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One-dimensional scaling limits in a planar Laplacian random growth model

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Abstract

We consider a family of growth models defined using conformal maps in which the local growth rate is determined by $|\Phi_n'|^{-\eta}$, where Φ_n is the aggregate map for n particles. We establish a scaling limit result in which strong feedback in the growth rule leads to one-dimensional limits in the form of straight slits. More precisely, we exhibit a phase transition in the ancestral structure of the growing clusters: for $\eta > 1$, aggregating particles attach to their immediate predecessors with high probability, while for $\eta < 1$ almost surely this does not happen.

Contents

1 Introduction **2**
1.1 Conformal aggregation processes 2
1.2 ALE: Aggregate Loewner Evolution 4

2 Overview of results **5**
2.1 A related Markovian model 7
2.2 Overview of the proof of Theorem 1 and organization of the paper 10

3 Loewner flows **12**
3.1 Loewner's equation 12
3.2 Reverse-time Loewner flow 14
3.3 Convergence of Loewner chains 15

4 Analysis of the slit map **16**
4.1 The half-plane 17
4.2 The exterior disk 18
4.3 Moment computations 21

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5 Estimates on conformal maps via Loewner's equation **23**
5.1 Near the tip 24
5.2 Away from the slits 29

6 Ancestral lines and convergence for ALE **36**
6.1 Combined derivative estimates 37
6.2 The ancestral lines and convergence theorem 37
6.3 Modifications of the model 41

1 Introduction

1.1 Conformal aggregation processes

Laplacian growth models describe processes where the local growth rate of a piece of the boundary of a growing compact cluster is determined by the Green's function of the exterior of the cluster. Such growth processes can be used to model a range of physical phenomena, including ones involving aggregates of diffusing particles. Discrete versions can be formulated on a lattice in all dimensions: some famous examples of this type of growth process include diffusion-limited aggregation (DLA) [30], the Eden model [4], or the more general dielectric breakdown model (DBM) [24]. Despite considerable numerical evidence suggesting that the clusters that arise in these processes exhibit fractal features, very few rigorous results are known (for DLA, see [16]) and it remains a formidable challenge to rigorously analyze long-term behavior such as sharp growth rates of the clusters.

One objection that can be leveled at lattice-based models is that the underlying discrete spacial structure could potentially introduce anisotropies in the growing clusters that are not present in the physical setting of the plane or three-space. Indeed, large-scale simulations in two dimensions demonstrate anisotropy along the coordinate axes [5]. This fact provides one motivation for the study of off-lattice versions of aggregation processes. In the plane, such off-lattice models can be formulated in terms of iterated conformal mappings, providing access to complex analytic machinery. Clusters produced by these conformal growth processes are initially isotropic by construction, but simulations suggest that in many instances, anisotropic structures appear on timescales where the number of aggregated particles become large compared to the size of the individual constituent particles. Nevertheless, proving the existence of such small-particle limits, whether anisotropic or not, has proved elusive, similarly to the case of lattice-based models.

A fascinating feature of Laplacian growth models is competition between concentration and dispersion of particle arrivals on the cluster boundary. Protruding structures ("branches") and their endpoints ("tips") tend to attract relatively many arrivals, but they compete with each other as well as the remainder of the boundary. (Kesten's discrete Beurling estimate gives an upper bound on the tip concentration in the case of DLA.) The degree to which tips are favored is determined by the exact choice of growth rule, and several models contain one or more parameters that affect concentration, dispersion, and competition [24, 8, 2, 18].

Previous work on small-particle limits of conformal aggregation models [25, 14, 29] has yielded growing discs, that is, smooth and isotropic shapes; the dispersion effect "wins" in the limit. In this paper, we study a particular instance of a conformal growth model, focusing instead on the concentration aspect of Laplacian growth and showing that anisotropic scaling limits arise in the presence of strong feedback in the growth rule. The scaling limits we exhibit are highly degenerate in the sense that growth, which is initially spread out, favors tips very strongly, and eventually collapses onto a single growing slit.

To state our results, we first describe the general class of processes our object of study fits into. Let $\mathbf{c} > 0$, and let $f_{\mathbf{c}}$ denote the unique conformal map

$$f_{\mathbf{c}} \colon \Delta = \{z \in \mathbb{C} \colon |z| > 1\} \cup \{\infty\} \to D_1 = \Delta \setminus (1, 1 + d]$$

having $f_{\mathbf{c}}(z) = e^{\mathbf{c}} z + \mathcal{O}(1)$ at infinity, and sending the exterior disk Δ to the complement of the closed unit disk with a slit of length $d = d(\mathbf{c})$ attached to the unit circle $\mathbb{T}$ at the point 1. The capacity increment $\mathbf{c}$ and the length d of the slit satisfy

$$e^{\mathbf{c}} = 1 + \frac{d^2}{4(1 + d)}; \tag{1}$$

in particular, $d \asymp \mathbf{c}^{1/2}$ as $\mathbf{c} \to 0$. In terms of aggregation, the closed unit disk can be viewed as a seed, while the slit represents an attached particle. Typically, we think of the particle as being small compared to the seed.

A general two-parameter framework to model random or deterministic aggregation, based on conformal maps, is given by the following construction. Pick a sequence $\{\theta_k\}_{k=1}^{\infty}$ in $[-\pi, \pi)$, and let $\{d_k\}_{k=1}^{\infty}$, or, equivalently, $\{c_k\}_{k=1}^{\infty}$, be a sequence of non-negative numbers connected via (1). From the two numerical sequences $\{\theta_k\}$ and $\{c_k\}$, we obtain a sequence $\{f_k\}_{k=1}^{\infty}$ of rotated and rescaled conformal maps, referred to as building blocks, via

$$f_k(z) = e^{i\theta_k} f_{c_k}(e^{-i\theta_k} z).$$

Finally, we set

$$\Phi_n(z) = f_1 \circ \cdots \circ f_n(z), \quad n = 1, 2, \ldots. \tag{2}$$

Each Φ_n is itself a conformal map sending the exterior disk onto the complement of a compact set $K_n \subset \mathbb{C}$, that is,

$$\Phi_n \colon \Delta \to \mathbb{C} \setminus K_n.$$

The sets $\{K_n\}_{n=1}^{\infty}$ are called clusters. They satisfy $K_{n-1} \subset K_n$, and model a growing two-dimensional aggregate formed of n particles. At infinity, we have

$$\Phi_n(z) = e^{C_n} z + \mathcal{O}(1),$$

where

$$\operatorname{cap}(K_n) = e^{C_n} = e^{\sum_{k=1}^{n} c_k} \tag{3}$$

is the total capacity of the nth cluster.

When modeling random aggregates formed via diffusion, one chooses the angles $\{\theta_k\}$ to be iid, and uniform in $[-\pi, \pi)$. Due to the conformal invariance of harmonic measure, this has the effect of attaching the nth particle at a point chosen according to harmonic measure (seen from infinity) on the boundary of K_{n-1}. This type of setup has been considered in a number of papers, see for instance [8, 19, 1, 21, 28, 10, 25, 13, 14, 29]; we shall only briefly mention models that are particularly pertinent to our study.

1.2 ALE: Aggregate Loewner Evolution

The main object of study in the present paper is a model we refer to as aggregate Loewner evolution, abbreviated $\mathrm{ALE}(\alpha, \eta)$, with parameters $\alpha \in \mathbb{R}$ and $\eta \in \mathbb{R}$. In $\mathrm{ALE}(\alpha, \eta)$, conformal maps Φ_n are defined as in (2) as follows.

Initialize by setting $\Phi_0(z) = z$ and letting $\mathcal{F}_0$ be the trivial σ-algebra.

- For $k = 1, 2, 3, \ldots$, we let θ_k have distribution conditional on $\mathcal{F}_{k-1} = \mathcal{F}(\theta_1, \ldots \theta_{k-1}; c_1, \ldots, c_{k-1})$ given by
$$h_k(\theta) = \frac{|\Phi'_{k-1}(e^{\boldsymbol{\sigma}+i\theta})|^{-\eta} \mathrm{d}\theta}{\int_{\mathbb{T}} |\Phi'_{k-1}(e^{\boldsymbol{\sigma}+i\theta})|^{-\eta} \mathrm{d}\theta}. \tag{4}$$
Here, $\boldsymbol{\sigma} > 0$ is a regularization parameter, which ensures that the angle distributions are well defined even though $\Phi'_{k-1}(e^{i\theta})$ has zeros and singularities on $\mathbb{T}$. The parameter $\boldsymbol{\sigma}$ is allowed to depend on the basic capacity parameter $\mathbf{c}$. Typically, we shall take
$$\boldsymbol{\sigma} = \boldsymbol{\sigma}(\mathbf{c}) = \mathbf{c}^{\gamma}$$
for some appropriate $\gamma > 0$.

- Next, we define a sequence of capacity increments for $k = 1, 2, 3, \ldots$ by taking
$$c_k = \mathbf{c} |\Phi'_{k-1}(e^{\boldsymbol{\sigma}+i\theta_k})|^{-\alpha}. \tag{5}$$

We note that $\mathrm{ALE}(\alpha, 0)$ is the same model as the Hastings-Levitov $\mathrm{HL}(\alpha)$ model studied in [8, 3, 28, 14], and in particular $\mathrm{ALE}(0, 0)$ coincides with the $\mathrm{HL}(0)$ model studied in depth in [25, 29]. The Hastings-Levitov model was introduced as a conformal mapping model of dielectric breakdown (DBM) [24], a discrete model in which vertices are added to a growing cluster by drawing bonds from among the neighboring lattice points. At stage n of $\mathrm{DBM}(\eta)$, a point is added to the cluster K_n by including a neighbor of $(j, k) \in K_n$ with probability
$$p_n\left((j,k) \to (j', k')\right) = \frac{\phi_n(j', k')^{\eta}}{\sum_{(l,m)} \phi_n(l, m)^{\eta}}.$$
Here, summation is over lattice neighbors of K_n and the function ϕ_n is discrete harmonic, that is $\Delta \phi_n = 0$ on $\mathbb{Z}^2 \setminus K_n$, and has $\phi_n = 0$ on K_n and $\phi_n = 1$ on some large external circle.

Off-lattice versions of DBM involving non-uniform angle choices determined by the derivative of a conformal map have been considered by several authors. Hastings [6], and subsequently Mathiesen and Jensen [21], study a model that essentially corresponds to $\mathrm{ALE}(2, \eta)$ modulo a slightly different parametrization in η. (In fact, an alternative name for the growth model in this paper could have been $\mathrm{DBM}(\alpha, \eta)$ or $\mathrm{HL}(\alpha, \eta)$, but we have opted for a different terminology to avoid confusion with lattice models, and also to emphasize connections with the Loewner equation, see below.) Hastings argues that large enough exponents, more precisely, for $\eta \geqslant 3$ in our parametrization, the corresponding clusters become one-dimensional; he also points out that the behavior of the models depends strongly on the choice of regularization.

Another model that fits into this general framework is the Quantum Loewner Evolution model ($\mathrm{QLE}(\gamma, \eta)$) of Miller and Sheffield [22, 23] which is proposed as a scaling limit of $\mathrm{DBM}(\eta)$ on a γ-Liouville quantum gravity surface. In the QLE model, particles are attached according to a distribution which depends on the power of the derivative of the cluster map, as in (4), but with an additional term involving the Gaussian Free Field due to the presence of Liouville quantum gravity. In the construction of QLE, capacity increments are kept constant, as for ALE$(0, \eta)$. However, each particle in QLE is constructed as an SLE curve, rather than the straight slits used in ALE.

Common to all conformal mapping models of Laplacian growth is the difficulty that derivatives of conformal mappings do not remain bounded away from 0 or ∞ as they approach the boundary and therefore the map $\theta \mapsto |\Phi_n'(e^{i\theta})|^{-1}$ can be very badly behaved. For instance, even when $n = 1$, $|\Phi_n'(e^{i\theta})|^{-\eta}$ is not integrable over $\mathbb{T}$ for certain values of η and hence the ALE(α, η) model would not be well defined if we were to use $|\Phi_n'(e^{i\theta})|^{-\eta}$ as angle density. As mentioned above, for this reason we define the model via the regularization parameter $\boldsymbol{\sigma}$ as in (4), and then let $\boldsymbol{\sigma} \to 0$ together with the (pre-image) particle size. A similar difficulty arises from the dependence of the particle sizes on the derivatives of the conformal mappings. Although in this case the model is well-defined without the need for a regularization parameter in (5), it is no longer possible to guarantee that the resulting clusters have total capacity bounded above and below. Indeed, even with the presence of a regularization parameter, it is not clear that the total capacity remains bounded as $\boldsymbol{\sigma} \to 0$. The exception is the ALE$(0, \eta)$ model: in light of (3), taking $n \asymp \mathbf{c}^{-1}$ is a natural choice of time-scaling in ALE$(0, \eta)$ as with this choice the resulting clusters have total capacity bounded above and below. This in turn means that the total diameter of the clusters K_n remains bounded as a consequence of Koebe's 1/4-theorem, see [27]. The fact that we have some a priori control over the global size of clusters is our main motivation for moving from studying HL(α) with α large to ALE$(0, \eta)$ with η large. Simulations suggest that one-dimensional limits are present also in HL(α) for large α but showing that this is the case seems technically more difficult.

In this paper, we mainly focus on ALE$(0, \eta)$ for $\eta > 1$, and show that the conformal maps Φ_n converge to a randomly oriented single-slit map in the regime where $n \asymp \mathbf{c}^{-1}$. This can be viewed as a rigorous version of Hastings' investigation [6] of ALE$(2, \eta)$ for the ALE$(0, \eta)$ model. To obtain our convergence results, we exploit what is in a way the most extreme mechanism that could lead to a single-slit limit, namely that of aggregated particles becoming attached to their immediate predecessors. The main difficulties in the proof are that the angle densitites induced by slit maps have maxima and minima of different orders, even in the presence of regularization, making it hard to show convergence to a point mass. Furthermore, the feedback mechanism in (4) is sensitive so that a single "bad" angle can destroy the genealogical structure of the growing slit by leading to the creation of a new, competing tip further down the slit, which could lead to a splitting of growth into two branches.

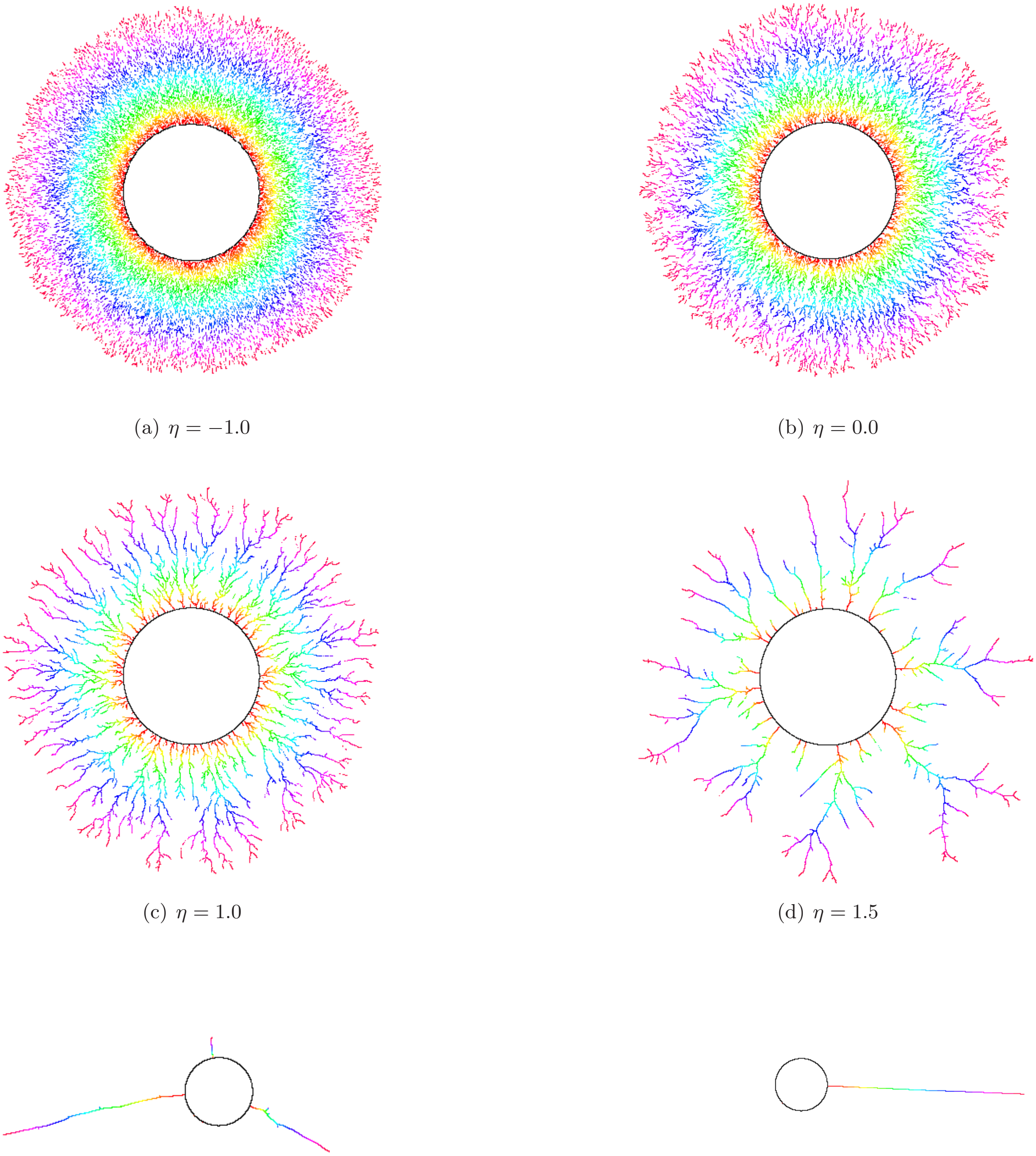
2 Overview of results

Clusters that are formed by successively composing slit maps come with a natural notion of ancestry for their constituent particles. We say that a particle j has parent 0 if it attaches directly to the unit disk and that the particle j has parent k if the jth particle is directly attached to the kth particle. More precisely, suppose that $\beta_{\mathbf{c}} \in (0, \pi)$ is defined by

$$f_{\mathbf{c}}^{-1}\left((1, 1 + d(\mathbf{c})]\right) = \{e^{i\theta} : |\theta| < \beta_{\mathbf{c}}\}$$

so $e^{\pm i\beta_{\mathbf{c}}}$ is mapped by the basic slit map to the base point of the slit i.e. $f_{\mathbf{c}}(e^{\pm i\beta_{\mathbf{c}}}) = 1$. Therefore particle j has parent 0 if $|\Phi_j(e^{i(\theta_j \pm \beta_{\mathbf{c}})})| = 1$ and particle j has parent $k \geqslant 1$ if

$$e^{-i\theta_k}\Phi_{k,j}(e^{i(\theta_j \pm \beta_{\mathbf{c}})}) \in (1, 1 + d(\mathbf{c})],$$



(a) $\eta = -1.0$

(b) $\eta = 0.0$

(c) $\eta = 1.0$

(d) $\eta = 1.5$

(e) $\eta = 2.0$

(f) $\eta = 4.0$

Figure 1: ALE$(0, \eta)$ *clusters with* $\mathbf{c} = 10^{-4}$, $\boldsymbol{\sigma} = \mathbf{c}^2$, *and* $n = 10,000$.

where $\Phi_{k,j}(z) = f_k \circ f_{k+1} \circ \cdots \circ f_j(z)$.

In the ALE$(0,\eta)$ model, each successive particle chooses its attachment point on the cluster according to the relative density of harmonic measure (as seen from infinity) raised to the power η. As the highest concentration of harmonic measure occurs at the tips of slits, intuitively one would expect that for sufficiently large values of η each particle is likely to attach near the tip of the previous particle. In this paper we show that this indeed happens, and we identify the values of η for which the above event occurs with high probability in the limit as $\mathbf{c} \to 0$. Figure 1 displays ALE$(0,\eta)$ clusters for different values of η.

