

# THE PARABOLIC ALGEBRA ON $L^p$ SPACES

E. KASTIS

ABSTRACT. The parabolic algebra was introduced by Katavolos and Power, in 1997, as the SOT - closed operator algebra acting on  $L^2(\mathbb{R})$  that is generated by the translation and multiplication semigroups. In particular, they proved that this algebra is reflexive and is equal to the Fourier binest algebra, that is, to the algebra of operators that leave invariant the subspaces in the Volterra nest and its analytic counterpart.

We prove that a similar result holds for the corresponding algebras acting on  $L^p(\mathbb{R})$ , where  $1 < p < \infty$ . In the last section, it is also shown that the reflexive closures of the Fourier binests on  $L^p(\mathbb{R})$  are all order isomorphic for  $1 < p < \infty$ .

## 1. INTRODUCTION

Let  $\{D_\mu, \mu \in \mathbb{R}\}$  and  $\{M_\lambda, \lambda \in \mathbb{R}\}$  be the groups of translation and multiplication respectively acting on the Hilbert space  $L^2(\mathbb{R})$ , given by

$$D_\mu f(x) = f(x - \mu), \quad M_\lambda f(x) = e^{i\lambda x} f(x).$$

It is well-known that these 1-parameter unitary groups are continuous in the strong operator topology (SOT), that they provide an irreducible representation of the Weyl-commutation relations,  $M_\lambda D_\mu = e^{i\lambda\mu} D_\mu M_\lambda$ , and that the SOT-closed operator algebra they generate is the von Neumann algebra  $B(L^2(\mathbb{R}))$  of all bounded operators. (See Taylor [24], for example.) On the other hand it was shown by Katavolos and Power in [11] that the strongly closed nonselfadjoint operator algebra generated by the semigroups for  $\mu \geq 0$  and  $\lambda \geq 0$  is a reflexive algebra, in the sense of Halmos [18], containing no self-adjoint operators, other than real multiples of the identity, and containing no nonzero finite rank operators. We consider here the operator algebras  $\mathcal{A}_{par}^p$  on  $L^p(\mathbb{R})$  for  $1 < p < \infty$ , which are similarly generated by the same semigroups, viewed now as bounded operators on  $L^p(\mathbb{R})$ . Our main result is that  $\mathcal{A}_{par}^p$  is also reflexive and, moreover, is equal to  $\mathcal{A}_{FB}^p$ , the algebra of operators that leave invariant each subspace in the Fourier binest  $\mathcal{L}_{FB}^p$  given by

$$\mathcal{L}_{FB}^p = \{0\} \cup \{L^p[t, \infty), t \in \mathbb{R}\} \cup \{e^{i\lambda x} H^p(\mathbb{R}), \lambda \in \mathbb{R}\} \cup \{L^p(\mathbb{R})\}$$

where  $H^p(\mathbb{R})$  is the usual Hardy space for the upper half plane. This lattice of closed subspaces is a binest equal to the union of two complete continuous nests of closed subspaces.

Although the reflexivity of non selfadjoint operator algebras has been studied intensively over the last fifty years, the developments have been largely confined within the limits of operator algebras acting on Hilbert spaces. For example, general nest algebras, being the most characteristic class of reflexive noncommutative non selfadjoint operator algebras

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since they were introduced by Ringrose in 1965 [20], have a well-developed general theory on Hilbert spaces (Davidson [4]). However, only sporadic results can be found for nest algebras on Banach spaces (see [25], [23], [3]).

On the other hand, the study of the reflexivity of non-selfadjoint algebras that are generated by semigroups of operators was begun by Sarason in 1966 [22], where he proved that  $H^\infty(\mathbb{R})$ , viewed as a multiplication algebra on  $H^2(\mathbb{R})$ , is reflexive. Since then, several results about 2-parameter Lie semigroup algebras have been obtained. One of the aims in the analysis of reflexivity and related properties is to understand better the algebraic structure of these somewhat mysterious algebras. Establishing reflexivity can provide a route to constructing operators in the algebra and thereby deriving further algebraic properties. As we stated above, the reflexivity of  $\mathcal{A}_{par}^2$ , known as the parabolic algebra, was obtained in [11]. Furthermore, Levene and Power have shown ([15]) the reflexivity of an analogous hyperbolic algebra [12], the algebra generated by the multiplication and dilation semigroups on  $L^2(\mathbb{R})$ . The latter semigroup is given by the operators  $V_t$ , with

$$V_t f(x) = e^{t/2} f(e^t x),$$

for  $t \geq 0$ . Recently, Power and the author proved that the triple semigroup operator algebra that is generated by the translation, dilation and multiplication semigroups is also reflexive [10].

A complication in establishing the reflexivity of the parabolic and hyperbolic algebras on Hilbert space is the absence of an approximate identity of finite rank operators, a key device in the theory of nest algebras (Erdos and Power [6], Davidson [4]). However, it was shown that the subspace of Hilbert-Schmidt operators is dense for both algebras and that these operators could be used as an alternative. In contrast Annoussis, Katavolos and Todorov [1] have shown that direct integral decomposition arguments provide a route to reflexivity for various discrete noncommutative semigroup algebras. In our case, we define a right ideal of what we refer to as  $(p, q)$ -integral operators which we show is able to play the role of the (two-sided) ideal of Hilbert - Schmidt operators. As a substitute for the techniques of Hilbert space geometry and tensor product identifications used in [10], [11], [15], we make use of more involved measure theoretic arguments appropriate for the  $(p, q)$ -integrable operators.

We also obtain a number of properties of the parabolic algebra on  $L^p(\mathbb{R})$ , that correspond to the classical case. Namely,  $\mathcal{A}_{par}^p$  is antisymmetric (or triangular [9]), in an appropriate sense, and  $\mathcal{A}_{par}^p$  contains no non-trivial finite rank operators. Furthermore, the lattice of  $\mathcal{A}_{par}^p$  is order isomorphic to the lattice of  $\mathcal{A}_{par}^2$  for all  $1 < p < \infty$ .

## 2. PRELIMINARIES

**2.1. The Hardy space  $H^p(\mathbb{R})$ ,  $p \in [1, \infty)$ .** We start with two elementary density lemmas for the Hardy spaces  $H^p(\mathbb{R})$  on the line, for  $p \in (1, \infty)$ . The details of the theory of Hardy spaces can be found in [8].

For each  $u$  in the open upper half plane  $\mathbb{H}^+$  of  $\mathbb{C}$  let

$$b_u(x) = \frac{1}{x + u}, \quad x \in \mathbb{R}.$$

Then  $b_u$  lies in  $H^p(\mathbb{R})$ , for every  $p \in (1, \infty)$ .

**Lemma 2.1.** *The linear spans of the sets  $\mathcal{D}_1 = \{b_u | u \in \mathbb{H}^+\}$ ,  $\mathcal{D}_2 = \{b_u b_w | u, w \in \mathbb{H}^+\}$  are both dense in  $H^p(\mathbb{R})$ , for  $1 < p < \infty$ .*

*Proof.* Fix some  $p \in (1, \infty)$  and suppose that there exists some  $f \in H^p(\mathbb{R})$  that does not lie in the closed linear span of  $\mathcal{D}_1$ . Then by the Hahn - Banach theorem, there is some function  $g \in L^q(\mathbb{R})$ , such that  $\int_{\mathbb{R}} b_u g = 0$ , for all  $u \in \mathbb{H}^+$ , and  $\int_{\mathbb{R}} f g \neq 0$ . But

$$\int_{\mathbb{R}} b_u g = 0, \forall u \in \mathbb{H}^+ \Leftrightarrow \int_{\mathbb{R}} \frac{g(x)}{x+u} dx = 0, \forall u \in \mathbb{H}^+ \Leftrightarrow g \in H^q(\mathbb{R})$$

Hence  $f g \in H^1(\mathbb{R})$ , so  $\int_{\mathbb{R}} f g = 0$ , which gives a contradiction.

