Accepted Manuscript

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PII: S0165-1889(18)30095-2
DOI: 10.1016/j.jedc.2018.03.004
Reference: DYNCON 3572

To appear in: *Journal of Economic Dynamics & Control*

Received date: 12 August 2017
Revised date: 27 January 2018
Accepted date: 18 March 2018

Please cite this article as: Nicolas de Roos, Alexander Matros, Vladimir Smirnov, Andrew Wait, Shipwrecks and treasure hunters, *Journal of Economic Dynamics & Control* (2018), doi: 10.1016/j.jedc.2018.03.004

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Shipwrecks and treasure hunters*

By Nicolas de Roos†, Alexander Matros‡, Vladimir Smirnov† and Andrew Wait†

March 21, 2018

Abstract

We examine dynamic search as a game in which two rivals explore (an island) for a hidden prize of known value. In every period until its discovery, the players decide how much of the unsearched area to comb. If a player finds the prize alone he wins it and the game ends. Players have a per-period discount factor and costs proportional to the area they search. First, as a benchmark for efficiency, we solve the one-player search problem. Second, in the two-player setting we show that typically there is inefficient over-search – a result akin to the tragedy of the commons. However, for players with intermediate levels of patience, there is the possibility of inefficient under-search as players incorporate the expected future payoffs in their current search decisions. Finally, with patient players, several counterintuitive results can arise: for example, players might be better off searching a larger island or looking for a less valuable prize. *Keywords*: R&D, search, uncertainty.
JEL classifications: D21; D32; O32.

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1 Introduction

In 1708 the *San José*, a Spanish galleon, was sunk in a battle with British warships near Barú, off the Colombian coast, carrying gold, silver and gems from the New World. Some three hundred years later, the Colombian government announced they located the shipwreck in December 2015, in a find estimated to be worth $1bn. Even if the potential bounties involved are smaller, the seas are littered with sunken treasure ships, such as; the S.S. Central America, sunk in 1857 along with 15 tons of gold off in the Carolinas; the S.S. Republic which sank with 51,000 U.S. silver and gold coins off the Georgian coast in 1865 (discovered in 2003); and the Whydah Gally, the 18th century galley discovered in 1984

*We would like to thank Murali Agastya, Natalia Ponomareva, Mark Melatos, Kunal Sengupta and Don Wright for their helpful comments.

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with loot valued at more than $400 million. Often these shipwreck treasure troves have similar characteristics. Firstly, while it might be broadly known where the ship sank – like a specific coastline, sea, or in the vicinity of a particularly treacherous pass – the exact location of shipwreck is unknown. This means, consequently, that there is uncertainty about the total search costs involved in discovering the wreck. Secondly, there is often quite accurate information about the value of the sunken treasure, with historical records detailing how much gold or other valuable items were on board; for example, the *San José* is thought to have been carrying 11 million 8-escudos gold coins. Thirdly, given the potential prize, often several interested rival teams are in a race to make the same find, each with the hope of being the first to discover the bounty and claim the prize.

Beyond looking for sunken shipwrecks laden with treasure, strategic search is ubiquitous: companies look for new profitable products and markets; prospectors search for unexploited deposits; and pharmaceutical companies engage in R&D in the hope that they discover the next generation of drugs. We analyze a dynamic model in which two players search for a treasure hidden somewhere in a given area (on an island or in a particular sea). The value of the treasure is common knowledge, and search is costly. Once the treasure is found the game ends. In each period, both players make their search decisions simultaneously. If the treasure is not discovered the game continues to the next period in which the players have to decide how much of the unsearched area to explore. If the two players find the treasure simultaneously (by both searching the correct location), whilst each incurs their own costs, the treasure is destroyed resulting in a payoff of zero for both parties.[1] We consider the case when search is observable; that is, players are informed about the areas that have already been searched by their opponents. For tractability, we also restrict the total search cost to be not more than the value of the treasure.

In this setup, each state can be described by the remaining unsearched area. While there could be multiple subgame perfect equilibria in the game, we show that, except for possibly one point, the symmetric Markov perfect equilibrium (SMPE) is unique.[2] Within this framework, we compare the two-player SMPE to the efficient search outcome (with one player). Relative to a single-player, multi-player search is typically inefficient, except for very small islands, when players behave as a cartel and search lasts just one period.[3] In the case of slightly larger but still small islands, multiple players search too fast, akin to the *standard tragedy of the commons*; players search too much in the current period, because the probability of finding a treasure is relatively high and each player does not consider how their search imposes a spillover cost on their rival. This leads to inefficient over-search. For example, in a race to be the first to discover a sunken treasure ship, each team might

[1] This assumption is standard in the R&D literature. Intuitively, if several players discover the treasure simultaneously, fierce competition (or a legal dispute) runs down the surplus to zero.

[2] As well as simplifying our analysis, imposing Markov perfection makes our results directly comparable to those in the previous literature. See Maskin and Tirole (1988), Bhaskar et al. (2010) and Battaglini et al. (2014) for a general discussion of why the use of SMPE is appropriate. The focus on symmetric MPE is supported by the fact that expected combined value in the asymmetric MPE is never more than in the SMPE; see Section 4.

[3] The rivals' search is efficient not only for small island sizes but also when the cost of searching the entire unexplored area is exactly equal to the value of the treasure, and the players have a discount factor that is sufficiently low. We discuss this case in Section 3.3.

be tempted to establish an intensive search program. While this can hasten discovery, it also dissipates potential rents in the process. In the case of large islands, the incentive to search more in the current period is not as strong. In fact, with a large search area the parties can even inefficiently under-search. The intuition for this is that with a large island, the parties shade their search, in the hope to free-ride on their rival's unsuccessful attempts. Moreover, the incentive to search in early periods could be further dampened if both parties anticipate the possibility of a competitive 'search war', which could arise as the unsearched area shrinks (and the island effectively becomes small). Note that this is a new result; in the present model, in contrast to much of the existing literature, both over- and under-search can endogenously arise along the equilibrium path within the same project.

As search is costly, it seems natural to conjecture that a smaller island (which lowers expected total search costs) is better than a bigger island for all players. This is not necessarily the case; in fact, expected payoffs can increase with the potential search area. If the island is of moderate size (not so small that all of the island is searched immediately), the tragedy of the commons effect is strong, and players over-search. If the island area is increased, the tragedy of the commons effect is diminished, and players search the island more efficiently. It turns out that this efficiency improvement may be large enough to outweigh the increase in the cost of searching the larger island. Similar non-monotonic results apply to the other key parameters as well; players can be better off with a smaller prize or with higher search costs. These results have implications for the design of search competitions for new pharmaceutical products or other innovations, and for mineral exploration.[4] The potential non-monotonicities could also explain why search for a shipwreck, for instance, can wax and wane over time.

The focus in this paper is on the dynamics of investment in relation to search for a private good. Previous work on private goods mostly deals with situations that are either static or involve complete information.[5] The typical outcome of these models is that firms overinvest. One of the few papers that considers the dynamics of investment is Reinganum (1981), who shows that in a dynamic R&D race where each of the two firm chooses a time path of expenditures, firms can either underinvest or overinvest (but not both in the same game) as compared with the choice of a single searcher. Aggregate expenditure on R&D may therefore be either too high or too low relative to the efficient outcome depending on the exogenous parameters. In her model, Reinganum assumes that the success function is exponential and the environment is stationary;[6] this means that equilibrium strategies can be represented as functions of time only, simplifying the analysis.[7]

There are many situations, however, where the memorylessness assumption is not sat-

[4] Examples of such research contests include the six Millennium Prize Problems, a 1992 refrigerator competition (see Taylor, 1995), a 1829 steam locomotion tournament (see Fullerton and McAfee, 1999), a 1714 British contest for a method of determining longitude at sea (see Che and Gale, 2003).

[5] See, for example, Loury (1979), Dasgupta and Stiglitz (1980a,b) and Lee and Wilde (1980), and Reinganum(1989) and Long (2010) for a survey of the literature.

[6] In the context of our model, this would be equivalent to assuming that the present value of the treasure is constant.

[7] Reinganum (1982) uses a similar framework to show how the availability of patent protection can accelerate development of the innovation.

isfactory; it is the welfare analysis of two-player search in a dynamic environment that is a key contribution of the model in this paper. The setup we use here is similar to Matros and Smirnov (2016), who analyze duplication in search.[8] They find that with independent search, all rents are dissipated. Here, our focus is on (in)efficiency, not duplication. We find that, while search is typically not efficient, not all rents are necessarily dissipated, and that both over- and under-search can be present. Furthermore, we find non-monotonic results with respect to island size, costs and the value of the prize.

It is worth mentioning the related dynamic model literature on public-good contributions, such as Georgiadis (2014).[9] In that model, a public-good project gradually progresses towards completion as agents make contributions. Despite the ever-present free-rider problem, members of a larger team work harder than those of a small team if and only if the project is sufficiently far from completion. Similar comparative statics can arise in our setting, although in our private-good framework both under- and over-investment are possible.

Finally, if there is a restriction that players in our model may only search the island in one period (a static model), our equilibrium results are equivalent to those in the contest (rent-seeking) game of Tullock (1980). This equivalence no longer holds when players are allowed to search in multiple periods in our dynamic setup as the searchable island size shrinks over time. The focus of the sequential rent-seeking literature is somewhat different, instead analyzing the Stackelberg leader-and-follower interaction where the players choose their (observable) one-shot rent-seeking expenditures sequentially.[10] In contrast, players in our game choose their investments in any given period simultaneously, but can invest in multiple search periods (if necessary).

2 The Model

Over an infinite horizon, $n = 1$ or $n = 2$ players (firms) search for a treasure hidden somewhere on an island of size $x_1 > 0$. For exposition, we concentrate on the $n = 2$ case. The treasure has the same value $R > 0$ for each player[11], and any particular point on the island is equally likely to be where the treasure is hidden. Payoffs are discounted at the common factor $\delta \in [0, 1]$. Throughout, we use the term 'area' to indicate the size of land searched and the term 'region' to indicate the location of the search.

[8]Two other papers that focus on duplicative search are Chatterjee and Evans (2004) and Fershtman and Rubinstein (1997). In Chatterjee and Evans (2004), in each period players decide whether to research one or two project, where only one project will eventually be successful. Inefficient duplication arises in that the players can search the same project too much or too little. In Fershtman and Rubinstein (1997) two players search for a treasure hidden in a finite number of boxes. Duplication arises as a rival's past search is unobservable. Also see Erat and Krishna (2012) and Konrad (2014).

[9]Also see, for example, Admati and Perry (1991), Marx and Matthews (2000), Lockwood and Thomas (2002), Compte and Jehiel (2004), Yildirim (2004, 2006), Bonatti and Hörner (2011), Matthews (2013) and Battaglini et al. (2014).

[10]For example, see Dixit (1987), Baik and Shogren (1992), Leininger (1993), Linster (1993), Glazer and Hassin (2000), Morgan (2003), Konrad (2009).

[11]We assume players are risk neutral, which implies that R could be replaced by $E[R]$ if returns are random, requiring no other changes to the model.

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In each period t, the following events unfold. First, each player learns the area and region of all prior searches.[12] Let x_t be the remaining unsearched area at the beginning of period t. Second, each player i simultaneously chooses an area $I_{i,t} \in [0, x_t]$ to search at a cost of $cI_{i,t}$, where $c > 0$. Finally, search plans are implemented. If the treasure is found, the game ends. If multiple players find the treasure simultaneously, each player incurs costs, but the treasure is destroyed.[13]

We assume that players are able to coordinate to avoid simultaneous search of the same region. Coordination might be feasible if there are some natural regions for each party to search (close to their home base, for example). It could also be possible for the players to coordinate when it is visibly evident to each party where its rival is searching if physical equipment is required, or a party can be tracked through some GPS system. Similarly, research institutes might be able to relatively easily ascertain their rival's research plans from the experts they hire and from large ticket-items of scientific hardware they purchase. Further to this, there might be some third party (regulator) implicit coordination of search. For instance, a regulator might require some official registration of exploration intent that aids coordination; a public announcement of grant recipients could aid in this process. Another economic scenario that allows the parties to coordinate could be as follows. Consider that each period is made up of several 'sub-periods'. At the start of each period both firms have to simultaneously decide on the size of their research budget and where they will search. Each firm's choice of research budget is publicly available to everyone (possibly through required public announcements for publicly listed firms). Given these budgetary commitments (which determine the area each firm searches in that period), one firm has an opportunity to reveal its search plans (the region of search). Following this disclosure, the other firm can adjust its region of search (but not the total area that it searches). As finding treasure simultaneously is costly, it is in the interests of the two rivals to coordinate in an attempt to avoid duplication. This process of revelation of search choices through sub-periods makes it possible for the firms to perfectly coordinate their search activities. This setup allows us to focus on the inefficiencies that can arise from the strategic interaction between the rivals, without the additional complications inherent in duplication.

If in any period t, each player i searches $I_{i,t}$, the two players together search an area of

$$J_t = \min \{x_t, I_{1,t} + I_{2,t}\} . \tag{1}$$

The expected payoff to player i in period t is

$$\frac{J_t - I_{j,t}}{x_t} R - cI_{i,t},$$

where $(J_t - I_{j,t})/x_t$ is the probability of finding the treasure alone. With probability J_t/x_t, the game ends in period t. With probability $1 - J_t/x_t$, the treasure is not found, the

[12] We follow Fudenberg and Tirole (1985) by considering a model of observable actions.

[13] This assumption, standard in the R&D literature, reflects the fierce competition that ensues between simultaneous claimants of the prize.

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unsearched area of the island shrinks to

$$x_{t+1} = x_t - J_t,$$

and the game continues in period $t + 1$.

We consider Markov strategies in which prior search influences current play only through its effect on the current unsearched area. A pure Markov strategy for player i is a time-invariant map $I_i : X \to X$, where $X = [0, x_1]$ and $I_i(x) \in [0, x]$. We restrict attention to symmetric equilibria, using *symmetric Markov perfect equilibrium* (SMPE) as the solution concept.[14] Moreover, we focus on *non-trivial* SMPE; that is, equilibria in which a positive amount of search occurs somewhere along the equilibrium path. Reflecting our consideration of symmetric Markov strategies, we omit the time subscript and indicate the search intensity function for firm i as $I_{i,t} = I_i(x)$. Where the meaning is clear, we also omit the subscript i in describing the equilibrium search and value functions.[15]

To solve for an SMPE we use the following approach. Given the state-contingent search plans of his rival, $I_j(x)$, player i solves a standard optimization problem, yielding the optimal search policy $I_i(x)$. Because the function $I_j(x)$ is endogenous, to identify a symmetric equilibrium we need to find the function $I_j(x)$ such that $I_i(x) \equiv I_j(x)$, where $I_i(x)$ is the optimal search for player i when he takes I_j as given.

