CONTRACTIBILITY OF THE STABILITY MANIFOLD FOR SILTING-DISCRETE ALGEBRAS

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ABSTRACT. We show that any bounded t-structure in the bounded derived category of a silting-discrete algebra is algebraic, i.e. has a length heart with finitely many simple objects. As a corollary, we obtain that the space of Bridgeland stability conditions for a silting-discrete algebra is contractible.

1. INTRODUCTION

Stability conditions on triangulated categories were introduced by Bridgeland in [12] as a means of extracting geometry from homological algebra with a view to constructing moduli spaces arising in the context of Homological Mirror Symmetry. They can be thought of as a continuous generalisation of bounded t-structures. The main result of [12] asserts that the space of stability conditions forms a complex manifold, the *stability manifold*. This can be thought of as geometrically encoding most of the cohomology theories on a given triangulated category.

Bounded t-structures admit a mutation theory given by HRS-tilts (see Proposition 2.3 below), giving rise to a graph that is closely related to the exchange graphs occurring in cluster combinatorics [24], which is the skeleton of the stability manifold in the Dynkin case. Despite being the focus of extensive investigation, for example [12, 14, 17, 18, 20, 26, 28, 31, 32], computations with stability conditions are difficult. For example, it is widely believed that whenever the stability manifold is nonempty it is contractible. However, this has been proved in only few cases, though the list is now growing, see [14, 18, 20, 26, 28, 32].

Silting objects are a generalisation of tilting objects due to Keller and Vossieck in [23]. In the context of bounded derived categories of finite-dimensional algebras, silting objects enable the detection of t-structures whose hearts are equivalent to module categories of finite-dimensional algebras [25]. *Silting-discreteness* [4] is a finiteness condition on a triangulated category that says there are only finitely many silting objects in any interval in the poset of silting objects [5]; see below for precise definitions. Examples of silting-discrete finite-dimensional algebras include hereditary algebras of finite representation type, derived-discrete algebras [14], preprojective algebras of Dynkin type [6], symmetric algebras of finite representation type [4], Brauer graph algebras whose Brauer graphs contain at most one cycle of odd length and no cycles of even length [1], and local algebras [5].

The purpose of this paper is to establish the following property of the bounded t-structures in the bounded derived category $\mathsf{D}^b(\Lambda)$ of a silting-discrete finite-dimensional

algebra Λ. We recall that a bounded t-structure is *algebraic* if it is given by a silting object; see Section 2 for the precise definition.

Theorem A. *If Λ is a silting-discrete finite-dimensional $\mathbf{k}$-algebra, then any bounded t-structure in $\mathsf{D}^b(\Lambda)$ is algebraic, i.e. has a length heart.*

In particular, this result means that any bounded t-structure (X, Y) in $\mathsf{D}^b(\Lambda)$ admits only finitely many HRS tilts. Hence, the techniques and methods used in [14, 32] to show that the stability manifold of a derived-discrete algebra is contractible can be applied here.

Corollary B. *If Λ is a silting-discrete finite-dimensional $\mathbf{k}$-algebra, then the stability manifold $\mathsf{stab}(\mathsf{D}^b(\Lambda))$ is contractible.*

In [3], Adachi, Mizuno and Yang have independently obtained similar results in the setting of silting-discrete triangulated categories.

The outline of this paper is as follows. In Section 2 we recall the concepts and results that will be necessary to establish Theorem A. In Section 3 we prove Theorem A. Once one has Theorem A the proof of Corollary B is implicit in [14, 32]. For the convenience of the reader we briefly sketch the narrative of the argument in [14, 32] in Section 4.

Convention. Throughout this paper all subcategories are full and strict, $\mathbf{k}$ is a field, and all algebras are finite-dimensional $\mathbf{k}$-algebras. For a finite-dimensional $\mathbf{k}$-algebra Λ we denote by $\mathsf{mod}(\Lambda)$ the category of finite-dimensional right Λ-modules; by $\mathsf{K}^b(\mathsf{proj}(\Lambda))$ we denote the bounded homotopy category of complexes of finite-dimensional projective Λ-modules; by $\mathsf{D}^b(\Lambda)$ we denote the bounded derived category of finite-dimensional right Λ-modules; and by $\mathsf{D}(\Lambda)$ we denote the unbounded derived category of right Λ-modules. Throughout D is a triangulated category and the shift functor is denoted by $[1] \colon \mathsf{D} \to \mathsf{D}$.

2. Background

For a subcategory S of a triangulated category D we define

$$\mathsf{S}[> n] = \{S[i] \mid S \in \mathsf{S}, i > n\} \text{ and } \mathsf{S}[< n] = \{S[i] \mid S \in \mathsf{S}, i < n\},$$

analogously for $\mathsf{S}[\leq n]$ and $\mathsf{S}[\geq n]$. For subcategories X and Y of D we define

$$\mathsf{X} * \mathsf{Y} = \{D \in \mathsf{D} \mid \text{there exists a triangle } X \to D \to Y \to X[1] \text{ with } X \in \mathsf{X} \text{ and } Y \in \mathsf{Y}\}.$$

A subcategory X is *extension closed* if $\mathsf{X} = \mathsf{X} * \mathsf{X}$. We shall denote the extension closure of X by $\langle \mathsf{X} \rangle$. We define the *right* and *left perpendicular categories* of X respectively by

$$
\begin{aligned}
\mathsf{X}^{\perp} &= \{D \in \mathsf{D} \mid \mathrm{Hom}_{\mathsf{D}}(X, D) = 0 \text{ for all } X \in \mathsf{X}\}; \\
{}^{\perp}\mathsf{X} &= \{D \in \mathsf{D} \mid \mathrm{Hom}_{\mathsf{D}}(D, X) = 0 \text{ for all } X \in \mathsf{X}\}.
\end{aligned}
$$

For subcategories of an abelian category H we use the same notation for the analogous definitions, using short exact sequences instead of triangles.

