Affirmative Action Through Extra Prizes*

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Abstract

Some affirmative action policies establish that a set of disadvantaged competitors has access to an extra prize. We analyse the effects of creating an extra prize by reducing the prize in the main competition. Contestants differ in ability and agents with relatively low ability belong to a disadvantaged minority. All contestants compete for the main prize, but only disadvantaged agents can win the extra prize. We show that an extra prize is a powerful tool to ensure participation of disadvantaged agents. Moreover, for intermediate levels of the disadvantage of the minority, introducing an extra prize increases total equilibrium effort compared to a standard contest. Thus, even a contest designer not interested in affirmative action might establish an extra prize in order to enhance competition.

Keywords: Asymmetric contest, equality of opportunity, affirmative action, discrimination, prize structure, exclusion principle

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1 Introduction

Some affirmative action policies establish that a set of disadvantaged competitors has access to an extra prize. Examples include regional Governments offering their own funding competitions for research projects, besides general funding opportunities from the central

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Government; international awards complemented by a prize for national competitors; the World Chess Championship offering additional competitions for specific groups; or a prize for the best academic paper by a young scientist. The purpose of the present paper is to investigate the incentive effects of this particular prize structure that is commonly used as an affirmative action instrument. Our main result is to show that this policy is not only appealing from a normative point of view but that it also has the potential to enhance competition. It can thus be desirable on efficiency grounds, fostering thereby the social acceptance of the policy.

We analyse the effects of extra prizes in a contest model. These models have been insightful in a variety of competitive situations, including rent-seeking, promotional competition, labour market tournaments, sports competitions or conflict. Following Stein (2002) or Franke et al. (2013), we investigate an asymmetric contest in which contestants differ in ability. Agents with relatively low ability belong to a 'disadvantaged minority'.[1]

A standard result in contest theory says that the most inefficient (or least able) agents might not actively participate in the competition (Stein, 2002). And indeed, 'minority representation' is an important concern in real competitions. For instance, in California the Disabled Veteran Business Enterprise and Small Business Certification Programs establish explicit target market shares for these disadvantaged groups. Similarly, the European Union has target shares for female representation on firms' boards and some universities in the U.K. have widening participation programs aiming at broadening the range of students who attend university so that they are representative of the home population. The challenge is then to design affirmative action policies that can reconcile the conflicting aims of reaching both (i) a sufficient level of minority representation and (ii) a sufficient level of competition. Avoiding trade-offs between these objectives is important because it influences the political support for and the prevalence of affirmative action policies. Ayres and Cramton (1996), for example, report that various California ballot initiatives tried to end state-sponsored affirmative action because of the belief that eliminating affirmative action could help to solve budget problems.

In our model the contest designer can create an extra prize at the cost of reducing the prize in the main competition. All contestants compete for the main prize, but

[1] For a survey of the contest literature see Corchón (2007), Konrad (2009) or Corchón and Serena (2017). See also Cornes and Hartley (2005) and Ryvkin (2013) for general models of asymmetric contests. Throughout the paper we follow the language of the affirmative action literature and use for example the term 'disadvantaged minority' for the agents favoured through affirmative action.

only disadvantaged agents can win the extra prize. This fits, for example, quotas for disadvantaged minorities, like gender quotas, in which the establishment of the quota reduces the budget available in the main competition. Disadvantaged agents thus should have an incentive to exert higher effort but it is far from obvious that the overall level of competition will be strengthened, as advantaged agents have lower incentives to invest.

We show that disadvantaged agents indeed do have an incentive to exert higher effort and that we can think of the effects of extra prizes 'as raising the *effective* ability of disadvantaged agents' thereby creating a 'level playing field'. In our model the introduction of an extra prize results in more homogeneous 'effective abilities' of contestants. This leads to our first major result that an extra prize is a powerful tool to ensure participation of disadvantaged agents. With an extra prize of moderate size both groups of agents are active; using the language of the affirmative action literature, there is diversity. Moreover, as the extra prize becomes larger, the incentives of advantaged agents to participate decline and the least able active advantaged agents drop out, while for disadvantaged agents the opposite happens. For these agents the incentives to participate increase and the most able inactive disadvantaged agents become active.

Our main result is to show that extra prizes have the potential to strengthen competition. The reason is that, as the disadvantaged minority competes stronger, advantaged agents might exert more effort than they otherwise would, resulting in a higher overall level of competition. More precisely, we show that for intermediate levels of the disadvantage of the minority, introducing an extra prize increases total equilibrium effort compared to a standard contest (for example in Stein, 2002). We also show by means of an example that the magnitude of the increase of total effort due to the extra prize might potentially be quite important. Thus, even a contest designer not interested in affirmative action might establish an extra prize in order to enhance competition.

We are not aware of an empirical study that fits exactly our model. The predictions of our model are, however, in line with empirical evidence. Brown (2011) shows that large differences in ability might reduce effort. Balafoutas and Sutter (2012) provide experimental evidence that related (but different) affirmative action policies can have an important impact on minority participation, while not harming the efficiency of the competition, as predicted by our model.[2]

[2]See also Schotter and Weigelt (1992), Corns and Schotter (1999), Franke (2012b) and Calsamiglia et al. (2013) for further evidence of performance enhancing incentive effects of affirmative action.

A distinctive feature of our model is that some agents might win more than one prize with a sole effort choice. In some situations this is a reasonable description of reality. For instance, in chess the World Championship does not exclude women, juniors or seniors, but each of these groups have in addition their separate championship. In Spain some (but not all) regional levels of Government (Comunidades Autónomas) provide research funding, in addition to funding from the central Government. In 2011, the Catalan film 'Black Bread' won both the (Spanish) Goya Award and the (Catalan) Gaudí Award in the category of Best Film.[3] In 2009, a local firm won both the main (international) prize and the prize for Catalan competitors in the fireworks contest organized yearly by the City Council of Tarragona.[4] In 2013, a prominent firm organized a photo competition in Germany that awarded both a main annual prize and a secondary monthly prize, based on a single submission. Currently, entrepreneurs younger than 40 years have access to a special competition in order to obtain funding for the establishment of companies, in addition to the main competition organized by the Spanish Ministry of Industry.[5]

In other situations it might not be true exactly that a contestant can be allocated two prizes. But it might be the case that at the time effort is chosen, the contestant might not know whether he will compete for the main or the extra prize. Consider a quota system. At the time of investment (for example in education) minority members might still have the option of participating as a minority member, in addition to participating in the main competition. Consider widening participation programs of universities. Universities prioritize applicants based on exam grades but might accept grades obtained in a less advantaged family environment, so that some applications compete in the main competition and in a widening participation program.[6] Consider scholarships for stu-

[3] See `http://www.premiosgoya.com/` and `http://academiadelcinema.cat/en/gaudi-awards`, accessed on 29/03/2018.

[4] See `http://www.tarragona.cat/cultura/festes-i-cultura-popular/concurs-internacional-de-focs-artificials-ciutat-de-tarragona`, accessed on 29/03/2018. In the current edition of the competition the extra prize is open to competitors from the Iberian peninsula.

[5] See `www.enisa.es/es/financiacion/info/lineas-enisa/jovenes-emprendedores`, accessed on 29/03/2018.

[6] See `www.nottingham.ac.uk/ugstudy/applying/ourpolicies.aspx`, accessed on 29/03/2018. Although the U.S. Supreme Court recently banned affirmative action in admissions to the state's public universities, it is still possible that applications from athletes, children of alumni and students from under-represented parts of the state are given special weight, see Liptak (2014). There is also a recent controversy about the use of holistic admissions criteria and race in admissions at UCLA, see Groseklose

dents from under-represented groups that aim at enhancing the diversity of the university community. These scholarships are open to students from minority groups and coexist with scholarships based on merit.[7] Consider the Spanish research programme 'Proyectos Europa Excelencia'. In order to compete in this programme a proposal must have competed unsuccessfully for a 'Starting Grant' of the European Research Council.[8] Consider the European Commission's recruitment procedure. Although the procedure is based on merit criteria, measures can be adopted in case of significant imbalance in the staff's nationalities under the principle of equality of EU citizens.[9]

Our paper relates to work analysing the incentive effects of affirmative action policies in competitive situations. A growing literature has determined policies that have the potential to create a 'level playing field' and thereby to lead to more intense competition, including subsidies to high-cost suppliers (Ewerhart and Fieseler, 2003; Rothkopf et al., 2003), bid preferences and other biases in the selection of the winner (Lazear and Rosen, 1981; Baik, 1994; Ayres and Cramton, 1996; Fryer and Loury, 2005; Fu, 2006; Fain, 2009; Franke, 2012a; Pastine and Pastine, 2012; Franke et al., 2013; Ridlon and Shin, 2013; Lee, 2013), share auctions (Alcalde and Dahm, 2013 and 2016), and head starts, handicaps or even exclusion of the most efficient participant (Baye et al., 1993; Che and Gale, 2003; Kirkegaard, 2012; Siegel, 2014).[10] But it should also be noted that in a biased contest model 'level playing field' policies do not always result in the most intense competition (Drugov and Ryvkin, 2017).[11] In this paper we investigate the incentive effects of an

(2014) and Kiewiet (2015).

[7]See `http://exchange.nottingham.ac.uk/blog/new-nottingham-masters-scholarship-scheme/`, accessed on 29/03/2018.

[8]There are also situations which might be interpreted as being the opposite to affirmative action, because the most efficient agents have access to extra prizes. Consider (European) football teams and their investment in players. All teams compete in the national leagues. In addition, however, the best teams compete in European competitions, like the Champions League. Or consider elite colleges that have a bias in favour of applicants who are children of alumni.

[9]See European Commission (2014), page 16.

[10]De Fraja (2005) also provides an efficiency argument for affirmative action. His argument, however, is not based on levelling the playing field but builds on asymmetric information of the individual's potential to benefit from education. We comment further on the relationship to Baye et al. (1993) in the concluding section.

[11]Kawamura and Moreno de Barreda (2014) and Pérez-Castrillo and Wettstein (2016) reach a similar conclusion in incomplete information all-pay auction settings in which the designer's payoffs increase in the winner's type. Brown and Chowdhury (2017) show that the creation of a 'level playing field' can also have the unintended consequence of increasing sabotage.

affirmative action policy that this previous literature has not yet considered and show that it has the potential to enhance competition.[12]

Our paper also contributes to the literature on the optimal prize structure in contests (e.g. Myerson, 1981; Glazer and Hassin, 1988; Clark and Riis, 1998b; Moldovanu and Sela, 2001; Moldovanu and Sela, 2006; Azmat and Möller, 2009; Fu and Lu, 2009; Möller, 2012). Our model, however, differs from this literature in two respects. First, the introduction of an extra prize establishes a novel prize structure that has not been analysed before. Since a player's prospects of receiving a reward for sunk investment depends not only on the magnitude of the investment but also on the player's identity, the competition is characterized by *targeted rewards for sunk investments.* Second, we allow for some contestants to win more than one prize with a sole effort choice.[13]

The extra prize explored in the present paper differs from a second prize not only because disadvantaged agents can win more than one prize but also because advantaged agents do not compete for it. Szymanski and Valletti (2005) investigate a three-person contest with a second prize where one contestant is strong and the other two are equally weak. They show that with a second prize total effort might increase, provided the disadvantage of the weak agents is large enough. Our Proposition 7 includes the three

[12]We are not claiming that the creation of an extra prize is an optimal mechanism for a contest organizer who is completely free to design the contest. But even if a better mechanism exists in theory, there are often legal or ethical restrictions in place that might constrain the organizer's policy choice. We build our contest model with the aim of capturing some aspects of real life competions and claim that it creates different incentives than other contests. For instance, a prominent model biases the contest success function towards some participant, see Franke et al. (2013) for a general model, Franke (2012a) for an affirmative action context and Farmer and Pecorino (1999), Clark and Riis (2000), Dahm and Porteiro (2008) or Esteve-González (2016) for other environments. This is different from creating an extra prize, because Franke et al. (2013) have shown that with the optimal bias there are at least three active players, whereas our Example 2 below shows that the optimal contest with extra prize might have only two active agents.

