Empirical safety stock estimation based on kernel and GARCH models

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Abstract

Supply chain risk management has drawn the attention of practitioners and academics alike. One source of risk is demand uncertainty. Demand forecasting and safety stock levels are employed to address this risk. Most previous work has focused on point demand forecasting, given that the forecast errors satisfy the typical normal i.i.d. assumption. However, the real demand for products is difficult to forecast accurately, which means that—at minimum—the i.i.d. assumption should be questioned. This work analyzes the effects of possible deviations from the i.i.d. assumption and proposes empirical methods based on kernel density estimation (non-parametric) and GARCH(1,1) models (parametric), among others, for computing the safety stock levels. The results suggest that for shorter lead times, the normality deviation is more important, and kernel density estimation is most suitable. By contrast, for longer lead times, GARCH models are more appropriate because the autocorrelation of the variance of the forecast errors is the most important deviation. In fact, even when no autocorrelation is present in the original demand, such autocorrelation can be present as a consequence of the overlapping process used to compute the lead time forecasts and the uncertainties arising in the estimation of the parameters of the forecasting

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model. Improvements are shown in terms of cycle service level, inventory investment and backorder volume. Simulations and real demand data from a manufacturer are used to illustrate our methodology.

1. Introduction

Supply chain risk management (SCRM) is becoming an area of interest for researchers and practitioners, and the number of related publications is growing [1]. A critical review of SCRM was presented in [2]. An important source of risk is the uncertainty of future demand. When demand is unusually large, a stockout may occur, with associated negative consequences. When demand is below expectations, the company is saddled with higher holding costs due to excess inventory. To mitigate such risks, companies use safety stocks, which are additional units beyond the stock required to meet a lead time forecasted demand. Different approaches can be utilized to calculate safety stock [3]. Although the most appropriate method depends on an organization's circumstances, calculating the safety stock based on customer service is widely used, because it does not require knowledge of the stockout cost, which can be very difficult to estimate.

Demand uncertainty has been estimated by using both demand variability [4] and demand forecast error variability ([3], [5]). In this work, we follow the latter approach, given that future demand is unknown and must be forecasted. Therefore, safety stocks are intended to prevent issues due to such demand forecast errors.

Demand forecast errors are typically assumed to be independent and identically distributed (i.i.d.), following a Gaussian distribution with zero mean and constant variance. However, the forecast errors often do not satisfy these assumptions, which can cause the achieved service levels to deviate from the target levels, with increased inventory costs [6, 7, 8, 9]. Even when the distribution assumptions are met, the uncertainties that arise when estimating the forecasting models are usually not considered [10, 11].

Two approaches can be used to address the problem of forecast errors that do not satisfy i.i.d. assumption. First, a theoretical approach, in which the demand forecasting model should be improved by modeling, for example, the mean demand autocorrelation [12, 13]; including exogenous variables such

as promotional variables [14, 9, 15] or weather variables [9] via regression-type models; as well as considering temporal heteroscedastic processes [16, 17, 18, 9, 19]. Under the assumption that the improved forecasting model captures the underlying data process, a theoretical formula for the cumulative variance of the forecast error can be determined [20]. Then, the safety stock can be computed using that theoretical formula and assuming a statistical distribution.

Nonetheless, it would be unrealistic to assume that we could find the "true" demand model for each SKU, and even if such models did exist, it would not always be possible for companies to implement them in practice. For instance, many companies judgmentally adjust statistical forecasts, which may induce forecast error biases [21, 22]. Furthermore, as pointed out by Fildes [23], companies often do not implement the latest research in forecasting, relying instead on software vendors who have been shown to be very conservative. Moreover, the choice of demand forecasting model may not be under the control of the operations/inventory planning department [6].

For the aforementioned reasons, an alternative to the theoretical approach is an empirical, data-driven counterpart. In a univariate framework, the empirical approach involves collecting the observed lead time forecast errors of whatever forecasting model a company has available and, subsequently calculating the quantiles of interest to determine the safety stock. Note that in this empirical approach, we need to know neither the point forecasting model nor its parameters. This is potentially advantageous in the case of forecasting support systems, which often do not provide such information to users. For the case in which exogenous variables are available, Beutel and Minner [9], working within a multivariate framework, proposed a data-driven linear programming approach in which the inventory level could be optimized by minimizing a cost objective or subject to a service level constraint.

Depending on which i.i.d. assumptions are violated, different options can be applied to improve the empirical safety stock calculation. For a multivariate case, Beutel and Minner [9] showed that the linear programming approach was robust to both unconditional bias (where demand is assumed to be independent of time) and unconditional heteroscedasticity as well as non-normal residuals. In the same reference, it was reported that in the case of heteroscedasticity due to independent variables, a multiple linear regression optimized via the ordinary least squares approach could be improved by relating standard deviations to the levels of explanatory variables (for example, price) in a parametric functional form.

3

Several methodologies have been proposed to correct the problem that arises in a univariate framework in the case that the lead time forecast errors are biased. The authors of [6] adjusted the standard deviation of the forecast errors and reference [24] proposed a kernel-smoothing technique to cope with such unconditional bias. If the mean demand is autocorrelated and is not properly modeled, then the forecast error bias (if any) may be autocorrelated as well. Lee [8] developed a semi-parametric approach to estimate the critical fractiles for autocorrelated demand, where the conditional lead time demand forecast bias is a parametric linear function of the recent demand history and the quantile is obtained non-parametrically from the empirical distribution of the regression errors.

Likewise, the forecast errors for a given lead time may be heteroscedastic and deviate from normality [24, 8]. Non-normal residuals have been addressed by means of kernel smoothing [24] and the empirical lead time error distribution quantile approach [8]. However, the case in which the lead time forecast errors exhibit conditional heteroscedasticity remains understudied.

In this work, the goal is to compute the safety stock for a given lead time using empirical methods based on the measured lead time forecast errors. Generalised AutoRegressive Conditional Heteroscedastic (GARCH) models [25] and exponential smoothing models [26] are investigated to overcome the assumption of the independence of the standard deviation of the lead time forecast errors and to exploit potential correlations.

To the best of the authors' knowledge, this is the first time that GARCH models have been applied empirically to forecast lead time error standard deviations and compute safety stocks. Furthermore, we explore the use of empirical non-parametric approaches such as kernel density estimation [27], which is commonly utilized in prediction interval studies [28], although its use in supply chain applications has been less frequent [24, 29]. The proposed empirical approaches are also compared with the traditional supply chain theoretical approaches, which are based on i.i.d. assumptions, using both simulated data and real data from a manufacturing company. The performance of the different approaches are measured using inventory investment, service level and backorder metrics. First, a newsvendor-type model is assumed, and subsequently, the results are compared with those of an order-up-to-level stock control policy.

The remainder of this paper is organized as follows: Section 2 formulates the safety stock problem. Section 3 reviews the theoretical approach for computing the safety stock. Section 4 describes the empirical approximation

4

proposed in this work. Section 5 explains the performance metrics used to assess the different methods. This section also provides implementation details for the alternative approaches. Simulation experiments are reported in Section 6. A case study based on shipment data from a real manufacturer is presented in Section 7. Finally, Section 8 presents concluding remarks.

2. Problem formulation

When the demand forecasting error is Gaussian i.i.d. with zero mean and constant variance, the safety stock (SS) for a target cycle service level (CSL), expressed as the target probability of no stockout during the lead time, can be computed as follows:

$$SS = k\sigma_L, \tag{1}$$

where $k = \Phi^{-1}(CSL)$ is the safety factor, $\Phi(\cdot)$ is the standard normal cumulative distribution function, and σ_L is the standard deviation of the forecast errors for a certain lead time L, which is assumed to be constant and known.