Now, for any distinct  $u, w \in \mathbb{H}^+$ , observe that

$$b_u(x)b_w(x) = \frac{b_u(x) - b_w(x)}{w - u}.$$

Define

$$h_n = (ni - u)b_u b_{ni} = b_u - b_{ni}.$$

Since  $h_n \rightarrow b_u$  pointwise, as  $n \rightarrow \infty$ , and  $|h_n(x)| \leq |b_u(x)|$  for sufficiently large  $n$ , for all  $x \in \mathbb{R}$ , it follows from dominated convergence that  $h_n \xrightarrow{\|\cdot\|_p} b_u$ . Therefore, given  $u \in \mathbb{H}^+$ , the function  $b_u$  lies in the closed linear span of  $\mathcal{D}_2$ , so by the first part of the lemma, the proof is complete.  $\square$

**Lemma 2.2.** *Let  $\mathbb{H}_{\mathbb{Q}}^+ = \{u \in \mathbb{H}^+ : u = x + iy, \text{ where } x, y \in \mathbb{Q}\}$ . For every  $t \in \mathbb{R}$  the countable set*

$$\Lambda_t = \{b_u D_t b_w | u, w \in \mathbb{H}_{\mathbb{Q}}^+\}$$

*is dense in  $H^p(\mathbb{R})$ , for every  $p \in (1, \infty)$ .*

*Proof.* Observe first that

$$D_t b_w(x) = \frac{1}{(x-t) + w} = \frac{1}{x + (w-t)} = b_{w-t}(x), x \in \mathbb{R}.$$

Since  $\mathbb{Q}$  is dense in  $\mathbb{R}$ , the rest of the proof is a simple application of dominated convergence.  $\square$

The Beurling theorem in the  $L^p$ -setting will be useful for the determination of the lattice of  $A_{par}^p$  ([7], [19]). Applying the isometric isomorphism

$$\Phi_p : L^p(\mathbb{T}) \rightarrow L^p(\mathbb{R}) : (\Phi_p f)(t) = \left( \frac{1}{\sqrt{\pi}(1-it)} \right)^{2/p} f \left( \frac{1+it}{1-it} \right), t \in \mathbb{R},$$

we can state the theorem on the real line, instead of the unit circle. Note that  $\Phi_p$  restricts to an isomorphism of  $H_p$ -spaces [13].

**Theorem 2.3.** *Given  $p \in (1, \infty)$ , let  $\mathcal{M}$  be a closed subspace of  $L^p(\mathbb{R})$  which is invariant under the semigroup  $\{M_\lambda, \lambda \geq 0\}$ . Then  $\mathcal{M}$  is either of the form  $L^p(E)$  for some Borel subset  $E \subseteq \mathbb{R}$  or  $\mathcal{M}$  is equal to  $\phi H^p(\mathbb{R})$  for some unimodular function  $\phi$ .*

**2.2. The space  $L^p(\mathbb{R}; L^q(\mathbb{R}))$ , for  $1 < p, q < \infty$ .** We now introduce some notation and terminology associated with the classical space  $L^p(\mathbb{R}; L^q(\mathbb{R}))$ . This space is a space of kernel functions for what we refer to as the  $(p, q)$ -integrable operators. For more details, we refer the reader to [16],[17].

Let  $p, q \in [1, +\infty]$ . Define  $\mathcal{S}(\mathbb{R}; L^q(\mathbb{R}))$  be the space of measurable simple functions; i.e. the functions  $f : \mathbb{R} \rightarrow L^q(\mathbb{R})$  taking only finitely many values :

$$f(x) = \sum_{k=1}^n \chi_{A_k}(x) g_k,$$

where  $\{A_k\}_{k=1, \dots, n}$  is a finite family of Borel measurable pairwise disjoint sets and where  $g_k \in L^q(\mathbb{R})$ .

**Definition 2.4.** A function  $f : \mathbb{R} \rightarrow L^q(\mathbb{R})$  is said to be strongly measurable if there is a sequence  $(f_n)$  in  $\mathcal{S}(\mathbb{R}; L^q(\mathbb{R}))$ , tending to  $f$  pointwise a.e.. Also,  $f$  is weakly measurable, if given  $\omega \in (L^q(\mathbb{R}))^*$  the function  $t \mapsto \omega(f(t))$  is Borel measurable.

The relationship between strong and weak measurability is given by the following theorem of Pettis [16], who introduced the notion of almost separably valued functions.

**Definition 2.5.** Let  $1 \leq q \leq \infty$ . A function  $f : \mathbb{R} \rightarrow L^q(\mathbb{R})$  is almost separably valued, if there is a conull Borel set  $A \subseteq \mathbb{R}$ , such that  $f(A)$  is separable.

**Theorem 2.6.** A function  $f : \mathbb{R} \rightarrow L^q(\mathbb{R})$  is strongly measurable if and only if it is weakly measurable and almost separably valued.

**Example 2.7.** Define  $f : \mathbb{R} \rightarrow L^\infty(\mathbb{R})$  by  $f(x) = \chi_{(-\infty, x]}$ . Then  $f$  is not almost separably valued, and hence not strongly measurable, since  $\|f(x) - f(t)\|_\infty = 1$  for  $x \neq t$ . However, for  $q \in (1, \infty)$ , the function  $g : \mathbb{R} \rightarrow L^q(\mathbb{R})$ , given by  $g(x) = \chi_{(-\infty, x]} f$ , where  $f \in L^q(\mathbb{R})$ , is strongly measurable. To see this, note that  $L^q(\mathbb{R})$  is separable and given  $\omega \in L^p(\mathbb{R})$ , where  $p$  is the conjugate exponent of  $q$ , we have

$$\omega(g(x)) = \int_{\mathbb{R}} \omega(y) \chi_{(-\infty, x]}(y) f(y) dy = \int_{-\infty}^x \omega(y) f(y) dy,$$

which is measurable, being the limit of absolutely continuous functions.

The definition of  $L^p$  spaces of  $L^q$ -valued functions is analogous to the case of scalar valued functions. Define  $L^p(\mathbb{R}; L^q(\mathbb{R}))$  as the set of equivalence classes (modulo equality for almost every  $x \in \mathbb{R}$ ) of strongly measurable functions  $f$  that satisfy  $(\int_{\mathbb{R}} \|f(x)\|_q^p dx)^{1/p} < \infty$  for  $1 \leq p < \infty$ , and  $\text{esssup} \|f(\cdot)\|_q$  for  $p = \infty$ . Each of the above spaces endowed with the respective norm

$$\|f\|_{p,q} = \left( \int_{\mathbb{R}} \|f(x)\|_q^p dx \right)^{1/p}, \text{ for } p \in [1, \infty),$$

$$\|f\|_{\infty,q} = \text{esssup} \|f(\cdot)\|_q, \text{ for } p = \infty,$$

becomes a Banach space.

**Remark 2.8.** In the case  $p = q = 2$  we have the natural isomorphisms

$$L^2(\mathbb{R}; L^2(\mathbb{R})) \cong L^2(\mathbb{R}) \otimes L^2(\mathbb{R}) \cong L^2(\mathbb{R}^2)$$

For the rest of the subsection, the exponents  $p, q$  lie on the open interval  $(1, \infty)$ . Given  $f_1, f_2, \dots, f_n \in L^p(\mathbb{R})$  and  $g_1, g_2, \dots, g_n \in L^q(\mathbb{R})$ , define

$$f : \mathbb{R} \rightarrow L^q(\mathbb{R}) : f(x) \mapsto \sum_{k=1}^n f_k(x)g_k.$$

We denote this function by  $\sum_{k=1}^n f_k \otimes g_k$  and we write  $\mathcal{F}(\mathbb{R}; L^q(\mathbb{R}))$  for the subspace of  $L^p(\mathbb{R}; L^q(\mathbb{R}))$  formed by such functions. Finally, we write  $\mathcal{F}(\mathbb{R}; \mathcal{S}(\mathbb{R}))$  for the set of functions  $\sum_{k=1}^n f_k \otimes \chi_{A_k}$ , where  $\{A_k\}_{k=1, \dots, n}$  is a partition of the real line.

**Proposition 2.9.** *The following sets are dense in  $L^p(\mathbb{R}; L^q(\mathbb{R}))$ .*

- (1)  $\mathcal{S}(\mathbb{R}; L^q(\mathbb{R})) \cap L^p(\mathbb{R}; L^q(\mathbb{R}))$ ;
- (2)  $\mathcal{F}(\mathbb{R}; L^q(\mathbb{R})) \cap L^p(\mathbb{R}; L^q(\mathbb{R}))$ ;
- (3)  $\mathcal{F}(\mathbb{R}; \mathcal{S}(\mathbb{R})) \cap L^p(\mathbb{R}; L^q(\mathbb{R}))$ .