The limiting behavior of the model is quite sensitive to the rate at which $\boldsymbol{\sigma} \to 0$ as $\mathbf{c} \to 0$. Figure 2 show how the angle sequences $\{\theta_k\}$ in ALE$(0,4)$ are affected by the choice of exponent γ when regularizing by $\boldsymbol{\sigma} = \mathbf{c}^\gamma$; this phenomenon is also observed by Hastings in [6] for a related model. In [14], which deals with slow-decaying $\boldsymbol{\sigma}$ scaling limits in a strongly regularized version of HL(α), it is shown that the scaling limits of the clusters are disks. By choosing $\boldsymbol{\sigma}$ to decay sufficiently slowly compared to $\mathbf{c}$, once can prove that the corresponding scaling limits in ALE$(0,\eta)$ are again disks, this is the topic of forthcoming work of Norris, Silvestri, and Turner. As we seek results which do not strongly depend on the choice of regularisation parameter, part of our objective is to identify the minimal value of η for which there exists some σ_0 (dependent on $\mathbf{c}$ and η) such that, provided $\boldsymbol{\sigma} < \sigma_0$, with high probability each particle lands on the tip of the previous particle.

The following is the main result of the paper and shows that the ALE$(0,\eta)$ model exhibits a phase transition at $\eta = 1$ in the genealogy of the growing cluster in the small-particle limit. See Theorem 16 for a complete statement and proof, in particular we give sufficient conditions on γ.

Theorem 1 (ALE$(0,\eta)$ model)**.** *For* ALE$(0,\eta)$*, let* $\Omega_N = \Omega_N^{\eta,\mathbf{c},\boldsymbol{\sigma}}$ *be the event defined by*

$$\Omega_N = \{\textit{Particle } j \textit{ has parent } j-1 \textit{ for all } j = 1, \dots, N\}.$$

For each $\eta > 1$*, there exists some* $\gamma = \gamma(\eta)$ *such that if* $\sigma_0 = \mathbf{c}^\gamma$ *and if* $N = n(T) = \lfloor T\mathbf{c}^{-1} \rfloor$ *for some fixed* $T > 0$*, then*

$$\lim_{\mathbf{c}\to 0} \inf_{0<\boldsymbol{\sigma}<\sigma_0} \mathbb{P}(\Omega_N) = 1,$$

whereas if $\eta < 1$*, then for any* $N > 1$*,*

$$\lim_{\mathbf{c}\to 0} \sup \sup_{\boldsymbol{\sigma}>0} \mathbb{P}(\Omega_N) = 0.$$

In the case when $\eta > 1$ *and* $\boldsymbol{\sigma} < \sigma_0$*, it follows that, for any* $r > 1$*,*

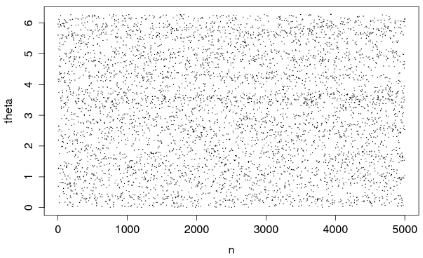
$$\sup_{t\leqslant T} \sup_{\{|z|>r\}} |\Phi_{n(t)}(z) - e^{i\theta_1} f_t(e^{-i\theta_1} z)| \to 0 \quad \textit{in probability as} \quad \mathbf{c} \to 0,$$

and the cluster $K_{n(t)}$ *converges in the Hausdorff topology to a slit of capacity* t *at position* $z = e^{i\theta_1}$*.*

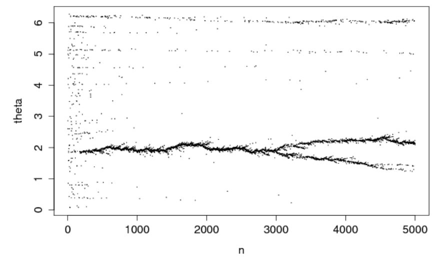
2.1 A related Markovian model

Observe that, for each k, we are free to specify the interval of length 2π in which to sample θ_k, and this choice does not have any effect on the maps Φ_n. It is convenient to choose to sample θ_k from the interval $[\theta_{k-1} - \pi, \theta_{k-1} + \pi)$. In this case, we can express the event as

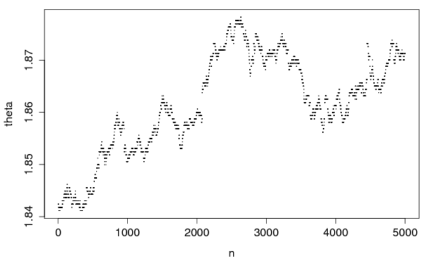
$$\Omega_N = \left\{ \sup_{2\leqslant j\leqslant N} |\theta_j - \theta_{j-1}| < \beta_{\mathbf{c}} \right\}.$$



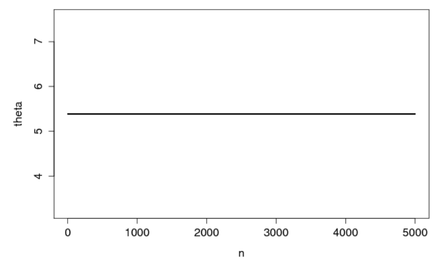
(a) $\boldsymbol{\sigma} = \mathbf{c}^{1/4}$



(b) $\boldsymbol{\sigma} = \mathbf{c}^{1/2}$



(c) $\boldsymbol{\sigma} = \mathbf{c}$ (Note different spatial scale.)



(d) $\boldsymbol{\sigma} = \mathbf{c}^2$

Figure 2: ALE$(0,4)$ *angle sequences with* $\mathbf{c} = 10^{-4}$ *and* $n = 5,000$, *with varying regularization* $\boldsymbol{\sigma}$.

(By definition, $\beta_{\mathbf{c}} \in (0,\pi)$ and $e^{i\pm\beta_{\mathbf{c}}}$ is mapped by the basic slit map to the base point of the slit i.e. $f_{\mathbf{c}}(e^{\pm i\beta_{\mathbf{c}}}) = 1$.) One of the main difficulties in analysing this event is that the distribution of θ_k conditional on $\mathcal{F}_{k-1}$ (as defined in (4)), depends non-trivially on the entire sequence $\theta_1, \ldots, \theta_{k-1}$. In this subsection, we introduce an auxiliary model for random growth in the exterior unit disk in which the sequence of attachment angles is Markovian. The auxiliary model is relatively straightforward to analyse and we show below that it exhibits an analogous phase transition to that described above. The remainder of the paper is concerned with examining how ALE$(0,\eta)$ and the auxiliary model relate to each other.

Set $\Phi_0^*(z) = z$ and let $\{\Phi_n^*\}$ be conformal maps obtained through composing

$$\Phi_n^* = f_1^* \circ \cdots \circ f_n^*,$$

where each f_k^* is a building block with $c_k = \mathbf{c}$, and rotation angle θ_k^* having conditional distribution with density

$$h_k^*(\theta|\theta_{k-1}^*) = \frac{1}{Z_{k-1}^*}|f'_{\mathbf{c}(k-1)}(e^{\boldsymbol{\sigma}+i(\theta-\theta_{k-1}^*)})|^{-\eta}, \quad k = 1,2,3,\ldots. \tag{6}$$

Here, we have set

$$Z_k^* = \int_{\mathbb{T}} |f'_{\mathbf{c}k}(e^{\boldsymbol{\sigma}+i\theta})|^{-\eta}\mathrm{d}\theta$$

and suppressed the dependence on $\mathbf{c}$, $\boldsymbol{\sigma}$ and η to ease notation. In order for the measure to be well-defined when $\eta \geqslant 1$, we require $\boldsymbol{\sigma} > 0$. In words, the density of the kth angle distribution in this model is obtained by replacing the complicated $(k-1)$th cluster map of ALE by a simple slit map "centered" at θ^*_{k-1}, and with deterministic capacity $\mathbf{c}(k-1)$.

For this model we obtain the following theorem: we again set $n(t) = \lfloor t/\mathbf{c} \rfloor$.

Theorem 2 (Auxiliary model)**.** *Set $\sigma_0 = \mathbf{c}^{\gamma^*}$ where*

$$\gamma^* > \frac{\eta+1}{2(\eta-1)}$$

Then

$$\lim_{\mathbf{c}\to 0} \inf_{0<\boldsymbol{\sigma}<\sigma_0} \mathbb{P}(\Omega_N) = 1 \quad \textit{if } \eta > 1$$
$$\limsup_{\mathbf{c}\to 0} \sup_{\boldsymbol{\sigma}>0} \mathbb{P}(\Omega_N) = 0 \quad \textit{if } \eta < 1.$$

Furthermore, when $\eta > 1$ and $\boldsymbol{\sigma} < \sigma_0$, for any $r > 1$,

$$\sup_{t\leqslant T} \sup_{\{|z|>r\}} |\Phi^*_{n(t)}(z) - e^{i\theta^*_1} f_t(e^{-i\theta^*_1} z)| \to 0 \quad \textit{in probability as} \quad \mathbf{c} \to 0,$$

and the cluster $K_{n(t)}$ converges in the Hausdorff topology to a slit of capacity t at position $z = e^{i\theta^*_1}$.

Remark. It can also be shown that $\lim_{\mathbf{c}\to 0} \inf_{0<\boldsymbol{\sigma}<\sigma_0} \mathbb{P}(\Omega_N) = 1$ when $\eta = 1$, provided $\sigma_0 \to 0$ exponentially fast as $\mathbf{c} \to 0$, but we omit the details here.

Proof. Since we can always rotate the clusters K_n by a fixed angle, without loss of generality, we assume that the initial angle $\theta^*_1 = 0$. As above, we choose to sample θ^*_k from the interval $[\theta^*_{k-1} - \pi, \theta^*_{k-1} + \pi)$. This means that we can write $\theta^*_n = u_2 + \cdots + u_n$ where the u_k are independent $[-\pi, \pi)$-valued random variables and $u_k = \theta^*_k - \theta^*_{k-1}$ has symmetric distribution $h^*_k(\theta|0)$.

First suppose $\eta > 1$. Then by (19) and Lemma 8 below there exists some constant A (which may change from line to line), depending only on T and η, such that

$$A^{-1}(k\mathbf{c})^{1/2} < \beta_{k\mathbf{c}} < A(k\mathbf{c})^{1/2},$$

$$\frac{A^{-1}}{\boldsymbol{\sigma}}\left(1 + \frac{\theta^2}{\boldsymbol{\sigma}^2}\right)^{-\eta/2} \leqslant h^*_k(\theta|0) \leqslant \frac{A}{\boldsymbol{\sigma}}\left(1 + \frac{\theta^2}{\boldsymbol{\sigma}^2}\right)^{-\eta/2} \quad \text{for } |\theta| < \tfrac{\beta_{k\mathbf{c}}}{2},$$

and

$$h^*_k(\theta|0) \leqslant A\boldsymbol{\sigma}^{\eta-1}(\mathbf{c}k)^{-\eta/2} \quad \text{for } |\theta| > \tfrac{\beta_{k\mathbf{c}}}{2}.$$

Therefore

$$\mathbb{P}\left(|u_k| \geqslant \frac{\beta_{\mathbf{c}}}{2}\right) = 2\int_{\frac{\beta_{\mathbf{c}}}{2}}^{\frac{\beta_{k\mathbf{c}}}{2}} h^*_k(\theta|0)\mathrm{d}\theta + 2\int_{\frac{\beta_{k\mathbf{c}}}{2}}^{\pi} h^*_k(\theta|0)\mathrm{d}\theta \leqslant A(\boldsymbol{\sigma}^{\eta-1}\mathbf{c}^{\frac{1}{2}(1-\eta)} + \boldsymbol{\sigma}^{\eta-1}(\mathbf{c}k)^{-\eta/2}).$$

Hence, for $\eta > 1$,

$$\mathbb{P}(\Omega^c_N) \leqslant \mathbb{P}\left(\sup_{2\leqslant k\leqslant N} |\theta^*_k - \theta^*_{k-1}| \geqslant \frac{\beta_{\mathbf{c}}}{2}\right) \leqslant \sum_{k=2}^{N} \mathbb{P}\left(|u_k| \geqslant \frac{\beta_{\mathbf{c}}}{2}\right) \leqslant A\boldsymbol{\sigma}^{\eta-1}\mathbf{c}^{-\frac{1}{2}(\eta-1)}\mathbf{c}^{-1} \longrightarrow 0$$

as $\mathbf{c} \to 0$.

Now suppose that $\eta < 1$. Then, using Lemmas 8 and 9 and letting $\mathbf{c} \to 0$, we get

$$\mathbb{P}(\Omega_N) \leqslant \mathbb{P}\left(|\theta_2^*| < \beta_{\mathbf{c}}\right) \leqslant A \left(\int_0^{\frac{\beta_{\mathbf{c}}}{2}} \frac{\mathbf{c}^{\eta/2} \mathrm{d}\theta}{(\boldsymbol{\sigma}^2 + \theta^2)^{\eta/2}} + \int_{\frac{\beta_{\mathbf{c}}}{2}}^{\beta_{\mathbf{c}}} \mathrm{d}\theta \right) \leqslant A\mathbf{c}^{1/2} \longrightarrow 0.$$

To show convergence of $\Phi^*_{n(t)}(z)$ to $f_t(z)$ for $t < T$ when $\eta > 1$ and $\boldsymbol{\sigma} < \sigma_0$, by Proposition 3 it is enough to show that $\sup_{n \leqslant N} |\theta^*_n| \to 0$ with high probability as $\mathbf{c} \to 0$. To do this, we write

$$\theta^*_n = \sum_{k=2}^{n} u_k \mathbf{1}_{\{|u_k| < \beta_{\mathbf{c}}/2\}} + \sum_{k=2}^{n} u_k \mathbf{1}_{\{|u_k| \geqslant \beta_{\mathbf{c}}/2\}}$$

and note that $M^*_n = \sum_{k=2}^n u_k \mathbf{1}_{\{|u_k| < \beta_{\mathbf{c}}/2\}}$ is a martingale. Since $\theta^*_n = M^*_n$ with high probability, convergence of $\sup_{n \leqslant N} |\theta^*_n|$ to 0 follows from moment bounds in Lemma 9 together with standard martingale arguments. □

2.2 Overview of the proof of Theorem 1 and organization of the paper

The main idea for the proof is to show that the Markovian model of the previous section is a good approximation of the ALE$(0, \eta)$ process. In order to do this one approach would be to try to argue that $|\Phi'_n(e^{\boldsymbol{\sigma} + i\theta})|$ can be globally well approximated by $|(f^{\theta_n}_{n\mathbf{c}})'(e^{\boldsymbol{\sigma} + i\theta})|$. However, this seems difficult to make work to sufficient precision when evaluating the maps close to the boundary. Specifically, the map $\Phi'_n(z)$ has zeros (respectively singularities) at each of the points on the boundary of the unit disk which are mapped to the tip (respectively to the base) of one of the slits corresponding to an individual particle. In contrast, for the map $(f^{\theta_n}_{n\mathbf{c}})'(z)$, the points corresponding to tips and bases of successive particles coincide and therefore the singularities and zeros corresponding to intermediate particles cancel each other out, leaving only a zero at the point mapped to the tip of the last particle and singularities at the two points which are mapped the base of the first particle (see Figure 3).

Interactions between nearby tips can be subtle and are in general hard to analyze [2]. Our strategy is instead to establish two properties of the distribution function $h_n(\theta)$.

- The first is to show that near the tip of the last particle to arrive the derivatives are in fact very close and so for very small values of θ, $h_n(\theta + \theta_{n-1})$ can be well approximated by $h^*_n(\theta|0)$.
- The second property is to show that $h_n(\theta)$ concentrates the measure so close to θ_{n-1} that even though the probability of attaching to earlier particles is higher than for the Markovian model, Ω_N still occurs with high probability, provided we now require

$$\gamma > \begin{cases} (2\eta^2 + \eta - 1)/[2(\eta - 1)^2] & \text{if } 1 < \eta < 3; \\ \frac{3\eta + 1}{2(\eta - 1)} & \text{if } \eta \geqslant 3 \end{cases}$$

  when regularizing by $\boldsymbol{\sigma} < \sigma_0 = \mathbf{c}^\gamma$; see Figure 4 for plots of the lower bounds on γ and γ^*.

We now give a brief overview of the structure of the paper. In Section 3 we provide some background information on the Loewner differential equation, which allows us to represent the aggregate maps Φ_n as solutions corresponding to a $[-\pi, \pi)$-valued driving process with equally spaced jump times and positions given by the random angles (4). In particular, we explain how convergence of an angle sequence $\{\theta_k\}$ allows us to deduce convergence of the corresponding conformal maps Φ_n.

Figure 3: *Diagram illustrating the presence of zeros and singularities in the derivative at each successive particle tip and base in $\Phi_n(z)$ (left). These zeros and singularities are absent in $f_{n\mathbf{c}}(z)$ except at the tip of the final particle and base of the first particle (right).*

In Section 4 we obtain estimates on the slit map used to construct ALE, as well as estimates on its derivatives. These latter estimates lead to moment bounds for $[-\pi, \pi)$-valued random variables constructed from slit map derivatives. The arguments used are elementary in nature, and heavily use the explicit form of the slit map.

Section 5 contains most of the technical machinery needed for the proof. In this section, we obtain estimates on the distance between two solutions to the Loewner equation in terms of the distance between their respective driving functions in the case where we know that one of the solutions is a slit map. These estimates, which we believe may be of independent interest, enable us to obtain much more precise estimates than exist for generic solutions. In particular, our estimates give very good approximations when the conformal mappings are quite close to the boundary, whereas generic estimates blow up in this region. We perform this analysis by splitting the Loewner equation into radial and angular parts, and linearizing the resulting differential equations.

In Section 6, we use our estimates on Loewner derivatives at the tip and away from the approximate slit to show that $h_n(\theta)$, the density function for the nth angle θ_n, has the required behaviour. Then, similar arguments to those used in the proof of Theorem 2 are used to establish Theorem 1, but since $\{\theta_k\}$ does not have a Markovian structure, there are further terms to control. Finally, we discuss some extensions of our results, valid for certain instances of the ALE(α, η) model as well as related models.

Notation

Many of the estimates presented in this paper, especially in Section 5, are more precise than what is strictly needed for the proof of our main theorem, in that we frequently keep track of the dependence of constants on parameters, and similar. We have opted to record detailed versions to enable potential further applications where such dependencies may be important. Generic constants, which may change from line to line, will mainly be denoted by the capital letters A and B.

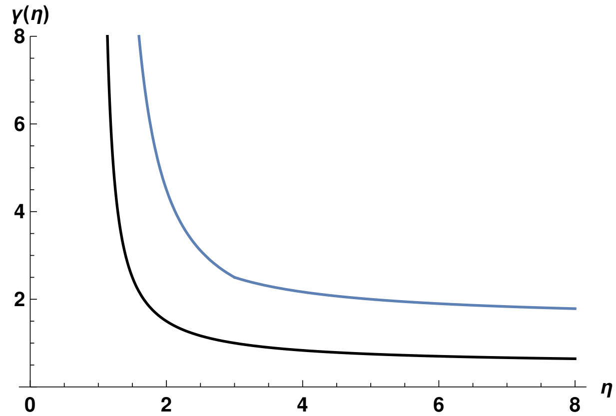


Figure 4: *Lower bounds on regularization exponents for* ALE *(blue) and the Markov model (black).*

3 Loewner flows

We shall make extensive use of Loewner techniques in this paper. Loewner equations describe the flow of families $\{\Psi_t\}_{t\geqslant 0}$ of conformal maps of a reference domain in $\mathbb{C}\cup\{\infty\}$ onto evolving domains in the plane in terms of measures on the boundary. We only give a very brief overview here, and refer the reader to [17] and the references therein for a discussion of Loewner theory.