[13]There are a few models in which a contestant can win multiple prizes. But this requires allocating resources to different contests, as in Gradstein and Nitzan (1989), or choosing effort twice, as in Sela (2012). To the best of our knowledge only in Matros and Rietzke (2017) a contestant can win more than one prize with a sole effort choice. Matros and Rietzke study a class of contests on networks where the network is defined by the participation of contestants in contests. Our prize structure with contestants participating in a main and extra prize contest can be reinterpreted as such a network structure. Their focus, however, is not on affirmative action considerations and thus different from ours. In addition, we allow a contestant's targeted rewards for sunk investments to vary in a more flexible way than they do and asymmetries come not only from access to prizes but also through heterogeneous abilities.

agent case with one advantaged and two equally disadvantaged contestants and reduces in this case to the requirement that the disadvantage of the weak agents should be large enough. Since in this example the incentives under both prize structures are similar, our results invite the conjecture that in more general settings with a second prize total effort might increase, provided the disadvantage of the weak agents is *intermediate*—but this is to the best of our knowledge still an open question.

The paper is organized as follows. The next section collects our assumptions and introduces the notation used. We conduct our strategic analysis in Section 3. The last section contains concluding remarks. All proofs are relegated to an Appendix.

2 The model

A set of risk-neutral contestants $N = \{1, 2, ..., n\}$ competes for a budget B.[14] An agent i's share of the budget depends on his effort exerted, which is denoted by e_i. Expenditures are not recovered. Players have different abilities $\alpha_i > 0$ that are reflected in heterogeneous effort costs $c_i(e_i) = e_i/\alpha_i$. Without loss of generality assume that lower indexed agents have higher ability, so that $\alpha_i \geq \alpha_{i+1}$ for all $i \in \{1, 2, ..., n-1\}$. Agent's know each other's abilities.

The contest designer does not observe the agent's abilities but there is an observable characteristic that distinguishes agents in such a way that they can be partitioned into two groups, $N = M \cup D$. We interpret D as the disadvantaged group that is the objective of affirmative action. In order to simplify notation we assume that agents in $M = \{1, 2, ..., m\}$ have higher ability than agents in $D = \{m + 1, ..., n\}$. Since our equilibrium analysis, however, only requires that the agent with the highest ability is not disadvantaged, this assumption could be substantially relaxed, for instance to a setting in which the average ability of advantaged agents is higher than that of disadvantaged contestants.[15] To

[14] In the literature, the outcome of contests has been interpreted to capture either win probabilities or shares of a prize, see Corchón and Dahm (2010). Since we assume that agents are risk neutral, we do not distinguish between both interpretations. Our model thus allows for contestants winning prizes with some probability, as in the case of the aforementioned film awards, or for agents winning shares of a overall budget, as in the case of quotas.

[15] While it seems feasible to analyse our model under the assumption that agents with high ability qualify for an extra prize, such a setting does not seem interesting from an affirmative action point of view. See Holzer and Neumark (2000), Fang and Moro (2010) and Niederle and Vesterlund (2011) for an assessment of the disadvantage of minority agents in different contexts.

distinguish our setting from a standard contest we suppose $1 \leq m \leq n-2$.

The aim of the contest designer is to maximize total effort by choosing $\beta \in [0,1]$.[16] The parameter β divides the budget into two prizes: $B_1 = (1-\beta)B$ and $B_2 = \beta B$. Members of group M only compete for prize B_1, while members of group D compete for both prizes. In this sense prize B_2 is an extra prize for group D.[17] Notice that although an agent in D exerts effort only once, he might win both prizes. Note also that when $\beta = 0$ or $\beta = 1$ we have a standard contest without extra prize.[18] We follow most of the literature and consider for each prize an imperfectly discriminating contest in which an agent i's share of the budget is proportional to his effort expended, see Tullock (1980).[19]

We introduce the following notation. A vector of individual efforts is denoted by $e = (e_1, e_2, \ldots, e_n)$ and total effort of a group G of agents is $E_G = \sum_{k \in G} e_k$. Similarly, the vector of abilities is denoted by $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$. Given a group of agents G with cardinality $|G|$, the harmonic mean of abilities is given by

$$\Gamma_G \equiv \frac{|G|}{\sum_{k \in G} \frac{1}{\alpha_k}}.$$

This harmonic mean of abilities plays in our model the same role as the harmonic mean of valuations in Hillman and Riley (1989). Since most of the formulas derived below are expressed in terms of Γ_G, it is convenient to note that Γ_G has an alternative interpretation

[16] We formalize the designer's objectives as maximizing total effort and not valuing participation by disadvantaged agents. This makes our main result that it is often in the interest of the designer to establish an affirmative action policy in the form of an extra prize more surprising. Besides these two objectives a designer could also be interested in competitive balance or the quality of the winner, see Serena (2017) for an insightful discussion.

[17] An alternative assumption postulates that introducing the extra prize does not affect the main prize. Our formulation makes it more difficult that the contest designer prefers to introduce an extra prize, because it has the opportunity cost of reducing the main prize.

[18] With the term standard contest we refer to a situation in which (given the value of β or the behaviour of other agents) the objective function of active agents reduces to the one in Stein (2002). For simplicity of the exposition we exclude the case $\beta = 1$ from most of our derivations. Sometimes, however, it is convenient to include this case. The statements referring to $\beta = 1$ follow from Stein (2002).

[19] We discuss generalisations of this function in the Conclusions and remark here only that in an all-pay auction the main forces of our model seem to be even stronger, see the recent analysis in Dahm (2017). We also follow most of the literature and assume that when in the competition for a prize none of k agent exerts effort, each agent wins the prize with probability $1/k$. Skaperdas (1996) and Clark and Riis (1998a) provide axiomatic characterizations of the contest success function employed, while Corchón and Dahm (2010) give a micro-foundation for the interpretation that the outcome is the choice of a designer.

in terms of effort costs. More precisely, the harmonic mean of abilities of the contestants in G is simply the reciprocal of the average of marginal effort costs.

In the contest with extra prize the expected payoff of player i is

$$EU_i(e) = \frac{e_i B_1}{E_N} + \frac{e_i B_2 z_i}{E_D} - \frac{e_i}{\alpha_i}, \tag{1}$$

where $z_i \in \{0, 1\}$ takes value 1 if and only if $i \in D$, and value 0 otherwise. Notice that in this formulation the win probabilities of the two prizes are independent. It is interesting to observe that our model also captures a situation in which a disadvantaged agent who wins the main prize also wins the extra prize. In such a situation the expected payoff of player i is

$$EU_i(e) = \frac{e_i B_1}{E_N} + \frac{e_i B_2 z_i}{E_N} + \frac{E_M}{E_N}\frac{e_i B_2 z_i}{E_D} - \frac{e_i}{\alpha_i},$$

which is equivalent to (1).

3 Strategic analysis

Our first result establishes that contests with extra prizes are a powerful tool to make sure that there will be minority representation, since the extra prize will not be uncontested.

Lemma 1. *For any (α, B) at least two contestants participate in the contest. Moreover, if $\beta > 0$, at least one agent $i \in D$ is active.*

One might think that with an extra prize there are always two disadvantaged contestants active. While this is intuitive, because otherwise there is no competition for the extra prize, it turns out not always to be true. Example 5 in Subsection 3.6 presents a counterexample in which the weaker of two disadvantaged contestants is not active for small extra prizes.

For later reference we observe that Lemma 1 implies that $E_D > 0$, if $\beta > 0$. In order to analyse participation in the contest further, we take the derivative of (1) and obtain

$$\frac{\partial EU_i(e)}{\partial e_i} = \frac{E_N - e_i}{(E_N)^2} B_1 + \frac{E_D - e_i}{(E_D)^2} B_2 z_i - \frac{1}{\alpha_i}. \tag{2}$$

Given that (1) is concave in e_i, the first-order conditions require that $\partial E_i(e)/\partial e_i = 0$ if $e_i > 0$ and $\partial E_i(e)/\partial e_i \leq 0$ if $e_i = 0$. The former implies that

$$\frac{E_N - e_i}{(E_N)^2} B_1 + \frac{E_D - e_i}{(E_D)^2} B_2 z_i = \frac{1}{\alpha_i}. \tag{3}$$

Consider the agents in set M. For these agents condition (3) reduces to the familiar expression

$$e_i = E_N \left(1 - \frac{E_N}{B_1 \alpha_i}\right), \tag{4}$$

see Stein (2002). This implies that comparing two advantaged agents with different abilities, the agent with higher ability exerts higher equilibrium effort. If the ability of an agent is low enough, he does not participate. We denote by $m^* \in M$ the agent with $e_{m^*} > 0$ such that for all $i < m^*$, $e_i > 0$ and for all $i \in M$ with $i > m^*$, $e_i = 0$. We denote by $M_{m^*} \subseteq M$ or $M^* \subseteq M$ the set of active advantaged agents, depending on whether we wish to stress the identity of the active agents. If no advantaged agent is active, we set $M^* = \emptyset$ and $m^* = 0$.

Consider the agents in set D. Here condition (3) becomes

$$e_i = E_N E_D \frac{B_2 E_N + B_1 E_D \left(1 - \frac{E_N}{B_1 \alpha_i}\right)}{B_1 (E_D)^2 + B_2 (E_N)^2}. \tag{5}$$

Again, equilibrium effort is ordered by ability. We define $D_{d^*}, D^* \subseteq D$, and d^* analogously to M_{m^*}, M^* and m^*.[20] If no disadvantaged agent is active, we set $D^* = \emptyset$ and $d^* = 0$.

The aim of our strategic analysis is to show that contests with extra prize admit a unique equilibrium (a formal statement will be provided in Proposition 4 in Subsection 3.3), and to investigate the effects of an extra prize on participation (in Subsection 3.3) as well as on total effort (in Subsections 3.4, 3.5 and 3.6). Doing so, however, requires looking first at the two different types of equilibria that might arise. In the first type of equilibrium, only one group is active, and behaviour is similar to that in a standard contest. In the second, there is diversity and complex effects emerge. We start by analysing each type of equilibrium successively in Subsections 3.1 and 3.2.

3.1 Standard equilibria

There are two situations in which equilibria that appear in a contest with extra prize are similar to those in a standard contest. The first is the trivial case when $\beta = 0$; when there is no extra prize. In the second, members of the advantaged group are discouraged from participating because the extra prize is sufficiently large. As we will formalize in

[20] Notice that d^* does not indicate the cardinality of the set of active agents but the index of the most disadvantaged active agent: $|D^*| = d^* - m$. Note also that if $\beta = 0$ or no advantaged agent is active, (5) reduces to (4), as the contest with extra prize becomes a standard contest.

Subsection 3.3, the introduction of an extra prize strengthens disadvantaged agents and weakens advantaged contestants. This strengthens the participation incentives of the former and weakens those of the latter. Once the extra prize is large enough, in equilibrium no advantaged agent is active and our model reduces to a variation of a standard contest. In order to maintain a consistent notation we summarize the results for this case by Hillman and Riley (1989) or Stein (2002) as follows.[21]

Lemma 2. *[Hillman and Riley, 1989; Stein, 2002] In a standard contest in which a group of agents $P = \{1, 2, \ldots, p\}$ competes for a prize B, the number of active players $|P^*|$ is larger than two and total equilibrium effort is given by*

$$E_N = \frac{|P^*| - 1}{|P^*|} B\Gamma_{P^*}. \tag{6}$$

Total effort is increasing in the number of active participants and the value of the prize, and is decreasing in the average of marginal effort costs of active contestants.