3 Analysis of the Model

Let us now analyze the equilibrium of the model. Following some preliminaries, in Section 3.1 we examine the problem of a single player, which in our setup is equivalent to the problem of a social planner. In Section 3.2, we consider the two-player problem. In Section 3.3, we discuss welfare issues. As a convention, we use superscripts in parentheses to indicate the number of searching players. For example, a superscript of (1) indicates there is one player searching.

We first establish several preliminary results. We begin with the Bellman equation faced by player i.

$$V(x, c, R) = \max_{I_i \in [0,x]} \left\{ -cI_i + \frac{J - I_j}{x} R + \delta \left(1 - \frac{J}{x}\right) V(x - J, c, R) \right\}, \tag{2}$$

where x is the area of the island that is unsearched before the current period, $V(x, c, R)$ is the value function for each player (we use the symmetry assumption here), and J is the aggregate search this period, given by equation (1). In equation (2): the first term is player i's current search costs; the second term describes her expected value from a successful current search; and the last term is her expected discounted value from future search.

The following lemma is a useful normalization that will be utilized in the analysis below.

[14] In Section 4 we discuss asymmetric Markov perfect equilibria while alternative equilibrium concepts are discussed in Section 5.

[15] We define equilibria in terms of the functions I_1 and I_2 that describe the area but not the region of search. An equilibrium is therefore consistent with a multiplicity of outcomes relating to the search region.

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Lemma 1. *(Matros and Smirnov(2016)) The following is an identity:*

$$V(x,c,R)/R = V(cx/R,1,1). \tag{3}$$

Lemma 1 implies that we can normalize $R = c = 1$ when solving (2) without loss of generality. Using this result, we relabel $V(x) = V(x,1,1)$ and rewrite

$$V(x) = \max_{I_i \in [0,x]} \left\{ -I_i + \frac{J - I_j}{x} + \delta \left(1 - \frac{J}{x}\right) V(x - J) \right\}. \tag{4}$$

The Lemma allows us to conduct comparative-static analysis as follows. An increase in the cost parameter c in the original problem (2) is equivalent to an increase in the unsearched area x in problem (4). An increase in the treasure value R has two effects; it decreases the unsearched area x, and directly changes the value function. We discuss comparative statics with respect to x, c and R in Sections 3.1 and 3.2. In terms of the new value function, the assumption $cx_1 \le R$ implies $x_1 \le 1$.

3.1 Single-player search

In this section we give a complete characterization of the unique solution to problem (4) when $n = 1$. Because costs are linear, the single-player case is equivalent to the multi-player scenario where the search of all players is coordinated by a social planner. We first define the polynomials $I_s^{(1)}(x)$, $P_{(s)}^{(1)}(x)$, and critical island sizes $\chi_s^{(1)}$,

$$I_{(s)}^{(1)}(x) = (x-1)(1 - \frac{\sin{(s-1)\varphi}}{\delta^{1/2}\sin s\varphi}) + \frac{\delta^{(s-1)/2}\sin\varphi}{\sin s\varphi},$$

$$P_{(s)}^{(1)}(x) = -\frac{\sin{(s+1)\varphi}}{2\sin s\varphi}\delta^{-1/2}(1-x)^2 + \frac{\sin\varphi}{\sin s\varphi}\delta^{(s-1)/2}(1-x) + \frac{\sin{(s-1)\varphi}}{2\sin s\varphi}\delta^{(2s-1)/2},$$

$$\chi_s^{(1)} = 1 - \delta^{s/2}\cos s\varphi, \quad \varphi = \arccos\sqrt{\delta}, \quad \text{for } s \in \{1,\ldots,m\}, \quad \text{where } m = \lceil \pi/(2\varphi) \rceil.$$

We derive the following theorem.

Theorem 1. *When $n = 1$ and $x \le 1$, the unique solution to problem (4) is described as follows. The search intensity is*[16]

$$I^{(1)}(x) = \begin{cases} I_{(1)}^{(1)}(x), & \text{if} \quad x \le \chi_1^{(1)}, \\ \vdots \\ I_{(s)}^{(1)}(x), & \text{if} \quad \chi_{s-1}^{(1)} < x \le \chi_s^{(1)}, \\ \vdots \\ I_{(m)}^{(1)}(x), & \text{if} \quad \chi_{m-1}^{(1)} < x \le 1; \end{cases}$$

[16]For $s \in \{1,\ldots,m\}$, $\sin s\varphi \le 0$ is not possible because it implies that the optimal sequence includes less than s steps; for details see Appendix A.

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and the value function is

$$V^{(1)}(x) = \begin{cases} P^{(1)}_{(1)}(x)/x, & \text{if} \quad x \leq \chi^{(1)}_1, \\ \vdots & \\ P^{(1)}_{(m)}(x)/x, & \text{if} \quad \chi^{(1)}_{m-1} < x \leq 1. \end{cases}$$

Proof. See Appendix A.

We indicate the maximum number of remaining periods of search with a subscript in parentheses. For example, $I^{(1)}_{(3)}(x)$ describes the search intensity of a single player when search will be concluded within three periods.

The optimal strategy for a single player is to plan to search the island for at most m periods, where m is an increasing function of δ. There are m possibilities, depending on the island size. First, for sufficiently small islands, $x \leq \chi^{(1)}_1$, the single player searches the entire island in a single period to avoid the possibility of waiting an extra period to find the prize; that is $I^{(1)}_{(1)}(x) = x$. The second possibility is that the firm plans to search the island for at most two periods. This happens for larger values of $x \in (\chi^{(1)}_1, \chi^{(1)}_2]$. In general, the single player plans to search the island for at most s periods for values of $x \in (\chi^{(1)}_{s-1}, \chi^{(1)}_s]$, where $s = 1, \ldots, m-1$. Finally, the firm plans to search the island for at most m periods when $x \in (\chi^{(1)}_{m-1}, 1]$. Overall, this means the optimal search intensity $I^{(1)}(x)$ is a spline of degree one on the interval $[0, 1]$ with knots $\chi^{(1)}_1, \ldots, \chi^{(1)}_{m-1}$.[17] We characterize the optimal search intensity $I^{(1)}(x)$ in the following corollary.

Corollary 1. *When $n = 1$ and $x \leq 1$, (a) the optimal search intensity $I^{(1)}(x)$ is a piece-wise linear, continuous and concave (hump-shaped) function; and (b) $I^{(1)}_{(s)}(x) \geq x/s$.*

Proof. See Appendix A.

The optimal search profile depends on a trade-off between impatience and the total cost of search. By Theorem 1, the single player will find the treasure in finite time. Impatience provides an incentive for more aggressive search in the current period, in order to bring forward the likely time of discovery. For $x \leq \chi^{(1)}_1 = 1 - \delta$, impatience drives the single player to search the entire island to obtain the prize immediately. On the other hand, consider the countervailing incentive. Recall that the searcher must commit to a search plan at the beginning of each period. This leads to the possibility that the player discovers the treasure with a higher level of planned search activity in that period when a smaller quantity of effort would have sufficed. By searching less intensively in the current period, she reduces the likely total cost of search.

Figure 1 illustrates a single player's optimal search intensity $I^{(1)}(x)$ for islands of size $x \in [0, 1]$ and discount factors $\delta = 0.4$ and $\delta = 0.9$. Panel (a) illustrates the area searched in the current period, and Panel (b) illustrates the fraction of the island searched in the current

[17] A spline is a special function defined piecewise by polynomials, see for example Ahlberg, Nielson, and Walsh (1967). The number of subintervals of the spline varies from $m = 2$ when $\delta \leq 1/2$, to $m \to \infty$ as $\delta \to 1$.

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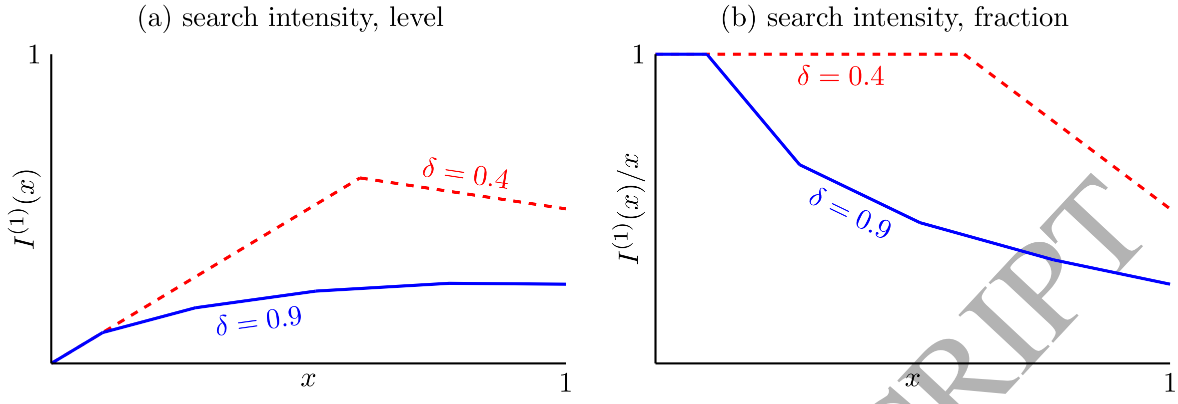
Figure 1: The optimal search intensity for a single searcher



period. The kinks in each line represent the knots, $\chi_s^{(1)}$, that determine the maximum number of searches for each island size. Comparison of the two lines highlights the role of impatience. For lower δ, the player searches more aggressively in the current period, conditional on any island size. For $\delta = 0.4$, she anticipates completing her search within two periods, while for $\delta = 0.9$, her search could last up to five periods.

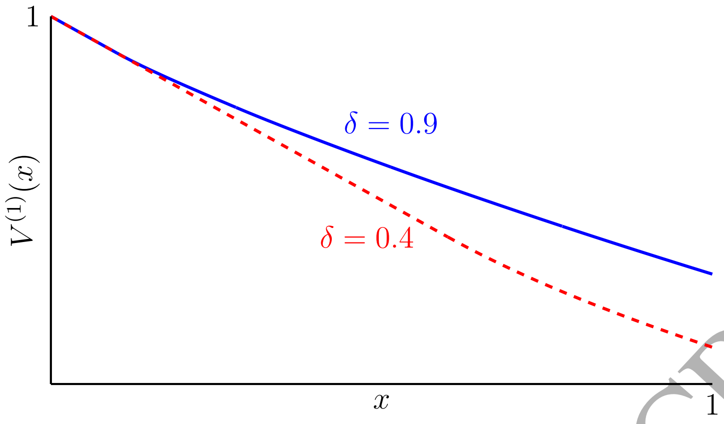
Now, let us consider the shape of the search profile when $\delta = 0.4$. For $x \leq \chi_1^{(1)}$, search intensity is directly constrained by the size of the island, and therefore increases one-for-one with island size. For $x > \chi_1^{(1)}$, out of concern for the total cost of search, the single player plans to split her search over two periods. Due to her impatience, she always plans to search more than half the island in the initial period – see Panel (b). However, as shown in Panel (a), the slope of her search intensity profile is actually negative for larger islands. This is because the probability of finding the treasure in the current period declines with the size of the island, and therefore the expected current payoff of search also declines with x. For $\delta < 0.5$ and $x > 1 - \delta$, this results in a decreasing intensity profile. For the extreme case of $x = 1$, the expected current payoff of search is zero for any search level, and the single player optimally splits her search evenly across two periods.

Next, consider $\delta = 0.9$. The player searches the entire island for $x \leq 1 - \delta$. For a more patient single player, this corresponds to a much smaller island size. For slightly larger islands, $x \in (\chi_1^{(1)}, \chi_2^{(1)}] = (1 - \delta, 1 - \delta(2\delta - 1)]$, she anticipates a search that may last up to two periods. As she crosses each knot, she plans to apportion her search across a greater number of periods, and the slope of her search profile falls. Even for this relatively patient player, we can see evidence of impatience in Panel (b). For any x, if the player to search the island within s periods, then she will search a fraction at least as great as $1/s$ in the current period, illustrating part (b) of the corollary.

The fact that the optimal search intensity is continuous implies that $\chi_s^{(1)} - I^{(1)}(\chi_s^{(1)}) = \chi_{s-1}^{(1)}$ for any $s > 1$. Thus, starting from any knot, the remaining island size jumps between successive knots with each period in the sole player's search procedure. Therefore, there is no discontinuity in the continuation value for the single searcher, and no discontinuity in search intensity.

Now we characterize the value function $V^{(1)}(x)$ derived in Theorem 1 in the following

Figure 2: The value function of the single searcher



corollary.

Corollary 2. *When $n = 1$ and $x \leq 1$, the value function $V^{(1)}(x)$ is continuously differentiable and monotonically decreasing.*

Proof. See Appendix A.

Figure 2 illustrates the value function $V^{(1)}(x)$ for $\delta = 0.9$ and $\delta = 0.4$. The value function is a smooth monotonically decreasing function. This is an intuitive result. Value decreases monotonically with island size because it is harder to find the treasure on a larger island. It follows directly from Lemma 1 that the general value function $V(x, c, R)$ is monotonic in all three arguments. The general value function is smooth because the strategy of the single searcher is unimpeded by the constraints potentially imposed by other competitors. As we show later, neither monotonicity nor smoothness (nor even continuity) are always satisfied with two firms. We turn to this case now.

3.2 Two-player search

In this section we give a complete characterization of all SMPE when $n = 2$ and $x \leq 1$. We first define the polynomials $I_s^{(2)}(x)$, $P_{(s)}^{(2)}(x)$, and critical island sizes $\chi_s^{(2)}$,[18]

$$I_{(1)}^{(2)} = \frac{x}{2}, \quad I_{(2)}^{(2)} = \frac{(1-\delta)(1-x) + \delta/2}{2\delta}, \quad I_{(3)}^{(2)} = \frac{(1-\delta)(1-x) + \delta/4}{2\delta},$$

$$P_{(s)}^{(2)}(x) = -\frac{1}{2}(1-x)^2 + \frac{1}{2^s}(1-x) + \frac{(4\delta)^s - 4\delta}{2(4\delta-1)4^s} \quad \text{for} \quad s = 1, 2, 3,$$

[18]Some of the expressions below are presented as ratios for expositional purposes. In particular, note that both numerators and denominators in $P_{(s)}^{(2)}$ contain a $(1-4\delta)$ term. When $\delta = 1/4$, these expressions are defined by their limits as $\delta \to 1/4$. Note also that this theorem corrects the critical points derived in Proposition 3 of Matros and Smirnov (2016).