*Proof.* The argument for the density of the first two sets can be found in [17]. For the last set it suffices to prove that given  $f \in L^p(\mathbb{R}), g \in L^q(\mathbb{R})$ , we can find a sequence  $(f_n)$  of elements in  $\mathcal{F}(\mathbb{R}; \mathcal{S}(\mathbb{R})) \cap L^p(\mathbb{R}; L^q(\mathbb{R}))$ , that converges to  $f \otimes g$  with respect to the  $\|\cdot\|_{p,q}$  norm. By the classical theory of  $L^q$  spaces, there is a sequence of simple functions

$$g_n = \sum_{k=1}^n a_n \chi_{A_n}, \quad a_n \in \mathbb{C},$$

such that  $g_n \rightarrow g$  in  $L^q(\mathbb{R})$ . Then the functions  $f \otimes g_n$  lie in  $\mathcal{F}(\mathbb{R}; \mathcal{S}(\mathbb{R})) \cap L^p(\mathbb{R}; L^q(\mathbb{R}))$  and

$$\|f \otimes g_n - f \otimes g\|_{p,q} = \|f\|_p \|g_n - g\|_q \rightarrow 0.$$

□

The characterization of the dual space of  $L^p(\mathbb{R}; L^q(\mathbb{R}))$  is again analogous to the scalar valued case, after we take account of duality in the range space  $L^q(\mathbb{R})$  (see [17]).

**Proposition 2.10.** *Let  $p, q \in (1, \infty)$  be conjugate exponents. The dual space of  $L^p(\mathbb{R}; L^q(\mathbb{R}))$  is isometrically isomorphic to  $L^q(\mathbb{R}; L^p(\mathbb{R}))$ .*

**Lemma 2.11.** *Given an operator  $T \in B(L^q(\mathbb{R}))$ , there is a unique bounded linear operator*

$$\tilde{T} : L^p(\mathbb{R}; L^q(\mathbb{R})) \rightarrow L^p(\mathbb{R}; L^q(\mathbb{R}))$$

*such that given  $f \otimes g \in \mathcal{F}(\mathbb{R}; L^q(\mathbb{R}))$*

$$\tilde{T}(f \otimes g) = f \otimes Tg.$$

*Moreover, the map  $T \mapsto \tilde{T}$  is isometric.*

*Proof.* Let  $f = \sum_{k=1}^n \chi_{A_k} \otimes g_k$ , such that  $g_k \in L^q(\mathbb{R})$  and  $\{A_k\}_{k=1,\dots,n}$  are pairwise disjoint Borel sets. By linearity, calculate

$$\begin{aligned}
\|\tilde{T}f\|_{p,q}^p &= \left\| \sum_{k=1}^n \chi_{A_k} \otimes Tg_k \right\|_{p,q}^p = \int_{\mathbb{R}} \left\| \sum_{k=1}^n \chi_{A_k}(x) Tg_k \right\|_q^p dx = \\
&= \int_{\mathbb{R}} \left( \int_{\mathbb{R}} \left| \sum_{k=1}^n \chi_{A_k}(x) Tg_k(y) \right|^q dy \right)^{p/q} dx = \int_{\mathbb{R}} \left( \int_{\mathbb{R}} \sum_{k=1}^n \chi_{A_k}(x) |Tg_k(y)|^q dy \right)^{p/q} dx = \\
&= \int_{\mathbb{R}} \left( \sum_{k=1}^n \chi_{A_k}(x) \|Tg_k\|_q^q \right)^{p/q} dx \leq \int_{\mathbb{R}} \left( \sum_{k=1}^n \chi_{A_k}(x) \|T\|^q \|g_k\|_q^q \right)^{p/q} dx = \\
&= \|T\|^p \int_{\mathbb{R}} \left( \sum_{k=1}^n \chi_{A_k}(x) \|g_k\|_q^q \right)^{p/q} dx = \|T\|^p \int_{\mathbb{R}} \left( \sum_{k=1}^n \chi_{A_k}(x) \int_{\mathbb{R}} |g_k(y)|^q dy \right)^{p/q} dx = \\
&= \|T\|^p \int_{\mathbb{R}} \left( \int_{\mathbb{R}} \left| \sum_{k=1}^n \chi_{A_k}(x) g_k(y) \right|^q dy \right)^{p/q} dx = \|T\|^p \|f\|_{p,q}^p.
\end{aligned}$$

Since the set  $\mathcal{S}(\mathbb{R}; L^q(\mathbb{R}))$  is dense in  $L^p(\mathbb{R}; L^q(\mathbb{R}))$ , the operator  $\tilde{T}$  is bounded. To show that the mapping  $T \mapsto \tilde{T}$  is isometric, check that given  $g \in L^q(\mathbb{R})$

$$\|\chi_{[0,1]} \otimes g\|_{p,q}^p = \int_{\mathbb{R}} \|\chi_{[0,1]}(x)g\|_q^p dx = \int_0^1 dx \|g\|_q^p = \|g\|_q^p$$

This yields an upper bound for the norm of the operator  $T$

$$\|Tg\|_q^p = \|\chi_{[0,1]} \otimes Tg\|_{p,q}^p = \|\tilde{T}(\chi_{[0,1]} \otimes g)\|_{p,q}^p \leq \|\tilde{T}\|^p \|\chi_{[0,1]} \otimes g\|_{p,q}^p = \|\tilde{T}\|^p \|g\|_q^p,$$

so the proof is complete.  $\square$

**Lemma 2.12.** *Let  $p, q \in (1, \infty)$ . The linear map*

$$\Theta : L^p(\mathbb{R}; L^q(\mathbb{R})) \rightarrow L^p(\mathbb{R}; L^q(\mathbb{R})) : \Theta(f)(x)(y) \mapsto f(x)(x - y)$$

*is a bijective isometry onto  $L^p(\mathbb{R}; L^q(\mathbb{R}))$ .*

*Proof.* It suffices again to consider  $f \in \mathcal{S}(\mathbb{R}; L^q(\mathbb{R}))$ . Let  $f(x) = \sum_{k=1}^n \chi_{A_k} \otimes g_k$  as before.

First, in order to obtain that  $\Theta f$  is strongly measurable, it suffices to show that given  $\omega \in L^p(\mathbb{R})$ , the function

$$\omega(\Theta f(\cdot)) = \mathbb{R} \rightarrow \mathbb{C} : x \mapsto \omega(\Theta f(x)) = \int_{\mathbb{R}} \omega(y)(\Theta f)(x)(y) dy$$

is measurable. This is trivial to prove, since

$$\begin{aligned}
\omega(\Theta f(x)) &= \int_{\mathbb{R}} \omega(y) \sum_{k=1}^n \chi_{A_k}(x) g_k(x - y) dy = \\
&= \sum_{k=1}^n \chi_{A_k}(x) \int_{\mathbb{R}} \omega(y) g_k(x - y) dy = \sum_{k=1}^n \chi_{A_k}(x) (\omega * g_k)(x),
\end{aligned}$$

and applying Young's inequality, the function  $\omega * g_k$  lies in  $L^\infty(\mathbb{R})$ . Now

$$\begin{aligned} \|\Theta f\|_{p,q}^p &= \int_{\mathbb{R}} \|\Theta(f)(x)\|_q^p dx = \int_{\mathbb{R}} \left( \int_{\mathbb{R}} \left| \sum_{k=1}^n \chi_{A_k}(x) g_k(x-y) \right|^q dy \right)^{p/q} dx = \\ &= \int_{\mathbb{R}} \left( \int_{\mathbb{R}} \left| \sum_{k=1}^n \chi_{A_k}(x) g_k(y) \right|^q dy \right)^{p/q} dx = \int_{\mathbb{R}} \|f(x)\|_q^p dx = \|f\|_{p,q}^p \end{aligned}$$

Since  $\Theta^{-1} = \Theta$ , the map is bijective  $\square$

**2.3. The Fourier binest algebra  $\mathcal{A}_{FB}^p$ .** In this subsection, we give the natural generalization of the Fourier binest algebra on  $L^p$  spaces. The Volterra nest  $\mathcal{N}_v^p$  is the continuous nest consisting of the subspaces  $L^p([t, +\infty))$ , for  $t \in \mathbb{R}$ , together with the trivial subspaces  $\{0\}, L^p(\mathbb{R})$ . The analytic nest  $\mathcal{N}_a^p$  is defined to be the chain of subspaces

$$e^{i\lambda x} H^p(\mathbb{R}), \quad \lambda \in \mathbb{R},$$

together with the trivial subspaces. These nests determine the Volterra nest algebra  $\mathcal{A}_v^p = \text{Alg}\mathcal{N}_v^p$  and the analytic nest algebra  $\mathcal{A}_a^p = \text{Alg}\mathcal{N}_a^p$ , both of which are reflexive operator algebras.

