

STRONG CONVERGENCE OF QUANTUM RANDOM WALKS VIA SEMIGROUP DECOMPOSITION

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ABSTRACT. We give a simple and direct treatment of the strong convergence of quantum random walks to quantum stochastic operator cocycles, via the semigroup decomposition of such cocycles. Our approach also delivers convergence of the pointwise product of quantum random walks to the quantum stochastic Trotter product of the respective limit cocycles, thereby revealing the algebraic structure of the limiting procedure. The repeated quantum interactions model is shown to fit nicely into the convergence scheme described.

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INTRODUCTION

Quantum random walks have been a feature of noncommutative probability for over twenty-five years; as emphasised in [BvH], “the convergence of discrete quantum Markov chains to continuous ones is a fundamental problem in quantum probability”. In Meyer’s book ([Me₂]), Journé is credited as the first to use discrete approximations to the relevant symmetric Fock space and to quantum stochastic processes. Around the same time a central-limit theorem, yielding the quantum harmonic oscillator as a limit of quantum Bernoulli processes, was proved ([AcB]; see [Me₁]). One should also mention von Waldenfels’ earlier use of discrete approximation to define quantum Lévy processes on unitary matrix groups as multiplicative Itô integrals ([vWa]). In further early work, it was shown

2010 *Mathematics Subject Classification.* 46L53 (primary); 46N50, 81S25, 82C10, 60F17 (secondary).

Key words and phrases. Quantum random walk; repeated interactions; noncommutative Markov chain; toy Fock space; quantum stochastic cocycle; series product; quantum stochastic Trotter product.

that certain quantum stochastic flows, which are generalisations of classical diffusions, may be approximated by so-called spin random walks ([LiP]); see also [Pa₁], and [Sin]. More recently, a theory of quantum random walks generated by completely bounded maps on operator spaces was developed, in an approach which admits the treatment of particle algebras in an arbitrary normal state ([B₁₋₃]). The theory was then extended to quantum random walks in Banach algebras, further elucidating the way in which the limits arise ([DL₂]). The approach to discrete approximation in [BvH] is in the spirit of the current paper, and may be viewed as an unbounded-generator counterpart in which the Trotter-Kato theorem is exploited in place of Euler's exponential formula.

These convergence theorems are analogues of Donsker's invariance principle, with the limit process being a quantum stochastic cocycle, *i.e.* the quantum stochastic analogue of a stochastic semigroup in the sense of Skorohod ([Sko]), rather than a classical Wiener process. As well as their probabilistic interpretation as noncommutative Markov chains, quantum random walks may also be seen as models for the dynamics of a quantum-mechanical system undergoing repeated interactions with an environment composed of an infinite number of identical particles. This point of view was adopted in [AtP] and [AtJ]; links between the repeated-interactions model and time-ordered exponentials ([Ho₁]) were demonstrated in [Gou]. Our approach is readily modified to the convergence of mapping-valued (as opposed to operator-valued) quantum random walks, and thereby to the discrete approximation of quantum Lévy processes.

There have been many applications of quantum random walks: to quantum filtering and quantum feedback control ([GoS], [Bv+]); to the approximation of Lévy processes on quantum groups ([FrS], [LiS]); to the construction of dilations of quantum dynamical semigroups ([Sah], [B₁]). Repeated-interactions models for the one-atom maser, an important system in quantum optics ([GaZ]), have been investigated in [BJM] and [BrP]; in contrast to the results we prove below, the convergence theorems obtained in these papers give only the reduced dynamics of the limit system and disregard the limit behaviour of the environment. Interesting connections between noncommutative Markov chains and multivariate operator theory were explored in [Goh]. We should alert the reader to the fact that there are several other notions of quantum random walk in the literature, for example 'quantum walk on a graph', 'unitary random walk', in particular 'Hadamard walk' ([AA+], [Kon], [Kem]), and 'open quantum random walk' ([At+]). The approximation of continuous-time quantum random walks by discrete-time walks is addressed in [Chi], for the former type, and in [Pel], for the latter.

For us here, a quantum random walk is a discrete-time, bi-adapted covariant quantum stochastic evolution, or discrete-time quantum stochastic cocycle (Definition 2.1). Adaptedness and covariance of the quantum random walk are with respect to the natural operator filtration of, and the time shift on, the algebra of bounded operators on a toy Fock space (introduced in Section 2). The limiting objects are (continuous-time) bi-adapted covariant quantum stochastic evolutions, or quantum stochastic cocycles (Definition 1.1). Adaptedness and covariance of the QS cocycle are with respect to the natural operator filtration of, and the time shift on, the algebra of bounded operators on a symmetric Fock space with test functions from an L^2 -space of Hilbert space-valued functions on the half-line. Thus the notion of independence implicit here is that of tensor independence, as opposed to free independence, or freeness ([VDN]), for example.

A central feature of this work is the exploitation of what has come to be known as the 'semigroup approach' ([LW_{2,3}]; see [L₁]). Specifically, we use the semigroup decomposition of continuous-time quantum stochastic cocycles (given in (1.4)) and Euler's exponential formula (4.8) to give a new, direct, and considerably simplified proof of the convergence of suitably scaled quantum random walks to quantum stochastic cocycles. Properties of a certain nonlinear transformation on block matrix operators which we refer to as the

Holevo transform (Theorem 6.2 and Proposition 6.1), and a key observation on compositions (Theorem 5.1), accompany our main convergence theorem (Theorem 4.3). Together these lead to the realisation of a general class of quantum stochastic cocycles as scaled limits of quantum random walks of the corresponding kind, that is, contractive, isometric, or unitary (Theorem 7.4). They also yield short and transparent demonstrations of strengthened forms of results on the repeated-interactions model ([AtP], [ADP]). Specifically, in Theorem 8.1 we generalise Theorem 19 of [AtP], dispensing with underlying Hilbert–Schmidt-type assumptions on the components of the generator of the limiting stochastic cocycle, and in Theorem 8.2 we generalise Theorem 3.1 of [ADP] by avoiding any restriction on the dimension of the noise whilst allowing scattering in the interaction Hamiltonians. Our results are coordinate-free throughout.

Outline. The structure of the paper is as follows. Following a background section on quantum stochastic operator cocycles, Section 2 describes the very close analogy between such cocycles and quantum random walks on a Hilbert space. After a short section on the scaled embedding of QRWs as continuous-time processes on a Fock space, Section 4 contains the new proof of our central result, and its corollary on the approximation of quantum stochastic flows by QRWs on the algebra of bounded operators on a Hilbert space, *i.e.* the Heisenberg picture. The algebraic structure of the approximation scheme is exposed in Section 5. In Section 6, we discuss a basic (nonlinear) transformation on block matrix operators which we refer to as *the Holevo transform*; it provides means for some of the realisations of the approximation scheme given in Section 7. In Section 8 we show how the repeated quantum interactions model, and entanglement of bipartite systems, fit nicely into the general scheme developed here.

In a sister paper ([BG+]), we consider embeddings of toy Fock space appropriate to faithful states on a particle algebra, and obtain quasifree stochastic cocycles, in the sense of [LiM], as limits of scaled random walks in that setting.

Notation. For a vector-valued function $g : S \rightarrow V$ and subset A of S , g_A denotes the function $S \rightarrow V$ which agrees with g on A and vanishes elsewhere, extending the standard notation 1_A for the indicator function of A . We make extensive use of the following extension to (the mathematician’s version of) the Dirac bra-ket notation. For a vector u in a Hilbert space \mathfrak{h} , the operator $I_{\mathfrak{H}} \otimes |u\rangle : \mathfrak{H} \rightarrow \mathfrak{H} \otimes \mathfrak{h}$ given by $\xi \mapsto \xi \otimes u$, is denoted E_u ; its adjoint is denoted E^u . The Hilbert space \mathfrak{H} is always clear from the context. We denote the space of bounded operators from \mathfrak{H} to a Hilbert space \mathfrak{K} by $B(\mathfrak{H}; \mathfrak{K})$, abbreviating $B(\mathfrak{H}; \mathfrak{H})$ to $B(\mathfrak{H})$, and write $B(\mathfrak{H})_{\text{sa}}$ for the space of selfadjoint operators on \mathfrak{H} , and $\text{Re} T$, respectively $\text{Im} T$, for the real part $\frac{1}{2}(T + T^*)$ and imaginary part $\frac{1}{2i}(T - T^*)$ of an operator $T \in B(\mathfrak{H})$. The algebraic and ultraweak tensor products are denoted $\underline{\otimes}$ and $\overline{\otimes}$ respectively and, for vectors ζ and η in a Hilbert space \mathfrak{h} , the vector functional $T \mapsto \langle \zeta, T\eta \rangle$ on $B(\mathfrak{h})$ is denoted $\omega_{\zeta, \eta}$, or ω_{ζ} if $\eta = \zeta$. As usual, $B(\mathfrak{h})_*$ denotes the space of ultraweakly continuous functionals on $B(\mathfrak{h})$. We write Ran , Spec and Conv respectively for range, spectrum and convex hull. For the symmetric Fock space over a Hilbert space, exponential vectors, and second quantisation we use the following notations. Let $\mathfrak{h}^{\vee n}$ denote the n -fold symmetric tensor power of a Hilbert space \mathfrak{h} , with the convention $\mathfrak{h}^{\vee 0} := \mathbb{C}$, then, for $u \in \mathfrak{h}$, Hilbert spaces \mathfrak{h}_1 and \mathfrak{h}_2 and $C \in B(\mathfrak{h}_1; \mathfrak{h}_2)$,

$$\Gamma(\mathfrak{h}) := \bigoplus_{n \geq 0} \mathfrak{h}^{\vee n}, \quad \varepsilon(u) := ((n!)^{-1/2} u^{\otimes n})_{n \geq 0} \quad \text{and} \quad \Gamma(C) := \bigoplus_{n \geq 0} C^{\otimes n}, \quad (0.1)$$

where the latter is viewed as an operator from $\Gamma(\mathfrak{h}_1)$ to $\Gamma(\mathfrak{h}_2)$. Since $\|C^{\otimes n}\| = \|C\|^n$ for all $n \in \mathbb{Z}_+$, $\Gamma(C)$ is a contraction if C is, and is unbounded otherwise. Second quantisation enjoys the following functorial properties: for compatible contraction operators C_1 and

C_2 ,

$$\Gamma(I_{\mathfrak{h}}) = I_{\Gamma(\mathfrak{h})}, \quad \Gamma(C^*) = \Gamma(C)^*, \quad \Gamma(C_1 C_2) = \Gamma(C_1)\Gamma(C_2) \quad \text{and} \quad \Gamma(C)\varepsilon(u) = \varepsilon(Cu).$$

Fix Hilbert spaces \mathfrak{h} and \mathfrak{k} , referred to as the ‘initial space’ or ‘system space’, and the ‘noise dimension space’ respectively. The following notations are used throughout:

$$\widehat{c} := \begin{pmatrix} 1 \\ c \end{pmatrix} \in \widehat{\mathfrak{k}} := \mathbb{C} \oplus \mathfrak{k} \quad (c \in \mathfrak{k}), \quad \text{thus} \quad \widehat{0} := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \text{and}$$

$$\Delta := I_{\mathfrak{h}} \otimes (0_{\mathbb{C}} \oplus I_{\mathfrak{k}}) = \begin{bmatrix} 0 & 0 \\ 0 & I_{\mathfrak{h} \otimes \mathfrak{k}} \end{bmatrix}, \quad \text{thus} \quad \Delta^\perp = \begin{bmatrix} I_{\mathfrak{h}} & 0 \\ 0 & 0 \end{bmatrix}.$$

The Hilbert spaces $\mathfrak{h} \otimes \widehat{\mathfrak{k}}$ and $\mathfrak{h} \oplus (\mathfrak{h} \otimes \mathfrak{k})$ are identified, so that each operator $Q \in B(\mathfrak{h} \otimes \widehat{\mathfrak{k}})$ has a block matrix form $\begin{bmatrix} A & C \\ B & D \end{bmatrix}$.

1. QUANTUM STOCHASTIC COCYCLES

In this section we briefly recall the basic facts that are needed concerning quantum stochastic (QS) analysis, and specifically operator cocycles and their generation via QS differential equations. We emphasise that by *quantum stochastic process* we mean here time-indexed family of operators adapted to the natural filtration of subalgebras of the algebra of bounded operators on a symmetric Fock space over an L^2 -space of vector-valued functions, as in Definition 1.1 below. For more detail, see [L₁] which is our basic reference, and [L₂] where an exposition of the relevant quantum Itô algebra may be found. For further background, see [Pa₂], [Me₂] and [Fa₂].

For any subinterval I of \mathbb{R}_+ , set

$$\mathcal{F}_I = \mathcal{F}_I^{\mathfrak{k}} := \Gamma(L^2(I; \mathfrak{k})),$$

abbreviating to $\mathcal{F} = \mathcal{F}^{\mathfrak{k}}$ when $I = \mathbb{R}_+$. For any subset \mathbb{T} of \mathfrak{k} , let $\mathbb{S}_{\mathbb{T}}$ denote the subset of $L^2(\mathbb{R}_+; \mathfrak{k})$ consisting of \mathbb{T} -valued step functions, whose right-continuous versions we always take, and set $\mathcal{E}_{\mathbb{T}} := \text{Lin}\{\varepsilon(f) : f \in \mathbb{S}_{\mathbb{T}}\}$. (When $\mathbb{T} = \mathfrak{k}$ we abbreviate to \mathbb{S} and \mathcal{E} .) The subspace $\mathcal{E}_{\mathbb{T}}$ is dense in \mathcal{F} if and only if the set \mathbb{T} is total and contains 0 ([Ske]; see [L₁], Proposition 2.1). A typical example of \mathbb{T} is an orthonormal basis augmented by the vector 0. The natural identification

$$\mathcal{F} = \mathcal{F}_{[0,r[} \otimes \mathcal{F}_{[r,t[} \otimes \mathcal{F}_{[t,\infty[} \quad (r, t \in \mathbb{R}_+, r \leq t) \quad (1.1)$$

witnessed by exponential vectors, $\varepsilon(f) = \varepsilon(f|_{[0,r[}) \otimes \varepsilon(f|_{[r,t[}) \otimes \varepsilon(f|_{[t,\infty[})$, is frequently invoked. We use the notation $I_{[r,t[}^{\mathcal{F}}$ for the identity operator on $\mathcal{F}_{[r,t[}$.

Two families of endomorphisms of $B(\mathcal{F})$ are defined by

$$\sigma_t^{\mathcal{F}}(T) := I_{[0,t[}^{\mathcal{F}} \otimes S_t T S_t^* \quad \text{and} \quad \rho_t^{\mathcal{F}}(T) := R_t T R_t \quad (t \in \mathbb{R}_+)$$

where S_t is the shift operator $\Gamma(s_t) : \mathcal{F} \rightarrow \mathcal{F}_{[t,\infty[}$ and R_t is the time-reversal operator $\Gamma(r_t) : \mathcal{F} \rightarrow \mathcal{F}$, for the unitary operator $s_t : L^2(\mathbb{R}_+; \mathfrak{k}) \rightarrow L^2([t, \infty[; \mathfrak{k})$ and selfadjoint unitary operator $r_t : L^2(\mathbb{R}_+; \mathfrak{k}) \rightarrow L^2(\mathbb{R}_+; \mathfrak{k})$ defined by

$$(s_t f)(s) = f(s-t) \text{ for } s \in [t, \infty[\quad \text{and} \quad (r_t f)(s) = \begin{cases} f(t-s) & \text{if } 0 \leq s \leq t, \\ f(s) & \text{if } s > t. \end{cases}$$

Definition 1.1. A QS bounded-operator (left) cocycle on \mathfrak{h} with noise dimension space \mathfrak{k} is a family of operators $X = (X_t)_{t \geq 0}$ in $B(\mathfrak{h} \otimes \mathcal{F})$ satisfying the following adaptedness and cocycle conditions:

$$X_0 = I_{\mathfrak{h} \otimes \mathcal{F}}, \quad X_{r+t} = X_r \sigma_r(X_t) \quad \text{and} \quad X_t \in B(\mathfrak{h} \otimes \mathcal{F}_{[0,t[}) \otimes I_{[t,\infty[}^{\mathcal{F}} \quad (r, t \in \mathbb{R}_+),$$

where $\sigma_r := \text{id}_{B(\mathfrak{h})} \otimes \sigma_r^{\mathcal{F}}$. A QS cocycle X is called *elementary*, or *Markov regular*, if

$$s \mapsto X_s^{f,g} \text{ is continuous} \quad (f, g \in L_{\text{loc}}^2(\mathbb{R}_+; \mathfrak{k})).$$

The notation here is as follows. For a QS process X ,

$$X_s^{f,g} := E^\varepsilon(f_{[0,s]}) X_s E_\varepsilon(g_{[0,s]}) \quad \text{and} \quad f_{[0,s]} := 1_{[0,s]} f. \quad (1.2)$$

A QS cocycle X is called *contractive*, *isometric*, or *unitary* if each operator X_t has that property; it is called *quasicontractive* if, for some $\beta \in \mathbb{R}_+$, the QS cocycle $(e^{-\beta t} X_t)_{t \geq 0}$ is contractive; in this case

$$\beta_0(X) := \min \{ \beta \in \mathbb{R} : \|e^{-\beta t} X_t\| \leq 1 \text{ for all } t \in \mathbb{R}_+ \}$$

is referred to as the *exponential growth bound* of X .