The main challenge associated with (1) is how to estimate σ_L. To do this, we consider two approaches: a theoretical approach based on a forecasting model, in which an estimate of σ_1 (one-step-ahead standard deviation of the forecast error) is provided and an analytical expression that relates σ_L to the forecasting model parameters, the lead time and σ_1 is employed. The estimation of σ_1 is possible because the forecast updating step is usually smaller than the lead time. Alternatively, an empirical parametric approach can also be employed in which σ_L is estimated directly from the lead time forecast error. Here, the term parametric indicates that a known statistical distribution is assumed; normality is traditionally chosen for this purpose.

If the statistical distribution of the error is unknown, empirical non-parametric methods can be used instead. In that case, the safety stock calculation should be reformulated as follows:

$$SS = Q_L(CSL), \tag{2}$$

where $Q_L(CSL)$ is the forecast error quantile at the probability defined by CSL. We discuss the theoretical approach in detail below and the empirical parametric and non-parametric approaches in the subsequent section.

3. Theoretical approach

3.1. Estimation of σ_1

Traditional textbooks suggest computing σ_1 based on forecast error metrics. For instance, the mean squared error (MSE) is used in [3], and the mean absolute deviation (MAD) is used in [5]. However, the conversion factor from the MAD to σ_1 depends on the assumed statistical distribution, and an inappropriate choice can result in a safety stock level that is inadequate to satisfy the required service level [3]. Therefore, hereafter, we use the MSE to estimate σ_1, i.e. $\sigma_1 = \sqrt{MSE}$. Moreover, the MSE estimation can be updated as new observations become available, as follows:

$$MSE_{t+1} = \alpha'(y_t - F_t)^2 + (1 - \alpha')MSE_t, \tag{3}$$

where y_t is the actual value at time t, F_t is the forecast value for the same time period, MSE_{t+1} is the MSE forecast for period $t + 1$ and α' is a smoothing constant that varies between 0 and 1. In this work, α' and the initial value are optimized by minimizing the in-sample squared error following the suggestion by [30]. Note that (3) is the well-known single exponential smoothing (SES) method.

3.2. Estimation of σ_L

A theoretical formula for σ_L can be obtained given the following: σ_1; the forecast updating procedure; the value of the lead time (L), which is assumed to be constant and known; and the values of the smoothing constants used. According to [3], the exact relationship can be complicated. If we disregard the fact that the errors usually increase for longer forecast horizons and also assume that the forecast errors are independent over time [31], then

$$\sigma_L = \sqrt{L}\sigma_1. \tag{4}$$

It should be noted that expression (4) has been strongly criticized in [20] because there is no theoretical justification for its use and it can produce very inadequate results.

The authors of [32, 33, 34, 35] showed that when the demand can be modeled using a local level model (i.e., the model that underlies SES) with the parameter α, the conditional variance for the lead time demand is

$$\sigma_L = \sigma_1\sqrt{L}\sqrt{1 + \alpha(L - 1) + \frac{1}{6}\alpha^2(L - 1)(2L - 1)}. \tag{5}$$

Interestingly, although (5) provides an exact relationship, the literature often relies on the approximation in (4) (see, for example, [6, 7]).

4. Empirical approach

The theoretical approach represented in (5) assumes that forecast errors are i.i.d., which means that we know the "true" model of the SKU's demand. However, if there are doubts about the validity of the "true" model, then empirical approaches can be a useful alternative [20]. Given the complex nature of markets, clients, promotions, economic situation and so forth, assuming that we know the true demand model for each SKU would be unrealistic; consequently, empirical approaches must at least be tested.

Note that although empirical approaches have been shown to yield good results in other applications, such as the calculation of prediction intervals [20], they have rarely been used in SCRM [8].

4.1. Parametric approach

In these empirical parametric approaches, we retain the assumption that lead time forecast errors are normally distributed, but we relax the independent variance assumption by allowing σ_L to vary over time. An empirical estimate of σ_L can be calculated as follows:

$$\sigma_L = \sqrt{\frac{\sum_{t=1}^{n}(e_t(L) - \bar{e}(L))^2}{n}}, \tag{6}$$

where $e_t(L) = y_L - F_L = \sum_{h=1}^{L} y_{t+h} - \sum_{h=1}^{L} F_{t+h}$ is the lead time forecast error and $\bar{e}(L)$ is the average error for the L under consideration.

4.1.1. Single exponential smoothing (SES)

Instead of using (6) to update σ_L each time a new observation is available, we can use SES to directly update the cumulative MSE over the lead time [36, 26]. Likewise, we can update σ_L on the basis of the lead time forecast error instead of the one-step-ahead forecasting error:

$$MSE_{L,t+1} = \alpha''(y_L - F_L)^2 + (1 - \alpha'')MSE_{L,t}, \tag{7}$$

where $\sqrt{MSE_{L,t+1}}$ is the forecast for σ_L at time $t+1$. Unlike in Eq. (3), the update step size does not match the lead time forecast error. The expression is updated after every period t. The process followed to calculate y_L and

F_L is known as the overlapping temporal demand aggregation process [37], which the authors recommend when a sufficient demand history is available (at least 24 observations). Similar to Eq. (3), both α'' and the initial value for Eq. (7) are optimized by minimizing the squared errors.

4.1.2. ARCH/GARCH models

Although exponential smoothing is the conventional approach in supply chain forecasting, when we are dealing with volatility forecasting other models have been developed that may be good candidates for enhancing risk estimation and, therefore, safety stock determination.

In particular, the AutoRegressive Conditional Heteroscedasticity (ARCH) model introduced in [38] is one of the most important developments in risk modeling, and it may be well suited for our application. In ARCH models, the forecast error is expressed as $\epsilon_t = \sigma_t \cdot v_t$, where

$$\sigma_{t+1}^2 = \omega + \sum_{i=1}^{p} a_i \epsilon_{t-i+1}^2, \tag{8}$$

where ω is a constant, and a_i are the respective coefficients of the p lags of v_t that form an i.i.d. process. ARCH models express the conditional variance as a linear function of the p lagged squared error terms. Bollerslev in [25] proposed the generalized autoregressive conditional heteroscedasticity (GARCH) models that represent a more parsimonious and less restrictive version of the ARCH(p) models. GARCH(p,q) models express the conditional variance of the forecast error (or return) (ϵ_t) at time t as a linear function of both q lagged squared error terms and p lagged conditional variance terms. For example, a GARCH(1,1) model is given by:

$$\sigma_{t+1}^2 = \omega + a_1 \epsilon_t^2 + \beta_1 \sigma_t^2. \tag{9}$$

It should be noted that exponential smoothing has the same formulation as the integrated GARCH model (IGARCH) [39], with $\beta_1 = 1 - a_1$ and $\omega = 0$. If we apply the GARCH(1,1) model to lead time forecasting error instead of to the one-step ahead forecasting error, Eq. (9) can be rewritten as follows:

$$\sigma_{L,t+1}^2 = \omega' + a_1' \epsilon_{L,t}^2 + \beta_1' \sigma_{L,t}^2. \tag{10}$$

In this work, we focus our analysis on the GARCH(1,1) model using an overlapping approach to compare it with the SES model shown in Eq. (7). Note that GARCH(1,1) also includes GARCH(0,1).

4.2. Non-parametric approach

Some demand distributions exhibit important asymmetries, particularly when they are subject to promotional periods or special events. In these cases, the typical normality assumption for forecasting errors does not hold. Non-parametric approaches, in which the safety stock is calculated in accordance with (2), are well suited for overcoming this problem. Here, we use two well-known non-parametric methods: (i) kernel density estimation and (ii) empirical percentile.