The Fourier binest is the subspace lattice

$$\mathcal{L}_{FB}^p = \mathcal{N}_v^p \cup \mathcal{N}_a^p$$

and the Fourier binest algebra  $\mathcal{A}_{FB}^p$  is the non-selfadjoint algebra  $\text{Alg}\mathcal{L}_{FB}^p$  of operators which leave invariant each subspace of  $\mathcal{L}_{FB}^p$ . It is elementary to check that  $\mathcal{A}_{FB}^p$  is a reflexive algebra, being the intersection of two reflexive algebras.

Given  $p \in (1, +\infty)$ , let  $J$  be the flip operator given by  $(Jf)(x) = f(-x)$ . Note that  $J$  is the isometric operator that takes the Volterra nest to its counterpart

$$(\mathcal{N}_v^p)^\perp = \{0\} \cup \{L^p(-\infty, t], t \in \mathbb{R}\} \cup \{L^p(\mathbb{R})\}$$

and the analytic nest to

$$(\mathcal{N}_a^p)^\perp = \{0\} \cup \{e^{-i\lambda x} \overline{H^p(\mathbb{R})}, \lambda \in \mathbb{R}\} \cup \{L^p(\mathbb{R})\}.$$

Hence  $J\mathcal{A}_{FB}^p J$  is the binest algebra generated by the lattice  $J\mathcal{L}_{FB}^p = (\mathcal{N}_v^p)^\perp \cup (\mathcal{N}_a^p)^\perp$ . Since the spaces  $e^{i\lambda x} H^p(\mathbb{R})$  and  $L^p[t, \infty)$  are naturally complemented and have trivial subspaces it is straightforward to adjust the Hilbert space arguments [11] to see that  $\mathcal{A}_{FB}^p$  is an antisymmetric operator algebra, meaning that  $\mathcal{A}_{FB}^p \cap J\mathcal{A}_{FB}^p J = \mathbb{C}I$ , and also that the algebra contains no non-zero finite rank operators.

**2.4. The parabolic algebra.** We first recall the definition of the strong operator topology (SOT). Given a net  $(T_i)_{i \in I}$  of bounded operators on a Banach space  $X$ , we say that  $T_i \xrightarrow{\text{SOT}} T$ , where  $T \in B(X)$ , if and only if  $T_i x \rightarrow T x$ , for every  $x \in X$ . In other words, the SOT topology on  $B(X)$  is defined as the topology of pointwise convergence on  $X$ .

The parabolic algebra  $\mathcal{A}_{par}^p$  is defined as the SOT-closed operator algebra on  $L^p(\mathbb{R})$  that is generated by the two isometric semigroups  $\{M_\lambda, \lambda \geq 0\}, \{D_\mu, \mu \geq 0\}$ . Since the generators of  $\mathcal{A}_{par}^p$  leave the subspaces of the binest  $\mathcal{L}_{FB}^p$  invariant, we have  $\mathcal{A}_{par}^p \subseteq \mathcal{A}_{FB}^p$ . Katavolos and Power showed in [11] that, in the case  $p = 2$ , these two algebras are equal.

**2.5. Integral Operators on  $L^p(\mathbb{R})$ .** Let  $p \in (1, \infty)$  and  $q$  be its conjugate exponent. Given  $k \in L^p(\mathbb{R}; L^q(\mathbb{R}))$ , the linear map

$$(Intk f)(x) = \int_{\mathbb{R}} k(x)(y) f(y) dy$$

defines a bounded operator on  $L^p(\mathbb{R})$ . Indeed, given  $f \in L^p(\mathbb{R})$ , applying the Hölder inequality we obtain

$$\int_{\mathbb{R}} \left| \int_{\mathbb{R}} k(x)(y) f(y) dy \right|^p dx \leq \|k\|_{p,q}^p \|f\|_p^p$$

We will refer to such an operator as  $(p, q)$ -integral operator and denote the set of  $(p, q)$ -integral operators by

$$\mathcal{G}^p = \{Intk : k \in L^p(\mathbb{R}; L^q(\mathbb{R}))\}$$

**Remark 2.13.** (1) The above calculation also proves that the norm  $\|\cdot\|_{p,q}$  dominates the operator norm, and so given  $(k_n)_{n \geq 1}, k \in L^p(\mathbb{R}; L^q(\mathbb{R}))$ , such that  $k_n \xrightarrow{\|\cdot\|_{p,q}} k$ , then  $Intk_n \rightarrow Intk$ .

(2) In the special case  $p = 2$ , then  $\mathcal{G}^2 = \mathcal{C}_2$ , where  $\mathcal{C}_2$  is the ideal of the Hilbert-Schmidt operators on  $L^2(\mathbb{R})$ .

**Lemma 2.14.**  $\mathcal{G}^p$  is a right ideal in  $B(L^p(\mathbb{R}))$ .

*Proof.* Let  $T \in B(L^p(\mathbb{R}))$ . Given  $f \in L^p(\mathbb{R})$  and  $k \in L^p(\mathbb{R}; L^q(\mathbb{R}))$ , such that

$$k = \sum_{\kappa=1}^n f_{\kappa} \otimes g_{\kappa}, \text{ we have}$$

$$\begin{aligned} (IntkTf)(x) &= \int_{\mathbb{R}} k(x, y)(Tf)(y) dy = \sum_{\kappa=1}^n f_{\kappa}(x) \int_{\mathbb{R}} g_{\kappa}(y)(Tf)(y) dy = \\ &= \sum_{\kappa=1}^n f_{\kappa}(x) \int_{\mathbb{R}} T^* g_{\kappa}(y) f(y) dy. \end{aligned}$$

where  $T^*$  is the adjoint operator of  $T$ . Therefore  $IntkT = Int\tilde{k}$ , where  $\tilde{k} = \sum_{\kappa=1}^n f_{\kappa} \otimes T^* g_{\kappa}$ .

In the general case, let  $k \in L^p(\mathbb{R}; L^q(\mathbb{R}))$  and  $k_m = \sum_{\kappa=1}^n f_{\kappa}^{(m)} \otimes g_{\kappa}^{(m)}$ , such that  $k_m \xrightarrow{\|\cdot\|_{p,q}} k$ .

Applying the above argument, we have  $Intk_m T = Int\tilde{k}_m$ , where  $\tilde{k}_m = \sum_{\kappa=1}^n f_{\kappa}^{(m)} \otimes T^* g_{\kappa}^{(m)}$ .



Then, by Lemma 2.11, there is a unique operator  $\widetilde{T}^* \in B(L^p(\mathbb{R}; L^q(\mathbb{R})))$ , such that

$$\begin{aligned} \|\widetilde{k}_m - \widetilde{k}_l\|_{p,q} &= \left\| \sum_{\kappa=1}^n f_{\kappa}^{(m)} \otimes T^* g_{\kappa}^{(m)} - \sum_{\kappa=1}^n f_{\kappa}^{(l)} \otimes T^* g_{\kappa}^{(l)} \right\|_{p,q} = \\ &= \left\| \widetilde{T}^* \left( \sum_{\kappa=1}^n f_{\kappa}^{(m)} \otimes g_{\kappa}^{(m)} - \sum_{\kappa=1}^n f_{\kappa}^{(l)} \otimes g_{\kappa}^{(l)} \right) \right\|_{p,q} \leq \\ &\leq \|\widetilde{T}^*\| \left\| \left( \sum_{\kappa=1}^n f_{\kappa}^{(m)} \otimes g_{\kappa}^{(m)} - \sum_{\kappa=1}^n f_{\kappa}^{(l)} \otimes g_{\kappa}^{(l)} \right) \right\|_{p,q} = \\ &= \|\widetilde{T}^*\| \|k_m - k_l\|_{p,q} \end{aligned}$$

It follows that the sequence  $(\widetilde{k}_m)_m$  is a Cauchy sequence, so by the completeness of  $L^p(\mathbb{R}; L^q(\mathbb{R}))$ , it converges to some  $\widetilde{k} \in L^p(\mathbb{R}; L^q(\mathbb{R}))$ . Since the  $\|\cdot\|_{p,q}$  norm dominates the operator norm, the sequence  $(Int\widetilde{k}_n)_n$  of  $(p,q)$ -integral operators converges to  $Int\widetilde{k}$ . Thus, by the uniqueness of the limit, we obtain  $IntkT = Int\widetilde{k}$ .  $\square$

### 3. REFLEXIVITY

In this section, we prove that the parabolic algebra  $\mathcal{A}_{par}^p$  is reflexive, given  $p \in (1, \infty)$ . In particular, we will show that  $\mathcal{A}_{par}^p = \mathcal{A}_{FB}^p$ . As we noted in the previous section, it suffices to prove that  $\mathcal{A}_{FB}^p \subseteq \mathcal{A}_{par}^p$ .