In order to describe when standard equilibria appear, it is useful to start with the definition of a threshold value for the size of the extra prize. To do so denote the set of active disadvantaged agents when no advantaged agent exerts effort by $D^*_{M^*=\emptyset}$.

Definition 1. *Let*

$$\overline{\beta} \equiv 1 - \frac{|D^*_{M^*=\emptyset}| - 1}{|D^*_{M^*=\emptyset}|} \frac{\Gamma_{D^*_{M^*=\emptyset}}}{\Gamma_{\{1\}}}.$$

Notice that $\Gamma_{\{1\}} = \alpha_1 > 0$ and $|D^*_{M^*=\emptyset}| \geq 2$, implying that $\overline{\beta}$ is well defined. The following result establishes that in any contest, in addition to the trivial case when $\beta = 0$, there are situations in which standard equilibria emerge.

Proposition 1. *For any (α, B) we have that $\overline{\beta} < 1$. Moreover, for any $\beta \in [\overline{\beta}, 1]$, it is an equilibrium that the set of active agents is $D^*_{M^*=\emptyset}$ and that these agents' equilibrium effort is as in a standard contest for a prize of size B.*

Intuitively, in the equilibria described in Proposition 1 the extra prize is too large. The extra prize reaches the aim of inducing participation of the disadvantaged group but it does so by discouraging participation of members of the advantaged group completely. We turn now to more moderate extra prizes which generate equilibria in which members of both groups are active.

[21] For a proof and the exact expressions of the equilibrium number of active players, individual efforts, win probabilities and expected utilities, see Stein (2002). Nti (1999) and Matros (2006) show uniqueness of equilibrium.

3.2 Equilibria with diversity

We start by defining a measure of minority representation, as the percentage of total effort that is expended by disadvantaged agents

$$\Omega \equiv \frac{E_D}{E_N}.$$

We say that there is diversity if $\Omega \notin \{0,1\}$.[22] The following result establishes that an extra prize of intermediate size guarantees diversity.

Proposition 2. *For any* (α, B) *we have that* $0 < \overline{\beta}$. *Moreover, for any* $\beta \in (0, \overline{\beta})$, *in equilibrium, there is diversity.*

For later reference, we observe that Proposition 2 implies that $E_N > E_D > 0$. In order to describe total equilibrium effort in an equilibrium with diversity, denote the set of active agents in the contest by $N^* = M^* \cup D^*$ and its cardinality by $|N^*| = |M^*| + |D^*|$.

Proposition 3. *Let* $\beta \in (0, \overline{\beta})$. *For any* (α, B), *total equilibrium effort is given by*

$$E_N = \Upsilon + \sqrt{\Upsilon^2 - \Phi}, \tag{7}$$

where

$$\begin{aligned} \Upsilon &\equiv \frac{B_1}{2}\left(\frac{|M^*|-1}{|M^*|}\Gamma_{M^*} + \frac{|N^*|-1}{|N^*|}\Gamma_{N^*}\right) \text{ and} \\ \Phi &\equiv \frac{B_1 \Gamma_{M^*}\Gamma_{N^*}}{|M^*||N^*|}\left((|N^*|-1)(|M^*|-1)B_1 - (|D^*|-1)B_2\right). \end{aligned}$$

Notice that (7) includes as a special case situations in which only one disadvantaged agent is active. In such a case increasing the extra prize establishes a transfer to the active disadvantaged agent and reduces the main prize in some proportion. Total effort, therefore, is like in a standard contest in which the main prize is reduced in some proportion to B_1 and declines with the average of marginal effort costs of active contestants.[23] In general, however, disadvantaged agents compete for two prizes, while advantaged contestants only compete for one prize. Hence it is not surprising that in (7) total effort

[22] This admittedly very limited measure of diversity is sufficient for our purpose. It will, however, follow from (8) and (11) below that $1-\Omega$ measures the equilibrium win probability of advantaged agents. Thus, in an affirmative action context our measure of diversity Ω can also be interpreted as saying that one situation is more diverse than another if the win probability of advantaged agents is lower, that is, Ω is higher.

[23] See Claim 1 in Appendix A.5.

depends on Γ_{M^*} and Γ_{N^*}, so that the averages of marginal effort costs of advantaged and disadvantaged contestants matter in different ways.

We complete now the description of the candidate equilibrium for $\beta \in (0, \overline{\beta})$. We will later confirm that this is indeed an equilibrium. A formal statement of existence and uniqueness of equilibrium will be provided in Proposition 4. Summing up (4) over all active advantaged agents and rearranging, we determine

$$\begin{aligned}
\Omega &= |M^*| \frac{\Upsilon + \sqrt{\Upsilon^2 - \Phi}}{B_1 \Gamma_{M^*}} - (|M^*| - 1) \in (0,1), \qquad (8) \\
E_M &= \left(\Upsilon + \sqrt{\Upsilon^2 - \Phi}\right)(1 - \Omega) \quad \text{and} \\
E_D &= \left(\Upsilon + \sqrt{\Upsilon^2 - \Phi}\right)\Omega.
\end{aligned}$$

The expressions for individual efforts of the active agents are obtained as follows.[24] First introducing (7) in (4) yields for $i \in M^*$ that

$$e_i = \left(\Upsilon + \sqrt{\Upsilon^2 - \Phi}\right)\left(1 - \frac{\Upsilon + \sqrt{\Upsilon^2 - \Phi}}{\alpha_i B_1}\right). \qquad (9)$$

For $i \in D^*$ we use (7) and (21), which we derive in Appendix A.4, in (5), obtaining

$$e_i = \left(\Upsilon + \sqrt{\Upsilon^2 - \Phi}\right)\Omega \frac{B_1 \Omega \left(1 - \frac{\Upsilon + \sqrt{\Upsilon^2 - \Phi}}{\alpha_i B_1}\right) + B_2}{B_1 \Omega^2 + B_2}. \qquad (10)$$

Since agents of the advantaged group only compete for one prize, only the win probability for prize B_1 is of interest. This is immediately determined. For $i \in M^*$ we have

$$p_i = 1 - \frac{\Upsilon + \sqrt{\Upsilon^2 - \Phi}}{\alpha_i B_1}. \qquad (11)$$

Members of the disadvantaged group, however, have the chance to obtain two prizes, and thus two win probabilities. The win probability of agent $i \in D^*$ for prize B_1 is

$$p_i = \Omega \frac{B_1 \Omega \left(1 - \frac{\Upsilon + \sqrt{\Upsilon^2 - \Phi}}{\alpha_i B_1}\right) + B_2}{B_1 \Omega^2 + B_2},$$

[24] When there is no extra prize ($\beta = 0$) or when $|D^*| = 1$, then expressions (9) and (10) below coincide and reduce to the description in Stein (2002). The first observation follows from the fact that for $\beta = 0$ we have that $B_2 = 0$. For the second observation, notice that when $|D^*| = 1$ by Claim 1 in Appendix A.5 we have that $E_N = |M^*|\Gamma_{N^*} B_1 / |N^*|$. This implies that $\Omega = 1 - \left(\Upsilon + \sqrt{\Upsilon^2 - \Phi}\right)/(B_1 \Gamma_{D^*})$ and (10) reduces to (9).

while the win probability of agent $i \in D^*$ for prize B_2 is

$$q_i = \frac{B_1\Omega\left(1 - \frac{\Upsilon+\sqrt{\Upsilon^2-\Phi}}{\alpha_i B_1}\right) + B_2}{B_1\Omega^2 + B_2}.$$

Lastly, we state the expected equilibrium utilities of the active agents. For $i \in M^*$ we have

$$EU_i = B_1\left(1 - \frac{\Upsilon + \sqrt{\Upsilon^2 - \Phi}}{\alpha_i B_1}\right)^2,$$

and for $i \in D^*$ one obtains

$$EU_i = \frac{\left[B_1\Omega\left(1 - \frac{\Upsilon+\sqrt{\Upsilon^2-\Phi}}{\alpha_i B_1}\right) + B_2\right]^2}{B_1\Omega^2 + B_2}.$$

3.3 The effects of the extra prize on participation

We are now in a position to investigate the effects of the extra prize on participation. Remember that with the help of condition (4) we have already established that $e_i > 0$ for $i \in M$ requires sufficient ability

$$\alpha_i > \frac{\Upsilon + \sqrt{\Upsilon^2 - \Phi}}{B_1}, \tag{12}$$

and thus $e_i > 0$ for $i \in M$ implies $e_j > 0$ for $j < i$. Moreover, for disadvantaged agents a similar property holds; $e_i > 0$ for $i \in D$ requires

$$\alpha_i > \frac{\Upsilon + \sqrt{\Upsilon^2 - \Phi}}{B_1}\frac{B_1\Omega}{B_1\Omega + B_2}. \tag{13}$$

Again $e_i > 0$ for $i \in D$ implies $e_j > 0$ for $m < j < i$. Hence it suffices to characterize candidate sets of active agents by the highest index of the agents in the sets: $M_{m^*} \subseteq M$ and $D_{d^*} \subseteq D$. The overall set of active agents can then be characterized with the help of these two indexes: $N_{d^*}^{m^*} \equiv M_{m^*} \cup D_{d^*}$.[25]

Example 1. *Consider $M = \{1,2\}$ and $D = \{3,4\}$. Since at least two contestants are active and participation is monotonic in the sense explained in the previous paragraph, the candidate sets of active agents are $N_0^2 = \{1,2\}$, $N_3^1 = \{1,3\}$, $N_4^0 = \{3,4\}$, $N_3^2 = \{1,2,3\}$, $N_4^1 = \{1,3,4\}$, and $N_4^2 = \{1,2,3,4\}$. When $\beta = 0$, it cannot be that agent 3 is active when agent 2 is not. Hence we exclude N_3^1, N_4^0, and N_4^1. When $\beta > 0$, Lemma 1 implies that N_0^2 is not relevant.*

[25]If one of these sets is empty, say $M^* = \emptyset$, we write $N_{d^*}^0$.

Lastly, we divide (13) by $B_1\Omega/(B_1\Omega + B_2)$ and define for the disadvantaged agents $i \in D$ the effective ability $\hat{\alpha}_i$ as follows

$$\hat{\alpha}_i \equiv \alpha_i \left(1 + \frac{B_2}{B_1\Omega}\right) \geq \alpha_i.$$

So we can think of the participation decision of an agent as determined by comparing the contestant's ability to the right hand side of (12). For disadvantaged agents, however, the extra prize raises ability from α_i to $\hat{\alpha}_i$, contributing thereby to a more level playing field. We summarize the preceding formally as

Corollary 1. *For any (α, B), the set of active contestants $N_{d^*}^{m^*}$ is found as the largest index $m^* = \{0, 1, 2, \ldots, m\}$ such that, given d^*, the ability of active advantaged agents is high enough, that is*

$$\alpha_{m^*} > \frac{\Upsilon + \sqrt{\Upsilon^2 - \Phi}}{B_1}, \tag{14}$$

and the largest index $d^* = \{0, m+1, m+2, \ldots, n\}$ such that, given m^*, the effective ability of active disadvantaged agents is high enough, that is

$$\hat{\alpha}_{d^*} > \frac{\Upsilon + \sqrt{\Upsilon^2 - \Phi}}{B_1}. \tag{15}$$

Corollary 1 complements Lemma 1 and Proposition 2. It shows from a different angle that the extra prize might have a strong effect on participation. When $\beta = 0$ and there is no extra prize, effective ability $\hat{\alpha}_i$ is equal to α_i. Disadvantaged agents have the lowest incentives among all contestants to participate, as their ability to compete is lowest. Introducing the extra prize, however, affects the participation conditions of both types of agents.