4.2.1. Kernel density estimation

This technique represents the probability density function $f(x)$ of the lead time forecast errors without requiring assumptions concerning the data distribution. The kernel density formula for a series X at a point x is given by

$$f(x) = \frac{1}{Nh} \sum_{j=1}^{N} K\left(\frac{x - X_j}{h}\right), \tag{11}$$

where N is the sample size, $K(\cdot)$ is the kernel smoothing function, which integrates to one and h is the bandwidth [27]. In the cited reference (on p. 43), it is shown that the optimal kernel function, often called the Epanechnikov kernel, is:

$$K_e(t) = \begin{cases} \frac{3}{4\sqrt{5}}\left(1 - \frac{1}{5}t^2\right) & -\sqrt{5} \leq t \leq \sqrt{5} \\ 0 & \text{otherwise} \end{cases} \tag{12}$$

Although the choice of h is still a topic of debate in the statistics literature, the following optimal bandwidth for a Gaussian kernel is typically chosen [27, 28]:

$$h_{opt} = 0.9AN^{-1/5}, \tag{13}$$

where A is an adaptive estimate of the spread given by

$$A = min(\text{standard deviation}, \text{interquantile range}/1.34) \tag{14}$$

The CSL quantile, denoted by $Q_L(CSL)$, can be estimated non-parametrically using the empirical distribution fitted by the kernel approach on the lead time forecast errors.

4.2.2. Empirical percentile

The well known percentile indicates the value below which a given percentage of observations fall. When calculating empirical percentiles the requested value is typically linearly interpolated from the available values.

5. Evaluation of the alternative approaches

5.1. Point forecasting

Before evaluating various approaches for measuring forecast error volatility, we must first define the point forecasting function. The point forecasting algorithm yields a measure of the central tendency of the forecast density function. This value and the resulting forecast errors are used as the inputs to the variability forecasting methods described in Sections 3 and 4. Note that the same point forecast is used in all cases, regardless of the variability forecasting approach considered.

In this work, SES is used to produce point forecasts for two reasons. First, SES is widely used in business applications [40, 41], and it is consistent with common practice, in which a company may not be using the optimal forecasting model due to either a lack of expertise or the availability of only a limited set of forecasting models in the software to which it has access. Second, we use SES as a benchmark because an analytical expression is available to compute the lead time forecasting error variance (see Eq. (5)), which requires an estimate of the value of α used in SES-based point forecasting. Such a forecasting method can be formulated as

$$F_{t+h} = \alpha y_t + (1 - \alpha)F_t, \tag{15}$$

where $0 \leq \alpha \leq 1$ and h is the forecasting horizon. Given the recursive nature of exponential smoothing, it is necessary to initialize the algorithm. We determine both the initial value and the α value by minimizing the in-sample mean squared error. Note that the lead time demand forecast is $F_L = \sum_{h=1}^{L} F_{t+h} = L \cdot F_{t+1}$.

5.2. Inventory performance metrics

In the supply chain context, as described in [42], [43] and [44], trade-off curves are employed to measure the forecasting performance of different techniques. These trade-off curves represent the pairs formed by the CSL and inventory investment plus backorders and inventory investment. Kourentzes

[43] transformed the inventory investment and backorder metrics into scale-independent measures that can easily summarize metrics across products by dividing them by the average in-sample demand, we follow the same approach here.

The way in which such inventory metrics are parameterized depends on the stock control policy. Here, we follow a newsvendor-type policy as in [9, 8], where the critical fractile minimizes the expected cost by balancing the costs of understocking and overstocking [31]. We also study the critical fractile estimation problem with a fixed replenishment lead time, in which such a critical fractile solution is repeatedly applied for a multi-period inventory model [8]. The inventory metrics are the achieved CSL, which is estimated as the percentage of iterations where the lead time demand realizations fall below the estimated lead time demand quantile for determined target CSL values (critical fractiles). Backorders, by which we mean the amount of unsatisfied demand, are calculated in two steps. First, we calculate the sum of the demand realizations that exceed the estimated target CSL for each SKU, and second, we calculate the average of that sum across SKUs. Since we are interested in the safety stock, the average inventory investment is calculated as the average value of the so-called scaled safety stock, which is the safety stock divided by the in-sample average demand. All three metrics are calculated on the hold-out sample. In Section 6.4, the implementation of a periodic-review order-up-to-level stock control policy is also considered, and its results are compared with those of the newsvendor policy. Note that an order-up-to-level stock control policy is more appropriate when products are sold over a long time horizon with numerous replenishment opportunities and, thus, there is no need to dispose of excess inventory [45].

In this work, the target CSL values are set to 85%, 90%, 95% and 99% for both simulations and real data.

5.3. Implementation of the approaches

The analysis was conducted using MATLAB. Here, we briefly present the implementation details of the approaches discussed above. We employed the econometric toolbox in MATLAB to implement the GARCH(1,1) model, selecting an interior-point optimization algorithm. We used the kernel density estimators in MATLAB with the default options selected for the bandwidth and the Epanechnikov kernel. The empirical percentiles were computed using the MATLAB function **prctile**.

6. Simulation results

In this section, we describe four simulation experiments conducted to explore the performance of the aforementioned empirical parametric and non-parametric approaches when there are deviations from the Gaussian i.i.d. assumptions. We study what happens when (i) the normality assumption does not hold for different sample sizes, (ii) the homoscedasticity and independent variance assumptions do not hold, and (iii) the lead time varies.

The simulations are assessed in terms of newsvendor inventory performance metrics (achieved CSL, scaled safety stock and backorders). To extend these simulation results to another well-known stock control policy, the last simulation study (iv) investigates the relationship between the newsvendor and order-up-to-level inventory performance metrics.

In these experiments, we divided the total sample into three parts. We used the first 20% of the data to estimate the parameters (initial value and smoothing parameter) for the point forecasting method and the next 50% to estimate the parameters for the volatility forecasting methods: SES, kernel density estimation, empirical percentile and GARCH(1,1). Note that empirical methods are estimated using data which have not been employed to produce the point forecasts, [46, 47, 48]. The remainder of the data was reserved as a hold-out sample for evaluation purposes.

6.1. Effects of sample size and demand distribution

To explore the influence of the demand distribution on the safety stock calculation, we employed three distributions: (i) normal, (ii) log-normal and (iii) gamma. Using this approach, we could observe any issues when the demand does not follow a normal distribution. Note that the log-normal distribution is reasonable when products are subject to promotional periods, during which the observed demand is higher than the baseline demand. The gamma distribution has previously been used in the literature [9, 29, 49] to assess violations of the normality assumption.

We conducted simulations for the following sample sizes: 50, 100 and 500. Studying the sample size is important because products often have short life cycles, and consequently, few historical observations are available. For each sample size and each distribution, we conducted Monte Carlo simulations with 100 repetitions. The simulated population values for the mean (μ) and standard deviation (σ) of the normal distribution were $\mu = 150$ and $\sigma = 25$. The parameters for the log-normal and gamma distributions were chosen to

provide a mean of 150 and a skewness of 9.6, with the latter being used as a non-normality measure. The resulting parameter values for the log-normal distribution were $\mu = 4.7$ and $\sigma = 0.7$, while for the gamma distribution they were $a = 0.04$ (shape parameter) and $b = 3,449$ (scale parameter).

Figure 1 shows the trade-off curves for a sample size of 50 observations and a lead time of 1 period. The upper plots show the deviations from the target CSL in percentage vs. the scaled safety stock. For example, with a normal distribution and a target CSL of 85%, denoted by the smallest marker, the CSL deviation for SES is approximately -1% (i.e., undercoverage with an achieved CSL of 84%). The closer the lines are to zero deviation, the better the performance is considered to be. The lower panels plot the relationship between backorders and scaled safety stocks. Note that the different target CSLs are organized from the smallest-size marker (85%) to the largest-size marker (99%).

The left column of Figure 1 (both upper and lower panels) shows the trade-off curves for the normally distributed demand. As expected, SES and GARCH perform reasonably well and produce a slight systematic undercoverage, whereas the kernel method produces a systematic overcoverage that is reduced for the highest target CSL. When the target is 99%, the empirical percentile method achieves a remarkable lower CSL, which also implies a higher volume of backorders.

The middle column of Figure 1 presents the corresponding trade-off curves for the log-normal demand distribution. The non-parametric approaches (kernel and percentile) successfully achieve the highest and lowest target CSLs (99% and 85%, respectively), whereas the parametric approaches (SES and GARCH) do not. Therefore, when the forecast errors are skewed, the typical assumption of normality results in CSL values that deviates from their targets, and the non-parametric approaches seem more robust in this case.