3.1 Loewner's equation

Let $\{\mu_t\}_{t>0}$ be a family of probability measures on the unit circle $\mathbb{T}$, in this context referred to as driving measures. Then the Loewner partial differential equation for the exterior disk,

$$\partial_t \Psi_t(z) = z\Psi'(z)\int_{\mathbb{T}} \frac{z+\zeta}{z-\zeta}\mathrm{d}\mu_t(\zeta), \tag{7}$$

with initial condition

$$\Psi_0(z) = z$$

admits a unique solution $\{\Psi_t\}_{t\geqslant 0}$ called a Loewner chain. Each $\Psi_t(z)$ is a conformal map of the exterior disk onto a simply connected domain,

$$\Psi_t \colon \Delta \to D_t = \mathbb{C}\cup\{\infty\}\setminus K_t$$

and at ∞ we have the power series expansion $\Psi_t(z) = e^t z + \mathcal{O}(1)$. The growing compact sets $\{K_t\}_{t\geqslant 0}$ are called hulls, satisfy $K_s \subsetneq K_t$ for $s<t$, and have $\mathrm{cap}(K_t) = e^t$ for $t \geqslant 0$, where $\mathrm{cap}(K)$ denotes the logarithmic capacity of a compact set $K \subset \mathbb{C}$.

The limit functions appearing in Theorem 1 can be realized in terms of Loewner chains, and in fact have a very simple Loewner representation.

Example 1 (Growing a slit). Let $\mu_t = \delta_1$, a point mass at $\zeta = 1$. Then (7) reads

$$\partial_t f_t(z) = z f_t'(z)\frac{z+1}{z-1}.$$

With initial condition $f_0(z) = z$, the solution has the explicit representation (viz. [20, p. 772])

$$f_t(z) = \frac{e^t}{2z}\left(z^2 + 2(1-e^{-t})z + 1 + (z+1)\sqrt{z^2 + 2(1-2e^{-t})z + 1}\right). \tag{8}$$

The solution precisely consists of the slit maps $f_t \colon \Delta \to \Delta \setminus (1, 1 + d(t)]$, where

$$d(t) = 2e^t(1 + \sqrt{1 - e^{-t}}) - 2, \quad t > 0. \tag{9}$$

This means that the growing hulls are $K_t = \overline{\mathbb{D}} \cup (1, 1 + d(t)]$, the closed unit disk plus a radial slit emanating from $\zeta = 1$.

In this paper, we are mainly concerned with the case $\mu_t = \delta_{e^{i\xi_t}}$ for some function $\xi_t \colon (0, T] \to \mathbb{R}$ and in that setting, we refer to ξ_t as a driving term.

The conformal maps arising in ALE(α, η) have the following simple Loewner representation. We first solve the Loewner equation with driving measure $\mu_t = \delta_{e^{i\xi_t}}$, where

$$\xi_t = \sum_{k=1}^{n} \theta_k \mathbf{1}_{(C_{k-1}, C_k]}(t), \tag{10}$$

with $C_k = \sum_{j=1}^{k} c_k$, and the angles $\{\theta_k\}$ and capacity increments $\{c_k\}$ given by (4) and (5), respectively. Explicitly then, the Loewner problem associated with ALE(α, η) reads

$$\partial_t \Psi_t(z) = z\Psi'(z)\frac{z + e^{i\xi_t}}{z - e^{i\xi_t}} \quad \text{where} \quad \Psi_0(z) = z. \tag{11}$$

To obtain the composite ALE(α, η)-maps Φ_n described in Section 1, we evaluate the solution to (11) at $t = \mathbf{c}n$; thus

$$\Phi_n = \Psi_{\mathbf{c}n}, \quad n = 1, 2, \dots .$$

The random driving function ξ_t can be viewed as a càdlàg jump process exhibiting a complicated dependence structure encoded through angles and capacity increments. When $\alpha = 0$, the dependence structure is only present in the distribution of the increments, as the jump times are deterministic, and equal to $\mathbf{c} \cdot k$ for $k = 1, 2, \dots$. We emphasize that this is the main technical reason why the ALE$(0, \eta)$ is easier to analyze then the general ALE(α, η) model or the Hastings-Levitov model HL(α).

Another version of Loewner's equation arises in the setting of the upper half-plane

$$\mathbb{H} = \{z \in \mathbb{C} \colon \mathrm{Im} z > 0\};$$

this is usually referred to as the chordal version of the Loewner equation. While the ALE model is defined in Δ, the half-plane formulas are simpler and are used in our analysis of the tip behavior, and so we give a quick overview here. The chordal version of Loewner's equation with driving function $\tilde{\xi}_t \colon (0, T] \to \mathbb{R}$ reads

$$\partial_t \tilde{\Psi}_t(z) = -\tilde{\Psi}_t'(z)\frac{2}{z - \tilde{\xi}_t}, \tag{12}$$

and we again consider the initial condition $\tilde{\Psi}_0(z) = z$. Solving (12), we obtain a family of conformal maps $\tilde{\Psi}_t \colon \mathbb{H} \to \mathbb{H} \setminus \tilde{K}_t$, mapping the half-plane to the half-plane minus a compact set and having expansion $\tilde{\Psi}_t(z) = z - \mathrm{hcap}(K_t)/z + \mathcal{O}(1/z^2)$ at infinity. The quantity $\mathrm{hcap}(K_t) = 2t$ is referred to as the half-plane capacity.

Example 2 (Growing a slit in the half-plane). In the case where $\tilde{\xi}_t = 0$, the chordal equation becomes

$$\partial_t F_t(z) = -\frac{2}{z} F_t'(z)$$

and we again obtain slit map solutions. In the half-plane, the corresponding closed formula is given by a substantially simpler expression than for Δ, namely

$$F_t(z) = \sqrt{z^2 - 4t}, \quad z \in \mathbb{H}.$$

3.2 Reverse-time Loewner flow

The Loewner equation (11) is a first-order partial differential equation, and in the ALE(α, η) model, it gives rise to a non-linear PDE problem since the driving measures depend on the maps f_t via their derivatives. As is common in Loewner theory, we shall analyze solutions by passing to the backwards flow associated with (11): this essentially entails employing the method of characteristics to obtain an ordinary differential equation that describes the evolution at hand. See [17, 1] for detailed derivations and discussions.

Let $T > 0$ be fixed. The equation for the backward or reverse-time flow in the exterior disk is

$$\partial_t u_t(z) = -u_t(z) \frac{e^{i\Xi_t} + u_t(z)}{e^{i\Xi_t} - u_t(z)}, \tag{13}$$

where we define

$$\Xi_t = \xi_{T-t}, \quad 0 \leqslant t \leqslant T.$$

Then, setting $u_0(z) = z$, we obtain (see [17, Chapter 4])

$$u_T(z) = f_T(z)$$

where f_t denotes the solution to the forward equation (11) with driving function ξ_t. Note that this holds in general only at the special time T.

In the half-plane, the corresponding reverse flow is governed by the equation

$$\partial_t h_t(z) = -\frac{2}{h_t(z) - \tilde{\Xi}_t}, \tag{14}$$

with driving function

$$\tilde{\Xi}_t = \tilde{\xi}_{T-t}$$

and initial condition $h_0(z) = z$. For each $T > 0$, we again have

$$h_T(z) = \tilde{\Psi}_T(z),$$

where $\tilde{\Psi}_T(z)$ is the solution to the Loewner PDE (12) for the upper half-place with driving function $\tilde{\xi}_t$, evaluated at time T.

The main advantage of the backward flow is the fact that, for each z, (13) and (14) are now formally ODEs, simplifying the problems of analyzing and estimating solutions to the corresponding flow problems. Such analyses are carried out in Section 5 and will be crucial in the proof of Theorem 1.

3.3 Convergence of Loewner chains

Our strategy will be to argue that the driving functions arising in the ALE process are close, in the regime where $n \asymp \mathbf{c}^{-1}$, to the constant driving function $\xi_t = \theta_1$. We would then like to argue that the resulting conformal maps are close. These kinds of continuity results have been established in several settings, see for instance [10, Proposition 3.1] and [13, Proposition 1], and [9] for a more systematic discussion.

Since the ALE driving processes exhibit synchronous jumps, it is natural to measure distances between them in the uniform norm $\|\cdot\|_\infty$. For $T > 0$, we denote the space of piecewise continuous functions $\xi\colon [0,T) \to \mathbb{R}$ endowed with this norm by D_T. We consider the space Σ consisting of conformal maps $f(z) = Cz + \mathcal{O}(1)$, with $C > 0$ uniformly bounded, and we endow Σ with the topology of uniform convergence on compact subsets of Δ. We then view the conformal maps Ψ_t, and hence the aggregate maps Φ_n, as random elements of Σ.

The following result is well-known, but we give a proof for completeness. (With additional work, one could obtain estimates on rates of convergence, but we do not pursue this direction here.)

Proposition 3. *Let $T > 0$ be given. For $j = 1, 2$ let $\Psi_t^{(j)}$, $0 \leqslant t \leqslant T$, be the solution to the Loewner equation (7) with driving term $\xi_t^{(j)}$. For every $\epsilon > 0$ there exists $\delta = \delta(\epsilon, T) > 0$ such that if $\|e^{i\xi^{(1)}} - e^{i\xi^{(2)}}\|_\infty < \delta$, then*

$$\sup_{0\leqslant t\leqslant T}\ \sup_{\{|z|\geqslant 1+\epsilon\}} \left|\Psi_t^{(1)}(z) - \Psi_t^{(2)}(z)\right| < \epsilon.$$

Proof. Fix $s \in [0,T]$ and consider the reverse-time Loewner equation (13). We let $u_t^{(j)}$ be the reverse flow driven by $\xi_{s-t}^{(j)}$ for $0 \leqslant t \leqslant s$. Write $W_t^{(j)} = e^{i\xi_{s-t}^{(j)}}$. Taking the difference and differentiating $H = u^{(1)} - u^{(2)}$ with respect to t gives

$$\dot{H} - Hv = (W^{(1)} - W^{(2)})w,$$

where

$$v = v(t) = \frac{u^{(1)}u^{(2)} - W^{(1)}W^{(2)} - (1/2)(u^{(1)} + u^{(2)})(W^{(1)} + W^{(2)})}{(u^{(1)} - W^{(1)})(u^{(2)} - W^{(2)})}$$

and

$$w = w(t) = \frac{(u^{(1)} + u^{(2)})^2}{2(u^{(1)} - W^{(1)})(u^{(2)} - W^{(2)})}.$$

Since the flows move away from the unit circle, these expressions show that there is a constant A depending only on T such that if $|z| \geqslant 1 + \epsilon$ then for all $0 \leqslant t \leqslant s \leqslant T$,

$$\operatorname{Re} v(t) \leqslant A/\epsilon^2, \qquad |w(t)| \leqslant A/\epsilon^2.$$

Since $H(0) = 0$,

$$H(t) = \int_0^t \left[e^{\int_s^t v(r)dr}(W^{(1)}(s) - W^{(2)}(s))w(s)\right] \mathrm{d}s$$

and consequently, for a different T-dependent A,

$$\sup_{\{|z|\geqslant 1+\epsilon\}} |\Psi_t^{(1)}(z) - \Psi_t^{(2)}(z)| = \sup_{\{|z|\geqslant 1+\epsilon\}} |H(t)| \leqslant \|W^{(2)} - W^{(1)}\|_\infty e^{A/\epsilon^2} A/\epsilon^2.$$

Hence we can take $\delta < e^{-A/\epsilon^2}\epsilon^3/A$ and this is clearly uniform in $0 \leqslant t \leqslant T$. ∎

Thus, we obtain convergence in law of conformal maps provided we can show convergence in law of driving processes. Note that in our main result we have convergence to a degenerate deterministic limit (modulo rotation). As is explained in [10, Section 4.2], we can strengthen the convergence that follows from Proposition 3 in this instance, and obtain convergence of K_n with respect to the Hausdorff metric in Δ.

4 Analysis of the slit map

In our arguments, we shall need effective bounds on the building blocks f_k making up the aggregated map $\Phi_n = f_1 \circ \cdots \circ f_n$, as well as on the derivatives f_k', in order to estimate moments of angle sequences, among other things. In this section, we present both global and local estimates for the slit map.

An explicit formula for the slit map $f_t \colon \Delta \to \Delta \setminus (1, 1+d(t)]$ was given in (8), while the length $d(t)$ of the growing slit is given by (9). When performing estimates, it is sometimes helpful to view f_t as a composition of Möbius maps with a slit map in the upper half-plane.

Define conformal maps $m_{\mathbb{H}} \colon \Delta \to \mathbb{H} = \{z \in \mathbb{C} \colon \mathrm{Im} z > 0\}$ and $m_\Delta \colon \mathbb{H} \to \Delta$ by setting

$$m_{\mathbb{H}}(z) = i\frac{z-1}{z+1}, \quad z \in \Delta \tag{15}$$

and

$$m_\Delta(z) = \frac{1-iz}{1+iz}, \quad z \in \mathbb{H}. \tag{16}$$

For $d > 0$, we check that

$$\tilde{f}_d(z) = 2\frac{\sqrt{d+1}}{d+2}\left(z^2 - \frac{d^2}{4(d+1)}\right)^{1/2}, \quad z \in \mathbb{H}, \tag{17}$$

maps the upper half-plane conformally onto the upper half-plane minus the slit $\left(0, i\frac{d}{d+2}\right]$. Note that (17) is a rescaling of the half-plane slit map from Section 3.

By following the image of the unit circle $\mathbb{T}$ under composition, and checking the trajectories of the point at infinity and the boundary point $1 \in \overline{\Delta}$, we verify that

$$f_t = m_\Delta \circ \tilde{f}_{d(t)} \circ m_{\mathbb{H}},$$

where $d(t)$ is the explicit formula in (9). We note that $f_t(1) = 1 + d(t)$, and that one can compute that $f_t(e^{i\beta_t}) = f_t(e^{-i\beta_t}) = 1$ for

$$\beta_t = 2\arctan\left(\frac{d(t)}{2\sqrt{d(t)+1}}\right). \tag{18}$$

We shall refer to $\exp(i\beta_t)$ and $\exp(-i\beta_t)$ as the base points of the slit. In our scaling limit results, we will make use of the facts that

$$\frac{\beta_t}{d(t)} \to 1 \quad \text{and} \quad \frac{d(t)}{2t^{1/2}} \to 1, \quad \text{as} \quad t \to 0. \tag{19}$$

4.1 The half-plane

We start by analyzing the half-plane slit map $F_t(z) = \sqrt{z^2 - 4t}$ in detail near the tip. We note that

$$F_t'(z) = \frac{z}{(z^2-4t)^{1/2}} \quad \text{and} \quad F_t''(z) = -\frac{4t}{(z^2-4t)^{3/2}}.$$

Thus, F_t' has a zero at $z = 0$, which gets mapped to the tip, and singularities at $\pm 2t^{1/2}$, the points that are mapped to the base of the slit.

Lemma 4 (Near the tip, half-plane version)**.** *Let $z = x + iy$, where $0 \leqslant y \leqslant t^{1/2}$ and $|x| \leqslant \frac{6}{5}t^{1/2}$. Then*

$$|\mathrm{Re}[F_t(z)]| \leqslant y. \tag{20}$$

and

$$\frac{7}{10}(2t^{1/2}) \leqslant \mathrm{Im}[F_t(z)] \leqslant \frac{\sqrt{5}}{2}(2t^{1/2}). \tag{21}$$

Proof. By symmetry, it suffices to establish these bounds for $x > 0$. The upper bound in (21) is obtained by evaluating $F_t(z)$ at $z = iy$, right above the tip. We now establish the lower bound. Note that $x \mapsto \mathrm{Im}[F_d(x+iy)]$ is decreasing on $[0, \frac{6}{5}t^{1/2}]$, and hence it suffices to estimate $\mathrm{Im}[F_d(\frac{6}{5}t^{1/2}+iy)]$. First, we compute

$$\left|F_t\left(\frac{6}{5}t^{1/2} + iy\right)\right|^4 = \left(\frac{16}{25}\right)^2 (2t^{1/2})^4 \left(1 + \frac{425}{64}\left(\frac{y}{2t^{1/2}}\right)^2 + \left(\frac{25}{16}\right)^2\left(\frac{y}{2t^{1/2}}\right)^4\right).$$

Thus, setting $y = 0$, we obtain

$$\left|F_t\left(\frac{6}{5}t^{1/2} + iy\right)\right| \geqslant \frac{8}{5}t^{1/2}.$$

Next,

$$\arg F_t\left(\frac{6}{5}t^{1/2} + iy\right) = \frac{\pi}{2} - \frac{1}{2}\arctan\left(\frac{6}{5}\frac{\frac{y}{2t^{1/2}}}{\frac{16}{25} + (\frac{y}{2t^{1/2}})^2}\right) \geqslant \frac{\pi}{2} - \frac{1}{2}\arctan\left(\frac{15}{16}\frac{y}{t^{1/2}}\right),$$

and by our assumption $y/t^{1/2} \leqslant 1$,

$$\mathrm{Im}\left[F_t\left(\frac{6}{5}t^{1/2} + iy\right)\right] \geqslant \frac{8}{5}\cos\left(\frac{1}{2}\arctan\left(\frac{15}{16}\right)\right)t^{1/2},$$

which leads to the lower bound in (21). A similar argument establishes (20). We first note that $\mathrm{Re}F_d(x+iy)$ is increasing in x on $[0, \frac{6}{5}t^{1/2}]$ and use our expression for $\arg F_t$ to obtain

$$\mathrm{Re}[F_t(6t^{1/2}/5 + iy)] \leqslant \frac{8}{5}t^{1/2}\left(1 + 8\left(\frac{y}{2t^{1/2}}\right)^2\right)^{\frac{1}{4}} \sin\left(\frac{1}{2}\arctan\left(\frac{15}{16}\frac{y}{2t^{1/2}}\right)\right) \leqslant 3^{1/4}\frac{3}{4}y,$$

where the last estimate (20) follows from the bound $\sin\left(\frac{1}{2}\arctan(x)\right) \leqslant \frac{x}{2}$. □

4.2 The exterior disk

We now transfer back to the exterior disk, the setting of the ALE model, using the decomposition $f_t = m_\Delta \circ \tilde{f}_{d(t)} \circ m_{\mathbb{H}}$. To do this, we reparametrize and rescale the half-plane slit map to obtain

$$\tilde{f}_{d(t)}(z) = C(t)\left(z^2 - \rho(t)^2\right)^{1/2},$$

where

$$C(t) = 2\frac{\sqrt{1+d(t)}}{2+d(t)} \quad \text{and} \quad \rho(t) = \frac{d(t)}{2\sqrt{d(t)+1}}.$$

We recall from (19) that $\rho(t)/t^{1/2} \to 1$ as $t \to 0$.

First, we describe the radial and angular effect of the building block map f_t near the tip for small values of t.

Lemma 5 (Near the tip, disk version). *For $|z| - 1 \leqslant t^{1/2}$ and $0 < t < 1/20$, and for $\arg z \leqslant \beta_t/2$,*

$$|f_t(z)| - 1 \geqslant \frac{1}{10}t^{1/2} \quad \textit{and} \quad |\arg f_t(z)| \leqslant 2(|z|-1).$$

Moreover,

$$\frac{|f_t(z) - 1|}{|f_t(z)| - 1} \leqslant 8.$$

Proof. Since $|f_t(z)|$ is non-decreasing in $|z|$, we may assume that $|z| - 1 < \frac{1}{10}t^{1/2}$.