To see this it is instructive to consider the derivative of the right hand side of both participation conditions (14) and (15) with respect to β. Since $\partial\left(E_N/B_1\right)/\partial\beta \geq 0$, the extra prize discourages participation of advantaged agents. For disadvantaged agents, however, there is a countervailing effect, because their effective ability $\hat{\alpha}_i$ is also raised, as $\partial\left(B_2/B_1\Omega\right)/\partial\beta > 0$.[26]

[26] For completeness we mention that $\partial\left(E_N/B_1\right)/\partial\beta > 0$ requires $|D^*| > 1$, see Appendix A.5. When only one disadvantaged agent is active, increasing the extra prize does not affect participation of active agents. The reason is that increasing the extra prize establishes a transfer to the active disadvantaged agent and reduces the main prize in some proportion. This does not affect participation of active agents, because in a Tullock contest participation is not affected when valuations are multiplied by a constant. Note also that when $E_N/B_1 = \alpha_1$ and the most efficient agent ceases to be active, condition (13) becomes condition (12), where B_1 is replaced by B.

These two countervailing effects imply that extra prizes have the potential to induce participation of disadvantaged agents who are not active without such a prize and to discourage participation of advantaged agents who are active without such a prize. We will observe these forces in later sections in more specific settings.

We are now in a position to state formally existence and uniqueness of equilibrium. For $\beta \notin (0, \overline{\beta})$ the result follows from Stein (2002) or Fang (2002). For $\beta \in (0, \overline{\beta})$ we provide in Appendix A.6 a proof proceeding in two steps. First, we confirm that the candidate strategies in (9) and (10) are indeed an equilibrium. Second, we prove uniqueness of the equilibrium in pure strategies by showing that the set of active contestants $N_{d^*}^{m^*}$ is unique.

Proposition 4. *For any (α, B), there is a unique equilibrium in which the set of active contestants $N_{d^*}^{m^*}$ employ the pure strategies described in (9) and (10), while the other agents exert no effort.*

3.4 The effects of the extra prize on total effort

We establish first the existence of an optimal size for the extra prize.

Proposition 5. *For any (α, B), there is $\beta^* \in [0, \overline{\beta})$ such that $E_N(\beta)$ attains a maximum at β^*.*

The previous proposition establishes that a maximum is well defined. The fact that $\beta^* < \overline{\beta}$ follows from Fang's well known result that it is never optimal to exclude all advantaged agents (Fang, 2002). Without further assumptions determining the optimal size of the extra prize is complex, because (as we will see for example in Example 5) total effort is not differentiable with respect to β when β reaches values for which the set of active contestants changes. In light of this problem, we derive now a sufficient condition for $\beta^* > 0$ that is based on small extra prizes for which we know that the set of active contestants does not vary.

Given the participation condition (4), we can always find $\beta = \epsilon$ with $\epsilon > 0$ but sufficiently close to zero such that the set of active agents will consist of the same set of agents as for $\beta = 0$, except when there was no minority representation (Proposition 2). In the latter case, at least the most efficient disadvantaged agent also becomes active. Formally, we define the following sets of agents. Let $M^*_{\beta=0} \subseteq M$ and $D^*_{\beta=0} \subseteq D$ be the sets of advantaged and disadvantaged agents that are active for $\beta = 0$, respectively. Let

$$D^*_{\alpha_{m+1}} = \{i \in D : \alpha_i = \alpha_{m+1}\}$$

be the most able of the disadvantaged agents. Notice that this set has at least cardinality one. The cardinality is higher when there is more than one agent with the highest ability. In the following Proposition, M^*_ϵ, D^*_ϵ, and N^*_ϵ refer to $M^*_{\beta=0}$, $D^*_{\beta=0} \cup D^*_{\alpha_{m+1}}$, and $M^*_\epsilon \cup D^*_\epsilon$, respectively. Lastly, we define the following numbers

$$\gamma \equiv \frac{|M^*_\epsilon| - 1}{|M^*_\epsilon|}, \quad \delta \equiv \frac{|N^*_\epsilon| - 2}{|N^*_\epsilon| - 1} \quad \text{and } \zeta \equiv \frac{|N^*_\epsilon|}{|N^*_\epsilon| - 2}.$$

Notice that, since $|N^*_\epsilon| = |M^*_\epsilon| + |D^*_\epsilon|$, it is not difficult to see that $\gamma = \delta$ holds for $|D^*_\epsilon| = 1$, while $\gamma < \delta$ holds for $|D^*_\epsilon| > 1$.

Proposition 6. *Let $|D^*_\epsilon| > 1$, that is, assume that for very small but strictly positive extra prizes at least two disadvantaged agents are active. Then for any (α, B), the optimal extra prize increases total equilibrium effort compared to a standard contest, that is, $\beta^* > 0$ if*

$$\gamma^2 \zeta < \frac{\Gamma_{N^*_\epsilon}}{\Gamma_{M^*_\epsilon}} < \delta^2 \zeta. \tag{16}$$

The previous Proposition provides a condition assuring that a revenue maximizing contest organizer finds it in her interest to establish an extra prize. Why does an extra prize have the potential to increase total effort? The reason is that an extra prize increases the effective ability $\hat{\alpha}_i$ of disadvantaged agents (as formalized in Condition (15) of Corollary 1). This balances the competition and results in higher total effort, provided condition (16) is fulfilled. This condition requires that the ratio between the average of marginal effort costs of advantaged agents and the average of marginal effort costs of all contestants—which under our assumptions is a number smaller than one—must lie in some interval which is limited by bounds that are smaller than one. Thus, for condition (16) to hold, the minority must be disadvantaged but not too much. The disadvantage of the minority must be of an intermediate level.

Notice that Proposition 6 allows for situations in which without extra prize there is no minority representation and disadvantaged agents only become active with a small extra prize. Also, the requirement that $|D^*_\epsilon| > 1$ is a weaker condition than asking to have more than one agent with exact ability α_{m+1}. It rules out situations in which only one disadvantaged agent is active. This is intuitive, because when $|D^*_\epsilon| = 1$, increasing the extra prize establishes a transfer to the active disadvantaged agent and reduces the main prize in some proportion. Thus the contested rent is reduced and total effort declines.[27]

[27] As already explained, for $|D^*_\epsilon| = 1$ both bounds in condition (16) coincide. Example 5 below illustrates a situation with $|D^*_\epsilon| = 1$. In Claim 1 in Appendix A.5 we show that when $|D^*_\epsilon| = 1$, total effort is given by $E_N = (|N^*_\epsilon| - 1)B(1-\beta)\Gamma_{N^*_\epsilon}/|N^*_\epsilon|$, as prescribed by (6).

Note also that condition (16) depends on the level of competition. Both the upper and lower bound increase in the number of active advantaged agents ($|M_\epsilon^*|$). However, increasing the number of active disadvantaged agents ($|D_\epsilon^*|$), decreases the lower bound and increases the upper bound so that the interval defined by condition (16) becomes wider. Also, the lower bound vanishes when there is only one active advantaged agent ($|M_\epsilon^*| = 1$), while the upper bound tends to one when $|N_\epsilon^*|$ becomes large. Therefore, we can say that when there is only one advantaged agent and many disadvantaged agents of similar ability, then the introduction of an extra prize is beneficial, unless there is not much difference between advantaged and disadvantaged agents. We will gain further intuition into condition (16) and some insights into the optimal size of the extra prize in what follows.

3.5 Homogeneity within groups

We consider now the case in which all agents in a given group have the same ability.[28] Such a situation represents the benchmark of minimal heterogeneity in our model. We will see that condition (16) is also a necessary condition under this assumption.

Suppose that $\alpha_i = \overline{\alpha}$ for all $i \in M$ and $\alpha_i = \underline{\alpha}$ for all $i \in D$. We denote $\underline{\alpha}/\overline{\alpha} \equiv \alpha \in (0, 1]$.

Because of the symmetry within groups, conditions (4) and (5) imply that either all group members are active, or none is. So for $\beta = 0$ the set of active agents is either M or N. For $\beta \in (0, \overline{\beta})$, the set of active agents is N, while for larger values for β only group D is active. Moreover, $\overline{\beta}$ simplifies to $1 - (n - m - 1)\alpha/(n - m)$.

Consider now Proposition 6. Notice that if D is active, then by assumption at least two disadvantaged agents are expending effort; thus $|D_\epsilon^*| > 1$ is always fulfilled. Straightforward algebra yields

$$\frac{\Gamma_{N_\epsilon^*}}{\Gamma_{M_\epsilon^*}} = \frac{\alpha}{1 - \frac{m}{n}(1 - \alpha)}.$$

The right hand side of this expression is strictly increasing in α. Moreover, it tends to zero when α goes to zero, and goes to one when α approaches one. Thus condition (16) is fulfilled for intermediate values of α. The next result states the precise condition.

[28] Lee (2013) makes a similar assumption. Szymanski and Valletti (2005) restrict in addition to three contestants.

Proposition 7. *For any* (α, B)*,* $\beta^* > 0$ *if and only if*

$$\frac{(m-1)^2(n-m)}{m(m(n-m)-1)} < \alpha < \frac{(n-m)(n-2)}{1+(n-m)(n-2)}. \tag{17}$$

Moreover, for any n *and* m*, there exists* α *such that* $\beta^* > 0$*.*

The assumption of homogeneity within groups allows to strengthen Proposition 6 considerably. On one hand, condition (16) becomes also a necessary condition. On the other, for any configuration of groups there exist intermediate levels of disadvantage such that a revenue maximizing contest organizer finds it in her interest to establish an extra prize.

Notice that in the special case of $m = 1$ the left hand side of condition (17) is zero. This parallels a result in Szymanski and Valletti (2005) in their analysis of a three player contest with a second prize. We provide now an example with $m = 2$ that provides further intuition into condition (17). We also set $B = 1$ and normalize $\overline{\alpha} = 1$, so that $\underline{\alpha} = \alpha$.

Example 2. *Let* $M = \{1, 2\}$ *and* $D = \{3, 4\}$*. Consider a standard contest with* $\beta = 0$*. If all agents are active, total effort is given by* $E_N = 3\alpha/(2+2\alpha)$*. If only advantaged agents are active, total effort is given by* $E_M = 1/2$*. In both cases it follows from condition (4) that disadvantaged agents are active if* $\alpha > 1/2$*, and total effort is continuous in* α*. Introducing a sufficiently small extra prize,* $\beta < \overline{\beta} = 1 - \alpha/2$*, assures that all agents are active. Condition (17) becomes* $1/3 < \alpha < 4/5$*. Once the extra prize becomes sufficiently large,* $\beta > \overline{\beta}$*, only disadvantaged agents are active and total effort is given by* $E_D = \alpha/2$*.*

We display in Figure 1 several cases. The highest curve assumes $\alpha = 0.9$*, while the lowest supposes* $\alpha = 0.2$*. For both values of* α *condition (17) does not hold. The curve in the middle displays* $\alpha = 0.55$*, for which the condition holds. In the case of* $\alpha = 0.2$ *minority participation has increased.*[29]

Notice that increasing the total number of agents by increasing the number of disadvantaged agents, widens the interval defined by condition (17) and makes it easier that the introduction of an extra prize is beneficial. In the following example we maintain the total number of agents constant and vary the proportion of advantaged agents to provide some further intuitions into the size of the optimal extra price and its effect on total effort. We maintain that $B = 1$, $\overline{\alpha} = 1$ and $\underline{\alpha} = \alpha$.