**Proposition 3.1.** *Let  $Intk \in \mathcal{G}^p \cap \mathcal{A}_{FB}^p$ . Then  $k$  satisfies the following properties*

- (1)  $\Theta(k) \in L^p(\mathbb{R}; L^q(\mathbb{R}_+))$
- (2) For every Borel set  $A$  of finite measure and  $h \in H^q(\mathbb{R})$ , we have

$$\int_{\mathbb{R}} \left( \int_{\mathbb{R}} \Theta(k)(x)(y) h(x) \chi_A(y) dy \right) dx = 0.$$

*Proof.* Let  $Intk \in \mathcal{G}^p \cap \mathcal{A}_{FB}^p$ .

- (1) Since  $IntkL^p[t, \infty) \subseteq L^p[t, \infty)$ , for every  $t \in \mathbb{R}$ , it follows that  $k(x)(y) = 0$ , for almost every  $(x, y) \in \mathbb{R}^2$ , such that  $y > x$ . Therefore,  $\Theta(k)(x) \in L^q(\mathbb{R}_+)$  for almost every  $x \in \mathbb{R}$ .
- (2) Since  $IntkD_{\mu}M_{\lambda}H^p(\mathbb{R}) \subseteq M_{\lambda}H^p(\mathbb{R})$ , for every  $\lambda, \mu \in \mathbb{R}$ , given functions  $f \in H^p(\mathbb{R})$ ,  $g \in H^q(\mathbb{R})$ , we have

$$(IntkD_{\mu}M_{\lambda}f)(M_{-\lambda}g) \in H^1(\mathbb{R})$$

so

$$\int_{\mathbb{R}} (IntkD_{\mu}M_{\lambda}f)(x)(M_{-\lambda}g)(x) dx = 0.$$

Therefore, for every  $q \in L^1(\mathbb{R})$ , we obtain

$$\int_{\mathbb{R}} \left( \int_{\mathbb{R}} (IntkD_{\mu}M_{\lambda}f)(x)(M_{-\lambda}g)(x) dx \right) q(\mu) d\mu = 0.$$

Take  $q(\mu) = \chi_A(-\mu)$ , where  $A$  is a Borel set of finite measure. Then, by Fubini's theorem

$$\begin{aligned} & \int_{\mathbb{R}} \left( \int_{\mathbb{R}} \left( \int_{\mathbb{R}} k(x)(y) e^{i\lambda(y-\mu)} f(y-\mu) e^{-i\lambda x} g(x) dy \right) dx \right) \chi_A(-\mu) d\mu = 0 \Leftrightarrow \\ & \int_{\mathbb{R}} \left( \int_{\mathbb{R}} \left( \int_{\mathbb{R}} e^{i\lambda(y-\mu)} f(y-\mu) \chi_A(-\mu) d\mu \right) k(x)(y) e^{-i\lambda x} g(x) dy \right) dx = 0 \stackrel{\mu \rightarrow \mu+y-x}{\Leftrightarrow} \\ & \int_{\mathbb{R}} \left( \int_{\mathbb{R}} \left( \int_{\mathbb{R}} e^{i\lambda(x-\mu)} f(x-\mu) \chi_A(x-y-\mu) d\mu \right) k(x)(y) e^{-i\lambda x} g(x) dy \right) dx = 0 \Leftrightarrow \\ & \int_{\mathbb{R}} \left( \int_{\mathbb{R}} \left( \int_{\mathbb{R}} k(x)(y) D_{\mu} f(x) g(x) D_{\mu} \chi_A(x-y) dy \right) dx \right) e^{-i\lambda \mu} d\mu = 0 \end{aligned}$$

We claim that the function

$$\Phi : \mathbb{R} \rightarrow \mathbb{C} : \mu \mapsto \int_{\mathbb{R}} \left( \int_{\mathbb{R}} k(x)(y) D_{\mu} f(x) g(x) D_{\mu} \chi_A(x-y) dy \right) dx$$

is a well defined  $L^1$  function. By Tonelli's theorem, it suffices to show that

$$\int_{\mathbb{R}} \left( \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |k(x)(y) D_{\mu} f(x) g(x) D_{\mu} \chi_A(x-y)| d\mu \right) dy \right) dx < \infty$$

. We have

$$\begin{aligned} & \int_{\mathbb{R}} \left( \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |k(x)(y) D_{\mu} f(x) g(x) D_{\mu} \chi_A(x-y)| d\mu \right) dy \right) dx = \\ & = \int_{\mathbb{R}} \left( \int_{\mathbb{R}} \left( |k(x)(y)| \int_{\mathbb{R}} |f(x-\mu) \chi_A(x-y-\mu)| d\mu \right) dy \right) |g(x)| dx = \\ & = \int_{\mathbb{R}} \left( \int_{\mathbb{R}} \left( |k(x)(y)| \int_{\mathbb{R}} |f(\mu) \chi_A(\mu-y)| d\mu \right) dy \right) |g(x)| dx = \\ & = \int_{\mathbb{R}} \left( \int_{\mathbb{R}} \left( |k(x)(y)| \int_{\mathbb{R}} |f(\mu) J \chi_A(y-\mu)| d\mu \right) dy \right) |g(x)| dx, \end{aligned}$$

where  $J$  is again the flip operator. By Young's inequality the function  $h := |f| * J \chi_A$  lies in  $L^p(\mathbb{R})$ , so the expression above is equal to

$$\int_{\mathbb{R}} \left( \int_{\mathbb{R}} |k(x)(y) h(y)| dy \right) |g(x)| dx$$

which by Holder's inequality is bounded by  $\|h\|_p \|k\|_{p,q} \|g\|_q$ , so our claim is proven. Since the Fourier transform of the function  $\Phi$  is the zero function, it follows that for almost every  $\mu \in \mathbb{R}$

$$\int_{\mathbb{R}} \left( \int_{\mathbb{R}} k(x)(y) D_{\mu} f(x) g(x) D_{\mu} \chi_A(x-y) dy \right) dx = 0$$

Hence by Lemma 2.2

$$\int_{\mathbb{R}} \left( \int_{\mathbb{R}} k(x)(y) h(x) D_{\mu} \chi_A(x-y) dy \right) dx = 0$$

for every  $h \in H^q(\mathbb{R})$ . Moreover, since the Borel set  $A$  was freely chosen and  $D_\mu \chi_A = \chi_{A+\mu}$ , where  $A + \mu = \{x + \mu : x \in A\}$ , we have the equivalence

$$\int_{\mathbb{R}} \left( \int_{\mathbb{R}} k(x)(y)h(x)\chi_A(x-y)dy \right) dx = 0 \stackrel{y \rightarrow x-y}{\Leftrightarrow} \int_{\mathbb{R}} \left( \int_{\mathbb{R}} \Theta(k)(x)(y)h(x)\chi_A(y)dy \right) dx = 0$$

□

Our next goal is to determine a dense set of  $\mathfrak{G}^p \cap \mathcal{A}_{FB}^p$ . We start with an approximation lemma.

