tr}(Q_\lambda Q_\lambda^*) = 2 + |\lambda|^2$. Therefore Q_λ is unitarily equivalent to Q_2 for all $|\lambda| = 2$.

Conjecture 3.3.16. *The deformed biunitary* $A^2\left(\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}, 0, 0, 0\right)_0$ *for* $|\lambda| > 2$ *does not have a universal C*-completion.*

The evidence for this conjecture being true comes from the construction of the representations. When $\lambda = 2$ the representations are given by the solutions to the parabolic equations $x^2 + 2xy + y^2 = 1$. Therefore, solutions can go to infinity in some sense. For example the solution given by $(t, 1-t)$ goes to infinity in both coordinates and this is how representations are constructed in Non-Example 3.3.15. Some intuition behind why $-2 < \lambda < 2$ does allow a completion is because the solutions to $x^2 + \lambda xy + y^2 = 1$ make an ellipse, which can not go to infinity in the same way as is demonstrated in Example 3.3.14. The case $|\lambda| > 2$ corresponds to hyperbolas which similarly to the parabolic situation should allow for infinite limits.

As we can see by the preceding examples and non-examples what choice of matrices $\underline{Q}$ allows the *-bialgebra $A^d(\underline{Q})_0$ a universal C*-completion is not entirely clear. The following proposition gives a large class of *-bialgebras that do allow the universal completion.

For $z \in \mathbb{C} \setminus \{0\}$ we call a matrix A **z-(semi)definite** if $z^{-1}A$ is positive (semi)definite. For example a negative definite matrix is a (-1)-definite matrix.

Proposition 3.3.17. *Let $Q = \operatorname{diag}(q_1, \ldots, q_d)$ be z-definite for some $z \neq 0$ then $A^d\left(Q, 0, 0, 0\right)_0$ has a universal C*-completion.*

Proof. Without loss of generality assume $q_l > 0$ for all l then

$$q_i 1 = \sum u_{ki}^* q_k u_{ki} \geq u_{ji}^* q_j u_{ji}$$

for all i and j. Therefore, $u_{ji}^* u_{ji} \leq \frac{q_i}{q_j} 1$ for all i and j. □

Corollary 3.3.18. *The algebra $A^d\left(Q, 0, 0, 0\right)_0$ has a universal C* completion for any z-definite matrix Q.*

Non-Example 3.3.19. Let $Q = \operatorname{diag}(q_1, \ldots, q_d)$ such that $q_l = 0$ for at least one $l \in \{1, \ldots d\}$ then

$$\pi_x(u_{ij}) = \begin{cases} \delta_{i,j} & i \neq l \\ x & i = l \end{cases}$$

can be extended to a unital *-homomorphism for all $x \in \mathbb{R}$ from $A^d\left(Q, 0, 0, 0\right)_0$ to $\mathbb{C}$ and therefore a universal C*-completion is not possible.

Proposition 3.3.20. *Let $Q = \operatorname{diag}(q_1, \ldots, q_d)$ and $R = \operatorname{diag}(r_1, \ldots, r_d)$ be semi-definite and let $I_Q = \{i; q_i \neq 0\}$ and $I_R = \{i; r_i \neq 0\}$. If $I_Q \cup I_R = \{1, \ldots, d\}$ then $A^d\,(Q, 0, 0, R)_0$ has a universal C^* completion.*

Proof. Without loss of generality assume Q and R are positive semi-definite then

$$q_i 1 = \sum_{k \in I_Q} u_{ki}^* q_k u_{ki} \geq u_{ji}^* q_j u_{ji} \quad \text{ or } \quad r_i 1 = \sum_{k \in I_R} u_{ki} r_k u_{ki}^* \geq u_{mi} r_n u_{mi}^*$$

for all $i \in \{1, \ldots, d\}$,$j \in I_Q$ and $m \in I_R$. This establishes an upper bound for the norm for all generators. □

Corollary 3.3.21. *The algebra $A^d\,(Q, 0, 0, R)_0$ has a universal C^* completion for any commuting normal semi-z-definite matrices Q and R such that the simultaneous diagonalisations of Q and R satisfy the condition of Proposition 3.3.20.*

Lévy Processes on C*-Bialgebras

Using similar arguments to those of Chapter 2 we construct Lévy processes on C*-bialgebras. We prove a correspondence between algebraic Lévy processes and Lévy processes on C*-bialgebras when our algebras are generated by partial isometry matrices as in Corollary 2.2.11. Let $\sigma_t : \mathcal{B}(\mathcal{F}) \to \mathcal{B}(\mathcal{F}_{[t,\infty)})$ denote the natural shift operators on the Fock space for all $t \geq 0$ from the beginning of Section 2.2. From here on A refers to a C*-bialgebra and A_0 a dense unital sub-*-bialgebra of A. What follows is a slight alteration to [Fra06, Definition 1.2].

Definition 3.3.22. Let A be a C*-bialgebra. A family of unital *-homomorphisms $(j_t : A \to \mathcal{B}(\mathcal{F}))_{t \in \mathbb{R}_+}$ such that

$$j_0(a) = \epsilon(a)\,\mathrm{id}_{\mathcal{F}} \quad \text{ and } \quad j_{s+t} = (j_s \otimes (\sigma_s \circ j_t)) \circ \Delta$$

for all $s, t \in \mathbb{R}_+$ is called a **Fock space Lévy process on** A.

As opposed to the quantum stochastic flows discussed in Chapter 2 we do not require any restriction on growth similar to Proposition 2.1.9 as all quantum stochastic flows associated to Lévy processes on *-bialgebras automatically satisfy this condition as a result of the fundamental theorem of coalgebras [LS05, Proposition 2.2].

Theorem 3.3.23. *Let A_0 be a *-bialgebra generated by partial isometry matrices and A be its universal C^*-completion. Fock space Lévy processes on A are in one-to-one correspondence with Schürmann triples on A_0.*

Proof. First let $(j_t)_{t\in\mathbb{R}_+}$ be a Fock space Lévy process on A. It is straightforward to see that $\phi_t(a) = \langle e(0), j_t(a)e(0)\rangle$ defines a convolution semigroup of states on A. The restriction of this convolution semigroup of states to the dense *-bialgebra is a convolution semigroup of states in the algebraic sense and therefore we can construct a Schürmann triple using Proposition 3.1.27.

Starting with a Schürmann triple and by [Sch93, Theorem 2.5.3] we have a quantum Lévy process $j_t : A_0 \to \mathcal{L}^\dagger(\mathcal{F})$. Using the generation properties of the algebra and by the same type of calculations in Proposition 2.2.10 we have an extension of these maps to $j_t : A \to \mathcal{B}(\mathcal{F})$. □

As a result any of the Lévy processes previously discussed that were on any *-bialgebra with the partial isometry matrix generation properties can be extended to the universal C*-bialgebra associated to it. This includes all universal compact quantum group, which includes the universal rotation algebra and the isometry C*-bialgebras.

Limit Theorem on Partial Isometry Matrix C*-Bialgebras

We generalise Theorem 6.1.2 in [Sch93]. This is modified to act as a noncommutative analogue of the central limit theorem for C*-bialgebras.

Theorem 3.3.24. *Let A be a C^*-bialgebra and A_0 be a sub *-bialgebra. Let $(\phi_n)_{n\in\mathbb{N}} \subseteq A^*$ be a family of states such that the functional $L(a) = \lim_{n\to\infty} n(\phi_n - \epsilon)(a)$ is well defined on A_0 then L is a generating functional on A_0.*

Furthermore, $\lim_{n\to\infty} \phi_n^{*\lfloor nt\rfloor}|_{A_0} = \exp_*(tL)$.