[29] Unlike the biased contest model in Franke et al. (2013) the optimal contest in case of $\alpha = 0.2$ has only two active agents. This shows that creating an extra prize is conceptually different from biasing the contest success function.

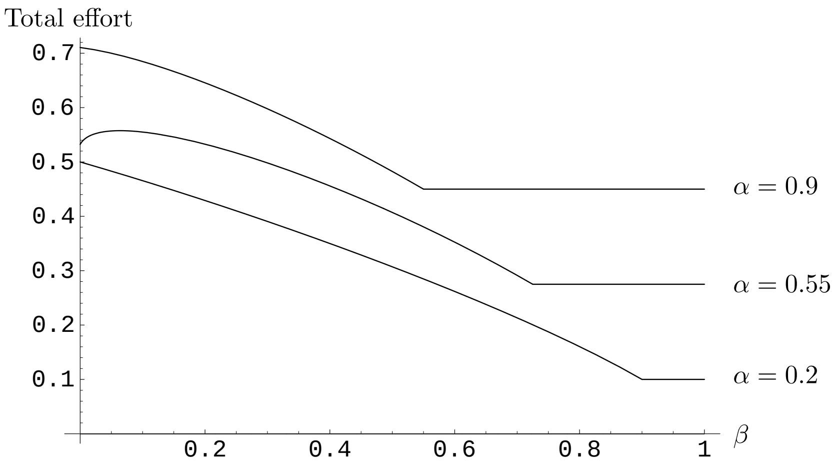

Figure 1: Total effort in Example 2; Condition (17) holds for $\alpha = 0.55$.

Example 3. *Let $n = 10$. Figure 2 displays on the horizontal axis the proportion (number) of advantaged agents m. In the left diagram the vertical axis indicates the optimal size β^* of the extra prize. In the right diagram the vertical axis shows total effort with the optimal extra prize $E_N(\beta^*)$ and without extra prize $E_N(\beta = 0)$; the difference is shaded in grey. Both diagrams show five levels of ability of the disadvantaged agents; $\alpha \in \{0.4^{\bullet}, 0.6^{\blacksquare}, 0.8^{\blacklozenge}, 0.9^{\blacktriangle}, 0.95^{\blacktriangledown}\}$. Each function in Figure 2 maintains a level of ability and the total number of agents, while varying the proportion of advantaged agents.*[30]

The example indicates the following pattern. First, as (for a given level of ability) the proportion of advantaged agents increases, the size of the optimal extra prize declines (left diagram), while total effort increases both with optimal extra prize and without extra prize (right diagram). The effect on the difference in total effort is ambiguous. Second, as (for a given proportion of advantaged agents) abilities become more homogeneous, the size of the optimal extra prize is a 'concave function' of the ability level (left diagram), while total effort increases both with optimal extra prize and without extra prize (right diagram). Again the effect on the difference in total effort is ambiguous but overall seems to decline in α. When the ability levels of the two groups are homogeneous, say $\alpha \in \{0.8, 0.9, 0.95\}$,

[30] For visual clarity we connect the points corresponding to a given ability level, even though the number of advantaged agents is a natural number. For the same reason we include the case $m = 9$, even though then an extra prize is never profitable, as there is only one disadvantaged agent. Detailed calculations are available upon request.

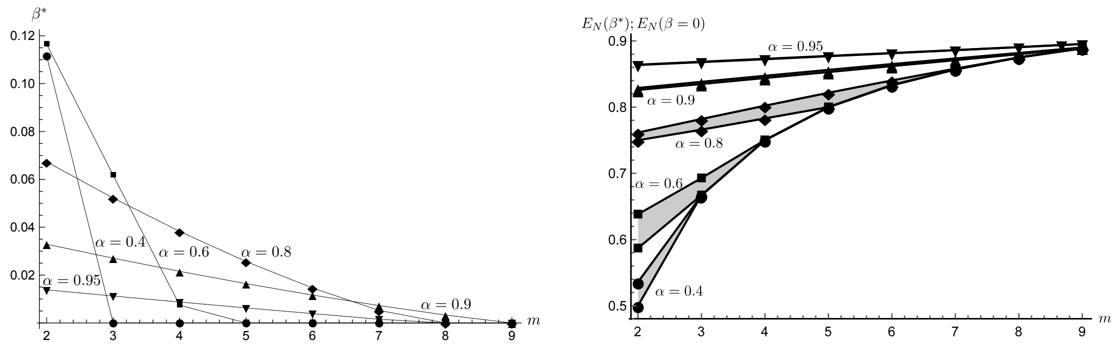

Figure 2: Illustration of Example 3. Left: optimal size of the extra prize. Right: total effort with the optimal extra prize and with no extra prize (difference in grey).

establishing an optimal extra prize is almost always profitable (left diagram) but it makes little difference in terms of total effort (right diagram). When, however, the ability levels of the two groups are different, say $\alpha \in \{0.4, 0.6, 0.8\}$*, establishing an optimal extra prize is only profitable when the number of advantaged agents is relatively low but in this case it makes a difference in terms of total effort.*

In Example 3 the maximal value for the optimal size β^* of the extra prize is below 12% of the total budget. Overall, the examples we have calculated seem to indicate that the optimal size declines with the number of agents. In the next subsection we look at examples with three agents and heterogeneous ability levels within groups in which the optimal extra prize is much larger than 12%.

3.6 Heterogeneity within groups

We consider now very briefly the case in which agents in a group have different abilities. We look at the simplest three agent setting in order to make the following two points. Example 4 proves that Condition (16) can be informative in such a setting, while Example 5 shows that Condition (16) is no longer a necessary condition under this assumption. Example 5 also shows that for small extra prizes only one disadvantaged contestant might be active and hence Lemma 1 cannot be strengthened. For small extra prizes only one disadvantaged agent might be active. In both examples we set $B = 1$.

Example 4. *Let* $M = \{1\}$ *and* $D = \{2, 3\}$*. Consider the following vector of abilities* $\alpha = (1, 0.105, 0.1)$*. Here disadvantaged agents are strong enough and always active.*

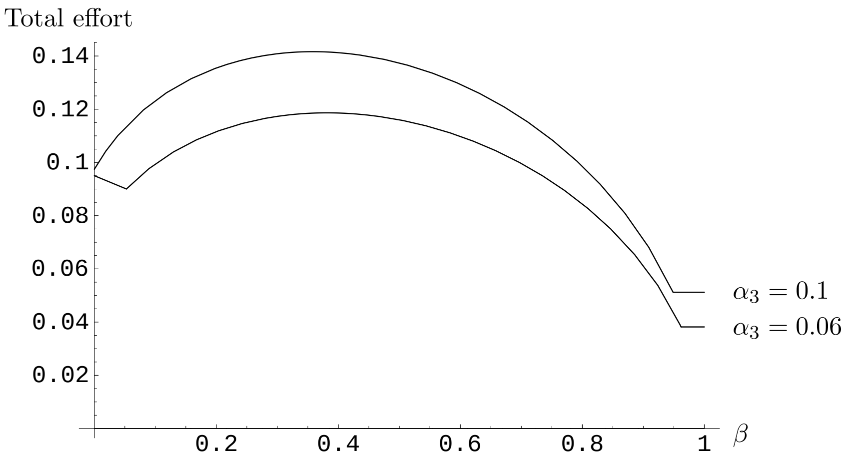

Figure 3: Total effort in Examples 4 and 5; Condition (16) holds for $\alpha_3 = 0.1$.

Straightforward calculation reveals that Condition (16) holds. Indeed the optimal extra prize is $\beta^* = 0.35$*, inducing an almost 45% increase of total effort with respect to a standard contest. Once the extra prize is large enough (*$\beta = 0.948$*), agent 1 ceases to be active. The upper curve in Figure 3 displays this case.*

Example 5. *Let* $M = \{1\}$ *and* $D = \{2, 3\}$*. Consider the same abilities as in Example 4 but set* $\alpha_3 = 0.06$*. This weakens agent 3 so that he is not active without an extra prize. Notice that Condition (16) does not hold. Once the extra prize becomes large enough, however, agent 3 becomes active (*$\beta = 0.0526$*). This fosters competition and increases total effort. The optimal extra prize is* $\beta^* = 0.38$*, inducing an almost 45% increase of total effort with respect to a standard contest. Once the extra prize is large enough (*$\beta = 0.96$*) agent 1 ceases to be active. The lower curve in Figure 3 visualizes this case.*

We have seen that the introduction of an extra prize might increase total effort and participation. The preceding examples, however, have also shown that all combinations of these consequences are possible: both total effort and participation increase in Example 5; neither total effort nor participation increase in Example 2 for $\alpha = 0.9$; total effort decreases while participation increases in Example 2 for $\alpha = 0.2$; and lastly, total effort increases while participation remains unaffected in Example 4.

4 Conclusions

This paper has analysed the effects of establishing an extra prize for disadvantaged agents in a contest model. We have shown that even very small extra prizes are very effective in making sure that there is minority representation in the competition. Moreover, for intermediate levels of the disadvantage of the minority, establishing an extra prize increases total equilibrium effort compared to a standard contest. Extra prizes might therefore be designed purely on efficiency grounds, which should facilitate the social acceptance of this affirmative action policy.

While our model is quite general in many respects, it relies on a specific contest success function and it is an important question how our results generalize to more general specifications. For instance, raising the individual efforts to the power of some number (larger than one), makes the contest success function more responsive to effort, until in the extreme the all-pay auction is reached. Recent work in Dahm (2017) has shown that in this latter case the main forces of our model are even stronger, because almost any extra prize is preferable to a standard all-pay auction without extra prize. This robustness seems to indicate that for intermediate exponents similar forces might be at work.

Related to this is the fact that our analysis allows a deeper understanding of the forces underlying the exclusion principle (see Baye et al. 1993). The principle says that a contest designer might sometimes strengthen competition and increase total effort by excluding the contestant with the highest valuation from participating in the competition. The exclusion principle applies when the contest success function is responsive enough to effort and includes the function introduced by Tullock for exponents larger than two (see Baye et al., 1993; Alcalde and Dahm, 2007; and Alcalde and Dahm, 2010). For the function employed in this paper, however, Fang (2002) has shown that the exclusion principle does not apply. This is consistent with our analysis, because establishing an extra prize reduces the main prize and *partially* excludes the most efficient competitor(s). As Fang (2002) has shown, complete exclusion is never beneficial. *Partial* exclusion, however, might foster competition and increase total effort. In this sense, our results show that a *partial* exclusion principle applies to Tullock contests.

Our analysis suggests several avenues for future research. A first avenue looks more systematically at the optimal size of the extra prize. How does the optimal size depend on the distribution of agents and abilities? A second avenue generalizes the contest structure to several extra prizes. Think of research funding that can be allocated through a single centralized contest or through several local contests. Beviá and Corchón (2015) investigate these issues for a wide class of contest success functions, including the one we employ. The authors provide conditions for the contest organizer to prefer centralized or decentralized contests under the assumption that the organizer must choose between the two structures. Researchers, however, compete often with very similar research proposals for funding both from local and central governments, where the former can be understood as extra prizes and the latter as the main prize. The results of our paper suggest that it might be optimal to combine local and central funding, but what is the optimal degree of decentralization of research funds? Another avenue endows the contest designer not only with the opportunity to create an extra prize but also with the power to choose the contestants that qualify for it. What is the optimal set of agents competing for the extra prize? These questions are outside the scope of the present paper and left for future research.