**Lemma 3.2.** *Let  $\phi \in L^1(\mathbb{R})$ . Then, given  $p \in [1, \infty)$ , the convolution operator*

$$\Delta_\phi : L^p(\mathbb{R}) \rightarrow L^p(\mathbb{R}) : f \mapsto \phi * f,$$

*is bounded. Furthermore, if  $\phi$  has essential support in  $\mathbb{R}_+$ , then  $\Delta_\phi$  belongs to the SOT-closed algebra generated by  $\{D_t \mid t \in \mathbb{R}_+\}$ .*

*Proof.* The continuity of  $\Delta_\phi$  is immediate by Young's inequality, which also gives  $\|\Delta_\phi\| \leq \|\phi\|_1$ . The argument of the second claim is similar to that for  $p = 2$  [14]. Suppose first that  $\phi$  has compact support  $[a, b]$ , for some  $b > a \geq 0$ . Given  $n \in \mathbb{N}$  and  $m \in 0, 1, \dots, n-1$ , define  $\alpha_{m,n} = \int_{\tau(m,n)}^{\tau(m+1,n)} \phi(s)ds$ , where  $\tau(m,n) = a + \frac{m}{n}(b-a)$ . We claim that the sequence  $(T_n)_n$  given by

$$T_n = \sum_{m=0}^{n-1} \alpha_{m,n} D_{\tau(m,n)}$$

converges in the SOT-topology to  $\Delta_\phi$ . Consider  $f \in L^p$ . Then by Hahn - Banach theorem

$$\begin{aligned} \|(D_\phi - T_n)f\|_p &= \sup \left\{ \left| \int_{\mathbb{R}} (D_\phi - T_n)f(x)g(x)dx \right| : \|g\|_q = 1 \right\} = \\ &= \sup \left\{ \left| \int_{\mathbb{R}} \sum_{m=0}^{n-1} \int_{\tau(m,n)}^{\tau(m+1,n)} \phi(t) ((D_t - D_{\tau(m,n)})f(x)) dtg(x)dx \right| : \|g\|_q = 1 \right\} = \\ &= \sup \left\{ \left| \int_{\mathbb{R}} \int_{\mathbb{R}} \phi(t) ((D_t - D_{\rho_n(t)})f(x)) dtg(x)dx \right| : \|g\|_q = 1 \right\} \leq \\ &\leq \sup \left\{ \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |\phi(t) ((D_t - D_{\rho_n(t)})f(x)) g(x)| dx \right) dt : \|g\|_q = 1 \right\} \end{aligned}$$

where  $\rho_n(t) = a + \frac{b-a}{n} \left\lfloor \frac{(t-a)n}{b-a} \right\rfloor$ ,  $t \in [a, b]$ . Now

$$\int_{\mathbb{R}} |\phi(t) ((D_t - D_{\rho_n(t)})f(x)) g(x)| dx \leq \|(D_t - D_{\rho_n(t)})f\|_p \|g\|_q,$$

so it follows that

$$\|(D_\phi - T_n)f\|_p \leq \int_{\mathbb{R}} |\phi(t)| \|(D_t - D_{\rho_n(t)})f\|_p dt.$$

Since  $\|(D_t - D_{\rho_n(t)})f\|_p \rightarrow 0$  as  $n \rightarrow \infty$  and  $|\phi(t)| \|(D_t - D_{\rho_n(t)})f\|_p \leq 2|\phi(t)|\|f\|_p$ , we get that  $\|(D_\phi - T_n)f\|_p \rightarrow 0$ , by dominated convergence theorem.

The general case, is a simple application of Young's inequality. □

**Remark 3.3.** In the  $L^2(\mathbb{R})$  case, there is a simpler proof, using the unitary Fourier-Plancherel transform  $F$ . Note that

$$\Delta_\phi = F^* M_{\hat{\phi}} F$$

Since  $\phi \in L^1(\mathbb{R}_+)$ , it follows that  $\hat{\phi} \in \overline{H^\infty(\mathbb{R})}$ . Therefore, the multiplication operator  $M_{\hat{\phi}}$  lies in the SOT-closed algebra generated by  $\{M_{-\lambda} : \lambda \in \mathbb{R}_+\}$ . Hence, using the fact that  $D_\lambda = F^* M_{-\lambda} F$ , the proof is complete.

**Lemma 3.4.** *Let  $h \in H^p(\mathbb{R})$ ,  $\phi \in L^q(\mathbb{R}_+)$ , where  $p \in (1, \infty)$  and  $q$  is its conjugate exponent. Define  $k = \Theta^{-1}(h \otimes \phi)$ . Then, the operator  $\text{Int}k = M_h \Delta_\phi$  lies in  $\mathcal{G}^p \cap \mathcal{A}_{par}^p$ .*

*Proof.* First, consider  $h \in H^\infty(\mathbb{R})$ ,  $\phi \in L^1(\mathbb{R}_+)$ . Then

$$\begin{aligned} (\text{Int}k f)(x) &= \int_{\mathbb{R}} \Theta^{-1}(h \otimes \phi)(x)(y) f(y) dy = \\ &= \int_{\mathbb{R}} h(x) \phi(x-y) f(y) dy = (M_h \Delta_\phi f)(x), \end{aligned}$$

so  $\|\text{Int}k\| \leq \|h\|_\infty \|\phi\|_1$ . By the previous lemma  $\Delta_\phi \in \text{SOT} - \text{alg}\{D_t : t \in \mathbb{R}_+\}$ , hence  $\text{Int}k \in \mathcal{A}_{par}^p$ . Take now  $h \in H^p(\mathbb{R})$  and  $\phi \in L^q(\mathbb{R}_+)$ . Then there exist  $h_m \in (H^\infty \cap H^p)(\mathbb{R})$ ,  $\phi_m \in (L^1 \cap L^q)(\mathbb{R}_+)$ , such that  $h_m \xrightarrow{\|\cdot\|_p} h$  and  $\phi_m \xrightarrow{\|\cdot\|_q} \phi$ . Now it is straightforward to show that  $h_m \otimes \phi_m \xrightarrow{\|\cdot\|_{p,q}} h \otimes \phi$ . Since the norm  $\|\cdot\|_{p,q}$  dominates the operator norm and  $\mathcal{A}_{par}^p$  is norm closed,

$$\text{Int}k = \text{Int}(\Theta^{-1}(h \otimes \phi)) \in \mathcal{A}_{par}^p.$$

Moreover, the fact that  $h$  and  $\phi$  lie in  $H^p(\mathbb{R})$  and  $L^q(\mathbb{R}_+)$  respectively implies that  $\text{Int}k \in \mathcal{G}^p$ .  $\square$

**Proposition 3.5.**  $\mathcal{G}^p \cap \mathcal{A}_{FB}^p = \mathcal{G}^p \cap \mathcal{A}_{par}^p$ , for every  $p \in (1, \infty)$ .

*Proof.* Let  $\mathcal{G}^p \cap \mathcal{A}_{FB}^p$  be strictly larger than  $\mathcal{G}^p \cap \mathcal{A}_{par}^p$ . Then by Riesz's lemma [21], there exists an element  $\text{Int}k_0 \in \mathcal{G}^p \cap \mathcal{A}_{FB}^p$ , where  $k_0$  lies in the unit sphere of  $L^p(\mathbb{R}; L^q(\mathbb{R}))$ , such that

$$\inf\{\|k_0 - k\|_{p,q} : \text{Int}k \in \mathcal{G}^p \cap \mathcal{A}_{par}^p\} > \frac{9}{10}.$$

Since  $\mathcal{F}(\mathbb{R}; L^q(\mathbb{R}))$  is dense in  $L^p(\mathbb{R}; L^q(\mathbb{R}))$  and  $\Theta$  is a bijective isometry, there exists  $a \in \Theta^{-1}(\mathcal{F}(\mathbb{R}; L^q(\mathbb{R})))$ , such that

$$\|k_0 - a\|_{p,q} < \frac{1}{10}$$

Also, by the boundedness of the Riesz projection from  $L^p(\mathbb{R})$  to  $H^p(\mathbb{R})$  [8], we can write the element  $a$ , as  $a = b + c + d + e$  where

$$\begin{aligned} b &= \Theta^{-1}\left(\sum_{k=1}^{N_1} h_k^1 \otimes g_k^1\right), \text{ with } h_k^1 \in H^p(\mathbb{R}), g_k^1 \in L^q(\mathbb{R}_+) \\ c &= \Theta^{-1}\left(\sum_{k=1}^{N_2} h_k^2 \otimes g_k^2\right), \text{ with } h_k^2 \in H^p(\mathbb{R}), g_k^2 \in L^q(\mathbb{R}_-) \\ d &= \Theta^{-1}\left(\sum_{k=1}^{N_3} h_k^3 \otimes g_k^3\right), \text{ with } h_k^3 \in \overline{H^p(\mathbb{R})}, g_k^3 \in L^q(\mathbb{R}_+) \\ e &= \Theta^{-1}\left(\sum_{k=1}^{N_1} h_k^4 \otimes g_k^4\right), \text{ with } h_k^4 \in \overline{H^p(\mathbb{R})}, g_k^4 \in L^q(\mathbb{R}_-) \end{aligned}$$