The right column of Figure 1 shows the results for the gamma-distributed simulated demands. The conclusions here are similar to those for the log-normal demands, although they are exacerbated. In general terms, the parametric approaches result in overcoverage for lower CSL targets (85%-90%); in fact, the lower the target is, the higher the overcoverage is. Similarly, for high CSL targets, the higher the target is (95%–99%), the higher the undercoverage is. By contrast, the empirical non-parametric approaches achieve CSL values that are closer to the targets. This difference among the target CSLs is also reflected in the lower plot, where the non-parametric approaches
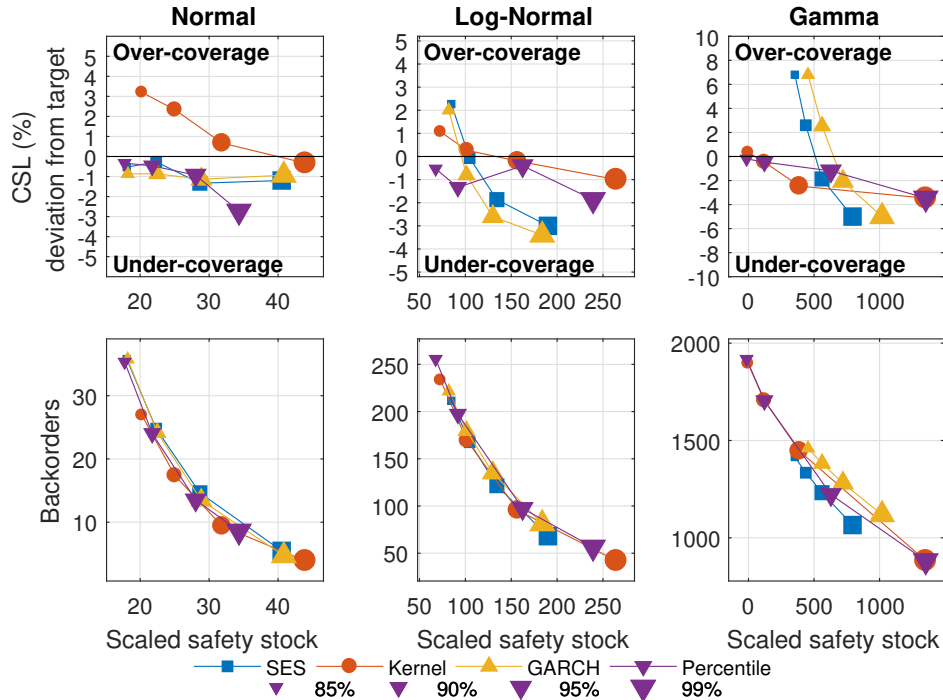
Figure 1: Trade-off curves for a sample size of 50 observations. Left column: normal distribution; middle column: log-normal distribution; right column: gamma distribution.

yield high backorder values for low CSL values and vice versa. By contrast, the parametric approaches yield backorder values that vary less strongly with the CSL value.

Figure 2 shows the corresponding trade-off curves when the sample size is increased to 100 observations. Again, the non-parametric approaches are better able to capture the asymmetry of the forecast errors at the highest and lowest CSLs (99% and 85%) for both the log-normal and gamma simulated demands. When the distribution is normal, all methods show similar performance in achieving values close to the target CSLs, although the kernel method results in systematically higher CSLs than expected.

The same conclusions are drawn when the sample size is further increased from 100 to 500 observations.

To summarize, non-parametric methods such as the kernel and percentile methods produce better trade-off curves for the log-normal and gamma distributions since the parametric methods assume that the forecast errors are normal. However, the performance of the non-parametric methods deterio-
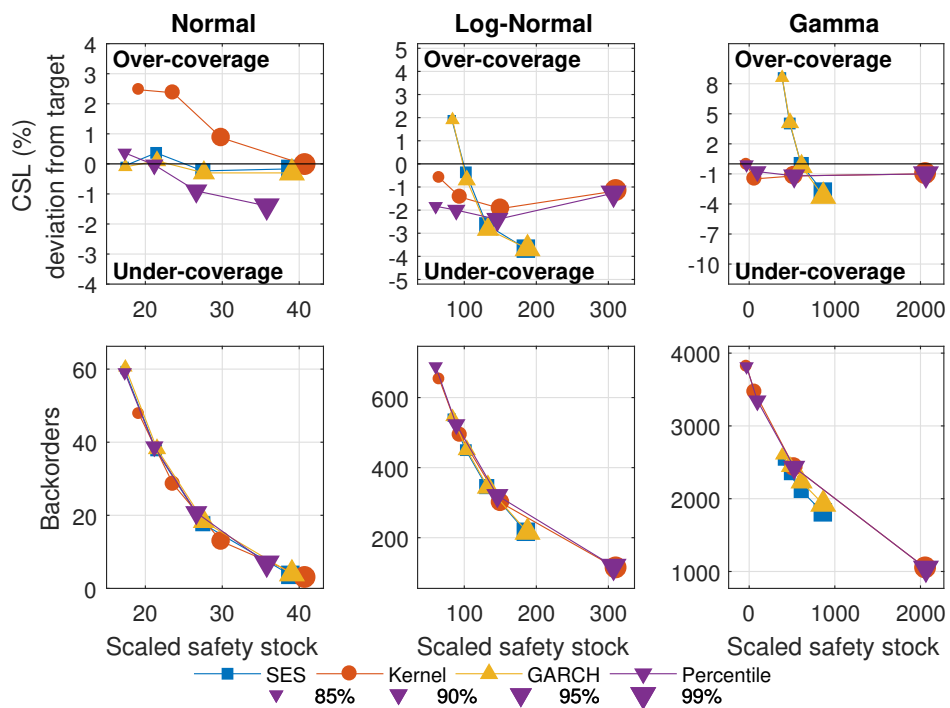
Figure 2: Trade-off curves for a sample size of 100 observations. Left column: normal distribution; middle column: log-normal distribution; right column: gamma distribution.

rates when the sample size is small since they are data-driven approaches. When the distribution is normal, the parametric approaches work well, as expected, although there is undercoverage when the sample size is small because the parameters are not known and must be estimated [10]. The kernel method results in overcoverage, and the percentile method is more sensitive to small sample sizes. The overcoverage of the kernel method is due to the bandwidth selection, which establishes a compromise between bias and variance [27]. Because correlations of the forecast error variance are not considered in these simulations, the GARCH model does not exhibit any significant advantage over the SES model.

6.2. Demand with time-varying volatility

In this experiment, we performed two sets of simulations with time-varying volatility. In the first case, the demand followed a normal distribution with a constant mean ($\mu = 150$) and two different standard deviations ($\sigma_1 = 25$ and $\sigma_2 = 50$); the sample size was 500 observations, and the lead time was 1 period. σ_1 was used for the time periods corresponding to 1:50, 101:226 and 351:425, with the purpose of ensuring the occurrence of volatility changes in both the in-sample periods and the hold-out sample. In the second case, the demand was simulated with a constant mean of 50 and a stochastic term that followed a GARCH(1,1) model, with the parameters $\omega = 0.01$, $a_1 = 0.4$ and $\beta_1 = 0.5$. In both cases, 100 simulations were run.

Figure 3 depicts the average trade-off curves obtained by each considered technique for both experiments. The right column shows the trade-off curves for the first experiment with volatile abrupt changes. Regarding backorders, the SES and GARCH methods yield slightly lower backorders for similar scaled safety stocks. With respect to CSL deviations, the SES and GARCH methods provide overcoverage for lower targets and undercoverage for the highest target. The kernel method produces overcoverage for each target, which is reduced for larger target CSLs, whereas the Percentile method achieves a CSL deviation close to zero.