Proof. Let $\phi'_n := n(\phi_n - \epsilon)$ then $\phi'_n(1) = 0, \phi'_n(a^*) = \overline{\phi'_n(a)}$ and $\phi'_n(b^*b) \geq 0$ for all $n \in \mathbb{N}, a \in A_0$ and $b \in \ker \epsilon$ then $L = \lim_{n\to\infty} \phi'_n$ has the properties of a generating functional.

Using the fundamental theorem of coalgebras we have that for every $a \in A_0$ there is a finite dimensional subcoalgebra B_0 that contains a. Therefore given any linear functional ϕ the operator $T_\phi := (\mathrm{id} \otimes \phi) \circ \Delta$ leaves B_0 invariant. We can then view the statement in terms of matrices.

The conditions of the theorem can be rephrased $\lim_{n\to\infty} n(T_{\phi_n}|_{B_0} - \mathrm{id}_{B_0}) = T_L|_{B_0}$ and the statement as $T_{\phi_n}^{\lfloor nt \rfloor} \to \exp(tT_L)$ as $n \to \infty$ for all $t \in \mathbb{R}$.

$$\begin{aligned}
T_{\phi_n}^{\lfloor nt \rfloor}|_{B_0} &= \left(\mathrm{id}_{B_0} + \frac{nt(T_{\phi_n}|_{B_0} - \mathrm{id}_{B_0})}{nt} \right)^{\lfloor nt \rfloor} \\
&= \sum_{k=0}^{\lfloor nt \rfloor} \binom{\lfloor nt \rfloor}{k} \frac{1}{(nt)^k} (ntT_{\phi_n}|_{B_0})^k \\
&= \sum_{k=0}^{\infty} S_k(nt)
\end{aligned}$$

where

$$S_k(nt) = \begin{cases} \frac{(nt(T_{\phi_n}|_{B_0} - \mathrm{id}_{B_0}))^k}{k!} \left(\frac{\lfloor nt \rfloor}{nt} \right) \left(\frac{\lfloor nt \rfloor}{nt} - \frac{1}{nt} \right) \cdots \left(\frac{\lfloor nt \rfloor}{nt} - \frac{k+1}{nt} \right) & k \leq \lfloor nt \rfloor \\ 0 & k > \lfloor nt \rfloor. \end{cases}$$

Since $nt(T_{\phi_n}|_{B_0} - \mathrm{id}_{B_0})$ converges as $n \to \infty$ there exists $C \geq 0$ such that $\|nt(T_{\phi_n}|_{B_0} - \mathrm{id}_{B_0})\| \leq C$ and therefore $\|S_k(nt)\| \leq C^k/k!$. Also note that the condition of the theorem implies that $\lim_{n\to\infty} nt(T_{\phi_n}|_{B_0} - \mathrm{id}_{B_0}) = tT_L|_{B_0}$ and $\lim_{n\to\infty} \lfloor nt \rfloor / nt = 1$ implies that $\lim_{n\to\infty} S_k(nt) = \frac{(tT_L|_{B_0})^k}{k!}$.

We have that $\|T_{\phi_n}^{\lfloor nt \rfloor}|_{B_0}\| \leq e^C$ and

$$\lim_{n\to\infty} T_{\phi_n}^{\lfloor nt \rfloor}|_{B_0} = \sum_{k=0}^{\infty} \lim_{n\to\infty} S_k(nt) = \sum_{k=0}^{\infty} \frac{(tT_L|_{B_0})^k}{k!} = \exp(tT_L|_{B_0})$$

for all $t \geq 0$. The result then follows by use of the counit. $\square$

Remark 3.3.25. The previous theorem can easily be made into a purely *-bialgebra result and in that setting it is a direct extension of [Sch93, Theorem 6.1.2] that allows for the "time" parameter t.

Corollary 3.3.26. *If A is a C*-bialgebra generated by partial isometry matrices and A_0 the associated dense *-bialgebra then $(\phi_n)_{n\in\mathbb{N}} \subseteq A^*$ such that $L(a) = \lim_{n\to\infty} n(\phi_n - \epsilon)(a)$ is well defined on A_0 then $\lim_{n\to\infty} \phi_n^{*\lfloor nt\rfloor}(a) = \omega_t(a)$ for all $a \in A$ where $(\omega_t)_{t\in\mathbb{R}_+} \subseteq A^*$ is the associated convolution semigroup of states to the generating functional L.*

Proof. For all $a \in A, n \in \mathbb{N}$ and $t \in \mathbb{R}_+$

$$\begin{aligned}
|\omega_t(a) - \phi_n^{*\lfloor nt\rfloor}(a)| &\leq |\omega_t(a) - \omega_t(a_0)| + |\omega_t(a_0) - \phi_n^{*\lfloor nt\rfloor}(a_0)| + |\phi_n^{*\lfloor nt\rfloor}(a_0) - \phi_n^{*\lfloor nt\rfloor}(a)| \\
&\leq |\omega_t(a) - \omega_t(a_0)| + |\omega_t(a_0) - \phi_n^{*\lfloor nt\rfloor}(a_0)| + |\phi_n^{*\lfloor nt\rfloor}(a_0) - \phi_n^{*\lfloor nt\rfloor}(a)| \\
&\leq \|a - a_0\| + |\omega_t(a_0) - \phi_n^{*\lfloor nt\rfloor}(a_0)| + \|a - a_0\|
\end{aligned}$$

for all $a_0 \in A_0$. Taking the limit as $n \to \infty$ and using the density of A_0 in A we get the result. $\square$

Example 3.3.27. Consider the Toeplitz algebra, i.e. the C*-algebra generated by the right shift operator $T \in \mathcal{B}(\ell^2)$. Let $\phi_k(T^nT^{*m}) = \cos\left(\frac{(n-m)}{\sqrt{k}}\right)$, then

$$\lim_{k\to\infty} k(\phi_k - \epsilon)(T^nT^{*m}) = \frac{d}{dx}|_{x=0} \cos(\sqrt{x}(n-m)) = -\frac{(n-m)^2}{2}.$$

Therefore $\lim_{k\to\infty} \phi_k^{\lfloor kt\rfloor}(T^nT^{*m}) = e^{-\frac{(n-m)^2 t}{2}}$. We see in the following section that this gives the standard random walk approximation of the Brownian motion on the circle.

3.4 The Toeplitz Algebra

We consider the Toeplitz algebra, which corresponds to $\mathcal{I}(1)$ from Example 3.3.10. We investigate some related commutative C*-algebras and look at the restriction of Lévy processes on these subalgebras concretely. This famous C*-algebra is a infinite dimensional C*-bialgebra [AGL11] and corresponds to the universal inverse semigroup C*-bialgebra for the bicyclic semigroup.

Let us remind ourselves that the Toeplitz algebra is isomorphic to the C*-algebra generated by the right shift operator $T \in \mathcal{B}(\ell^2)$, and we denote it by $\mathcal{T}$ [Mur90, 3.5.18 Theorem]. The elements

$$T_{n,m} := T^n T^{*m}$$

for $n, m \in \mathbb{Z}_+$ span a dense subalgebra $\mathcal{T}_0$. Further to the characterisation of Schürmann triples for the Toeplitz algebra given by Theorem 3.2.2 for $d = 1$ we can give a relatively straightforward description of how the associated maps act on the basis of the algebra.