If X is a QS cocycle then, for each $c, d \in \mathfrak{k}$,

$$P^{c,d} := (X_t^{c_{[0,t]}, d_{[0,t]}})_{t \geq 0} \quad (1.3)$$

defines a semigroup on \mathfrak{h} . Here a vector c in \mathfrak{k} is viewed as an element of $L_{\text{loc}}^2(\mathbb{R}_+; \mathfrak{k})$, with $c_{[0,t]}$ denoting the function equal to c on the interval $[0, t[$ and zero outside, for each $t \in \mathbb{R}_+$. If X is quasicontractive then X is elementary if and only if each of these *associated semigroups* is norm continuous. Moreover, QS cocycles are characterised (amongst adapted QS processes with exponential domain) by the semigroup-decomposition property:

$$X_t^{f,g} = P_{t_1-t_0}^{f(t_0), g(t_0)} \dots P_{t_{n+1}-t_n}^{f(t_n), g(t_n)} \quad (f, g \in \mathbb{S}, t \in \mathbb{R}_+) \quad (1.4)$$

in which the set $\{0 = t_0 < t_1 < \dots < t_n < t_{n+1} = t\}$ contains the points of discontinuity of $f_{[0,t]}$ and $g_{[0,t]}$ ([LW₂], Proposition 3.2). The *vacuum expectation semigroup* is the associated semigroup $P^{0,0}$, and the following conditions on a quasicontractive QS cocycle X are all equivalent:

- (i) X is strongly continuous;
- (ii) X^* (defined below) is strongly continuous;
- (iii) X is weak operator continuous;
- (iv) X has strongly continuous expectation semigroup

([LW₃], Lemma 1.2). Here we must mention an important symmetry of the theory. Given a QS cocycle X , its *dual cocycle* is defined by

$$X^\sharp := (\rho_t(X_t^*))_{t \in \mathbb{R}_+} = (\rho_t(X_t)^*)_{t \in \mathbb{R}_+},$$

where $\rho_t := \text{id}_{B(\mathfrak{h})} \otimes \overline{\rho_t^{\mathcal{F}}}$ ([Jou]). It is easily verified that X^\sharp is indeed a QS cocycle, and that the dual cocycle of X^\sharp is X . Given a QS cocycle X on \mathfrak{h} with noise dimension space \mathfrak{k} ,

$$(X_{r,t} := \sigma_r(X_{t-r}))_{0 \leq r \leq t}$$

defines a (*continuous-time*) *bi-adapted covariant (left) evolution*, that is

$$\begin{aligned} X_{r,t} &\in (B(\mathfrak{h}) \otimes I_{[0,r]}^{\mathcal{F}}) \overline{\otimes} (B(\mathcal{F}_{[r,t]}) \otimes I_{[t,\infty]}^{\mathcal{F}}), \\ X_{r+u,t+u} &= \sigma_u(X_{r,t}), \\ X_{t,t} &= I_{\mathfrak{h} \otimes \mathcal{F}}, \quad \text{and} \quad X_{r,t} = X_{r,s} X_{s,t} \end{aligned}$$

for $r, s, t, u \in \mathbb{R}_+$ with $r \leq s$ and $s \leq t$. Furthermore, every such evolution arises in this way. Extending the notation (1.2) as follows,

$$X_{r,t}^{f,g} := E^\varepsilon(f_{[r,t]}) X_{r,t} E_\varepsilon(g_{[r,t]}), \quad (1.5)$$

the family $(X_{r,t}^{f,g})_{0 \leq r \leq t}$ forms an evolution in $B(\mathfrak{h})$, for each $f, g \in L_{\text{loc}}^2(\mathbb{R}_+; \mathfrak{k})$.

Remark. In this paper we deal with QS left cocycles throughout. There are also QS right cocycles, defined in the same way as left cocycles except that the cocycle identity now reads $X_{r+t} = \sigma_r(X_t) X_r$ ($r, t \in \mathbb{R}_+$). The adjoint and time-reversal operations, given respectively by $X^* := ((X_t)^*)_{t \in \mathbb{R}_+}$ and $X^{\text{r}} := (\rho_t(X_t))_{t \in \mathbb{R}_+}$, turn QS left cocycles into right ones, and vice-versa. Note that $X^\sharp = X^{*\text{r}} = X^{\text{r}*}$.

For future reference (in Sections 6 and 7) we next discuss the stochastic generation of QS cocycles and the associated Itô algebra of generators, in particular we give a sample decomposition of the generator of an isometric QS cocycle; here we are summarising results from [L₂], where further detail may be found. In Section 6 this is related to the compilation of QRW generators, and in Section 7 it is shown how to tailor these compositions/decompositions for convergence to a given QS cocycle.

As will be increasingly clear, the crucial composition law for (bounded) generators of QS cocycles is the *series product*. This is the composition on $B(\mathfrak{h} \otimes \widehat{\mathfrak{k}})$ defined by

$$F_1 \triangleleft F_2 := F_1 + F_2 + F_1 \Delta F_2. \quad (1.6)$$

In [L₂] it is shown that $(B(\mathfrak{h} \otimes \widehat{\mathfrak{k}}), \triangleleft, *)$ is a *-monoid, that is, an involutive semigroup-with-identity. A significant representation of this *-monoid is given in Section 6 below.

For operators $Z \in B(\mathfrak{h})$, $L \in B(\mathfrak{h}; \mathfrak{h} \otimes \mathfrak{k})$, $M \in B(\mathfrak{h} \otimes \mathfrak{k}; \mathfrak{h})$ and $W \in B(\mathfrak{h} \otimes \mathfrak{k})$, we set

$$F_{Z,L,W} := \begin{bmatrix} Z - \frac{1}{2}L^*L & -L^*W \\ L & W - I \end{bmatrix}. \quad (1.7)$$

Note that

$$(F_{Z,L,W})^* \triangleleft F_{Z,L,W} = \begin{bmatrix} Z^* + Z & 0 \\ 0 & W^*W - I \end{bmatrix},$$

and, for $Z_i \in B(\mathfrak{h})$, $L_i \in B(\mathfrak{h}; \mathfrak{h} \otimes \mathfrak{k})$, and $W_i \in B(\mathfrak{h} \otimes \mathfrak{k})$ ($i = 1, 2$),

$$F_{Z_1, L_1, W_1} \triangleleft F_{Z_2, L_2, W_2} = F_{Z, L, W},$$

where

$$\begin{aligned} W &= W_1 W_2, \\ L &= L_1 + W_1 L_2, \quad \text{and} \\ Z &= Z_1 + Z_2 - \frac{1}{2}L_2^*(I - W_1^*W_1)L_2 - i \operatorname{Im} L_1^*W_1 L_2. \end{aligned} \quad (1.8)$$

In particular, for $Z = Z_0 + \dots + Z_5$ where $Z_0, \dots, Z_5 \in B(\mathfrak{h})$,

$$F_{Z_0, 0, I} \triangleleft F_{Z_1, 0, I} \triangleleft F_{Z_2, L, I} \triangleleft F_{Z_3, 0, I} \triangleleft F_{Z_4, 0, W} \triangleleft F_{Z_5, 0, I} = F_{Z, L, W}. \quad (1.9)$$

For us here, the following properties (all proved in [L₂]) are key; they should be read in conjunction with Theorem 1.2 below.

(i) The isometric structure relation $F^* \triangleleft F = 0$ is equivalent to F being of the form $F_{Z,L,W}$ with Z skewadjoint and W isometric, and the coisometric structure relation $F \triangleleft F^* = 0$ is equivalent to F being of the form

$$\begin{bmatrix} Z - \frac{1}{2}MM^* & M \\ -WM^* & W - I \end{bmatrix},$$

with Z skewadjoint and W coisometric.

(ii) For $\beta \in \mathbb{R}$, the relations $F^* \triangleleft F \leq 2\beta\Delta^\perp$ and $F \triangleleft F^* \leq 2\beta\Delta^\perp$ are equivalent.

(iii) Setting $F = F_1 \triangleleft F_2$,

$$\text{if } F_i^* \triangleleft F_i \leq 2\beta_i\Delta^\perp \text{ for } i = 1, 2, \text{ then } F^* \triangleleft F \leq (\beta_1 + \beta_2)\Delta^\perp;$$

$$\text{if } F_i^* \triangleleft F_i = 0 \text{ for } i = 1, 2, \text{ then } F^* \triangleleft F = 0.$$

By a *weak solution* of the QS differential equation $dX_t = X_t d\Lambda_F(t)$ with $X_0 = I_{\mathfrak{h} \otimes \mathcal{F}}$, is meant a family of operators $X = (X_t)_{t \geq 0}$ on $\mathfrak{h} \otimes \mathcal{F}$ with domain $\mathfrak{h} \otimes \mathcal{E}$ such that, for all $f, g \in \mathbb{S}$, $u, v \in \mathfrak{h}$ and $t \in \mathbb{R}_+$,

- (a) $\langle u\varepsilon(f), X_t v\varepsilon(g) \rangle = \langle u\varepsilon(f_{[0,t]}), X_t v\varepsilon(g_{[0,t]}) \rangle \langle \varepsilon(f_{[t,\infty[)}], \varepsilon(g_{[t,\infty[)} \rangle$,
- (b) $s \mapsto \langle u\varepsilon(f), X_s v\varepsilon(g) \rangle$ is continuous, and
- (c) $\langle u\varepsilon(f), (X_t - I_{\mathfrak{h} \otimes \mathcal{F}})v\varepsilon(g) \rangle = \int_0^t ds \langle u\varepsilon(f), X_s E^{\hat{f}(s)} F E_{\hat{g}(s)} v\varepsilon(g) \rangle$.

A *strong solution* is a weak solution that is sufficiently regular that the QS integrals $\int_0^t X_s d\Lambda_F(s)$ are defined and (c) holds in integrated form:

(c)' $X_t - I_{\mathfrak{h} \otimes \mathcal{F}} = \int_0^t X_s d\Lambda_F(s)$ ($t \in \mathbb{R}_+$)
 (see [L₁]). *Weak regularity* means, for all $f, g \in \mathbb{S}$,

$(E^{\varepsilon(f)} X_t E_{\varepsilon(g)})_{t \geq 0}$ is bounded operator valued and locally uniformly bounded.

Theorem 1.2 ([Fa₁], Proposition 3.1; [LW₁], Theorem 7.5). *Let X be an elementary quasicontractive QS cocycle on \mathfrak{h} with noise dimension space \mathfrak{k} , and let \mathbb{T} be a total subset of \mathfrak{k} containing 0. Then there is a unique operator $F \in B(\mathfrak{h} \otimes \widehat{\mathfrak{k}})$, called its stochastic generator, such that X weakly satisfies the QS differential equation*

$$X_0 = I_{\mathfrak{h} \otimes \mathcal{F}}, \quad dX_t = X_t d\Lambda_F(t) \quad (1.10)$$

on the exponential domain $\mathfrak{h} \otimes \mathcal{E}_{\mathbb{T}}$. The QS cocycle X is strongly continuous and satisfies (1.10) strongly on $\mathfrak{h} \otimes \mathcal{E}$, moreover

$F^* \triangleleft F \leq 2\beta\Delta^\perp$ if and only if $(e^{-\beta t} X_t)_{t \geq 0}$ is contractive ($\beta \in \mathbb{R}$);

$F^* \triangleleft F = 0$ if and only if X is isometric;

$F \triangleleft F^* = 0$ if and only if X is coisometric.

Conversely, let $F \in B(\mathfrak{h} \otimes \widehat{\mathfrak{k}})$. Then the QS differential equation (1.10) has a unique weakly regular, weak solution, denoted X^F . Moreover, if F satisfies $F^* \triangleleft F \leq 2\beta\Delta^\perp$ for some $\beta \in \mathbb{R}$, then X^F is an elementary quasicontractive QS cocycle.

Remarks. (i) In the converse part, the QS process X^F need not be bounded if the constraint $F^* \triangleleft F \leq 2\beta\Delta^\perp$ is not imposed.

(ii) QS generation and duality are related in the following simple way ([L₁], p. 252): if $X = X^F$ then $X^\sharp = X^{F^*}$.

(iii) Suppose that, for $i = 1, 2$, $F_i \in B(\mathfrak{h} \otimes \widehat{\mathfrak{k}})$ satisfies $F_i^* \triangleleft F_i \leq 2\beta_i\Delta^\perp$ for some $\beta_i \in \mathbb{R}$. Then the quasicontractive QS cocycle $X^{F_1 \triangleleft F_2}$ is expressible in terms of limits of QS Trotter products of the cocycles X^{F_1} and X^{F_2} ([L₂], Proposition 3.4).

2. QUANTUM RANDOM WALKS

In this section we register the basic facts about quantum random walks on a Hilbert space. One aim here is to emphasise the very close analogy between quantum random walks and QS cocycles. Indeed we show how QS cocycles may naturally be viewed as the continuous-time counterpart to quantum random walks.

For any $m, n \in \mathbb{Z}_+$ with $m \leq n$, set

$$\Upsilon_{[m,n]} = \Upsilon_{[m,n]}^{\mathfrak{k}} := \widehat{\mathfrak{k}}_{(m)} \otimes \cdots \otimes \widehat{\mathfrak{k}}_{(n-1)} \quad \text{and} \quad \Upsilon_{[n,\infty]} := \bigotimes_{p=n}^{\infty} \widehat{\mathfrak{k}}_{(p)}$$

where $\widehat{\mathfrak{k}}_{(n)} = \widehat{\mathfrak{k}}$ for each $n \in \mathbb{Z}_+$ and the infinite tensor product is with respect to the constant stabilising sequence of unit vectors $\widehat{0}$; also set

$$\Upsilon := \Upsilon_{[0,\infty]}.$$

Whether intervals are discrete or continuous will always be clear from context. The ‘toy Fock space’ identifications

$$\Upsilon = \Upsilon_{[0,m]} \otimes \Upsilon_{[m,n]} \otimes \Upsilon_{[n,\infty]} \quad (m, n \in \mathbb{Z}_+, m \leq n)$$

are discrete analogues of the continuous tensor decompositions (1.1) of \mathcal{F} . We use the notation $I_{[m,n]}^\Upsilon$ for the corresponding identity operators.

Two families of endomorphisms of $B(\Upsilon)$ are defined by

$$\sigma_n^\Upsilon : T \mapsto I_{[0,n]}^\Upsilon \otimes S_n T S_n^* \quad \text{and} \quad \rho_n^\Upsilon : T \mapsto R_n T R_n \quad (n \in \mathbb{Z}_+),$$

where S_n is the unitary shift operator $\Upsilon \rightarrow \Upsilon_{[n,\infty[}$, and $R_n := R_n \otimes I_{[n,\infty[}^\Upsilon$ for the selfadjoint unitary operator R_n on $\Upsilon_{[0,n]}$ determined by $R_n(\zeta_1 \otimes \cdots \otimes \zeta_n) = \zeta_n \otimes \cdots \otimes \zeta_1$. Also define the embedding

$$j^\Upsilon : B(\widehat{\mathfrak{k}}) \rightarrow B(\Upsilon), \quad T \mapsto T \otimes I_{[1,\infty[}^\Upsilon.$$

Definition 2.1. A discrete-time QS (left) cocycle on \mathfrak{h} with noise dimension space \mathfrak{k} is a family $(W_n)_{n \in \mathbb{Z}_+}$ in $B(\mathfrak{h} \otimes \Upsilon)$ such that

$$W_0 = I_{\mathfrak{h} \otimes \Upsilon}, \quad W_{l+n} = W_l \sigma_l(W_n) \quad \text{and} \quad W_n \in B(\mathfrak{h} \otimes \Upsilon_{[0,n]}) \otimes I_{[n,\infty[}^\Upsilon \quad (l, n \in \mathbb{Z}_+)$$

where $\sigma_l := \text{id}_{B(\mathfrak{h})} \overline{\otimes} \sigma_l^\Upsilon$. We refer to these as (*left*) *quantum random walks* (QRW).

