UNIQUENESS OF THE MAXIMAL IDEAL OF OPERATORS ON THE ℓ_p-SUM OF ℓ_∞^n $(n \in \mathbb{N})$ FOR $1 < p < \infty$

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ABSTRACT. A recent result of Leung (*Proceedings of the American Mathematical Society* 2015) states that the Banach algebra $\mathscr{B}(X)$ of bounded, linear operators on the Banach space $X = \big(\bigoplus_{n\in\mathbb{N}} \ell_\infty^n\big)_{\ell_1}$ contains a unique maximal ideal. We show that the same conclusion holds true for the Banach spaces $X = \big(\bigoplus_{n\in\mathbb{N}} \ell_\infty^n\big)_{\ell_p}$ and $X = \big(\bigoplus_{n\in\mathbb{N}} \ell_1^n\big)_{\ell_p}$ whenever $p \in (1,\infty)$.

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1. Introduction and statement of main results

For $p \in [1,\infty)$, consider the Banach space

$$W_p = \Big(\bigoplus_{n\in\mathbb{N}} \ell_\infty^n\Big)_{\ell_p}.$$

Denny Leung [12] has recently proved that the Banach algebra $\mathscr{B}(W_1)$ of all (bounded, linear) operators acting on W_1 has a unique maximal ideal, thus establishing the dual version of [11, Theorem 3.2]. We shall show that Leung's conclusion extends to $\mathscr{B}(W_p)$ for $p \in (1,\infty)$ and to $\mathscr{B}(W_p^*)$, where $W_p^* \cong \big(\bigoplus_{n\in\mathbb{N}} \ell_1^n\big)_{\ell_q}$ is the dual Banach space of W_p, with $q \in (1,\infty)$ denoting the conjugate exponent of p. More precisely, using the following piece of notation

$$\mathscr{M}_X = \{T \in \mathscr{B}(X) : \text{the identity operator on } X \text{ does not factor through } T\} \qquad (1.1)$$

for a Banach space X, we can state our main result as follows.

Theorem 1.1. *For each $p \in (1,\infty)$, the sets $\mathscr{M}_{W_p}$ and $\mathscr{M}_{W_p^*}$ given by* (1.1) *are the unique maximal ideals of the Banach algebras $\mathscr{B}(W_p)$ and $\mathscr{B}(W_p^*)$, respectively.*

This theorem adds the spaces W_p and W_p^* for $p \in (1,\infty)$ to the already substantial list, summarized in [8, p. 4832], of Banach spaces X for which the set $\mathscr{M}_X$ is known to be the unique maximal ideal of $\mathscr{B}(X)$.

In general, Dosev and Johnson [6, p. 166] observed that, for a Banach space X, the set $\mathscr{M}_X$ given by (1.1) is an ideal of $\mathscr{B}(X)$ if (and only if) $\mathscr{M}_X$ is closed under addition, and in the positive case, $\mathscr{M}_X$ is automatically the unique maximal ideal of $\mathscr{B}(X)$. Thus, to prove Theorem 1.1, it suffices to show that the sets $\mathscr{M}_{W_p}$ and $\mathscr{M}_{W_p^*}$ are closed under addition.

Our approach is completely different from Leung's. Let us here describe the two most important results that we establish en route to Theorem 1.1, as they outline our strategy, and they may be of some independent interest. First, in Section 2, we introduce a new operator

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ideal in the following way. For $p \in [1,\infty]$ and Banach spaces X and Y, define

$$\mathscr{S}_{\{\ell_p^n\,:\,n\in\mathbb{N}\}}(X,Y) = \big\{T \in \mathscr{B}(X,Y) : T \text{ does not fix the family } \{\ell_p^n : n \in \mathbb{N}\} \text{ uniformly}\big\}. \tag{1.2}$$

(Details of this terminology can be found in Definitions 2.2 and 2.9.)

Theorem 1.2. *The class $\mathscr{S}_{\{\ell_p^n\,:\,n\in\mathbb{N}\}}$ given by (1.2) is a closed operator ideal in the sense of Pietsch for each $p \in [1,\infty]$.*

Second, in Section 3, we show that the ideal $\mathscr{S}_{\{\ell_\infty^n\,:\,n\in\mathbb{N}\}}(W_p)$ is equal to the set $\mathscr{M}_{W_p}$.