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$$\chi_1^{(2)} = 1 - \frac{\delta}{2}, \;\; \chi_2^{(2)} = 1 + \frac{3}{8}\delta - \frac{1}{2}\delta^2.$$

We obtain the following theorem.

Theorem 2. *Suppose $n = 2$ and $x \leq 1$. Then, (a) non-trivial SMPE exist as described below; and (b) all search is conducted within three periods. If $\delta \leq 0.75$, then the search intensity of each firm is*

$$I^{(2)}(x) = \begin{cases} I_{(1)}^{(2)}, & \text{if} \quad x \leq \chi_1^{(2)}, \\ I_{(2)}^{(2)}, & \text{if} \quad \chi_1^{(2)} < x \leq 1; \end{cases}$$

and the value function for each firm is

$$V^{(2)}(x) = \begin{cases} P_{(1)}^{(2)}(x)/x, & \text{if} \quad x \leq \chi_1^{(2)}, \\ P_{(2)}^{(2)}(x)/x, & \text{if} \quad \chi_1^{(2)} < x \leq 1. \end{cases}$$

If $\delta > 0.75$, then the search intensity of each firm is[19]

$$I^{(2)}(x) = \begin{cases} I_{(1)}^{(2)}, & \text{if} \quad x \leq \chi_1^{(2)}, \\ I_{(2)}^{(2)}, & \text{if} \quad \chi_1^{(2)} < x \leq \chi_2^{(2)}, \\ I_{(3)}^{(2)}, & \text{if} \quad \chi_2^{(2)} \leq x \leq 1; \end{cases}$$

and the value function for each firm is

$$V^{(2)}(x) = \begin{cases} P_{(1)}^{(2)}(x)/x, & \text{if} \quad x \leq \chi_1^{(2)}, \\ P_{(2)}^{(2)}(x)/x, & \text{if} \quad \chi_1^{(2)} < x \leq \chi_2^{(2)}, \\ P_{(3)}^{(2)}(x)/x, & \text{if} \quad \chi_2^{(2)} \leq x \leq 1. \end{cases}$$

Proof. See Appendix A.

In a non-trivial SMPE, the two firms plan to search the island for at most 2 or 3 periods. If the island is sufficiently small ($x \leq \chi_1^{(2)}$), the players search the entire island in a single period. For $x \in (\chi_1^{(2)}, \min\{\chi_2^{(2)}, 1\}]$, the firms plan to search the island for at most two periods. Finally, the firms plan to search the island for at most three periods when $\delta > 0.75$ and $x \in [\chi_2^{(2)}, 1]$. In sum, the equilibrium search intensity $I^{(2)}(x)$ is a spline of degree one on the interval $[0, 1]$ with knot $\chi_1^{(2)}$ and possibly $\chi_2^{(2)}$.

Comparison of Theorems 1 and 2 reveals a qualitative difference between the search plans of a single player and two rivals. A patient firm searching alone plans to apportion search over many periods, and in the limit as δ approaches 1, she is prepared to continue her search indefinitely. By contrast, for $x \leq 1$ and any δ, two rivals will search for at most three periods. To gain intuition, consider the marginal incentive for a sole searcher to search more intensively in the current period if she plans to search for at most two periods. If she must wait until next period to discover the prize, she discounts the value at the rate

[19]Note that when $\delta > 0.75$ at the point $x = \chi_2^{(2)}$, the SMPE is not unique.

δ. The knot $\chi_1^{(1)} = 1 - \delta$ reflects this trade-off. Now consider a firm competing to find the treasure with a rival anticipating up to two periods of search. If she searches more aggressively today, she increases the probability that she will claim the prize for herself today. If instead the search is not concluded until the following period, she anticipates sharing the prize (in expectation) and she discounts this value. This trade-off results in the knot $\chi_1^{(2)} = 1 - \delta/2$. As δ approaches 1, $\chi_1^{(1)}$ approaches 0 while $\chi_1^{(2)}$ approaches 1/2. With patient players, this results in a dramatic difference in the search profile with a sole searcher rather than two rivals.

When $\delta \leq 0.75$, search is conducted within two periods. We characterize the equilibrium search intensity and the value function in this case with the following corollary.

Corollary 3. *When $n = 2$, $x \leq 1$ and $\delta \leq 0.75$, the equilibrium search intensity $I^{(2)}(x)$ is a unique, piece-wise linear, continuous and quasiconcave function, while the corresponding value function $V^{(2)}(x)$ is a unique, continuous and monotonically decreasing function.*

Proof. See Appendix A.

Panel (a) of Figure 3 illustrates the equilibrium search intensity in the SMPE for each of the two rivals when $\delta = 0.75$ and $\delta = 0.99$. Consider first $\delta = 0.75$. In this case search is concluded within two periods, and there is only one knot of the spline, $\chi_1^{(2)}$. There is a kink in the figure around the knot, but search intensity is continuous. This is because searching for one period (which occurs for $x \leq \chi_1^{(2)}$) is equivalent to searching for two periods with all the search conducted in the first period. As in the single-player case, equilibrium search intensity is a non-monotonic function of island size. For sufficiently small islands, the firms search the entire island in a single period, and search intensity is therefore increasing in island size for $x \leq \chi_1^{(2)}$. Beyond the knot, the intensity of search decreases with island size, reflecting the diminished probability of successful search in the current period.

Panel (b) of Figure 3 illustrates the value function when $\delta = 0.75$ and $\delta = 0.99$. Consider $\delta = 0.75$. As in the single-player case, the value function is monotonically decreasing with the size of the unsearched area because it is harder to find the treasure on a larger island. However, contrary to the single-player case, the value function is not smooth; the slope of the value function changes at knot $\chi_1^{(2)}$, coinciding with the discontinuity of the slope of the equilibrium search intensity. The discontinuity relates to the inefficiency of rival search for $x > \chi_1^{(1)}$, which we discuss in Section 3.3.

Next we characterize the equilibrium search intensity and the value function when $\delta > 0.75$.

Corollary 4. *When $n = 2$, $x \leq 1$ and $\delta > 0.75$, the equilibrium search intensity $I^{(2)}(x)$ is a piece-wise linear, discontinuous and quasiconcave function, while the corresponding value function $V^{(2)}(x)$ is neither a continuous nor monotonically decreasing function; specifically, there exists $\widetilde{x} > \chi_2^{(2)}$ and $\widehat{x} < \chi_2^{(2)}$ such that $V^{(2)}(\widetilde{x}) > V^{(2)}(\widehat{x})$.*

Proof. See Appendix A.

Return to Panel (a) of Figure 3. When $\delta = 0.99$, each firm searches for at most three periods. The most striking feature of the Figure is the discontinuity in the equilibrium

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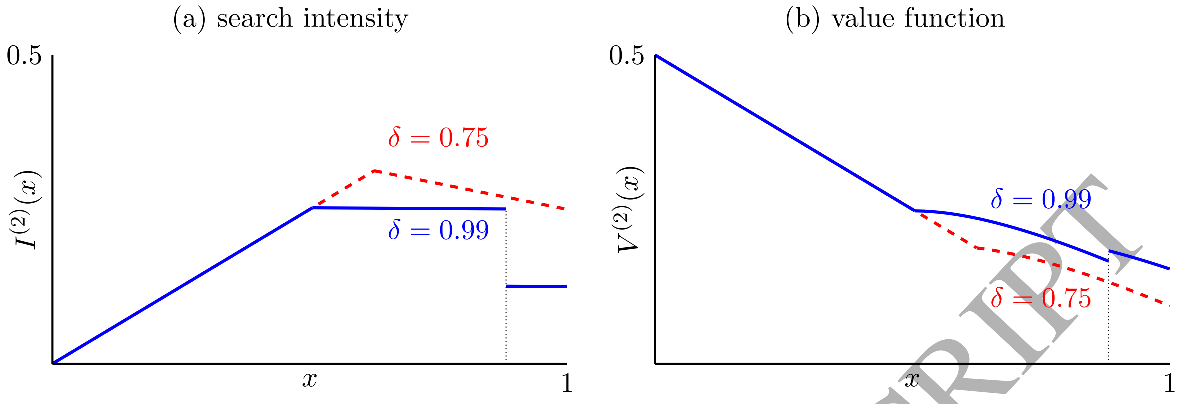
Figure 3: Two-player search intensity and value



search intensity at the knot $\chi_2^{(2)}$. To the left of the knot, the firms intend to search the island for up to two periods. In fact, the collective search plans of the two players will reduce the island size strictly below $\chi_1^{(2)}$. That is, $\chi_2^{(2)} - 2I^{(2)}(\chi_2^{(2)}) < \chi_1^{(2)}$.[20] By contrast, for $x > \chi_2^{(2)}$, each firm plans to search for up to three periods. As a result, there is a discontinuous drop in the continuation value of search at the knot, and in equilibrium each firm searches less aggressively beyond the knot. This is in stark contrast with the continuity result for the social-planner's problem.

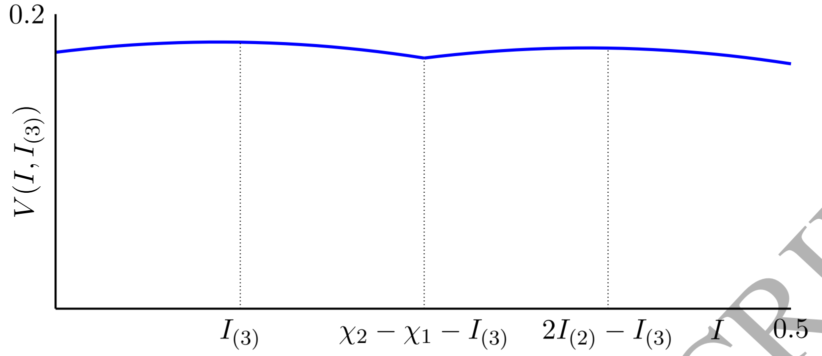
A related feature evident in the figure is the multiplicity of equilibria at the knot $\chi_2^{(2)}$. At this point, there are two equilibria, corresponding to planned search for up to two and for up to three periods, respectively. Turning to Panel (b) of Figure 3, we see that the two equilibria yield different values to the two firms. In the equilibrium in which firms intend to search the island in up to three periods, search is less aggressive in the initial period and the associated value is higher. This leads to a surprising feature of the value function: it is non-monotonic in island size. For $x < \chi_2^{(2)}$, only the two-period equilibrium exists, while for $x > \chi_2^{(2)}$, only the three-period equilibrium exists. The less aggressive search plans of the three-period equilibrium yields higher value and therefore the value function jumps up at the knot $\chi_2^{(2)}$. Hence, it is possible that a larger island size could make the two firms better off. Corollary 4 suggests that this feature holds whenever $\delta > 0.75$ and the two players intend to search an island of size $x \leq 1$ for up to 3 periods.

Figure 4 provides additional detail for the multiple equilibria at the knot $\chi_2^{(2)}$. Each panel depicts the value to player 1 as a function of her current search intensity, holding fixed the intensity of player 2. In the figure, we use the shorthand $I_{(s)}$ to denote the intensity of each player at $x = \chi_2^{(2)}$ in the equilibrium in which search lasts at most s periods, and for convenience we drop the superscripts when referring to knot points and search intensities. In panel (a), we fix the intensity of player 2 at the level consistent with the equilibrium in which search is concluded within three periods, $I_{(3)}$, and plot the value to player 1 as a function of her own intensity, I. In panel (b), we conduct the analogous exercise when player 2's intensity is fixed at $I_{(2)}$.

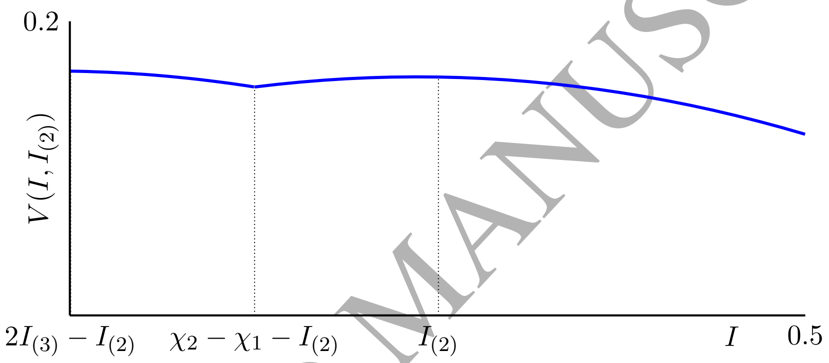
[20] Note that from equation (1), in equilibrium two players together search an area of $2I^{(2)}(\chi_2^{(2)})$.

Figure 4: The value for two searchers when $\delta = 0.99$.

(a) Rival plans to search within three periods



(b) Rival plans to search within two periods



Consider first panel (a). If player 1 searches with intensity $I \in [0, \chi_2 - \chi_1 - I_{(3)})$, then search will be concluded within three periods. Within this range, her optimal search level is $I_{(3)}$, consistent with the three-period search equilibrium. However, if she searches more aggressively ($I > \chi_2 - \chi_1 - I_{(3)}$), then search will be completed within two periods. Her optimal search intensity in this case is $I = 2I_{(2)} - I_{(3)}$. She is indifferent between these two search intensities ($I_{(3)}$ and $2I_{(2)} - I_{(3)}$), and therefore there is a symmetric equilibrium in which each player searches $I_{(3)}$. Now consider panel (b). Search will conclude within three periods if player 1 searches with intensity $I < \chi_2 - \chi_1 - I_{(2)}$, and within two periods if $I > \chi_2 - \chi_1 - I_{(2)}$. Player 1 is indifferent between the search intensities $I = 2I_{(3)} - I_{(2)}$ and $I_{(2)}$.[21] Thus, there is also a symmetric equilibrium with search intensity $I_{(2)}$. Finally, note that for $x \neq \chi_2^{(2)}$, player 1 is no longer indifferent between the two investment levels depicted in each panel. For $x < \chi_2^{(2)}$, only the two-period search equilibrium survives; for $x > \chi_2^{(2)}$, only the three-period search equilibrium remains.

[21]Note that Theorem 2 implies that both $2I_{(2)} - I_{(3)}$ and $2I_{(3)} - I_{(2)}$ are always positive.