Thus QRWs are determined by the family $(W_n) \in B(\mathfrak{h} \otimes \Upsilon_{[0,n]})_{n \in \mathbb{N}}$ for which

$$W_n = W_n \otimes I_{[n,\infty[}^\Upsilon \quad (n \in \mathbb{N}). \quad (2.1)$$

Given a QRW W , the *dual* QRW is defined by

$$W^\sharp := (\rho_n(W_n^*))_{n \in \mathbb{Z}_+} = (\rho_n(W_n)^*)_{n \in \mathbb{Z}_+},$$

where $\rho_n := \text{id}_{B(\mathfrak{h})} \overline{\otimes} \rho_n^\Upsilon$. As with QS cocycles, it is easily verified that W^\sharp is indeed a QRW, and that its dual is W .

Let $G \in B(\mathfrak{h} \otimes \widehat{\mathfrak{k}})$. Then the family $(W_n)_{n \in \mathbb{Z}_+}$ in $B(\mathfrak{h} \otimes \Upsilon)$ defined by

$$W_0 = I_{\mathfrak{h} \otimes \Upsilon} \quad \text{and} \quad W_n := \prod_{0 \leq i < n}^{\rightarrow} G_i, \quad \text{where} \quad G_i := \sigma_i((\text{id}_{B(\mathfrak{h})} \overline{\otimes} j^\Upsilon)(G)) \quad \text{for} \quad n \in \mathbb{N}$$

is readily seen to define a QRW on \mathfrak{h} , which is denoted W^G , and, since $B(\mathfrak{h} \otimes \Upsilon_{[0,1]}) \otimes I_{[1,\infty[}^\Upsilon = \text{Ran}(\text{id}_{B(\mathfrak{h})} \overline{\otimes} j^\Upsilon)$, it is clear that every QRW arises in this way. The operator G is referred to as the *generator* of the QRW. Generation and duality are related in the following simple way: if $W = W^G$ then $W^\sharp = W^{G^*}$.

Given a left QRW W on \mathfrak{h} with noise dimension space \mathfrak{k} ,

$$(W_{l,n} := \sigma_l(W_{n-l}))_{0 \leq l \leq n}$$

defines a *discrete-time bi-adapted covariant evolution*, that is

$$\begin{aligned} W_{l,n} &\in (B(\mathfrak{h}) \otimes I_{[0,l]}^\Upsilon) \overline{\otimes} (B(\Upsilon_{[l,n]}) \otimes I_{[n,\infty[}^\Upsilon), \\ W_{l+p,n+p} &= \sigma_p(W_{l,n}), \\ W_{n,n} &= I_{\mathfrak{h} \otimes \Upsilon} \quad \text{and} \quad W_{l,n} = W_{l,m} W_{m,n}, \end{aligned}$$

for $l, m, n, p \in \mathbb{Z}_+$ with $l \leq m$ and $m \leq n$. Conversely, every such evolution $(W_{l,n})_{0 \leq l \leq n}$ is so determined by the left QRW $(W_{0,n})_{n \in \mathbb{Z}_+}$. In view of the covariance property,

$$W_{l,n} = \prod_{l \leq m < n}^{\rightarrow} W_{m,m+1},$$

and, in terms of its generator G ,

$$W_{m,m+1} = (\text{id}_{B(\mathfrak{h})} \overline{\otimes} (\sigma_m^\Upsilon \circ j^\Upsilon))(G) \quad (m \in \mathbb{Z}_+).$$

Remark. Here, as for QS cocycles, we deal with left QRWs throughout. There are also right QRWs, defined in the same way as left QRWs except that the cocycle identity is switched to $W_{l+n} = \sigma_l(W_n) W_l$ ($l, n \in \mathbb{Z}_+$). The adjoint and time-reversal operations, given respectively by $W^* := ((W_n)^*)_{n \in \mathbb{Z}_+}$ and $W^r := (\rho_n(W_n))_{n \in \mathbb{Z}_+}$, turn left QRWs into right ones, and vice-versa. Note that, for a left or right QRW, $W^\sharp = W^{*r} = W^{r*}$.

3. EMBEDDING

Suitably scaled discrete QS cocycles converge to continuous QS cocycles in the sense made precise in Theorem 4.3 below. This entails embedding QRWs into the habitat of continuous-time processes, for which the relevant definition follows.

Definition 3.1. Let $h > 0$. The h -scale embedded left QRW generated by $G \in B(\mathfrak{h} \otimes \widehat{\mathfrak{k}})$, is the bounded-operator QS process X on \mathfrak{h} , with noise dimension space \mathfrak{k} , defined by $X_t := X_{0,h\lfloor t/h \rfloor}$ where

$$X_{hl,hn} := \prod_{l \leq m < n}^{\rightarrow} X_{hm,h(m+1)} \quad (l, n \in \mathbb{Z}_+)$$

and

$$X_{hm,h(m+1)} := (\text{id}_{B(\mathfrak{h})} \otimes (\sigma_{hm}^{\mathcal{F}} \circ J_h^{\mathcal{F}}))(G) \quad (m \in \mathbb{Z}_+),$$

through the embedding

$$J_h^{\mathcal{F}} : B(\widehat{\mathfrak{k}}) \rightarrow B(\mathcal{F}), \quad T \mapsto J_h T (J_h)^* \otimes I_{[h,\infty[}^{\mathcal{F}}$$

in which $J_h : \widehat{\mathfrak{k}} \rightarrow \mathcal{F}_{[0,h[}$ denotes the isometry determined by the prescription

$$\widehat{c} \mapsto \widetilde{\varepsilon}(h^{-1/2}c).$$

Here the vector $h^{-1/2}c$ is considered as the corresponding constant function in $L^2([0, h[; \mathfrak{k})$, and the following *truncated exponential vectors* are employed

$$\widetilde{\varepsilon}(g) := (1, g, 0, 0, \dots) \quad (g \in L^2(I; \mathfrak{k}), I \text{ a subinterval of } \mathbb{R}_+). \quad (3.1)$$

Notation. The h -scale embedded left QRW generated by $G \in B(\mathfrak{h} \otimes \widehat{\mathfrak{k}})$ is denoted $X^{h,G}$.

Remark. For future reference, we note the following elementary estimate on embedded quantum random walks:

$$\|X_t^{h,G}\| \leq \|G\|^{\lfloor t/h \rfloor} \quad (t \in \mathbb{R}_+). \quad (3.2)$$

In particular, the process $X^{h,G}$ is contractive if the QRW generator G is. It is obviously isometric or coisometric if and only if G has the same property.

4. CONVERGENCE

In this section we show that suitably scaled families of QRWs converge to QS cocycles, in analogy with the Donsker invariance principle.

For $n \in \mathbb{Z}_+$, and for g in either $L_{\text{loc}}^2(\mathbb{R}_+; \mathfrak{k})$ or $L^2([hn, h(n+1)[; \mathfrak{k})$, let $g[n, h]$ denote the average of g over the interval $[hn, h(n+1)[$:

$$g[n, h] := h^{-1} \int_{hn}^{h(n+1)} g. \quad (4.1)$$

Thus, for $g \in L^2([0, h[; \mathfrak{k})$, $(J_h)^* \varepsilon(g) = \sqrt{h} \widehat{g[0, h]}$.

Remark. Observe that, in the notation

$$X_{mh,nh}^{f,g} := E^{\varepsilon(f_{[hm,hn[})} X_{mh,nh} E_{\varepsilon(g_{[hm,hn[})} \quad (f, g \in \mathbb{S}, m, n \in \mathbb{Z}_+, m \leq n), \quad (4.2)$$

where $X = X^{h,G}$, we have discrete evolutions for each $f, g \in \mathbb{S}$:

$$X_{hn,hn}^{f,g} = I_{\mathfrak{h}}, \quad X_{hl,hm}^{f,g} X_{hm,hn}^{f,g} = X_{hl,hn}^{f,g} \quad (l, m, n \in \mathbb{Z}_+, l \leq m \leq n). \quad (4.3)$$

For $h > 0$, define the standard scaling matrix (*cf.* [LiP])

$$\mathcal{S}_h^k := \begin{bmatrix} h^{-1/2} & 0 \\ 0 & I_k \end{bmatrix} \in B(\widehat{\mathbf{k}}),$$

and let s_h denote conjugation by $I_{\mathfrak{h}} \otimes \mathcal{S}_h^k$ on $B(\mathfrak{h} \otimes \widehat{\mathbf{k}})$, thus

$$s_h \left(\begin{bmatrix} A & C \\ B & D \end{bmatrix} \right) = \begin{bmatrix} h^{-1}A & h^{-1/2}C \\ h^{-1/2}B & D \end{bmatrix} \quad (h > 0). \quad (4.4)$$

On the one hand the scaling is motivated by purely (quantum) probabilistic considerations via Donsker's functional central limit theorem, and on the other hand it is related to the weak coupling and low density limits of statistical physics ([vHo], [Dav], [Düm]). The connection is emphasised in [AFL], for example; for further detailed discussion on this, see [AtJ]. In Section 8 we see how the scaling operates in the important example of repeated quantum interactions.

Lemma 4.1. *Set $X = X^{h,G}$ where $G \in B(\mathfrak{h} \otimes \widehat{\mathbf{k}})$ and $h > 0$. Let $f, g \in \mathbb{S}$ and $m, n \in \mathbb{Z}_+$ with $m \leq n$.*

(a) *Then*

$$X_{hn, h(n+1)}^{f,g} = I_{\mathfrak{h}} + h E^{\widehat{f[n, h]}} s_h(G - \Delta^\perp) E_{\widehat{g[n, h]}} \quad (4.5)$$

and

$$\|X_{hn, h(n+1)}^{f,g} - I_{\mathfrak{h}}\| \leq h \max_{c \in \text{Ran } f, d \in \text{Ran } g} \|E_{\widehat{c}} s_h(G - \Delta^\perp) E_{\widehat{d}}\|. \quad (4.6)$$

(b) *Suppose that f and g are constant, with values c and d respectively, on the interval $[hm, hn]$. Then*

$$X_{hm, hn}^{f,g} = (I_{\mathfrak{h}} + h E_{\widehat{c}} s_h(G - \Delta^\perp) E_{\widehat{d}})^{n-m}. \quad (4.7)$$

Proof. (a) Since $\widehat{\sqrt{h}c} = \sqrt{h} \mathcal{S}_h^k \widehat{c}$ for $c \in \mathbf{k}$, the first identity follows from the definition:

$$\begin{aligned} X_{hn, h(n+1)}^{f,g} - I_{\mathfrak{h}} &= E^{\widehat{\sqrt{h}f[n, h]}} (G - \Delta^\perp) E_{\widehat{\sqrt{h}g[n, h]}} \\ &= h E^{\widehat{f[n, h]}} s_h(G - \Delta^\perp) E_{\widehat{g[n, h]}}. \end{aligned}$$

Since

$$\widehat{f[n, h]} = h^{-1} \int_{hn}^{h(n+1)} \widehat{f} \in \text{Conv Ran } \widehat{f},$$

and similarly for g , (4.6) follows from (4.5).

(b) Since $\widehat{f[j, h]} = \widehat{c}$ and $\widehat{g[j, h]} = \widehat{d}$ for $j \in \{m, \dots, n-1\}$, this follows from the factorisation

$$X_{hm, hn}^{f,g} := X_{hm, h(m+1)}^{f,g} \cdots X_{h(n-1), hn}^{f,g}$$

and identity (4.5). \square

In order to obtain the approximation result below in its proper form, we need a lemma.

Lemma 4.2. *For a Hilbert space \mathbf{H} and compact subinterval J of \mathbb{R}_+ , let $(a_\lambda)_{\lambda \in \Lambda}$ be a net of contraction-operator-valued maps from J to $B(\mathbf{H})$, let $a : J \rightarrow B(\mathbf{H})$ be isometry valued and strongly continuous, and suppose that $\langle \zeta, a_\lambda(\cdot)\eta \rangle \rightarrow \langle \zeta, a(\cdot)\eta \rangle$ uniformly, for all $\zeta, \eta \in \mathbf{H}$. Then $a_\lambda(\cdot)\eta \rightarrow a(\cdot)\eta$ uniformly, for all $\eta \in \mathbf{H}$.*

Proof. Let $\eta \in \mathbf{H}$ and $\epsilon > 0$. Since a is strongly continuous and J is compact, there is an \mathbf{H} -valued step function $\varphi = \sum_{j=1}^N \zeta_j 1_{J_j}$ such that $\sup_{t \in J} \|a(t)\eta - \varphi(t)\| < \epsilon$. Therefore, for all $t \in J$,

$$\begin{aligned} & \|(a_\lambda(t) - a(t))\eta\|^2 \\ & \leq 2 \operatorname{Re} \langle a(t)\eta, (a(t) - a_\lambda(t))\eta \rangle \\ & = 2 \operatorname{Re} \langle a(t)\eta - \varphi(t), (a(t) - a_\lambda(t))\eta \rangle + \sum_{j=1}^N 1_{J_j}(t) \langle \zeta_j, (a(t) - a_\lambda(t))\eta \rangle \\ & = 4\|\eta\|\epsilon + \max_{j=1}^N |\langle \zeta_j, (a(t) - a_\lambda(t))\eta \rangle|. \end{aligned}$$

Since the second term tends to zero uniformly, the result follows. \square

In the proof of Theorem 4.3 below, we use Euler's exponential formula in the following form. Let $a, a(h) \in B(\mathfrak{h})$, for $h > 0$, and let $T \in \mathbb{R}_+$; if $a(h) \rightarrow a$ as $h \rightarrow 0$ then

$$\sup_{[r,t] \subset [0,T]} \|(I_{\mathfrak{h}} + ha(h))^{[t/h] - [r/h]} - e^{(t-r)a}\| \rightarrow 0 \quad \text{as } h \rightarrow 0. \quad (4.8)$$

Theorem 4.3. *Let \mathbf{T}' and \mathbf{T} be total subsets of \mathfrak{k} containing 0, and let $F, G_h \in B(\mathfrak{h} \otimes \widehat{\mathfrak{k}})$ ($h > 0$) satisfy*

$$E_{\widehat{d}}[s_h(G_h - I_{\mathfrak{h} \otimes \widehat{\mathfrak{k}}}) - F]E_{\widehat{d}} \rightarrow 0 \quad \text{as } h \rightarrow 0 \quad (c \in \mathbf{T}', d \in \mathbf{T}). \quad (4.9)$$

Then

$$\sup_{t \in [0,T]} \|E^{\varepsilon'}(X_t^{h,G_h} - X_t^F)E_{\varepsilon}\| \rightarrow 0 \quad \text{as } h \rightarrow 0 \quad (\varepsilon' \in \mathcal{E}_{\mathbf{T}'}, \varepsilon \in \mathcal{E}_{\mathbf{T}}, T \in \mathbb{R}_+). \quad (4.10)$$

Moreover, the following refinements hold.