2.1. Torsion pairs and t-structures. The general notion of a torsion pair on an abelian category goes back to [16].

Definition 2.1. A torsion pair in an abelian category H consists of a pair of full subcategories $(\mathcal{T}, \mathcal{F})$ such that $\mathcal{T}^{\perp} = \mathcal{F}$, $\mathcal{T} = {}^{\perp}\mathcal{F}$, and $\mathsf{H} = \mathcal{T} * \mathcal{F}$. We call $\mathcal{T}$ the *torsion class* and $\mathcal{F}$ the *torsionfree class* of the torsion pair.

If the abelian category H is $\mathsf{mod}(\Lambda)$ for a finite-dimensional algebra Λ, then any sub-category $\mathcal{T}$ closed under extensions, factor objects and direct summands gives rise to a torsion class of a torsion pair; see, e.g. [8, Ch. VI]. A dual statement holds for *torsionfree classes*. For $M \in \mathsf{mod}(\Lambda)$ we write $\mathsf{Fac}(M)$ for the full subcategory of all factor modules of finite direct sums of copies of M. When M is a support τ-tilting module (see Section 2.2 below), $\mathsf{Fac}(M)$ is the smallest torsion class containing M.

The analogue of a torsion pair in a triangulated category is a t-structure [10].

Definition 2.2. A *t-structure* on a triangulated category D consists of a pair of full subcategories (X, Y) such that $\mathsf{X}^{\perp} = \mathsf{Y}$, $\mathsf{X} = {}^{\perp}\mathsf{Y}$, $\mathsf{D} = \mathsf{X} * \mathsf{Y}$ and $\mathsf{X}[1] \subseteq \mathsf{X}$ (equivalently, $\mathsf{Y}[-1] \subseteq \mathsf{Y}$). The subcategory $\mathsf{H} = \mathsf{X} \cap \mathsf{Y}[1]$ is an abelian subcategory of D called the *heart* of (X, Y). A t-structure is called *bounded* if

$$\mathsf{D} = \bigcup_{n \in \mathbb{Z}} \mathsf{X}[n] = \bigcup_{n \in \mathbb{Z}} \mathsf{Y}[n].$$

For a bounded t-structure (X, Y) we have $\mathsf{X} = \langle \mathsf{H}[\geq 0] \rangle$ and $\mathsf{Y} = \langle \mathsf{H}[< 0] \rangle$. A t-structure is called *algebraic* if it is bounded and H is a length category, i.e. H has finitely many isomorphism classes of simple objects and each object of H is both Artinian and Noetherian.

There is a close connection between torsion pairs and t-structures.

Proposition 2.3 ([11, 29, 34]). *Suppose (X, Y) is a t-structure on D with heart H. Then there is a bijection*

$$\{ \textit{t-structures } (\mathsf{X}', \mathsf{Y}') \textit{ with } \mathsf{X}[1] \subseteq \mathsf{X}' \subseteq \mathsf{X} \} \overset{1-1}{\longleftrightarrow} \{ \textit{torsion pairs } (\mathcal{T}, \mathcal{F}) \textit{ in } \mathsf{H} \};$$
$$(\mathsf{X}', \mathsf{Y}') \longmapsto (\mathcal{T} = \mathsf{H} \cap \mathsf{X}', \mathcal{F} = \mathsf{H} \cap \mathsf{Y}');$$
$$(\mathsf{X}' = \langle \mathcal{T}, \mathsf{X}[1] \rangle, \mathsf{Y}' = \langle \mathsf{Y}, \mathcal{F} \rangle) \longleftarrow (\mathcal{T}, \mathcal{F}).$$

The t-structure $(\mathsf{X}', \mathsf{Y}')$ in Proposition 2.3 is called a left *HRS-tilt* of (X, Y) at the torsion pair $(\mathcal{T}, \mathcal{F})$ and is called *intermediate with respect to* (X, Y); see [21, Proposition I.2.1]. Note that $\mathsf{X}' = \mathsf{X}[1] * \mathcal{T}$ and $Y' = \mathcal{F} * \mathsf{Y}$.

2.2. **Silting, t-structures and τ-tilting.** Silting was first introduced in [23]; however, we follow the treatment of [5].

Definition 2.4. A subcategory S of D is *silting* if $\mathsf{thick}(\mathsf{S}) = \mathsf{D}$ and $\mathrm{Hom}_{\mathsf{D}}(S, S'[i]) = 0$ for each $S, S' \in \mathsf{S}$ and $i > 0$, where $\mathsf{thick}(\mathsf{S})$ is the smallest triangulated subcategory of D containing S that is closed under direct summands. An object S of D is a *silting object* if $\mathsf{add}(S)$ is a silting subcategory, where $\mathsf{add}(S)$ consists of the direct summands of finite coproducts of copies of S.

For a finite-dimensional algebra Λ we shall freely abuse notation and identify silting subcategories with silting objects, since any silting subcategory in $\mathsf{K}^b(\mathsf{proj}(\Lambda))$ is of the form $\mathsf{add}(S)$, for some silting object uniquely determined up to additive closure.