The left column of Figure 3 shows the trade-off curves for the second simulation case based on a GARCH model. In general, the GARCH method produces low CSL deviations with low scaled safety stocks. Although SES achieves backorder values similar to those of GARCH, its CSL performance is much worse than that of either GARCH or the non-parametric approaches. Additionally, the backorder volumes produced by the parametric approaches

16

are lower than those of their non-parametric counterparts, even for lower scaled safety stocks.

These results can be interpreted as follows: (i) When the forecast errors are subject to variance autocorrelation, GARCH models are a promising alternative for computing safety stocks that deserves further exploration. The best option that has been commonly applied thus far is SES [50, 3], which can be seen as a particular case of GARCH models [39]. (ii) These results are also important because they suggest that the level of backorders can provide an indication of potential autocorrelation in the forecast error variance. In other words, when parametric approaches provide a level of backorders lower than non-parametric methods do, this may indicate a potential variance autocorrelation. To support this argument, Figure 4 shows the actual values and point forecasts for one Monte Carlo simulation on the hold-out sample for a CSL target of 95%. The same plot also depicts the safety stocks computed using the kernel and GARCH methods. We can see that the GARCH method is able to adapt to periods of high/low volatility, whereas the kernel method is not. Interestingly, although the kernel and GARCH methods result in similar numbers of stockouts during high-volatility periods and consequently produce similar achieved CSLs, the level of backorders is much higher when the kernel approach is used. Similarly, during periods of low volatility, the average safety stock yielded by the kernel method is greater than that of the GARCH method, and thus, the inventory investment is also higher with the kernel approach.

6.3. Influence of the lead time

In the simulations reported above, the lead time was set to 1 period. In this section, we investigate the influence of the lead time on the safety stock computation. For this experiment, we set the lead time to 4 periods, the sample size to 500 observations and each simulation was repeated 100 times. We used a demand process that followed an ARIMA(0,1,1) model with additional Gaussian innovations of zero mean and a standard deviation of $\sigma = 2$, where $\theta = -0.75$. We used that ARIMA model because SES is optimal for such a model; thus, all differences among the evaluated methods are independent of the point forecasting model. The value of $\theta = -0.75$ corresponds to a theoretical exponential smoothing constant of $\alpha = (1 + \theta) = 0.25$. To ensure positive demand values, a constant value of 50 units was added to the signal. Because the lead time was greater than 1 period, in addition to the previously considered empirically based approaches, we
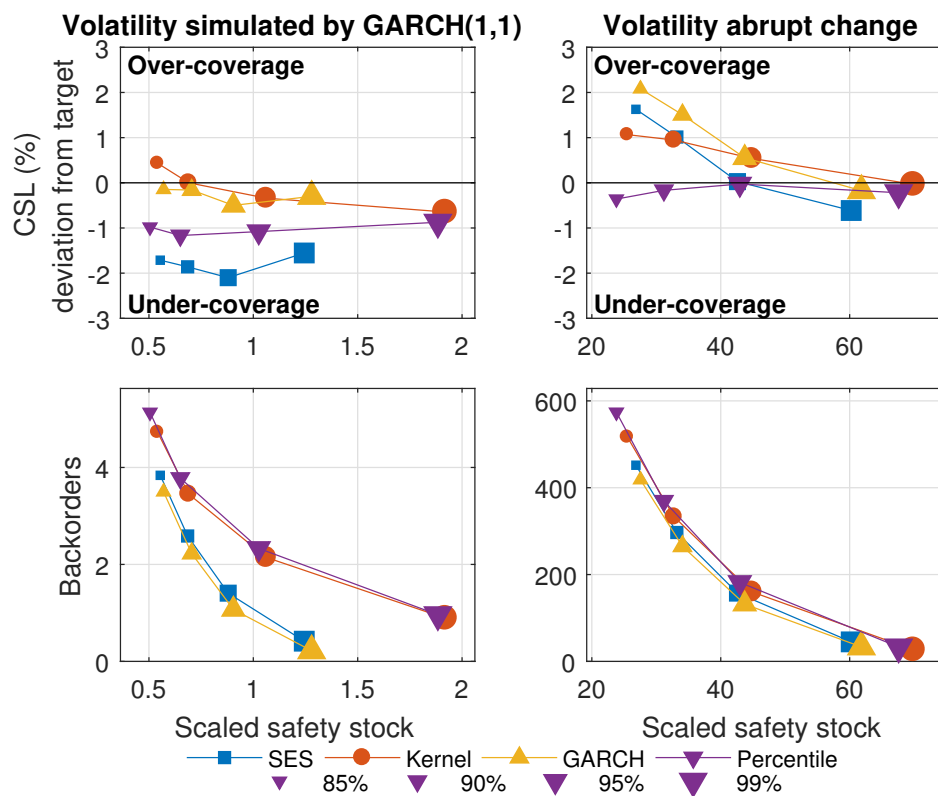
Figure 3: Trade-off curves for simulations with demand subject to time-varying volatility
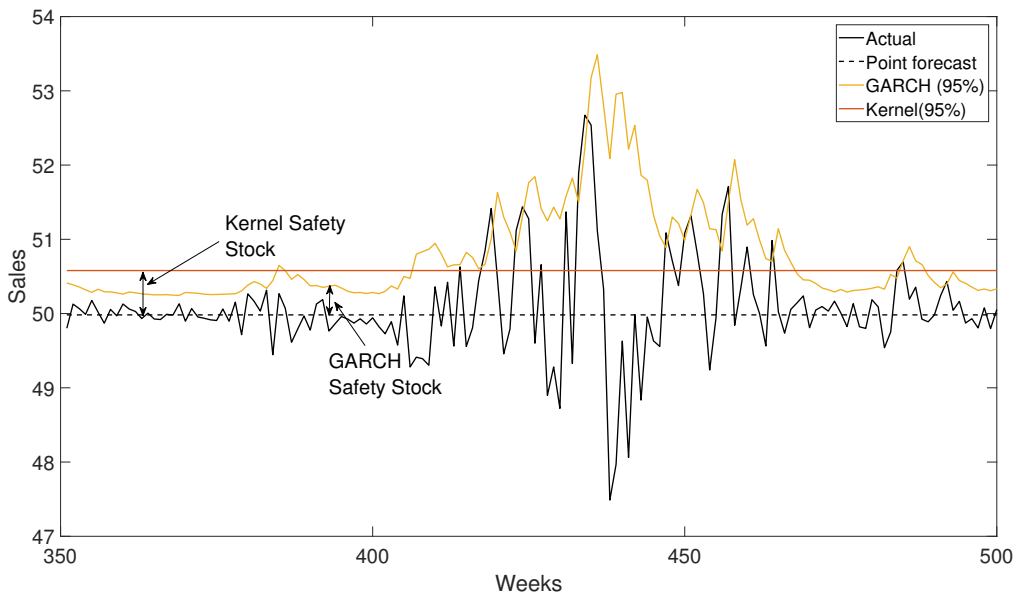
Figure 4: Example results from one Monte Carlo simulation

could also apply the theoretical approaches represented in Eqs. (4) and (5), hereafter denoted by $\sigma_L(4)$ and $\sigma_L(5)$, respectively. Recall that the SES, kernel, GARCH and percentile methods are based on empirical lead time forecast errors.