We first note that

$$m_{\mathbb{H}}(re^{i\theta}) = -\frac{2r\sin(\theta)}{r^2 + 2r\cos(\theta) + 1} + i\frac{r^2-1}{r^2 + 2r\cos(\theta)+1}.$$

Using the fact that the real part is maximized on the unit circle, at $\arg z = \beta_t/2$, together with the identities $\sin(\arctan(x)) = \frac{x}{\sqrt{1+x^2}}$ and $\cos(\arctan(x)) = \frac{1}{\sqrt{1+x^2}}$, we find that for $1 < r < 1 + \frac{1}{4}t^{1/2}$ and $|\theta| \leqslant \beta_t/2$,

$$|\mathrm{Re}[m_{\mathbb{H}}(re^{i\theta})]| \leqslant \frac{2\sin(\frac{\beta_t}{2})}{2 + 2\cos(\frac{\beta_t}{2})} = \frac{\rho(t)}{1 + \sqrt{1+\rho(t)^2}} \leqslant \frac{1}{2}\rho(t).$$

We also have

$$\mathrm{Im}[m_{\mathbb{H}}(re^{i\theta})] \geqslant \frac{r^2-1}{r^2+2r+1} \geqslant \frac{r-1}{r+1} \geqslant \frac{1}{3}(r-1),$$

and by a similar argument to the one used to estimate the real part,

$$\mathrm{Im}[m_{\mathbb{H}}(re^{i\theta})] \leqslant \frac{r+1}{r^2 + 2r\cos(\frac{\beta_t}{2}) + 1}(r-1) \leqslant \frac{2}{3}(r-1).$$

Thus, we can apply Lemma 4 to obtain

$$|\mathrm{Re}[\tilde{f}_{d(t)}(m_{\mathbb{H}}(z))] \leqslant \frac{2}{3}(|z|-1) \quad \text{and} \quad \frac{7}{10}\rho(t) \leqslant \mathrm{Im}[\tilde{f}_{d(t)}(m_{\mathbb{H}}(z))] \leqslant \frac{5}{4}\rho(t). \tag{22}$$

To finish the proof, note that if $w = x + iy \in \mathbb{H}$,

$$m_\Delta(x+iy) = \frac{1 - x^2 - y^2}{x^2 + (1-y)^2} - i\frac{2x}{x^2+(1-y)^2}, \quad z = x + iy \in \mathbb{H}.$$

For $y > x$, which is satisfied when $x = \mathrm{Re}[\tilde{f}_{d(t)}(m_{\mathbb{H}}(z))]$ and $y = \mathrm{Im}[\tilde{f}_{d(t)}(m_{\mathbb{H}}(z)]$,

$$|\arg[m_\Delta(x+iy)]| = \left|\arctan\left(-\frac{2x}{1-x^2-y^2}\right)\right| \leqslant \left|\arctan\left(-\frac{2x}{1-2y^2}\right)\right| \leqslant \frac{2x}{1-2y^2}$$

and

$$|m_\Delta(x+iy)| - 1 = \frac{((1-x^2-y^2)^2+4x^2)^{1/2}}{x^2+(y-1)^2} - 1 \geqslant \frac{2y-2x^2-2y^2}{x^2+(1-y)^2} \geqslant 2\frac{1-2y}{x^2+(1-y)^2}y.$$

We also have

$$|m_\Delta(x+iy)-1| = 2\frac{(x^2+y^2)^{1/2}}{(x^2+(1-y)^2)^{1/2}}$$

so that

$$\frac{|m_\Delta(x+iy)-1|}{|m_\Delta(x+iy)|-1} \leqslant \frac{\left(1+\left(\frac{x}{y}\right)^2\right)^{1/2}(x^2+(1-y)^2)^{1/2}}{1-2y} \leqslant \frac{2}{1-2y}.$$

The assertion of the Lemma now follows from the bounds in (22) and the fact that $2\mathrm{Im}[\tilde{f}_{d(t)}(m_{\mathbb{H}}(z)] \leqslant \frac{5}{2}\rho(t) \leqslant \frac{3}{4}$, by our assumption on t. □

We now examine the derivative of the slit map in Δ.

Lemma 6. *For $0 < t \leqslant 1$ and $1 < |z| \leqslant 2$, we have*

$$f_t'(z) = H_t(z)\frac{e^t(z-1)}{(z-e^{i\beta_t})^{1/2}(z-e^{-i\beta_t})^{1/2}} \tag{23}$$

where $H_t(z)$ is holomorphic in z, has $\lim_{z\to\infty} H_t(z) = 1$, and satisfies

$$A^{-1} \leqslant |H_t(z)| \leqslant A$$

for an absolute constant $A > 0$.

Proof. Since the slit map $f_t(z)$ solves the Loewner equation

$$\partial_t f_t(z) = z f_t'(z)\frac{z+1}{z-1}$$

we have

$$f_t'(z) = \frac{z-1}{z(z+1)}\partial_t f_t(z). \tag{24}$$

Differentiating the explicit expression (8) with respect to t, we find that

$$\partial_t f_t(z) = \frac{e^t}{2z}\frac{z+1}{\sqrt{(z+1)^2-4e^{-t}z}}\left((z+1)\sqrt{(z+1)^2-4e^{-t}z}+(z+1)^2-2e^{-t}z\right).$$

Inserting this into (24), we obtain

$$f_t'(z) = H_t(z)\frac{e^t(z-1)}{\sqrt{(z+1)^2-4e^{-t}z}}$$

with

$$H_t(z) = \frac{1}{2z^2}\left[(z+1)\left(z+1+\sqrt{(z+1)^2-4e^{-t}z}\right)-2e^{-t}z\right].$$

It remains to show that H_t is bounded above and below. But this follows immediately upon writing $H_t(z) = z^{-1}e^{-t}f_t(z)$, where f_t is the slit map itself. Finally, we verify that $z_t = e^{i\beta_t}$ solves $(z+1)^2 - 4e^{-t}z = 0$, and this leads to the factorization $(z+1)^2 - 4e^{-t}z = (z - e^{i\beta_t})(z - e^{-i\beta_t})$. $\square$

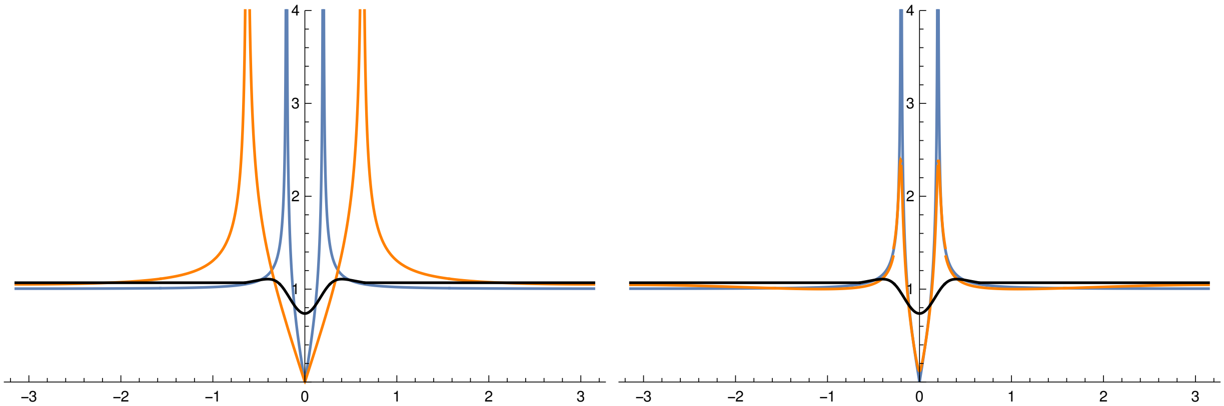


Figure 5: *Plots of* $\theta \mapsto |f_t'(e^{\boldsymbol{\sigma}+i\theta})|$. *Left:* $\boldsymbol{\sigma} = 0.0001$ *fixed,* $t = 0.01$ *(blue) and* $t = 0.1$ *(orange).* *Right:* $t = 0.01$ *fixed,* $\boldsymbol{\sigma} = 0.0001$ *(blue) and* $\boldsymbol{\sigma} = 0.02$ *(orange).*
Plot with $t = 0.1$ and $\boldsymbol{\sigma} = 0.2$ (black) shown in both pictures for comparison.

Our analysis of the ALE model will require local estimates on the derivative of the slit map. Representative graphs of how $\theta \mapsto |f_t'(e^{\boldsymbol{\sigma}+i\theta})|$ varies with t and $\boldsymbol{\sigma}$ are shown in Figure 5.

Lemma 7. *Let* $0 < t \leqslant 1$ *and suppose* $|z| - 1 \leqslant d(t)$*. Then the derivative of the slit map admits the following estimates:*

1. *(Near the tip) For* $|\arg z| < \frac{1}{2}\beta_t$,
$$A_1 e^t \frac{|z-1|}{d(t)} \leqslant |f_t'(z)| \leqslant A_2 e^t \frac{|z-1|}{d(t)}.$$

2. *(Near the base) For* $|\arg z \pm \beta_t| \leqslant \frac{1}{2}\beta_t$,
$$A_1 e^t \leqslant |f_t'(z)| \leqslant A_2 e^t \frac{d(t)}{|z|-1}.$$

3. *(Away from tip and base) For* $|\arg z| > \frac{3}{2}\beta_t$,
$$A_1 e^t \leqslant |f_t'(z)| \leqslant A_2 e^t.$$

Proof. We treat the case $|\arg z| < \frac{1}{2}\beta_t$ first. In light of the global bounds on the function $H_t(z)$ from Lemma 6, it suffices to estimate the square root expressions appearing in the denominator in (23). Noting that $0 < |z| - 1 \leqslant d(t)$, we have

$$|z - e^{i\beta_t}| = |e^{\log|z| + i(\arg z - \beta_t)} - 1| \asymp \left((\log|z|)^2 + (\arg(z) - \beta_t)^2\right)^{1/2} \asymp d(t),$$

and hence

$$|z-e^{i\beta_t}|^{1/2}|z-e^{-i\beta_t}|^{1/2}\asymp d(t),$$

as claimed.

Near the base, the same reasoning as before shows that $|z-1|\asymp d(t)$. On the other hand,

$$|z|-1\leqslant|z-e^{i\beta_t}|\leqslant|e^{\log|z|+i(\beta_t+\frac{1}{2}\beta_t)}-e^{i\beta_t}|\leqslant Ad(t),$$

where the lower bound is attained when $\arg(z)=\beta_t$. Combining these bounds leads to the claimed estimates for $|\arg z\pm\beta_t|\leqslant\frac{\beta_t}{2}$.

On each fixed radius, the function $v\colon \arg(z)\mapsto\left|\frac{z-1}{(z-e^{i\beta_t})^{1/2}(z-e^{-i\beta_t})^{1/2}}\right|$ is decreasing on $[\frac{3}{2}\beta_t,\pi]$, with $v(\pi)=(e^{\log|z|}+1)/((e^{\log|z|}+\cos\beta_t)^2+\sin^2\beta_t)^{1/2}\geqslant 1$. So in order to obtain the last set of estimates, it suffices to note that v remains bounded above and below as $\arg(z)\to\frac{3}{2}\beta_t$, by the same arguments as before. □

4.3 Moment computations

We now return to random growth models and present the moment bounds used in Section 2. As before, $\boldsymbol{\sigma}>0$ is our regularization parameter, while $\eta>0$ is a model parameter.

For $t>0$, define the normalization factor

$$Z_t^*=Z_t^*(\eta,\boldsymbol{\sigma})=\int_{\mathbb{T}}|f_t'(e^{\boldsymbol{\sigma}+is})|^{-\eta}\mathrm{ds}. \tag{25}$$

Lemma 8. *Let $0<t<1$ and suppose $\boldsymbol{\sigma}\leqslant t^{1/2}$. The total mass Z_t^* satisfies the following.*

- *($\eta<1$) There are constants A_1 and A_2 such that*

$$A_1\leqslant Z_t^*\leqslant A_2. \tag{26}$$

  In particular, Z_t^* remains finite as $\boldsymbol{\sigma}\to 0$.

- *($\eta>1$) There are constants A_1 and A_2 such that*

$$A_1 d(t)^{\eta}\boldsymbol{\sigma}^{-(\eta-1)}\leqslant Z_t^*\leqslant A_2 d(t)^{\eta}\boldsymbol{\sigma}^{-(\eta-1)}. \tag{27}$$

  In particular, if $\boldsymbol{\sigma}\leqslant t^{\frac{\eta}{2(\eta-1)}}$, then Z_t^* diverges as $\boldsymbol{\sigma}\to 0$.

Moreover, for $\eta>1$ we have the following estimates:

1. *(Near the tip) For $|\theta|<\frac{\beta_t}{2}$,*

$$A_1\frac{1}{\boldsymbol{\sigma}}\left(1+\left(\frac{\theta}{\boldsymbol{\sigma}}\right)^2\right)^{-\eta/2}\leqslant\frac{1}{Z_t^*}|f_t'(e^{\boldsymbol{\sigma}+i\theta})|^{-\eta}\leqslant A_2\frac{1}{\boldsymbol{\sigma}}\left(1+\left(\frac{\theta}{\boldsymbol{\sigma}}\right)^2\right)^{-\eta/2}.$$

2. *(Near the base) For $|\theta-\beta_t|\leqslant\frac{1}{2}\beta_t$,*

$$A_1\boldsymbol{\sigma}^{2\eta-1}d(t)^{-2\eta}\leqslant\frac{1}{Z_t^*}|f_t'(e^{\boldsymbol{\sigma}+i\theta})|^{-\eta}\leqslant A_2\boldsymbol{\sigma}^{\eta-1}d(t)^{-\eta}.$$

3. (Away from the tip and base) For $|\theta| > \frac{3}{2}\beta_t$,

$$A_1\boldsymbol{\sigma}^{\eta-1}d(t)^{-\eta} \leqslant \frac{1}{Z_t^*}|f_t'(e^{\boldsymbol{\sigma}+i\theta})|^{-\eta} \leqslant A_2\boldsymbol{\sigma}^{\eta-1}d(t)^{-\eta}.$$

Proof. We begin by treating the case $\eta < 1$. In light of Lemma 7, non-trivial global bounds on Z_t^* from above and below follow immediately from the bounds for $|s| > \frac{3}{2}\beta_t$ provided the contribution from $(-\frac{\beta_t}{2}, \frac{\beta_t}{2})$ is finite. Hence it suffices to estimate the integral

$$\int_{-\frac{\beta_t}{2}}^{\frac{\beta_t}{2}} |f_t'(e^{\boldsymbol{\sigma}+is})|^{-\eta}\mathrm{d}s \asymp Ad(t)\int_0^{\frac{\beta_t}{2}} \frac{1}{(\boldsymbol{\sigma}^2+s^2)^{\eta/2}}\mathrm{d}s,$$

where we have used that $A_1 < |e^{\boldsymbol{\sigma}+is}-1|/(\boldsymbol{\sigma}^2+s^2)^{1/2} < A_2$ for $s, \boldsymbol{\sigma}$ small. Next, we note that

$$\int_0^{\frac{\beta_t}{2}} \frac{1}{(\boldsymbol{\sigma}^2+s^2)^{\eta/2}}\mathrm{d}s \leqslant \int_0^{\frac{\beta_t}{2}} \frac{1}{s^\eta}\mathrm{d}s,$$

and the latter integral is bounded for $0 < t < 1$ since $\eta < 1$.

We turn to the case $\eta > 1$. Since the integral $\int |f_t'(e^{is})|^{-\eta}\mathrm{d}s$ now diverges due to the singularity at $s = 0$, it again suffices to estimate the contribution coming from $|\theta| < \beta_t/2$ in order to establish (27). We have

$$\begin{aligned}\int_0^{\frac{\beta_t}{2}} |f_t(e^{\boldsymbol{\sigma}+is})|^{-\eta}\mathrm{d}s &\leqslant Ad(t)^\eta \int_0^{\frac{\beta_t}{2}} (\boldsymbol{\sigma}^2+s^2)^{-\eta/2}\mathrm{d}s \\ &\leqslant Ad(t)^\eta\boldsymbol{\sigma}^{-\eta}\int_0^{\frac{\beta_t}{2\boldsymbol{\sigma}}} \boldsymbol{\sigma}(1+u^2)^{-\eta/2}\mathrm{d}u\end{aligned}$$

after a change of variables. Since $\int_0^\infty (1+u^2)^{-\eta/2}\mathrm{d}u$ is now finite, the upper bound follows. Similar reasoning together with the assumption that $\boldsymbol{\sigma} \leqslant t^{1/2}$ yields the lower bound on the integral. The estimates on the normalized derivative for $\eta > 1$ follow upon dividing through in Lemma 7 by the total mass Z_t^*. □

We now turn to moment bounds for $\eta > 1$.

Lemma 9. *Under the hypotheses of Lemma 8, we have, for each $\eta > 0$,*

$$\int_{\mathbb{T}} \theta\,\frac{1}{Z_t^*}|f_t'(e^{\boldsymbol{\sigma}+i\theta})|^{-\eta}\,d\theta = 0.$$

Suppose $x \in (\boldsymbol{\sigma}, \frac{\beta_t}{2})$. Then, for $1 < \eta < 3$, we have

$$A_1x^{3-\eta}\boldsymbol{\sigma}^{\eta-1} \leqslant \int_{-x}^{x} \theta^2\frac{1}{Z_t^*}|f_t'(e^{\boldsymbol{\sigma}+i\theta})|^{-\eta}\,d\theta \leqslant A_2x^{3-\eta}\boldsymbol{\sigma}^{\eta-1},$$

and for $\eta = 3$, we have

$$A_1\boldsymbol{\sigma}^2\log(x\boldsymbol{\sigma}^{-1}) \leqslant \int_{-x}^{x} \theta^2\frac{1}{Z_t^*}|f_t'(e^{\boldsymbol{\sigma}+i\theta})|^{-3}\,d\theta \leqslant A_2\boldsymbol{\sigma}^2\log(x\boldsymbol{\sigma}^{-1}).$$

For $\eta > 3$, we have

$$A_1\boldsymbol{\sigma}^2 \leqslant \int_{-x}^{x} \theta^2\,\frac{1}{Z_t^*}|f_t'(e^{\boldsymbol{\sigma}+i\theta})|^{-\eta}\,d\theta \leqslant A_2\boldsymbol{\sigma}^2.$$

Proof. The statement that $\int \theta |f_t'(e^{\boldsymbol{\sigma}+i\theta})|^{-\eta} \mathrm{d}\theta = 0$ follows immediately from symmetry of the function $\theta \mapsto |f_t'(e^{\boldsymbol{\sigma}+i\theta})|$ for each $\boldsymbol{\sigma}$ and t.

We turn to second moments, and deal with the parameter range $1 < \eta \leqslant 3$ first. By Lemma 8,

$$\int_{-x}^{x} \theta^2 \frac{1}{Z_t^*} |f_t'(e^{\boldsymbol{\sigma}+i\theta})|^{-\eta} \mathrm{d}\theta = 2\int_{0}^{x} \theta^2 \frac{1}{Z_t^*} |f_t'(e^{\boldsymbol{\sigma}+i\theta})|^{-\eta} \mathrm{d}\theta \asymp \boldsymbol{\sigma}^2 \int_0^x \frac{\left(\frac{\theta}{\boldsymbol{\sigma}}\right)^2}{\left(1+(\frac{\theta}{\boldsymbol{\sigma}})^2\right)^{\eta/2}} \frac{\mathrm{d}\theta}{\boldsymbol{\sigma}}.$$

Performing a change of variables, and assuming $\eta < 3$, we obtain the integral

$$\begin{aligned}
\boldsymbol{\sigma}^2 \int_0^{\frac{x}{\boldsymbol{\sigma}}} u^2(1+u^2)^{-\eta/2} \mathrm{d}u &= \boldsymbol{\sigma}^2 \int_0^1 u^2(1+u^2)^{-\eta/2}\mathrm{d}u + \boldsymbol{\sigma}^2 \int_1^{\frac{x}{\boldsymbol{\sigma}}} u^2(1+u^2)^{-\eta/2}\mathrm{d}u \\
&\asymp A_1\boldsymbol{\sigma}^2 + A_2\boldsymbol{\sigma}^2 \int_1^{\frac{x}{\boldsymbol{\sigma}}} u^{2-\eta}\mathrm{d}u \\
&= A_1\boldsymbol{\sigma}^2 + A_2\frac{1}{3-\eta}\boldsymbol{\sigma}^2 x^{3-\eta}\boldsymbol{\sigma}^{\eta-3} \\
&= A_1\boldsymbol{\sigma}^2 + A_2 x^{3-\eta}\boldsymbol{\sigma}^{\eta-1},
\end{aligned}$$

as claimed. An obvious modification of the argument leads to bounds for $\eta = 3$.