There is a partial order on silting subcategories of D (see [5]): for silting subcategories S and T,

$$\mathsf{S} \geq \mathsf{T} \text{ if and only if } \mathrm{Hom}_{\mathsf{D}}(S, T[i]) = 0 \text{ for all } S \in \mathsf{S}, T \in \mathsf{T} \text{ and } i > 0,$$

which is equivalent to $\mathsf{T} \subseteq (\mathsf{S}[< 0])^{\perp}$. We denote this poset by $\mathbb{P}_1(\mathsf{D})$. A silting subcategory T is called *two term with respect to* S if $\mathsf{S} \geq \mathsf{T} \geq \mathsf{S}[1]$, which happens if and only if $\mathsf{T} \in \mathsf{S} * \mathsf{S}[1]$; see, for example, [22].

Definition 2.5 ([4, Definition 3.6 & Proposition 3.8]). A finite-dimensional algebra Λ is *silting-discrete* if for any silting object S and any natural number n there are only finitely many silting objects T such that $S \geq T \geq S[n]$. Note that, via [32, Lemma 2.14], this is equivalent to there being only finitely many silting objects T such that $S \geq T \geq S[1]$.

In the case that $\mathsf{D} = \mathsf{D}^b(\Lambda)$ for a finite-dimensional algebra Λ, there is a correspondence between silting subcategories and algebraic t-structures.

Theorem 2.6 ([25] & [22]). *Let Λ be a finite-dimensional $\mathbf{k}$-algebra. Then there is a bijection*

$$\{silting\ subcategories\ of\ \mathsf{K}^b(\mathsf{proj}(\Lambda))\} \quad \overset{1-1}{\longleftrightarrow} \quad \{algebraic\ t\text{-}structures\ on\ \mathsf{D}^b(\Lambda)\};$$
$$\mathsf{S} \quad \longmapsto \quad \left(\mathsf{X_S} = (\mathsf{S}[<0])^\perp, \mathsf{Y_S} = (\mathsf{S}[\geq 0])^\perp\right).$$

Moreover, this restricts to a bijection with intermediate algebraic t-structures,

$$\{silting\ subcategories\ \mathsf{T} \subseteq \mathsf{S}*\mathsf{S}[1]\} \overset{1-1}{\longleftrightarrow} \{algebraic\ t\text{-}structures\ (\mathsf{X},\mathsf{Y})\ with\ \mathsf{X_S}[1] \subseteq \mathsf{X} \subseteq \mathsf{X_S}\}.$$

Definition 2.7. Let Λ be a finite-dimensional $\mathbf{k}$-algebra and denote the Auslander–Reiten translation on $\mathsf{mod}(\Lambda)$ by τ. Write $|M|$ for the number of nonisomorphic indecomposable summands of a Λ-module M.

(1) ([2, Definitions 0.1 & 0.3]) A pair $(M, P) \in \mathsf{mod}(\Lambda) \times \mathsf{proj}(\Lambda)$ is a *τ-rigid pair* if $\mathrm{Hom}_\Lambda(M, \tau M) = 0$ and $\mathrm{Hom}_\Lambda(P, M) = 0$. A τ-rigid pair is a *support τ-tilting pair* if $|M| + |P| = |\Lambda|$. In this case M is called a *support τ-tilting module*. If in a support τ-tilting pair $P = 0$, we call M a *τ-tilting module*.

(2) ([15, Definition 1.1]) The algebra Λ is *τ-tilting finite* if there are only finitely many isomorphism classes of basic τ-tilting Λ-modules.

The following characterisation of support τ-tilting pairs will be useful.

Lemma 2.8 ([2, Corollary 2.13], see also [7, Theorem 2.5(3)]). *Let $M \in \mathsf{mod}(\Lambda)$ and*

$$P_1 \overset{\sigma}{\to} P_0 \to M \to 0$$

be its minimal projective presentation. The pair (M, P) is support τ-tilting if and only if $\mathsf{Fac}(M)$ consists of the $N \in \mathsf{mod}(\Lambda)$ such that $\mathrm{Hom}_\Lambda(\widetilde{\sigma}, N)$ is surjective, where $\widetilde{\sigma} = \begin{bmatrix} \sigma & 0 \end{bmatrix}$ in the projective presentation

$$P_1 \oplus P \overset{\widetilde{\sigma}}{\to} P_0 \to M.$$

A result of [15] relates τ-tilting finiteness with functorial finiteness of torsion classes; we refer the reader to, for example [8], for the definition of functorial finiteness.

Theorem 2.9 ([15, Theorem 3.8]). *A finite-dimensional algebra Λ is τ-tilting finite if and only if every torsion class (equivalently, every torsionfree class) in $\mathsf{mod}(\Lambda)$ is functorially finite.*

The results of [2] combined with [22] give the following theorem.