Figure 5 shows that $\sigma_L(4)$ produces a lower CSL and a higher number of backorders than $\sigma_L(5)$ does because of the simplifications assumed in the case of $\sigma_L(4)$; thus, our findings agree with those of [20] regarding the inadequacy of $\sigma_L(4)$. The percentile, kernel and GARCH methods, along with $\sigma_L(5)$, achieve CSLs very close to the targets, although the kernel method does so with a higher scaled safety stock. Interestingly, in terms of the CSL, SES performs worse than the non-parametric approaches, although GARCH performs well. Regarding backorders, GARCH achieves very good performance—even better than that of $\sigma_L(5)$. In the discussion of the time-varying simulations in the previous section, we concluded that when a parametric model such as GARCH provided lower backorder levels than the rest of the methods, this could indicate possible autocorrelation. Figure 5 shows that GARCH achieves superior performance, which could indicate that there is some autocorrelation that is not considered in the theoretical Eq. (5) that GARCH can capture empirically. We conjecture that such a potential autocorrelation
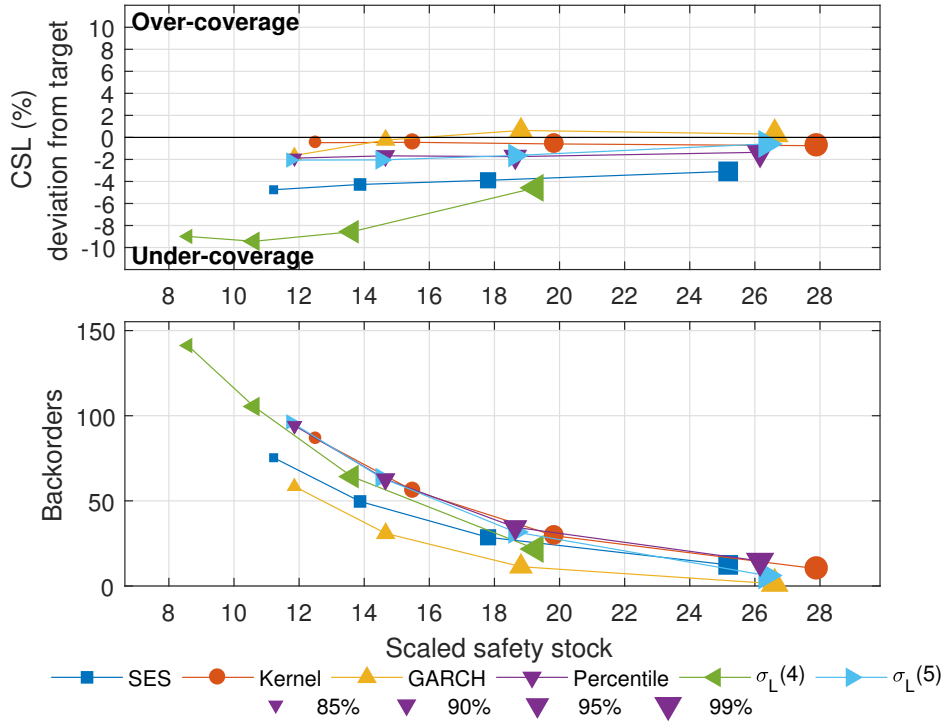
19

Figure 5: Trade-off curves for a demand that follows an ARIMA(0,1,1) model and a lead time of 4 periods

may originate from two sources: (i) The first possible source is overlapping temporal demand aggregation, as suggested by the fact that our results agree with those reported by [37], which indicate that the overlapping approach reduces inventory backorder volumes while maintaining the same volumes of held inventory. Note that such a variance autocorrelation due to overlapping may be present even when the original demand is independent; (ii) Similarly, the authors of [10] showed that when the parameters of the forecasting model are unknown and must be estimated, the forecast errors are correlated even when the demand is independent.

6.4. Stock control policies

In the previous simulation experiments, we analyzed the results from the perspective of a newsvendor inventory policy. Then, the question is whether the previous simulations results still hold for different stock control policies. In this section, we answer this question by simulating a periodic-

review, order-up-to stock control policy following the work of [51], as shown in Figure 6.

As in the previous experiment, the demand was described by a constant component (50 units) plus an ARIMA(0,1,1) component with $\theta = -0.75$, a standard deviation of $\sigma = 2$ and a sample size of 500 observations. Each simulation was repeated 100 times. The ordering decision for an order-up-to-level policy is expressed as follows:

$$O_t = S_t - IP_t, \tag{16}$$

where O_t, S_t and IP_t denote the ordering decision, order-up-to level and inventory position, respectively, at time t. Note that the review interval is 1. The inventory position can be defined in terms of the net stock (NS_t) and outstanding orders (OO_t):

$$IP_t = NS_t + OO_t, \tag{17}$$

where NS_t is

$$NS_t = \frac{1}{1 - z^{-1}}(R_t - y_t), \tag{18}$$

where Z^{-1} is a Z-transform such as $Z^{-1}y_t = y_{t-1}$ and R_t represents the received orders. We use the variable NS_t to compute the typical stock control metrics. In other words, the achieved CSL is calculated as the proportion of the time periods in which NS_t is greater than zero given that a new replenishment order is sent every time period, meaning that the cycle is unity. Backorders are calculated by summing the negative values of NS_t across time and subsequently averaging across SKUs. Finally, the inventory investment is estimated as the average of the positive NS_t values across time periods and SKUs.

The order-up-to level is updated every period:

$$S_t = F_L + k\sigma_L, \tag{19}$$

where F_L is the lead time forecast over L periods. Thus, in every period, the retailer updates the order-up-to level with the current estimates of F_L and σ_L. Note that S_t requires the estimation of the safety stock $k\sigma_L$.

The main difference between the block diagram model presented in [51] and the one shown in Figure 6 lies in σ_L. In [51], σ_L is not computed, and the uncertainty is considered as the forecasting demand in one period (i.e.,
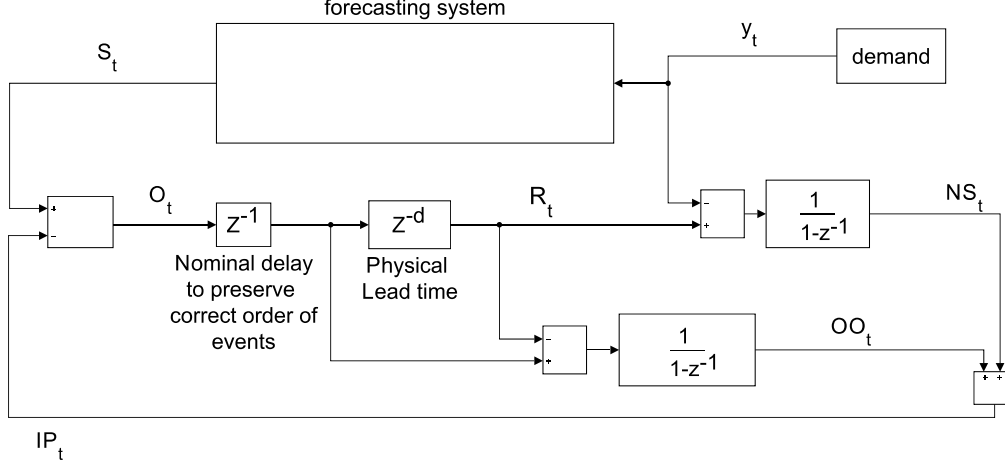
Figure 6: Order-up-to-level stock control policy plus a forecasting system based on Eq. (5)

increasing the lead time by one period). In our work, we focus on different ways to compute σ_L and how those affect the safety stock performance. For this simulation, we use Eq. (5) to compute σ_L, although any of the methods presented could be used because we are analyzing the relations between the newsvendor and order-up-to stock control metrics. The described simulation is implemented in SIMULINK. Recall that L already includes a nominal one-period order delay (Z^{-1}) because of the sequence of events [51]; therefore, we now have $L = d + 1$, where Z^{-d} is the physical lead time delay. To be coherent with the previous section, the lead time is set to $L = 4$; therefore $d = 3$. The sequence of events is the following [51]: 1. receive, 2. satisfy demand, 3. count inventory, 4. place order.