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Comparative statics for value functions

Given the non-monotonicity of $V^{(2)}(x)$, $V^{(2)}(x, c, R)$ is also non-monotonic in x. In addition, from the non-monotonicity of $V^{(2)}(x)$ and equation (3), it follows that $V^{(2)}(x, c, R)$ does not monotonically decrease with respect to c. We summarize these results in the following corollary.

Corollary 5. *When $n = 2$, $cx/R \leq 1$ and $\delta > 0.75$, $V^{(2)}(x, c, R)$ is not a monotonically decreasing function with respect to either x or c; specifically, there exists $\tilde{x} > x$ such that $V^{(2)}(\tilde{x}, c, R) > V^{(2)}(x, c, R)$ and there exists $\tilde{c} > c$ such that $V^{(2)}(x, \tilde{c}, R) > V^{(2)}(x, c, R)$.*

The comparative statics with respect to R do not automatically follow. Rather, from Lemma 1, $V(x, c, R) = V(cx/R, 1, 1)R$, so a change in R could have two opposing effects. However, as one effect is finite, while the other one could be infinitely small, it is possible to show that there is a non-monotonicity in this case as well.

Corollary 6. *When $n = 2$, $cx/R \leq 1$ and $\delta > 0.75$, $V^{(2)}(x, c, R)$ is not a monotonically increasing function with respect to R; specifically, there exists $\tilde{R} > R$ such that $V^{(2)}(x, c, \tilde{R}) < V^{(2)}(x, c, R)$.*

In Section 5, we discuss policy implications of these results.

3.3 Over- and under-search

Proposition 1 discusses over- and under-search in the two-player problem. We first introduce the aggregate n-player search intensity and value as $J^{(n)}(x) = nI^{(n)}(x)$ and $W^{(n)}(x) = nV^{(n)}(x)$, respectively. For a given x, we associate over- and under-search in the current period with $J^{(2)}(x) > J^{(1)}(x)$ and $J^{(2)}(x) < J^{(1)}(x)$, respectively. Search is said to be efficient in the two-player problem if $J^{(2)}(x) = J^{(1)}(x)$ for every x reached on the equilibrium path, which means that $W^{(2)}(x) = W^{(1)}(x)$. To help state the proposition, we define the critical values of the discount factor $\delta^* \approx 0.9$, $\delta^{**} \approx 0.9064$ and $\delta^{***} \approx 0.9065$, derived in the proof of Proposition 1.

Proposition 1. *The two-player problem exhibits*

1. *over-search if ($\delta \leq 0.75$ and $x \in (1-\delta, 1)$) or ($\delta \in (0.5, 0.75]$ and $x = 1$) or ($\delta \in (0.75, \delta^{***}]$ and $x \in (1-\delta, x^*(\delta))$) or ($\delta \in (\delta^{***}, 1]$ and $x \in (1-\delta, 1]$);*

2. *under-search if $\delta \in (0.75, \delta^{***})$ and $x \in (x^*(\delta), 1]$;*

3. *efficient search if ($x \in (0, 1-\delta]$) or ($x = 1$ and $\delta \leq 0.5$);*

where

$$x^*(\delta) = \begin{cases} 1 + \frac{3}{8}\delta - \frac{1}{2}\delta^2, & \text{if } \delta \in (0.75, \delta^*], \\ \frac{1-7.75\delta+5\delta^2+3\delta^3}{1-8\delta+8\delta^2}, & \text{if } \delta \in (\delta^*, \delta^{**}], \\ \frac{5-18.5\delta+8\delta^2+7\delta^3}{5-20\delta+16\delta^2}, & \text{if } \delta \in (\delta^{**}, \delta^{***}]. \end{cases}$$

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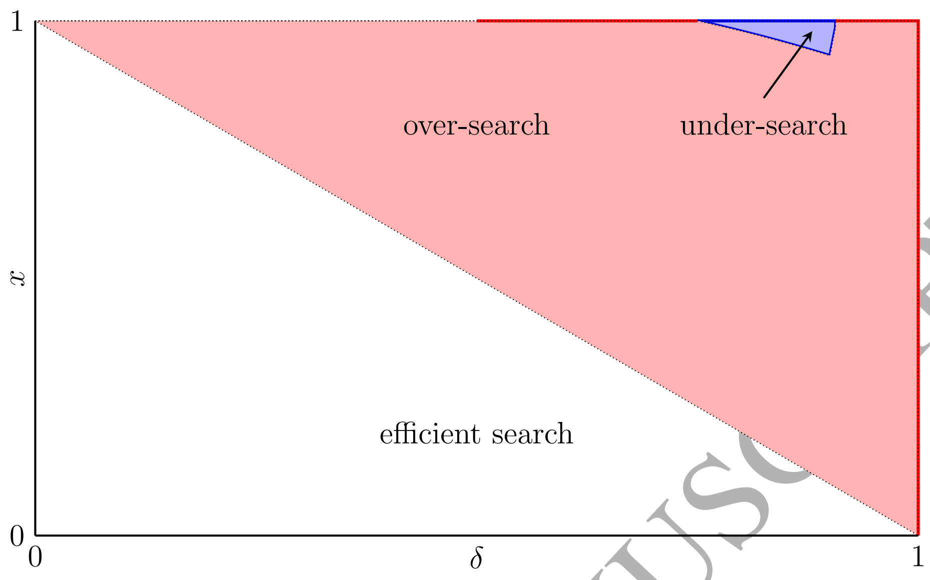
Figure 5: Area of under-search and over-search



Proof. See Appendix A.

Figure 5 illustrates the regions of under- and over-search described in the Proposition. Let us first consider over-search. The two-player model exhibits an inefficiency akin to the tragedy of the commons. When the value of x is relatively large, $x > 1 - \delta$, the two rivals can over-search. In this case the current payoff, the first term in equation (4), is positive; that is, $(1 - x)I/x > 0$. Under rivalrous search, each player has an incentive to increase her current expected payoff by searching more. In equilibrium players over-search, which is similar to the standard tragedy of the commons; each player's individually rational search causes a spillover on their rival, decreasing the total surplus.

Figure 6 illustrates the above discussion when $\delta = 0.5$. Panel (a) depicts aggregate search with both one and two searchers. For small values of x, aggregate search $J^{(n)}(x)$ is the same for $n = 1$ and $n = 2$; players search the whole island in just one period. On this part of the graph, both curves coincide. For larger values of x, the curve for $n = 2$ is above the curve for $n = 1$. This means that the two firms suffer from the tragedy of the commons; the players search the whole island inefficiently quickly.

Panel (a) also illustrates the special case of $\delta \leq 0.5$. When $\delta \leq 0.5$, two-player search is efficient for $x = 1$. This result has the following intuition. For $\delta \leq 0.5$, in both the two-player and single-player games, players plan to search the island for at most two periods. When $x = 1$, players get zero expected payoff in period one of their two-period search. Consequently, the objective function of each of the players is to maximize their expected payoffs from the second period only. In the second period of the symmetric equilibrium in the two-player game, each player receives a payoff proportional to the single-player payoff. This guarantees that multi-player search reproduces the single-player outcome.

Panel (b) of Figure 6 shows the combined value function when one or two players search the island and $\delta = 0.5$. Consistent with the previous discussion, multi-player search

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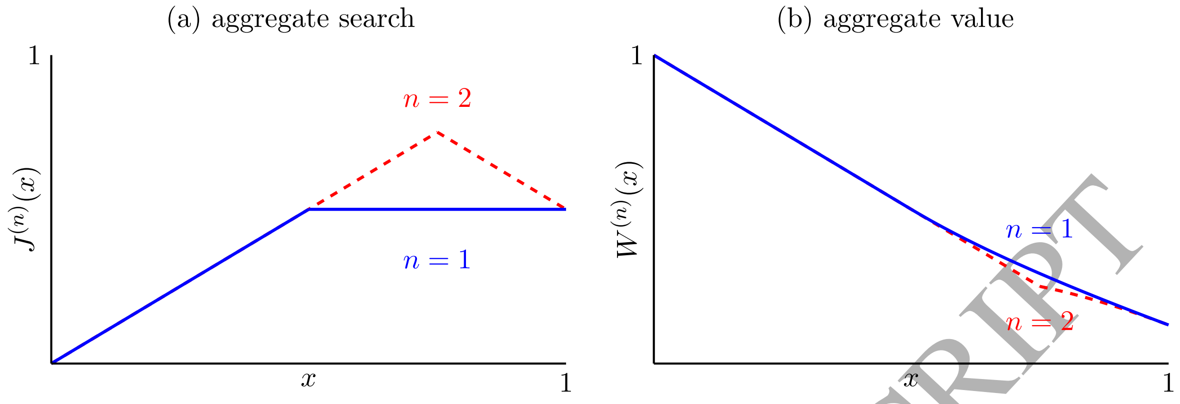
Figure 6: Aggregate search intensity and value, $\delta = 0.5$



reproduces the single-player outcome either when: the island is small and players search the whole island in just one period; or when $x = 1$.

According to Part 2 of the Proposition, under-search is possible when $\delta > 0.75$ but not too close to one. Figure 7 illustrates the outcome when $\delta = 0.85$. As one can see from the picture, when x is close to one, aggregate two-player search, $J^{(2)}(x)$, is less than the single-player's search intensity $J^{(1)}(x)$. This result has the following intuition. When x is close to one, the current payoff, $(1 - x)I/x$, could be arbitrarily small. Therefore, the principal motivation of a player's current search is to increase their continuation value. In the case of two rivals, the players anticipate wasteful over-search in the following period. This provides a disincentive to search in the current period.

Consider again Figure 5. When $\delta \in (0.75, \delta^{***})$ there will be under-search for islands that are sufficiently large; the shape of the region of under-search in the Figure is determined by δ^*, δ^{**} and δ^{***}. It is in this region that players shade their search efforts in the current period in anticipation of future over-search. The non-monotonicity of this result is illustrated in the Figure as δ increases. Under-search is only possible for intermediate patience levels. For relatively impatient players ($\delta < 0.75$), there is no under-search. The intuition for this result is as follows. Relatively impatient rivals tend to scramble to find the prize too quickly; this means that the prospect of a low continuation value in future search periods is not a concern for them. Myopic players potentially suffer from over- not under-search. For high discount factors, the possibility of under-search again disappears, but for different reasons; a single player who is very patient will plan to extend search over many periods, while two players plan to complete their search within at most three periods. As a consequence, over-search again is the issue in strategic search with patient players, not under-search.

4 Asymmetric equilibria

The equilibria derived in Section 3.2 are restricted to be symmetric – in equilibrium both firms are required to have equal search intensities in each period. In this section we relax

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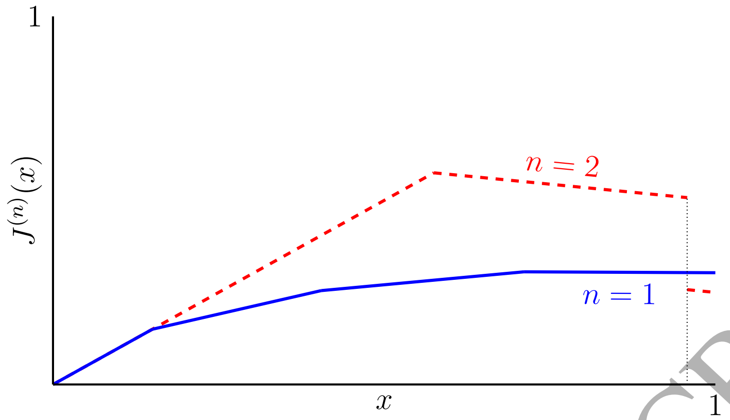
Figure 7: Aggregate search intensity, $\delta = 0.85$



this assumption.

With asymmetry there are multiple equilibria. To illustrate this consider search in the last period. Given there is no duplication, in this period each player has an incentive to search as much as possible, provided that $I_1 + I_2 = x_1$. This will be true in any MPE. Additional multiplicity arises in a game with more than one search period because it is always the case that the first order condition for each firm requires that the sum of the players' search, $I_1 + I_2$, must be a constant. To sidestep the issue of multiple equilibria, we assume that whenever multiplicity of equilibria is present, $I_1 + I_2$ are shared in proportions $\beta < 0.5$ and $1 - \beta$ for players 1 and 2, respectively. Here, β and $(1 - \beta)$ reflect some preordained bargaining strengths or competitive advantage.[22]

The following proposition characterizes the asymmetric MPE, given each player's relative bargaining strength.

Proposition 2. *When $n = 2$, $x \leq 1$ and $\beta < 0.5$, the unique non-trivial SMPE exists and is described as follows. The search intensities of each firm are*

$$I_1^{(asym)}(x) = \begin{cases} \beta x, & \text{if} \quad x \leq 1 - \delta\beta, \\ x - 0.5 + \frac{1-x}{2\delta\beta}, & \text{if} \quad 1 - \delta\beta < x \leq 1; \end{cases}$$

and

$$I_2^{(asym)}(x) = \begin{cases} (1 - \beta)x, & \text{if} \quad x \leq 1 - \delta\beta, \\ 0, & \text{if} \quad 1 - \delta\beta < x \leq 1; \end{cases}$$

and the value functions for each firm are

$$V_1^{(asym)}(x) = \begin{cases} \beta(1 - x), & \text{if} \quad x \leq 1 - \delta\beta, \\ \frac{\delta^2\beta^2 + 2\delta\beta(1-x) + (1-4\delta\beta)(1-x)^2}{4\delta\beta x}, & \text{if} \quad 1 - \delta\beta < x \leq 1; \end{cases}$$

[22]An alternative way of introducing asymmetry would be to allow the players to have different discount factors or search costs. Similar results and intuition apply in both of these cases.

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$$V_2^{(asym)}(x) = \begin{cases} (1-\beta)(1-x), & \text{if} \quad x \leq 1-\delta\beta, \\ \frac{(\delta^2\beta^2-(1-x)^2)(1-\beta)}{4\delta\beta^2 x}, & \text{if} \quad 1-\delta\beta < x \leq 1; \end{cases}$$

Proof. See Appendix A.

This Proposition has a similar intuition to the symmetric case outlined above (Theorem 2). In a non-trivial asymmetric MPE, two rivals plan to search the island for at most two periods. If the island is sufficiently small ($x \leq 1-\delta\beta$), the two firms search the entire island in a single period. For $x \in (1-\delta\beta, 1]$, the firms plan to search the island for at most two periods. When $x > 1-\delta\beta$ in the case of the first player, the preferred combined search in the first period of a two-period path is larger than in the case of the second player. Consequently, for a given value of β there is a unique asymmetric MPE that involves only the first player searching in the first period and both players searching in the second period. The intuition is that as the first player expects a smaller payoff in the second period, he is willing to search more in the first period. This suggests that the intrinsically weaker player searches more early to preempt their disadvantage in the final search period.