Finally, we treat the case $\eta > 3$ and show that the second moment decays like $\boldsymbol{\sigma}^2$ independently of η. It now suffices to examine

$$\int_{|\theta|<x} \theta^2 \frac{1}{Z_t} |f_t'(e^{\boldsymbol{\sigma}+i\theta})|^{-\eta} \mathrm{d}\theta \asymp 2\boldsymbol{\sigma}^2 \int_0^{\frac{x}{\boldsymbol{\sigma}}} u^2(1+u^2)^{-\eta/2}\mathrm{d}u.$$

The integral on the right now converges since $\eta > 3$, and in fact

$$\int_0^\infty u^2(1+u^2)^{-\eta/2}\mathrm{d}u = \frac{\sqrt{\pi}}{4}\frac{\Gamma(\frac{\eta-3}{2})}{\Gamma(\frac{\eta}{2})}.$$

To get the lower bound, we use the assumption $1 < x/\boldsymbol{\sigma}$ to bound the integral from below. The second assertion of the Lemma follows. $\square$

5 Estimates on conformal maps via Loewner's equation

We now obtain refined estimates on the distance between solutions to the Loewner equation in terms of the distance between their driving functions, in the special case when the driving functions are close to constant. Generic estimates between conformal maps tend to blow up close to the boundary (as seen in, for example, Proposition 3).

As we wish to compare $|\Phi_n'(e^{\boldsymbol{\sigma}+i\theta})|$ to $|(f_{n\boldsymbol{c}}^{\theta_n})'(e^{\boldsymbol{\sigma}+i\theta})|$ when $\boldsymbol{\sigma}$ is typically much smaller than the difference between the respective driving functions, we need bespoke estimates which behave well close to the boundary.

Suppose $\Psi_t(z)$ is the solution to the Loewner equation (11). In this section, we compare $\Psi_t(z)$ and $\Psi_t'(z)$ to $f_t(z)$ and $f_t'(z)$ in the case when ξ, the driving function of Ψ, is close to zero. For fixed $T > 0$, let

$$\|\xi\|_T = \sup_{t\leqslant T} |\xi_t|.$$

Writing $u_t^\xi = r_t^\xi e^{i\vartheta_t^\xi}$, substituting into (13) and separating $\mathrm{Re}[(u_t^\xi+1)/(u_t^\xi-1)]$ and $\mathrm{Im}[(u_t^\xi+1)/(u_t^\xi-1)]$ we obtain the two differential equations

$$\partial_t r_t^\xi = r_t^\xi \frac{(r_t^\xi)^2 - 1}{(r_t^\xi)^2 - 2r_t^\xi \cos(\vartheta_t^\xi - \xi_{T-t}) + 1} \tag{28}$$

and

$$\partial_t \vartheta_t^\xi = -2\frac{r_t^\xi \sin(\vartheta_t^\xi - \xi_{T-t})}{(r_t^\xi)^2 - 2r_t^\xi \cos(\vartheta_t^\xi - \xi_{T-t}) + 1}. \tag{29}$$

In the special case when $\xi \equiv 0$ it is straightforward to see that if $|z| > 1$ and $\arg z = 0$, then $\vartheta_t^0 = 0$ for all t, whereas if $|z| = 1$ and $|\arg z| > 0$, then $r_t^0 = 1$ for all $t < \inf\{s > 0 : \vartheta_s^0 = 0\}$. Therefore, in these two cases, solving the pair of differential equations above reduces to solving a single ordinary differential equation, and we are able to obtain explicit solutions. We establish our bespoke estimates by linearising the differential equations around these explicit solutions.

In Section 5.1 we perform the analysis in the case when $\arg z$ is close to 0, and in Section 5.2 we carry out the case when $|z|$ is close to 1. We also obtain cruder estimates which apply in the intermediate regime between these two cases and will be used in the next section to "glue" the two results together.

5.1 Near the tip

In this subsection, we analyse $\Psi_t(z)$ in the case when $\arg z$ is close to zero. We begin by computing an explicit expression for $f_t(r)$ when $r > 1$. Consider the reverse-time Loewner equations above for $z = r$, with $\xi_t = 0$ for all t. Since there is no dependency on T, we have that $f_t(r) = u_t^0$ for all $t > 0$. From (28) it is immediate that $\vartheta_t^0 = 0$ for all $t > 0$. Substituting this into (28) we get

$$\partial_t r_t^0 = r_t^0 \frac{r_t^0 + 1}{r_t^0 - 1}. \tag{30}$$

Solving this gives

$$\log\left(\frac{(r_t^0+1)^2 r}{r_t^0 (r+1)^2}\right) = t$$

or

$$r_t^0 = \frac{(r+1)^2 e^t}{2r}\left(1 + \sqrt{1 - \frac{4re^{-t}}{(r+1)^2}}\right) - 1. \tag{31}$$

Observe that if $r = 1$, then $r_t^0 = d(t) + 1$.

Now consider Ψ_t having general driving function ξ_t which is assumed to be small.

Proposition 10. *There exists some absolute constant A such that, for all $1 < |z| < 2$ and $T > 0$ satisfying $\|\xi\|_T + |\arg z| \leqslant A^{-1} e^{-T}(|z| - 1)$, we have*

$$|\arg \Psi_T(z) - \arg z| \leqslant \|\xi - \arg z\|_T.$$

and

$$0 \leqslant r_T^0 - |\Psi_T(z)| \leqslant \frac{Ae^T \|\xi - \arg z\|_T^2}{|z| - 1}$$

where r_t^0 is given by (31) with $r = |z|$.

Hence A can be chosen so that

$$|\Psi_T(z) - f_T(z)| \leqslant Ae^T(\|\xi\|_T + |\arg z|).$$

Proof. First suppose that $z = r_0 \in \mathbb{R}$ and let $u_t^\xi = r_t^\xi e^{i\vartheta_t^\xi}$ be the solution to (13) starting from z. We begin by showing that $|\vartheta_t^\xi| \leqslant \|\xi\|_T$ for all $t \leqslant T$. Suppose that there exists some $t_1 \leqslant T$ such that $\vartheta_{t_1}^\xi > \|\xi\|_T$. Since $\vartheta_0^\xi = 0$ and $t \mapsto \vartheta_t^\xi$ is continuous, there exists $s_1 < t_1$ such that $\vartheta_{s_1}^\xi = \|\xi\|_T$ and $\vartheta_t^\xi > \|\xi\|_T$ for all $t \in (s_1, t_1]$ (take $s_1 = \sup\{t \leqslant t_1 : \vartheta_t^\xi < \|\xi\|_T\}$). Then since $\xi_{T-t} \leqslant \|\xi\|_T$ for all $t \leqslant T$, $\sin(\vartheta_t^\xi - \xi_{T-t}) > 0$ for all $t \in (s_1, t_1]$. Therefore, by (29),

$$\vartheta_{t_1}^\xi - \vartheta_{s_1}^\xi = -2\int_{s_1}^{t_1} \frac{r_t^\xi \sin(\vartheta_t^\xi - \xi_{T-t})}{(r_t^\xi)^2 - 2r_t^\xi\cos(\vartheta_t^\xi - \xi_{T-t}) + 1}\mathrm{d}t < 0.$$

Hence $\vartheta_{t_1}^\xi < \vartheta_{s_1}^\xi = \|\xi\|_T$ which contradicts our assumption that $\vartheta_{t_1}^\xi > \|\xi\|_T$. By symmetry, there is also no $t \leqslant T$ for which $\vartheta_t^\xi < -\|\xi\|_T$ and hence $|\vartheta_t^\xi| < \|\xi\|_T$ for all $t \leqslant T$.

We now turn to r_t^ξ. Note that by (28), $r_0 \leqslant r_t^\xi \leqslant r_t^0$. Set $\delta_r(t) = r_t^0 - r_t^\xi$ and let

$$T^\xi = \inf\{t > 0 : \delta_r(t) > \|\xi\|_T\} \wedge T.$$

Let

$$h_1(r, x) = \frac{r(r^2-1)}{r^2 - 2r\cos x + 1}. \tag{32}$$

Then

$$\frac{\mathrm{d}\delta_r(t)}{\mathrm{d}t} = \frac{\mathrm{d}r_t^0}{\mathrm{d}t} - \frac{\mathrm{d}r_t^\xi}{\mathrm{d}t} = h_1(r_t^0, 0) - h_1(r_t^\xi, \vartheta_t^\xi - \xi_{T-t}).$$

Suppose that $t \leqslant T^\xi$. Set

$$\begin{aligned} H_1(t) &= h_1(r_t^0 - \delta_r(t), \vartheta_t^\xi - \xi_{T-t}) - \left(h_1(r_t^0, 0) - \frac{\partial h_1}{\partial r}(r_t^0, 0)\delta_r(t) + \frac{\partial h_1}{\partial x}(r_t^0, 0)(\vartheta_t^\xi - \xi_{T-t})\right) \\ &= -\frac{\mathrm{d}\delta_r(t)}{\mathrm{d}t} + \frac{((r_t^0)^2 - 2r_t^0 - 1)\delta_r(t)}{(r_t^0 - 1)^2}. \end{aligned}$$

Here we have used

$$\begin{aligned} \frac{\partial h_1}{\partial r}(r, x) &= \frac{r^4 - 4r^3\cos x + 4r^2 - 1}{(r^2 - 2r\cos x + 1)^2} \\ \frac{\partial h_1}{\partial x}(r, x) &= -\frac{-2r^2(r^2-1)\sin x}{(r^2 - 2r\cos x + 1)^2}. \end{aligned} \tag{33}$$

Then, using (30),

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{r_t^0 - 1}{r_t^0(r_t^0+1)}\delta_r(t)\right) &= \frac{r_t^0 - 1}{r_t^0(r_t^0+1)}\frac{\mathrm{d}\delta_r(t)}{\mathrm{d}t} - \delta_r(t)\frac{((r_t^0)^2 - 2r_t^0 - 1)}{(r_t^0)^2(r_t^0+1)^2}\frac{r_t^0(r_t^0+1)}{r_t^0 - 1} \\ &= \frac{r_t^0 - 1}{r_t^0(r_t^0+1)}\left(\frac{\mathrm{d}\delta_r(t)}{\mathrm{d}t} - \delta_r(t)\frac{(r_t^0)^2 - 2r_t^0 - 1}{(r_t^0 - 1)^2}\right) \\ &= -\frac{r_t^0 - 1}{r_t^0(r_t^0+1)}H_1(t) \end{aligned}$$

and hence

$$\delta_r(t) = -\frac{r_t^0(r_t^0+1)}{r_t^0-1}\int_0^t \frac{r_s^0-1}{r_s^0(r_s^0+1)} H_1(s)\mathrm{d}s. \tag{34}$$

By Taylor's Theorem,

$$H_1(t) = \frac{1}{2}\left(\delta_r(t)^2\frac{\partial^2 h_1}{\partial r^2} - 2\delta_r(t)(\vartheta_t^\xi - \xi_{T-t})\frac{\partial^2 h_1}{\partial r\partial x} + (\vartheta_t^\xi - \xi_{T-t})^2\frac{\partial^2 h_1}{\partial x^2}\right)\Bigg|_{(r_t^0 - a_t\delta_r(t), a_t(\vartheta_t^\xi - \xi_{T-t}))}$$

for some $a_t \in (0,1)$. Now

$$\begin{aligned}
\frac{\partial^2 h_1}{\partial r^2}(r,x) &= \frac{4(r^3\cos(2x) - 3r^2\cos x + 3r - \cos x)}{(r^2-2r\cos x+1)^3},\\
\frac{\partial^2 h_1}{\partial r\partial x}(r,x) &= \frac{4r\sin x(2r^3\cos x - 3r^2+1)}{(r^2-2r\cos x+1)^3},\\
\frac{\partial^2 h_1}{\partial x^2}(r,x) &= -\frac{2r^2(r^2-1)(r^2\cos x + r(\cos(2x)-3)+\cos x)}{(r^2-2r\cos x+1)^3}.
\end{aligned} \tag{35}$$

We now bound these functions for $r > 1$, seeking to reduce the singularities in the denominators as much as possible by exploiting simultaneous vanishing in the numerators. By using the fact that $2r(1-\cos x) \leqslant r^2 - 2r\cos x + 1$, we have

$$\begin{aligned}
\left|\frac{\partial^2 h_1}{\partial r^2}(r,x)\right| &\leqslant \frac{2(r^2+3r+4)}{(r-1)^3},\\
\left|\frac{\partial^2 h_1}{\partial r\partial x}(r,x)\right| &\leqslant \frac{4r\sin|x|(r+1)^2}{(r-1)^4},\\
\left|\frac{\partial^2 h_1}{\partial x^2}(r,x)\right| &\leqslant \frac{8r^2(r+1)}{(r-1)^3}.
\end{aligned}$$

Observe that since $\|\xi\|_T \leqslant A^{-1}(r_0-1) \leqslant A^{-1}(r_t^0-1)$, provided $A \geqslant 2$, we obtain the crude estimates

$$(r_t^0-1)/2 \leqslant (1-A^{-1})(r_t^0-1) \leqslant r_t^0 - a_t\delta_r(t) - 1 \leqslant r_t^0 - 1,$$

and

$$\sin|a_t(\vartheta_t^\xi - \xi_{T-t})| \leqslant |a_t(\vartheta_t^\xi - \xi_{T-t})| \leqslant 3\|\xi\|_T.$$

Therefore, using the fact that $t \leqslant T^\xi$, there exists some absolute constant A' such that

$$|H_1(t)| \leqslant \frac{A'\|\xi\|_T^2 r_t^0(r_t^0+1)^2}{(r_t^0-1)^3}.$$

Substituting this bound into (34), we get

$$\begin{aligned}
\delta_r(t) &\leqslant A'\|\xi\|_T^2\frac{r_t^0(r_t^0+1)}{r_t^0-1}\int_0^t \frac{r_s^0-1}{r_s^0(r_s^0+1)}\frac{r_s^0(r_s^0+1)^2}{(r_s^0-1)^3}\mathrm{d}s\\
&= A'\|\xi\|_T^2\frac{r_t^0(r_t^0+1)}{r_t^0-1}\int_0^t \frac{1}{r_s^0(r_s^0-1)}\partial_s r_s^0\mathrm{d}s\\
&= A'\|\xi\|_T^2\frac{r_t^0(r_t^0+1)}{r_t^0-1}\log\frac{(r_t^0-1)r_0}{r_t^0(r_0-1)}\\
&\leqslant \frac{10A'\|\xi\|_T^2 e^T}{r_0-1}.
\end{aligned}$$

Here we have used that $r_t^0 + 1 \leqslant 5e^t$ and that $x \log(1/x)$ is bounded by 1.

Therefore, taking $A > 10A'$, we have $\delta_r(t) \leqslant \|\xi\|_T$ and hence $T^\xi = T$. It follows immediately that

$$|\arg \Psi_T(z)| \leqslant \|\xi\|_T \quad \text{and} \quad 0 \leqslant r_T^0 - |\Psi_T(z)| \leqslant \frac{A\|\xi\|_T^2 e^T}{|z|-1}. \tag{36}$$

Now suppose that $\arg z = \theta \neq 0$. It is straightforward to show that

$$\Psi_T(z) = e^{i\theta}\psi_T(|z|)$$

where $\psi_T(z)$ solves the Loewner equation with driving function $\xi_t - \theta$. It follows that

$$|\Psi_T(z)| = |\psi_T(|z|)| \quad \text{and} \quad \arg \Psi_T(z) = \arg \psi_T(|z|) + \theta.$$

The result follows by replacing $\Psi_T(z)$ by $\psi_T(|z|)$ and ξ_t by $\xi_t - \theta$ in (36).

Finally, for the last part of the theorem, note that, if $\xi_t \equiv 0$, then clearly the conditions of the theorem are satisfied. Therefore, $|f_T(z)| - 1$ and $\arg f_T(z)$ also satisfy the above bounds from which the second statement of the theorem follows. □

We now turn to comparing the derivatives $\Psi_t'(z)$ and $f_t'(z)$.

Proposition 11. *There exists some absolute constant A (taken to be at least as large as the absolute constant in Proposition 10), such that for all $1 < |z| < 2$ and $T > 0$ satisfying $\|\xi\|_T + |\arg z| \leqslant A^{-1}e^{-T}(|z|-1)$ we have*

$$\left|\log\left(|\Psi_T'(z)| \frac{|z|(|z|+1)(r_T^0 - 1)}{r_T^0(r_T^0+1)(|z|-1)}\right)\right| \leqslant \frac{Ae^T\|\xi - \arg z\|_T^2}{(|z|-1)^2}$$

and hence

$$\left|\log\left|\frac{\Psi_T'(z)}{f_T'(z)}\right|\right| \leqslant \frac{2Ae^T(\|\xi\|_T + |\arg z|)^2}{(|z|-1)^2}.$$

Proof. Recall that $\Psi_T(z) = u_T^\xi(z)$ where $u_t^\xi(z)$ is the solution to (13) starting from $u_0^\xi(z) = z$. Hence $\Psi_T'(z) = \partial_z u_T^\xi(z)$. Set $y^\xi(t) = \partial_z u_t^\xi(z)$. Differentiating both sides of (13) gives

$$\partial_t y^\xi(t) = y^\xi(t)\left(1 - \frac{2e^{2i\xi_{T-t}}}{\left(u_t^\xi - e^{i\xi_{T-t}}\right)^2}\right).$$

Hence, using $y^\xi(0) = 1$, we get

$$y^\xi(t) = \exp\left(\int_0^t \left(1 - \frac{2e^{2i\xi_{T-s}}}{\left(u_s^\xi - e^{i\xi_{T-s}}\right)^2}\right) \mathrm{d}s\right)$$

and so

$$|y^\xi(t)| = \exp\left(\int_0^t \left(1 - \operatorname{Re}\frac{2}{\left(u_s^\xi e^{-i\xi_{T-s}} - 1\right)^2}\right) \mathrm{d}s\right). \tag{37}$$

We evaluate the right hand side using the estimates from Proposition 10. Setting $u_t^\xi = r_t^\xi e^{i\vartheta_t^\xi}$, we have

$$\operatorname{Re}\frac{2}{\left(u_s^\xi e^{-i\xi_{T-s}}-1\right)^2} = \frac{2((r_s^\xi)^2\cos(2(\vartheta_s^\xi-\xi_{T-s})) - 2r_s^\xi\cos(\vartheta_s^\xi-\xi_{T-s})+1)}{\left((r_s^\xi)^2 - 2r_s^\xi\cos(\vartheta_s^\xi-\xi_{T-s})+1\right)^2}.$$

Let

$$h(r,x) = 1 - \frac{2(r^2\cos(2x) - 2r\cos x + 1)}{(r^2-2r\cos x+1)^2}. \tag{38}$$

Define $\delta_r(t) = r_t^0 - r_t^\xi$ as in Proposition 10. Then

$$\begin{aligned}\log|y^\xi(t)| &= \int_0^t \left(\frac{(r_s^0)^2 - 2r_s^0 - 1}{(r_s^0-1)^2} + H(s)\right)\mathrm{d}s \\ &= \log\left(\frac{r_t^0(r_t^0+1)(r_0-1)}{r_0(r_0+1)(r_t^0-1)}\right) + \int_0^t H(s)\mathrm{d}s,\end{aligned}$$

where

$$\begin{aligned}H(t) &= h(r_t^\xi, \vartheta_t^\xi - \xi_{T-t}) - h(r_t^0, 0) \\ &= -\frac{\partial h}{\partial r}(r_t^0 - a_t\delta_r(t), a_t(\vartheta_t^\xi - \xi_{T-t}))\delta_r(t) + \frac{\partial h}{\partial x}(r_t^0 - a_t\delta_r(t), a_t(\vartheta_t^\xi - \xi_{T-t}))(\vartheta_t^\xi - \xi_{T-t}),\end{aligned}$$

for some $a_t \in (0,1)$. Now

$$\begin{aligned}\frac{\partial h}{\partial r}(r,x) &= \frac{4(r^3\cos(2x) - 3r^2\cos x + 3r - \cos x)}{(r^2-2r\cos x+1)^3}, \\ \frac{\partial h}{\partial x}(r,x) &= \frac{4r\sin x(2r^3\cos x - 3r^2+1)}{(r^2-2r\cos(x)+1)^3}.\end{aligned} \tag{39}$$

These are the same expressions as in (35) and hence if $r > 1$, from the computation in the previous theorem,

$$\begin{aligned}\left|\frac{\partial h}{\partial r}(r,x)\right| &\leqslant \frac{2(r^2+3r+4)}{(r-1)^3}, \\ \left|\frac{\partial h}{\partial x}(r,x)\right| &\leqslant \frac{4r\sin|x|(r+1)^2}{(r-1)^4}.\end{aligned}$$

Hence, as in the proof of Proposition 10, for some absolute constant A,

$$|H(t)| \leqslant \frac{Ae^T\|\xi - \arg z\|_T^2 r_t^0(r_t^0+1)}{(r_0-1)(r_t^0-1)^3}.$$

Using exactly the same argument as in Proposition 10, we obtain

$$\left|\log\left(|\Psi_T'(z)|\frac{r_0(r_0+1)(r_T^0-1)}{r_T^0(r_T^0+1)(r_0-1)}\right)\right| \leqslant \frac{Ae^T\|\xi-\arg z\|_T^2}{(r_0-1)^2}.$$

The same bound holds for $|f_T'(z)|$ and hence

$$\left|\log\left|\frac{\Psi_T'(z)}{f_T'(z)}\right|\right| \leqslant \frac{2Ae^T(\|\xi\|_T + |\arg z|)^2}{(|z|-1)^2}.$$

□

5.2 Away from the slits

We now analyse $\Psi_t(z)$ in the case when $|z|-1$ is are close to zero, and t is small enough to ensure that $\Psi_s(z)$ does not get too close to the growing slits so long as $s \leqslant t$.