Note that at least one of the elements  $c, d, e$  has norm bigger than  $\frac{1}{4}$ . For otherwise, since  $\|k_0 - b\|_{p,q} > \frac{9}{10}$ , we have

$$\|k_0 - a\|_{p,q} = \|k_0 - b - (c + d + e)\|_{p,q} \geq \|k_0 - b\|_{p,q} - \|c + d + e\|_{p,q} > \frac{3}{20}.$$

Without loss of generality, let  $\|c\|_{p,q} > \frac{1}{4}$ . By the Hahn - Banach theorem and Proposition 2.10, there exists  $\omega \in L^q(\mathbb{R}; L^p(\mathbb{R}))$ , such that  $|\omega(c)| > \frac{1}{4}$  and  $\|\omega\| = 1$ . Hence, by Proposition 2.9 and the definition of the element  $c$ , we may assume that  $\|\omega\| \leq \frac{3}{2}$  and  $\omega$  is given by the formula

$$\begin{aligned} \omega(k) &= \sum_{m=1}^n \int_{\mathbb{R}} \left( \int_{\mathbb{R}} k(x)(y) f_m(x) \chi_{A_m}(x-y) dy \right) dx = \\ &= \sum_{m=1}^n \int_{\mathbb{R}} \left( \int_{\mathbb{R}} \Theta k(x)(y) f_m(x) \chi_{A_m}(y) dy \right) dx \end{aligned}$$

where  $f_m \in \overline{H^q(\mathbb{R})}$  and  $\{A_m\}_{m=1,\dots,n}$  is a family of Borel subsets of  $\mathbb{R}_-$ . So it follows from Proposition 3.1 that

$$\frac{3}{2} \|k_0 - a\|_{p,q} \geq |\omega(k_0 - a)| = |\omega(k_0) - \omega(b) - \omega(c) - \omega(d) - \omega(e)| = |\omega(c)| > \frac{1}{4},$$

which is a contradiction.  $\square$

The following proposition and proof follow the pattern for the case  $p = 2$ , given in [14].

**Proposition 3.6.** *For every  $p \in (1, \infty)$ , the algebra  $\mathcal{A}_{par}^p$  contains a bounded approximate identity of elements in  $\mathcal{G}^p$ .*

*Proof.* Take  $h_n(x) = \frac{ni}{x+ni}$  and  $\phi_n(y) = n\chi_{[0,1/n]}(y)$ . It is trivial to see that  $h_n$  and  $\phi_n$  lie in  $H^r(\mathbb{R})$  and  $L^r(\mathbb{R}_+)$ , respectively, for every  $r \in (1, \infty)$ . Let  $k_n = \Theta^{-1}(h_n \otimes \phi_n)$ . Then  $\|Intk_n\| \leq \|h_n\|_{\infty} \|\phi_n\|_1 \leq 1$ . Since  $h_n \rightarrow 1$  uniformly on compact sets of the real line, it

follows that  $M_{h_n} \xrightarrow{SOT} I$ . Now given  $f \in C_C(\mathbb{R})$ , note that

$$\begin{aligned} \|\Delta_{\phi_n} f - f\|_p^p &= \int_{\mathbb{R}} \left| \int_{\mathbb{R}} n\chi_{[0,1/n]}(y)f(x-y)dy - f(x) \right|^p dx = \\ &= \int_{\mathbb{R}} \left| \int_0^{1/n} n f(x-y)dy - f(x) \right|^p dx \end{aligned}$$

Check that  $\left| \int_0^{1/n} n f(x-y)dy - f(x) \right|^p \leq 2^p \|f\|_{\infty}^p \chi_S(x)$ , where  $S$  is the compact set

$$S = \{x + \tau \mid x \in \text{supp} f, \tau \in [0, 1]\}.$$

Hence by dominated convergence  $\Delta_{\phi_n} f \rightarrow f$ . Since  $C_C(\mathbb{R})$  is dense in  $L^p(\mathbb{R})$  it follows that  $\Delta_{\phi_n} \xrightarrow{SOT} I$ . Multiplication is  $SOT$ -continuous on the closed unit ball of bounded operators, so  $M_{h_n} \Delta_{\phi_n} \xrightarrow{SOT} I$ .  $\square$

**Theorem 3.7.** *For every  $p \in (1, \infty)$ , the parabolic algebra  $\mathcal{A}_{par}^p$  is equal to the Fourier binest algebra  $\mathcal{A}_{FB}^p$ .*

*Proof.* As we have noted before, it suffices to prove that  $\mathcal{A}_{FB}^p \subseteq \mathcal{A}_{par}^p$ . Let  $T \in \mathcal{A}_{FB}^p$  and  $(X_n)_{n \geq 1}$  be the bounded approximate identity of the previous proposition. By Lemma 2.14 and Proposition 3.5, the operators  $X_n T$  lie in  $\mathfrak{G}^p \cap \mathcal{A}_{FB}^p = \mathfrak{G}^p \cap \mathcal{A}_{par}^p$ . Since  $\mathcal{A}_{par}^p$  is  $SOT$ -closed, the given operator  $T = \underset{n}{SOT - \lim} X_n T$  lies in  $\mathcal{A}_{par}^p$ .  $\square$

**Proposition 3.8.** *The Fourier binest algebra  $\mathcal{A}_{FB}^{\infty}$  is strictly larger than the parabolic algebra  $\mathcal{A}_{par}^{\infty}$ .*

*Proof.* Note first that the non-selfadjoint  $\|\cdot\|_{\infty}$ -closed algebra of the trigonometric polynomials  $\{e^{i\lambda x} : \lambda \geq 0\}$  is the algebra  $AAP(\mathbb{R})$  of analytic almost periodic functions [2], which is strictly smaller than  $H^{\infty}(\mathbb{R})$ . Take a function  $\phi$  that lies in  $H^{\infty}(\mathbb{R})$  and it is not an element of  $AAP(\mathbb{R})$ . It suffices to show that  $M_{\phi} \notin \mathcal{A}_{par}^{\infty}$ . If this is not the case, there is some sequence  $p_n(M_{\lambda}, D_{\mu})$  in the non-closed algebra generated by  $\{M_{\lambda}, D_{\mu} : \lambda, \mu \geq 0\}$  which converges strongly to  $M_{\phi}$ . Thus for any  $f \in L^{\infty}(\mathbb{R})$ , we have

$$\left\| p_n(M_{\lambda}, D_{\mu})f - M_{\phi}f \right\|_{\infty} \rightarrow 0, \text{ as } n \rightarrow \infty.$$

Choosing  $f \equiv 1$ , it follows that

$$\left\| p_n(M_{\lambda}, I)f - M_{\phi}f \right\|_{\infty} \rightarrow 0, \text{ as } n \rightarrow \infty.$$

and so  $\phi \in AAP(\mathbb{R})$ , a contradiction.  $\square$

**Remark 3.9.** It remains unclear to the author whether the parabolic operator algebras  $\mathcal{A}_{par}^1$  and  $\mathcal{A}_{par}^{\infty}$  acting on the respective Banach spaces  $L^1(\mathbb{R})$  and  $L^{\infty}(\mathbb{R})$  are reflexive operator algebras.