In the following corollary we compare the asymmetric equilibrium from Proposition 2 with the symmetric equilibria derived in Theorem 2.

Corollary 7. *The expected combined value accruing to the players in the SMPE is always greater than or equal to the value generated in any asymmetric MPE.*

Proof. See Appendix A.

The number of search periods is always weakly longer in the SMPE than in the asymmetric MPE. Firstly, when $x \in (\chi_1^{(2)}, 1-\beta\delta)$ there are two search periods in the SMPE but only one in the asymmetric equilibrium. Secondly, the maximum number of search periods in the symmetric case is three whereas, in contrast, the maximum number of periods of search in the asymmetric case is two. The intuition for this result is that asymmetry worsens the incentive to over-search for the relatively weak player, leading to lower total surplus. The first firm in the asymmetric case, being in the relatively weaker position, is more desperate to search than either firm in the symmetric case, and this over-search in the early period reduces overall surplus available to both parties.

5 Discussion

Profitable opportunities however manifested – new markets, products or clients, a cache of jewels or some sort of reward or prize – do not just materialize; they need to be found. Consequently, search is an intrinsic part of business. But few firms have the luxury of seeking out such opportunities on their own without rivals, or in an environment that is not changing as search proceeds. In this paper we examine the efficiency of search by two rivals in a strategic and dynamic environment.

As it captures all of the relevant gains from trade, a firm that has the luxury of searching the island on its own does so efficiently. We find that two rivals can also achieve the welfare-maximizing outcome when the search area (the island) is relatively small and search is

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completed in one period. In this case, the two rivals act like they are in a cartel, each searching their share of the island. When the search area is slightly larger (but not too large), two-player search is typically inefficient; in a similar manner to the tragedy of the commons, the likelihood of capturing the prize tempts the rivals into oversearching, reducing total surplus. This behavior mirrors cut-throat races to locate sunken treasures or the actions of prospectors in unregulated gold fields. We also show that total search effort of the two firms can be too low – this arises when the search area is relatively large and the players have intermediate levels of patience. In these instances, each player, expecting future inefficiencies, has an incentive to shade their search effort below the efficient level. Notably, these inefficiencies – both too little and too much search – can occur even without search duplication, studied elsewhere (for example, Fershtman and Rubinstein (1997) and Chatterjee and Evans (2004)).

A government, either directly or through the agency of a regulator, is often an interested party in these search environments: a government can restrict the number of actors; it can manipulate the potential costs and benefits; and it can even occasionally have some influence on the search area.[23] This is where our counterintuitive comparative-statics results are important. For example, the parties can be worse off searching a smaller island. Similarly, the parties might be better off looking for a less valuable prize. These non-monotonic results arise because of the interaction between multiple potential market failures. Moreover, they suggest that a government needs to take care regarding any policy intervention in search markets so as not to worsen the existing welfare distribution. In some cases, where feasible, a government might consider restricting the number of potential searchers to just one, through a licensing arrangement or some other mechanism, so as to avoid the inefficiencies that arise due to the strategic search of oligopolist rivals. This is, in fact, common practice in mineral exploration, for example.

As always, our analysis has several caveats. First, we assume that a sole searcher acts like a social planner in that it captures all of the surplus. In our model, however, adding consumer welfare would not give rise to a richer set of results. A similar point could be made about incorporating more than two rivals. Second, while we make the natural assumption that there is an equal probability that the treasure will be located at any given point on the island, Matros and Smirnov (2016) considered the case when the probability of its location is not uniformly distributed across the possible search area. Third, whilst Matros and Smirnov (2011) explore non-linear search costs, here for tractability we assume linear costs. Again, this allows us to focus on the potential inefficiencies that arise from the rivals' strategic interaction.

Finally, as is typical in the literature, our focus is on the SMPE. As discussed, multiple asymmetric MPE exist, but as it turns out the SMPE is somewhat of a focal point as it ensures a total payoff that cannot be bettered in any other MPE. An alternative equilibrium concept is, of course, subgame perfection.[24] Subgame perfection is particularly useful in infinite horizon games to study punishment and rewards. In equilibrium, the game presented here has a finite end, which makes it less amenable to trigger strategies. Moreover, it is

[23]A government might be able to affect the search area if it has the findings of previous unsuccessful attempts of looking for a sunken ship, in mineral exploration or even in the case of medical research.

[24]Matros and Smirnov (2016) discuss the application of subgame perfection to their search game.

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strategic search competition, rather than collusion, that is our focus here. Rivalry in our environment, even with just two players, can be intense, potentially leading to inefficient search outcomes.

Appendix A

We begin with some further notation. When convenient we omit: firm and time subindexes; and superscripts indicating how many firms searching.

It proves convenient to introduce the following function:

$$\Psi(x) \equiv xV(x). \tag{5}$$

From equation (4) and definition (5), it follows that $\Psi(x) \geq 0$ for any x and

$$\Psi(x) = \max_{I_i \in [0,x]} \{J - I_j - xI_i + \delta\Psi(x - J)\}. \tag{6}$$

Define the value-iteration operator B as follows:

$$(B\Psi)(x) \equiv \max_{I_i \in [0,x]} \{J - I_j - xI_i + \delta\Psi(x - J)\}. \tag{7}$$

The following Lemma proved in Matros and Smirnov (2016) implies value iteration can be used to solve (6).

Lemma 2. *There exists finite T such that:*
(a) any non-trivial SMPE involves $J_t > 0$ for $0 \leq t \leq T$, and $x_{T+1} = 0$; and
(b) all SMPE can be obtained in T steps. Specifically,

$$\Psi_0 \equiv 0, \qquad \Psi_s \equiv B\Psi_{s-1} \quad \textit{for} \quad s = 1, 2, \ldots, T, \qquad \textit{and} \qquad \Psi = \Psi_T.$$

With the help of Lemma 2, one can construct the sequence $\{\Psi_s\}$ and find all SMPE. This value-iteration procedure is equivalent to backward induction.

Proof of Theorem 1

Suppose the single player is restricted to searching the island for at most k periods, and define the associated value as $V_k(x) \equiv \Psi_k(x)/x$, for any $x \geq 0$.

Construction of Ψ_1 and V_1

Let us start from the end of the search process. What will be the value if the single searcher could only search for at most one period? Equation (6) transforms into

$$\Psi_1(x) = \max_{I \in [0,x]} \{(1 - x)I\}. \tag{8}$$

It is evident that in the unique equilibrium the equilibrium I can be described in the following way[25]:

[25] Note that if $x = 1$, then *any* $I \in [0, x]$ is optimal. We assume that the single player chooses $I = x$ in this case.

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$$I = x, \text{ if } x \leq 1.$$

If $x \leq 1$, then in equilibrium the single player searches the whole island, $I = x$. Consequently, the solution of (8) is

$$\Psi_1(x) = P_{(1)}(x), \text{ if } x \leq 1; \tag{9}$$

where $P_{(1)}(x) = x(1-x)$. For future reference note that

$$P_{(1)}(x) = a_1(1-x)^2 + b_1(1-x) + c_1, \tag{10}$$

where

$$a_1 = -1, \quad b_1 = 1, \quad c_1 = 0. \tag{11}$$

From the above definition, it follows that

$$V_1(x) = 1 - x, \text{ if } x \leq 1.$$

Construction of Ψ_2 and V_2

What will be the value if the single player can search the island for at most two periods? In general there could be two possibilities, depending on the island size. The first possibility is that the single player searches the whole island in just one period. Intuitively this happens for small values of x because it is too costly to wait for another period when the island is very small. The second possibility is that the single firm searches the island for at most two periods. This happens for large values of x.

We have already considered the first possibility in the previous section. Now we analyze the situation when the single player plans to search for at most two periods. The first step is to construct $\Psi_2(x)$. Equation (6) in this case transforms into

$$\Psi_2(x) = \max_{I \in [0,x]} \{(1-x)I + \delta \Psi_1(x-I)\}. \tag{12}$$

The necessary condition for I to be optimal in the interior of $[0, x]$ is

$$(1-x) - \delta \Psi_1'(x-I) = 0. \tag{13}$$

In order to continue the search for the second period, the remaining island size has to satisfy

$$x - I \geq 0. \tag{14}$$

The sufficient condition for I to be optimal in the interior of $[0, x]$ is satisfied because

$$\Psi_1''(x-I) = 2a_1 < 0.$$

The way to proceed is to construct the equilibrium with the help of condition (13), and then show that the derived equilibrium satisfies condition (14).

From expressions (13) and (9), it follows that

$$(1-x) - \delta\,(1 - 2(x-I)) = 0.$$

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Consequently,

$$I = \frac{(2\delta - 1)x + 1 - \delta}{2\delta}. \tag{15}$$

Substituting (15) into equation (12), we obtain a *spline* of degree two on the interval $[0, 1]$:

$$\Psi_2(x) = \begin{cases} \Psi_1(x), & \text{if} \quad 0 \le x \le \chi_1, \\ P_{(2)}(x), & \text{if} \quad \chi_1 < x \le 1; \end{cases} \tag{16}$$

where

$$P_{(2)}(x) = a_2(1-x)^2 + b_2(1-x) + c_2, \tag{17}$$

with

$$a_2 = -1 + \frac{1}{4\delta}, \quad b_2 = \frac{1}{2}, \quad c_2 = \frac{\delta}{4}. \tag{18}$$

The point $x = \chi_1$ is the first knot of the spline. When $x = \chi_1$ the sole searcher is indifferent between searching the island for two periods or for just one period:

$$\Psi_1(\chi_1) = \Psi_2(\chi_1). \tag{19}$$

From (10) and (17), we get

$$\chi_1 = 1 - \delta. \tag{20}$$

It is straightforward to check that the solution given by (16) satisfies condition (14) for any $x \in [\chi_1, 1]$. Therefore, if the single player can search the island for at most two periods, the equilibrium is a spline of degree one on the interval $[0, 1]$ with one knot $x = \chi_1$:

$$I(x) = \begin{cases} x, & \text{if} \quad x \le \chi_1, \\ \frac{(2\delta-1)x+1-\delta}{2\delta}, & \text{if} \quad \chi_1 < x \le 1; \end{cases}$$

and the value function is

$$V_2(x) = \begin{cases} V_1(x), & \text{if} \quad x \le \chi_1, \\ P_{(2)}(x)/x, & \text{if} \quad \chi_1 < x \le 1. \end{cases}$$

We can describe the construction of Ψ_k and V_k now.

Construction of Ψ_k and V_k

What will be the value if the single player can search the whole island for at most $k \ge 3$ periods? In general there could be k possibilities, depending on the island size x_1. The sole searcher can plan to search the island for at most $1, 2, ..., k$ periods in a non-trivial equilibrium.

Let us construct $\Psi_k(x)$. Equation (6) in this case transforms into

$$\Psi_k(x) = \max_{I \in [0,x]} \{(1-x)I + \delta \Psi_{k-1}(x - I)\}. \tag{21}$$

A necessary condition for I to be the optimal value in the interior of $[0, x]$ is

$$(1 - x) = \delta \Psi'_{k-1}(x - I). \tag{22}$$

In order to continue search for the next period, the new value of x has to satisfy

$$x - I \geq \chi_{k-2}, \tag{23}$$

where $\chi_0 = 0$. The sufficient condition for I to be the optimal value in the interior of $[0, x]$ is satisfied if

$$\Psi''_{k-1}(x - I) < 0. \tag{24}$$

We will use condition (22) to find I, and then show that it satisfies conditions (23) and (24). Note that if function $\Psi_{k-1}(x)$ in (21) is a quadratic polynomial, then $\Psi_k(x) = B\Psi_{k-1}(x)$ has to be a quadratic polynomial as well. Since $P_{(1)}(x)$ and $P_{(2)}(x)$ are quadratic polynomials by (10) and (17), any $P_{(k)}(x)$ can be represented in the following form:

$$P_{(k)}(x) = a_k(1-x)^2 + b_k(1-x) + c_k, \quad k \geq 1. \tag{25}$$

Let $I_{(k)}$ be the optimal search intensity of the single player who anticipates search for up to k periods. From condition (22) and expression (25), it follows that

$$I_{(k)} = -\frac{(1-x)(1+2\delta a_{k-1})}{2\delta a_{k-1}} - \frac{b_{k-1}}{2a_{k-1}}. \tag{26}$$

Define that value of x such that the single searcher is indifferent between planning to search the area for k periods or for $k-1$ periods as knot χ_{k-1}:

$$\Psi_{k-1}(\chi_{k-1}) = \Psi_k(\chi_{k-1}). \tag{27}$$

For the moment, let us assume that equation (27) has a unique solution - we prove the uniqueness when we explicitly derive χ_{k-1}. Substituting (26) into equation (21), we obtain a *spline* of degree two on the interval $[0, 1]$ with knots $\chi_1, ..., \chi_{k-1}$:

$$\Psi_k(x) = \begin{cases} \Psi_{k-1}(x), & \text{if} \quad 0 \leq x \leq \chi_{k-1}, \\ P_{(k)}(x), & \text{if} \quad \chi_{k-1} < x \leq 1; \end{cases} \tag{28}$$

where $P_{(k)}(x)$ is defined in (25). Therefore, if the sole searcher plans to search the island for at most k periods, then $I(x)$ is a spline of degree one on the interval $[0, 1]$ with knots $\chi_1, ..., \chi_{k-1}$:

$$I(x) = \begin{cases} x, & \text{if} \quad x \leq \chi_1, \\ -\frac{(1-x)(1+2\delta a_1)}{2\delta a_1} - \frac{b_1}{2a_1}, & \text{if} \quad \chi_1 < x \leq \chi_2, \\ \vdots & \\ -\frac{(1-x)(1+2\delta a_{k-1})}{2\delta a_{k-1}} - \frac{b_{k-1}}{2a_{k-1}}, & \text{if} \quad \chi_{k-1} < x \leq 1; \end{cases} \tag{29}$$

and the value function is

$$V_k(x) = \begin{cases} V_{k-1}(x), & \text{if} \quad 0 \leq x \leq \chi_{k-1}, \\ P_{(k)}(x)/x, & \text{if} \quad \chi_{k-1} < x \leq 1. \end{cases}$$

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Derivation of a_k, b_k and c_k

