Microfoundations for stochastic frontiers

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Highlights

- We provide microfoundations for stochastic frontier analysis.
- We revise previous work showing that a simple Bayesian learning model supports gamma distributions.
- The conclusion depends on problem formulation and assumptions about the sampling process and the prior.
- After a new formulation of the problem the distribution of one-sided error component does not belong to known family.
- More doubt is cast using expected utility of profit maximization.

Microfoundations for stochastic frontiers

Mike G. Tsionas*

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Abstract

The purpose of the paper is to propose microfoundations for stochastic frontier models. Previous work shows that a simple Bayesian learning model supports *gamma* distributions for technical inefficiency in stochastic frontier models. The conclusion depends on how the problem is formulated and what assumptions are made about the sampling process and the prior. After the new formulation of the problem it turns out that the distribution of the one-sided error component does not belong to a known family. Moreover, we find that without specifying a utility function or even the cost inefficiency function, the relative effectiveness of managerial input can be determined using only cost data and estimates of the returns to scale. The point of this construction is that features of the inefficiency function $u(z)$ *can be recovered from the data*, based on the solid microfoundation of expected utility of profit maximization but the model does not make a prediction about the distribution.

Keywords: Economics; Stochastic frontier analysis; microfoundations; Bayesian learning; Learning-by-Doing.

JEL Codes: C13, D83.

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1 Introduction

Technical efficiency derived from stochastic frontier or envelopment methods has a long history in operations research and econometrics. See, for example Sun, Kumbhakar and Tveterås (2015), Keshvari and Kuosmanen (2013), Olesen and Petersen (2016), Lee (2006, 2010), Ondrich and Ruggiero (2001) etc. The application of stochastic frontier models in operations research has been growing steadily over the years and it has reached a stage of maturity. Among the many applications we can single out Berger and Humphrey (1997) on the efficiency of financial institutions, Lovell (1995) for a general overview on policy analysis perspectives, Bos, Koetter, Kolari and Kool (2009) on bank heterogeneity and efficiency scores, Lozano (1997) and Dong et al (2016) on banking profit efficiency, Resti (2000) on a comparison of parametric and DEA techniques using simulated data. Moreover, see Kumbhakar (2011) on estimating production technologies under the assumption of maximizing returns to outlay, Assaf and Gillen (2012) on airport efficiency under different forms of governance, Drake and Simper (2003) on law enforcement efficiency in England and Wales, Behr (2010) on robust quantile-based measurement of efficiency.

Other areas of interest include but are not limited to incorporation of MIMIC models in stochastic frontier models (Chaudhuri, Kumbhakar and Sundaram, 2016), Sudit (1995) on productivity measurement in industrial operations, environmental efficiency (Reinhard et al, 2000), Bolt and Humphrey (2015) on measuring banking competition, Annaert et al (2003) on evaluating mutual funds etc. Badunenko and Kumbhakar (2016) examine the circumstances under which one should measure persistent and transient or "short-run" inefficiency which is a quite flexible state-of-the art model proposed by Tsionas and Kumbhakar (2014). Some innovative applications of SFM are in general dental practices in the U.S (Chen and Ray, 2013), evaluation of technical efficiency and managerial correlates of solid waste management by Welsh SMEs (Cordeiro et al, 2012) etc.

The purpose of this paper is to examine the microfoundations upon which such analyses rest and can be meaningful. As there *ex ante* and *ex post* views of production, it becomes clear that statistical uncertainty at the firm level plays a critical role in the formulation of a model for and estimation of technical efficiency. For example, Oikawa (2016) shows that a simple Bayesian learning model supports *gamma* distributions for technical inefficiency. We find that his conclusion is not robust to ex ante views of production and is not even the unique or "correct" view of formulating models of technical inefficiency.

We derive some new results and we examine key aspects of the model paying particular attention to the *ex ante* nature of production and the reasonable assumptions that can be made to derive implications for technical inefficiency in stochastic frontier models.