Theorem 1.3. *Let $p \in (1,\infty)$. An operator $T \in \mathscr{B}(W_p)$ fixes the family $\{\ell_\infty^n : n \in \mathbb{N}\}$ uniformly if and only if the identity operator on W_p factors through T.*

Ultraproducts play a key role in the proofs of both of these theorems.

2. Operators fixing certain Banach spaces and the proof of Theorem 1.2

Throughout this paper, all Banach spaces are supposed to be over the same scalar field $\mathbb{K}$, either the real or the complex numbers. By an *ideal*, we understand a two-sided, algebraic ideal. The term *operator* means a bounded, linear mapping between Banach spaces. Given two Banach spaces X and Y, we write $\mathscr{B}(X,Y)$ for the Banach space of all operators from X to Y, and we set $\mathscr{B}(X) = \mathscr{B}(X,X)$.

An operator $T\colon X \to Y$ is *bounded below* by a constant $c > 0$ if $\|Tx\| \geqslant c\|x\|$ for each $x \in X$. This is equivalent to saying that T is an isomorphism onto its range $T[X]$, which is closed, and the inverse operator from $T[X]$ onto X has norm at most c^{-1}. The class of operators which are bounded below is open in the norm topology; more precisely, we have the following estimate, which is an immediate consequence of the subadditivity of the norm.

Lemma 2.1. *Let X and Y be Banach spaces, let $c > \varepsilon \geqslant 0$ be constants, and let $S, T\colon X \to Y$ be operators such that $\|S - T\| \leqslant \varepsilon$ and T is bounded below by c. Then S is bounded below by $c - \varepsilon$.*

Definition 2.2. Let E, X and Y be Banach spaces, let $T\colon X \to Y$ be an operator, and let $C \geqslant 1$ be a constant. We say that T *C-fixes a copy of E* if there is an operator $S\colon E \to X$ of norm at most C such that the composite operator TS is bounded below by $1/C$. In the case where the value of the constant C is not important, we shall simply say that T *fixes a copy of E*.

An operator which does not fix a copy of E is called *E-strictly singular;* the set of E-strictly singular operators from X to Y is denoted by $\mathscr{S}_E(X,Y)$.

We note for later reference that an operator which C-fixes a non-zero Banach space for some constant $C \geqslant 1$ must have norm at least $1/C^2$.

A straightforward application of Lemma 2.1 leads to the following conclusion.

Corollary 2.3. *Let E, X and Y be Banach spaces, let $C' \geqslant C \geqslant 1$ be constants, and let $S, T\colon X \to Y$ be operators such that T C-fixes a copy of E and $\|S - T\| \leqslant (C' - C)/C^2C'$. Then S C'-fixes a copy of E.*

It follows in particular that the set $\mathscr{S}_E(X,Y)$ is norm-closed in $\mathscr{B}(X,Y)$ for all Banach spaces E, X and Y. Moreover, the class $\mathscr{S}_E$ is clearly closed under arbitrary compositions, in the sense that $STR \in \mathscr{S}_E(W,Z)$ whenever $R \in \mathscr{B}(W,X)$, $T \in \mathscr{S}_E(X,Y)$ and $S \in \mathscr{B}(Y,Z)$

(and W, X, Y and Z are Banach spaces). Thus $\mathscr{S}_E$ is a closed operator ideal in the sense of Pietsch if (and only if) it is closed under addition. We shall now show that this is the case provided that the Banach space E is *minimal*, in the sense that E is infinite-dimensional and each of its closed, infinite-dimensional subspaces contains a further subspace which is isomorphic to E. Examples of minimal Banach spaces include the classical sequence spaces c_0 and ℓ_p for $1 \leqslant p < \infty$ (Pełczyński [14]), the dual of Tsirelson's space (Casazza, Johnson and Tzafriri [3]; note that we follow the convention, originating from [7], that the term 'Tsirelson's space' refers to the dual of the space originally constructed by Tsirelson) and Schlumprecht's space (Schlumprecht [2]). On the other hand, we note in passing that Tsirelson's space is not itself minimal [4].