Figure 7 shows the relations between the newsvendor performance metrics and the order-up-to stock control metrics. In the three plots, the x and y axes represent the order-up-to metrics and the newsvendor metrics, respectively, for target CSLs of 85%, 90%, 95% and 99%, as in the previous simulations. We observe a high level of correlation between the metrics, as shown by the linear fits in all plots. In other words, the achieved CSL and backorder metrics achieved by the two stock control policies are approximately equal for every target CSL. The main difference lies in the inventory investment of the two techniques; the inventory investment of the newsvendor policy, as measured
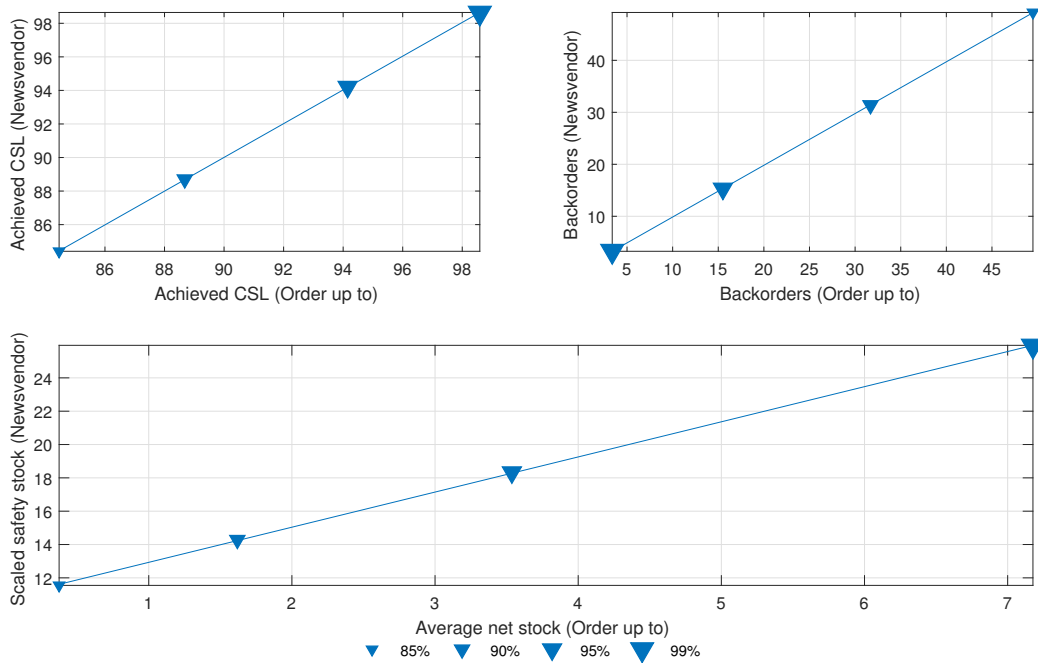
Figure 7: Relationships between the newsvendor and order-up-to-level stock control policy metrics for target CSL values of 85%, 90%, 95% and 99%

by the scaled safety stock, is greater than the inventory investment of the order-up-to policy measured by the average net stock. Nevertheless, as in the case of the other metrics, the correlation between the scaled safety stock and the average net stock for different target CSLs is very high. Therefore, the simulation results reported in the previous sections for the newsvendor stock policy can be easily extrapolated to another well-known policy, namely, the order-up-to-level stock control policy.

7. Case study data

The data employed in this paper were previously used by [48]. These data come from a major UK fast-moving consumer goods manufacturer that specializes in the production of household and personal care products. In total, 229 products with 173 weekly observations per product are available. According to [48] the SKUs did not present seasonality, and approximately 21% of them presented trend. We employed SES to produce point forecasts, although that may not always be the best possible forecasting method. This

can be used to assess the robustness of the proposed methods to point forecasts obtained using a method that does not perfectly match the unknown underlying data-generating process. This is a common situation both in research and in practice because identifying the true process is not trivial, as discussed before. To verify that SES is a reasonable method to provide point forecasts for this dataset, we compared the forecasting accuracy of the SES with that of $ARIMA(p, d, q)$ models automatically identified by minimizing the Bayesian information criterion (BIC) [52] for $p = (1, 2, 3)$, $q = (1, 2, 3)$, and $d = (0, 1)$. For the point forecasting exercise, we designated 70% of the data as the hold-in sample and the rest as the hold-out sample. We normalized the sales for each SKU with respect to the corresponding mean in the hold-in sample to aggregate the results for all SKUS. The results are summarized in Table 1.

Table 1: Comparison of forecasting accuracy between SES and ARIMA models

Error metric	SES	Automatic ARIMA
Mean(RMSE)	0.64	0.72
Median(RMSE)	0.60	0.65

In Table 1, Mean(RMSE) denotes the result of computing the root mean square error for each SKU on the hold-out sample and then averaging across all SKUs. By computing the median instead of the mean of the per-SKU RMSEs, we obtain the Median(RMSE) results reported in the second row. Overall, the SES method achieves a lower forecasting error than the automatic ARIMA models. This is not surprising since, as mentioned previously, the data do not exhibit any evident trends and/or seasonality.

Regarding the volatility forecasting exercise, as in the simulations, the data were split into three subsets. Again, SKU sales were normalized with respect to the hold-in sample means to aggregate the results for all SKUS in the trade-off curves.

To test the normality and variance independence of the forecasting errors, Table 2 shows two statistical metrics that test the residuals normality (the Jarque-Bera test) and the conditional heteroscedasticity (the Engle test), both at the 5% significance level. These tests were applied for different lead times ranging from 1 to 4 weeks. The values in each column represent the percentages of SKUs that do not pass the null hypotheses of normality and no conditional heteroscedasticity. These values were computed using the forecast errors obtained after the removal of the training data used to produce

the point forecasts. For the shortest lead time of 1 week, the null hypothesis of normality is rejected for 88.2% of the SKUs, and the null hypothesis of no conditional heteroscedasticity is rejected for 28.2%. Note the influence of the lead time on the statistical tests. As the lead time increases, the percentage of SKUs that do not pass the normality test decreases, as a consequence of the central limit theorem. By contrast, the percentage of SKUs that exhibit heteroscedasticity increases with the lead time. This may be due to the overlapping aggregation process and the estimation of the parameters, as explained in Section 6.3, where the influence of different lead times was analyzed.

Table 2: Percentages of the SKUs that do not pass the Jarque-Bera and Engle tests at the 5% significance level for different lead times

Lead time	Jarque-Bera test (%)	Engle test (%)
1	88.2	28.4
2	82.1	91.7
3	74.7	98.7
4	69.4	98.7

To test the adequacy of the GARCH(1,1) model on this real dataset, we computed the BIC for a general GARCH(p,q) model with p=1,2,3 and q=1,2,3 using the second part of the data devoted to optimizing the volatility models and a lead time of 4 weeks. The results show that GARCH(1,1) is a valid model because it minimizes the BIC for 97% of the SKUs.

Figures 8 and 9 show the trade-off curves of the manufacturer data for lead times of 1 and 4 weeks, respectively. In Figure 8, because the lead time is 1 week, SES, $\sigma_L(4)$ and $\sigma_L(5)$ methods all produce the same results; thus, for the sake of clarity, only the SES results are plotted. Because the conditional heteroscedasticity effect is lower for this lead time, the non-parametric kernel method produces lower deviations from the target for most CSLs (except 90%). For the highest target CSL (99%), the kernel method achieves the lowest CSL deviation and the lowest level of backorders, although at the expense of an increased scaled safety stock. This shows evidence of significant skewness in the data and is consistent with the results of the simulations, in which non-parametric approaches such as the kernel method produced better results for non-normal residuals.