Proposition 3.4.1. *Let (V, A, λ) be a Lévy process triple on $\mathcal{T}_0$ then the associated Schürmann triple satisfies*

$$\begin{aligned}
\rho(T_{n,m}) =& V_{n,m} \\
\eta(T_{n,m}) =& \sum_{i=0}^{n-1} V_{i,0}A - \sum_{i=1}^{m} V_{n,i}A \\
L(T_{n,m}) =& i(n-m)\lambda - \frac{n+m}{2}\|A\|^2 \\
&+ \sum_{i=1}^{n}\sum_{j=1}^{m} \langle((1-\delta_{n,0})(1-\delta_{m,0})V_{j,i} - (n-i)V_{0,i} + (m-j)V_{j,0})A, A\rangle
\end{aligned}$$

for all $n, m \in \mathbb{Z}_+$ where $V_{i,j} := V^i V^{*j}$ for all $i, j \in \mathbb{Z}_+$.

Proof. The statement about ρ is trivial. Let us consider the cocycle, η we prove the statement true for $T_{n,0}$ and $T_{0,m}$ inductively and then use the product rule to find $\eta(T_{n,m})$. By definition $\eta(T) = A$ and $\eta(T^*) = -V^*A$ so both base cases are satisfied. Assume the statements are true for $n = r$ then using the product rule we get

$$\eta(T^{r+1}) = \rho(T)\eta(T^r) + \eta(T) \qquad \eta(T^{*(r+1)}) = \rho(T^*)\eta(T^{*r}) + \eta(T^*)$$
$$= V\sum_{i=0}^{r-1} V_{i,0}A + A \qquad = V^*\left(\sum_{i=1}^{r} -V_{0,i}A\right) - V^*A$$
$$= \sum_{i=0}^{r} V_{i,0}A \qquad = -\sum_{i=1}^{r+1} V_{0,i}A$$

and the statements hold by induction. For $n, m \in \mathbb{Z}_+$

$$\begin{aligned}\eta(T_{n,m}) &= \rho(T^n)\eta(T^{*m}) + \eta(T^n)\\ &= V^n\left(\sum_{i=1}^{m} -V_{0,i}A\right) + \sum_{i=0}^{n-1} V_{i,0}A.\end{aligned}$$

We repeat this method for L. The base case is true by definition $L(T) = i\lambda - \|A\|^2/2$. Assume the statement

$$L(T^r) = ir\lambda - \frac{r}{2}\|A\|^2 - \sum_{i=1}^{r} \langle (r-i)V_{0,i}A, A\rangle .$$

Using the product for L we get

$$\begin{aligned}L(T^{r+1}) =&L(T) + L(T^r) + \langle \eta(T^*), \eta(T^r)\rangle\\ =&i\lambda - \frac{1}{2}\|A\|^2 + ir\lambda - \frac{r}{2}\|A\|^2 - \sum_{i=1}^{r} \langle (r-i)V_{0,i}A, A\rangle\\ &+ \left\langle -V^*A, \sum_{i=0}^{r-1} V^iA\right\rangle\\ =&i(r+1)\lambda - \frac{r+1}{2}\|A\|^2 - \sum_{i=1}^{r+1} \langle (r+1-i)V_{0,i}A, A\rangle .\end{aligned}$$

Therefore we have proven the statement about $L(T_{n,0})$ for all $n \in \mathbb{Z}_+$. By use of the *-linear property of L we get the $L(T_{0,n})$ statement for all $n \in \mathbb{Z}_+$:

$$L(T^{*r}) = -ir\lambda - \frac{r}{2}\|A\|^2 - \sum_{i=1}^{r} \langle (r-i)V_{i,0}A, A\rangle .$$

To conclude we consider $L(T_{n,m})$. Using the product rule we get

$$\begin{aligned}L(T_{n,m}) =& L(T^n) + L(T^{*m}) + \langle \eta(T^{*n}), \eta(T^{*m})\rangle \\ =& i(n-m)\lambda - \frac{n+m}{2}\|A\|^2 - \sum_{i=1}^{n} \langle (n-i)V_{0,i}A, A\rangle - \sum_{j=1}^{m} \langle (m-j)V_{j,0}A, A\rangle \\ &+ \left\langle -\sum_{i=1}^{n} V_{0,i}A, -\sum_{j=1}^{m} V_{0,j}A \right\rangle \\ =& i(n-m)\lambda - \frac{n+m}{2}\|A\|^2 + \sum_{i=1}^{n}\sum_{j=1}^{m} \langle V_{j,i} - (n-i)V_{0,i} - (m-j)V_{j,0}A, A\rangle\end{aligned}$$

for $n, m \in \mathbb{N}$. □

The Integers with Infinity

It is easily checked that $T_{n,n}$ is a projection for each $n \in \mathbb{Z}_+$ and these projections strictly decrease with respect to n. That is for $n < m$ we have $T_{n,n}T_{m,m} = T_{m,m}T_{n,n} = T_{m,m}$. For the dense subalgebra we see

$$\begin{aligned}\left(\sum_{i=0}^{n} \alpha_i T_{i,i}\right)\left(\sum_{i=0}^{n} \beta_i T_{i,i}\right) &= \sum_{k=0}^{n} \sum_{\max(i,j)=k} \alpha_i\beta_j T_{k,k} \\ &= \sum_{k=0}^{n} \left[\left(\sum_{i=0}^{k} \alpha_i\right)\left(\sum_{i=0}^{k} \beta_i\right) - \left(\sum_{i=0}^{k-1} \alpha_i\right)\left(\sum_{i=0}^{k-1} \beta_i\right)\right] T_{k.k}.\end{aligned}$$

Let the linear span of this family be denoted $\mathcal{T}_{P,0}$ and the closed linear span of this family be denoted $\mathcal{T}_P$.

Proposition 3.4.2. *The set $\mathcal{T}_P$ is a commutative sub C*-bialgebra of the Toeplitz algebra. Moreover,*

$$\mathcal{T}_P \cong C((\mathbb{Z}_+ \cup \{\infty\}, \min)).$$

Proof. The fact $\mathcal{T}_P$ is a commutative sub-C*-bialgebras is obvious. The isomorphism is given by $\pi : \sum_{i=0}^{\infty} \alpha_i T_{i,i} \mapsto (\sum_{i=0}^{k} \alpha_i)_{k\in\mathbb{Z}_+\cup\{\infty\}}$. Let $A = \sum_{i=0}^{\infty} \alpha_i T_{i,i} \in \mathcal{B}(\ell^2(\mathbb{Z}_+))$ and $(e_k)_{k\in\mathbb{Z}_+}$ the standard basis of $\ell^2(\mathbb{Z}_+)$, we see that π is isometric because for each $k \in \mathbb{Z}_+$ we have that

$$Ae_k = \sum_{i=0}^{k} \alpha_i e_k \implies \|A\| = \sup_{k\in\mathbb{Z}_+} \left|\sum_{i=0}^{k} \alpha_i\right|$$

by using the spectral radius formula. The set $\mathbb{Z}_+ \cup \{\infty\}$ can be given the structure of a semigroup with identity where multiplication is given by the minimum and the ∞ is the identity. Hence we can define a comultiplication and counit on $C(\mathbb{Z}_+ \cup \{\infty\})$ given by

$$\Delta'(f)(m,n) = f(\min\{m,n\}) \text{ and } \epsilon'(f) = f(\infty)$$

respectively. Note that $\pi(T_{i,i})(j) = 1_{i\leq j}$

$$(\pi \otimes \pi)\Delta(T_{i,i})(k,l) = 1_{i\leq k}1_{i\leq l} = 1_{i\leq \min\{k,l\}} = \Delta' \circ \pi(T_{i,i})(k,l)$$

and similarly we can show that $\epsilon = \epsilon' \circ \pi$. $\square$

We can characterise the positive elements of this sub C*-bialgebra

Proposition 3.4.3. *An element $A = \sum_{i=0}^{\infty} \alpha_i T_{i,i}$ is positive in $\mathcal{T}$ if and only if $\sum_{i=0}^{k} \alpha_i \geq 0$ for all $k \in \mathbb{Z}_+ \cup \{\infty\}$.*