2 Some new results

Oikawa (2016) shows that a simple Bayesian learning model supports *gamma* distributions for technical inefficiency. This conclusion depends very much on the prior of the critical variable θ. The essence of his analysis is that prior to observing output in a stochastic frontier model[1]

$$y = \alpha_0 \prod_{k=1}^{K} x_t^{\alpha_k} e^v e^{-(\theta - z)^2}, \tag{1}$$

where $v \sim \mathcal{N}(0, \sigma_v^2)$, θ is a "target", z is a managerial variable, and $u = (\theta - z)^2$ is technical inefficiency. Building on Jovanovic and Nyarko (1996), Oikawa (2016) assumes that each firm draws θ at random from $\mathcal{N}(\bar{\theta}, \sigma_\theta^2)$ where $\bar{\theta}, \sigma_\theta^2$ are "known and common to all firms. Hence, this is the common prior distribution for each θ." To reduce technical inefficiency as much as possible, the manager infers the true θ by using information on the past realization of outputs and sets z to minimize the expected loss. So the problem of the manager is to choose z so that $E\left(e^{-u}\right) = E\left(e^{-(\theta-z)^2}\right)$ is maximized by choice of z. The argument in Oikawa (2016) is quite simple and rests on the well known result that if θ is normal then $\frac{\theta - E(\theta)}{\sqrt{Var(\theta)}} \sim \mathcal{N}(0,1)$ and therefore the square follows a $\chi^2(1)$, where the first and second moments are appropriately defined in the posterior sense.

The following generalizes Lemma 1 in Oikawa (2016).

LEMMA 1. Suppose $f(\theta)$ is a prior for θ. Then the optimal value of z satisfies: $\tilde{z} = \frac{\int \theta e^{-(\tilde{z}-\theta)^2} f(\theta) d\theta}{\int e^{-(\tilde{z}-\theta)^2} f(\theta) d\theta}$, if the integrals exist.

PROOF. To maximize $V(z) \equiv E\left(e^{-u}\right) = E\left(e^{-(\theta-z)^2}\right)$ we have

$$V(z) = \int e^{-(\theta-z)^2} f(\theta) d\theta. \tag{2}$$

The first order condition of the problem is $\int (\theta - \tilde{z}) e^{-(\theta-\tilde{z})^2} f(\theta) d\theta = 0$. The result follows immediately by solving with respect to $\tilde{z}$. $\square$

In Oikawa (2016) the fundamental result is that if $\theta \sim \mathcal{N}(\mu, \sigma^2)$ then $\tilde{z} = \mu$. In general, the result

[1] Clearly, the Cobb-Douglas specification is not essential to the problem as Oikawa (2016) sets it out.

cannot hold for non-elliptical distributions. Moreover, it does not appear correct that the support of θ should be $\Re$ instead of the positive half-line or even a finite interval. Even if we abstract from such considerations, we formulate what we believe is a correct version of the problem in section 3.

3 Some more results

Continuing on some results of interest, using simple change of variables it is easy to show that if $\mathcal{Z} \sim \mathcal{N}(0,1)$ then the expression $\int e^{-(\tilde{z}-\theta)^2} f(\theta) d\theta = \sqrt{\pi} E f\left(z + \mathcal{Z}/\sqrt{2}\right)$. The relation between θ and $\mathcal{Z}$ is the following:

$$\theta = \tilde{z} + \frac{\mathcal{Z}}{\sqrt{2}}.$$

Therefore, we can simplify the result in Lemma 1 as follows:

LEMMA 2. The optimal value of z satisfies: $Ef'\left(\tilde{z} + \mathcal{Z}/\sqrt{2}\right) = 0$ provided $Ef''(\tilde{z} + \mathcal{Z}/\sqrt{2}) < 0$.