Figure 9 shows the results for a lead time of 4 weeks and also includes the results for the theoretical approaches ($\sigma_L(4)$ and $\sigma_L(5)$; $\sigma_L(5)$ achieves
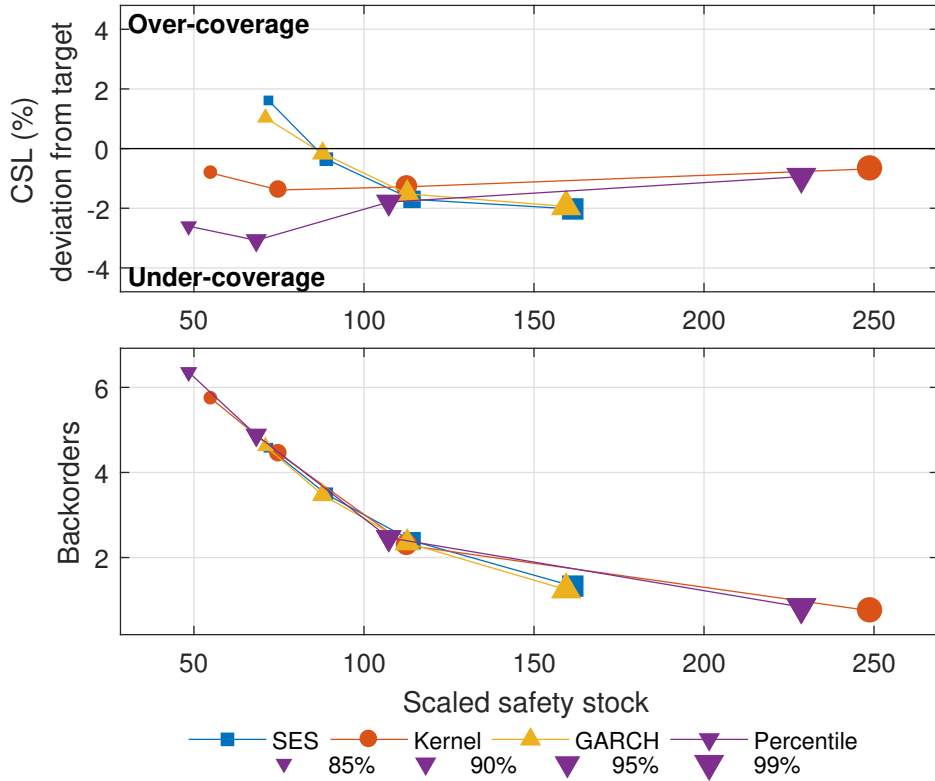
Figure 8: Trade-off curves for the manufacturer data for a lead time of 1 week

a lower CSL deviation and a lower level of backorders than $\sigma_L(4)$ does, as expected. Regarding the empirical non-parametric approaches, the kernel method results in a lower deviation from the target CSL and a lower level of backorders with a higher scaled safety stock. Finally, the empirical parametric approaches (SES and GARCH) demonstrate very good performances, especially the GARCH model. Clearly, the GARCH method achieves the lowest CSL deviation among all methods at a similar level of scaled safety stock. Additionally, the GARCH method performs very well with respect to backorders. Note that this superior performance of the GARCH method indicates that the SKUs in this dataset, in general, are subject to lead time forecast errors with important variance autocorrelations. This finding is corroborated by the Engle test results reported in Table 2, and is consistent with the simulation results shown in Figure 3.
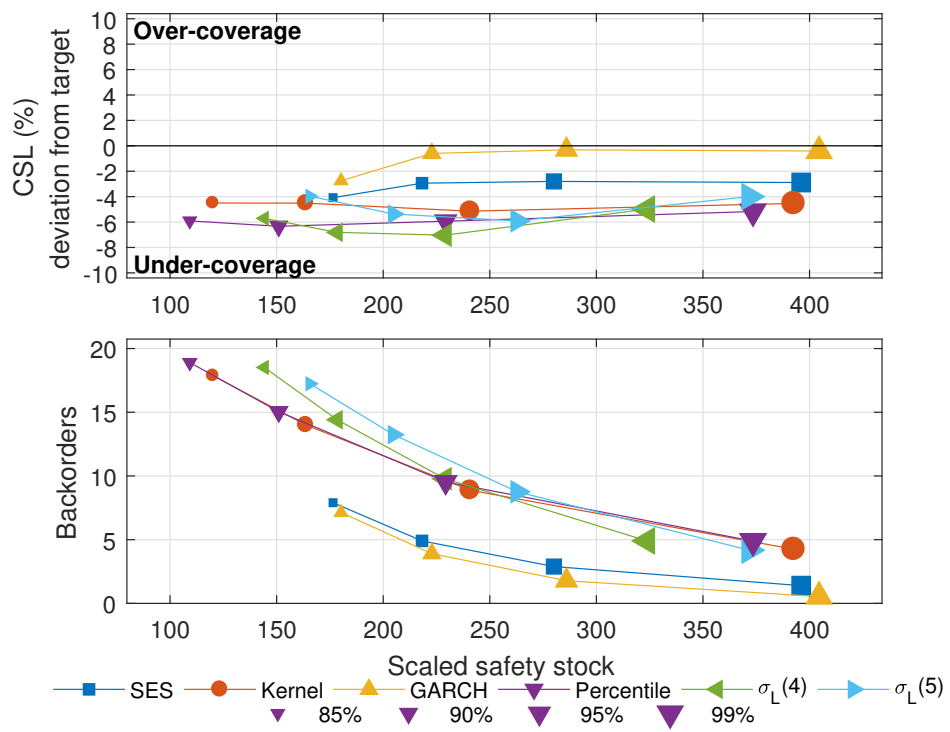
Figure 9: Trade-off curves for the manufacturer data for a lead time of 4 weeks

8. Conclusions

Despite the attention paid by both academics and practitioners to SCRM, the links between demand uncertainty and risks are still under-researched. One tool that supply chains typically employ to prevent further risks is safety stocks. This work examined empirical approaches, both parametric and non-parametric, for estimating the variability of the forecast errors and determining the appropriate levels of safety stocks. In addition, these empirical methods were compared with the traditional theoretical methods described in the supply chain literature. Our intent was to provide recommendations for cases in which the assumption of normal i.i.d. forecast errors does not hold, which is common in practice.

The results of simulations show that empirical non-parametric approaches such as the kernel method are suitable when the statistical distribution of the forecast errors cannot be assumed to be normal. Additionally, if the variance independence assumption cannot be guaranteed, empirical parametric approaches such as SES and GARCH offer a promising alternative. Moreover, we found that when the GARCH or SES methods yields improved level of backorders, this improvement indicates the possibility of temporal heteroscedasticity in the forecast errors. Under such circumstances, the GARCH method captures that heterocedasticity and achieves better performance in terms of the achieved CSL and backorder volume.

The results of the simulation experiments were evaluated for a newsvendor stock policy, and the findings were compared with the outcomes obtained when the newsvendor policy was replaced with an order-up-to-level policy. The results showed high correlations between the inventory performance metrics for the different stock control policies.

We also conducted experiments on real data and analyzed the influence of the lead time on the i.i.d. assumptions. For shorter lead times, the results were predominantly influenced by non-normality, and the kernel method produced better results, particularly for higher target CSLs. As the lead time increased, conditional heteroscedasticity became dominant, which caused the empirical parametric methods (GARCH and SES) to produce better results. GARCH outperformed SES in terms of the achieved CSL, because SES is simply a particular case of the more general GARCH framework. These results on real data validate the conclusions obtained from the simulation exercises.

The managerial implications of this work are as follows. Although the

28

theoretical Eq. (4) is widely used because of its simplicity, if the forecasting model and parameters are known for a certain lead time, we recommend employing a more precise expression, such as Eq. (5), which is valid for the particular case of the SES forecasting model. Nevertheless, Eq. (5) does not consider the case in which the forecasting model parameters are not known and must be estimated, nor how the cumulative lead time demand forecasts are computed (using an overlapping or non-overlapping approach).

If sufficient data are available but the forecasting model and its parameters cannot be determined (either because the forecasting support system does not provide such information, because the forecasts have been obtained using a combination of forecasting methods, or because there are doubts regarding whether the forecast errors satisfy the i.i.d. assumptions), then we suggest the use of empirical approaches. For shorter lead times, the kernel method can effectively capture deviations from normality; for longer lead times, GARCH models are highly suitable and are a good alternative to SES, which has been traditionally utilized since the nineteen-sixties [50]. In this case, GARCH models can be a good choice even when the demand variance is uncorrelated.

Future research should address some of the limitations of this work. We analyzed SES and the GARCH(1,1) model for point and volatility forecasts, respectively. Future works should extend the applied GARCH models to incorporate exogenous variables that are available in advance, such as price discounts due to marketing campaigns. With regard to non-parametric methods, the optimal selection of the kernel function and/or bandwidth also deserves further investigation.

Acknowledgment

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