Proof. If A is positive in $\mathcal{T}$ then $\pi(A)$ is positive in $C(\mathbb{Z}_+ \cup \{\infty\})$ this gives us that $\sum_{i=0}^{k} \alpha_i \geq 0$ for all $k \in \mathbb{Z}_+ \cup \{\infty\}$.

Conversely if $\sum_{i=0}^{k} \alpha_i \geq 0$ for all $k \in \mathbb{Z}_+ \cup \{\infty\}$ then

$$\sum_{i=0}^{\infty} \alpha_i T_{i,i} = \left(\sum_{i=0}^{\infty} \beta_i T_{i,i}\right)^2$$

where $\beta_i = \sqrt{\sum_{k=0}^{i} \alpha_k} - \sqrt{\sum_{k=0}^{i-1} \alpha_k}$ for each i. $\square$

Using this we characterise the generating functionals on $\mathcal{T}_{P,0}$. Note that as $\epsilon(T_{n,m}) = 1$ for all $n, m \in \mathbb{Z}_+$ we have that $A = \sum_{i=0}^{N} \alpha_i T_{i,i} \in \ker\epsilon$ if and only if $\sum_{i=1}^{N} \alpha_i = 0$.

Proposition 3.4.4. *A linear functional $L : \mathcal{T}_{P,0} \to \mathbb{C}$ is a generating functional if and only if there exists a decreasing sequence $0 = \lambda_0 \geq \lambda_1 \geq \dots$ such that $L(T_{i,i}) = \lambda_i$.*

Proof. First let L be a generating functional. By definition $L(T_{0,0}) = 0$ and $L(T_{i,i}) = L(T_{i,i}^*) = \overline{L(T_{i,i})}$. Therefore $\lambda_0 = 0$ and $\lambda_i \in \mathbb{R}$ for all $i \in \mathbb{Z}_+$. By conditional positivity we have that if $A = \sum_{i=0}^N \alpha_i T_{i,i}$ such that $\sum_{i=0}^N \alpha_i = 0$ then $L(A^*A) \geq 0$. Fix $k \in \mathbb{Z}_+$ and consider $\alpha_k = -\alpha_{k+1} = 1$ and $\alpha_i = 0$ for all other i then $A^*A = A$ and $L(A^*A) = \lambda_k - \lambda_{k+1} \geq 0$ and therefore $\lambda_k \geq \lambda_{k+1}$ for all k.

Conversely if $L(T_{i,i}) = \lambda_i$ as above then this clearly defines a functional such that $L(1) = 0$ and $L(A^*) = \overline{L(A)}$. Finally we need to prove conditional positivity. That is if for any N and $(\alpha_i)_{i=0}^N$ such that $\sum_{i=1}^N \alpha_i = 0$ we have that $L(A^*A) \geq 0$ where $A = \sum_{i=0}^N \alpha_i T_{i,i}$. Then

$$\begin{aligned}
L(A^*A) &= \sum_{i=0}^N \left[\left| \sum_{k=0}^i \alpha_i \right|^2 - \left| \sum_{k=0}^{i-1} \alpha_i \right|^2 \right] \lambda_i \\
&= \sum_{i=1}^{N-1} \left[\left| \sum_{k=0}^i \alpha_i \right|^2 - \left| \sum_{k=0}^{i-1} \alpha_i \right|^2 \right] \lambda_i - \left| \sum_{k=0}^{N-1} \alpha_i \right|^2 \lambda_N \\
&= -|\alpha_0|^2 \lambda_1 + \sum_{i=1}^{N-1} \left| \sum_{k=0}^i \alpha_i \right|^2 (\lambda_i - \lambda_{i+1}) \\
&\geq 0.
\end{aligned}$$

□

We can similarly describe algebraic states.

Proposition 3.4.5. *A linear functional $\phi : \mathcal{T}_{P,0} \to \mathbb{C}$ is a state if and only if there exists a decreasing sequence $1 = \lambda_0 \geq \lambda_1 \geq \dots \geq 0$ such that $L(T_{i,i}) = \lambda_i$.*

Proof. First let ϕ be a state. By definition $\phi(T_{0,0}) = 1$ and $\phi(T_{i,i}) \geq 0$. Therefore $\lambda_0 = 1$ and $\lambda_i \in \mathbb{R}_+$ for all $i \in \mathbb{Z}_+$. By positivity we have that if $A = \sum_{i=0}^N \alpha_i T_{i,i}$ then $\phi(A^*A) \geq 0$. Fix $k \in \mathbb{Z}_+$ and consider $\alpha_k = -\alpha_{k+1} = 1$ and $\alpha_i = 0$ for all other i then $A^*A = A$ and $\phi(A^*A) = \lambda_k - \lambda_{k+1} \geq 0$ and therefore $\lambda_k \geq \lambda_{k+1}$ for all $k \in \mathbb{Z}_+$. Also considering $\alpha_k = 1$ and $\alpha_i = 0$ for all other i by the same method we see that $\lambda_k \geq 0$ for all $k \in \mathbb{Z}_+$.

Conversely if $\phi(T_{i,i}) = \lambda_i$ as above then this clearly defines a functional such that $\phi(1) = 1$. We thus need to prove positivity. That is if for any N and $(\alpha_i)_{i=0}^N$ we have that $\phi(A^*A) \geq 0$ where $A = \sum_{i=0}^N \alpha_i T_{i,i}$. Then

$$\begin{aligned}
\phi(A^*A) &= \sum_{i=0}^N \left[\left| \sum_{k=0}^i \alpha_i \right|^2 - \left| \sum_{k=0}^{i-1} \alpha_i \right|^2 \right] \lambda_i \\
&= |\alpha_0|^2 \lambda_0 + \sum_{i=1}^N \left| \sum_{k=0}^i \alpha_i \right|^2 (\lambda_i - \lambda_{i+1}) \\
&\geq 0. \qquad \square
\end{aligned}$$

Corollary 3.4.6. *A generating functional L on $\mathcal{T}_{P,0}$ is Poisson if and only if its associated sequence from Proposition 3.4.4 is bounded below.*

Proof. If $L = \lambda(\phi - \epsilon)$ for some state ϕ and $\lambda > 0$ where the associated sequence for the state from Proposition 3.4.5 is $(\mu_i) \subseteq [0, 1]$ then $L(T_{i,i}) = \lambda(\mu_i - 1) \in [-\lambda, 0]$ for all i

Conversely let $0 = \lambda_0 \geq \lambda_1 \geq \cdots \geq -\lambda$ for some $\lambda > 0$ then

$$1 = \frac{\lambda}{\lambda} \geq \frac{\lambda_1 + \lambda}{\lambda} \geq \cdots \geq 0$$

and

$$L(T_{i,i}) = \lambda \left(\frac{(\lambda_i + \lambda)}{\lambda} - 1 \right).$$

Therefore, $L = \lambda(\phi - \epsilon)$ where ϕ is the state associated to the sequence $((\lambda + \lambda_i)/\lambda)_{i \in \mathbb{Z}_+}$ by Proposition 3.4.5. $\square$

The following shows that we can construct all Poisson processes on $\mathcal{T}_P$ by using Lévy process triples on $\mathcal{T}_0$.