(a) *Suppose that, for some $\beta \in \mathbb{R}$ and all $T \in \mathbb{R}_+$,*

$$\sup_{h>0, t \in [0,T]} \|G_h\|^{[t/h]} < \infty \quad \text{and} \quad F^* \triangleleft F \leq 2\beta\Delta^\perp.$$

Then (4.10) may be strengthened to

$$\sup_{t \in [0,T]} \|(\operatorname{id}_{B(\mathfrak{h})} \otimes \varphi)(X_t^{h,G_h} - X_t^F)\| \rightarrow 0 \quad \text{as } h \rightarrow 0 \quad (\varphi \in B(\mathcal{F})_*, T \in \mathbb{R}_+).$$

(b) *Suppose that each G_h is a contraction and F satisfies $F^* \triangleleft F = 0$. Then also*

$$\sup_{t \in [0,T]} \|(X_t^{h,G_h} - X_t^F)\xi\| \rightarrow 0 \quad \text{as } h \rightarrow 0 \quad (\xi \in \mathfrak{h} \otimes \mathcal{F}, T \in \mathbb{R}_+).$$

(c) *Suppose that each G_h is a contraction and F satisfies $F \triangleleft F^* = 0$. Then also*

$$\sup_{t \in [0,T]} \|(X_t^{h,G_h} - X_t^F)^*\xi\| \rightarrow 0 \quad \text{as } h \rightarrow 0 \quad (\xi \in \mathfrak{h} \otimes \mathcal{F}, T \in \mathbb{R}_+).$$

Proof. Fix $T \in \mathbb{R}_+$ and set $X^{(h)} := X^{h,G_h}$ and $X := X^F$. The first part amounts to fixing $f \in \mathbb{S}_{\mathbf{T}'}$ and $g \in \mathbb{S}_{\mathbf{T}}$, and showing that

$$E^{\varepsilon(f_{[0,t]})}(X_t^{(h)} - X_t)E_{\varepsilon(g_{[0,t]})} \rightarrow 0 \quad \text{uniformly on } [0, T], \quad \text{as } h \rightarrow 0.$$

Fix f and g accordingly, and set

$$\begin{aligned} Q_{hm,hn}^{(h)} & := E^{\varepsilon(f_{[hm,hn]})} X_{hm,hn}^{(h)} E_{\varepsilon(g_{[hm,hn]})} \quad (m, n \in \mathbb{Z}_+, m < n) \\ Q_t^{(h)} & := E^{\varepsilon(f_{[0,t]})} X_t^{(h)} E_{\varepsilon(g_{[0,t]})} \quad \text{and} \quad Q_t := E^{\varepsilon(f_{[0,t]})} X_t E_{\varepsilon(g_{[0,t]})} \quad (t > 0). \end{aligned}$$

Choose $T_+ > \max(D \cup \{T\})$ where D is the union of the sets of points of discontinuity of f and g , let

$$\{t_0 < \dots < t_{N+1}\} = \{0\} \cup D \cup \{T_+\}.$$

Henceforth $h > 0$ is assumed to be smaller than mesh D . By the discrete evolution property (4.3), for $t > 0$,

$$Q_t^{(h)} = \sum_{k=0}^N 1_{[t_k, t_{k+1}[}(t) \overrightarrow{\prod}_{1 \leq j < k} A_j(h) B_k(h, t) C(h[t/h], t)$$

where, when $j \in \{1, \dots, N-1\}$,

$$A_j(h) := Q_{h[t_j/h], h(1+[t_j/h])}^{(h)} Q_{h(1+[t_j/h]), h[t_{j+1}/h]}^{(h)}$$

and, when $k \in \{0, \dots, N\}$, and $t \in [t_k, t_{k+1}[$,

$$B_k(h, t) = \begin{cases} Q_{h[t_k/h], h(1+[t_k/h])}^{(h)} Q_{h(1+[t_k/h]), h[t/h]}^{(h)} & \text{if } [t_k/h] < [t/h], \\ I_{\mathfrak{h}} & \text{if } [t_k/h] = [t/h], \end{cases}$$

for the operators defined by $C(u, v) := \langle \varepsilon(f_{[u, v]}, \varepsilon(g_{[u, v]})) I_{\mathfrak{h}} \rangle$ ($0 \leq u \leq v$). On the other hand, by the semigroup decomposition of QS cocycles,

$$Q_t = \sum_{k=0}^N 1_{[t_k, t_{k+1}[}(t) P_{t_1-t_0}^{(0)} \cdots P_{t_k-t_{k-1}}^{(k-1)} P_{t-t_k}^{(k)},$$

where, for $i = 0, \dots, N$, $P^{(i)}$ denotes the $(f(t_i), g(t_i))$ -associated semigroup of the QS cocycle X , defined in (1.3).

Now set

$$F_h := s_h(G_h - I) = s_h(G_h - \Delta^\perp) - \Delta, \quad \text{and} \quad (4.11)$$

$$M_h := \max \{ \|E^{\widehat{c}}(F_h + \Delta)E_{\widehat{d}}\| : c \in \text{Ran } f, d \in \text{Ran } g \} \quad (h > 0).$$

Then, since $E^{\widehat{c}}(F_h + \Delta)E_{\widehat{d}} \rightarrow E^{\widehat{c}}(F + \Delta)E_{\widehat{d}}$ as $h \rightarrow 0$, for all $c \in \Gamma'$ and $d \in \Gamma$, $\limsup_{h \rightarrow 0} M_h < \infty$. Lemma 4.1 implies that, for all $n \in \{0, \dots, N\}$ and $h > 0$,

$$\|Q_{h[t_n/h], h(1+[t_n/h])}^{(h)} - I_{\mathfrak{h}}\| = h \|E^{\widehat{f}[n, h]}(F_h + \Delta)E_{\widehat{g}[n, h]}\| \leq h M_h,$$

since $\widehat{f}[n, h] \in \text{Conv Ran } \widehat{f}$, and similarly for g . Also, since $\|C(h[t/h], t) - I_{\mathfrak{h}}\| \leq h \exp\|f\| \|g\|$,

$$C(h[t/h], t) \rightarrow I_{\mathfrak{h}} \text{ uniformly in } t \text{ as } h \rightarrow 0.$$

Therefore, since the generator of $P^{(j)}$ is $E^{\widehat{f}(t_j)}(F + \Delta)E_{\widehat{g}(t_j)}$, Lemma 4.1 and Euler's formula imply that, as $h \rightarrow 0$

$$A_j(h) \rightarrow P_{t_{j+1}-t_j}^{(j)} \quad \text{and} \quad \sup_{t \in [t_k, t_{k+1}[} \|B_k(h, t) - P_{t-t_k}^{(k)}\| \rightarrow 0$$

for $j \in \{0, \dots, N-1\}$ and $k \in \{0, \dots, N\}$. It follows that $Q_{0, t}^{(h)} \rightarrow Q_t$ uniformly on $[0, T]$, as required.

(a) By the basic estimate (3.2), $\{X_t^{(h)} : h > 0, t \in [0, T]\}$ is uniformly bounded and, by the characterisation of quasicontractivity of elementary QS cocycles recalled in Theorem 1.2, $\|X_t\| \leq e^{\beta t}$ ($t \in \mathbb{R}_+$) so $\{X_t : t \in [0, T]\}$ is uniformly bounded too. The result therefore follows from the first part, by the norm totality of the family $\{\omega_{\varepsilon', \varepsilon} : \varepsilon' \in \mathcal{E}_{\Gamma'}^t, \varepsilon \in \mathcal{E}_{\Gamma}\}$ in $B(\mathcal{F})_*$ and the well-known fact (*e.g.* [Efr], Corollary 2.2.3) that $\|\varphi\|_{\text{cb}} = \|\varphi\|$ for any $\varphi \in B(\mathcal{F})_*$.

(b & c) It follows from (a) that

$$\sup_{t \in [0, T]} |\langle \zeta, (X_t^{(h)} - X_t)\eta \rangle| \rightarrow 0 \text{ as } h \rightarrow 0 \quad (\zeta, \eta \in \mathfrak{h} \otimes \mathcal{F}).$$

By Theorem 1.2 and Remark (ii) following it, X and X^* are both strongly continuous. Since X is isometric if $F^* \triangleleft F = 0$ and X^* is coisometric if $F \triangleleft F^* = 0$, (b) and (c) follow from Lemma 4.2. \square

Remarks. (i) A useful generalisation arises from the introduction of an extra parameter from a directed set Λ . Thus, for a net $(G_{h,\lambda})_{h>0,\lambda\in\Lambda}$ in $B(\mathfrak{h} \otimes \widehat{\mathfrak{k}})$, Theorem 4.3 holds with respect to the net of processes $(X^{h,G_{h,\lambda}})_{h>0,\lambda\in\Lambda}$. This is exploited (with $\Lambda = \mathbb{N}$) in Theorem 6.2 below.

(ii) The above theorem may be derived from Theorem 7.6 of [B₁], and corresponds to Theorem 5.1 of [DL₂]. Theorem 13 of [AtP] is a version of this result, established under hypotheses which are much stronger when \mathfrak{k} is infinite dimensional. For an earlier form, see [Pa₁], Theorem 4.1. Here, by direct application of the semigroup decomposition of QS cocycles, we avoid the approximation of quantum Wiener integrals by their discrete analogues; we also avoid any need to appeal to vacuum-adapted QS calculus or sesquilinear QS calculus, thereby achieving a much simpler and more direct proof. Furthermore, unlike in [DL₂], there is no restriction to dyadic rational discretisation.

(iii) Theorem 4.3 may also be profitably viewed in terms of the approximation of *elementary evolutions*, in the sense of [DL₁], by discrete evolutions. More specifically, Lemma 4.1 yields the alternative representation

$$Q_t^{(h)} = C_t^h \prod_{0 \leq n < \lfloor t/h \rfloor}^{\rightarrow} (I_{\mathfrak{h}} + h a(n, h)) \quad (t \in \mathbb{R}_+),$$

where $C_t^h = \langle \varepsilon(f_J), \varepsilon(g_J) \rangle$ for some subset J of $[h(\lfloor t/h \rfloor - 1), t[$ such that $|J| \leq 2h$, and

$$a(n, h) := E^{\widehat{f[n,h]}}(F_h + \Delta) E_{\widehat{g[n,h]}} \quad (n \in \mathbb{Z}_+),$$

where F_h is given by (4.11). On the other hand, $E = (E_{r,t})_{0 \leq r \leq t}$ is an elementary evolution whose generator takes the form

$$a : s \mapsto E^{\widehat{f(s)}}(F + \Delta) E_{\widehat{g(s)}},$$

and Theorem 3.2 of [DL₁] implies that, in the notation (4.1),

$$\prod_{0 \leq n < \lfloor t/h \rfloor}^{\rightarrow} (I_{\mathfrak{h}} + h a[n, h]) \rightarrow Q_t$$

uniformly for $0 \leq t \leq T$ ($T \in \mathbb{R}_+$). The main part of Theorem 4.3 then follows from the fact that

$$\max_{n \in \mathbb{N}} \left\{ \|a(n, h) - a[n, h]\| : f \text{ and } g \text{ are constant on } [hn, h(n+1)[\right\} \rightarrow 0 \text{ as } h \rightarrow 0,$$

and

$$\#\left\{ n \in \mathbb{N} : f \text{ or } g \text{ is not constant on } [hn, h(n+1)[\right\} \leq \#D \quad (0 < h < \text{mesh } D),$$

where D is the union of the sets of points of discontinuity of f and g .

(iv) The strategy of employing semigroup decomposition in the proof of Theorem 4.3 adapts well from operator cocycles to QS mapping cocycles, and more generally to Banach-algebra-valued QS sesquilinear cocycles; specifically, Theorem 3.6 of [DL₂] may be proved directly, along the lines of Remark (iii) above.

(v) The basic hypothesis of Theorem 4.3 is equivalent to the condition

$$E^{\widehat{c}}[s_h(G_h^* - I) - F^*] E_{\widehat{d}} \rightarrow 0 \text{ as } h \rightarrow 0 \quad (c \in \mathbb{T}, d \in \mathbb{T}'),$$

with corresponding refinements. Therefore, the theorem also yields convergence of the embedded QRWs $(X^{h,G_h^*})_{h>0}$ to $X^{F^*} = X^\sharp$, the dual cocycle of X .

(vi) The sets T' and T typically each consist of vectors from an orthonormal basis for \mathfrak{k} augmented by the vector 0.

(vii) In (a) the limit QS cocycle X^F is quasicontractive, with $\|X_t^F\| \leq e^{\beta t}$; in (b) the cocycle X^F is isometric, and is unitary if also $F \triangleleft F^* = 0$. These follow from the characterisations of quasicontractivity, isometry and unitarity of elementary QS cocycles listed in Theorem 1.2.

(viii) The basic hypothesis (4.9) is usefully expressed in the following equivalent form:

$$E_{\widehat{d}}^{\widehat{c}}[s_h(G_h - \Delta^\perp) - (F + \Delta)] \rightarrow 0 \quad \text{as } h \rightarrow 0 \quad (c \in \mathsf{T}', d \in \mathsf{T}).$$

Then, writing in the block-matrix forms

$$G_h = \begin{bmatrix} I_{\mathfrak{h}} + hK_h & \sqrt{h}M_h \\ \sqrt{h}L_h & C_h \end{bmatrix} \quad \text{and} \quad F = \begin{bmatrix} K & M \\ L & C - I \end{bmatrix}, \quad (4.12)$$

it follows that

$$s_h(G_h - \Delta^\perp) = \begin{bmatrix} K_h & M_h \\ L_h & C_h \end{bmatrix} \quad \text{and} \quad F + \Delta = \begin{bmatrix} K & M \\ L & C \end{bmatrix}.$$

In these terms (4.9) amounts to the following more transparent condition:

$$E_{\widehat{d}}^{\widehat{c}} \begin{bmatrix} K_h - K & M_h - M \\ L_h - L & C_h - C \end{bmatrix} \rightarrow 0 \quad \text{as } h \rightarrow 0 \quad (c \in \mathsf{T}', d \in \mathsf{T}). \quad (4.13)$$

When $\dim \mathfrak{k} < \infty$, this is equivalent to the simple norm-convergence conditions

$$K_h \rightarrow K, \quad L_h \rightarrow L, \quad M_h \rightarrow M \quad \text{and} \quad C_h \rightarrow C \quad \text{as } h \rightarrow 0.$$

However, when \mathfrak{k} is infinite dimensional, it is only the *components* of L_h , M_h and C_h with respect to some total families T and T' in \mathfrak{k} (such as orthogonal bases) that need to converge to the corresponding components of L , M and C .

(ix) The theorem begs two questions. The first is, given a generator F of a QS cocycle, can such a family of operators $(G_h)_{h>0}$ be found? This is resolved in Theorem 7.4 where it is shown that it easily can, and that moreover, the operators may be chosen to be respectively isometric, coisometric, or unitary if the QS cocycle generated by F has that property. The second question is, what families of, say unitary, operators $(G_h)_{h>0}$ have scaled limits F , in the sense of (4.9), which generate unitary QS cocycles? In Section 8, the repeated quantum interaction model is shown to provide a wide source of examples of such families.

The next example is instructive.

Example 4.4 (Preservation-type QS cocycles). Let $F = \begin{bmatrix} 0 & 0 \\ 0 & C - I \end{bmatrix} \in B(\mathfrak{h} \otimes \widehat{\mathfrak{k}})$, and $G = F + I = \begin{bmatrix} I & 0 \\ 0 & C \end{bmatrix} \in B(\mathfrak{h} \otimes \widehat{\mathfrak{k}})$, for a contraction $C \in B(\mathfrak{h} \otimes \mathfrak{k})$. Note the identification

$$\mathfrak{h} \otimes \mathcal{F}^{\mathfrak{k}} = \bigoplus_{n \in \mathbb{Z}_+} \left(\mathfrak{h} \otimes \mathfrak{k}^{\otimes n} \otimes \mathcal{F}_{[0,t]}^{(n)} \right) \otimes \mathcal{F}_{[t,\infty[}^{\mathfrak{k}} \quad (t \in \mathbb{R}_+),$$

in which $\bigoplus_{n \in \mathbb{Z}_+} \mathcal{F}_{[0,t]}^{(n)}$ is the chaos decomposition of $\mathcal{F}_{[0,t]}^{\mathbb{C}}$.