A Appendix

In this Appendix we provide a proof for the results stated in the main text. In addition to the notation introduced there, we find it convenient to use the following notation

$$\Sigma \equiv \frac{1}{4}\left(\frac{|M^*|-1}{|M^*|}\Gamma_M^* - \frac{|N^*|-1}{|N^*|}\Gamma_N^*\right)^2 + \frac{\Gamma_{M^*}\Gamma_{N^*}}{|M^*||N^*|}(|D^*|-1)\frac{\beta}{1-\beta}. \tag{18}$$

Also, given a group $G \in \{N, M, D\}$, it will prove useful to define a weighted harmonic mean of abilities as

$$\Lambda_{G^*} \equiv \frac{|G^*|-1}{|G^*|}\Gamma_{G^*}. \tag{19}$$

In addition, we simplify vectors of individual efforts when we focus on an agent i using the shorter notation $e = (e_i, e_{-i})$, where $e_{-i} = (\ldots, e_{i-1}, e_{i+1}, \ldots)$.

A.1 Proof of Lemma 1

For $\beta = 0$, the statement follows from Stein (2002). By way of contradiction let $\beta > 0$ and suppose that there is an equilibrium in which $e_i = 0$ for all agents $i \in D$. Suppose some advantaged agent exerts effort (otherwise a standard argument applies and proves the lemma). Notice that $EU_i(e_i, e_{-i}) = B_2/(n-m)$. Consider the alternative effort $\tilde{e}_i = \epsilon > 0$. This deviation yields

$$EU_i(\tilde{e}_i, e_{-i}) > B_2 - \frac{\epsilon}{\alpha_i},$$

where the inequality comes from the fact that agent i might win the main prize. Since $B_2 - \epsilon/\alpha_i$ is larger than $B_2/(n-m)$ for ϵ small enough, an equilibrium cannot have $e_i = 0$ for all agents $i \in D$. Q.E.D.

A.2 Proof of Proposition 1

Consider the first statement. Since $|D^*_{M^*=\emptyset}| \geq 2$, we have that $\Gamma_{D^*_{M^*=\emptyset}} > 0$. Thus we have $\overline{\beta} < 1$.

Consider now the second statement. It follows from Stein (2002) that if $M^* = \emptyset$ no agent $i \in D$ can profitably deviate from the strategies described in the statement. So let the agents $i \in D$ use these strategies and assume that $j \in M$ deviates to $e_j > 0$. Since the payoffs are concave, $\partial E_j(e)/\partial e_j|_{e_j=0} > 0$ must hold. This implies $\alpha_j B_1 > E_D$, where $D = D^*_{M^*=\emptyset}$, or equivalently

$$1 - \beta > \frac{|D^*_{M^*=\emptyset}| - 1}{|D^*_{M^*=\emptyset}|} \frac{\Gamma_{D^*_{M^*=\emptyset}}}{\Gamma_{\{j\}}}.$$

But since $\alpha_1 \geq \alpha_j$ and $\beta \in [\overline{\beta}, 1]$, we have

$$\frac{|D^*_{M^*=\emptyset}| - 1}{|D^*_{M^*=\emptyset}|} \frac{\Gamma_{D^*_{M^*=\emptyset}}}{\Gamma_{\{j\}}} \geq \frac{|D^*_{M^*=\emptyset}| - 1}{|D^*_{M^*=\emptyset}|} \frac{\Gamma_{D^*_{M^*=\emptyset}}}{\Gamma_{\{1\}}} \geq 1 - \beta,$$

a contradiction. Q.E.D.

A.3 Proof of Proposition 2

Consider the first statement. We have that $\overline{\beta} > 0$ if and only if

$$\alpha_1 > \frac{|D^*_{M^*=\emptyset}| - 1}{|D^*_{M^*=\emptyset}|} \Gamma_{D^*_{M^*=\emptyset}}.$$

This inequality holds, because

$$\alpha_1 \geq \Xi_{D^*_{M^*=\emptyset}} \geq \Gamma_{D^*_{M^*=\emptyset}} > \frac{|D^*_{M^*=\emptyset}| - 1}{|D^*_{M^*=\emptyset}|} \Gamma_{D^*_{M^*=\emptyset}},$$

where $\Xi_{D^*_{M^*=\emptyset}}$ is the arithmetic mean of abilities of the agents in $D^*_{M^*=\emptyset}$.[31]

Now consider the second statement and suppose $\beta \in (0, \overline{\beta})$. From Lemma 1 we know that $D^* \neq \emptyset$. By way of contradiction suppose that $M^* = \emptyset$. Then $e_1 = 0$, implying that $\partial E_1(e)/\partial e_1 \leq 0$ must hold. This implies $\alpha_1 B_1 \leq E_N$. On the other hand, $M^* = \emptyset$ implies that $E_N = E_D$, where $D = D^*_{M^*=\emptyset}$. Therefore the following must hold

$$1 - \beta \leq \frac{|D^*_{M^*=\emptyset}| - 1}{|D^*_{M^*=\emptyset}|} \frac{\Gamma_{D^*_{M^*=\emptyset}}}{\Gamma_{\{1\}}}.$$

Since, however $\beta < \overline{\beta}$ we obtain

$$\frac{|D^*_{M^*=\emptyset}| - 1}{|D^*_{M^*=\emptyset}|} \frac{\Gamma_{D^*_{M^*=\emptyset}}}{\Gamma_{\{1\}}} < 1 - \beta,$$

a contradiction. Q.E.D.

A.4 Proof of Proposition 3

We prove the statement with the help of two lemmatas.

Lemma 3. *For any (α, B), if $\beta \in (0, \overline{\beta})$, then*

$$E_N \in \{\Upsilon - \sqrt{\Upsilon^2 - \Phi}, \Upsilon + \sqrt{\Upsilon^2 - \Phi}\}.$$

Proof: First notice that the two candidate expressions for E_N are well defined, because we can write

$$\begin{aligned} \Upsilon^2 - \Phi \quad = \quad & \frac{(B_1)^2}{4} \left(\frac{|M^*| - 1}{|M^*|} \Gamma_{M^*} - \frac{|N^*| - 1}{|N^*|} \Gamma_{N^*} \right)^2 \\ & + \frac{B_1 \Gamma_{M^*} \Gamma_{N^*}}{|M^*||N^*|} (|D^*| - 1) B_2 \geq 0. \end{aligned} \quad (20)$$

Summing up (3) over all $i \in M^*$ and rearranging yields

$$E_D = \frac{(E_N)^2}{B_1} \frac{|M^*|}{\Gamma_{M^*}} - (|M^*| - 1) E_N. \quad (21)$$

[31] It is well known that the the arithmetic mean is not smaller than the harmonic mean and a proof is thus omitted.

Summing up (3) over all $i \in D^*$, inserting (21) and rearranging, yields the following quadratic equation

$$\begin{aligned} 0 &= (E_N)^2 \frac{\left(\sum_{i\in N^*} \frac{1}{\alpha_i}\right)\left(\sum_{i\in M^*} \frac{1}{\alpha_i}\right)}{B_1} \\ &\quad -E_N\left(\left(\sum_{i\in N^*}\frac{1}{\alpha_i}\right)(|M^*|-1)+\left(\sum_{i\in M^*}\frac{1}{\alpha_i}\right)(|N^*|-1)\right) \\ &\quad +(|N^*|-1)(|M^*|-1)B_1-(|D^*|-1)B_2. \end{aligned} \tag{22}$$

From here we obtain

$$(E_N)^2 - E_N 2\Upsilon + \Phi = 0,$$

implying the statement. Q.E.D.

Lemma 4. *For any (α, B), if $\beta \in (0, \overline{\beta})$, then*

$$E_N \neq \Upsilon - \sqrt{\Upsilon^2 - \Phi}.$$

Proof: Suppose $M^* \neq \emptyset$ and $E_N = \Upsilon - \sqrt{\Upsilon^2 - \Phi}$. Following (20), we can write $E_N = \Upsilon - \sqrt{X^2 + Y}$, where

$$\begin{aligned} X &\equiv \frac{B_1}{2}\left(\frac{|M^*|-1}{|M^*|}\Gamma_{M^*} - \frac{|N^*|-1}{|N^*|}\Gamma_{N^*}\right) \\ Y &\equiv \frac{B_1\Gamma_{M^*}\Gamma_{N^*}}{|M^*||N^*|}(|D^*|-1)B_2. \end{aligned}$$

Since the function $f(x) = \sqrt{x}$ is increasing in its argument and $Y \geq 0$, we have $E_N \leq \Upsilon - \sqrt{X^2}$, where

$$\sqrt{X^2} = \begin{cases} \frac{B_1}{2}\left(\Lambda_{M^*} - \Lambda_{N^*}\right) & \text{if} \quad \Lambda_{M^*} \geq \Lambda_{N^*} \\ \\ \frac{B_1}{2}\left(\Lambda_{N^*} - \Lambda_{M^*}\right) & \text{if} \quad \Lambda_{M^*} < \Lambda_{N^*} \end{cases} \tag{23}$$

and Λ_{M^*} and Λ_{N^*} are defined in (19). The remainder of the proof distinguishes these two cases and shows that each of them leads to $E_D \leq 0$, contradicting Lemma 1.

Suppose $\Lambda_{M^*} \geq \Lambda_{N^*}$, which implies that $E_N \leq \Lambda_{N^*} B_1$. Using (21) we obtain

$$E_D \leq E_N\left(\Lambda_{N^*}\frac{|M^*|}{\Gamma_{M^*}} - (|M^*|-1)\right) \leq 0,$$

where the last inequality comes from the fact that

$$\Lambda_{M^*} \geq \Lambda_{N^*} \Leftrightarrow \Lambda_{N^*}\frac{|M^*|}{\Gamma_{M^*}} \leq |M^*| - 1.$$

Suppose $\Lambda_{M^*} < \Lambda_{N^*}$, which implies that $E_N \leq \Lambda_{M^*} B_1$. Using (21) we obtain

$$E_D = E_N \left(\frac{E_N}{B_1}\frac{|M^*|}{\Gamma_{M^*}} - (|M^*| - 1) \right) \leq 0,$$

where the last inequality comes from the fact that

$$E_N \leq \Lambda_{M^*} B_1 \Leftrightarrow \frac{E_N}{B_1}\frac{|M^*|}{\Gamma_{M^*}} \leq |M^*| - 1.$$

Q.E.D.

Proposition 3 follows directly from Lemmatas 3 and 4. Q.E.D.

A.5 The derivatives mentioned in Subsection 3.3

Remember that by Proposition 2, $|D^*| \geq 1$ holds.

Claim 1. $\frac{\partial(E_N/B_1)}{\partial\beta} = 0$ *if* $|D^*| = 1$ *and* $\frac{\partial(E_N/B_1)}{\partial\beta} > 0$ *if* $|D^*| > 1$.

Proof: Let $\beta \in (0, \overline{\beta})$. Suppose $|D^*| = 1$. From (20), we have $E_N = \Upsilon + \sqrt{X^2}$, where $\sqrt{X^2}$ is defined in (23). This implies that

$$E_N = \begin{cases} \Lambda_{M^*} B_1 & \text{if} \quad \Lambda_{M^*} \geq \Lambda_{N^*} \\ \\ \Lambda_{N^*} B_1 & \text{if} \quad \Lambda_{M^*} < \Lambda_{N^*} \end{cases}, \tag{24}$$

where Λ_{M^*} and Λ_{N^*} are defined in (19). Assume $E_N = \Lambda_{M^*} B_1$. Inserting this expression in (21) implies $E_D = 0$, contradicting Proposition 2. Thus $E_N = \Lambda_{N^*} B_1$ and $\partial\left(E_N/B_1\right)/\partial\beta = 0$.