PROOF. The proof is immediate. □

In fact the optimal solution, if it exists, satisfies:

$$\tilde{z} = \frac{E\left\{\left(\tilde{z} + \mathcal{Z}/\sqrt{2}\right) f\left(\tilde{z} + \mathcal{Z}/\sqrt{2}\right)\right\}}{Ef\left(\tilde{z} + \mathcal{Z}/\sqrt{2}\right)} \tag{3}$$

We now consider the crux of the matter in Oikawa (2016) who considers exclusively the case of a normal prior for θ: $\theta \sim N(\mu, \sigma^2)$. Under this assumption he shows that $\tilde{z} = \mu$ but we are not informed how μ and σ are related to $\bar{\theta}$ and σ_θ; presumably they are the same.[2] Prior to observing output y the firm observes a sequence $\{\theta_1, \ldots, \theta_T\}$ of random variables. It is not entirely clear how the manager observes the target θ without actual production operations but we bypass this point.

Suppose now the prior $f(\theta)$ is not necessarily normal. We have the following:

LEMMA 3. Suppose $f(\theta)$ is flat over $\Re$ or $\Re_+$ or $f(\theta) \propto \theta^{-1}$, $\theta \in \Re_+$. In the first case $V(z)$ does not depend on z. In the second case, $V(z) = 2\sqrt{\pi}\Phi(\sqrt{2}z)$ where $\Phi(z)$ is the standard normal distribution function. In the third case, $V(z) = E\left(\frac{\mathbb{Z}}{\sqrt{2}} + z\right)^{-1}$ where $\mathbb{Z} \sim N(0,1)$ truncated below at $-\sqrt{2}z$. Additionally, suppose θ is uniformly distributed over a finite interval $[a, b]$. Then $V(z) = \frac{\sqrt{\pi}}{b-a}\left\{\Phi\left(\sqrt{2}(b-z)\right) - \Phi\left(\sqrt{2}(a-z)\right)\right\}$.

[2] In p. 16 of Oikawa (2016) $d\eta$ should be $d\theta$.

PROOF. (i) Since $V(z) = \int e^{-(\theta-z)^2} f(\theta) d\theta$, if $f(\theta) \propto$ const. it is clear that $V(z) = 2\sqrt{\pi}$. (ii) If $f(\theta)$ is flat over $\Re_+$then $V(z) = \int_0^\infty e^{-(\theta-z)^2} f(\theta) d\theta$. Performing the integration we obtain $V(z) = 2\sqrt{\pi}\Phi(\sqrt{2}z)$. (iii) In the third case, which uses the "standard" Jeffreys' prior for a positive parameter, $V(z) = \int_0^\infty \theta^{-1} e^{-(\theta-z)^2} f(\theta) d\theta$. The integral is $V(z) = \sqrt{2\pi}\Phi\left(\sqrt{2}z\right) \int_{-\sqrt{2}z}^\infty \left(\frac{t}{\sqrt{2}} + z\right)^{-1} \frac{1}{\sqrt{2\pi}\Phi\left(\sqrt{2}z\right)} e^{-t^2/2} dt$. This expression is the expectation $E\left(\frac{\mathbb{Z}}{\sqrt{2}} + z\right)^{-1}$ when $\mathbb{Z} \sim \mathcal{N}(0,1)$ truncated below at $-\sqrt{2}z$. (iv) When $f(\theta) = \frac{1}{b-a}$, $\theta \in [a,b]$ we have $V(z) = \int_a^b e^{-(\theta-z)^2} \frac{1}{b-a} d\theta$ which yields

$$V(z) = \frac{1}{b-a} \int_a^b e^{-\frac{1}{2}\left[\sqrt{2}(\theta-z)\right]^2} d\theta.$$

Let $t = \sqrt{2}\,(\theta - z)$, then $\theta = z + \frac{t}{\sqrt{2}}$. Since $\theta \in [a,b]$, then $t \in \left[\sqrt{2}(a-z), \sqrt{2}(b-z)\right]$. Therefore,

$$V(z) = \frac{1}{b-a} \int_{\sqrt{2}(a-z)}^{\sqrt{2}(b-z)} e^{-\frac{1}{2}t^2} d\left(z + \frac{t}{\sqrt{2}}\right) = \frac{1}{b-a} \int_{\sqrt{2}(a-z)}^{\sqrt{2}(b-z)} e^{-\frac{1}{2}t^2} \frac{1}{\sqrt{2}} dt = \frac{\sqrt{\pi}}{b-a} \int_{\sqrt{2}(a-z)}^{\sqrt{2}(b-z)} \varphi(t) dt,$$

where $\varphi(t)$ is the standard normal probability density function. Therefore,

$$V(z) = \frac{\sqrt{\pi}}{b-a} \left\{ \Phi\left(\sqrt{2}(b-z)\right) - \Phi\left(\sqrt{2}(a-z)\right) \right\} \qquad \square.$$