Theorem 2.10 ([22, Theorem 4.6] and [2]). *Let D be a Krull-Schmidt, Hom-finite, $\mathbf{k}$-linear triangulated category and let $\mathsf{S} = \mathsf{add}(S)$ for a silting object S. Let $\Gamma = \mathrm{End}_\mathsf{D}(S)$. Then there is a bijection between the following sets.*

(1) *Basic silting objects T of D with $T \in S * S[1]$, modulo isomorphism.*
(2) *Basic support τ-tilting pairs of $\mathsf{mod}(\Gamma)$, modulo isomorphism.*
(3) *Torsion pairs $(\mathcal{T}, \mathcal{F})$ in $\mathsf{mod}(\Gamma)$ in which $\mathcal{T}$ and $\mathcal{F}$ are functorially finite.*

Remark 2.11. Suppose we are in the setup of Theorem 2.10. Recall from [22, Remark 4.1(ii)] that there is an equivalence $\mathsf{Mod}(\mathsf{S}) \simeq \mathsf{Mod}(\Gamma)$, where $\mathsf{Mod}(\mathsf{S})$ is the category of contravariant functors from S to the category of abelian groups. Let (M, P) be a support τ-tilting pair of $\mathsf{mod}(\Gamma)$ with minimal projective presentation

$$P_1 \xrightarrow{\sigma} P_0 \to M \to 0$$

and the 'extended' presentation

$$\widetilde{P} = P_1 \oplus P \xrightarrow{\widetilde{\sigma}} P_0 \to M \to 0$$

of Lemma 2.8. One can uniquely lift this presentation to $\mathsf{Mod}(\mathsf{S})$ as

$$\mathrm{Hom}_{\mathsf{D}}(-, \widetilde{S})|_{\mathsf{S}} \xrightarrow{(-,f)} \mathrm{Hom}_{\mathsf{D}}(-, S_0)|_{\mathsf{S}},$$

where $\mathrm{Hom}_{\mathsf{D}}(-, D)|_{\mathsf{S}}$ denotes the image of D under the restricted Yoneda functor [9]; cf. [22, Remark 3.1]. The corresponding silting object $T \in \mathsf{S} * \mathsf{S}[1]$ is then the mapping cone of $f \colon \widetilde{S} \to S_0$ in D.

3. Proof of Theorem A

We start by showing that when Λ is silting-discrete any HRS-tilt of an algebraic t-structure is again algebraic.

Proposition 3.1. *Let Λ be a silting-discrete finite-dimensional algebra. Let $\mathsf{S} \subseteq \mathsf{K}^b(\mathsf{proj}(\Lambda))$ be a silting subcategory and let $(\mathsf{X}_\mathsf{S}, \mathsf{Y}_\mathsf{S})$ be the corresponding algebraic t-structure on $\mathsf{D}^b(\Lambda)$. If (X, Y) is a t-structure intermediate with respect to $(\mathsf{X}_\mathsf{S}, \mathsf{Y}_\mathsf{S})$ then (X, Y) is algebraic.*

Proof. Suppose (X, Y) is a t-structure intermediate with respect to an algebraic t-structure $(\mathsf{X}_\mathsf{S}, \mathsf{Y}_\mathsf{S})$, where $\mathsf{S} = \mathsf{add}(S)$ for some basic silting object S. First observe that since (X, Y) is intermediate with respect to a bounded t-structure $(\mathsf{X}_\mathsf{S}, \mathsf{Y}_\mathsf{S})$ it is automatically bounded. Let $\Gamma = \mathrm{End}_{\mathsf{K}^b(\mathsf{proj}(\Lambda))}(S)$ and note that $\mathsf{H}_\mathsf{S} \simeq \mathsf{mod}(\Gamma)$ by [25]. Since Λ is silting-discrete, there are finitely many silting objects in $\mathsf{S} * \mathsf{S}[1]$, and therefore, by Theorem 2.10, finitely many support τ-tilting pairs in $\mathsf{mod}(\Gamma)$, whence Γ is τ-tilting finite.

By Proposition 2.3, there exists a torsion pair $(\mathcal{T}, \mathcal{F})$ on H_S such that $\mathsf{X} = \langle \mathcal{T}, \mathsf{X}_\mathsf{S}[1] \rangle$ and $\mathsf{Y} = \langle \mathsf{Y}_\mathsf{S}, \mathcal{F} \rangle$. By Theorem 2.9, $\mathcal{T}$ and $\mathcal{F}$ are functorially finite, so that by the correspondence in Theorem 2.10, $\mathcal{T} = \mathsf{Fac}(M)$ for some support τ-tilting pair (M, P) of $\mathsf{mod}(\Gamma)$, which in turn corresponds to some silting object $T \in \mathsf{S} * \mathsf{S}[1]$. By Theorem 2.6, this corresponds to an algebraic t-structure $(\mathsf{X}_\mathsf{T}, \mathsf{Y}_\mathsf{T})$ that is intermediate with respect to $(\mathsf{X}_\mathsf{S}, \mathsf{Y}_\mathsf{S})$. Invoking Proposition 2.3 again, there is a torsion pair $(\mathcal{T}_\mathsf{T}, \mathcal{F}_\mathsf{T})$ on H_S such that $\mathsf{X}_\mathsf{T} = \langle \mathcal{T}_\mathsf{T}, \mathsf{X}_\mathsf{S}[1] \rangle$ and $\mathsf{Y}_\mathsf{T} = \langle \mathsf{Y}_\mathsf{S}, \mathcal{F} \rangle$. Furthermore, $\mathcal{T}_\mathsf{T} = \mathsf{X}_\mathsf{T} \cap \mathsf{H}_\mathsf{S}$.