On the one hand the QS cocycle X^F has the explicit description ([L₁], Example 5.3)

$$X_t^F = \bigoplus_{n \in \mathbb{Z}_+} \left(\widetilde{G}_n \otimes I_{[0,t]}^{(n)} \right) \otimes I_{[t,\infty[}^{\mathcal{F}^{\mathfrak{k}}} \quad (t \in \mathbb{R}_+),$$

where $I_{[0,t]}^{(n)}$ denotes the identity operator on $\mathcal{F}_{[0,t]}^{(n)}$ and, in the notation (2.1),

$$\widetilde{G}_n := (I_{\mathfrak{h}} \otimes J^{\otimes n})^* W_n^G (I_{\mathfrak{h}} \otimes J^{\otimes n}) \quad (n \in \mathbb{Z}_+)$$

in which J denotes the natural isometry $\mathfrak{k} \rightarrow \widehat{\mathfrak{k}}$. On the other hand, since in this case

$$F = s_h(G - I) \quad \text{for all } h > 0,$$

Theorem 4.3 implies that

$$\sup_{t \in [0, T]} \left\| (\text{id}_{B(\mathfrak{h})} \otimes \varphi)(X^{h, G_t} - X_t^F) \right\| \rightarrow 0 \quad \text{as } h \rightarrow 0 \quad (\varphi \in B(\mathcal{F}^{\mathfrak{k}})_*, T \in \mathbb{R}_+).$$

Remark. In the pure-noise case, when the initial space is just \mathbb{C} , X^F is given by the formula

$$X_t^F = \Gamma(M_{[0, t[} \otimes C + M_{[t, \infty[} \otimes I_{\mathfrak{k}}) \quad (t \in \mathbb{R}_+)$$

where, for any subinterval J of \mathbb{R}_+ , $M_J \in B(L^2(\mathbb{R}_+))$ denotes the multiplication operator by the indicator function of J , and Γ denotes the second quantisation operation defined in (0.1).

As a fast corollary to Theorem 4.3 we obtain (pace Remark (iv) above) a basic QRW approximation result for inner QS flows on a full operator algebra.

Corollary 4.5. *Let j be the QS flow on $B(\mathfrak{h})$ induced by an elementary unitary QS cocycle X on \mathfrak{h} with noise dimension space \mathfrak{k} , thus*

$$j_t(x) = X_t(x \otimes I_{\mathfrak{h} \otimes \mathcal{F}})X_t^* \quad (x \in B(\mathfrak{h}), t \in \mathbb{R}_+),$$

and suppose that X has stochastic generator F . For $h > 0$, let $j^{(h)}$ be the mapping process on $B(\mathfrak{h})$ given by

$$j_t^{(h)}(x) = X_t^{h, G_h}(x \otimes I_{\mathfrak{h} \otimes \mathcal{F}})(X_t^{h, G_h})^* \quad (x \in B(\mathfrak{h}), t \in \mathbb{R}_+),$$

where $(G_h)_{h>0}$ is a family of contractions in $B(\mathfrak{h} \otimes \widehat{\mathfrak{k}})$ satisfying $s_h(G_h - I) \rightarrow F$ in norm as $h \rightarrow 0$. Then

$$\sup_{t \in [0, T]} \left\| (j_t(x) - j_t^{(h)}(x))\xi \right\| \rightarrow 0 \quad \text{as } h \rightarrow 0 \quad (x \in B(\mathfrak{h}), \xi \in \mathfrak{h} \otimes \mathcal{F}, T \in \mathbb{R}_+). \quad (4.14)$$

Proof. Fix $T \in \mathbb{R}_+$ and set $X^{(h)} := X^{h, G_h}$. Since $B(\mathfrak{h})$ is linearly spanned by its isometries, it suffices to prove (4.14) for x isometric. Accordingly, let $x \in B(\mathfrak{h})$ be isometric. By Theorem 4.3,

$$\sup_{t \in [0, T]} \left\| (X_t^{(h)} - X_t)\xi \right\| \rightarrow 0 \quad \text{as } h \rightarrow 0 \quad (\xi \in \mathfrak{h} \otimes \mathcal{F}, T \in \mathbb{R}_+).$$

Since X and $X^{(h)}$ are contraction processes and thus both locally uniformly bounded, it follows that

$$\sup_{t \in [0, T]} \left| \langle \zeta, (j_t(x) - j_t^{(h)}(x))\eta \rangle \right| \rightarrow 0 \quad \text{as } h \rightarrow 0 \quad (\zeta, \eta \in \mathfrak{h} \otimes \mathcal{F}, T \in \mathbb{R}_+).$$

Now, for each $t \in \mathbb{R}_+$, $j_t(x)$ is isometric and $j_t^{(h)}(x)$ is a contraction, therefore (4.14) follows from Lemma 4.2. \square

5. COMPOSITIONS

In this section we show how, under the convergence scheme of Section 4, pointwise products of QRWs converge to QS Trotter products of the limiting QS cocycles ($[L_2]$). This specialises nicely to the case where the initial space \mathfrak{h} is a tensor product and the two cocycles live on separate tensor components.

Recall the series-product notation (1.6).

Theorem 5.1. *Let $F_i, G_i(h) \in B(\mathfrak{h} \otimes \widehat{\mathfrak{k}})$ ($h > 0$), for $i = 1, 2$, let $c, d \in \mathfrak{k}$ and suppose that*

$$E^{\widehat{c}}[s_h(G_1(h) - I) - F_1] \rightarrow 0 \quad \text{and} \quad [s_h(G_2(h) - I) - F_2]E_{\widehat{d}} \rightarrow 0 \quad \text{as } h \rightarrow 0. \quad (5.1)$$

Then

$$E^{\widehat{c}}[s_h(G_1(h)G_2(h) - I) - F_1 \triangleleft F_2]E_{\widehat{d}} \rightarrow 0 \quad \text{as } h \rightarrow 0.$$

Also, if $s_h(G_i(h) - I) \rightarrow F_i$ (in norm) for $i = 1, 2$, then $s_h(G_1(h)G_2(h) - I) \rightarrow F_1 \triangleleft F_2$.

Proof. Let $h > 0$, and set

$$F_1(h) := s_h(G_1(h) - I), \quad F_2(h) := s_h(G_2(h) - I) \quad \text{and} \quad F(h) := s_h(G_1(h)G_2(h) - I).$$

Then, from the identity

$$G_1(h)G_2(h) - I = (G_1(h) - I) + (G_2(h) - I) + (G_1(h) - I)(G_2(h) - I),$$

we see that

$$\begin{aligned} F(h) &= F_1(h) + F_2(h) + F_1(h)(h\Delta^\perp + \Delta)F_2(h) \\ &= F_1(h) + F_2(h) + F_1(h)\Delta F_2(h) + h F_1(h)\Delta^\perp F_2(h), \end{aligned}$$

from which both conclusions follow. \square

Remarks. (i) Given an elementary QS operator cocycle X , a series decomposition of its stochastic generator

$$F = F_1 \triangleleft \cdots \triangleleft F_n,$$

and families $(G_i(h))_{h>0}$ in $B(\mathfrak{h} \otimes \widehat{\mathfrak{k}})$ ($i = 1, \dots, n$) satisfying

$$s_h(G_i(h) - I) \rightarrow F_i \quad (i = 1, \dots, n),$$

Theorems 5.1 and 4.3 give that X is the limit of the embedded QRWs $(X^{h, G_h})_{h>0}$, where $G_h := G_1(h) \cdots G_n(h)$. This fact is exploited in the proof of Proposition 7.2, and in Remark (ii) following Theorem 6.2.

(ii) If $F_i^* \triangleleft F_i \leq 2\beta_i \Delta^\perp$, where $\beta_i \in \mathbb{R}$ (respectively $F_i^* \triangleleft F_i = 0$ or $F_i \triangleleft F_i^* = 0$), for $i = 1, 2$, then the QS cocycle $X^{F_1 \triangleleft F_2}$ may be realised as a QS Trotter product of the quasicontractive (respectively isometric or coisometric) QS cocycles X^{F_1} and X^{F_2} ([L2], Theorem 3.4).

The following observation, in which the Riesz–Nagy symbol \smile denotes ‘commutes with’ ([RSz]), is relevant here.

Proposition 5.2 ([JuL]). *Let X^1 and X^2 be quasicontractive QS cocycles on \mathfrak{h} with noise dimension space \mathfrak{k} . Suppose that X^1 and X^2 commute on \mathfrak{h} , meaning that*

$$E^{\zeta_1} X_s^1 E_{\eta_1} \smile E^{\zeta_2} X_t^2 E_{\eta_2} \quad (s, t \in \mathbb{R}_+, \zeta_1, \eta_1, \zeta_2, \eta_2 \in \mathcal{F}).$$

Then the QS process $X^1 X^2 := (X_t^1 X_t^2)_{t \geq 0}$ is also a quasicontractive QS cocycle. Moreover, if the QS cocycles X^1 and X^2 are both elementary then $X^1 X^2 = X^{F_1 \triangleleft F_2}$ where F_1 and F_2 are the stochastic generators of X^1 and X^2 respectively.

Example 5.3. Let $X^{(i)}$ be a quasicontractive QS cocycle on \mathfrak{h}_i with noise dimension space \mathfrak{k} , for $i = 1, 2$. These amplify to QS cocycles on $\mathfrak{h} := \mathfrak{h}_1 \otimes \mathfrak{h}_2$, by setting $I_1 := I_{\mathfrak{h}_1}$, $I_2 := I_{\mathfrak{h}_2}$,

$$X_t^2 := I_1 \otimes X_t^{(2)} \quad \text{and} \quad X_t^1 := I_2 \widetilde{\otimes} X_t^{(1)},$$

where $B(\mathfrak{h}_1) \overline{\otimes} B(\mathfrak{h}_2 \otimes \mathcal{F})$ is identified with $B(\mathfrak{h} \otimes \mathcal{F})$ and the notation $\widetilde{\otimes}$ incorporates the tensor flip from $B(\mathfrak{h}_2) \overline{\otimes} B(\mathfrak{h}_1 \otimes \mathcal{F})$ to $B(\mathfrak{h} \otimes \mathcal{F})$. Since the \mathcal{F} -slices of X_s^1 and X_t^2 belong to $B(\mathfrak{h}_1) \otimes I_2$ and $I_1 \otimes B(\mathfrak{h}_2)$ respectively, the cocycles manifestly commute on \mathfrak{h} . Therefore, by Proposition 5.2, the product $X^1 X^2$ is a quasicontractive QS cocycle.

Suppose now that, for $i = 1, 2$, $X^{(i)}$ is elementary with stochastic generator $F_{(i)}$, and the family $(G_h^{(i)})_{h>0}$ satisfies $s_h(G_h^{(i)} - I_i) \rightarrow F_{(i)}$, and set

$$F_2 := I_1 \otimes F_{(2)}, \quad F_1 := I_2 \tilde{\otimes} F_{(1)}, \quad G_2(h) := I_1 \otimes G_h^{(2)} \quad \text{and} \quad G_1(h) := I_2 \tilde{\otimes} G_h^{(1)},$$

in which the tilde now incorporates the tensor flip from $B(\mathfrak{h}_2) \bar{\otimes} B(\mathfrak{h}_1 \otimes \widehat{\mathfrak{k}})$ to $B(\mathfrak{h} \otimes \widehat{\mathfrak{k}})$. Then

$$X^1 X^2 = X^{F_1 \triangleleft F_2} \quad \text{and} \quad s_h(G_i(h) - I_{\mathfrak{h} \otimes \widehat{\mathfrak{k}}}) \rightarrow F_i \quad (i = 1, 2).$$

In terms of the block matrix decompositions $F_{(i)} = \begin{bmatrix} K_i & M_i \\ L_i & C_{i-I} \end{bmatrix}$ ($i = 1, 2$),

$$F_1 \triangleleft F_2 = \begin{bmatrix} K_1 \otimes I_2 + I_1 \otimes K_2 + (I_2 \tilde{\otimes} M_1)(I_1 \otimes L_2) & (I_2 \tilde{\otimes} M_1)(I_1 \otimes C_2) + I_1 \otimes M_2 \\ I_2 \tilde{\otimes} L_1 + (I_2 \tilde{\otimes} C_1)(I_1 \otimes L_2) & (I_2 \tilde{\otimes} C_1)(I_1 \otimes C_2) - I \end{bmatrix}.$$

In the case of one-dimensional noise this simplifies to

$$\begin{bmatrix} K_1 \otimes I_2 + I_1 \otimes K_2 + M_1 \otimes L_2 & M_1 \otimes C_2 + I_1 \otimes M_2 \\ L_1 \otimes I_2 + C_1 \otimes L_2 & C_1 \otimes C_2 - I \end{bmatrix},$$

whereas the quantum random walk generator satisfies

$$G_1(h)G_2(h) = \begin{bmatrix} I + hK_h & \sqrt{h}M_h \\ \sqrt{h}L_h & C_h \end{bmatrix} + hO(h)$$

where, writing $G_i(h)$ in the form $\begin{bmatrix} I + hK_i(h) & \sqrt{h}M_i(h) \\ \sqrt{h}L_i(h) & C_i(h) \end{bmatrix}$ for $i = 1, 2$, $\begin{bmatrix} K_h & M_h \\ L_h & C_h \end{bmatrix}$ equals

$$\begin{bmatrix} K_1(h) \otimes I_2 + I_1 \otimes K_2(h) + M_1(h) \otimes L_2(h) & M_1(h) \otimes C_2(h) + I_1 \otimes M_2(h) \\ L_1(h) \otimes I_2 + C_1(h) \otimes L_2(h) & C_1(h) \otimes C_2(h) \end{bmatrix}$$

and

$$O(h) = \begin{bmatrix} h(K_1(h) \otimes K_2(h)) & \sqrt{h}(K_1(h) \otimes M_2(h)) \\ \sqrt{h}(L_1(h) \otimes K_2(h)) & L_1(h) \otimes M_2(h) \end{bmatrix}.$$

Remark. Example 5.3 specialises to the entanglement of bipartite systems under repeated interactions, as considered in [ADP]. This is fully elaborated upon in Section 8 below.

6. HOLEVO TRANSFORM

In this section we consider a nonlinear transformation of block operator matrices in $B(\mathfrak{h} \otimes \widehat{\mathfrak{k}}) = B(\mathfrak{h} \oplus (\mathfrak{h} \otimes \mathfrak{k}))$ intimately related to the convergence to QS cocycles for two classes of QRW. Its origin lies in Holevo's approach to realising QS cocycles, and more general solutions of QS differential equations, as time-ordered exponentials ([Ho_{1,2}]; see [Ho_{3,4}]). Since time-ordered exponentials may be seen as explicit continuous counterparts to QRWs, the appearance of this transform is, with hindsight, not unexpected. On the one hand the transform is key to the realisation of QS cocycles as limits of scaled QRWs described in Section 7. On the other hand it delivers convergence of QRWs to unitary QS cocycles in the repeated quantum interaction model, as shown in Section 8. In particular, part (a) of Proposition 6.1 and part (b) of Theorem 6.2 show clearly the origin of what, in [AtP], is referred to as the 'surprising term' in the 'effective Hamiltonian' arising in the limit of repeated quantum interactions (see Theorem 8.1 below).