Suppose $|D^*| > 1$. We have that

$$\frac{E_N}{B_1} = \frac{1}{2}\left(\frac{|M^*| - 1}{|M^*|}\Gamma_M^* + \frac{|N^*| - 1}{|N^*|}\Gamma_N^* \right) + \sqrt{\Sigma},$$

with $\sqrt{\Sigma} > 0$, where Σ is defined in (18). Taking the derivative we obtain

$$\frac{\partial\left(E_N/B_1\right)}{\partial\beta} = \frac{\Gamma_{M^*}\Gamma_{N^*}(|D^*| - 1)}{2|M^*||N^*|(1-\beta)^2\sqrt{\Sigma}} > 0.$$

Q.E.D.

Claim 2. $\frac{\partial(B_2/(B_1\Omega))}{\partial\beta} > 0.$

Proof: Let $\beta \in (0, \overline{\beta})$. We have that

$$\frac{\partial\left(\frac{B_2}{B_1\Omega}\right)}{\partial\beta} = \frac{\partial\left(\frac{\beta}{1-\beta}\right)}{\partial\beta}\frac{1}{\Omega} + \frac{\beta}{1-\beta}\frac{\partial\left(\frac{1}{\Omega}\right)}{\partial\beta}.$$

Notice that

$$\frac{\partial\left(\frac{\beta}{1-\beta}\right)}{\partial\beta}\frac{1}{\Omega} = \frac{1}{(1-\beta)^2\Omega} > 0,$$

as $\Omega \in (0,1)$. Suppose $|D^*| = 1$. From Claim 1, we know that $E_N = \Lambda_{N^*}B_1$. This implies that Ω is independent of β. Therefore $\frac{\partial(1/(\Omega))}{\partial\beta} = 0$ and $\frac{\partial(B_2/(B_1\Omega))}{\partial\beta} > 0$.

Suppose $|D^*| > 1$ and remember that this implies that $\sqrt{\Sigma} > 0$. Using Claim 1, we obtain

$$\frac{\beta}{1-\beta}\frac{\partial\left(\frac{1}{\Omega}\right)}{\partial\beta} = -\frac{\beta}{(1-\beta)^3\Omega^2}\frac{\Gamma_{N^*}(|D^*|-1)}{2|N^*|\sqrt{\Sigma}}.$$

Hence we have that $\frac{\partial(B_2/(B_1\Omega))}{\partial\beta} > 0$ if and only if

$$1 - |M^*| + \frac{E_N}{B_1}\frac{|M^*|}{\Gamma_{M^*}} > \frac{\beta}{(1-\beta)}\frac{\Gamma_{N^*}(|D^*|-1)}{2|N^*|\sqrt{\Sigma}}.$$

Introducing E_N/B_1 from Claim 1 and rearranging yields

$$\frac{1}{2}\left(\frac{|N^*|-1}{|N^*|}\Gamma^*_N - \frac{|M^*|-1}{|M^*|}\Gamma^*_M\right) + \sqrt{\Sigma} > \frac{\beta}{(1-\beta)}\frac{|D^*|-1}{2\sqrt{\Sigma}}\frac{\Gamma_{N^*}}{|N^*|}\frac{\Gamma_{M^*}}{|M^*|}.$$

Multiplying by $2\sqrt{\Sigma}$ and collecting terms we obtain

$$\begin{aligned}&\frac{1}{2}\left(\frac{|M^*|-1}{|M^*|}\Gamma^*_M - \frac{|N^*|-1}{|N^*|}\Gamma^*_N\right)^2 + \frac{\beta(|D^*|-1)}{(1-\beta)}\frac{\Gamma_{N^*}}{|N^*|}\frac{\Gamma_{M^*}}{|M^*|}\\ > &\left(\frac{|M^*|-1}{|M^*|}\Gamma^*_M - \frac{|N^*|-1}{|N^*|}\Gamma^*_N\right)\sqrt{\Sigma}.\end{aligned}$$

Squaring and cancelling terms yields finally that $\frac{\partial(B_2/(B_1\Omega))}{\partial\beta} > 0$ if and only if

$$\frac{\beta(|D^*|-1)}{(1-\beta)}\frac{\Gamma_{N^*}}{|N^*|}\frac{\Gamma_{M^*}}{|M^*|} > 0,$$

which is true for $\beta > 0$. Q.E.D.

A.6 Proof of Proposition 4

We prove Proposition 4 in two steps. In Claim 3 we confirm that the candidate strategies in (9) and (10) are indeed an equilibrium. Second, we prove in Claim 4 uniqueness of the equilibrium in pure strategies.

Claim 3. *The candidate efforts described in Proposition 4 constitute an equilibrium.*

Proof: Denote the vector of individual candidate equilibrium efforts by e^*. Consider an agent i and assume that all other agents exert the candidate equilibrium effort e_j^* for any $j \neq i$. Agent i chooses $e_i \geq 0$ in order to maximize (1). Since we already know that active agents do not have an incentive to deviate from the candidate equilibrium effort, consider inactive agents.

Consider first agent $i \in \{m^* + 1, \ldots, m\}$. The first order condition evaluated at $e_i^* = 0$ is

$$\frac{1}{E_N(e^*)} B_1 - \frac{1}{\alpha_i} \leq 0,$$

which by (12) is negative. Hence $e_i^* = 0$ is indeed a best response.

Consider next agent $i \in \{d^* + 1, \ldots, n\}$. The first order condition evaluated at $e_i^* = 0$ is

$$\frac{1}{E_N(e^*)} B_1 + \frac{1}{E_D(e^*)} B_2 - \frac{1}{\alpha_i} \leq 0,$$

which using the definition of Ω and (13) can be shown to be negative. Thus $e_i^* = 0$ is indeed a best response. Q.E.D.

Claim 4. *There is no other pure strategy equilibrium.*

Proof: Proceeding by contradiction, suppose that for a given β there are two different sets of active contestants $H \equiv N_{d-j}^{m}$ and $J \equiv N_{d}^{m-k}$. Notice that j and k must both be strictly larger than zero.[32] Moreover, Lemma 1 implies that we can focus on $1 \leq k \leq m$ and $1 \leq j < d$. In each equilibrium we indicate total effort by E_H and E_J; and distinguish similarly Ω_H and Ω_J.

In equilibrium H the following participation conditions must hold

$$\alpha_m > \frac{E_H}{B_1}, \quad \alpha_d \leq \frac{E_H}{B_1} \frac{B_1 \Omega_H}{B_1 \Omega_H + B_2},$$

[32] If both are zero, the sets are the same. By construction of the sets, if $j = 0$ and $k > 0$, then, given d^*, $m - k$ is not the largest index, as $m > m - k$; and similarly for $j > 0$ and $k = 0$.

while in equilibrium J we must have, when $k < m$,

$$\alpha_m \leq \frac{E_J}{B_1}, \quad \alpha_d > \frac{E_J}{B_1}\frac{B_1\Omega_J}{B_1\Omega_J + B_2},$$

and for $k = m$, as $\Omega_J = 1$,

$$\alpha_d > \frac{E_J}{B_1}\frac{B_1}{B_1 + B_2}.$$

Notice that the conditions referring to agent m imply that $E_J > E_H$. We establish now that $\Omega_J > \Omega_H$ holds.

$$\begin{aligned}
\Omega_J - \Omega_H &= m\left(1 - \frac{E_H}{B_1\Gamma_{N^m_{d-j}\cap M}}\right) - (m-k)\left(1 - \frac{E_J}{B_1\Gamma_{N^{m-k}_d\cap M}}\right) \\
&= k - \frac{mE_H}{B_1\Gamma_{N^m_{d-j}\cap M}} + \frac{(m-k)E_J}{B_1\Gamma_{N^{m-k}_d\cap M}} \\
&> k - \frac{mE_H}{B_1\Gamma_{N^m_{d-j}\cap M}} + \frac{(m-k)E_H}{B_1\Gamma_{N^{m-k}_d\cap M}}.
\end{aligned}$$

This last expression is strictly larger than zero if and only if

$$\frac{B_1}{E_H}k - \sum_{i\leq m}\frac{1}{\alpha_i} + \sum_{i\leq m-k}\frac{1}{\alpha_i} = \frac{B_1}{E_H}k - \sum_{i=m-k+1}^{m}\frac{1}{\alpha_i} > 0.$$

Since

$$\sum_{i=m-k+1}^{m}\frac{1}{\alpha_i} \leq k\frac{1}{\alpha_m},$$

the last expression is implied by the participation condition of agent m in equilibrium H.

Lastly, notice that $E_J > E_H$ and $\Omega_J > \Omega_H$, on one hand, and the participation conditions of agent d in both equilibria, on the other, imply the following. For $k < m$,

$$\alpha_d > \frac{E_J}{B_1}\frac{B_1\Omega_J}{B_1\Omega_J + B_2} > \frac{E_H}{B_1}\frac{B_1\Omega_H}{B_1\Omega_H + B_2} \geq \alpha_d,$$

and for $k = m$, as $1 = \Omega_J > \Omega_H$ and $\frac{B_1}{B_1+B_2} > \frac{B_1\Omega_H}{B_1\Omega_H+B_2}$,

$$\alpha_d > \frac{E_J}{B_1}\frac{B_1}{B_1 + B_2} > \frac{E_H}{B_1}\frac{B_1\Omega_H}{B_1\Omega_H + B_2} \geq \alpha_d.$$

In both cases we reach the desired contradiction. Q.E.D.

The above two claims complete the proof of Proposition 4. Q.E.D.

A.7 Proof of Proposition 5

We start proving that E_N is continuous on $[0,1]$.

Lemma 5. *$E_N(\beta)$ is continuous on $[0,1]$.*

Proof: The statement follows from four claims.

Claim 5. *For $\beta = 0$, (7) reduces to (6).*

Proof: Let $\beta = 0$. As in the proof of Claim 1, from (20), we have $E_N = \Upsilon + \sqrt{X^2}$, where $\sqrt{X^2}$ is defined in (23). Again, total effort is given in (24). Assume $E_N = \Lambda_{M^*} B_1$, where Λ_{M^*} is defined in (19). (21) implies that $E_D = 0$ and thus $\Lambda_{M^*} B_1 = \Lambda_{N^*} B_1$. Summarizing, for $\beta = 0$ (7) reduces to $E_N = \Lambda_{N^*} B_1$, the desired expression (6). Q.E.D.

Claim 6. *For $\beta = \overline{\beta}$, (7) reduces to (6), which is constant on $[\overline{\beta}, 1]$.*

Proof: Let $\beta = \overline{\beta} - \epsilon$, for $\epsilon > 0$ arbitrarily close to zero. If $i \in M^*$ then $\alpha_i = \alpha_1$. This implies that $\Gamma_{M^*} = \alpha_1$. Moreover, when β goes to $\overline{\beta}$ we have that $e_i = 0$ for all $i \in M$ and $|N^*| = |D^*|$ (and $\Gamma_{N^*} = \Gamma_{D^*}$). Using these simplifications and the definition of $\overline{\beta}$ in (20), we have

$$\begin{aligned} \Upsilon^2 - \Phi &= \left(B\frac{\Lambda_{D^*}}{\alpha_1}\right)^2 \left(\frac{1}{4}\Lambda_{M^*}^2 + \frac{1}{4}\Lambda_{D^*}^2 - \frac{1}{2}\Lambda_{M^*}\Lambda_{D^*} + \frac{\alpha_1^2}{|M^*|} - \frac{\alpha_1}{|M^*|}\Lambda_{D^*}\right) \\ &= \left(B\frac{\Lambda_{D^*}}{\alpha_1}\right)^2 \frac{1}{4}\left(\frac{(|M^*|+1)}{|M^*|}\alpha_1 - \Lambda_{D^*}\right)^2. \end{aligned}$$

Since $(|M^*|+1)\alpha_1/|M^*| > \Lambda_{D^*}$,

$$\begin{aligned} E_N &= \frac{B}{2}\frac{\Lambda_{D^*}}{\alpha_1}\left(\frac{|M^*|-1}{|M^*|}\alpha_1 + \Lambda_{D^*} + \frac{|M^*|+1}{|M^*|}\alpha_1 - \Lambda_{D^*}\right) \\ &= \frac{|D^*|-1}{|D^*|}\Gamma_{D^*} B. \end{aligned}$$

Q.E.D.