We claim that $\mathcal{T}_\mathsf{T} = \mathcal{T}$. First observe that any $N \in \mathsf{H}_\mathsf{S}$ satisfies $N \in (\mathsf{T}[< -1])^\perp$ because $\mathsf{T} \subseteq \mathsf{S} * \mathsf{S}[1]$. Therefore $N \in \mathsf{H}_\mathsf{S}$ lies in $\mathcal{T}_\mathsf{T}$ if and only if $N \in (\mathsf{T}[-1])^\perp$. By Lemma 2.8, $N \in \mathsf{Fac}(M)$ if and only if $\mathrm{Hom}_{\mathsf{H}_\mathsf{S}}(\widetilde{\sigma}, N)$ is surjective, where we use the notation of Remark 2.11. By Remark 2.11, we can lift $\widetilde{\sigma}$ to the functor category as $\mathrm{Hom}_{\mathsf{D}}(-, \widetilde{S})|_{\mathsf{S}} \xrightarrow{(-,f)} \mathrm{Hom}_{\mathsf{D}}(-, S_0)|_{\mathsf{S}}$, and note that via the restricted Yoneda functor (e.g. [22, Remark 3.1]), $\mathrm{Hom}_{\mathsf{H}_\mathsf{S}}(\widetilde{\sigma}, N)$ is surjective if and only if

$$
\begin{array}{ccc}
\mathrm{Hom}_{\mathsf{Mod}(\mathsf{S})}(\mathrm{Hom}_{\mathsf{D}}(-, S_0)|_{\mathsf{S}}, \mathrm{Hom}_{\mathsf{D}}(-, N)|_{\mathsf{S}}) & \xrightarrow{(f^*, (-,N))} & \mathrm{Hom}_{\mathsf{Mod}(\mathsf{S})}(\mathrm{Hom}_{\mathsf{D}}(-, \widetilde{S})|_{\mathsf{S}}, \mathrm{Hom}_{\mathsf{D}}(-, N)|_{\mathsf{S}}) \\
\uparrow{\scriptstyle\sim} & & \uparrow{\scriptstyle\sim} \\
\mathrm{Hom}_{\mathsf{D}}(S_0, N)|_{\mathsf{S}} & \xrightarrow{\quad(f,N)\quad} & \mathrm{Hom}_{\mathsf{D}}(\widetilde{S}, N)|_{\mathsf{S}}
\end{array}
$$

is surjective, where the vertical arrows are given by the Yoneda embedding. But since an additive generator of T is given as the mapping cone

$$\widetilde{S} \xrightarrow{f} S_0 \to T \to \widetilde{S}[1]$$

and $\mathrm{Hom}_{\mathsf{D}}(S_0[-1], N)|_{\mathsf{S}} = 0$ since $N \in \mathsf{H_S}$, we have $N \in (\mathsf{T}[-1])^\perp$ if and only if $N \in \mathsf{Fac}(M)$. Hence $\mathcal{T}_\mathsf{T} = \mathcal{T}$. It follows that $(\mathsf{X}, \mathsf{Y}) = (\mathsf{X_T}, \mathsf{Y_T})$, i.e. any t-structure intermediate with respect to $(\mathsf{X_S}, \mathsf{Y_S})$ is algebraic. $\qquad\square$

In order to prove the key lemma, Lemma 3.5, we shall need two auxiliary results. The first one is the following straightforward observation; cf. [32, Lemma 2.9].

Lemma 3.2. *Suppose* (X, Y) *is a bounded t-structure on* $\mathsf{D}^b(\Lambda)$ *and* $(\mathsf{X_S}, \mathsf{Y_S})$ *is an algebraic t-structure on* $\mathsf{D}^b(\Lambda)$. *There exist integers* $m \geq n$ *such that* $\mathsf{X_S}[m] \subseteq \mathsf{X} \subseteq \mathsf{X_S}[n]$.

Proof. Note that $\mathsf{X_S}[m] \subseteq \mathsf{X} \subseteq \mathsf{X_S}[n]$ is equivalent to $\mathsf{Y_S}[m] \supseteq \mathsf{Y} \supseteq \mathsf{Y_S}[n]$. Since $(\mathsf{X_S}, \mathsf{Y_S})$ is algebraic, there exist finitely many simple objects $X_1, \ldots, X_t \in \mathsf{H_S}$ such that $\langle X_1, \ldots, X_t \rangle = \mathsf{H_S}$. The boundedness of (X, Y) asserts the existence of an integer k such that $X_i \in \mathsf{X}[k]$ for each $1 \leq i \leq t$, whence $\mathsf{H_S}[\geq 0] \subseteq \mathsf{X}[k]$. Thus, $(\mathsf{H_S}[\geq 0])^\perp = \mathsf{Y_S} \supseteq \mathsf{Y}[k]$, and we can take $m = -k$. Analogously, there also exists an l such that $X_i \in \mathsf{Y}[l]$ for each $1 \leq i \leq t$, so that $\mathsf{H_S}[< 0] \in \mathsf{Y}[l-1]$ and $^\perp(\mathsf{H_S}[< 0]) = \mathsf{X_S} \supseteq \mathsf{X}[l-1]$, and we can take $n = 1 - l$. $\qquad\square$

Our second auxiliary result is the following lifting and restriction lemma.

Lemma 3.3. *If* (X, Y) *is a t-structure on* $\mathsf{D}^b(\Lambda)$, *then* $(\widetilde{\mathsf{X}}, \widetilde{\mathsf{Y}}) := (^\perp(\mathsf{X}^\perp), \mathsf{X}^\perp)$ *is a t-structure on* $\mathsf{D}(\Lambda)$ *such that* $(\widetilde{\mathsf{X}} \cap \mathsf{D}^b(\Lambda), \widetilde{\mathsf{Y}} \cap \mathsf{D}^b(\Lambda)) = (\mathsf{X}, \mathsf{Y})$. *Moreover, if there are integers* $m \geq n$ *such that* $\mathsf{X_S}[m] \subseteq \mathsf{X} \subseteq \mathsf{X_S}[n]$ *for some algebraic t-structure* $(\mathsf{X_S}, \mathsf{Y_S})$, *then we also have* $\widetilde{\mathsf{X_S}}[m] \subseteq \widetilde{\mathsf{X}} \subseteq \widetilde{\mathsf{X_S}}[n]$.