Set

$$A := \{T \in B(\mathfrak{h} \oplus (\mathfrak{h} \otimes \mathfrak{k}) \oplus \mathfrak{h}) : P_{\mathfrak{h} \oplus (\mathfrak{h} \otimes \mathfrak{k}) \oplus \{0\}} T = T = T P_{\{0\} \oplus (\mathfrak{h} \otimes \mathfrak{k}) \oplus \mathfrak{h}}\},$$

thus \mathbf{A} consists of the elements having block matrix form $\begin{bmatrix} 0 & * & * \\ 0 & 0 & 0 \end{bmatrix}$. The unital Banach algebra $B(\mathfrak{h} \oplus (\mathfrak{h} \otimes \mathfrak{k}) \oplus \mathfrak{h})$ has an involution given by

$$T \mapsto T^* := \Xi T^* \Xi, \quad \text{where } \Xi := \begin{bmatrix} 0 & 0 & I_{\mathfrak{h}} \\ 0 & I_{\mathfrak{h} \otimes \mathfrak{k}} & 0 \\ I_{\mathfrak{h}} & 0 & 0 \end{bmatrix}$$

with respect to which \mathbf{A} is a closed $*$ -subalgebra. The prescription

$$\begin{bmatrix} A & C \\ B & D \end{bmatrix} \mapsto \begin{bmatrix} 0 & C & A \\ 0 & D & B \\ 0 & 0 & 0 \end{bmatrix}$$

defines an involutive linear homeomorphism $\tau : (B(\mathfrak{h} \otimes \widehat{\mathfrak{k}}), *) \rightarrow (\mathbf{A}, *)$, and the involutive continuous (nonlinear) injection

$$B(\mathfrak{h} \oplus (\mathfrak{h} \otimes \mathfrak{k}) \oplus \mathfrak{h}) \rightarrow B(\mathfrak{h} \oplus (\mathfrak{h} \otimes \mathfrak{k}) \oplus \mathfrak{h}), \quad T \mapsto e^T - I,$$

restricts to a map $\eta : \mathbf{A} \rightarrow \mathbf{A}$. The *Holevo transform* is the map

$$\tau^{-1} \circ \eta \circ \tau : B(\mathfrak{h} \otimes \widehat{\mathfrak{k}}) \rightarrow B(\mathfrak{h} \otimes \widehat{\mathfrak{k}}), \quad Q \mapsto F[Q] := \tau^{-1}(e^{\tau(Q)} - I).$$

If we set $\mathcal{S} := I_{\mathfrak{h} \oplus (\mathfrak{h} \otimes \mathfrak{k}) \oplus \mathfrak{h}} + \mathbf{A}$ and let $\tilde{\tau}$ denote the map $Q \mapsto \tau(Q) + I$, then, recalling the remark below (1.6), $(\mathcal{S}, \cdot, \star)$ is a $*$ -monoid, $\tilde{\tau}$ is a $*$ -monoid isomorphism from $(B(\mathfrak{h} \otimes \widehat{\mathfrak{k}}), \triangleleft, \star)$ to $(\mathcal{S}, \cdot, \star)$ ([Bel], [L2, Proposition 1.5]), and $F[Q] = \tilde{\tau}^{-1}(e^{\tilde{\tau}(Q)} - I)$. From these representations it is readily verified that $F[Q]^* = F[Q^*]$ and

$$F[Q]^* \triangleleft F[Q] = 0 \iff e^{\tau(Q^*)} = e^{\tau(-Q)} \iff F[Q] \triangleleft F[Q]^* = 0, \quad (6.1)$$

so that $F[Q]$ satisfies the unitary structure relations (see Theorem 1.2) if Q is skewadjoint.

In order to examine the Holevo transform in more detail, and to show its use, we need to introduce some functions and give various relations they enjoy. Thus, let e_0, e_1, e_2 and e be the entire functions whose values at $z \neq 0$ are given respectively by

$$e^z, \quad \frac{e^z - 1}{z}, \quad \frac{e^z - 1 - z}{z^2} \quad \text{and} \quad \frac{e^z - e^{-z} - 2z}{2z^2} = \frac{\sinh z - z}{z^2}, \quad (6.2)$$

and, for $n \in \mathbb{N}$, let p_n denote the polynomial whose value at $z \neq 0$ is given by

$$\frac{(1 + z/n)^n - 1 - z}{z^2}.$$

Thus $e_0(0) = e_1(0) = 1$, $e_2(0) = 1/2$, $e(0) = 0$, $p_n \rightarrow e_2$ uniformly on bounded subsets of \mathbb{C} and the following identities hold, for all $z \in \mathbb{C}$ and $n \in \mathbb{N}$:

$$z^n - 1 = n(z - 1) + n(z - 1)p_n(n(z - 1))n(z - 1), \quad (6.3a)$$

$$1 + ze_2(z) = e_1(z) \quad \text{and} \quad z + z^2e_2(z) = e_0(z) - 1, \quad (6.3b)$$

$$n(e_0(z/n) - 1) - z = e_2(z/n)z^2/n, \quad (6.3c)$$

$$e_1(-z)e_0(z) = e_1(z) \quad \text{and} \quad \frac{1}{2}e_1(-z)e_1(z) + e(z) = e_2(z). \quad (6.3d)$$

In terms of these, the Holevo transform is given by

$$Q = \begin{bmatrix} A & C \\ B & D \end{bmatrix} \mapsto F[Q] := \begin{bmatrix} A + Ce_2(D)B & Ce_1(D) \\ e_1(D)B & e_0(D) - I \end{bmatrix}. \quad (6.4)$$

Thus, by (6.3b),

$$F[Q] - Q = \begin{bmatrix} C \\ D \end{bmatrix} e_2(D) \begin{bmatrix} B & D \end{bmatrix} = Q \begin{bmatrix} 0 & 0 \\ 0 & e_2(D) \end{bmatrix} Q. \quad (6.5)$$

We next look at how parameterisations of Q are reflected in parameterisations of $F[Q]$. For operators $A \in B(\mathfrak{h})$, $B \in B(\mathfrak{h}; \mathfrak{h} \otimes \mathfrak{k})$, $C \in B(\mathfrak{h} \otimes \mathfrak{k}; \mathfrak{h})$ and $D \in B(\mathfrak{h} \otimes \mathfrak{k})$, set

$$Q_{A,B,D} := \begin{bmatrix} A & -B^* \\ B & D \end{bmatrix}. \quad (6.6)$$

In terms of the notation (1.7), note the relations

$$F[Q_{A,0,D}] = F_{A,0,e^D} \quad \text{and} \quad F_{Z,L,I} = F[Q_{Z,L,0}].$$

Below we extend the scope of this correspondence (for skewadjoint D). To this end let $e_a, e_b : [0, 2\pi[\rightarrow \mathbb{C}$ denote the continuous functions whose values at $t \in]0, 2\pi[$ are given respectively by

$$\frac{i \sin t - t}{2 \cos t - 1} \quad \text{and} \quad \frac{it}{e^{it} - 1},$$

and note the following easily verified identities, for $t \in [0, 2\pi[$,

$$\overline{e_a(t)} = -e_a(t), \quad e_1(it)e_b(t) = 1, \quad \overline{e_b(t)}e_1(it) = e_0(it), \quad |e_b(t)|^2 e(it) = e_a(t). \quad (6.7)$$

Proposition 6.1.

- (a) Let $A \in B(\mathfrak{h})$, $B \in B(\mathfrak{h}; \mathfrak{h} \otimes \mathfrak{k})$ and $D \in B(\mathfrak{h} \otimes \mathfrak{k})$, with D skewadjoint. Then

$$F[Q_{A,B,D}] = F_{Z,L,W}$$

where

$$W = e_0(D), \quad L = e_1(D)B \quad \text{and} \quad Z = A - B^*e(D)B$$

(so that W is unitary and $Z - A$ is skewadjoint). In particular, if $Q \in B(\mathfrak{h} \otimes \widehat{\mathfrak{k}})$ is skewadjoint then

$$F[Q]^* \triangleleft F[Q] = 0 = F[Q] \triangleleft F[Q]^* \quad (6.8)$$

and so the QS cocycle $X^{F[Q]}$ is unitary.

- (b) Conversely, let $Z \in B(\mathfrak{h})$, $L \in B(\mathfrak{h}; \mathfrak{h} \otimes \mathfrak{k})$ and $W \in B(\mathfrak{h} \otimes \mathfrak{k})$, with W unitary and satisfying $\text{Spec } W \subset \{e^{it} : t \in [0, 2\pi[\}$. Then

$$F_{Z,L,W} = F[Q_{A,B,D}],$$

where

$$A = Z + L^*e_a(R)L, \quad B = e_b(R)L \quad \text{and} \quad D = iR,$$

for the unique selfadjoint operator $R \in B(\mathfrak{h} \otimes \mathfrak{k})$ satisfying $e^{iR} = W$ and $\text{Spec } R \subset [0, 2\pi[$ (so that $A - Z$ is skewadjoint). In particular, if Z is skewadjoint then so is $Q_{A,B,D}$, and so the operator $e^{Q_{A,B,D}}$ is unitary.

Proof. (a) By the skewadjointness of D , and oddness of the function e , W is unitary and Z is skewadjoint. Moreover, by (6.3d),

$$-L^*W = -B^*e_1(D)^*e_0(D) = -B^*e_1(-D)e_0(D) = -B^*e_1(D),$$

and

$$\begin{aligned} -\frac{1}{2}L^*L - B^*e(D)B &= -B^*[\frac{1}{2}e_1(D)^*e_1(D) + e(D)]B \\ &= -B^*[\frac{1}{2}e_1(-D)e_1(D) + e(D)]B = -B^*e_2(D)B, \end{aligned}$$

so $F[Q_{A,B,D}]$ has the claimed form.

Now suppose that $Q \in B(\mathfrak{h} \otimes \widehat{\mathfrak{k}})$ is skewadjoint. Then Q is of the form $Q_{A,B,D}$ with A and D skewadjoint. so $F[Q]$ is of the form $F_{Z,L,W}$ with Z skewadjoint and W unitary, and thus (6.8) holds by property (i) above Theorem 1.2 (confirming the remark which follows observation in (6.1)).

(b) Let R be as specified. Using the identities (6.7), and Part (a), we see that $A - Z = L^*e_a(R)L$ is skewadjoint and, since $D = iR$ is skewadjoint, $F[Q_{A,B,D}] = F_{\tilde{Z},\tilde{L},W}$ where

$$\begin{aligned}\tilde{L} &= e_1(iR)e_b(R)L = L, \text{ and} \\ \tilde{Z} &= A - L^*e_b(R)^*e(iR)e_b(R)L = A - L^*e_a(R)L = Z.\end{aligned}$$

Thus $F[Q_{A,B,D}] = F_{Z,L,W}$.

Now suppose that Z is skewadjoint. Then $A = Z + L^*e_a(R)L$ is skewadjoint and so $Q_{A,B,D}$ is too. \square

We may now see precisely how the Holevo transform relates to the convergence of scaled quantum random walks.

Theorem 6.2. *Let $Q \in B(\mathfrak{h} \otimes \widehat{\mathfrak{k}})$.*

(a) *Suppose that $(P(h, n))_{h>0, n \in \mathbb{N}}$ is a family in $B(\mathfrak{h} \otimes \widehat{\mathfrak{k}})$ satisfying*

$$ns_h(P(h, n) - I) \rightarrow Q \text{ as } h \rightarrow 0 \text{ and } n \rightarrow \infty.$$

Then

$$s_h(P(h, n)^n - I) \rightarrow F[Q] \text{ as } h \rightarrow 0 \text{ and } n \rightarrow \infty, \quad (6.9)$$

and so if Q is skewadjoint and each $P(h, n)$ is contractive then, for all $\xi \in \mathfrak{h} \otimes \mathcal{F}$, $T \in \mathbb{R}_+$,

$$\sup_{t \in [0, T]} \left(\| (X_t^{h, P(h, n)^n} - X_t^{F[Q]})\xi \| + \| (X_t^{h, P(h, n)^n} - X_t^{F[Q]})^*\xi \| \right) \rightarrow 0$$

as $h \rightarrow 0$ and $n \rightarrow \infty$.

(b) *Suppose that $(Q_h)_{h>0}$ is a family in $B(\mathfrak{h} \otimes \widehat{\mathfrak{k}})$ satisfying*

$$s_h(Q_h) \rightarrow Q \text{ as } h \rightarrow 0.$$

Then

$$s_h(e^{Q_h} - I) \rightarrow F[Q] \text{ as } h \rightarrow 0, \quad (6.10)$$

and so if Q is skewadjoint and each Q_h is dissipative then, for all $\xi \in \mathfrak{h} \otimes \mathcal{F}$, $T \in \mathbb{R}_+$,

$$\sup_{t \in [0, T]} \left(\| (X_t^{h, e^{Q_h}} - X_t^{F[Q]})\xi \| + \| (X_t^{h, e^{Q_h}} - X_t^{F[Q]})^*\xi \| \right) \rightarrow 0 \text{ as } h \rightarrow 0.$$

Proof. Let $\begin{bmatrix} A & C \\ B & D \end{bmatrix}$ be the block matrix form of Q and, for $h > 0$, set

$$\Delta_h := I_{\mathfrak{h}} \otimes (\mathcal{S}_h^{\mathfrak{k}})^{-1} = \begin{bmatrix} \sqrt{h}I_{\mathfrak{h}} & 0 \\ 0 & I_{\mathfrak{h} \otimes \mathfrak{k}} \end{bmatrix}.$$

In both (a) and (b) the final claims follow from Proposition 6.1 and Theorem 4.3 along with Remark (i) following it.

(a) For $h > 0$ and $n \in \mathbb{N}$ define the operator

$$Q_{h, n} := ns_h(P(h, n) - I) \in B(\mathfrak{h} \otimes \widehat{\mathfrak{k}}).$$

Then, invoking the identity (6.3a), we see that

$$s_h(P(h, n)^n - I) = Q_{h, n} + Q_{h, n}\Delta_h P_n(\Delta_h Q_{h, n}\Delta_h)\Delta_h Q_{h, n}.$$

Now, as $h \rightarrow 0$ and $n \rightarrow \infty$, $Q_{h, n} \rightarrow Q$ and $\Delta_h \rightarrow \Delta$, so

$$\Delta_h Q_{h, n} \rightarrow \begin{bmatrix} 0 & 0 \\ B & D \end{bmatrix}, \quad Q_{h, n}\Delta_h \rightarrow \begin{bmatrix} 0 & C \\ 0 & D \end{bmatrix} \text{ and } \Delta_h Q_{h, n}\Delta_h \rightarrow \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix}.$$

Thus, since $p_n \rightarrow e_2$ uniformly on compact subsets of \mathbb{C} , identity (6.5) implies that

$$s_h(P(h, n)^n - I) \rightarrow Q + Q \begin{bmatrix} 0 & 0 \\ 0 & e_2(D) \end{bmatrix} Q = F[Q] \quad \text{as } h \rightarrow 0 \text{ and } n \rightarrow \infty.$$

(b) It follows from the identity (6.3c) that

$$\begin{aligned} ns_h(e^{Q_h/n} - I) - s_h(Q_h) &= s_h(Q_h e_2(Q_h/n) Q_h) / n \\ &= s_h(Q_h) \Delta_h e_2(\Delta_h s_h(Q_h) \Delta_h / n) \Delta_h s_h(Q_h) / n. \end{aligned}$$

Now e_2 is continuous at 0 and, as $h \rightarrow 0$, $s_h(Q_h) \rightarrow Q$ and $\Delta_h \rightarrow \Delta$, therefore

$$ns_h(e^{Q_h/n} - I) \rightarrow Q \quad \text{as } h \rightarrow 0 \text{ and } n \rightarrow \infty,$$

and so (6.10) holds by (a). \square

Remark. Part (b) may be compared with (22) in [Gou] and Theorem 19 of [AtP].

Appealing to Theorem 5.1, we see that Theorem 6.2 has the following consequence.

Corollary 6.3. *Suppose that, for $i = 1, 2$, $s_h(Q_i(h)) \rightarrow Q_i$ as $h \rightarrow 0$, for operators $Q_i, Q_i(h) \in B(\mathfrak{h} \otimes \widehat{\mathfrak{k}})$ ($h > 0$). Then*

$$s_h(e^{Q_1(h)+Q_2(h)} - I) \rightarrow F[Q_1 + Q_2] \quad \text{and} \quad s_h(e^{Q_1(h)} e^{Q_2(h)} - I) \rightarrow F[Q_1] \triangleleft F[Q_2].$$

7. REALISATIONS

In this section we give a variety of ways of implementing the approximation schemes of Theorems 4.3 and 6.2, with the assistance of Theorem 5.1. In particular, we show that each kind of QS cocycle (quasicontractive, isometric, coisometric or unitary) may be obtained as a limit of QRWs of the same kind.

Corresponding to the assemblies (1.7), set

$$V_{Z,L,W} := (e^Z \oplus I) V_L^\ell (I \oplus W),$$

where

$$V_L := \begin{bmatrix} (I + L^*L)^{-1/2} & -L^*(I + LL^*)^{-1/2} \\ L(I + L^*L)^{-1/2} & (I + LL^*)^{-1/2} \end{bmatrix},$$

for $Z \in B(\mathfrak{h})$, $L \in B(\mathfrak{h}; \mathfrak{h} \otimes \mathfrak{k})$, $M \in B(\mathfrak{h} \otimes \mathfrak{k}; \mathfrak{h})$ and $W \in B(\mathfrak{h} \otimes \mathfrak{k})$. Note that V_L is unitary.