Let us now find a_k, b_k, and c_k for any $k \geq 2$. Using (21), (25) and (26), we get the following system of difference equations:

$$a_k = -1 - \frac{1}{4\delta a_{k-1}}, \quad b_k = -\frac{b_{k-1}}{2a_{k-1}}, \quad c_k = \delta\left(c_{k-1} - \frac{b_{k-1}^2}{4a_{k-1}}\right), \quad k \geq 2, \tag{30}$$

where

$$a_1 = -1, \quad b_1 = 1, \quad c_1 = 0. \tag{31}$$

Let us solve the system of difference equations (30). Define

$$R_k = v^k \cdot \prod_{j=1}^{k} a_j \qquad k = 1, 2, \dots, \tag{32}$$

where $v = \sqrt{\delta}$. Using (30), one gets the following linear second-order difference equation

$$R_{k+1} = vR_k \cdot \left(-1 - \frac{1}{4\delta a_k}\right) = -vR_k - \frac{1}{4}R_{k-1} \quad k \geq 2. \tag{33}$$

The initial conditions are $R_0 = 1$ and $R_1 = -v$. The characteristic equation $4z^2 + 4vz + 1 = 0$ has two complex roots

$$z_1 = \frac{-v + ir}{2}, \quad z_2 = \frac{-v - ir}{2}, \qquad r = \sqrt{1 - v^2} > 0.$$

Denote $\varphi = \{\arg z_1 \in [0, \pi/2]\} = \arccos v$; then $z_{1,2} = -\frac{e^{\pm i\varphi}}{2}$. Further, write the solutions to equation (33) in the form $R_k = Az_1^{k+1} - Bz_2^{k+1}$, and use the initial conditions to get $A = B = -\frac{i}{\sin\varphi}$. Consequently

$$R_k = -\frac{i}{(-2)^{k+1}\sin\varphi}(e^{i(k+1)\varphi} - e^{-i(k+1)\varphi}) = -\frac{\sin[(k+1)\varphi]}{(-2)^k \sin\varphi}. \tag{34}$$

Apply (32) and (30) to get

$$a_k = \frac{R_k}{vR_{k-1}} = -\frac{\sin(k+1)\varphi}{2v\sin k\varphi}, \quad b_k = -\frac{b_{k-1}}{2a_{k-1}} = \frac{v^{k-1}\sin\varphi}{\sin k\varphi}, \tag{35}$$

$$\text{and } c_k = \delta\left[c_{k-1} - \frac{b_{k-1}^2}{4a_{k-1}}\right] = \frac{v^{2k-1}\sin(k-1)\varphi}{2\sin k\varphi}. \tag{36}$$

Derivation of χ_k

To find χ_k, one needs to solve the quadratic equation $P_{(k)}(\chi_k) = P_{(k+1)}(\chi_k)$; namely

$$(a_{k+1} - a_k)(1 - \chi_k)^2 + (b_{k+1} - b_k)(1 - \chi_k) + c_{k+1} - c_k = 0, \ \ k \geq 1. \tag{37}$$

Substitute a_k from (35) to derive

$$a_{k+1} - a_k = \frac{\sin(k+1)\varphi}{2v\sin k\varphi} - \frac{\sin(k+2)\varphi}{2v\sin(k+1)\varphi} = \frac{\sin^2(k+1)\varphi - \sin k\varphi\sin(k+2)\varphi}{2v\sin k\varphi\sin(k+1)\varphi} = \frac{\sin^2\varphi}{2v\sin k\varphi\sin(k+1)\varphi}. \tag{38}$$

Substitute b_k from (35), and note that $v = \cos\varphi$ to derive

$$b_{k+1} - b_k = \frac{v^k\sin\varphi}{\sin(k+1)\varphi} - \frac{v^{k-1}\sin\varphi}{\sin k\varphi} = \frac{2v^k\sin\varphi(\cos\varphi\sin k\varphi - \sin(k+1)\varphi)}{2v\sin k\varphi\sin(k+1)\varphi} = \frac{-2v^k\sin^2\varphi\cos k\varphi}{2v\sin k\varphi\sin(k+1)\varphi}. \tag{39}$$

Substitute c_k from (36), and note that $v = \cos\varphi$ to derive

$$c_{k+1} - c_k = \frac{v^{2k+1}\sin k\varphi}{2\sin(k+1)\varphi} - \frac{v^{2k-1}\sin(k-1)\varphi}{2\sin k\varphi} = \frac{v^{2k}(\cos^2\varphi\sin^2 k\varphi - \sin(k+1)\varphi\sin(k-1)\varphi)}{2v\sin k\varphi\sin(k+1)\varphi} = \frac{v^{2k}\sin^2\varphi\cos^2 k\varphi}{2v\sin k\varphi\sin(k+1)\varphi}. \tag{40}$$

Substitute the above relationships into (37), and cancel the non-zero common term $\frac{\sin^2\varphi}{2v\sin k\varphi\sin(k+1)\varphi}$ to derive

$$(1-\chi_k)^2 - 2v^k\cos k\varphi(1-\chi_k) + v^{2k}\cos^2 k\varphi = (1-\chi_k - v^k\cos k\varphi)^2 = 0.$$

Consequently,

$$\chi_k = 1 - v^k\cos k\varphi. \tag{41}$$

Note that both solutions to $P_{(k)}(\chi_k) = P_{(k+1)}(\chi_k)$ coincide, which means that the solution is unique.

Derivation of m

The way to find the maximum number of search periods is to write the condition that $\chi_m = 1$. This condition gives a critical value of δ; for slightly larger values of δ, there is an additional search period. This means $\varphi = \pi/(2m)$, which gives

$$m = \lceil \pi/(2\varphi) \rceil. \tag{42}$$

Constraints

Let us prove that for any value of x, (28) satisfies conditions (23) and (24). In addition, if $a_{k-1} \geq 0$, the optimal path includes $k-1$ steps only. Define that part of x which the single player does not search in the current period by $y = x - I_{(k)}$.

Next, show that condition (23) is satisfied; that is, $y \geq \chi_{k-2}$. Let us prove this inequality by contradiction, assuming that $y < \chi_{k-2}$. Note that by construction, when $y < \chi_{k-2}$, the following condition holds: $P_{(k-1)}(y) < P_{(k-2)}(y)$. That implies that instead of using the original k-period path (searching $x - y$ in the first period and making a further $k-1$ searches according to $P_{(k-1)}(y)$), the sole firm could use a $(k-1)$-period path (searching the same amount $x - y$ in the first period, and making a further $k-2$ searches according to $P_{(k-2)}(y)$), and increase the value. Refer to equation (21): both paths have the same first term, while the second term is larger for the $(k-1)$-period path. This implies that the k-period path does not improve the value in comparison with the optimal $(k-1)$-period path, which means that whenever $y < \chi_{k-2}$, the k-period path is not optimal. Condition $\chi_{k-2} \leq y$ is thus proved.

Let us show that condition (24) is satisfied; that is, $\Psi''_{k-1}(y) < 0$. From equation (28), it is clear that the sufficient condition for $\Psi''_{k-1}(y) < 0$ is that $P''_{(i-1)}(y) < 0 \;\forall\; i = 2, \ldots, k-1$. From equation (25), it is easy to see that the above condition is equivalent to $a_{i-1} < 0 \;\forall\; i = 2, \ldots, k-1$.

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Let us show that if $a_{k-1} \geq 0$, the optimal path includes $k-1$ steps only. Substitute χ_k from (41) into (25) to get

$$P_{(k)}(\chi_k) = \frac{v^{2k-1}}{2n \sin k\varphi}(-\sin(k+1)\varphi \cos^2 k\varphi + 2\sin\varphi \cos k\varphi + \sin(k-1)\varphi). \tag{43}$$

Use the fact that

$$\sin(k+1)\varphi = \sin(k-1)\varphi + 2\sin\varphi\cos k\varphi$$

and substitute into (43) to derive

$$P_{(k)}(\chi_k) = \frac{-\delta^k a_k \sin^2 k\varphi}{n}. \tag{44}$$

Wherever $a_{k-1} \geq 0$, then $P_{(k-1)}(\chi_{k-1}) \leq 0$, which means that the k-th step is unnecessary. Consequently, from (35) it follows that $\sin k\varphi > 0 \; \forall \; k = 1, \ldots, m$. This concludes the proof. □

Proof of Theorem 2

Construction of Ψ_1 and V_1

Let us start from the end of the search process. What will be the value if players could only search for at most one period? Equation (6) transforms into

$$\Psi_1(x) = \max_{I\in[0,x]}\{\min\{x, I + I_j\} - I_j - xI\}. \tag{45}$$

It is evident that in the unique non-trivial SMPE the equilibrium I can be described in the following way[26]:

$$I = x/2, \text{ if } x \leq 1. \tag{46}$$

Given $x \leq 1$, in the SMPE players search the whole island, $2I = x$. Consequently, the solution of (45) is

$$\Psi_1(x) = P_{(1)}(x), \tag{47}$$

where polynomial $P_{(1)}(x) = x(1-x)/2$. For future reference note that

$$P_{(1)}(x) = \frac{a_1}{2}(1-x)^2 + \frac{b_1}{2}(1-x) + \frac{c_1}{2}, \tag{48}$$

where

$$a_1 = -1, \quad b_1 = 1, \quad c_1 = 0. \tag{49}$$

Define the value for each player (if the players can search the island for at most k periods) as $V_k(x) \equiv \Psi_k(x)/x$, for any $x \geq 0$. From the above definition, it follows that

$$V_1(x) = (1-x)/2.$$

[26]Note that if $x = 1$, then *any* $I \in [0, x/2]$ is optimal. We assume that players choose $I = x/2$ in this case.

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Construction of Ψ_2 and V_2

What will be the value if players can search the island for at most two periods? In general there could be two possibilities, depending on the island size. The first possibility is that the players search the whole island in just one period. Intuitively this happens for small values of x because it is too costly to wait for another period when the island is very small. The second possibility is that the players search the island for at most two periods. This happens for large values of x.

We have already considered the first possibility in the previous section. Now we analyze the situation when players plan to search for at most two periods. The first step is to construct $\Psi_2(x)$. Equation (6) in this case transforms into

$$\Psi_2(x) = \max_{I \in [0,x]} \{\min\{x, I + I_j\} - I_j - xI + \delta \Psi_1 (\max\{0, x - I - I_j\})\}. \tag{50}$$

The necessary condition for I to be optimal in the interior of $[0, x]$, provided that search lasts for two periods, is

$$(1 - x) - \delta \Psi_1'(x - 2I) = 0. \tag{51}$$

In order to continue the search for the second period, the remaining island size has to satisfy

$$x - 2I \geq 0. \tag{52}$$

The sufficient condition for I to be optimal in the interior of $[0, x]$ is satisfied because

$$\Psi_1''(x - 2I) = a_1 < 0.$$

The way to proceed is to construct the equilibrium with the help of condition (51), and then show that the derived equilibrium satisfies condition (52).

From expressions (51) and (47), it follows that

$$(1 - x) - \delta \left(\frac{1 - 2(x - 2I)}{2} \right) = 0.$$

Consequently,

$$I = \frac{(2\delta - 2)x + 2 - \delta}{4\delta}. \tag{53}$$

Substituting (53) into equation (50), we obtain a *spline* of degree two on the interval $[0, 1]$:

$$\Psi_2(x) = \begin{cases} \Psi_1(x), & \text{if} \quad 0 \leq x \leq \chi_1, \\ P_{(2)}(x), & \text{if} \quad \chi_1 < x \leq 1; \end{cases} \tag{54}$$

where

$$P_{(2)}(x) = \frac{a_2}{2}(1 - x)^2 + \frac{b_2}{2}(1 - x) + \frac{c_2}{2}, \tag{55}$$

with

$$a_2 = -1, \quad b_2 = \frac{1}{2}, \quad c_2 = \frac{\delta}{4}. \tag{56}$$

The point $x = \chi_1$ is the first knot of the spline. When $x \geq \chi_1$ neither player has a unilateral incentive to switch from searching the island for two periods to searching the island for just one period; and $x = \chi_1$ is the point of indifference:

$$\Psi_2(\chi_1) = (1 - \chi_1)(\chi_1 - I). \tag{57}$$

On the right hand side of (57), $1 - \chi_1$ is the marginal payoff to search in the current period, and $\chi_1 - I$ is the additional search intensity required by any one player to complete search immediately. From (53) and (55), we get

$$\chi_1 = 1 - \frac{\delta}{2}. \tag{58}$$

Note that when $x = \chi_1$, one-period search is equivalent to two-period search, as in equilibrium players search nothing in the second period, values in (46) and (53) are the same.

It is straightforward to check that the solution given by (54) satisfies condition (52) for any $x \in [\chi_1, 1]$. Therefore, if the players can search the island for at most two periods, the SMPE is a spline of degree one on the interval $[0, 1]$ with one knot $x = \chi_1$:

$$I(x) = \begin{cases} \frac{x}{2}, & \text{if} \quad x \le \chi_1, \\ \frac{(2\delta-2)x+2-\delta}{4\delta}, & \text{if} \quad \chi_1 < x \le 1; \end{cases}$$

and the value function is

$$V_2(x) = \begin{cases} V_1(x), & \text{if} \quad x \le \chi_1, \\ P_{(2)}(x)/x, & \text{if} \quad \chi_1 < x \le 1. \end{cases}$$

We can describe the construction of Ψ_k and V_k now.

Construction of Ψ_k and V_k

What will be the value if players can search the whole island for at most $k \ge 3$ periods? In general there could be k possibilities, depending on the island size x_1. The players can plan to search the island for at most $1, 2, ..., k$ periods in a non-trivial SMPE.