Proof. Since $\mathsf{D}^b(\Lambda)$ is essentially small, [33, Corollary 3.5] says that $(\widetilde{\mathsf{X}}, \widetilde{\mathsf{Y}})$ is indeed a t-structure. Since $\mathrm{Hom}_{\mathsf{D}(\Lambda)}(\mathsf{X}, \widetilde{\mathsf{Y}}) = 0$ by definition, the inclusion $\mathsf{X} \subseteq \widetilde{\mathsf{X}}$ holds. Since $\mathrm{Hom}_{\mathsf{D}(\Lambda)}(\mathsf{X}, \mathsf{Y}) = 0$ we also get $\mathsf{Y} \subseteq \widetilde{\mathsf{Y}}$. Thus, $(\widetilde{\mathsf{X}} \cap \mathsf{D}^b(\Lambda), \widetilde{\mathsf{Y}} \cap \mathsf{D}^b(\Lambda)) = (\mathsf{X}, \mathsf{Y})$. For the final statement, note that $\mathsf{X_S}[m] \subseteq \mathsf{X} \subseteq \mathsf{X_S}[n]$ if and only if $(\mathsf{X_S}[m])^\perp \supseteq \mathsf{X}^\perp \supseteq (\mathsf{X_S}[n])^\perp$, that is $\widetilde{\mathsf{Y_S}}[m] \supseteq \widetilde{\mathsf{Y}} \supseteq \widetilde{\mathsf{Y_S}}[n]$. $\qquad\square$

Remark 3.4. We observe that the lift $(\widetilde{\mathsf{X_S}}, \widetilde{\mathsf{Y_S}})$ of an algebraic t-structure $(\mathsf{X_S}, \mathsf{Y_S})$ on $\mathsf{D}^b(\Lambda)$ is a silting t-structure on $\mathsf{D}(\Lambda)$ in the sense of [7]. Namely, let $\mathsf{U} = (\mathsf{S}[< 0])^\perp$ and $\mathsf{V} = (\mathsf{S}[\geq 0])^\perp$, where the orthogonals are taken in $\mathsf{D}(\Lambda)$. We claim that $(\mathsf{U}, \mathsf{V}) = (\widetilde{\mathsf{X_S}}, \widetilde{\mathsf{Y_S}})$. Since S is a silting subcategory we have $\mathsf{S}[\geq 0] \subseteq \mathsf{X_S}$. Thus, $(\mathsf{S}[\geq 0])^\perp \supseteq \mathsf{X_S}^\perp$, i.e. $\widetilde{\mathsf{Y_S}} \subseteq \mathsf{V}$. For the reverse inclusion, observe that $\mathsf{X_S} \subseteq \mathsf{U}$, so that $\widetilde{\mathsf{Y_S}} = \mathsf{X_S}^\perp \supseteq \mathsf{U}^\perp = \mathsf{V}$.

We are now ready for the key lemma.

Lemma 3.5. *Let* Λ *be a silting-discrete finite-dimensional algebra. Suppose* (X, Y) *is a bounded t-structure on* $\mathsf{D}^b(\Lambda)$. *Then there exists a silting subcategory* $\mathsf{S} = \mathsf{add}(S)$ *and an algebraic t-structure* $(\mathsf{X_S}, \mathsf{Y_S})$ *such that* $\mathsf{X_S}[1] \subseteq \mathsf{X} \subseteq \mathsf{X_S}$.

Proof. Without loss of generality, we may assume $\mathsf{X_T}[m] \subseteq \mathsf{X} \subseteq \mathsf{X_T}$ for some algebraic t-structure $(\mathsf{X_T}, \mathsf{Y_T})$. By Lemma 3.3, we can lift the t-structures and inclusions to $\mathsf{D}(\Lambda)$; these t-structures restrict to the given t-structures on $\mathsf{D}^b(\Lambda)$ and are decorated with tildes.

Following [19, Section 2], we set $\widetilde{\mathsf{Y}_1} = \widetilde{\mathsf{Y_T}}[1] \cap \widetilde{\mathsf{Y}}$, which, by [13, 33], gives rise to a t-structure $(\widetilde{\mathsf{X}_1}, \widetilde{\mathsf{Y}_1})$. It has the following properties: $\widetilde{\mathsf{Y_T}}[1] \supseteq \widetilde{\mathsf{Y}_1} \supseteq \widetilde{\mathsf{Y_T}}$ (equivalently,

$\widetilde{X_T}[1] \subseteq \widetilde{X_1} \subseteq \widetilde{X_T}$); by [19, Lemma 2.12] we have $\widetilde{Y_1}[m-1] \supseteq \widetilde{Y} \supseteq \widetilde{Y_1}$, i.e. $\widetilde{X_1}[m-1] \subseteq \widetilde{X} \subseteq \widetilde{X_1}$.