Proposition 3.4.7. *Let L be a Poisson generating functional on $\mathcal{T}_{P,0}$ with associated sequence $0 = \lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda$ then the Levy triple $(R, Rw - w, 0)$ where $R \in \mathcal{B}(\ell^2(\mathbb{Z}_+))$ is the right shift operator and $w = (\sqrt{\lambda_i - \lambda_{i+1}}) \in \ell^2(\mathbb{Z}_+)$ has an associated generating functional L' on $\mathcal{T}_0$ such that $L'|_{\mathcal{T}_{P,0}} = L$.*

Proof. Without loss of generality assume $\lim_{n\to\infty}\lambda_n = \lambda$, and as a result $\|w\|^2 = -\lambda$. By Proposition 3.2.10 $L'(T_{n,m}) = \langle w, (R_{n,m} - I)w\rangle$. Note that

$$(R_{n,n} - I)w = (-\sqrt{\lambda_0 - \lambda_1}, -\sqrt{\lambda_1 - \lambda_2}, \dots, -\sqrt{\lambda_{n-1} - \lambda_n}, 0 \dots)$$

for all $n \in \mathbb{Z}_+$. Therefore $L'(T_{n,n}) = \sum_{i=0}^{n-1} -(\lambda_i - \lambda_{i+1}) = \lambda_n$ for all $n \in \mathbb{Z}_+$. $\square$

We can also prove that the only Gaussian processes on $\mathcal{T}_P$ are trivial.

Proposition 3.4.8. *If L is a Gaussian generating functional on $\mathcal{T}_{P,0}$ then $L = 0$.*

Proof. A generating functional L is Gaussian if $L(abc) = 0$ for all $a, b, c \in \ker\epsilon$. Since the projection $T_{k,k} - T_{k+1,k+1}$ is in $\ker\epsilon$ for all $k \in \mathbb{Z}_+$ we see that $\lambda_k - \lambda_{k+1} = 0$ for all k and therefore $\lambda_k = 0$ for all $k \in \mathbb{Z}_+$. $\square$

Proposition 3.4.9. *Let (V, A, λ) be a Lévy process triple on $\mathcal{T}_0$ then the associated Schürmann triple on $\mathcal{T}_0$ satisfies*

$$\begin{aligned}
\rho(T_{n,n}) &= V_{n,n} \\
\eta(T_{n,n}) &= \sum_{i=0}^{n-1} (V_{i,0} - V_{n,i+1})\, A \\
L(T_{n,n}) &= -n\|A\|^2 + \left\langle \left[\sum_{i,j=1}^{n} V_{i,j} - \sum_{i=1}^{n} (n-i)(V_{i,0} + V_{0,i}) \right] A, A \right\rangle
\end{aligned}$$

for all $n \in \mathbb{N}$ where $V_{i,j} := V^i V^{*j}$ for all $i, j \in \mathbb{Z}_+$.

Proof. This is immediate from Proposition 3.4.1. $\square$

Proposition 3.4.10. *Let $\phi : \mathcal{T} \to \mathbb{C}$ be a continuous linear functional, then $\phi|_{\mathcal{T}_P}$ corresponds to a complex measure μ on $\mathbb{Z}_+ \cup \{\infty\}$ where*

$$\mu(n) = \phi(T_{n,n}) - \phi(T_{n+1,n+1}), \text{ and } \mu(\infty) = \lim_{k\to\infty} \phi(T_{k,k})$$

for $n \in \mathbb{Z}_+$.

Proof. For all $i \in \mathbb{Z}_+$ the element $T_{i,i}$ is positive in $\mathcal{T}$ using the representation $\pi : \mathcal{T}_P \to C(\mathbb{Z}_+ \cup \{\infty\})$ we have that $\pi(T_{i,i}) = 1_{\{j;j\geq i\}}$ and $\pi(T_{i,i} - T_{i+1,i+1}) = 1_{\{i\}}$. Clearly

$$\phi(T_{i,i} - T_{i+1,i+1}) = \phi \circ \pi^{-1}(1_{\{i\}}) = \mu(i)$$

for some measure μ by the Markov-Riesz-Kakutani theorem.

Finally $\pi(T_{k,k}) = 1_{\{j;j\geq k\}}$ so by continuity $\mu(\infty) = \lim_{k\to\infty} \phi(T_{k,k})$ □

Example 3.4.11. What follows is an example of a Lévy process on $\mathcal{T}_P$ with

$$H = \ell^2(\mathbb{Z}_+), \quad V = R, \quad A = (-1, 0, 0, 0, 0, 0, 1, 0, \dots) \quad \text{and} \quad \lambda = 0$$

as in Theorem 3.2.2. This is a Poisson process where $w = (1, 1, 1, 1, 1, 1, 0, \dots)$ as in Proposition 3.2.10. Its associated generating functional is given by

$$L(T_{i,i}) = \begin{cases} -i & i < 6 \\ -6 & i \geq 6. \end{cases}$$

As the comultiplication on $\mathcal{T}$ is of the form $\Delta(T_{n,m}) = T_{n,m} \otimes T_{n,m}$ the associated semigroup of states is given by straightforward exponentiation;

$$\phi_t(T_{i,i}) = \begin{cases} e^{-it} & i < 6 \\ e^{-6t} & i \geq 6. \end{cases}$$

Finally by Proposition 3.4.10 we have the associated probability measures to ϕ_t on $\mathbb{Z}_+ \cup \{\infty\}$ are given by

$$\mu_t(i) = \begin{cases} e^{-it} - e^{-(i+1)t} & i < 6 \\ 0 & i \geq 6 \\ e^{-6t} & i = \infty. \end{cases}$$

A sample path is as follows: the process starts at infinity and moves to a number

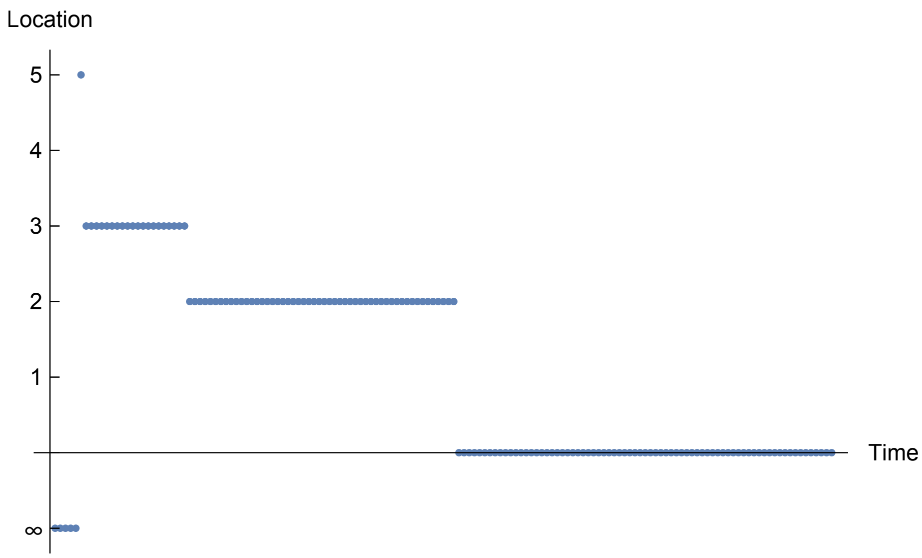


Figure 3.1: Sample path of Lévy process from Example 3.4.11.

between zero and five, it then gradually decrease until it reaches zero. For an illustration of this sample path see Figure 3.1.