We use the following abbreviation below:

$$T_+ := (\operatorname{Re} T)_+ \quad \text{for } T \in B(\mathfrak{H}). \quad (7.1)$$

Lemma 7.1. *Let $Z \in B(\mathfrak{H})$ for a Hilbert space \mathfrak{H} . Then*

$$\|e^Z\| \leq e^{\|Z_+\|}.$$

Proof. The operator $Y := Z - Z_+$ is dissipative and therefore

$$\|(e^{Y/n} e^{Z_+/n})^n\| \leq (\|e^{Y/n}\| \|e^{Z_+/n}\|)^n \leq \|e^{Z_+/n}\|^n \leq (e^{\|Z_+\|/n})^n = e^{\|Z_+\|} \quad (n \in \mathbb{N}),$$

so the result follows from the Lie–Trotter product formula ([ReS]). \square

Proposition 7.2. *Let $T \in B(\mathfrak{h} \otimes \widehat{\mathfrak{k}})$, $Z \in B(\mathfrak{h})$, $L \in B(\mathfrak{h}; \mathfrak{h} \otimes \mathfrak{k})$, $M \in B(\mathfrak{h} \otimes \mathfrak{k}; \mathfrak{h})$ and $W, R \in B(\mathfrak{h} \otimes \mathfrak{k})$, where W is contractive and R is selfadjoint with $\operatorname{Spec} R \subset [0, 2\pi]$. Set*

$$T_0^0 := E^{\widehat{0}} T E_{\widehat{0}} \quad \text{and} \quad Q_h := Q_{h(Z+L^*e_a(R)L), \sqrt{h}e_b(R)L, iR} \quad (h > 0),$$

in the notation (6.6). Then, for $h > 0$ and $t \in \mathbb{R}_+$,

$$\|e^{hT}\|^{[t/h]} \leq e^{t\|T_+\|}, \quad \|e^{Q_h}\|^{[t/h]} \leq e^{t\|Z_+\|} \quad \text{and} \quad \|V_{hZ, \sqrt{h}L, W}\|^{[t/h]} \leq e^{t\|Z_+\|}. \quad (7.2)$$

Moreover, as $h \rightarrow 0$,

$$s_h(e^{Q_h} - I) \rightarrow F_{Z,L,e^{iR}} \quad \text{and} \quad s_h(e^{hT}V_{hZ,\sqrt{h}L,W} - I) \rightarrow F_{T_0^0+Z,L,W}$$

Proof. Since $(Q_h)_+ = \begin{bmatrix} hZ_+ & 0 \\ 0 & 0 \end{bmatrix}$ and $h\lfloor t/h \rfloor \leq t$ ($h > 0$, $t \in \mathbb{R}_+$), the inequalities (7.2) follow from Lemma 7.1 and the unitarity of V_L ,

As $h \rightarrow 0$

$$s_h(V_{\sqrt{h}L} - I) = \begin{bmatrix} h^{-1}[(I + hL^*L)^{-1/2} - I] & -L^*(I + hLL^*)^{-1/2} \\ L(I + hL^*L)^{-1/2} & (I + hLL^*)^{-1/2} - I \end{bmatrix} \rightarrow F_{0,L,I}$$

and

$$s_h(e^{hT} - I) = s_h(hT) + O(h) \rightarrow F_{T_0^0,0,I}.$$

Moreover, for all $h > 0$,

$$s_h(I \oplus W - I) = F_{0,0,W}.$$

Therefore, by Theorem 5.1 and identity (1.9)

$$\begin{aligned} s_h(e^{hT}V_{hZ,\sqrt{h}L,W} - I) &= s_h(e^{hT}e^{h(Z \oplus 0)}V_{\sqrt{h}L}(I \oplus W) - I) \\ &\rightarrow F_{T_0^0,0,I} \triangleleft F_{Z,0,I} \triangleleft F_{0,L,I} \triangleleft F_{0,0,W} = F_{T_0^0+Z,L,W}. \end{aligned}$$

Finally, since for all $h > 0$

$$s_h(Q_h) = Q_{Z+L^*e_a(R)L,e_b(R)L,iR},$$

Theorem 6.2 (b) and Proposition 6.1 (b) imply that

$$s_h(e^{Q_h} - I) \rightarrow F_{Z,L,e^{iR}}. \quad \square$$

We next show that, given an elementary QS cocycle X which is either quasicontractive with exponential growth bound β , isometric, coisometric, or unitary, we may easily construct (from its stochastic generator) QRWs which are respectively quasicontractive with exponential growth bound β , isometric, coisometric, or unitary, and enjoy locally uniformly strong convergence to X . We need the following lemma, which expresses a functorial property common to the generation of QS cocycles and that of embedded QRWs.

Lemma 7.3. *Let $J \in B(\mathfrak{k}; \mathbb{K})$ be an isometry into a Hilbert space \mathbb{K} , and set*

$$\begin{aligned} J_{\mathfrak{k}} &:= I_{\mathfrak{h}} \otimes (I_{\mathbb{C}} \oplus J) \in B(\mathfrak{h} \otimes \widehat{\mathfrak{k}}; \mathfrak{h} \otimes \widehat{\mathbb{K}}) \quad \text{and} \\ J_{\mathcal{F}} &:= I_{\mathfrak{h}} \otimes \Gamma(I_{L^2(\mathbb{R}_+)} \otimes J) \in B(\mathfrak{h} \otimes \mathcal{F}^{\mathfrak{k}}; \mathfrak{h} \otimes \mathcal{F}^{\mathbb{K}}). \end{aligned}$$

Let $F, G \in B(\mathfrak{h} \otimes \widehat{\mathbb{K}})$. Then

$$J_{\mathcal{F}}^* X_t^F J_{\mathcal{F}} = X_t^{J_{\mathfrak{k}}^* F J_{\mathfrak{k}}} \quad \text{and} \quad J_{\mathcal{F}}^* X_t^{h,G} J_{\mathcal{F}} = X_t^{h, J_{\mathfrak{k}}^* G J_{\mathfrak{k}}} \quad (t \in \mathbb{R}_+, h > 0).$$

Proof. Let us adopt the following notation for $c \in \mathfrak{k}$, $f \in L^2(\mathbb{R}_+; \mathfrak{k})$ and $Q \in B(\mathfrak{h} \otimes \widehat{\mathbb{K}})$:

$$Jf := Jf(\cdot), \quad \text{and} \quad Q' := J_{\mathfrak{k}}^* Q J_{\mathfrak{k}}.$$

Thus

$$E_{\widehat{d}}^{\widehat{c}} Q' E_{\widehat{d}} = E_{\widehat{d}}^{\widehat{Jc}} Q E_{\widehat{Jd}} \quad (c, d \in \mathfrak{k}).$$

Let $F, G \in B(\mathfrak{h} \otimes \widehat{\mathbb{K}})$ and $h > 0$.

(a) Set $Y := (J_{\mathcal{F}}^* X_t^F J_{\mathcal{F}})_{t \geq 0}$. The first identity follows from uniqueness for weakly regular weak solutions of the QS differential equation $dX_t = X_t d\Lambda_{F'}(t)$ with $X_0 = I_{\mathfrak{h} \otimes \mathcal{F}}$

(Theorem 1.2) since, for $u, v \in \mathfrak{h}$, $f, g \in L^2(\mathbb{R}_+; \mathfrak{k})$ and $t \in \mathbb{R}_+$,

$$\begin{aligned} \langle u\varepsilon(f), (Y_t - I)v\varepsilon(g) \rangle &= \langle J_{\mathcal{F}}u\varepsilon(f), (X_t^F - I)J_{\mathcal{F}}v\varepsilon(g) \rangle \\ &= \langle u\varepsilon(Jf), (X_t^F - I)v\varepsilon(Jg) \rangle \\ &= \int_0^t ds \langle u\varepsilon(Jf), X_s^F E^{\widehat{Jf(s)}} F E_{\widehat{Jg(s)}} v\varepsilon(Jg) \rangle \\ &= \int_0^t ds \langle J_{\mathcal{F}}u\varepsilon(f), X_s^F E^{\widehat{Jf(s)}} F' E_{\widehat{g(s)}} J_{\mathcal{F}}v\varepsilon(g) \rangle \\ &= \int_0^t ds \langle u\varepsilon(f), Y_s E^{\widehat{f(s)}} F' E_{\widehat{g(s)}} v\varepsilon(g) \rangle, \end{aligned}$$

so $Y = X^{F'}$.

(b) Set $Y := (J_{\mathcal{F}}^* X^{h, G_t} J_{\mathcal{F}})_{t \geq 0}$ and let $f, g \in L^2(\mathbb{R}_+; \mathfrak{k})$ and $t \in \mathbb{R}_+$. Then

$$E^{\varepsilon(f)} Y_t E_{\varepsilon(g)} = \alpha \prod_{0 \leq p < \lfloor t/h \rfloor}^{\rightarrow} A_p \quad \text{and} \quad E^{\varepsilon(f)} X_t^{h, G'} E_{\varepsilon(g)} = \alpha \prod_{0 \leq p < \lfloor t/h \rfloor}^{\rightarrow} B_p,$$

where

$$\begin{aligned} A_p &:= E^{\varepsilon(Jf_{[hp, h(p+1)[})} X^{h, G_{hp, h(p+1)}} E_{\varepsilon(Jg_{[hp, h(p+1)[})}, \\ B_p &:= E^{\varepsilon(f_{[hp, h(p+1)[})} X_{hp, h(p+1)}^{h, G'} E_{\varepsilon(g_{[hp, h(p+1)[})}, \quad \text{and} \\ \alpha &:= \langle \varepsilon(f_{[h, \infty[}), \varepsilon(g_{[h, \infty[}) \rangle \quad \text{where } j = \lfloor t/h \rfloor. \end{aligned}$$

These products coincide since, for $p \in \mathbb{Z}_+$ and $u, v \in \mathfrak{h}$, setting $K = [hp, h(p+1)[$ and recalling the notation (4.1),

$$\begin{aligned} \langle u\varepsilon(f_K), (X_{hp, h(p+1)}^{h, G'} - I)v\varepsilon(g_K) \rangle &= \langle u\sqrt{h}Jf[p, h], (G - \Delta^\perp)v\sqrt{h}Jg[p, h] \rangle \\ &= \langle u\sqrt{h}f[p, h], (G' - \Delta'^\perp)v\sqrt{h}g[p, h] \rangle \\ &= \langle u\varepsilon(Jf_K), (X_{hp, h(p+1)}^{h, G} - I)v\varepsilon(Jg_K) \rangle \end{aligned}$$

(cf. the proof of Lemma 4.1), and

$$\langle u\varepsilon(Jf_K), v\varepsilon(Jg_K) \rangle = \langle u, v \rangle e^{\langle f_K, g_K \rangle} = \langle u\varepsilon(f_K), v\varepsilon(g_K) \rangle,$$

so $Y = X^{h, G'}$. □

We are now ready to fulfill the promise contained in the remark following Corollary 4.5.

Theorem 7.4. *Let X be an elementary QS cocycle on \mathfrak{h} with noise dimension space \mathfrak{k} .*

(a) *Suppose that X is quasicontractive with exponential growth bound β and stochastic generator F . Then there is a family $(G_h)_{h>0}$ in $B(\mathfrak{h} \otimes \widehat{\mathfrak{k}})$ such that*

$$\|G_h\|^{\lfloor t/h \rfloor} \leq e^{t\beta} \quad (h > 0, t \in \mathbb{R}_+), \quad (7.3a)$$

$$s_h(G_h - I) \rightarrow F \quad \text{as } h \rightarrow 0, \quad (7.3b)$$

$$\sup_{t \in [0, T]} \|(X_t - X_t^{h, G_h})\xi\| \rightarrow 0 \quad \text{as } h \rightarrow 0 \quad (\xi \in \mathfrak{h} \otimes \mathcal{F}, T \in \mathbb{R}_+) \quad \text{and} \quad (7.3c)$$

$$\sup_{t \in [0, T]} \|(X_t - X_t^{h, G_h})^* \xi\| \rightarrow 0 \quad \text{as } h \rightarrow 0 \quad (\xi \in \mathfrak{h} \otimes \mathcal{F}, T \in \mathbb{R}_+). \quad (7.3d)$$

(b) *Suppose that X is isometric, coisometric, or unitary. Then there is a family $(G_h)_{h>0}$ in $B(\mathfrak{h} \otimes \widehat{\mathfrak{k}})$ satisfying (7.3b) such that, respectively, each G_h is isometric and (7.3c) holds, each G_h is coisometric and (7.3d) holds, or each G_h is unitary and (7.3c), or equivalently (7.3d), holds.*

Proof. The proof is in three parts.

(1) Suppose first that X is isometric. Then, by Theorem 1.2 and Remark (i) preceding it, $F = F_{Z,L,W}$ for a skewadjoint operator $Z \in B(\mathfrak{h})$, an operator $L \in B(\mathfrak{h}; \mathfrak{h} \otimes \mathfrak{k})$ and an isometry $W \in B(\mathfrak{h} \otimes \mathfrak{k})$. Set $G_h := V_{hZ, \sqrt{h}L, W}^\ell$ ($h > 0$). Then, by Proposition 7.2 and Theorem 4.3 (b), the family of isometries $(G_h)_{h>0}$ in $B(\mathfrak{h} \otimes \widehat{\mathfrak{k}})$ satisfies (7.3b) and (7.3c). Moreover, if X is unitary then W is unitary and so each G_h is too, and (7.3d) also holds.

(2) Suppose next that X is coisometric. Then the dual cocycle X^\sharp is isometric with stochastic generator F^* . Therefore, by (1), there is a family of isometries $(\widetilde{G}_h)_{h>0}$ in $B(\mathfrak{h} \otimes \widehat{\mathfrak{k}})$ satisfying $s_h(\widetilde{G}_h - I) \rightarrow F^*$ as $h \rightarrow 0$. Setting $G_h = (\widetilde{G}_h)^*$ ($h > 0$), each G_h is coisometric and (7.3b) holds, so Theorem 4.3 (c) implies that (7.3d) holds.

(3) Suppose finally that X is quasicontractive with exponential growth bound β . Then the contraction cocycle $(e^{-\beta t} X_t)_{t \geq 0}$ has stochastic generator $F - \beta \Delta^\perp$. Set $\mathbb{K} := \mathfrak{k} \oplus \mathfrak{k} \oplus \mathbb{C}$ and let J be the isometry $\begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix} \in B(\mathfrak{k}; \mathbb{K})$. Then, by Theorem 1.3 of [L₂], there is an operator $\widetilde{F} \in B(\mathfrak{h} \otimes \mathbb{K})$ such that, in the notation of Lemma 7.3,

$$\widetilde{F}^* \triangleleft \widetilde{F} = 0 = \widetilde{F} \triangleleft \widetilde{F}^* \quad \text{and} \quad F - \beta \Delta^\perp = J_{\mathfrak{k}}^* \widetilde{F} J_{\mathfrak{k}}.$$

It follows from part (1) that there is a family of unitaries $(\widetilde{G}_h)_{h>0}$ in $B(\mathfrak{h} \otimes \widehat{\mathfrak{k}})$ such that, as $h \rightarrow 0$,

$$s_h(\widetilde{G}_h - I_{\widehat{\mathfrak{k}}}) \rightarrow \widetilde{F}, \quad \text{and} \\ \sup_{t \in [0, T]} \left(\| (X_t^{\widetilde{F}} - X_t^{h, \widetilde{G}_h}) \xi \| + \| (X_t^{\widetilde{F}} - X_t^{h, \widetilde{G}_h})^* \xi \| \right) \rightarrow 0 \quad (\xi \in \mathfrak{h} \otimes \mathcal{F}, T \in \mathbb{R}_+).$$

Now set $G_h = e^{\beta h} J_{\mathfrak{k}}^* \widetilde{G}_h J_{\mathfrak{k}}$ ($h > 0$). Then

$$\|G_h\|^{[t/h]} \leq e^{\beta h [t/h]} \leq e^{t\beta} \quad (h > 0, t \in \mathbb{R}_+),$$

so (7.3a) holds, and (7.3b) holds since

$$s_h(G_h - I) = s_h(e^{\beta h} J_{\mathfrak{k}}^* \widetilde{G}_h J_{\mathfrak{k}} - I) \\ = e^{\beta h} J_{\mathfrak{k}}^* s_h(\widetilde{G}_h - I) J_{\mathfrak{k}} + (e^{\beta h} - 1) s_h(I) \rightarrow J_{\mathfrak{k}}^* \widetilde{F} J_{\mathfrak{k}} + \beta \Delta^\perp = F.$$

Moreover (7.3c) and (7.3d) hold since, by Lemma 7.3,

$$X_t^F - X_t^{h, G_h} = X_t^{J_{\mathfrak{k}}^* (\widetilde{F} + \beta \Delta^\perp) J_{\mathfrak{k}}} - X_t^{h, J_{\mathfrak{k}}^* e^{\beta h} \widetilde{G}_h J_{\mathfrak{k}}} \\ = J_{\mathcal{F}}^* (X_t^{\widetilde{F} + \beta \Delta^\perp} - X_t^{h, e^{\beta h} \widetilde{G}_h}) J_{\mathcal{F}} = J_{\mathcal{F}}^* (e^{\beta t} X_t^{\widetilde{F}} - e^{\beta h [t/h]} X_t^{h, \widetilde{G}_h}) J_{\mathcal{F}}$$

for $t \in \mathbb{R}_+$ and $h > 0$, and $h[t/h] \rightarrow t$ locally uniformly as $h \rightarrow 0$. \square

8. REPEATED QUANTUM INTERACTIONS

In this section we consider repeated quantum interactions and the entanglement of bipartite systems, and prove two theorems demonstrating how these fit into the theory developed in the preceding sections. Whereas in the previous section we started with a quantum stochastic evolution and showed how to realise it as a limit of quantum random walks, in this section we travel in the opposite direction and, starting with a discrete quantum dynamics we first show how, through the appropriate scaling, one obtains a limiting continuous-time dynamics. We then treat the case where the discrete dynamics is given by a composition consisting of two systems which are physically independent and are separately interacting with a common environment.