Now, by Proposition 2.3, there exists a torsion pair $(\widetilde{\mathcal{T}}, \widetilde{\mathcal{F}})$ on $\widetilde{H_T}$ such that $\widetilde{X_1} = \langle \widetilde{\mathcal{T}}, \widetilde{X_T}[1] \rangle$ and $\widetilde{Y_1} = \langle \widetilde{Y_S}, \widetilde{\mathcal{F}} \rangle$. By [27, Corollary 3] or [30, Corollary 4.7], $H_T \simeq \mathsf{Mod}(\Gamma)$, where $\Gamma = \mathrm{End}(T)$. Since any torsion pair on $\mathsf{Mod}(\Gamma)$ restricts to a torsion pair on $\mathsf{mod}(\Gamma)$, the t-structure $(\widetilde{X_1}, \widetilde{Y_1})$ restricts to a t-structure (X_1, Y_1) on $\mathsf{D}^b(\Lambda)$ such that $X_T[1] \subseteq X_1 \subseteq X_T$. By Proposition 3.1, (X_1, Y_1) is an algebraic t-structure with $X_1[m-1] \subseteq X \subseteq X_1$. The lemma now follows by induction. $\square$

Proof of Theorem A. Let (X, Y) be a bounded t-structure in $\mathsf{D}^b(\Lambda)$ for a silting-discrete finite-dimensional algebra Λ. By Lemma 3.5, (X, Y) is intermediate with respect to an algebraic t-structure (X_S, Y_S). By Proposition 3.1, (X, Y) is therefore algebraic. $\square$

4. STABILITY CONDITIONS

4.1. Stability conditions.
Rather than give a formal definition of stability conditions, we give an equivalent formulation; see [12]. Let $\mathbb{H} = \{r \exp(i\pi\varphi) \mid r > 0 \text{ and } 0 < \varphi \leq 1\}$. A *stability function* on an abelian category H consists of a group homomorphism $Z \colon K_0(\mathsf{H}) \to \mathbb{C}$ such that $Z(H) \in \mathbb{H}$ for each $H \in \mathsf{H}$. If H is a length category then a stability function is uniquely determined by its action on the simple objects.

Proposition 4.1 ([12, Proposition 5.3]). *Specifying a stability condition $\sigma = (Z, \mathsf{H})$ on a triangulated category D is equivalent to specifying a bounded t-structure on D together with a stability function $Z \colon K_0(\mathsf{H}) \to \mathbb{C}$ on its heart H that satisfies the Harder-Narasimhan (HN) property.*

Since any stability function on a length heart satisfies the HN property and, by Theorem A, all the bounded t-structures in $\mathsf{D}^b(\Lambda)$ are algebraic when Λ is silting-discrete, we refrain from defining the HN property and refer the reader to [12]. From now on, since a bounded t-structure is determined by its heart we shall identify it with its heart.

Definition 4.2 ([12, Theorem 1.2]). Let D be a triangulated category. The set $\mathsf{stab}(\mathsf{D})$ of stability conditions $\sigma = (Z, \mathsf{H})$ forms a complex manifold called the *stability manifold of* D.

The set of stability conditions $\sigma = (Z, \mathsf{H})$ for some fixed H identifies a 'chamber' C_H of $\mathsf{stab}(\mathsf{D})$. If H is algebraic, then $\mathsf{C}_\mathsf{H} \cong \mathbb{H}^t$, where t is the number of nonisomorphic simple objects of H; see [34].

4.2. Finite-type components.
In [32] the *tilting poset* of a triangulated category D, denoted $\mathsf{tilt}(\mathsf{D})$, consists of bounded t-structures in D with the following partial order,

$(X, Y) \leq (X', Y') \iff (X', Y')$ is obtained from (X, Y) by finitely many left HRS-tilts.

The subposet $\mathsf{tilt}_{\mathsf{alg}}(\mathsf{D})$ consists of algebraic t-structures, where $(X, Y) \leq (X', Y')$ if and only if (X', Y') is obtained from (X, Y) through finitely many left HRS-tilts, where each intermediate t-structure in the sequence of tilts is again algebraic. A component $\mathsf{tilt}^\circ(\mathsf{D})$ of the tilting poset has *finite type* if each t-structure in $\mathsf{tilt}^\circ(\mathsf{D})$ is algebraic and has only finitely many torsion pairs in its heart. The following proposition is an immediate consequence of Theorem A and Theorem 2.6.

Proposition 4.3. *Let Λ be a silting-discrete finite-dimensional algebra. Then $\mathsf{tilt}(\mathsf{D}^b(\Lambda)) = \mathsf{tilt}_{\mathsf{alg}}(\mathsf{D}^b(\Lambda))$ and each component $\mathsf{tilt}^\circ(\mathsf{D}^b(\Lambda))$ of $\mathsf{tilt}(\mathsf{D}^b(\Lambda))$ has finite type.*

By [32, Lemma 4.3], if $\mathsf{tilt}^\circ(\mathsf{D})$ is a finite-type component of $\mathsf{tilt}(\mathsf{D})$ then

$$\mathsf{stab}^\circ(\mathsf{D}) = \bigcup_{\mathsf{H} \in \mathsf{tilt}^\circ(\mathsf{D})} \mathsf{C}_\mathsf{H}$$

is a component of $\mathsf{stab}(\mathsf{D})$ called a *finite-type component*. The main theorem of [32] is the following.