We shall require the following lemma (see [13, Proposition 2.c.4], where it is attributed to Kato [9]), whose statement involves the following standard piece of terminology: an operator is *approximable* if it belongs to the norm-closure of the set of finite-rank operators.

Lemma 2.4. *Let X and Y be infinite-dimensional Banach spaces, and let $T\colon X \to Y$ be an operator which is not bounded below on any finite-codimensional subspace of X. Then, for each $\varepsilon > 0$, X contains a closed, infinite-dimensional subspace W such that the restriction of T to the subspace W is approximable and has norm at most ε.*

Proposition 2.5. *Let E be a minimal Banach space. Then the class $\mathscr{S}_E$ of E-strictly singular operators is a closed operator ideal in the sense of Pietsch.*

Proof. By the remarks above, it suffices to show that, for each pair X, Y of Banach spaces, the set $\mathscr{S}_E(X,Y)$ is closed under addition. To verify this, suppose that $S \in \mathscr{S}_E(X,Y)$ and $T \in \mathscr{B}(X,Y)$ are operators such that $S+T \notin \mathscr{S}_E(X,Y)$; we must show that $T \notin \mathscr{S}_E(X,Y)$. Choose an operator $R\colon E \to X$ such that $(S+T)R$ is bounded below by $c > 0$, say. Since $S \in \mathscr{S}_E(X,Y)$ and E is minimal, the restriction of SR to any infinite-dimensional subspace of E is not bounded below. Hence Lemma 2.4 implies that E contains a closed, infinite-dimensional subspace F such that $\|SR|_F\| \leqslant c/2$. After replacing F with a suitably chosen subspace, we may in addition suppose that F is isomorphic to E. Lemma 2.1 shows that $TR|_F$ is bounded below by $c/2$, and so $T \notin \mathscr{S}_E(X,Y)$. □

Remark 2.6. A more general version of Proposition 2.5 can be deduced from a result of Stephani [15, Theorem 2.1], as Rosenberger observed in his Mathematical Review (MR582517) of Stephani's paper.

The connection between Proposition 2.5 and Theorem 1.2 goes via ultraproducts. We refer the reader to [1, Section 11.1] or [5, Chapter 8] for basic facts and notation involving ultraproducts. The following lemma is essentially a quantitative version of the fact that each ultrapower of a Banach space X is finitely representable in X.

Lemma 2.7. *Let E, X and Y be Banach spaces, where E is finite-dimensional, let $C' > C \geqslant 1$ be constants, let $T\colon X \to Y$ be an operator, and let $\mathfrak{U}$ be a free ultrafilter on $\mathbb{N}$ such that the ultrapower $T_{\mathfrak{U}}\colon X_{\mathfrak{U}} \to Y_{\mathfrak{U}}$ C-fixes a copy of E. Then T C'-fixes a copy of E.*

To prove it, we shall require the following simple variant of [1, Lemma 11.1.11], where we keep record of the constants involved.

Lemma 2.8. *Let T be an operator from a non-zero, finite-dimensional Banach space E into a Banach space X, let N be a finite ε-net in the unit sphere of E for some $\varepsilon \in (0,1)$, and let*

$\eta \leqslant \min_{x\in N}\|Tx\|$ and $\xi \geqslant \max_{x\in N}\|Tx\|$. Then

$$\frac{\eta - \varepsilon(\xi+\eta)}{1-\varepsilon}\|x\| \leqslant \|Tx\| \leqslant \frac{\xi}{1-\varepsilon}\|x\| \qquad (x \in E).$$