Since the exit of the contestant(s) with the highest ability is covered by Claim 6, we next show continuity of total effort when, given a set of active agents, the most inefficient advantaged agent with ability different from contestant 1 ceases to be active.

Claim 7. *Let $m^* > 1$ and consider $E_N(\beta)$ given by (7). When β is such that $1/\alpha_{m^*} = B_1/E_N$, $E_N(\beta)$ is continuous.*

Proof: From the proof of Lemma 3, we know that (7) is a root of (22). We show that at the value for β in question, (22) is the same as a similar expression defined for the case in which agent m^* has exited. Thus the roots must be the same, too. Using $1/\alpha_{m^*} = B_1/E_N$ in (22) we obtain

$$\frac{(E_N)^2}{B_1}\left(\left(\sum_{i\in N^*\backslash m^*}\frac{1}{\alpha_i}\right)+\frac{B_1}{E_N}\right)\left(\left(\sum_{i\in M^*\backslash m^*}\frac{1}{\alpha_i}\right)+\frac{B_1}{E_N}\right)$$
$$-E_N\left(\left(\left(\sum_{i\in N^*\backslash m^*}\frac{1}{\alpha_i}\right)+\frac{B_1}{E_N}\right)(|M^*|-1)+\left(\left(\sum_{i\in M^*\backslash m^*}\frac{1}{\alpha_i}\right)+\frac{B_1}{E_N}\right)(|N^*|-1)\right)$$
$$+(|N^*|-1)(|M^*|-1)B_1-(|D^*|-1)B_2.$$

Rearranging yields

$$\frac{(E_N)^2}{B_1}\left(\sum_{i\in N^*\backslash m^*}\frac{1}{\alpha_i}\right)\left(\sum_{i\in M^*\backslash m^*}\frac{1}{\alpha_i}\right)$$
$$-E_N\left(\left(\sum_{i\in N^*\backslash m^*}\frac{1}{\alpha_i}\right)(|M^*|-2)+\left(\sum_{i\in M^*\backslash m^*}\frac{1}{\alpha_i}\right)(|N^*|-2)\right)$$
$$+(|N^*|-2)(|M^*|-2)B_1-(|D^*|-1)B_2,$$

that is, (22) for m^*-1 active agents. Q.E.D.

We turn now to show continuity of total effort when, given a set of active agents, a disadvantaged agent becomes active or ceases to be active.

Claim 8. *Let $j \in D$ and consider $E_N(\beta)$ given by (7). When β is such that $1/\alpha_j = B_1/E_N + B_2/E_D$, $E_N(\beta)$ is continuous.*

Proof: We proceed as in the proof of Claim 7. Notice that when $1/\alpha_j = B_1/E_N+B_2/E_D$, agent i's optimal effort choice is zero. Using the equilibrium values for E_D and Ω this condition can be written as

$$B_2 = \frac{(E_N)^2}{B_1}\frac{\left(\sum_{i\in M^*}\frac{1}{\alpha_i}\right)}{\alpha_j} - E_N\left(\sum_{i\in M^*}\frac{1}{\alpha_i}\right) - \frac{(|M^*|-1)E_N}{\alpha_j} + (|M^*|-1)B_1.$$

Suppose agent $j \in D$ becomes active. From (22) we obtain

$$\begin{aligned}
&\frac{(E_N)^2}{B_1}\left(\sum_{i\in M^*}\frac{1}{\alpha_i}\right)\left(\left(\sum_{i\in N^*\cup j}\frac{1}{\alpha_i}\right)-\frac{1}{\alpha_j}\right)\\
&-E_N\left(\left(\sum_{i\in N^*\cup j}\frac{1}{\alpha_i}\right)(|M^*|-1)+\left(\sum_{i\in M^*}\frac{1}{\alpha_i}\right)|N^*|\right)+E_N\left(\frac{|M^*|-1}{\alpha_j}+\left(\sum_{i\in M^*}\frac{1}{\alpha_i}\right)\right)\\
&+|N^*|(|M^*|-1)B_1-(|M^*|-1)B_1-|D^*|B_2+B_2.
\end{aligned}$$

Rearranging and using the above expression for B_2 yields

$$\begin{aligned}
&\frac{(E_N)^2}{B_1}\left(\sum_{i\in M^*}\frac{1}{\alpha_i}\right)\left(\sum_{i\in N^*\cup j}\frac{1}{\alpha_i}\right)-E_N\left(\left(\sum_{i\in N^*\cup j}\frac{1}{\alpha_i}\right)(|M^*|-1)+\left(\sum_{i\in M^*}\frac{1}{\alpha_i}\right)|N^*|\right)\\
&+|N^*|(|M^*|-1)B_1-|D^*|B_2,
\end{aligned}$$

that is, we obtain (22) for the set of contestants $N^*\cup j$. The proof of the case in which agent $i \in D$ ceases to be active proceeds along the same lines. Q.E.D.

We are now in a position to apply Weierstrass' extreme value theorem, which guarantees that for any (α, B), there is $\beta^* \in [0,1]$ such that $E_N(\beta)$ attains a maximum at β^*. Moreover, given that $E_N(\beta)$ is constant on $[\overline{\beta},1]$ and Proposition 1 in Fang (2002), which says that with the contest success function assumed the exclusion principle does not apply, we conclude that $\beta^* \in [0,\overline{\beta})$. Q.E.D.

A.8 Proof of Proposition 6

In order to simplify notation, in this proof we drop the subindex ϵ and use the expressions Σ and Λ_{G^*} from (18) and (19), respectively. Consider $\beta \in (0,\bar{\beta}]$ for which total effort is given by (7). In order to compute the derivative with respect to β rewrite (7) as follows

$$E_N = B_1\left[\frac{1}{2}\left(\Lambda_{M^*}+\Lambda_{N^*}\right)+\sqrt{\Sigma}\right].$$

The derivative of total effort with respect to β can then be expressed as

$$\frac{\partial E_N}{\partial\beta}=\frac{\partial B_1}{\partial\beta}\left[\frac{1}{2}\left(\Lambda_{M^*}+\Lambda_{N^*}\right)+\sqrt{\Sigma}\right]+B_1\frac{\partial\left[\frac{1}{2}\left(\Lambda_{M^*}+\Lambda_{N^*}\right)+\sqrt{\Sigma}\right]}{\partial\beta}.$$

Using the expression derived in the proof of Claim 1, we obtain

$$\frac{\partial E_N}{\partial\beta}=\frac{B\Gamma_{N^*}\Gamma_{M^*}(|D^*|-1)}{2(1-\beta)\,|N^*|\,|M^*|\sqrt{\Sigma}}-\frac{B}{2}\left(\Lambda_{M^*}+\Lambda_{N^*}\right)-B\sqrt{\Sigma}.$$

This expression is well defined, because $|D^*| > 1$ implies that $\sqrt{\Sigma} > 0$. On the other hand when β goes to zero, $\sqrt{\Sigma}$ goes to

$$\begin{cases} 0 & \text{if} \quad \Lambda_{M^*} = \Lambda_{N^*} \Leftrightarrow \frac{\Gamma_{N^*}}{\Gamma_{M^*}} = \gamma\delta\zeta \\ \\ \frac{\Lambda_{M^*} - \Lambda_{N^*}}{2} & \text{if} \quad \Lambda_{M^*} > \Lambda_{N^*} \Leftrightarrow \frac{\Gamma_{N^*}}{\Gamma_{M^*}} < \gamma\delta\zeta \\ \\ \frac{\Lambda_{N^*} - \Lambda_{M^*}}{2} & \text{if} \quad \Lambda_{M^*} < \Lambda_{N^*} \Leftrightarrow \frac{\Gamma_{N^*}}{\Gamma_{M^*}} > \gamma\delta\zeta \end{cases} .$$

Hence in the first case $\Lambda_{M^*} = \Lambda_{N^*}$, when β goes to zero, $\partial E_N / \partial\beta > 0$.

Assume the second case $\Lambda_{M^*} > \Lambda_{N^*}$. We have that $\partial E_N / \partial\beta|_{\beta=0} > 0$ if and only if

$$\frac{\Gamma_{N^*}\Gamma_{M^*}}{|N^*|\,|M^*|} \left(|D^*| - 1\right) > \Lambda_{M^*} \left(\Lambda_{M^*} - \Lambda_{N^*}\right).$$

Because of Proposition 2 and $|D^*| > 1$, $|N^*| = |M^*| + |D^*| > 2$,

$$(|D^*| - 1) + (|M^*| - 1)(|N^*| - 1) = |M^*| \left(|N^*| - 2\right) > 0,$$

the previous expression can be rewritten as,

$$\gamma^2\zeta < \frac{\Gamma_{N^*}}{\Gamma_{M^*}}. \tag{25}$$

Assume the third case $\Lambda_{M^*} < \Lambda_{N^*}$. We have that $\partial E_N / \partial\beta|_{\beta=0} > 0$ if and only if

$$\frac{\Gamma_{N^*}\Gamma_{M^*}}{|N^*|\,|M^*|} \left(|D^*| - 1\right) > \Lambda_{N^*} \left(\Lambda_{N^*} - \Lambda_{M^*}\right),$$

yielding through a similar transformation

$$\frac{\Gamma_{N^*}}{\Gamma_{M^*}} < \delta^2\zeta. \tag{26}$$

Summarizing, we have that $\beta^* > 0$ if one of the following holds:

$$\gamma^2\zeta < \frac{\Gamma_{N^*}}{\Gamma_{M^*}} < \gamma\delta\zeta; \qquad \frac{\Gamma_{N^*}}{\Gamma_{M^*}} = \gamma\delta\zeta; \qquad \gamma\delta\zeta < \frac{\Gamma_{N^*}}{\Gamma_{M^*}} < \delta^2\zeta. \tag{27}$$

The fact that $\gamma^2\zeta \leq \gamma\delta\zeta < \delta^2\zeta$ with the first inequality being strict whenever $|M^*| > 1$ implies the statement. Q.E.D.

A.9 Proof of Proposition 7

The fact that condition (17) implies that $\beta^* > 0$ follows from straightforward algebraic simplification of condition (16).

Suppose $\beta^* > 0$. By Fang's well known result that it is never optimal to exclude all advantaged agents (Fang, 2002), we have that $E_N(\beta = 0) > E_N(\beta = \overline{\beta})$. Note also that in between these values for β the function $E_N(\beta)$ is twice differentiable. The fact that when β goes to zero, $\partial E_N/\partial\beta > 0$ follows then from noticing that in between these values for β the function $E_N(\beta)$ is strictly concave and that the condition on the derivative implies condition (17).[33]

The last statement follows from observing that the upper bound of condition (17) is strictly larger than the lower bound. Q.E.D.

[33]The second derivative of total effort is negative. Using Σ defined in (18), we have

$$\frac{\partial^2 E_N}{\partial \beta^2} = -\frac{B}{(1-\beta)\sqrt{\Sigma}}\left(\frac{\bar{\alpha}\Gamma_N(|D^*|-1)}{2(1-\beta)|M^*||N^*|\sqrt{\Sigma}}\right)^2.$$

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