Theorem 4.4 ([32, Theorem 4.9]). *If* $\mathsf{stab}^\circ(\mathsf{D})$ *is a finite-type component of* $\mathsf{stab}(\mathsf{D})$ *then* $\mathsf{stab}^\circ(\mathsf{D})$ *is contractible.*

4.3. **Connectedness of the silting poset.** Recall that the *Hasse diagram (or quiver)* of a poset $(S, \leq)$ has as vertices the elements of S, and there is an arrow $s \to t$ in the diagram if and only if $s < t$ and there is no element $x \in S$ such that $s < x < t$. A poset is called *connected* if its Hasse quiver is connected.

By [25, Theorem 7.13] and [5, Theorem 2.35], there is an arrow $(\mathsf{X}, \mathsf{Y}) \to (\mathsf{X}', \mathsf{Y}')$ in the Hasse quiver of $\mathsf{tilt}(\mathsf{D}^b(\Lambda))$ if and only if the second t-structure can be obtained from the first by HRS tilt with respect to a torsion pair in the heart generated by a simple object. These are called *simple left tilts* in [32]. In particular, it follows that the poset $\mathsf{tilt}_{\mathsf{alg}}(\mathsf{D}^b(\Lambda))$ and the silting poset $\mathbb{P}_1(\mathsf{K}^b(\mathsf{proj}(\Lambda)))$ from Section 2.2 have isomorphic Hasse quivers.

Recall from [14] that a silting pair $(\mathsf{M}, \mathsf{M}')$ consists of a silting subcategory M of a triangulated category D and a functorially finite subcategory $\mathsf{M}' \subseteq \mathsf{M}$. The poset of silting pairs $\mathbb{P}_2(\mathsf{D})$ was defined in [14] via the opposite of the following partial order:

$$(\mathsf{N}, \mathsf{N}') \geq (\mathsf{M}, \mathsf{M}') \overset{\text{def}}{\iff} \mathsf{R}_{\mathsf{M}'}(\mathsf{M}) \geq \mathsf{R}_{\mathsf{N}'}(\mathsf{N}) \geq \mathsf{N} \geq \mathsf{M},$$

where on the right-hand side the partial order is that of $\mathbb{P}_1(\mathsf{D})$ defined in Section 2.2 and $\mathsf{R}_{\mathsf{M}'}(\mathsf{M})$ is the right mutation of M at M'; see [5] and [14, Section 5] for details. One gets the following theorem by observing that the proof in [14] works in this level of generality.

Theorem 4.5 ([14, Corollary 6.2 and Theorem 7.1]). *Suppose* Λ *is a silting-discrete finite-dimensional algebra. Then* $\mathbb{P}_2(\mathsf{K}^b(\mathsf{proj}(\Lambda)))$ *is a CW poset and* $B\mathbb{P}_2(\mathsf{K}^b(\mathsf{proj}(\Lambda)))$, *the classifying space of the poset, is contractible.*

Corollary 4.6. *If* Λ *is a silting-discrete finite-dimensional* $\mathbf{k}$-*algebra then* $\mathsf{tilt}_{\mathsf{alg}}(\mathsf{D}^b(\Lambda)) = \mathsf{tilt}(\mathsf{D}^b(\Lambda))$ *is connected.*

Proof. By Proposition 4.3, we have that $\mathsf{tilt}_{\mathsf{alg}}(\mathsf{D}^b(\Lambda)) = \mathsf{tilt}(\mathsf{D}^b(\Lambda))$, and we only need to check that

$$\mathsf{tilt}_{\mathsf{alg}}(\mathsf{D}^b(\Lambda)) \cong \mathbb{P}_1(\mathsf{K}^b(\mathsf{proj}(\Lambda)))$$

is connected. This is a generalisation of [14, Corollary 7.2]: the Hasse quiver of $\mathbb{P}_1(\mathsf{K}^b(\mathsf{proj}(\Lambda)))$ is the 1-skeleton of $B\mathbb{P}_2(\mathsf{K}^b(\mathsf{proj}(\Lambda)))$, which is contractible by Theorem 4.5, whence $\mathbb{P}_1(\mathsf{K}^b(\mathsf{proj}(\Lambda)))$ is connected. $\square$

Remark 4.7. We prove Corollary 4.6 using Theorem 4.5 above because we believe the stronger statement that $B\mathbb{P}_2(\mathsf{K}^b(\mathsf{proj}(\Lambda)))$ is contractible for any silting-discrete algebra is of independent interest. However, there is a direct argument, which we give below.

Given any $T \in \mathbb{P}_1(\mathsf{K}^b(\mathsf{proj}(\Lambda)))$ we have $\Lambda[m] \leq T \leq \Lambda[n]$, for some integers $m \geq n$. Silting discreteness together with [5, Proposition 2.36] give that $\Lambda[m]$ can be obtained from T by a finite number of irreducible left mutations. Therefore, each $T \in \mathbb{P}_1(\mathsf{K}^b(\mathsf{proj}(\Lambda)))$ is connected to some shift $\Lambda[m]$ in the Hasse quiver of $\mathbb{P}_1(\mathsf{K}^b(\mathsf{proj}(\Lambda)))$. But, applying this argument with $T = \Lambda[m-1]$, we get that $\Lambda[m]$ and $\Lambda[m-1]$ are connected in that Hasse quiver. Therefore $\mathbb{P}_1(\mathsf{K}^b(\mathsf{proj}(\Lambda)))$ is connected.

Corollary B now follows directly from Corollary 4.6 by applying Theorem 4.4.

Corollary 4.8. *If Λ is a silting-discrete finite-dimensional $\mathbf{k}$-algebra then the stability manifold $\mathsf{stab}(\mathsf{D}^b(\Lambda))$ is contractible.*

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