In the model developed by Attal and Pautrat ([AtP]), one has a family of discrete-time evolutions of an open quantum system consisting of a system S with its Hamiltonian H_S , coupled to an environment modeled by an infinite chain of identical particles with each

particle governed by a Hamiltonian $H_{\mathbb{P}}$, repeatedly interacting with the system \mathbb{S} over a short time period of length h , through an interaction Hamiltonian $H_{\mathbb{I}}(h)$. Specifically, one takes

$$G_h = e^{-ihH_{\mathbb{T}}(h)} \quad (h > 0),$$

where the *total Hamiltonian* decomposes as

$$H_{\mathbb{T}}(h) = H_{\mathbb{S}} \otimes I_{\widehat{\mathbf{k}}} + I_{\mathfrak{h}} \otimes H_{\mathbb{P}} + H_{\mathbb{I}}(h)$$

for a *system Hamiltonian* $H_{\mathbb{S}} \in B(\mathfrak{h})_{\text{sa}}$, a *particle Hamiltonian* $H_{\mathbb{P}} \in B(\widehat{\mathbf{k}})_{\text{sa}}$ and an *interaction Hamiltonian* taking the form

$$H_{\mathbb{I}}(h) = \frac{1}{h} \begin{bmatrix} 0 & \sqrt{h}V_{\text{Di}}^* \\ \sqrt{h}V_{\text{Di}} & H_{\mathbb{S}\mathfrak{c}} \end{bmatrix}$$

for operators $V_{\text{Di}} \in B(\mathfrak{h}; \mathfrak{h} \otimes \mathbf{k})$ and $H_{\mathbb{S}\mathfrak{c}} \in B(\mathfrak{h} \otimes \mathbf{k})_{\text{sa}}$.

This fits perfectly into the general scheme described here. Indeed,

$$-ihH_{\mathbb{T}}(h) = -i \begin{bmatrix} hH_{\mathbb{S}} & \sqrt{h}V_{\text{Di}}^* \\ \sqrt{h}V_{\text{Di}} & H_{\mathbb{S}\mathfrak{c}} \end{bmatrix} - ihI_{\mathfrak{h}} \otimes H_{\mathbb{P}} - ih(0_{\mathfrak{h}} \oplus (H_{\mathbb{S}} \otimes I_{\mathbf{k}}))$$

so, under the scaling (4.4),

$$s_h(-ihH_{\mathbb{T}}(h)) \rightarrow -i \begin{bmatrix} H_{\mathbb{S}} + \omega(H_{\mathbb{P}})I_{\mathfrak{h}} & V_{\text{Di}}^* \\ V_{\text{Di}} & H_{\mathbb{S}\mathfrak{c}} \end{bmatrix} \quad \text{as } h \rightarrow 0,$$

where $\omega := \omega_{\widehat{0}}$, the vector state corresponding to the vector $\widehat{0} \in \widehat{\mathbf{k}}$. Therefore, by Theorem 6.2 and Proposition 6.1, we have the following strong convergence of a scaled unitary QRW to a QS unitary cocycle.

Theorem 8.1. *Let $G_h = e^{-ihH_{\mathbb{T}}(h)}$ where $H_{\mathbb{T}}(h)$ is as above, for $h > 0$. Then*

$$\sup_{t \in [0, T]} \|(X_t^{h, G_h} - X_t^F)\xi\| \rightarrow 0 \quad \text{as } h \rightarrow 0 \quad (\xi \in \mathfrak{h} \otimes \mathcal{F}, T \in \mathbb{R}_+)$$

and, in the notations (1.7) and (6.2), $F = F_{-iH, L, W}$ for the operators

$$H = H_{\mathbb{S}} + \omega(H_{\mathbb{P}})I_{\mathfrak{h}} - iV_{\text{Di}}^* e^{-iH_{\mathbb{S}\mathfrak{c}}} V_{\text{Di}}, \quad L = -ie_1(-iH_{\mathbb{S}\mathfrak{c}})V_{\text{Di}}, \quad \text{and } W = e_0(-iH_{\mathbb{S}\mathfrak{c}}).$$

Remarks. This result implies Theorem 19 of [AtP] in coordinate-free form, with the difference that here no Hilbert–Schmidt-type conditions need be imposed on the matrix components of L and W with respect to some fixed orthonormal basis of the noise dimension space \mathbf{k} .

For a discussion of the physical origins of the components of the interaction Hamiltonian see [BrP]. In brief, the scaling order \sqrt{h} corresponds to a *weak coupling limit*, or van Hove limit ([vHo], [Dav]), whereas the scaling order h corresponds to a *low density limit* ([Düm]).

In case the interaction Hamiltonian has no scattering component, F takes the following simpler form

$$\begin{bmatrix} -i(H_{\mathbb{S}} + \omega(H_{\mathbb{P}})I_{\mathfrak{h}}) - \frac{1}{2}(V_{\text{Di}})^*V_{\text{Di}} & -i(V_{\text{Di}})^* \\ -iV_{\text{Di}} & 0 \end{bmatrix}.$$

On the other hand, in case there is no dipole term in the interaction Hamiltonian, so that it is purely scattering, the operators $H_{\mathbb{T}}(h)$ and F then take the respective forms

$$\begin{bmatrix} H_{\mathbb{S}} & 0 \\ 0 & h^{-1}H_{\mathbb{S}\mathfrak{c}} + (H_{\mathbb{S}} \otimes I_{\mathbf{k}}) \end{bmatrix} + I_{\mathfrak{h}} \otimes H_{\mathbb{P}} \quad \text{and} \quad \begin{bmatrix} -i(H_{\mathbb{S}} + \omega(H_{\mathbb{P}})I_{\mathfrak{h}}) & 0 \\ 0 & e^{-iH_{\mathbb{S}\mathfrak{c}}} - I_{\mathfrak{h} \otimes \mathbf{k}} \end{bmatrix}.$$

Thus, if also $H_{\mathbb{S}} = 0$, then $X^F = (e^{-it\omega(H_{\mathbb{P}})}U_t)_{t \in \mathbb{R}_+}$ where U is a unitary QS cocycle of preservation type, as described in Example 4.4.

We now turn to the model of entanglement of bipartite systems under repeated quantum interactions studied by Attal, Deschamps and Pelligrini ([ADP]). Here the system space \mathfrak{h} is a tensor product $\mathfrak{h}_1 \otimes \mathfrak{h}_2$ of constituent system spaces, and $G_h = G_1(h)G_2(h)$ where

$$\begin{aligned} G_2(h) &= e^{-ihI_1} \otimes H_{\mathfrak{T}}^{(2)}(h) = I_1 \otimes e^{-ihH_{\mathfrak{T}}^{(2)}(h)} \\ G_1(h) &= e^{-ihI_2} \tilde{\otimes} H_{\mathfrak{T}}^{(1)}(h) = I_2 \tilde{\otimes} e^{-ihH_{\mathfrak{T}}^{(1)}(h)}, \end{aligned}$$

in which $I_1 := I_{\mathfrak{h}_1}$, $I_2 := I_{\mathfrak{h}_2}$, with the tilde capturing the tensor flip from $B(\mathfrak{h}_2) \overline{\otimes} B(\mathfrak{h}_1 \otimes \mathfrak{k})$ to $B(\mathfrak{h} \otimes \mathfrak{k})$ (as in Example 5.3), and the total Hamiltonians decompose as

$$H_{\mathfrak{T}}^{(i)}(h) = H_{\mathfrak{S}}^{(i)} \otimes I_{\widehat{\mathfrak{k}}} + I_1 \otimes H_{\mathfrak{P}} + H_1^{(i)}(h) \quad (i = 1, 2)$$

for system Hamiltonians $H_{\mathfrak{S}}^{(i)} \in B(\mathfrak{h}_i)_{\text{sa}}$ ($i = 1, 2$), a single particle Hamiltonian $H_{\mathfrak{P}} \in B(\widehat{\mathfrak{k}})_{\text{sa}}$ and interaction Hamiltonians taking the form

$$H_1^{(i)}(h) = \frac{1}{h} \begin{bmatrix} 0 & \sqrt{h} \left(V_{\text{Di}}^{(i)} \right)^* \\ \sqrt{h} V_{\text{Di}}^{(i)} & H_{\text{Sc}}^{(i)} \end{bmatrix}$$

for operators $V_{\text{Di}}^{(i)} \in B(\mathfrak{h}_i; \mathfrak{h}_i \otimes \mathfrak{k})$ and $H_{\text{Sc}}^{(i)} \in B(\mathfrak{h}_i \otimes \mathfrak{k})_{\text{sa}}$ ($i = 1, 2$). From the preceding example we deduce that (again setting $\omega := \omega_{\widehat{\mathfrak{0}}}$),

$$s_h(-ihH_{\mathfrak{T}}^{(i)}(h)) \rightarrow -i \begin{bmatrix} H_{\mathfrak{S}}^{(i)} + \omega(H_{\mathfrak{P}})I_i & \left(V_{\text{Di}}^{(i)} \right)^* \\ V_{\text{Di}}^{(i)} & H_{\text{Sc}}^{(i)} \end{bmatrix} \quad \text{as } h \rightarrow 0 \quad (i = 1, 2).$$

Theorem 8.2. *Let $G_1(h)$ and $G_2(h)$ be as above, for $h > 0$. Then*

$$\sup_{t \in [0, T]} \left\| (X_t^{h, G_1(h)G_2(h)} - X_t^{F_1 \triangleleft F_2}) \xi \right\| \rightarrow 0 \quad \text{as } h \rightarrow 0 \quad (\xi \in \mathfrak{h} \otimes \mathcal{F}, T \in \mathbb{R}_+)$$

where $F_2 := I_1 \otimes F_{(2)}$, $F_1 := I_2 \tilde{\otimes} F_{(1)}$, and in the notations (6.2) and (1.7), $F_{(i)} = F_{-iH^{(i)}, L^{(i)}, W^{(i)}}$ for the operators

$$\begin{aligned} H^{(i)} &= H_{\mathfrak{S}}^{(i)} + \omega(H_{\mathfrak{P}})I_{\mathfrak{h}_i} - i(V_{\text{Di}}^{(i)})^* e(-iH_{\text{Sc}}^{(i)}) V_{\text{Di}}^{(i)}, \\ L^{(i)} &= -ie_1(-iH_{\text{Sc}}^{(i)}) V_{\text{Di}}^{(i)}, \quad \text{and } W^{(i)} = e_0(-iH_{\text{Sc}}^{(i)}). \end{aligned}$$

Moreover, $F_1 \triangleleft F_2 = F_{-iH, L, W}$ for the operators

$$\begin{aligned} H &= H^{(1)} \otimes I_2 + I_1 \otimes H^{(2)} + \text{Im} \left(\left[I_2 \tilde{\otimes} (V_{\text{Di}}^{(1)})^* e_1(-iH_{\text{Sc}}^{(1)}) \right] \left[I_1 \otimes e_1(-iH_{\text{Sc}}^{(2)}) V_{\text{Di}}^{(2)} \right] \right), \\ L &= -i \left(I_2 \tilde{\otimes} e_1(-iH_{\text{Sc}}^{(1)}) V_{\text{Di}}^{(1)} + \left[I_2 \tilde{\otimes} e_0(-iH_{\text{Sc}}^{(1)}) \right] \left[I_1 \otimes e_1(-iH_{\text{Sc}}^{(2)}) V_{\text{Di}}^{(2)} \right] \right), \quad \text{and} \\ W &= e_0 \left(-iI_2 \tilde{\otimes} H_{\text{Sc}}^{(1)} \right) e_0 \left(-iI_1 \otimes H_{\text{Sc}}^{(2)} \right). \end{aligned}$$

Proof. The first part follows from Theorem 6.2 and Propositions 6.1 and 5.1. The second part follows from identity (1.8) and the relation $e_1(-z)e_0(z) = e_1(z)$ ($z \in \mathbb{C}$). \square

Remarks. In view of Proposition 5.2, the limiting cocycle $X^{F_1 \triangleleft F_2}$ is actually the (point-wise) product of the individual cocycles X^{F_1} and X^{F_2} where $F_1, F_2 \in B(\mathfrak{h} \otimes \widehat{\mathfrak{k}})$ are as above.

If we assume that neither of the interaction Hamiltonians has a scattering component: $H_{\text{Sc}}^{(1)} = H_{\text{Sc}}^{(2)} = 0$, then F takes the form $\begin{bmatrix} -iH - \frac{1}{2}L^*L & -L^* \\ L & 0 \end{bmatrix}$ where

$$\begin{aligned} H &= H_{\mathfrak{S}}^{(1)} \otimes I_2 + I_1 \otimes H_{\mathfrak{S}}^{(2)} + 2\omega(H_{\mathfrak{P}})I_1 \otimes I_2 + \text{Im} \left(\left[I_2 \tilde{\otimes} (V_{\text{Di}}^{(1)})^* \right] \left[I_1 \otimes V_{\text{Di}}^{(2)} \right] \right) \quad \text{and} \\ L &= -i \left(I_2 \tilde{\otimes} V_{\text{Di}}^{(1)} + I_1 \otimes V_{\text{Di}}^{(2)} \right). \end{aligned}$$

Assuming further that the noise dimension space \mathfrak{k} is finite dimensional, with fixed orthonormal basis $(e_j)_{1 \leq j \leq d}$, and setting $I_A := I_{\mathfrak{h}_1}$, $I_B := I_{\mathfrak{h}_2}$, $H^A := H^{(1)}$, $H^B := H^{(2)}$,

$$\lambda_0 := \omega(H_P), \quad V_j := (I_{\mathfrak{h}_1} \otimes \langle e_j |) V_{D_i}^{(1)} \quad \text{and} \quad W_j := (I_{\mathfrak{h}_2} \otimes \langle e_j |) V_{D_i}^{(2)}, \quad \text{for } j = 1, \dots, d,$$

one gets

$$F_1 \triangleleft F_2 = \begin{bmatrix} K & L_1^* & \cdots & L_d^* \\ L_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ L_d & 0 & \cdots & 0 \end{bmatrix} \in B(\mathfrak{h} \otimes \mathbb{C}^{d+1}),$$

where

$$L_j := -i(V_j \otimes I_B + I_A \otimes W_j) \quad \text{for } j = 1, \dots, d, \quad \text{and}$$

$$K := -i(H^A \otimes I_B + I_A \otimes H^B + 2\lambda_0 I_A \otimes I_B)$$

$$- \frac{1}{2} \sum_{j=1}^d (V_j^* V_j \otimes I_B + I_A \otimes W_j^* W_j) + \sum_{j=1}^d V_j^* \otimes W_j$$

and so, modulo the fact that we work with left (rather than right) cocycles, we recover Theorem 3.1 of [ADP] as a special case of Theorem 8.2.

Acknowledgements. We are grateful to anonymous referees for insightful remarks and a very thorough engagement with the paper. This work benefited from the support of the UK-India Education and Research Initiative grant QP-NCG-QI: *Quantum Probability, Noncommutative Geometry and Quantum Information*.

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