Example 3.4.12. A non Poisson example can be achieved by considering any Lévy process triple on $\mathcal{T}_0$ (V, A, λ) such that V is not a unitary and A is not of the form $Vw - w$ for some $w \in H$. Such an example is $H = \ell^2(\mathbb{Z}_+), V = R$ the right shift, $A = e_0$ and $\lambda = 0$. By applying these choices to Corollary 3.4.9

$$\rho(T_{n,n}) = R_{n,n}, \quad \eta(T_{n,n}) = \sum_{i=0}^{n-1} e_i, \quad \text{and} \quad L(T_{n,n}) = -n$$

for all $n \in \mathbb{Z}_+$. Again the associated semigroup of states is given $\phi_t(T_{n,n}) = e^{-nt}$ and the associated measure on $\mathbb{Z}_+ \cup \{\infty\}$ is

$$\mu_t(i) = \begin{cases} e^{-it} - e^{-(i+1)t} & i \in \mathbb{Z}_+ \\ 0 & i = \infty. \end{cases}$$

for $t \neq 0$.

Note the lack of continuity $0 = \mu_t(\infty) \not\to \delta_\infty(\infty) = 1$ as $t \to 0$. This is because $1_{\{\infty\}} \notin C(\mathbb{Z}_+ \cup \{\infty\})$. The convolution semigroup of states is weakly continuous but does not satisfy the obvious stronger choices for continuity, i.e. $\mu_t(A) \not\to \delta_\infty(A)$

for all $A \in 2^{Z_+ \cup \{\infty\}}$ as $t \to 0$ and $\|\mu_t - \delta_\infty\| = 1 + \sum_{i=0}^{\infty} |e^{-it} - e^{-(i+1)t}| = 2$ which clearly does not tend to zero as $t \to 0$.

The Circle

The Toeplitz algebra has the compact operators as an ideal. The quotient $\mathcal{T}/\mathcal{K}(\ell^2)$ is isomorphic to $C(\mathbb{T})$. The quotient map on the dense sub *-bialgebra is given by $T_{n,m} \mapsto z^{n-m}$ [Mur90, Section 3.5].

Proposition 3.4.13. *The algebra of continuous functions on $\mathbb{T}$ is a quotient sub-C^*-bialgebra of $\mathcal{T}$.*

Proof. The quotient map $\theta : \mathcal{T} \to \mathbb{T}$ is clearly a surjective homomorphism. The C*-algebra $C(\mathbb{T})$ has C*-bialgebra structure given by its group structure $\Delta_{\mathbb{T}}(z^n)(\lambda_1, \lambda_2) = (\lambda_1\lambda_2)^n$ and $\epsilon_{\mathbb{T}}(z^n) = 1^n$ for all $\lambda_1, \lambda_2 \in \mathbb{T}$ and $n \in \mathbb{Z}_+$. Therefore $\Delta_{\mathbb{T}}(z^n) = z^n \otimes z^n$ and $\epsilon_{\mathbb{T}}(z^n) = 1$ for all $n \in \mathbb{Z}_+$. From this we easily see that $(\theta \otimes \theta) \circ \Delta_{\mathcal{T}} = \Delta_{\mathbb{T}} \circ \theta$ and $\epsilon_{\mathbb{T}} \circ \theta = \epsilon_{\mathcal{T}}$. $\square$

We can characterise the Lévy processes on $\mathcal{T}$ that are well defined on the C*-algebra $C(\mathbb{T})$.

Proposition 3.4.14. *Let H be a Hilbert space and (V, A, λ) a Lévy process triple on $\mathcal{T}_0$. Then the associated maps (ρ, η, L) on the basis elements $T_{n,m}$ of $\mathcal{I}(1)_0$ depend only the difference $n - m$ if and only if V is unitary.*

Proof. Firstly if $\rho(T_{n,m}) = \rho(T_{n+1,m+1})$ for all n, m then

$$1 = \rho(1) = \rho(TT^*) = \rho(T)\rho(T)^* = VV^*$$

which implies that V is unitary.

Assume V is unitary. The map ρ depends on the difference trivially as $\rho(T^*) =$

$V^* = V^{-1}$ so $\rho(T_{n,m}) = V^{n-m}$. Using Theorem 3.2.2 we can easily show that

$$\begin{aligned}
\eta(T_{n,m}) &= \sum_{k=0}^{n-1} V^k A - \sum_{k=1}^{m} V^{n-k} A \\
&= \sum_{k=0}^{n-1} V^k A - \sum_{k=1}^{m} V^{n-k} A + V^n(A) - V^n(A) \\
&= \sum_{k=0}^{n} V^k A - \sum_{k=1}^{m+1} V^{(n+1)-k} A = \eta(T_{n+1,m+1})
\end{aligned}$$

and the argument follows by induction. We have shown that $\eta : \mathcal{I}(1)_0 \to H$ depends only on $n-m$ so let $h : \mathbb{Z} \to H$ be given by $h(n-m) = \eta(T_{n,m})$. From the properties of η we get the following useful relations for h:

$$h(-n-1) = V^{-1}h(-n) + h(-1) \text{ and } h(n) = -V^{-n}h(-n) \tag{3.4.1}$$

and

$$0 = L(T^*T) = L(T^*) + L(T) + \langle h(-1), h(-1)\rangle$$

which implies that

$$L(T^*) + L(T) = -\langle h(-1), h(-1)\rangle . \tag{3.4.2}$$

Let us show that $L(T_{n,m})$ depends only on $n-m$ also. Using the product rule for L and equations (3.4.1) and (3.4.2) we get

$$\begin{aligned}
L(T_{n+1,m+1}) =& L(T^{n+1}) + L(T^{*(m+1)}) + \langle h(-n-1), h(-m-1)\rangle \\
=& L(T^n) + L(T) + \langle h(-n), h(1)\rangle + L(T^{*m}) + L(T^*) + \langle h(1), h(-m)\rangle \\
&+ \langle h(-n), h(-m)\rangle - \langle h(-n), h(1)\rangle - \langle h(1), h(-m)\rangle + \langle h(-1), h(-1)\rangle \\
=& L(T^n) + L(T^{*m}) + \langle h(-n), h(-m)\rangle = L(T_{n,m})
\end{aligned}$$

and again the argument follows by induction. □

Corollary 3.4.15. *If (V, A, λ) is a Lévy process triple on $\mathcal{T}_0$ such that V is unitary* *then*

$$\begin{aligned}\rho(T_{n,m}) =& V^{n-m}\\ \eta(T_{n,m}) =& \operatorname{sign}(n-m)\sum_{i=1}^{|n-m|} V^{\operatorname{sign}(n-m)i}V^{-1_{\mathbb{Z}_+}(n-m)}A\\ L(T_{n,m}) =& i(n-m)\lambda - \frac{|n-m|}{2}\|A\|^2 - \sum_{i=1}^{|n-m|}(|n-m|-i)\left\langle V^{\operatorname{sign}(n-m)i}A, A\right\rangle\end{aligned}$$

for all $n, m \in \mathbb{Z}_+$ *where* $\operatorname{sign}(x) = 1_{\mathbb{Z}_+}(x) - 1_{\mathbb{Z}/\mathbb{Z}_+}(x)$.