Let us construct $\Psi_k(x)$. Equation (6) in this case transforms into

$$\Psi_k(x) = \max_{I \in [0,x]} \{\min\{x, I + I_j\} - I_j - xI + \delta\Psi_{k-1}(\max\{0, x - I - I_j\})\}. \tag{59}$$

A necessary condition for I to be the optimal value in the interior of $[0, x]$, provided that search lasts for up to k periods, is

$$(1 - x) = \delta\Psi'_{k-1}(x - 2I). \tag{60}$$

In order to continue search for the next period, the new value of x has to satisfy

$$x - 2I \ge \chi_{k-2}, \tag{61}$$

where $\chi_0 = 0$. The sufficient condition for I to be the optimal value in the interior of $[0, x]$ is satisfied if

$$\Psi''_{k-1}(x - 2I) < 0. \tag{62}$$

We will use condition (60) to find I, and then show that it satisfies conditions (61) and (62). Note that if function $\Psi_{k-1}(x)$ in (59) is a quadratic polynomial, then $\Psi_k(x) = B\Psi_{k-1}(x)$ has to be a quadratic polynomial as well. Since $P_{(1)}(x)$ and $P_{(2)}(x)$ are quadratic polynomials by (48)

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and (55), any $P_{(k)}(x)$ can be represented in the following form:

$$P_{(k)}(x) = \frac{a_k}{2}(1-x)^2 + \frac{b_k}{2}(1-x) + \frac{c_k}{2}, \quad k \geq 1. \tag{63}$$

Let $I_{(k)}$ be the optimal search intensity of a rival who anticipates search for up to k periods. From condition (60) and expression (63), it follows that

$$I_{(k)} = -\frac{(1-x)(2+2\delta a_{k-1})}{4\delta a_{k-1}} - \frac{b_{k-1}}{4a_{k-1}}. \tag{64}$$

At the knot $x = \chi_{k-1}$, each player is indifferent between planning to search the area for up to k periods and unilaterally accelerating search to ensure that it is concluded within $k-1$ periods:[27]

$$\Psi_k(\chi_{k-1}) = \Psi_{k-1}(\chi_{k-1}) + 2(1-\chi_{k-1})(I_{(k-1)} - I_{(k)}). \tag{65}$$

On the right hand side of (65), $1 - \chi_{k-1}$ is the marginal payoff to search in the current period and $2(I_{(k-1)} - I_{(k)})$ is the additional search intensity required in the current period to unilaterally shift to an accelerated search plan that will be concluded within $k-1$ periods.

For the moment, let us assume that equation (65) has a unique solution. The uniqueness of the solution will be proved later. Note however that when $x = \chi_{k-1}$, there are two equilibria corresponding to planning to search the area for k periods or for $k-1$ periods as the condition (65) is the same condition that describes potential deviations from both equilibria.

Substituting (64) into equation (59), we obtain a *spline* of degree two on the interval $[0, 1]$ with knots $\chi_1, ..., \chi_{k-1}$:

$$\Psi_k(x) = \begin{cases} \Psi_{k-1}(x), & \text{if} \quad 0 \leq x \leq \chi_{k-1}, \\ P_{(k)}(x), & \text{if} \quad \chi_{k-1} < x \leq 1; \end{cases} \tag{66}$$

where $P_{(k)}(x)$ is defined in (63). Therefore, if players plan to search the island for at most k periods, then $I(x)$ is a spline of degree one on the interval $[0, 1]$ with knots $\chi_1, ..., \chi_{k-1}$:

$$I(x) = \begin{cases} \frac{x}{2}, & \text{if} \quad x \leq \chi_1, \\ -\frac{(1-x)(2+2\delta a_1)}{4\delta a_1} - \frac{b_1}{4a_1}, & \text{if} \quad \chi_1 < x \leq \chi_2, \\ \vdots & \\ -\frac{(1-x)(2+2\delta a_{k-1})}{4\delta a_{k-1}} - \frac{b_{k-1}}{4a_{k-1}}, & \text{if} \quad \chi_{k-1} < x \leq 1; \end{cases}$$

and the value function is

$$V_k(x) = \begin{cases} V_{k-1}(x), & \text{if} \quad 0 \leq x \leq \chi_{k-1}, \\ P_{(k)}(x)/x, & \text{if} \quad \chi_{k-1} < x \leq 1. \end{cases}$$

[27] An alternative deviation is to intensify search more dramatically so that search is completed within $k-2$ periods. It is straightforward to show that accelerating search to ensure that it is concluded within $k-2$ (or fewer) periods is dominated by searching within $k-1$ periods.

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Derivation of a_k, b_k and c_k

Let us now find a_k, b_k, and c_k for any $k \geq 2$. Using (59), (63) and (64), we get the following system of difference equations:

$$a_k = -1, \quad b_k = \frac{b_{k-1}}{2}, \quad c_k = \delta \left(c_{k-1} + \frac{b_{k-1}^2}{4} \right), \quad k \geq 2, \tag{67}$$

where

$$a_1 = -1, \quad b_1 = 1, \quad c_1 = 0. \tag{68}$$

Let us find the solution to the system of difference equations (67). It is straightforward to derive $a_k = -1$ and $b_k = \frac{1}{2^{k-1}}$. The expression for c_k in (67) can be simplified to

$$c_k = \delta(c_{k-1} + 1/4^{k-1}). \tag{69}$$

Introduce a new variable $e_k = c_k 4^k$. Equation (69) transforms to

$$e_k = 4\delta(e_{k-1} + 1),$$

where $e_1 = 0$. The solution to this linear difference equation is $e_k = \frac{4\delta - (4\delta)^k}{1-4\delta}$. Substitute $c_k = e_k/4^k$ to derive $c_k = \frac{4\delta-(4\delta)^k}{(1-4\delta)4^k}$. Consequently,

$$a_k = -1, \;\; b_k = \frac{1}{2^{k-1}}, \;\; c_k = \left(\frac{(4\delta)^{k-1} - 1}{4^{k-1}(4\delta - 1)} \right) \delta. \tag{70}$$

Derivation of χ_k

To find χ_k, one needs to solve the quadratic equation $P_{(k+1)}(\chi_k) = P_{(k)}(\chi_k) + 2(1-\chi_k)(I_{(k)} - I_{(k+1)})$, namely

$$a_{k+1}(1-\chi_k)^2 + b_{k+1}(1-\chi_k) + c_{k+1} = a_k(1-\chi_k)^2 + b_k(1-\chi_k) + c_k + 2(1-\chi_k)(I_{(k)} - I_{(k+1)}), \quad k > 1.$$

From equation (70), $a_k = a_{k+1} = -1$; consequently,

$$\chi_k = 1 + \frac{c_{k+1} - c_k}{2(b_{k+1} - b_k)}.$$

Substitute b_k and c_k from equation (70) to derive the following indifference points

$$\chi_k = 1 - \frac{3\delta + (4\delta)^k(\delta - 1)}{2^{k+1}(4\delta - 1)}. \tag{71}$$

Note that the solution exists and is always unique. In the case when $k = 2$, $\chi_2 = 1 + \frac{3}{8}\delta - \frac{1}{2}\delta^2$.

Constraints and maximum search time

Define that part of x which player i does not search in the current period by $y = x - I_i$ and the part of x no player searches in the current period by $z = x - (I_1 + I_2)$. Let d indicate the maximum number of search periods for the two-player problem.

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Let us prove that for any value of x, (66) satisfies condition(62); that is, $\Psi''_{k-1}(z) < 0$. From equation (66), it is clear that the sufficient condition for $\Psi''_{k-1}(z) < 0$ is that $P''_{(i-1)}(z) < 0\ \forall\ i = 2, \ldots, k-1$. From equation (63), it is easy to see that the above condition is equivalent to $a_{i-1} < 0\ \forall\ i = 2, \ldots, k-1$. It is satisfied as from (68) $a_{i-1} = -1 < 0$.

Next, let us see if condition (61) is satisfied; that is, $\chi_{k-2} \leq z$. Let us prove the above condition directly when $k = 2$ and $k = 3$. From equations (64) and (70), it follows that $z(x) = 1 - 2^{1-k} - \frac{1-x}{\delta}$. It is easy to see that $z(x)$ is a monotonically increasing function in x. Consequently, it is sufficient to prove the above condition for $x = \chi_1$ when $k = 2$ and for $x = \chi_2$ when $k = 3$. Substituting values derived in (70) directly shows that both conditions are satisfied.

To see that condition (61) is never satisfied for $k = 4$, substitute $x = 1$ into $z(x)$ and show that $\chi_2 > z(1)$, proving that four-period path does not exist for $x \leq 1$. Consequently, the maximum number of search periods, d, is always less than four. Specifically,

$$d = \begin{cases} 2, & \text{if} \quad \delta \leq 0.75, \\ 3, & \text{if} \quad \delta > 0.75. \end{cases} \tag{72}$$

This concludes the proof. □

Proof of Corollary 1

First, $I(x)$ presented in Theorem 1 is a piece-wise linear function. Second, let us prove that $I(x)$ is also a continuous function for $x \leq 1$. Given that $I(x)$ is piece-wise linear, discontinuities are only possible in the knots of the spline. Substitute $\chi_s = 1 - \delta^{s/2}\cos s\varphi$ into $I_{(s)}(x)$ to derive

$$I_{(s)}(\chi_s) = -\delta^{s/2}\cos s\varphi \left(1 - \frac{\sin{(s-1)\varphi}}{\delta^{1/2}\sin s\varphi}\right) + \frac{\delta^{(s-1)/2}\sin\varphi}{\sin s\varphi}. \tag{73}$$

Next, substitute $\chi_s = 1 - \delta^{s/2}\cos s\varphi$ into $I_{(s+1)}(x)$ to derive

$$I_{(s+1)}(\chi_s) = -\delta^{s/2}\cos s\varphi \left(1 - \frac{\sin s\varphi}{\delta^{1/2}\sin{(s+1)\varphi}}\right) + \frac{\delta^{s/2}\sin\varphi}{\sin{(s+1)\varphi}}. \tag{74}$$

Use the fact that

$$\sin(s-1)\varphi\sin(s+1)\varphi = \sin^2 s\varphi - \sin^2\varphi, \tag{75}$$

to show that the expressions in (73) and (74) are equal.

To prove that $I(x)$ is a concave (hump-shaped) function it is sufficient to show that the slope of $I(x)$ is decreasing. Specifically, we need to show that

$$\left(1 - \frac{\sin{(s-1)\varphi}}{\delta^{1/2}\sin s\varphi}\right) > \left(1 - \frac{\sin s\varphi}{\delta^{1/2}\sin{(s+1)\varphi}}\right).$$

The above inequality follows from (75).

Let us prove the inequality $I_{(s)}(x) \geq x/s$ now. First, to prove this condition when $x = 1$, substitute $x = 1$ into $I_{(s)}(x)$, to derive $I_{(s)}(1) = \frac{\cos^{s-1}\varphi\sin\varphi}{\sin s\varphi} \geq 1/s$. Simplifying leads to the following inequality

$$\sin s\varphi \leq s\cos^{s-1}\varphi\sin\varphi. \tag{76}$$

Let us prove inequality (76) by induction. The statement is trivially correct for $s = 1$. Next, we

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show that whenever the inequality holds for $s = k-1$, it also holds for $s = k$. Suppose (76) holds for $s = k-1$. That is,

$$\sin{(k-1)\varphi} \leq (k-1)\cos^{k-2}\varphi \sin\varphi. \tag{77}$$

Observe that

$$\sin k\varphi = \sin{(k-1)\varphi}\cos\varphi + \cos{(k-1)\varphi}\sin\varphi, \tag{78}$$

and notice that if (77) and (78) hold, then it is sufficient for

$$\cos{(k-1)\varphi} \leq \cos^{k-1}\varphi. \tag{79}$$

to also hold to ensure that (76) applies for $s = k$.

Let us prove inequality (79) by a second layer of induction. The statement is trivially correct for $k = 1$. We show that, whenever it holds for $k = l-1$, it also holds for $k = l$. Suppose (79) holds for $k = l-1$; that is,

$$\cos{(l-2)\varphi} \leq \cos^{l-2}\varphi. \tag{80}$$

Use condition (80) and the fact that

$$\cos l\varphi = \cos{(l-1)\varphi}\cos\varphi - \sin{(l-1)\varphi}\sin\varphi, \tag{81}$$

to show that $\sin{(l-1)\varphi}\sin\varphi > 0$ is sufficient for (79) to hold at $k = l$. At the end of the proof of Theorem 1 it was proved that $\sin s\varphi > 0 \ \forall \ s \leq m$. Hence, by induction, (79) holds for any k and, returning to the first layer of induction, (76) holds for any s. Consequently, the inequality $I_{(s)}(1) \geq 1/s$ is proved.

Next, to prove inequality $I_{(s)}(x) \geq x/s$ for any x, use $I_{(s)}(x)$ to derive

$$(x-1)(1 - \frac{\sin{(s-1)\varphi}}{\delta^{1/2}\sin s\varphi}) + \frac{\delta^{(s-1)/2}\sin\varphi}{\sin s\varphi} \geq x/s. \tag{82}$$

The condition proved earlier $I_{(s)}(1) \geq 1/s$ and the fact that

$$\sin(s-1)\varphi = \sin s\varphi\cos\varphi - \cos s\varphi\sin\varphi, \tag{83}$$

allows us to show that condition

$$\sin s\varphi\cos\varphi \geq s\cos s\varphi\sin\varphi \tag{84}$$

is sufficient for (82) to hold. When $\cos s\varphi \leq 0$, condition (84) trivially holds. When $\cos s\varphi > 0$, condition (84) transforms to

$$\tan s\varphi \geq s\tan\varphi. \tag{85}$$

One can use the fact that

$$\tan{(s+1)\varphi} = \frac{\tan s\varphi + \tan\varphi}{1 - \tan s\varphi\tan\varphi} \tag{86}$$

to show that condition (85) holds. This concludes the proof. □

Proof of Corollary 2

First, it is easy to see $V(x)$ presented in Theorem 1 is continuous. Remember that knot χ_s is defined as an area for which the single player is indifferent between planning to search the area for s periods or for $s+1$ periods.

Second, to prove that the value function is continuously differentiable for $x \leq 1$, we need to show that derivatives of $V(x)$ coincide at knots χ_s; that is,

$$\left(\frac{P_{(s)}(\chi_s)}{\chi_s}\right)' = a_s - \frac{a_s + b_s + c_s}{\chi_s^2} \sim a_{s+1} - \frac{a_{s+1} + b_{s+1} + c_{s+1}}{\chi_s^2} = \left(\frac{P_{(s+1)}(\chi_s)}{\chi_s}\right)'.$$

Simplify the above expression to

$$\chi_s^2 \sim 1 + \frac{b_3 - b_2 + c_3 - c_2}{a_3 - a_2}. \tag{87}$$

The substitution of values (38), (39), (40) and (41) shows the above expression (87) is satisfied with equality.