## 4. THE LATTICE OF THE PARABOLIC ALGEBRA

Let  $K_{\lambda,s}^p = M_\lambda M_{\phi_s} H^p(\mathbb{R})$  where  $\phi_s(x) = e^{-isx^2/2}$ . This is evidently an invariant subspace for the multiplication semigroup and for  $s \geq 0$  one can check that it is invariant for the translation semigroup. Thus for  $s \geq 0$  the nest  $\mathcal{N}_s = M_{\phi_s} \mathcal{N}_a$  is contained in  $Lat\mathcal{A}_{par}^p$  and these nests are distinct. In fact any two nontrivial subspaces from nests with distinct  $s$  parameter have trivial intersection.

Suppose now that  $p = 2$ . With the strong operator topology for the associated orthogonal subspace projections it can be shown ([11]) that the set of these nests for  $s \geq 0$ , together with the Volterra nest  $\mathcal{N}_v^2$ , is homeomorphic to the closed unit disc. A cocycle argument given in [11] leads to the fact that every invariant subspace for  $\mathcal{A}_{par}^2$  is of this form for  $p = 2$ . That is

$$(4.1) \quad Lat\mathcal{A}_{par}^2 = \{K_{\lambda,s}^2 | \lambda \in \mathbb{R}, s \geq 0\} \cup \mathcal{N}_v$$

We prove now the corresponding result for the general case of  $\mathcal{A}_{par}^p$ , where  $1 < p < \infty$ .

Let  $K$  be a non-trivial element of  $Lat\mathcal{A}_{par}^p$ . By the Beurling theorem, either  $K = L^p(E)$  for some Borel set  $E \subseteq \mathbb{R}$  or  $K = M_\phi H^p(\mathbb{R})$  for some unimodular function  $\phi$ . On the other hand, the subspace  $K \cap L^2(\mathbb{R})$  is invariant under the generators of the parabolic algebra. Therefore, the  $\|\cdot\|_2$ -closure of  $K \cap L^2(\mathbb{R})$  lies in  $Lat\mathcal{A}_{par}^2$ . Hence, according to the Beurling-Lax theorem, we have two cases. In the first case, where  $K = L^p(E)$ , then

$$\overline{L^p(E) \cap L^2(\mathbb{R})}^{\|\cdot\|_2} \in Lat\mathcal{A}_{par}^2 \Rightarrow L^2(E) \in Lat\mathcal{A}_{par}^2 \Rightarrow E = [t, \infty),$$

for some  $t \in \mathbb{R}$ . In the second case,  $K = M_\phi H^p(\mathbb{R})$ , which implies

$$\overline{M_\phi H^p(\mathbb{R}) \cap L^2(\mathbb{R})}^{\|\cdot\|_2} \in Lat\mathcal{A}_{par}^2 \Rightarrow M_\phi H^2(\mathbb{R}) \in Lat\mathcal{A}_{par}^2 \Rightarrow M_\phi = M_{\phi_s} M_\lambda,$$

for some  $s \in [0, +\infty)$ ,  $\lambda \in \mathbb{R}$ . Hence, we have the following result.

**Theorem 4.1.** *Given  $p \in (1, \infty)$ , the invariant subspace lattice of the algebra  $\mathcal{A}_{par}^p$  is*

$$Lat\mathcal{A}_{par}^p = \{K_{\lambda,s}^p | \lambda \in \mathbb{R}, s \geq 0\} \cup \mathcal{N}_v^p$$

Recall that the reflexive closure of a set of closed subspaces  $\mathcal{L}$  is the subspace lattice  $LatAlg\mathcal{L}$ . Thus the theorem identifies the reflexive closure of the binest  $\mathcal{L}_{FB}^p$ .

**Remark 4.2.** In [11], Katavolos and Power proved that  $Lat\mathcal{A}_{par}^2$ , viewed as a topological space of projections on  $L^2(\mathbb{R})$ , endowed with the strong operator topology, is homeomorphic to the closed unit disc. In particular, they obtained the so-called strange limit

$$P_{K_{\lambda,s}^2} \xrightarrow{SOT} P_{L^2[\lambda, +\infty)}, \text{ as } s \rightarrow \infty,$$

which relies on the Paley - Wiener theorem and the fact that the Fourier transform is unitary on  $L^2(\mathbb{R})$ . Even though the Riesz projection from  $L^p(\mathbb{R})$  onto  $H^p(\mathbb{R})$  remains bounded, it is unknown to the author if the above convergence still holds, for  $p \in (1, +\infty) \setminus \{2\}$ .

We expect that the operator algebras  $\mathcal{A}_{par}^p$ , for  $1 < p < \infty$ , are pairwise non isomorphic, even as rings of linear operators. However, the standard methods for such a demonstration (which go back to Eidelheit [5]) rely on exploiting the presence of rank one operators to

deduce an isomorphism between the underlying Banach spaces. Possibly the  $(p, q)$ -integral operators could once again play a substitute role in this demonstration.

#### REFERENCES

- [1] M. Anoussis, A. Katavolos and I.G. Todorov, Operator algebras from the Heisenberg semigroup, Proc. Edin. Math. Soc., 55 (2012), 1–22.
- [2] A. S. Besicovich, Almost periodic functions, Dover Publications Inc., New York, (1954).
- [3] N. Boudi and P. Šemrl, Semigroup automorphisms of nest algebras, Houston J. of Mathematics, Vol.38, No. 4 (2012), 1197–1206.
- [4] K. R. Davidson, Nest Algebras, Pitman Research Notes in Math., 191, Longman, 1988.
- [5] M. Eidelheit, On isomorphisms of rings of linear operators, Studia Math. 9 (1940), 97–105.
- [6] J. A. Erdos and S. C. Power, Weakly closed ideals of nest algebras, J. Operator Th., 7 (1982), 219–235.
- [7] H. Helson, Lectures on invariant subspaces, Academic Press, New York, (1964).
- [8] K. Hoffman, Banach spaces of analytic functions, Prentice-Hall, Englewood Cliffs, N. J., (1962).
- [9] R. V. Kadison and I. M. Singer, Triangular operator algebras, Amer. J. Math., 82 (1960), 227–259.
- [10] E. Kastis and S. C. Power, The operator algebra generated by the translation, dilation and multiplication semigroups. Journal of Functional Analysis, 269 (2015), p. 3316–3335.
- [11] A. Katavolos and S. C. Power, The Fourier binest algebra, Math. Proc. Cambridge Philos. Soc., 122 (1997), 525–539.
- [12] A. Katavolos and S. C. Power, Translation and dilation invariant subspaces of  $L^2(\mathbb{R})$ , J. Reine Angew. Math., 552 (2002), 101–129.
- [13] P. Koosis Introduction to  $H_p$  spaces (Second edition) Cambridge Tract in Math., 115, Cambridge, 1998.
- [14] R. H. Levene, Lie semigroup operator algebras, Ph.D. Thesis, Lancaster University, 2004.
- [15] R. H. Levene and S. C. Power, Reflexivity of the translation-dilation algebras on  $L^2(\mathbb{R})$ , International J. Math., 14 (2003), 1081–1090.
- [16] B. J. Pettis, On integration in vector spaces, Trans. Amer. Math. Soc. 44 (1938), 277–304.
- [17] G. Pisier, Martingales in Banach Spaces (in connection with Type and Cotype), Course IHP, 2011.
- [18] H. Radjavi and P. Rosenthal, Invariant subspaces, Dover, 2003.
- [19] D. A. Redett, S-invariant subspaces of  $L^p(\mathbb{T})$ , Proc. Amer. Math. Soc., Vol. 133, 5 (2004), 1459–1461.
- [20] J. R. Ringrose, On some algebras of operators, Proc. London Math. Soc., 15 (1965), 61–83.
- [21] B. P. Rynne and M. A. Youngson, Linear Functional Analysis (2nd ed.), Springer, London, (2008).
- [22] D. Sarason, Invariant subspaces and unstarred operator algebras, Pacific J. Math., 17 (1966), 511–517.
- [23] M. Spivack, Derivations and nest algebras on Banach Space, Israel J. Math., Vol.50. No. 3, (1985), 193–200.
- [24] M. E. Taylor, Noncommutative Harmonic Analysis, Math. Surveys and Monographs, vol 22, Amer. Math. Soc., 1986.
- [25] D. J. Westwood, Nests in Banach space, J. Math. Anal. and Applications, 156 (1991), 558–567.

DEPT. MATH. STATS., LANCASTER UNIVERSITY, LANCASTER LA1 4YF, U.K.  
*E-mail address:* l.kastis@lancaster.ac.uk