The function $V(z)$ for case (iii) is presented in Figure 1 using a Monte Carlo approach with 10^7 standard normal draws. For certain choices of a and b the function $V(z)$ is presented in Figure 2. In the following lemma we consider the more realistic case of an exponential prior with parameter $\lambda > 0$, viz. $f(\theta) = \lambda e^{-\lambda\theta}$, $\theta \geq 0$.

LEMMA 4. Suppose θ follows an an exponential prior with parameter $\lambda > 0$. Then

$$V(z) = \lambda e^{(z-\lambda/2)^2 - z^2} \sqrt{2\pi}\Phi\left(\sqrt{2}\,(z - \lambda/2)\right)$$

and the optimal value of z satisfies: $\lambda = \frac{\varphi\left(\sqrt{2}(\tilde{z}-\lambda/2)\right)}{\Phi\left(\sqrt{2}(\tilde{z}-\lambda/2)\right)}$, where $\varphi()$ is the standard normal density function.

PROOF. We have $V(z) = \int_0^\infty e^{-(\theta-z)^2} f(\theta) d\theta = V(z) = \lambda \int_0^\infty e^{-(\theta-z)^2 - \lambda\theta} d\theta$. Performing the integration using change of variables and completion of the square we get: $V(z) = e^{M^2 - z^2} \lambda\sqrt{\pi} \int_{-\sqrt{2}M}^\infty \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt$, where $M = z - \frac{\lambda}{2}$. Therefore, $V(z) = \lambda e^{\lambda^2/4 - \lambda z} \sqrt{\pi}\Phi\left(\sqrt{2}\,(z - \lambda/2)\right)$. Taking logs the first order condition yields easily the stated result for $\tilde{z}$. The proof of the result is as follows. Since $V(z) = \lambda \int_0^\infty e^{-(\theta-z)^2-\lambda\theta} d\theta = e^{M^2-z^2} \lambda \int_0^\infty e^{-(\theta-M)^2} d\theta = e^{M^2-z^2} \lambda \int_0^\infty e^{-\frac{1}{2}\left[\sqrt{2}(\theta-M)^2\right]} d\theta$. Let $t = \sqrt{2}(\theta - M)$, then $\theta = M + \frac{t}{\sqrt{2}}$. Since $\theta \geq 0$, then $t \geq -\sqrt{2}M$. Therefore, we obtain:

$$V(z) = e^{M^2-z^2} \lambda \int_{-\sqrt{2}M}^{\infty} e^{-\frac{1}{2}t^2} d\left(M + \frac{t}{\sqrt{2}}\right) = e^{M^2-z^2} \lambda \int_{-\sqrt{2}M}^{\infty} e^{-\frac{1}{2}t^2} dt = e^{M^2-z^2} \lambda \sqrt{\pi} \int_{-\sqrt{2}M}^{\infty} \varphi(t) dt,$$

where $\varphi(t)$ is the standard normal probability density function. In turn, this yields $V(z) = \lambda e^{\lambda^2/4-\lambda z} \sqrt{\pi} \Phi\left(\sqrt{2}\left(z - \lambda/2\right)\right)$.
□

In this case, the optimal z is determined through $\Lambda(z) = \frac{\varphi(z)}{\Phi(z}$, a form of Mills ratio.

4 Formulation of the problem

In this section we provide what we believe is a correct formulation of the problem. We are given a random sample $\mathscr{D} = \{\theta_1, \ldots, \theta_T\}$ from a distribution with density $p(\theta_t; \theta)$ prior to observing output. A prior $f(\theta)$ is placed on the parameter θ. Then we have to determine the distribution of $u = (\theta - z)^2$ conditional on z, which is determined by maximizing the expected value Ee^{-u}. We distinguish two cases, viz. when the random sample arises from an an exponential distribution or from a normal distribution. In this context, it is clear that the optimal value of z has to be determined from the posterior distribution $\theta|\mathscr{D}$ and cannot be set in advance.