We begin by computing an explicit expression for $f_t(e^{i\theta})$ when $|\theta| \in (0,\pi)$ and t is sufficiently small that $f_s(e^{i\theta})$ stays away from the slit for all $s \leqslant t$. Although $f_t(z)$ is not explicitly defined when $|z|=1$, $f_t(z)$ for $|z|>1$ can be continuously extended to the boundary of the unit disk in a well-defined way, so this is the interpretation we put on $f_t(e^{i\theta})$.

Set $f_t(e^{i\theta}) = r_t^0 e^{i\vartheta_t^0}$. From (28) it is immediate that $r_t^0 = 1$ for all $t \leqslant \inf\{t>0 : f_t(e^{i\theta})=1\}$. Substituting this into (29) we get

$$\partial_t \vartheta_t^0 = -\frac{\sin \vartheta_t^0}{1-\cos\vartheta_t^0} = -\cot\frac{\vartheta_t^0}{2}.$$

Solving this gives

$$\vartheta_t^0 = \vartheta_t^0(e^{i\theta}) = \cos^{-1}\big((1+\cos\theta)e^t - 1\big) \tag{40}$$

and hence

$$\inf\{t>0 : f_t(e^{i\theta})=1\} = \log\frac{2}{1+\cos\theta}.$$

Now consider Ψ_t having general driving function ξ_t.

Proposition 12. *There exists some absolute constant A such that, for all $|z|>1$ and $T>0$ satisfying*

$$T \leqslant \log\frac{2}{1+\cos(\arg z)}$$

and

$$\|\xi\|_T + |z| - 1 \leqslant A^{-1}(e^T-1)^{-1/2}(1-\cos\vartheta_T^0),$$

where ϑ_t^0 is defined as in (40) with $\theta = \arg z$, we have

$$\left|\log\left(\frac{(|\Psi_T(z)|-1)\tan\frac{\vartheta_T^0}{2}}{(|z|-1)\tan\frac{\arg z}{2}}\right)\right| \leqslant \frac{A(\|\xi\|_T + |z|-1)\sqrt{e^T-1}}{1-\cos\vartheta_T^0} \leqslant 1$$

and

$$\left|\arg\Psi_T(z) - \vartheta_T^0\right| \leqslant 2(\|\xi\|_T + |z|-1)\tan\frac{\arg z}{2}\cot\frac{\vartheta_T^0}{2}.$$

Hence

$$|\Psi_T(z) - f_T(z)| \leqslant 10(\|\xi\|_T + |z|-1)\tan\frac{\arg z}{2}\cot\frac{\vartheta_T^0}{2} \leqslant 30\left(1+\frac{\|\xi\|_T}{|z|-1}\right)(|f_T(z)|-1).$$

Proof. Set $\theta = \arg z$. To simplify expressions, assume that $0<\theta<\pi$; the corresponding result holds for $-\pi<\theta<0$ by symmetry.

Let $u_t^\xi = r_t^\xi e^{i\vartheta_t^\xi}$ be the solution to (13) starting from z. Set $\delta_r(t) = r_t^\xi - 1$ and $\delta_\theta(t) = \vartheta_t^\xi - \vartheta_t^0$ and let

$$\begin{aligned} T^\xi = &\inf\left\{t>0 : \delta_r(t) > 3(|z|-1)\tan(\theta/2)\cot(\vartheta_t^0/2)\right\} \wedge \\ &\inf\left\{t>0 : |\delta_\theta(t) - \xi_{T-t}| > 2(\|\xi\|_T + |z|-1)\tan(\theta/2)\cot(\vartheta_t^0/2)\right\} \wedge T. \end{aligned}$$

(Note that $\delta_r(t)$ and T^ξ as defined here are not the same as the $\delta_r(t)$ and T^ξ defined in Section 5.1).

We begin by obtaining some crude bounds on $\delta_r(t)$ and $|\delta_\theta(t) - \xi_{T-t}|$ that hold under the assumption that $t \leqslant T^\xi$. In what follows, let $A \geqslant 12$. Firstly, note that by standard trigonometric identities, and using the explicit value of ϑ_t^0 from (40),

$$\tan\frac{\theta}{2}\cot\frac{\vartheta_t^0}{2} = \sqrt{\frac{(1-\cos\theta)(1+\cos\vartheta_t^0)}{(1+\cos\theta)(1-\cos\vartheta_t^0)}} = \sqrt{\frac{(1-\cos\theta)e^t}{1-\cos\vartheta_t^0}} \leqslant \sqrt{\frac{2e^t}{1-\cos\vartheta_t^0}}. \tag{41}$$

Hence, using the fact that ϑ_t^0 is decreasing in t,

$$\delta_r(t) \leqslant (1-\cos\vartheta_t^0)^{1/2}/(2\sqrt{2}) \leqslant 1/2 \tag{42}$$

and

$$|\delta_\theta(t) - \xi_{T-t}| \leqslant (1-\cos\vartheta_t^0)^{1/2}/(3\sqrt{2}) \leqslant 1/3. \tag{43}$$

Then, for any $a \in (0,1)$,

$$\sin(\vartheta_t^0 + a(\delta_\theta(t) - \xi_{T-t})) \leqslant \sin\vartheta_t^0 + |\delta_\theta(t) - \xi_{T-t}| \tag{44}$$

and

$$\frac{1}{1-\cos(\vartheta_t^0 + a(\delta_\theta(t) - \xi_{T-t}))} \leqslant \frac{2}{1-\cos\vartheta_t^0}. \tag{45}$$

We now turn to (28) and (29). Define $h_1(r,x)$ as in (32) and

$$h_2(r,x) = -\frac{2r\sin x}{r^2 - 2r\cos x + 1}.$$

Then

$$\frac{\mathrm{d}\delta_r(t)}{\mathrm{d}t} = h_1(1+\delta_r(t), \vartheta_t^0 + \delta_\theta(t) - \xi_{T-t}) - h_1(1, \vartheta_t^0)$$

and

$$\frac{\mathrm{d}\delta_\theta(t)}{\mathrm{d}t} = h_2(1+\delta_r(t), \vartheta_t^0 + \delta_\theta(t) - \xi_{T-t}) - h_2(1, \vartheta_t^0).$$

We first analyse $\delta_r(t)$. Set

$$\begin{aligned} H_1(t) &= h_1(1+\delta_r(t), \vartheta_t^0 + \delta_\theta(t) - \xi_{T-t}) - \left(h_1(1,\vartheta_t^0) + \frac{\partial h_1}{\partial r}(1,\vartheta_t^0)\delta_r(t) + \frac{\partial h_1}{\partial x}(1,\vartheta_t^0)(\delta_\theta(t) - \xi_{T-t})\right) \\ &= \frac{\mathrm{d}\delta_r(t)}{\mathrm{d}t} - \frac{\delta_r(t)}{1-\cos\vartheta_t^0}. \end{aligned}$$

Here we have used the expressions for the partial derivatives of $h_1(r,x)$ in (33). By computing the integrating factor corresponding to this expression, we obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(\delta_r(t)\tan\frac{\vartheta_t^0}{2}\right) &= \frac{\mathrm{d}\delta_r(t)}{\mathrm{d}t}\tan\frac{\vartheta_t^0}{2} + \frac{\delta_r(t)}{2}\sec^2\frac{\vartheta_t^0}{2}\frac{\mathrm{d}\vartheta_t^0}{\mathrm{d}t} \\ &= \tan\frac{\vartheta_t^0}{2}\left(\frac{\mathrm{d}\delta_r(t)}{\mathrm{d}t} - \frac{\delta_r(t)}{1-\cos\vartheta_t^0}\right) \\ &= \tan\frac{\vartheta_t^0}{2}H_1(t). \end{aligned}$$

Hence, using that $\delta_r(0) = |z| - 1$ and $\vartheta_0^0 = \theta$,

$$\log \frac{\delta_r(t) \tan \frac{\vartheta_t^0}{2}}{(|z| - 1) \tan \frac{\theta}{2}} = \int_0^t \delta_r(s)^{-1} H_1(s) \mathrm{d}s. \tag{46}$$

By Taylor's Theorem,

$$H_1(t) = \frac{1}{2} \left(\delta_r(t)^2 \frac{\partial^2 h_1}{\partial r^2} + 2\delta_r(t)(\delta_\theta(t) - \xi_{T-t}) \frac{\partial^2 h_1}{\partial r \partial x} + (\delta_\theta(t) - \xi_{T-t})^2 \frac{\partial^2 h_1}{\partial x^2} \right) \Bigg|_{(1+a_t\delta_r(t), \vartheta_t^0 + a_t(\delta_\theta(t) - \xi_{T-t}))}$$

for some $a_t \in (0, 1)$.

Using (35), together with the fact that $(r-1)^2 \leqslant r^2 - 2r\cos x + 1$, for $r \in (1, 3/2)$ and $x \in (0, \pi)$ we have that

$$\left| \frac{\partial^2 h_1}{\partial r^2}(r, x) \right| \leqslant \frac{1}{1 - \cos x} + \frac{3(r-1)}{(1 - \cos x)^2},$$
$$\left| \frac{\partial^2 h_1}{\partial r \partial x}(r, x) \right| \leqslant \frac{75 \sin x}{8(1 - \cos x)^2},$$
$$\left| \frac{\partial^2 h_1}{\partial x^2}(r, x) \right| \leqslant \frac{45(r-1)}{4(1 - \cos x)^2}.$$

Hence, using that $t \leqslant T^\xi$ and the crude bounds (42) – (45),

$$\begin{aligned}
\delta_r(t)^{-1}|H_1(t)| \leqslant & \frac{\delta_r(t)}{1 - \cos \vartheta_t^0} + \frac{6\delta_r(t)^2}{(1 - \cos \vartheta_t^0)^2} + \frac{60(|\delta_\theta(t) - \xi_{T-t}|)(\sin \vartheta_t^0 + |\delta_\theta(t) - \xi_{T-t}|)}{(1 - \cos \vartheta_t^0)^2} \\
\leqslant & 3(|z| - 1) \tan(\theta/2) \frac{\cot(\vartheta_t^0/2)}{1 - \cos \vartheta_t^0} + 300(\|\xi\|_T + |z| - 1)^2 \tan^2(\theta/2) \frac{\cot^2(\vartheta_t^0/2)}{(1 - \cos \vartheta_t^0)^2} \\
& + 120(\|\xi\|_T + |z| - 1) \tan(\theta/2) \frac{\cot(\vartheta_t^0/2) \sin \vartheta_t^0}{(1 - \cos \vartheta_t^0)^2} \\
\leqslant & A'(\|\xi\|_T + |z| - 1) \frac{\tan^2(\theta/2)}{\sqrt{e^T - 1}} \frac{\cot(\vartheta_t^0/2) \sin \vartheta_t^0}{(1 - \cos \vartheta_t^0)^2}
\end{aligned}$$

for some absolute constant A', where we used the assumption on T to show that $\cot(\theta/2) \leqslant 1/\sqrt{e^T - 1}$.

Using the fact that

$$\partial_t \vartheta_t^0 = -\frac{\sin \vartheta_t^0}{1 - \cos \vartheta_t^0} = -\cot \frac{\vartheta_t^0}{2},$$

we evaluate the integral

$$\int_0^t \frac{\cot(\vartheta_s^0/2) \sin \vartheta_t^0}{(1 - \cos \vartheta_s^0)^2} \mathrm{d}s = \frac{1}{1 - \cos \vartheta_t^0} - \frac{1}{1 - \cos \theta} = \frac{\cot^2(\theta/2)(e^t - 1)}{1 - \cos \vartheta_t^0}.$$

We will also use later that

$$\int_0^t \frac{\cot(\vartheta_s^0/2)}{1 - \cos \vartheta_s^0} \mathrm{d}s = \frac{\sin \vartheta_t^0}{1 - \cos \vartheta_t^0} - \frac{\sin \theta}{1 - \cos \theta} = \cot \frac{\vartheta_t^0}{2} - \cot \frac{\theta}{2}.$$

Substituting the first identity into (46), we get

$$\left|\log\left(\frac{\delta_r(t)\tan(\vartheta_t^0/2)}{(|z|-1)\tan(\theta/2)}\right)\right| \leqslant \frac{A'(\|\xi\|_T+|z|-1)\sqrt{e^T-1}}{1-\cos\vartheta_t^0}.$$

We now consider $\delta_\theta(t)$. Set

$$\begin{aligned}H_2(t) &= h_2(1+\delta_r(t),\vartheta_t^0+\delta_\theta(t)-\xi_{T-t}) - \left(h_2(1,\vartheta_t^0)+\frac{\partial h_2}{\partial r}(1,\vartheta_t^0)\delta_r(t)+\frac{\partial h_2}{\partial x}(1,\vartheta_t^0)(\delta_\theta(t)-\xi_{T-t})\right)\\ &= \frac{\mathrm{d}\delta_\theta(t)}{\mathrm{d}t} - \frac{\delta_\theta(t)-\xi_{T-t}}{1-\cos\vartheta_t^0}.\end{aligned}$$

Here we have used

$$\begin{aligned}\frac{\partial h_2}{\partial r}(r,x) &= \frac{2(r^2-1)\sin x}{(r^2-2r\cos x+1)^2}\\ \frac{\partial h_2}{\partial x}(r,x) &= \frac{-2r((r^2+1)\cos x-2r)}{(r^2-2r\cos x+1)^2}.\end{aligned}$$

Then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\delta_\theta(t)\tan\frac{\vartheta_t^0}{2}\right) &= \frac{\mathrm{d}\delta_\theta(t)}{\mathrm{d}t}\tan\frac{\vartheta_t^0}{2} - \frac{\delta_\theta(t)}{2}\sec^2\frac{\vartheta_t^0}{2}\cot\frac{\vartheta_t^0}{2}\\ &= \tan\frac{\vartheta_t^0}{2}\left(\frac{\mathrm{d}\delta_\theta(t)}{\mathrm{d}t} - \frac{\delta_\theta(t)}{1-\cos\vartheta_t^0}\right)\\ &= -\frac{\xi_{T-t}}{1-\cos\vartheta_t^0} + \tan\frac{\vartheta_t^0}{2}H_2(t)\end{aligned}$$

and hence

$$\delta_\theta(t) = -\cot\frac{\vartheta_t^0}{2}\int_0^t \frac{\xi_{T-s}}{1-\cos\vartheta_s^0}\mathrm{d}s + \cot\frac{\vartheta_t^0}{2}\int_0^t \tan\frac{\vartheta_s^0}{2}H_2(s)\mathrm{d}s. \tag{47}$$

By Taylor's Theorem,

$$H_2(t) = \frac{1}{2}\left(\delta_r(t)^2\frac{\partial^2 h_2}{\partial r^2} + 2\delta_r(t)(\delta_\theta(t)-\xi_{T-t})\frac{\partial^2 h_2}{\partial r\partial x} + (\delta_\theta(t)-\xi_{T-t})^2\frac{\partial^2 h_2}{\partial x^2}\right)\Bigg|_{(1+b_t\delta_r(t),\vartheta_t^0+b_t(\delta_\theta(t)-\xi_{T-t}))}$$

for some $b_t \in (0,1)$.

Now

$$\begin{aligned}\frac{\partial^2 h_2}{\partial r^2}(r,x) &= \frac{-4\sin x(r^3-3r+2\cos x)}{(r^2-2r\cos x+1)^3},\\ \frac{\partial^2 h_2}{\partial r\partial x}(r,x) &= \frac{2(r^2-1)\left((r^2+1)\cos x+2r(\cos^2 x-2)\right)}{(r^2-2r\cos x+1)^3},\\ \frac{\partial^2 h_2}{\partial x^2}(r,x) &= \frac{2r\sin x(r^4+2(r^3+r)\cos x-6r^2+1)}{(r^2-2r\cos x+1)^3},\end{aligned}$$

and hence, if $r \in (1, 3/2)$ and $x \in (0, \pi)$, we have

$$\left|\frac{\partial^2 h_2}{\partial r^2}(r,x)\right| \leqslant \frac{7(r-1)^2 \sin x}{4(1-\cos x)^3} + \frac{\sin(x)}{(1-\cos x)^2},$$
$$\left|\frac{\partial^2 h_2}{\partial r \partial x}(r,x)\right| \leqslant \frac{5(r-1)^3}{8(1-\cos x)^3} + \frac{37(r-1)}{4(1-\cos x)^2},$$
$$\left|\frac{\partial^2 h_2}{\partial x^2}(r,x)\right| \leqslant \frac{111(r-1)^2 \sin x}{32(1-\cos x)^3} + \frac{117 \sin x}{32(1-\cos x)^2}.$$

Therefore

$$\begin{aligned}|H_2(t)| \leqslant& \frac{(40\delta_r(t)^2 + 10|\delta_\theta(t) - \xi_{T-t}|^2)(\sin\vartheta_t^0 + |\delta_\theta(t) - \xi_{T-t}|)}{(1-\cos\vartheta_t^0)^2} \\ \leqslant& \frac{A''(\|\xi\|_T + |z| - 1)^2 \tan^3(\theta/2)\cot^2(\vartheta_t^0/2)\sin\vartheta_t^0}{\sqrt{e^T - 1}(1-\cos\vartheta_t^0)^2}\end{aligned}$$

for some absolute constant A''. Here again we have used that $t \leqslant T^\xi$ and the crude estimates (42) – (45).

Substituting this bound into (47), we get

$$\begin{aligned}|\delta_\theta(t)| <& \|\xi\|_T \cot\frac{\vartheta_t^0}{2}\int_0^t \frac{\tan(\vartheta_s^0/2)}{1-\cos\vartheta_s^0}\mathrm{d}s + A''\frac{(\|\xi\|_T + |z| - 1)^2}{\sqrt{e^T - 1}}\tan^3\frac{\theta}{2}\cot\frac{\vartheta_t^0}{2}\int_0^t \frac{\cot(\vartheta_s^0/2)\sin\vartheta_s^0}{(1-\cos\vartheta_s^0)^2}\mathrm{d}s \\ =& \|\xi\|_T\left(\tan\frac{\theta}{2}\cot\frac{\vartheta_t^0}{2} - 1\right) + A''(\|\xi\|_T + |z| - 1)^2\frac{\tan(\theta/2)\cot(\vartheta_t^0/2)\sqrt{e^t - 1}}{1-\cos\vartheta_t^0} \\ \leqslant& \|\xi\|_T\left(\tan\frac{\theta}{2}\cot\frac{\vartheta_t^0}{2} - 1\right) + A''A^{-1}(\|\xi\|_T + |z| - 1)\tan\frac{\theta}{2}\cot\frac{\vartheta_t^0}{2}.\end{aligned}$$

Here we have used the condition on T to deduce that $\sqrt{e^T - 1} \leqslant \tan(\theta/2)$.