Finally, to show that the value function is monotonically decreasing we need to demonstrate that $\forall\ s \geq 2$ the following inequality holds:

$$V_s'(x) = \left(\frac{P_{(s)}(x)}{x}\right)' = a_s - \frac{a_s + b_s + c_s}{x^2} < 0. \tag{88}$$

To prove inequality (88), note from equation (44) it follows that $a_s < 0$. Consequently, it is sufficient to show that $a_s + b_s + c_s > 0$. From equation (37) and consecutive derivation of χ_s, $P_{(s)}(x) > P_{(s-1)}(x)\ \forall\ x \neq \chi_s,\ \ s \geq 2$. This means, in particular, that $P_{(s)}(0) > P_{(s-1)}(0)$. Given that $P_{(1)}(0) = a_1 + b_1 + c_1 = 0$, it follows that $a_s + b_s + c_s > 0$. This proves the corollary. □

Proof of Corollaries 3 and 4

First, $I(x)$ presented in Theorem 2 is a piece-wise linear function. Second, let us prove that $I(x)$ is a continuous function when $\delta \leq 0.75$ and discontinuous otherwise. Given that $I(x)$ is piece-wise linear, discontinuities are only possible in the knots of the spline. When $\delta \leq 0.75$ there is only one knot χ_1. Substitute $x = \chi_1 = 1 - \delta/2$ into $I(x)$, when the project is planned to be finished in $s = 1$ and $s = 2$ periods, to derive $I(\chi_1) = (1 - \delta/2)/2$ in both cases. This proves the continuity result. To prove the discontinuity result for $\delta > 0.75$ substitute $x = \chi_2$ into $I(x)$, when the project is planned to be finished in $s = 2$ periods, to derive

$$I_{(2)}(\chi_2) = \frac{2 - \delta - 2(1-\delta)\chi_2}{4\delta}. \tag{89}$$

Next, substitute $x = \chi_2$ into $I(x)$, when the project is planned to be finished in 3 periods, to derive

$$I_{(3)}(\chi_2) = \frac{2 - 1.5\delta - 2(1-\delta)\chi_2}{4\delta}. \tag{90}$$

The expression in (89) is larger than the expression in (90); specifically, the difference is $\Delta I(\chi_s) = 1/2^{s+1}$. This proves the discontinuity result.

Third, let us find the slope of $I(x)$. If the project is finished in one period then the slope is positive, $I'(x) = \frac{1}{2}$, otherwise it is negative, $I'(x) = \frac{\delta - 1}{2\delta}$. This proves quasiconcavity of $I(x)$.

Next, consider the value function for $0 < \delta \leq 0.75$,

$$V(x) = \begin{cases} (1-x)/2, & \text{if} \quad x \leq 1 - \delta/2, \\ (1.5 - x)/2 - (2-\delta)/(8x), & \text{if} \quad x \in (1 - \delta/2, 1]. \end{cases} \tag{91}$$

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First, let us prove that $V(x)$ is a continuous function when $0 < \delta \leq 0.75$ and $x \leq 1$. Given that $V(x)$ is piece-wise continuous, a discontinuity is only possible at χ_1. Substitute $\chi_1 = 1 - \delta/2$ into $V(x)$, when the project is planned to be finished in $s = 1$ and $s = 2$ periods, to derive $V(\chi_1) = \delta/4$ in both cases. This proves the continuity result. Second, to show that $V(x)$ is a monotonically decreasing function when $0 < \delta \leq 0.75$ and $x \leq 1$, take a derivative of $V(x)$ and show that this derivative is equal to $-1/2$ when the project is planned to be finished in one period. This derivative is equal to $-1/2 + \frac{2-\delta}{8x^2} < 0$ for $x > 1 - \delta/2$, when the project is planned to be finished in two periods. This proves the monotonicity result.

Next consider the value function for $\delta > 0.75$ and $x \leq 1$,

$$V(x) = \begin{cases} \frac{1-x}{2}, & \text{if} \quad x \leq 1 - \frac{\delta}{2}, \\ \frac{1.5-x}{2} - \frac{2-\delta}{8x}, & \text{if} \quad x \in (1 - \frac{\delta}{2}, 1 + \frac{3\delta}{8} - \frac{\delta^2}{2}], \\ \frac{1.75-x}{2} - \frac{3-\delta/4-\delta^2}{8x}, & \text{if} \quad x \in (1 + \frac{3\delta}{8} - \frac{\delta^2}{2}, 1]. \end{cases} \tag{92}$$

To show that $V(x)$ is both a discontinuous and non-monotonic function for $x \leq 1$, compare $V(\chi_2 - \varepsilon)$ and $V(\chi_2 + \varepsilon)$, where ε is infinitely small. One can see that $V(\chi_2 - \varepsilon) > V(\chi_2 + \varepsilon)$ when $\delta > 0.75$. This concludes the proof. □

Proof of Proposition 1

For $0 < \delta \leq 0.5$;

$$J^{(1)}(x) = \begin{cases} x, & \text{if} \quad x \leq 1 - \delta, \\ \frac{(2\delta-1)x+1-\delta}{2\delta}, & \text{if} \quad x \in (1 - \delta, 1]; \end{cases} \tag{93}$$

and

$$J^{(2)}(x) = \begin{cases} x, & \text{if} \quad x \leq 1 - \delta/2, \\ \frac{(2\delta-2)x+2-\delta}{2\delta}, & \text{if} \quad x \in (1 - \delta/2, 1]. \end{cases} \tag{94}$$

Directly comparing (93) with (94) shows that $J^{(2)}(x) > J^{(1)}(x)$ for $x \in (1-\delta, 1)$ and $J^{(2)}(x) = J^{(1)}(x)$ for $x \in (0, 1-\delta] \cup \{1\}$. This proves the first two parts of the Proposition when $\delta \leq 0.5$.

To prove Part 3) of the Proposition when $\delta \leq 0.5$, note that for $x < 1$ the result that $W^{(2)}(x) < W^{(1)}(x)$ follows directly as the solution to a single-player problem is unique. If $x = 1$, note that when $0 < \delta \leq 0.5$, from (42) and (72) it follows that $m = d = 2$. (Recall that m and d refer to the maximum number of search periods in the single-player and two-player problems, respectively). Applying (25) and (63) gives $V^{(1)}(1) = 2V^{(2)}(1) = c_2 = \frac{\delta}{4}$.

For $0.5 < \delta \leq 0.75$;

$$J^{(1)}(x) = \begin{cases} x, & \text{if} \quad x \leq 1 - \delta, \\ \frac{(2\delta-1)x+1-\delta}{2\delta}, & \text{if} \quad x \in (1 - \delta, 1 + \delta - 2\delta^2], \\ \frac{(4\delta-3)(x-1)+\delta}{4\delta-1}, & \text{if} \quad x \in (1 + \delta - 2\delta^2, 1]; \end{cases} \tag{95}$$

and

$$J^{(2)}(x) = \begin{cases} x, & \text{if} \quad x \leq 1 - \delta/2, \\ \frac{(2\delta-2)x+2-\delta}{2\delta}, & \text{if} \quad x \in (1 - \delta/2, 1]. \end{cases} \tag{96}$$

Directly comparing (95) with (96) shows that $J^{(2)}(x) > J^{(1)}(x)$ for $x \in (1-\delta, 1]$ and $J^{(2)}(x) = J^{(1)}(x)$ for $x \in (0, 1-\delta]$. This proves the Proposition when $0.5 < \delta \leq 0.75$.

For $\delta > 0.75$

$$J^{(1)}(x) = I^{(1)}(x) = \begin{cases} I^{(1)}_{(1)}(x), & \text{if} \quad x \le \chi^{(1)}_1, \\ \vdots & \\ I^{(1)}_{(s)}(x), & \text{if} \quad \chi^{(1)}_{s-1} < x \le \chi^{(1)}_s, \\ \vdots & \\ I^{(1)}_{(m)}(x), & \text{if} \quad \chi^{(1)}_{m-1} < x \le 1; \end{cases} \tag{97}$$

where $I^{(1)}_{(s)}(x) = (x-1)(1 - \frac{\sin(s-1)\varphi}{\delta^{1/2}\sin s\varphi}) + \frac{\delta^{(s-1)/2}\sin\varphi}{\sin s\varphi}$, and $\chi^{(1)}_s = 1 - \delta^{s/2}\cos s\varphi$, $\varphi = \arccos\sqrt{\delta}$, $m = \lceil \pi/(2\varphi) \rceil$, for $s \in \{1, \dots, m\}$.

Similarly for $\delta > 0.75$,

$$J^{(2)}(x) = \begin{cases} x, & \text{if} \quad x \le 1 - \delta/2, \\ \frac{(1-\delta)(1-x)+\delta/2}{\delta}, & \text{if} \quad x \in (1-\delta/2, 1+3\delta/8-\delta^2/2], \\ \frac{(1-\delta)(1-x)+\delta/2}{\delta}, & \text{if} \quad x \in [1+3\delta/8-\delta^2/2, 1]. \end{cases} \tag{98}$$

Directly comparing (97) with (98) yields that $J^{(2)}(x) > J^{(1)}(x)$ for $x \in (1-\delta, 1+3\delta/8-\delta^2/2]$ and $J^{(2)}(x) < J^{(1)}(x)$ for $x \in (1+3\delta/8-\delta^2/2, 1]$, provided that $I^{(1)}_{(5)}(\chi^{(2)}_2) \ge I^{(2)}_{(3)}(\chi^{(2)}_2)$. This gives the first critical discount factor $\delta^* = \frac{1}{12}(4 + (10)^{1/3} + 10^{2/3}) \approx 0.9$. Consequently for $\delta \in (0.75, \delta^*]$, the critical value of x is $x^*(\delta) = \chi^{(2)}_2 = 1 + 3\delta/8 - \delta^2/2$.

For $\delta > \delta^*$, $x^*(\delta)$ can be found from $I^{(1)}_{(5)}(x^*(\delta)) = I^{(2)}_{(3)}(x^*(\delta))$, which results in $x^*(\delta) = \frac{1-7.75\delta+5\delta^2+3\delta^3}{1-8\delta+8\delta^2}$. This holds while $x^*(\delta) \le \chi^{(1)}_5$, corresponding to the second critical $\delta^{**} \approx 0.9064$ that solves:

$$512\delta^6 - 1152\delta^5 + 864\delta^4 - 240\delta^3 + 32\delta^2 - 12\delta + 1 = 0.$$

For $\delta > \delta^{**}$, $x^*(\delta)$ can be found from the following condition $I^{(1)}_{(6)}(x^*(\delta)) = I^{(2)}_{(3)}(x^*(\delta))$, yielding $x^*(\delta) = \frac{5-18.5\delta+8\delta^2+7\delta^3}{5-20\delta+16\delta^2}$. Finally, the constraint that $x^*(\delta) \le 1$ gives the third critical $\delta^{***} = \frac{1}{14}(8 + (22)^{1/2}) \approx 0.9065$.

To prove the third part of the proposition when $\delta > 0.75$, note that for $x \le 1$ the result that $W^{(2)}(x) < W^{(1)}(x)$ follow directly as the solution to a single-player problem is unique.

This proves Proposition 1. □

Proof of Proposition 2

Let us solve this problem by backward induction. Consider the last period subgame in which the treasure was not found in the previous periods and the unsearched area of the island is $x \le x_1$. Each player has a dominant strategy to search as much as possible in the last period. Therefore in any SPE, $I_1 + I_2 = x$. Consider, an asymmetric equilibrium with player 1 searching $\beta < 0.5$ of the remaining unsearched area and player 2 searching the remainder, $1 - \beta$. The one-period values for asymmetric players in the subgame with the unsearched area of x are

$$V_1(x) = \beta(1-x) \text{ and } V_2(x) = (1-\beta)(1-x).$$

Now consider the second last period. Note that if player i searches $\tilde{I}_i$ in the first period of the two-period game, her expected payoff from the two-period game is

$$\tilde{I}_i/x - \tilde{I}_i + \delta(1 - (\tilde{I}_1 + \tilde{I}_2)/x)V_i(x - \tilde{I}_1 - \tilde{I}_2),$$

where player i incurs search cost of $\tilde{I}_i$, gets immediate expected payoff of $\tilde{I}_i/x$, and anticipates the expected payoff of $\delta(1-(\tilde{I}_1+\tilde{I}_2)/x)V_i(x-\tilde{I}_1-\tilde{I}_2)$ in the second period. The first-order conditions with respect to $\tilde{I}_i$ are

$$\tilde{I}_1 + \tilde{I}_2 = \frac{1-x_1}{2\delta\beta} + x_1 - 0.5.$$

Note that the preferred combined search in the first period is higher in the case of the first player as $\beta < 1-\beta$. Consequently, there is a unique asymmetric SPE that involves only the first player searching $\tilde{I}_1 = \frac{1-x_1}{2\delta\beta} + x_1 - 0.5$ in the first period and both players searching in the second period. The intuition is that as the first player expects a smaller payoff in the second period, she is willing to search more in the first period. The values in the two-period game for players 1 and 2 are $\tilde{V}_1(x) = \frac{\delta^2\beta^2+2\delta\beta(1-x)+(1-4\delta\beta)(1-x)^2}{4\delta\beta x}$ and $\tilde{V}_2(x) = \frac{(\delta^2\beta^2-(1-x)^2)(1-\beta)}{4\delta\beta^2 x}$, respectively.

Finally, let us prove by contradiction that the maximum number of search periods in the asymmetric case is 2. Consider a 3-period path that has only player 1 searching in period 2. If player 2 increases her search intensity in period 1 by the amount that player 1 is meant to search in period 2, she will unilaterally shift to a search plan with at most two periods, while increasing her payoff. Not only does she get an additional immediate payoff from searching more, but also the path is shorter and consequently more profitable for player 2. This proves that an equilibrium with a 3-period path is not feasible. The proposition is proved. □

Proof of Corollary 7

Directly comparing

$$W_1^{(2)}(x) = \begin{cases} 1-x, & \text{if} \quad x \le 1-\delta/2, \\ \frac{\delta^2+2\delta(1-x)-4\delta(1-x)^2}{4\delta x}, & \text{if} \quad 1-\delta/2 < x \le 1+\frac{3\delta}{8}-\frac{\delta^2}{2}; \\ \frac{\delta^3+\delta^2/4+\delta(1-x)-4\delta(1-x)^2}{4\delta x}, & \text{if} \quad 1+\frac{3\delta}{8}-\frac{\delta^2}{2} < x \le 1; \end{cases}$$

with

$$W_1^{(asym)}(x) = \begin{cases} 1-x, & \text{if} \quad x \le 1-\delta\beta, \\ \frac{\delta^2+2\delta(1-x)+((2\beta-1)/\beta^2-4\delta)(1-x)^2}{4\delta x}, & \text{if} \quad 1-\delta\beta < x \le 1; \end{cases}$$

shows that $W_1^{(2)}(x) \ge W_1^{(asym)}(x)$. This proves the corollary. □

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