Proof of Lemma 2.7. We may suppose that E is non-zero, so that E has a normalized basis $(e_j)_{j=1}^n$; denote by $(f_j)_{j=1}^n$ the corresponding coordinate functionals. Choose $C'' \in (C, C')$, and let N be a finite ε-net in the unit sphere of E, where

$$\varepsilon = \frac{C'-C''}{C'(C'')^2\|T\| + C' - C''} \in (0,1).$$

By the assumption, there is an operator $S\colon E \to X_{\mathfrak{U}}$ of norm at most C such that the composite operator $T_{\mathfrak{U}}S$ is bounded below by $1/C$. For each $j \in \{1,\ldots,n\}$, let $(x_{j,k})_{k\in\mathbb{N}} \in \ell_\infty(\mathbb{N},X)$ be a representative of the equivalence class of Se_j in $X_{\mathfrak{U}}$. Then, for each $x \in N$, we have

$$\lim_{k,\mathfrak{U}}\Big\|\sum_{j=1}^n \langle x, f_j\rangle x_{j,k}\Big\| = \|Sx\| \leqslant C < C'' \quad\text{and}\quad \lim_{k,\mathfrak{U}}\Big\|\sum_{j=1}^n \langle x, f_j\rangle Tx_{j,k}\Big\| = \|T_{\mathfrak{U}}Sx\| \geqslant \frac{1}{C} > \frac{1}{C''}.$$

Since N is finite and $\mathfrak{U}$ is closed under finite intersections, the set

$$M = \Big\{k \in \mathbb{N} : \Big\|\sum_{j=1}^n \langle x, f_j\rangle x_{j,k}\Big\| < C'' \text{ and } \Big\|\sum_{j=1}^n \langle x, f_j\rangle Tx_{j,k}\Big\| > \frac{1}{C''}\ \ (x\in N)\Big\} \tag{2.1}$$

belongs to $\mathfrak{U}$, and it is therefore non-empty; choose $k \in M$, and define a mapping $R\colon E \to X$ by setting $Re_j = x_{j,k}$ for each $j \in \{1,\ldots,n\}$ and extending by linearity. The estimates given in (2.1) together with Lemma 2.8 and the choice of ε imply that $\|R\| \leqslant C''/(1-\varepsilon) \leqslant C'$, where the final inequality follows from the fact that $\|T\| = \|T_{\mathfrak{U}}\| \geqslant 1/C^2 \geqslant 1/C'C''$, and TR is bounded below by

$$\frac{1/C'' - \varepsilon(\|T\|C'' + 1/C'')}{1-\varepsilon} = \frac{1}{C'},$$

so that T C'-fixes a copy of E. □

Definition 2.9. Let $\mathfrak{F}$ be a non-empty family of Banach spaces. We say that an operator T *fixes the family $\mathfrak{F}$ uniformly* if there is a constant $C \geqslant 1$ such that T C-fixes a copy of each Banach space in $\mathfrak{F}$.

To state our next result concisely, it is convenient to introduce the notation $E_p = \ell_p$ for $p \in [1,\infty)$ and $E_\infty = c_0$.

Corollary 2.10. *Let X and Y be Banach spaces, let $T \in \mathscr{B}(X,Y)$, and let $p \in [1,\infty]$. Then the following three conditions are equivalent:*

(a) *the operator T fixes the family $\{\ell_p^n : n \in \mathbb{N}\}$ uniformly;*
(b) *for every free ultrafilter $\mathfrak{U}$ on $\mathbb{N}$, the ultrapower $T_{\mathfrak{U}}\colon X_{\mathfrak{U}} \to Y_{\mathfrak{U}}$ fixes a copy of E_p;*
(c) *there exists a free ultrafilter $\mathfrak{U}$ on $\mathbb{N}$ such that the ultrapower $T_{\mathfrak{U}}\colon X_{\mathfrak{U}} \to Y_{\mathfrak{U}}$ fixes the family $\{\ell_p^n : n \in \mathbb{N}\}$ uniformly.*