Finally, observe that, provided $A \geqslant A' \vee A''$, this bound and the bound on $\delta_r(t)$ together imply that $T = T^\xi$. Hence

$$\begin{aligned}\left|\log\left(\frac{(|\Psi_T(z)| - 1)\tan(\vartheta_T^0/2)}{(|z| - 1)\tan(\theta/2)}\right)\right| =& \left|\log\left(\frac{\delta_r(T)\tan(\vartheta_T^0/2)}{(|z| - 1)\tan(\theta/2)}\right)\right| \\ \leqslant& \frac{A(\|\xi\|_T + |z| - 1)\sqrt{e^T - 1}}{1-\cos\vartheta_T^0}\end{aligned}$$

and

$$|\arg\Psi_T(z) - \vartheta_T^0| = |\delta_\theta(T)| \leqslant 2(\|\xi\|_T + |z| - 1)\tan\frac{\theta}{2}\cot\frac{\vartheta_T^0}{2}$$

as required.

Finally, for the last part of the theorem, note that, if $\xi_t \equiv 0$, then clearly the conditions of the theorem are satisfied. Therefore, $|f_T(z)| - 1$ and $\arg f_T(z)$ also satisfy the above bounds from which the second statement of the theorem follows. □

We now turn to comparing the derivatives $\Psi_t'(z)$ and $f_t'(z)$.

Proposition 13. *There exists some absolute constant A (taken to be at least as large as the absolute constant in Proposition 12), such that for all $|z| > 1$, $T > 0$ satisfying*

$$T \leqslant \log \frac{2}{1 + \cos(\arg z)}$$

and

$$\|\xi\|_T + |z| - 1 \leqslant A^{-1}(e^T - 1)^{-1/2}(1 - \cos \vartheta_T^0),$$

where ϑ_t^0 is defined as in (40) *with $\theta = \arg z$, we have*

$$\left| \log \left(|\Psi_T'(z)| \tan \frac{\vartheta_T^0}{2} \cot \frac{\arg z}{2} \right) \right| \leqslant \frac{A(\|\xi\|_T + |z| - 1)\sqrt{e^T - 1}}{1 - \cos \vartheta_T^0}$$

and hence

$$\left| \log \left| \frac{\Psi_T'(z)}{f_T'(z)} \right| \right| \leqslant \frac{2A(\|\xi\|_T + |z| - 1)\sqrt{e^T - 1}}{1 - \cos \vartheta_T^0} \leqslant 2.$$

Proof. Recall that $\Psi_T(z) = u_T^\xi(z)$ where $u_t^\xi(z)$ is the solution to (13) starting from $u_0^\xi(z) = z$ and hence $\Psi_T'(z) = \partial_z u_T^\xi(z)$. As in the previous section, $y^\xi(t) = \partial_z u_t^\xi(z)$ satisfies (37) and hence

$$|y^\xi(t)| = \exp\left(\int_0^t \left(1 - \operatorname{Re} \frac{2}{\left(u_s^\xi e^{-i\xi_{T-s}} - 1\right)^2} \right) \mathrm{d}s \right).$$

We evaluate the right hand side using the estimates from Proposition 12. Again, for simplicity, we assume that $\arg z = \theta \in (0, \pi)$. Setting $u_t^\xi = r_t^\xi e^{i\vartheta_t^\xi}$, $\delta_r(t) = r_t^\xi$ and $\delta_\theta(t) = \vartheta_t^\xi - \vartheta_t^0$, as in Proposition 12, we have

$$\begin{aligned} \log |y^\xi(t)| &= \int_0^t \left(\frac{1}{1 - \cos \vartheta_s^0} + H(s) \right) \mathrm{d}s \\ &= \log \left(\cot \frac{\vartheta_t^0}{2} \tan \frac{\theta}{2} \right) + \int_0^t H(s) \mathrm{d}s, \end{aligned}$$

where, with $h(r, x)$ is as in (38),

$$\begin{aligned} H(t) =& h(r_t^\xi, \vartheta_t^\xi - \xi_{T-t}) - h(1, \vartheta_t^0) \\ =& \frac{\partial h}{\partial r}(1 + a_t \delta_r(t), \vartheta_t^0 + a_t(\delta_\theta(t) - \xi_{T-t}))\delta_r(t) \\ &+ \frac{\partial h}{\partial x}(1 + a_t \delta_r(t), \vartheta_t^0 + a_t(\delta_\theta(t) - \xi_{T-t}))(\delta_\theta(t) - \xi_{T-t}), \end{aligned}$$

for some $a_t \in (0, 1)$. Using the expressions for the partial derivatives of $h(r, x)$ from (39), if $r \in (1, 3/2)$ and $x \in (0, \pi)$ then

$$\begin{aligned} \left| \frac{\partial h}{\partial r}(r, x) \right| &\leqslant \frac{27}{8(1 - \cos x)} + \frac{23(r - 1)}{4(1 - \cos x)^2} + \frac{(r - 1)^3}{2(1 - \cos x)^3}, \\ \left| \frac{\partial h}{\partial x}(r, x) \right| &\leqslant \frac{81 \sin x}{16(1 - \cos x)^2} + \frac{3(r - 1)^2 \sin x}{(1 - \cos x)^3}. \end{aligned}$$

Hence, using the crude estimates (42) – (45),

$$|H(t)| \leqslant \frac{27\delta_r(t)}{4(1-\cos\vartheta_t^0)} + \frac{55\delta_r(t)^2}{(1-\cos\vartheta_t^0)^2} + \frac{425|\delta_\theta(t) - \xi_{T-t}|\sin\vartheta_t^0}{(1-\cos\vartheta_t^0)^2}.$$

Using exactly the same argument as in Proposition 12, there exists some absolute constant A such that

$$\left|\log\left(|\Psi_T'(z)|\tan\frac{\vartheta_T^0}{2}\cot\frac{\theta}{2}\right)\right| \leqslant \frac{A(\|\xi\|_T + |z| - 1)\sqrt{e^T - 1}}{1 - \cos\vartheta_T^0}.$$

The same bound holds for $|f_T'(e^{\boldsymbol{\sigma}+i\theta})|$ and hence

$$\left|\log\left|\frac{\Psi_T'(e^{\boldsymbol{\sigma}+i\theta})}{f_T'(e^{\boldsymbol{\sigma}+i\theta})}\right|\right| \leqslant \frac{2A(\|\xi\|_T + |z| - 1)\sqrt{e^T - 1}}{1 - \cos\vartheta_T^0}.$$

□

Finally, we give a lower bound on the modulus of the derivative of a map associated with a general Loewner driving function.

Proposition 14. *There exists some absolute constant B such that, for all $T > 0$,*

$$|\Psi_T'(z)| \geqslant \frac{(|z| - 1)(1 - \cos(\arg(z)))}{B\sqrt{e^T - 1}(\|\xi\|_T + |z| - 1)}.$$

Proof. We first obtain a generic lower bound on $|\Psi_T'(z)|$, without making any assumptions on the driving function ξ or initial value z. Using the notation of the previous proof, recall that $\Psi_T'(z) = y^\xi(T)$ where

$$\log|y^\xi(t)| = \int_0^t h(r_s^\xi, \vartheta_s^\xi - \xi_{T-s})\mathrm{d}s.$$

Now

$$\begin{aligned} h(r,x) &= 1 - \frac{2}{r^2 - 2r\cos x + 1} + \frac{4r^2\sin^2 x}{(r^2 - 2r\cos x + 1)^2} \\ &\geqslant 1 - \frac{2}{r^2 - 2r\cos x + 1}. \end{aligned}$$

By (28),

$$\begin{aligned} \int_0^t \frac{2}{(r_s^\xi)^2 - 2r_s^\xi\cos(\vartheta_s^\xi - \xi_{T-s}) + 1}\mathrm{d}s &= \int_0^t \frac{2\partial_s r_s^\xi}{r_s^\xi\left((r_s^\xi)^2 - 1\right)}\mathrm{d}s \\ &= \log\frac{(r_t^\xi)^2 - 1}{(r_t^\xi)^2} - \log\frac{(r_0^\xi)^2 - 1}{(r_0^\xi)^2}. \end{aligned}$$

Therefore, using the fact that $|\Psi_T(z)| \geqslant |z|$,

$$|\Psi_T'(z)| \geqslant \frac{e^T|\Psi_T(z)|^2(|z|^2 - 1)}{|z|^2(|\Psi_T(z)|^2 - 1)} \geqslant \frac{e^T(|z| - 1)}{|\Psi_T(z)| - 1}.$$

Suppose T satisfies the conditions of Proposition 13. Then

$$|\Psi_T'(z)| \geqslant e^{-1}\tan(\arg(z)/2)\cot(\vartheta_t^0/2) \geqslant \frac{1}{3}$$

and hence, since $T \leqslant \log\frac{2}{1+\cos\arg z}$ implies that

$$1 - \cos\arg z \leqslant 2(1-e^{-T}),$$

we have

$$|\Psi_T'(z)| \geqslant \frac{(|z|-1)(1-\cos(\arg(z)))}{6\sqrt{e^T-1}(\|\xi\|_T + |z| - 1)},$$

so the required result holds provided $B \geqslant 6$.

If T does not satisfy the conditions from Proposition 13, then there exists some $0 < S_1 < T$ such that

$$\|\xi\|_T + |z| - 1 = A^{-1}(e^{S_1}-1)^{-1/2}(1-\cos\vartheta_{S_1}^0).$$

We can write $\Psi_T(z) = \Psi_{T-S_1}(\psi_{S_1}(z))$ where ψ_{S_1} is the solution to the Loewner equation for some driving function which is bounded by $\|\xi\|_T$. Using the generic estimate above, the results of Propositions 12 and 13 applied to $\psi_{S_1}(z)$, the identity in (41), and that $|\Psi_T(z)| - 1 \leqslant d(T) \leqslant 4e^T$,

$$\begin{aligned}
|\Psi_T'(z)| &\geqslant e^{T-S_1}\frac{|\psi_{S_1}(z)|-1}{|\Psi_T(z)|-1}|\psi_{S_1}'(z)| \\
&\geqslant \frac{e^{T-S_1}(|z|-1)\tan^2(\arg(z)/2)\cot^2(\vartheta_{S_1}^0/2)}{36e^T} \\
&\geqslant \frac{(|z|-1)(1-\cos(\arg z))}{36(1-\cos\vartheta_{S_1}^0)} \\
&\geqslant \frac{(|z|-1)(1-\cos(\arg z))}{36A\sqrt{e^T-1}(\|\xi\|_T + |z| - 1)}.
\end{aligned}$$

Taking the absolute constant $B = 36A$, gives the required result. □

6 Ancestral lines and convergence for ALE

We now return to the ALE$(0,\eta)$ process and show how the bounds obtained above allow us to prove the analogue of Theorem 2 for the Φ_n maps that generate ALE$(0,\eta)$ clusters.

Without loss of generality we may set $\theta_1 = 0$. Let

$$h_k(\theta) = \frac{1}{Z_k}|\Phi_{k-1}'(e^{\boldsymbol{\sigma}+i\theta})|^{-\eta}, \quad k = 2, 3, \ldots \tag{48}$$

denote the density functions conditional on $\mathcal{F}_{k-1}$ associated with the angle sequence $\{\theta_k\}$ of the ALE$(0,\eta)$-model with model parameter $\eta \in \mathbb{R}$, particle capacity $\mathbf{c} \in (0, 1/20)$ and regularization parameter $\boldsymbol{\sigma} \in (0,1)$. As usual, let $\mathcal{F}_k$ be the $\sigma-$algebra generated by $\theta_1, \ldots, \theta_k$.

6.1 Combined derivative estimates

We first begin by using the results of Section 5 to estimate how well $|\Phi_n'(e^{\boldsymbol{\sigma}+i\theta})|$ can be approximated by $|(f_{n\mathbf{c}}^{\theta_n})'(e^{\boldsymbol{\sigma}+i\theta})|$. In Section 2, we discussed how the intermediate particles are visible in the derivative of $\Phi_n(z)$ in a way they are not in $f_{n\mathbf{c}}^{\theta_n}(z)$ (see Figure 3). The estimates below capture this discrepancy.

Lemma 15. *Fix $T > 0$, let $n \leqslant \lfloor T/\mathbf{c} \rfloor$ and set $\epsilon_n = (e^{\boldsymbol{\sigma}} - 1) \vee \sup_{k \leqslant n} |\theta_k|$.*

(i) There exists some constant A, depending only on T, such that if $|\theta - \theta_n| < \mathbf{c}^{1/2}$ and $\epsilon_n < A^{-1}\mathbf{c}^{1/2}$, then

$$\left| \left| \frac{\Phi_n'(e^{\boldsymbol{\sigma}+i\theta})}{(f_{n\mathbf{c}}^{\theta_n})'(e^{\boldsymbol{\sigma}+i\theta})} \right| - 1 \right| < A\epsilon_n^2 \mathbf{c}^{-1}. \tag{49}$$

(ii) There exists a constant B only dependent on T, such that

$$\left| \Phi_n'(e^{\boldsymbol{\sigma}+i\theta}) \right| \geqslant B^{-1}(n\mathbf{c})^{-1/2}\epsilon_n^{-1}\boldsymbol{\sigma}(1 - \cos(\theta - \theta_n)).$$

Proof. (i) By the chain rule,

$$\frac{\Phi_n'(e^{\boldsymbol{\sigma}+i\theta})}{(f_{n\mathbf{c}}^{\theta_n})'(e^{\boldsymbol{\sigma}+i\theta})} = \frac{\Phi_{n-1}'(f_{\mathbf{c}}^{\theta_n}(e^{\boldsymbol{\sigma}+i\theta}))(f_{\mathbf{c}}^{\theta_n})'(e^{\boldsymbol{\sigma}+i\theta})}{(f_{(n-1)\mathbf{c}}^{\theta_n})'(f_{\mathbf{c}}^{\theta_n}(e^{\boldsymbol{\sigma}+i\theta}))(f_{\mathbf{c}}^{\theta_n})'(e^{\boldsymbol{\sigma}+i\theta})} = \frac{\Phi_{n-1}'(f_{\mathbf{c}}^{\theta_n}(e^{\boldsymbol{\sigma}+i\theta}))}{(f_{(n-1)\mathbf{c}}^{\theta_n})'(f_{\mathbf{c}}^{\theta_n}(e^{\boldsymbol{\sigma}+i\theta}))}.$$

We set

$$w = f_{\mathbf{c}}^{\theta_n}(e^{\boldsymbol{\sigma}+i\theta}) = e^{i\theta_n} f_{\mathbf{c}}(e^{\boldsymbol{\sigma}+i(\theta-\theta_n)}).$$

Then if $|\theta - \theta_n| \leqslant \mathbf{c}^{1/2}$, by Lemma 5, we have $|w| - 1 > A'\mathbf{c}^{1/2}$ and $|\arg w - \theta_n| < A'(e^{\boldsymbol{\sigma}} - 1)$ for some absolute constant A'.

Hence, by Proposition 11, there exists some constant A, depending only on T, such that

$$\left| \left| \frac{\Phi_{n-1}'(w)}{(f_{\mathbf{c}(n-1)}^{\theta_n})'(w)} \right| - 1 \right| \leqslant A\epsilon_n^2 \mathbf{c}^{-1}.$$

Note how this argument uses that Φ_n evolves in discrete steps, allowing us to invoke Lemma 5.

(ii) The conformal map $e^{-i\theta_n}\Phi_n(ze^{i\theta_n})$ has driving function bounded by $\sup_{k \leqslant n} |\theta_k - \theta_n| \leqslant 2\epsilon_n$. Setting $z = e^{\boldsymbol{\sigma}+i(\theta-\theta_n)}$, the result follows directly from Proposition 14.

□

6.2 The ancestral lines and convergence theorem

We now use the Lemma above to prove our main result. Fix $T > 0$ and set $N = \lfloor T/\mathbf{c} \rfloor$. Recall the definition of Ω_N from Section 2.

Theorem 16. *Set* $\sigma_0 = \mathbf{c}^\gamma$ *for*

$$\gamma > \frac{2(\lambda+1)\eta+1}{2(\eta-1)},$$

where

$$\lambda = \lambda(\eta) = \begin{cases} \frac{1}{\eta-1} & \text{if } 1 < \eta < 3; \\ \frac{1}{2} & \text{if } \eta \geqslant 3. \end{cases}$$

Then

$$\lim_{\mathbf{c}\to 0} \inf_{0<\boldsymbol{\sigma}<\sigma_0} \mathbb{P}(\Omega_N) = 1 \quad \text{if } \eta > 1$$
$$\lim_{\mathbf{c}\to 0} \sup \sup_{\boldsymbol{\sigma}>0} \mathbb{P}(\Omega_N) = 0 \quad \text{if } \eta < 1.$$

Furthermore, when $\eta > 1$ *and* $\boldsymbol{\sigma} < \sigma_0$, *for any* $r > 1$,

$$\sup_{t\leqslant T} \sup_{|z|>r} |\Phi_{n(t)}(z) - f_t(z)| \to 0 \qquad \text{in probability as} \qquad \mathbf{c} \to 0,$$

and hence the cluster $K_{n(t)}$ *converges in the Hausdorff topology to a slit of capacity* t *at position* 1.

Proof. Fix $\eta > 1$ and let

$$N_T = \inf\left\{k \geqslant 1\colon |\theta_k| > \boldsymbol{\sigma} k^\lambda (\log \mathbf{c}^{-1})^{6\lambda}\right\} \wedge N. \tag{50}$$

We shall first show that $\mathbb{P}(N_T = N) \to 1$ as $\mathbf{c} \to 0$.

Suppose that $n < N_T$. Using the fact that $\boldsymbol{\sigma} < \sigma_0$, we have

$$\epsilon_n \leqslant \boldsymbol{\sigma} n^\lambda (\log \mathbf{c}^{-1})^{6\lambda} \leqslant \left(T^\lambda \mathbf{c}^{\gamma-(\lambda+1/2)} (\log \mathbf{c}^{-1})^{6\lambda}\right) \mathbf{c}^{1/2}.$$

Hence, using the fact that $\gamma > \lambda + 1/2$, there exists some $c_1 > 0$, dependent only on T and η, such that if $\mathbf{c} < c_1$, then ϵ_n satisfies the conditions of Lemma 15. From now on assume that $\mathbf{c} < c_1$. Then, by Lemma 15, there exists A_n such that, if $|\theta - \theta_n| \leqslant \mathbf{c}^{1/2}$

$$(1 - A_n)|f'_{n\mathbf{c}}(e^{\boldsymbol{\sigma}+i(\theta-\theta_n)})|^{-\eta} < |\Phi'_n(e^{\boldsymbol{\sigma}+i\theta})|^{-\eta} < (1 + A_n)|f'_{n\mathbf{c}}(e^{\boldsymbol{\sigma}+i(\theta-\theta_n)})|^{-\eta},$$

and furthermore $A_n = A_\eta \boldsymbol{\sigma}^2 \mathbf{c}^{-1} n^{2\lambda} (\log \mathbf{c}^{-1})^{12\lambda}$ for A_η that depends only on η and T.

Also,

$$\begin{aligned} \int_{\mathbb{T}} |\Phi'_n(e^{\boldsymbol{\sigma}+i\theta})|^{-\eta} \mathbf{1}_{\{\mathbf{c}^{1/2}<|\theta-\theta_n|<\pi\}} \mathrm{d}\theta &\leqslant 2B^\eta (n\mathbf{c})^{\eta/2} n^{\lambda\eta} (\log \mathbf{c}^{-1})^{6\lambda\eta} \int_{\mathbf{c}^{1/2}}^{\pi} (1 - \cos u)^{-\eta} \mathrm{d}u \\ &\leqslant B'(n\mathbf{c})^{\eta/2} n^{\lambda\eta} \mathbf{c}^{-(2\eta-1)/2} (\log \mathbf{c}^{-1})^{6\lambda\eta} \end{aligned}$$

for some B' that depends only on η and T.