CASE I. We have a random sample $\mathscr{D} = \{\theta_1, \ldots, \theta_T\}$ from an exponential distribution with parameter θ and the prior is gamma with parameters α and β ,viz. $f(\theta) \propto \theta^{\alpha-1} e^{-\beta\theta}$. The posterior distribution of the parameter is

$$f(\theta|\mathscr{D}) \propto \theta^{T+\alpha-1} e^{-\left(\beta+\sum_{t=1}^T \theta_t\right)\theta} \; \theta \geq 0, \tag{4}$$

which is clearly a gamma distribution with shape parameter $T + \alpha$ and scale $\beta + \sum_{t=1}^T \theta_t = \beta + T\theta^a$:

$$\theta|\mathscr{D} \sim \mathrm{Ga}\left(T + \alpha,\ \beta + T\theta^a\right), \tag{5}$$

where $\theta^a = T^{-1} \sum_{t=1}^{T} \theta_t$ is the average. The distribution of $u = (\theta - z)^2$ is given by change of variables:

$$p(u|z, \mathscr{D}) \propto \left(z + \sqrt{u}\right)^{T+\alpha-1} \tfrac{1}{\sqrt{u}} e^{-(T\theta^a+\beta)(z+\sqrt{u})}. \tag{6}$$

This kernel density does not correspond to a known distribution but it also depends on the control variable, z. A change of variables shows that the distribution of $\sqrt{u}$ (*not* u itself) is "close" to a gamma distribution, as it is proportional to $(z+q)^{T+\alpha-1} q^{-3/2} e^{-(T\theta^a+\beta)(z+q)}$, $q = \sqrt{u}$. However, the distribution depends on z so it has little to do with the arguments of Oikawa (2016).

To determine z we need to maximize $V(z) = Ee^{-u}$. The distribution of $h = e^{-u}$ has density:

$$p(h|z, \mathscr{D}) \propto \left(z + \sqrt{-\ln h}\right)^{T+\alpha-1} \frac{1}{h\sqrt{-\ln h}} e^{-(T\theta^a+\beta)\left(z+\sqrt{-\ln h}\right)},\ 1 \geq h > 0. \tag{7}$$

Although we cannot determine $\tilde{z}$ analytically by maximizing the expected value of (7), given the solution it is clear that

$$p(u|\mathscr{D}) \propto \left(\tilde{z} + \sqrt{u}\right)^{T+\alpha-1} \tfrac{1}{\sqrt{u}} e^{-(T\theta^a+\beta)(\tilde{z}+\sqrt{u})},\ u \geq 0. \tag{8}$$

From the form of the distribution it does not appear that we have any reason to believe that u follows a *gamma* distribution.

CASE II. Suppose we have a random sample $\mathscr{D} = \{\theta_1, \ldots, \theta_T\}$ from a normal distribution $\mathcal{N}\left(\theta, \sigma_\theta^2\right)$ and the prior is normal with parameters μ and σ^2 . The posterior distribution of the parameter is

$$f(\theta|\mathscr{D}) \propto e^{-\frac{1}{2\sigma_\theta^2}\sum_{t=1}^{T}(\theta_t-\theta)^2 - \frac{1}{2\sigma^2}(\theta-\mu)^2}. \tag{9}$$

After elementary operations we have:

$$\theta|\mathscr{D} \sim \mathcal{N}\left(\hat{\theta},\ \sigma_*^2\right), \tag{10}$$

where $\hat{\theta} = \frac{T\sigma^2\theta^a + \mu\sigma_\theta^2}{T\sigma^2+\sigma_\theta^2}$ and $\sigma_*^2 = \frac{\sigma^2\sigma_\theta^2}{T\sigma^2+\sigma_\theta^2}$. The distribution of $u = (\theta - z)^2$ has density

$$f(u|z, \mathscr{D}) = \tfrac{1}{2}\left(2\pi\sigma_*^2\right)^{-1/2} \frac{1}{\sqrt{u}} e^{-\frac{1}{2\sigma_*^2}\left(z+\sqrt{u}-\hat{\theta}\right)^2},\ u > 0. \tag{11}$$