Proof. This is immediate from Proposition 3.4.1. □

We can use this characterisation of Lévy processes on the Toeplitz algebra and describe concrete Lévy processes on the circle. This is achieved by analysing the convolution semigroup of states, comparing with characteristic functions of probability distributions and as a result "wrapping" these distributions around the circle [Fis93, Section 3.3.3].

Proposition 3.4.16. *Given any probability measure* μ *on* $\mathbb{R}$ *the mapping*

$$A \mapsto \sum_{k=-\infty}^{\infty} \mu(A + 2\pi k)$$

for all A *Borel measurable subsets of* $(-\pi, \pi]$ *defines a probability measure on* $(-\pi, \pi]$.

For any regular probability measure μ the associated measure on $\mathbb{T}$ is called the **wrapped probability measure** and is denoted μ_w.

Definition 3.4.17. Given any probability distribution μ on $\mathbb{R}$ the function

$$t \mapsto \int_{\mathbb{R}} e^{itx}\mu(dx)$$

is called **the characteristic function of** μ and is denoted $\Phi_\mu(t)$.

The following is a straightforward relationship between the characteristic function and the moments of the wrapped probability measure for a probability measure on $\mathbb{R}$.

Proposition 3.4.18. *Let μ be a probability measure on $\mathbb{R}$ then*

$$\Phi_\mu(n) = \int_{\mathbb{T}} z^n \mu_w(dz) = \mu_w(z^n)$$

for all $n \in \mathbb{Z}$.

Proof. This follows directly from the definitions. $\square$

We have the following examples of Lévy processes on the circle with the appropriate choice of (V, A, λ). In each case the distribution on the circle is determined by the wrapping procedure.

Example 3.4.19. The triple $(V, A, \lambda) = (\mathrm{id}_{\mathbb{C}}, 1, 0)$ corresponds to a Gaussian process on the circle. The associated Schürmann triple is

$$\rho(R_{n,m}) = 1, \quad \eta(R_{n,m}) = n - m, \quad \text{and } L(R_{n,m}) = -\frac{(n-m)^2}{2}$$

by Corollary 3.4.15. Therefore the associated convolution semigroup of states on $\mathbb{T}$ is given by $\phi_t(z^n) = e^{-\frac{n^2 t}{2}}$. If we let Φ_t be the characteristic function of the normal distribution on the real line with mean zero and variance t then it is easily seen that $\Phi_t(n) = \phi_t(z^n)$. Therefore, we can conclude the Lévy process obtained is a wrapped version of the standard one dimensional Brownian motion. For a sample path of this process see Figure 3.2.

Example 3.4.20. The triple $(V, A, \lambda) = (e^i \,\mathrm{id}_{\mathbb{C}}, e^i - 1, \sin(1))$ corresponds to a Poisson process on $C(\mathbb{T})$. The associated Schürmann triple is

$$\rho(R_{n,m}) = e^{i(n-m)}, \quad \eta(R_{n,m}) = e^{i(n-m)} - 1, \quad \text{and } L(R_{n,m}) = e^{i(n-m)} - 1$$