We begin by getting estimates on Z_n. Using the notation of Section 2, recall from Lemma 8 that there exist A', A'' depending only on η and T such that

$$A'(n\mathbf{c})^{\eta/2} \boldsymbol{\sigma}^{-(\eta-1)} \leqslant Z^*_{n\mathbf{c}} \leqslant A''(n\mathbf{c})^{\eta/2} \boldsymbol{\sigma}^{-(\eta-1)}.$$

Hence,

$$(Z_{n\mathbf{c}}^*)^{-1}\int_{\mathbb{T}}|\Phi_n'(e^{\boldsymbol{\sigma}+i\theta})|^{-\eta}\mathbf{1}_{\{\mathbf{c}^{1/2}<|\theta-\theta_n|<\pi\}}\mathrm{d}\theta \leqslant B_\eta\boldsymbol{\sigma}^{\eta-1}n^{\lambda\eta}\mathbf{c}^{-(2\eta-1)/2}(\log\mathbf{c}^{-1})^{6\lambda\eta}$$

for some B_η that depends only on η and T. Set $B_n = B_\eta\boldsymbol{\sigma}^{\eta-1}n^{\lambda\eta}\mathbf{c}^{-(2\eta-1)/2}(\log\mathbf{c}^{-1})^{6\lambda\eta}$.

Observe that the choice of γ ensures that, provided $\boldsymbol{\sigma}<\sigma_0$, we have $\mathbf{c}^{-1/2}A_n\to 0$ and $NB_N\to 0$. We shall see that these conditions are sufficient to prove our result.

Now

$$\begin{aligned}Z_n &= \int_{\mathbb{T}}|\Phi_n'(e^{\boldsymbol{\sigma}+i\theta})|^{-\eta}\left(\mathbf{1}_{\{|\theta-\theta_n|\leqslant\mathbf{c}^{1/2}\}}+\mathbf{1}_{\{\mathbf{c}^{1/2}<|\theta-\theta_n|<\pi\}}\right)\mathrm{d}\theta\\ &\leqslant 2(1+A_n)\int_0^{\mathbf{c}^{1/2}}|f_{n\mathbf{c}}'(e^{\boldsymbol{\sigma}+i\theta})|^{-\eta}\mathrm{d}\theta + B_nZ_{n\mathbf{c}}^*\\ &\leqslant (1+A_n+B_n)Z_{n\mathbf{c}}^*.\end{aligned}$$

Similarly, we can show that

$$Z_n \geqslant (1-A_n-B_n)Z_{n\mathbf{c}}^*.$$

Since $A_n+B_n\to 0$ as $\mathbf{c}\to 0$ there exists $0<c_2\leqslant c_1$, depending only on T and η, such that $A_n+B_n<1/2$ provided $\mathbf{c}<c_2$. Assume from now on that $\mathbf{c}<c_2$. Hence, if $|\theta-\theta_n|<\mathbf{c}^{1/2}$ then,

$$(1-\alpha_n)h_{n+1}^*(\theta|\theta_n) < h_{n+1}(\theta) < (1+\alpha_n)h_{n+1}^*(\theta|\theta_n)$$

where $\alpha_n = 7(A_n+B_n)$. Equivalently

$$(1-\alpha_n)h_{n+1}^*(\theta|0) < h_{n+1}(\theta+\theta_n) < (1+\alpha_n)h_{n+1}^*(\theta|0).$$

As in Section 2, we choose to sample θ_k from the interval $[\theta_{k-1}-\pi,\theta_{k-1}+\pi)$ and so we can write $\theta_n = u_2+\cdots+u_n$ where the u_k are $[-\pi,\pi)$-valued random variables and, conditional on $\mathcal{F}_{k-1}$, $u_k=\theta_k-\theta_{k-1}$ has distribution function $h_k(\theta+\theta_{k-1})$. We write

$$\theta_n = M_n + \sum_{k=1}^n \mathbb{E}\left(u_k\mathbf{1}_{\{|u_k|\leqslant k^\lambda\boldsymbol{\sigma}(\log\mathbf{c}^{-1})^{2\lambda}\}}\middle|\mathcal{F}_{k-1}\right)+\sum_{k=1}^n u_k\mathbf{1}_{\{|u_k|>k^\lambda\boldsymbol{\sigma}(\log\mathbf{c}^{-1})^{2\lambda}\}}, \tag{51}$$

where

$$M_n = \sum_{k=1}^n\left(u_k\mathbf{1}_{\{|u_k|\leqslant k^\lambda\boldsymbol{\sigma}(\log\mathbf{c}^{-1})^{2\lambda}\}}-\mathbb{E}\left(u_k\mathbf{1}_{\{|u_k|\leqslant k^\lambda\boldsymbol{\sigma}(\log\mathbf{c}^{-1})^{2\lambda}\}}\middle|\mathcal{F}_{k-1}\right)\right)$$

is a martingale.

We first show M_n is small with high probability. By Lemma 9,

$$\begin{aligned}\mathbb{E}\left(|u_k|^2\mathbf{1}_{\{|u_k|\leqslant k^\lambda\boldsymbol{\sigma}(\log\mathbf{c}^{-1})^{2\lambda}\}}\middle|\mathcal{F}_{k-1}\right) &\leqslant (1+\alpha_{k-1})\int_{|\theta|\leqslant k^\lambda\boldsymbol{\sigma}(\log\mathbf{c}^{-1})^{2\lambda}}|\theta|^2h_k^*(\theta|0)\mathrm{d}\theta\\ &\leqslant \begin{cases}A\boldsymbol{\sigma}^2k^{(3-\eta)\lambda}(\log\mathbf{c}^{-1})^{2\lambda(3-\eta)} & \text{if } 1<\eta<3\\ A\boldsymbol{\sigma}^2(\log\mathbf{c}^{-1})^2 & \text{if } \eta\geqslant 3,\end{cases}\end{aligned}$$

for some constant A depending only on T and η. Hence M_n is a martingale with quadratic variation

$$\langle M_{n\wedge N_T}\rangle \leqslant An^{2\lambda}\boldsymbol{\sigma}^2(\log\mathbf{c}^{-1})^{4\lambda}.$$

By Bernstein's inequality, we obtain that

$$\mathbb{P}\left(|M_n| > \boldsymbol{\sigma} n^{\lambda}(\log \mathbf{c}^{-1})^{6\lambda}/2 \text{ for all } n \leqslant N_T\right) \leqslant AN \exp\left(-A(\log \mathbf{c}^{-1})^{4\lambda}\right) \to 0 \quad \text{as} \quad \mathbf{c} \to 0$$

as desired.

We next turn to the second term in (51). We use that

$$\begin{aligned}
&\mathbb{E}\left(u_k \mathbf{1}_{\{|u_k| \leqslant k^{\lambda}\boldsymbol{\sigma}(\log \mathbf{c}^{-1})^{2\lambda}\}} \middle| \mathcal{F}_{k-1}\right) \\
&= \int_{|\theta| \leqslant k^{\lambda}\boldsymbol{\sigma}(\log \mathbf{c}^{-1})^{2\lambda}} \theta h_k(\theta + \theta_{k-1}) \mathrm{d}\theta \\
&= \int_{|\theta| \leqslant k^{\lambda}\boldsymbol{\sigma}(\log \mathbf{c}^{-1})^{2\lambda}} \theta h_k^*(\theta|0) \mathrm{d}\theta + \int_{|\theta| \leqslant k^{\lambda}\boldsymbol{\sigma}(\log \mathbf{c}^{-1})^{2\lambda}} \theta(h_k(\theta + \theta_{k-1}) - h_k^*(\theta|0)) \mathrm{d}\theta \\
&= \int_{|\theta| \leqslant k^{\lambda}\boldsymbol{\sigma}(\log \mathbf{c}^{-1})^{2\lambda}} \theta(h_k(\theta + \theta_{k-1}) - h_k^*(\theta|0)) \mathrm{d}\theta,
\end{aligned}$$

by the symmetry of $h_k^*(\theta|0)$. Hence, by a similar computation to that in Lemma 9,

$$\begin{aligned}
\left|\mathbb{E}\left(u_k \mathbf{1}_{\{|u_k| \leqslant k^{\lambda}\boldsymbol{\sigma}(\log \mathbf{c}^{-1})^{2\lambda}\}} \middle| \mathcal{F}_{k-1}\right)\right| &\leqslant \int_{|\theta| \leqslant k^{\lambda}\boldsymbol{\sigma}(\log \mathbf{c}^{-1})^{2\lambda}} |\theta||h_k(\theta + \theta_{k-1}) - h_k^*(\theta|0)| \mathrm{d}\theta \\
&\leqslant \alpha_{k-1} \int_{|\theta| \leqslant k^{\lambda}\boldsymbol{\sigma}(\log \mathbf{c}^{-1})^{2\lambda}} |\theta| h_k^*(\theta|0) \mathrm{d}\theta \\
&\leqslant \begin{cases} A\alpha_{k-1}\boldsymbol{\sigma} k^{(2-\eta)\lambda}(\log \mathbf{c}^{-1})^{2\lambda(2-\eta)} & \text{if } 1 < \eta < 2 \\ A\alpha_{k-1}\boldsymbol{\sigma}(\log \mathbf{c}^{-1})^2 & \text{if } \eta \geqslant 2, \end{cases}
\end{aligned}$$

for some constant A depending only on T and η. Therefore, if $1 < \eta < 2$,

$$\begin{aligned}
\left|\sum_{k=1}^{n} \mathbb{E}\left(u_k \mathbf{1}_{\{|u_k| \leqslant k^{\lambda}\boldsymbol{\sigma}(\log \mathbf{c}^{-1})^{2\lambda}\}} \middle| \mathcal{F}_{k-1}\right)\right| &\leqslant A\boldsymbol{\sigma}(\log \mathbf{c}^{-1})^{2\lambda(2-\eta)} \sum_{k=1}^{n} \alpha_{k-1} k^{(2-\eta)\lambda} \\
&\leqslant \boldsymbol{\sigma} n^{\lambda}(\log \mathbf{c}^{-1})^{6\lambda} \left(An^{-(\eta-1)\lambda}\alpha_n(\log \mathbf{c}^{-1})^{-2\lambda(1+\eta)}\right),
\end{aligned}$$

and if $\eta \geqslant 2$,

$$\begin{aligned}
\left|\sum_{k=1}^{n} \mathbb{E}\left(u_k \mathbf{1}_{\{|u_k| \leqslant k^{\lambda}\boldsymbol{\sigma}(\log \mathbf{c}^{-1})^{2\lambda}\}} \middle| \mathcal{F}_{k-1}\right)\right| &\leqslant A\boldsymbol{\sigma}(\log \mathbf{c}^{-1})^{2} \sum_{k=1}^{n} \alpha_{k-1} \\
&\leqslant \boldsymbol{\sigma} n^{\lambda}(\log \mathbf{c}^{-1})^{6\lambda} \left(An^{1-\lambda}\alpha_n(\log \mathbf{c}^{-1})^{-2(3\lambda-1)}\right).
\end{aligned}$$

By our choice of γ, there exists $0 < c_3 \leqslant c_2$, depending only on T and η, such that

$$\left|\sum_{k=1}^{n} \mathbb{E}\left(u_k \mathbf{1}_{\{|u_k| \leqslant k^{\lambda}\boldsymbol{\sigma}(\log \mathbf{c}^{-1})^{2\lambda}\}} \middle| \mathcal{F}_{k-1}\right)\right| < \boldsymbol{\sigma} n^{\lambda}(\log \mathbf{c}^{-1})^{6\lambda}/2$$

provided $\mathbf{c} < c_3$. From now on assume that $\mathbf{c} < c_3$.

Finally, we deal with the last term in (51). The same computation as used to bound Z_n can be used to show that

$$\mathbb{P}(|u_k| \geqslant \mathbf{c}^{1/2};\ k \leqslant N_T) \leqslant B_k.$$

We also have

$$\begin{aligned}\mathbb{P}(k^\lambda \boldsymbol{\sigma}(\log \mathbf{c}^{-1})^{2\lambda} < |u_k| \leqslant \mathbf{c}^{1/2}) &\leqslant A(1+\alpha_{k-1}) \int_{k^\lambda \boldsymbol{\sigma}(\log \mathbf{c}^{-1})^{2\lambda}}^{\mathbf{c}^{1/2}} \frac{1}{\boldsymbol{\sigma}} \left(1+\left(\frac{\theta}{\boldsymbol{\sigma}}\right)^2\right)^{-\eta/2} \mathrm{d}\theta \\ &\leqslant A \int_{k^\lambda (\log \mathbf{c}^{-1})^{2\lambda}}^{\infty} \left(1+\theta^2\right)^{-\eta/2} \mathrm{d}\theta \\ &\leqslant A k^{-\lambda(\eta-1)} (\log \mathbf{c}^{-1})^{-2\lambda(\eta-1)}.\end{aligned}$$

Hence, putting these two bounds together,

$$\begin{aligned}&\mathbb{P}\left(\sum_{k=1}^{n} u_k \mathbf{1}_{\{|u_k|>k^\lambda \boldsymbol{\sigma}(\log \mathbf{c}^{-1})^{2\lambda}\}} \neq 0 \text{ for some } n \leqslant N_T\right) \\ \leqslant &\mathbb{P}(|u_k| > k^\lambda \boldsymbol{\sigma}(\log \mathbf{c}^{-1})^{2\lambda} \text{ for some } k \leqslant N_T) \\ \leqslant &A \sum_{k=1}^{N} \left(k^{-\lambda(\eta-1)} (\log \mathbf{c}^{-1})^{-2\lambda(\eta-1)} + B_k\right) \\ \leqslant &A \left((\log \mathbf{c}^{-1})^{-1} + N B_N\right) \to 0\end{aligned}$$

since $\boldsymbol{\sigma} < \sigma_0$.

But on the high probability event

$$\left\{|M_n| < \boldsymbol{\sigma} n^\lambda (\log \mathbf{c}^{-1})^{6\lambda}/2 \text{ for all } n \leqslant N_T\right\} \cap \left\{\sum_{k=1}^{n} u_k \mathbf{1}_{\{|u_k|>k^\lambda \boldsymbol{\sigma}(\log \mathbf{c}^{-1})^{2\lambda}\}} = 0 \text{ for all } n \leqslant N_T\right\},$$

$$\sup_{n \leqslant N_T} |\theta_n| < \boldsymbol{\sigma} n^\lambda (\log \mathbf{c}^{-1})^{6\lambda}$$

and hence $N_T = N$. Furthermore, it is immediately clear that $\Omega_N \subset \{N_T = N\}$ and therefore, we have shown that if $\eta > 1$,

$$\lim_{\mathbf{c}\to 0} \inf_{0<\boldsymbol{\sigma}<\sigma_0} \mathbb{P}(\Omega_N) = 1.$$

Exactly the same argument as Theorem 2 can be used to show that

$$\lim_{\mathbf{c}\to 0} \sup \sup_{\boldsymbol{\sigma}>0} \mathbb{P}(\Omega_N) = 0$$

if $\eta < 1$, and that when $\eta > 1$ and $\boldsymbol{\sigma} < \sigma_0$, for any $r > 1$,

$$\sup_{t \leqslant T} \sup_{|z|>r} |\Phi_{n(t)}(z) - f_t(z)| \to 0 \quad \text{in probability as} \quad \mathbf{c} \to 0,$$

and hence the cluster $K_{n(t)}$ converges in the Hausdorff topology to a slit of capacity t at 1. □

6.3 Modifications of the model

One criticism that can be levelled at the ALE$(0, \eta)$ model, from the point of view of modelling physical phenomena, is that the conformal mappings distort the sizes of particles as they are added to the growing cluster. Using the result proved above that the scaling limit of the ALE$(0, \eta)$

cluster is a growing slit, it can be shown that the size of the nth particle is approximately equal to $d(\mathbf{c}n) - d(\mathbf{c}(n-1))$. Using the expression for $d(t)$ in (9), we obtain

$$d(\mathbf{c}n) - d(\mathbf{c}(n-1)) \asymp \begin{cases} \frac{2\mathbf{c}^{1/2}}{n^{1/2}+(n-1)^{1/2}} & \text{if} \quad \mathbf{c}n \ll 1; \\ 2\mathbf{c}e^{\mathbf{c}n} & \text{if} \quad \mathbf{c}n \gg 1. \end{cases}$$

In particular, the first particle is of size approximately $2\mathbf{c}^{1/2}$, whereas all subsequent particles are strictly smaller.

A number of modifications to the model are possible which result in clusters where all of the particles are roughly the same size. The simplest modification is to recursively choose a deterministic sequence of capacities with $c_1 = \mathbf{c}$ and c_n satisfying

$$d(C_n) - d(C_{n-1}) = d(\mathbf{c}) \quad \text{where} \quad C_n = \sum_{j=1}^{n} c_j.$$

Another modification (see [6, 21]) is to take the capacity of the nth particle to be

$$c_n = \mathbf{c}|\Phi'_{n-1}(e^{\tilde{\boldsymbol{\sigma}}+i\theta_n})|^{-2}$$

for some regularization parameter $\tilde{\boldsymbol{\sigma}} > 0$, not necessarily equal to the angular regularization parameter $\boldsymbol{\sigma}$. Closely related (see [1, 28]), is to choose capacity c_n corresponding to slit length

$$d_n = \inf\{d > 0 : d|\Phi'_{n-1}((1+d)e^{i\theta_n})| = d(\mathbf{c})\}.$$

In each of these modified models, the total capacity of the cluster no longer grows linearly in the number of particles and is potentially random. It is therefore necessary to modify the timescale in which to obtain scaling results. More precisely, given some fixed $T > 0$, let

$$n(t) = \sup\{n : C_n < t\} \quad \text{for} \quad t \leqslant T,$$

and set $N = n(T)$. The event Ω_N can then be defined as before.

It is relatively straightforward to verify that the proof and conclusion of Theorem 16 still hold for these modified models (and further generalisations). We only state the modified result for $\eta > 1$, as the case $\eta < 1$ is identical to that for the Markov model, for any choice of capacity sequence.

Corollary 17. *For $\eta > 1$ and $\mathbf{c} > 0$, define σ_0 as in Theorem 16 and take $\boldsymbol{\sigma} < \sigma_0$. Consider a sequence of conformal mappings, constructed as in* (2) *from sequences $\{\theta_k\}_{k=1}^{\infty}$ and $\{c_k\}_{k=1}^{\infty}$, where (without loss of generality) $\theta_1 = 0$ and, conditional on $\mathcal{F}_{n-1} = \sigma(\theta_k, c_k : 1 \leqslant k \leqslant n-1)$, θ_n are given by* (4)*.*

Provided there exists some constant $A > 0$, depending only on T and η, such that

$$\mathbb{P}(c_k \geqslant A\mathbf{c} \textit{ for all } k = 1, \dots N) \to 1$$

as $\mathbf{c} \to 0$, it holds that $\mathbb{P}(\Omega_N) \to 1$ as $\mathbf{c} \to 0$. Furthermore, such a constant A exists for the three modifications defined above as well as for ALE(α, η) *for any $\alpha > 0$.*

In this case, for any $r > 1$,

$$\sup_{t \leqslant T} \sup_{|z|>r} |\Phi_{n(t)}(z) - f_t(z)| \to 0 \quad \textit{in probability as} \quad \mathbf{c} \to 0,$$

and hence the cluster $K_{n(t)}$ converges in the Hausdorff topology to a slit of capacity t at position 1*.*

Proof. The proof consists of checking step by step that each inequality in the proofs of Lemma 15 and Theorem 16 still holds (possibly with new constants). The only changes are that we compare Φ_n to

$$f_{C_n}^{\theta_n} = f_{c_1}^{\theta_n} \circ \cdots \circ f_{c_n}^{\theta_n}$$

instead of $f_{\mathbf{c}n}^{\theta_n}$ and we need to define

$$N_T = \inf\left\{k \geqslant 1 \colon |\theta_k| > \boldsymbol{\sigma} k^{\lambda}(\log \mathbf{c}^{-1})^{6\lambda} \text{ or } c_k < A\mathbf{c}\right\} \wedge N$$

and then use the additional assumption in the statement of the corollary to show that $N_T = N$ with high probability.

To show that the additional assumption holds for the modified models defined above, it is enough to show that, so long as $n \leqslant N_T$, there exists some constant A (depending only on T and η), such that

$$|\Phi'_{n-1}(e^{\tilde{\boldsymbol{\sigma}}+i\theta_n})|^{-1} > A.$$

But this follows by using the (analogous) estimates in Lemma 15 for the modified model and observing that there exists some constant A' (depending only on T) such that

$$|f'_t(z)| < A'.$$

whenever $|\arg(z)| \leqslant \beta_t/2$ and $t \leqslant T$. □

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