The distribution of $h = e^{-u}$ has density

$$f(h|z,\mathscr{D}) \propto \frac{1}{\sqrt{-\ln h}} h^{-1} e^{-\frac{1}{2\sigma_*^2}\left(z+\sqrt{-\ln h}-\hat{\theta}\right)^2},\ 1 \geq h > 0. \tag{12}$$

Again, as we cannot determine $\tilde{z}$ analytically by maximizing $E(h)$, we have

$$f(u|\mathscr{D}) = \tfrac{1}{2}\left(2\pi\sigma_*^2\right)^{-1/2} \frac{1}{\sqrt{u}} e^{-\frac{1}{2\sigma_*^2}\left(\tilde{z}+\sqrt{u}-\hat{\theta}\right)^2},\ u > 0. \tag{13}$$

Again, the density does not correspond to a distribution in a known family -although the distribution of $q = \sqrt{u}$ has density proportional to $q^{-3/2} e^{-\frac{1}{2\sigma_*^2}\left(\tilde{z}+q-\hat{\theta}\right)^2}$. This *cannot* be a *gamma* density. In the form (11) or (13) the distribution cannot easily match a gamma distribution: Obviously, the shape parameter would be $\frac{1}{2}$ but it is difficult to match the quadratic term with a linear form in u. To do so, we should have $z \simeq \hat{\theta}$ which can be easily violated. If this is the case, however, the density would be a gamma with shape $\frac{1}{2}$ and scale $\frac{1}{2\sigma_*^2}$. If we expand the square in () and drop terms not related to u, it becomes clear that the term does not admit a Taylor series expansion around $u = 0$. This further complicates the matter of approximating the quadratic exponential term in (13) by a linear term in u around zero. A second difficulty in a linear expansion around another value is that we must have $z > \hat{\theta}$ which, again, it can be easily violated.

This density is presented in Figure 3 for $\tilde{z} = -1, 0, 1$ in its standard form with $\hat{\theta} = 0$ and $\sigma_* = 1$. For $\hat{\theta} = 1$ and $\hat{\theta} = -1$ the density is presented in Figures 4 and 5. However, it turns out that (12) is monotonically increasing. Therefore, we cannot determine an optimal value of $\tilde{z}$ based on the first order condition, $V'(\tilde{z}) = 0$; rather the optimal value of $\tilde{z}$ corresponds to a value such that $V(\tilde{z}) = 1$. This case is clearly empirically uninteresting. Assuming there is a cost of using the managerial resource which is $\rho \in (0, 1]$ per unit, the optimal solution satisfies $V(\tilde{z}) = \rho$. If instead the cost of the managerial resource is $\frac{1}{2}\rho z^2$ the first order condition gives $V'(\tilde{z}) - \rho\tilde{z} = 0$. Then the objective function is: $W(z) = Ee^{-u} - \frac{1}{2}\rho z^2$. For various values of the parameters we present the optimal solution in Table 1. The optimal solution is determined using numerical integration and a direct search procedure.

The objective function $W(z)$ is well behaved and the maximum is unique. The objective function $V(z) - \rho z$ also has a unique maximum but its shape is not quadratic far from the optimum. Relative to Oikawa (2016) *we do need adjustment costs* in the managerial resource z in order to be able to find a non-trivial solution for z and optimal inefficiency $V(\tilde{z})$.

Table 1: Optimal solution

$\hat{\theta}$	σ_*	ρ	$\tilde{z}$	optimal efficiency
0	1	0.2	0.642	0.901
		1	0.592	0.608
		10	0.577	0.579
1	0.5	0.2	1.065	0.828
		1	0.462	0.633
		10	0.419	0.437

Notes: Optimal efficiency is $V(\tilde{z})$ and the objective function is $W(z) = V(z) - \frac{1}{2}\rho z^2$. The optimal solution is determined using numerical integration and a direct search procedure.