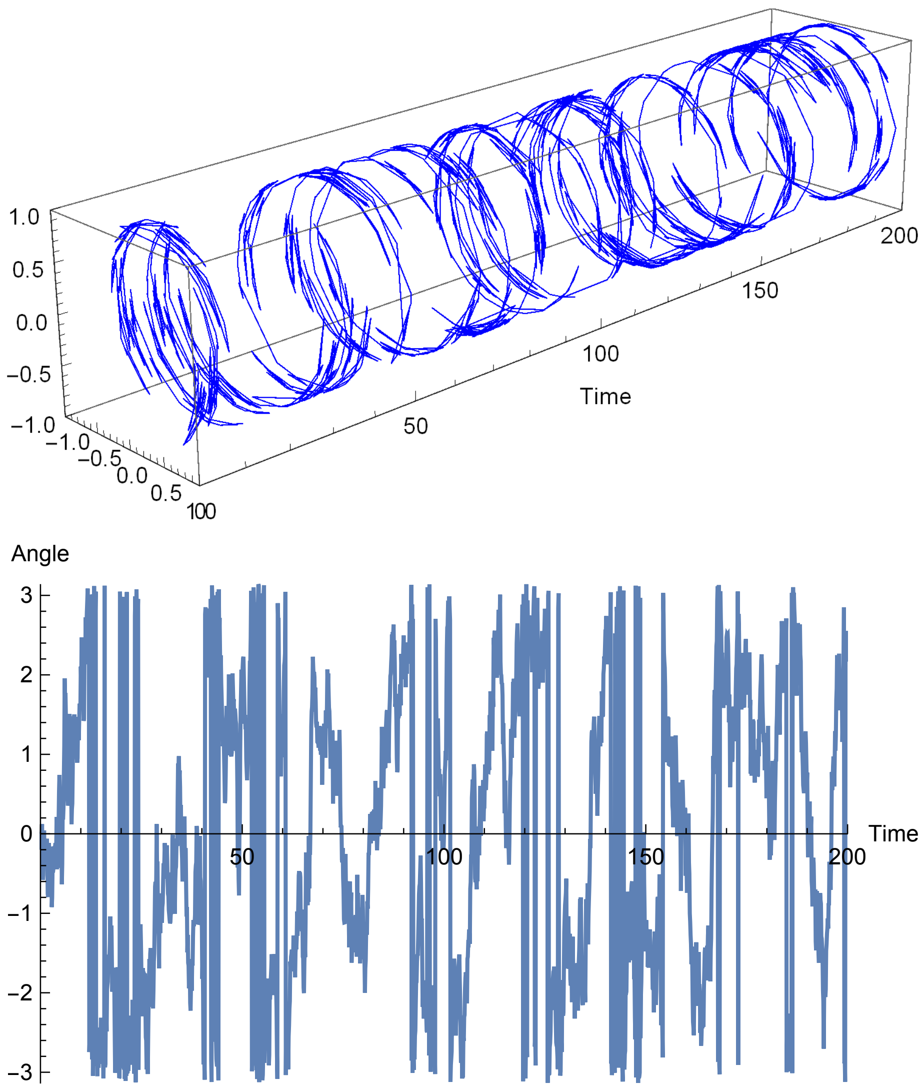


Figure 3.2: A sample path for Example 3.4.19

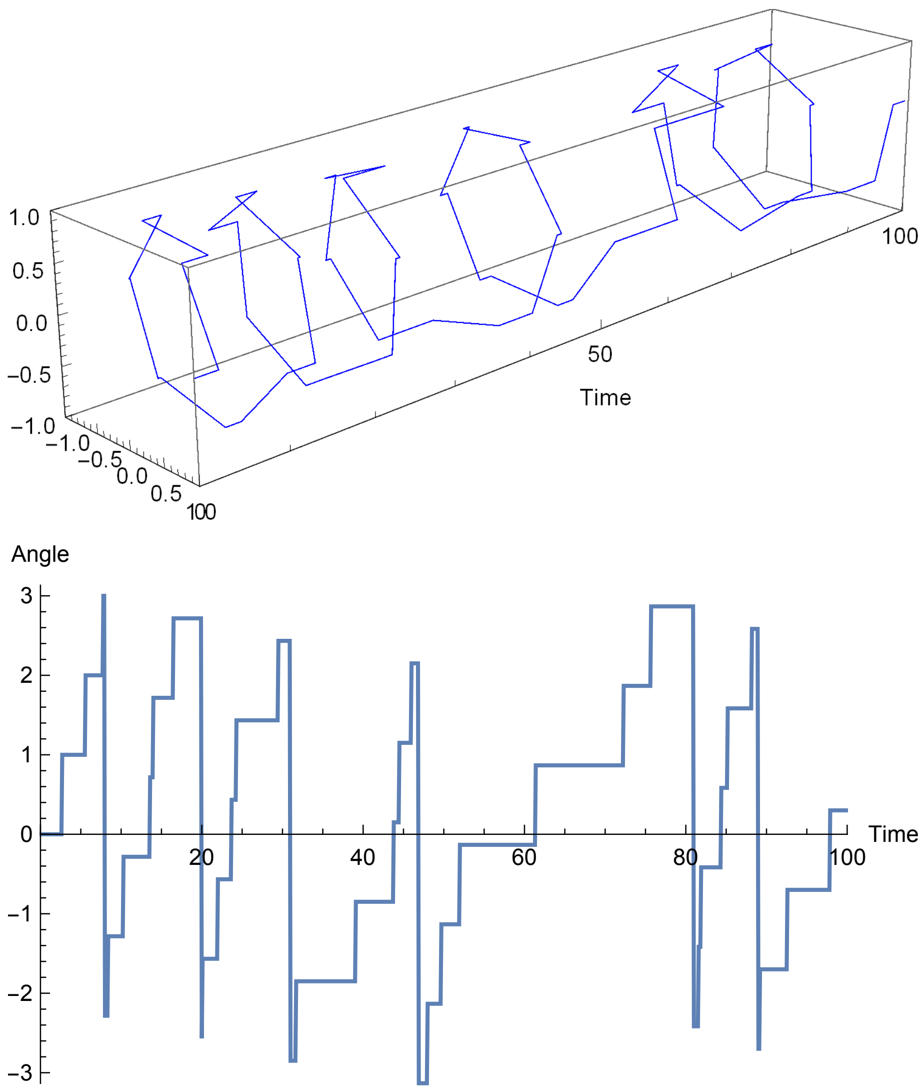
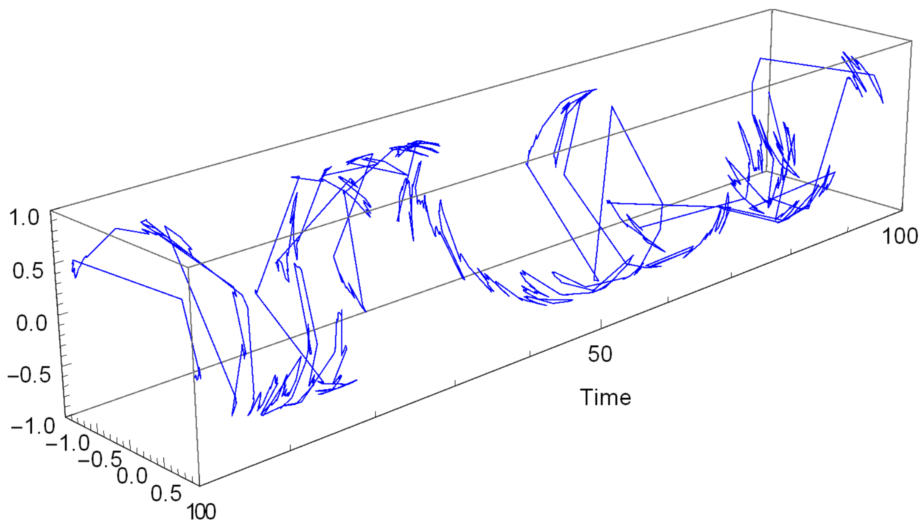


Figure 3.3: A sample path for Example 3.4.20

by Corollary 3.4.15. Therefore the associated convolution semigroup of states on $\mathbb{T}$ is given by $\phi_t(z^n) = e^{(e^{in}-1)t}$. If we let Φ_t be the characteristic function of the Poisson distribution on the real line with mean t then it is easily seen that $\Phi_t(n) = \phi_t(z^n)$. Therefore, we can conclude the Lévy process obtained is a wrapped version of the standard Poisson process which jumps with intensity one with rate one. For a sample path of this process see Figure 3.3.

Example 3.4.21. The triple $(V, A, \lambda) = (U, 2^{1/4}e_0, 0)$ where $U : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ is the unitary given by $Ue_k = e_{k+1}$ where $(e_k)_{k\in\mathbb{Z}}$ is the canonical basis of $\ell^2(\mathbb{Z})$



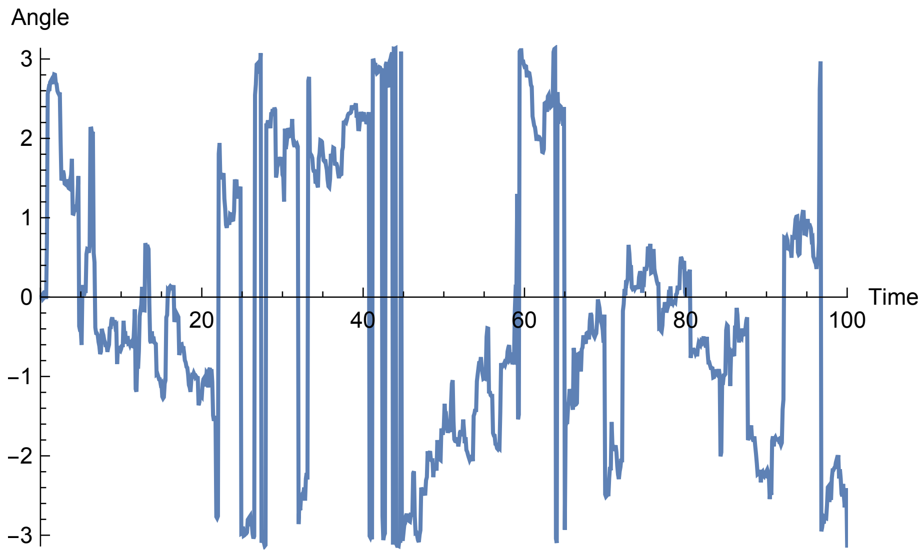


Figure 3.4: A sample path for Example 3.4.21

corresponds to a Cauchy process. The associated Schürmann triple is

$$\rho(R_{n,m}) = U^{n-m}, \quad \eta(T_{n,m}) = \operatorname{sign}(n-m)2^{1/4}\sum_{i=1}^{n-m} e_{(\operatorname{sign}(n-m)i-1_{\mathbb{Z}_+}(n-m))}$$

and $L(R_{n,m}) = -|n-m|$ by Corollary 3.4.15. Therefore the associated convolution semigroup of states on $\mathbb{T}$ is given by $\phi_t(z^n) = e^{-|n|t}$. If we let Φ_t be the characteristic function of the Cauchy distribution on the real line with mean zero and variance t then it is easily seen that $\Phi_t(n) = \phi_t(z^n)$. Therefore, we can conclude the Lévy process obtained is a wrapped version of the Cauchy process. For a sample path of this process see Figure 3.4.

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