5 Yet another formulation

Suppose we have the specification in (1) and, although the error term v is not under the control of the firm, we ignore this problem and focus on the one-sided error component u for which we assume it follows either i) a lognormal distribution $\ln u \sim N\left(\gamma z,\, \sigma_u^2\right)$ or ii) an exponential specification, $u \sim \text{Exp}(\lambda)$, viz. $f(u; z, \gamma) = \lambda^{-1} e^{-\lambda^{-1} u}$ where $\ln \lambda = \gamma z$.[3] Here, z plays again the role of a managerial variable and $\gamma < 0$ is a certain parameter which measures the effectiveness of z on reducing inefficiency. We want to minimize Ee^{-u} for given γ and σ_u^2 which have been learned, presumably, during a thought experiment before output is actually observed. Using the moment generating functions of the lognormal[4] and the exponential distributions, we have i) $V(z) = Ee^{-u} = 1 - e^{\gamma z + \frac{\sigma_u^2}{2}}$ for the lognormal and ii) $V(z) = \frac{e^{-\gamma z}}{1+e^{-\gamma z}}$ for the exponential.

As both are strictly increasing in z we have to set $z = \infty$ which is clearly absurd. If we assume that there is a cost $\rho > 0$ per unit of the managerial resource then the first order conditions are: i) $1 - e^{\gamma \tilde{z} + \frac{\sigma_u^2}{2}} = \rho$ for the lognormal and ii) $\frac{\gamma e^{-\gamma \tilde{z}}}{(1+e^{-\gamma \tilde{z}})^2} = -\rho$. In case (i) for a solution to exist, we need $\max\left\{0,\, 1 - e^{\sigma_u^2/2}\right\} < \rho < 1$. In the second case we need $0 < \rho < 1$.

From these two cases it is clear that deviating from the assumption of a *gamma* distribution (as the exponential is a special case of the *gamma*) does not produce a result that comes even close to a *gamma* distribution. For the reader thinking that we started from a lognormal assumption and, therefore, we obtain lognormality again, the answer is that she is right but Oikawa (2016) started from a normality assumption about θ to obtain a *gamma* or *chi*-square distribution for $(\theta - \tilde{z})^2$ which is equally obvious!

[3] A constant or other variables can be included without changing the results.

[4] The moment generating function $Ee^{t\mathbb{X}}$ for a lognormal random variable is defined only for $t \leq 0$ which is not a problem here as $t = -1$.

6 Relative comparisons of managerial inefficiency

Suppose $y = F(x)e^{v-u(z)}$ where $x \in \Re^K$ is a vector of inputs whose prices are $w \in \Re^K_{++}$and $u(z)$ is technical inefficiency with $u'(z) < 0$. We assume $u(z)$ is a deterministic function. Under an expected utility of real profits specification the objective of the firm is to maximize:

$$EU(\Pi),\ \Pi = F(x)e^{v-u(z)} - w^\top x - \rho z, \tag{14}$$

for some utility function $U(\Pi)$. In fact we can redefine v to be a random error in technical inefficiency.

The first order conditions are:

$$\begin{aligned} F_j(x) \cdot E\left\{U'(\Pi)e^{v-u(z)}\right\} &= w_j EU'(\Pi),\ j = 1, ..., K, \\ F(x)u'(z) \cdot E\left\{U'(\Pi)e^{v-u(z)}\right\} &= -\rho EU'(\Pi), \end{aligned} \tag{15}$$

where $F_j(x) = \frac{\partial F(x)}{\partial x_j}$. From these conditions, it is easy to see that $\frac{F_j(x)}{F_1(x)} = \frac{w_j}{w_1}$, $j = 2, ..., K$ which implies that behavior is consistent with cost minimization. See also Kumbhakar (2002) and Kumbhakar and Tveterås (2003). Multiplying the first set of equations by x_j in (15), dividing the two equations and summing up we have:

$$\sum_{j=1}^{K} F_j(x)\frac{x_j}{F(x)} = -\tfrac{1}{\rho}u'(z)\sum_{j=1}^{K} w_j x_j. \tag{16}$$

As the left-hand-side is returns to scale ($RTS = \sum_{j=1}^{K} \frac{\partial \log F(x)}{\partial \log x_j}$) we have the formula:

$$\frac{RTS}{TC} = -\tfrac{1}{\rho}u'(z), \tag{17}$$

where TC is the cost of non-managerial inputs. If unit managerial compensation (ρ) is common across firms[5] then, relative to another firm "o", we have:

$$\frac{u'(z)}{u'_o(z_o)} = \frac{RTS}{RTS_o}\frac{TC_o}{TC}. \tag{18}$$

Therefore, w*ithout specifying a utility function or even the cost inefficiency function, the relative effectiveness of managerial input* (measured by $u'(z)$) can be determined using only cost data and estimates of the returns to scale. The point of this construction is that features of the inefficiency function $u(z)$ *can be recovered from the data*, based on the solid microfoundation of expected utility of

[5] Or at least, for a common benchmark shadow value of managerial compensation.

profit maximization but the model does not make a prediction about the distribution. To make this point more clear we can generalize (14) as follows: Maximize $EU(\Pi)$, $\Pi = F(x)e^{v-u(z)\xi} - w^\top x - \rho z$, where ξ is a random error in inefficiency whose support is the positive half-line; for example $\xi \sim \mathcal{N}_+(0,1)$. The first order conditions will change to $\frac{RTS}{TC} \cdot \frac{E\left[U'(\Pi)e^{v-u(z)\xi}\right]}{E\left[U'(\Pi)e^{v-u(z)\xi}\xi\right]} = -\frac{1}{\rho}u'(z)$. The ratio of expectations depends on risk aversion and downside risk aversion as well as higher moments of v and ξ along with $u(z)$, see Kumbhakar (2002). Further analysis of this problem is beyond the scope of this paper.

7 Concluding remarks

In this paper we provided what we believe is a correct solution to the problem of microfoundations for stochastic frontier analysis. Contrary to Oikawa (2016) it turns out that we do not have a reason to favor a *gamma* distribution for the one-sided error component of the stochastic frontier model. Another conclusion of Oikawa (2016) is found not to hold, viz. we *do* need adjustment costs in the managerial input to determine an optimal non-trivial value for the input and optimal inefficiency. Moreover, we find that without specifying a utility function or even the cost inefficiency function, the relative effectiveness of managerial input (measured by $\partial u(z)/\partial z$) can be determined using only cost data and estimates of the returns to scale. The point of this construction is that features of the inefficiency function $u(z)$ *can be recovered from the data*, based on the solid microfoundation of expected utility of profit maximization but the model does not make a prediction about the distribution. This result is likely to be of considerable value in the applied analysis of production and efficiency.

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Figure 1: Function $V(z)$, case (iii) of Lemma 3

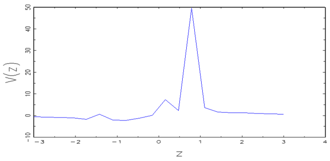

Figure 2: Function $V(z)$, case (iv) of Lemma 3

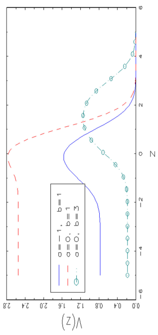

Figure 3: Density $f(u)$ for various values of $\tilde{z}$

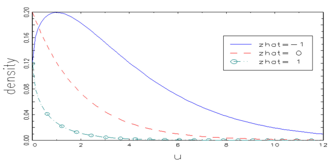

Figure 4: Density $f(u)$ for various values of $\tilde{z}$, $\hat{\theta} = 1$

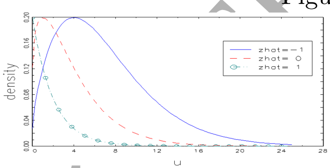

Figure 5: Density $f(u)$ for various values of $\tilde{z}$, $\hat{\theta} = -1$