Proof. (a)⇒(b). Suppose that there exists a constant $C \geqslant 1$ such that, for each $n \in \mathbb{N}$, we can find an operator $S_n\colon \ell_p^n \to X$ of norm at most C such that the composite operator TS_n is bounded below by $1/C$, and let $\mathfrak{U}$ be a free ultrafilter on $\mathbb{N}$. Then we have an operator $S = (\prod S_n)_{\mathfrak{U}}$ of norm at most C from the ultraproduct $(\prod \ell_p^n)_{\mathfrak{U}}$ into the ultrapower $X_{\mathfrak{U}}$, and the composite operator $T_{\mathfrak{U}}S$ is bounded below by $1/C$. For each $n \in \mathbb{N}$, ℓ_p^n is an $L_p(\mu)$-space for $p < \infty$ and a $C(K)$-space for $p = \infty$, and these classes are preserved by ultraproducts (see, *e.g.*, [5, Theorem 8.7]). Thus the domain of S is an infinite-dimensional $L_p(\mu)$-space for $p < \infty$ and an infinite-dimensional $C(K)$-space for $p = \infty$, so that in either case it contains an isomorphic copy of E_p. Taking an operator $R\colon E_p \to (\prod \ell_p^n)_{\mathcal{U}}$ which is bounded below, we see that $T_{\mathcal{U}}SR$ is also bounded below, so that $T_{\mathcal{U}}$ fixes a copy of E_p.

The implication (b)⇒(c) is obvious, while (c)⇒(a) follows from Lemma 2.7. □

Proof of Theorem 1.2. The class $\mathscr{S}_{\{\ell_p^n\,:\,n\in\mathbb{N}\}}$ is clearly closed under arbitrary compositions and contains all finite-rank operators, while Corollary 2.3 shows that it is closed in the operator norm. Now suppose that $S, T \in \mathscr{S}_{\{\ell_p^n\,:\,n\in\mathbb{N}\}}(X, Y)$ for some Banach spaces X and Y. Corollary 2.10 implies that $S_{\mathcal{U}}, T_{\mathcal{U}} \in \mathscr{S}_{E_p}(X_{\mathcal{U}}, Y_{\mathcal{U}})$ for every free ultrafilter $\mathcal{U}$ on $\mathbb{N}$, where we recall that $E_p = \ell_p$ for $p < \infty$ and $E_p = c_0$ for $p = \infty$. Consequently, we have $(S+T)_{\mathcal{U}} = S_{\mathcal{U}} + T_{\mathcal{U}} \in \mathscr{S}_{E_p}(X_{\mathcal{U}}, Y_{\mathcal{U}})$ by Proposition 2.5, and hence another application of Corollary 2.10 shows that $S + T \in \mathscr{S}_{\{\ell_p^n\,:\,n\in\mathbb{N}\}}(X, Y)$. □

3. The proofs of Theorems 1.3 and 1.1

We begin by establishing some lemmas and introducing some notation that will be required in the proof of Theorem 1.3. Our first lemma needs no proof: it follows immediately from the 1-injectivity of the Banach space ℓ_∞^n.

Lemma 3.1. *Let $n \in \mathbb{N}$, let X be a Banach space, and let $T\colon \ell_\infty^n \to X$ be an operator which is bounded below by $c > 0$. Then T has a left inverse $X \to \ell_\infty^n$ of norm at most c^{-1}.*

Our second lemma concerns strictly singular perturbations of operators that fix ℓ_p for some $p \in [1, \infty)$ or c_0.

Lemma 3.2. *Let X and Y be Banach spaces, let $E = \ell_p$ for some $p \in [1, \infty)$ or $E = c_0$, let $C' > C \geqslant 1$ be constants, and let $S, T\colon X \to Y$ be operators, where S is strictly singular and T C-fixes a copy of E. Then $S + T$ C'-fixes a copy of E.*

Proof. By the assumption, we can choose an operator $R\colon E \to X$ such that $\|R\| \leqslant C$ and TR is bounded below by $1/C$. Set $\varepsilon = (C' - C)/C'(C + 1) \in (0, 1)$. Since SR is strictly singular, Lemma 2.4 implies that E contains a closed, infinite-dimensional subspace F such that $\|SR|_F\| \leqslant \varepsilon$. Keeping careful track of the constants in the proof of Pełczyński's theorem that E is minimal, as it is given in [1, Proposition 2.2.1], for instance, as well as in the proof of [1, Theorem 1.3.9], we see that in fact every closed, infinite-dimensional subspace of E contains almost isometric copies of E. We can therefore find an operator $U\colon E \to F$ such that $(1 - \varepsilon)\|x\| \leqslant \|Ux\| \leqslant \|x\|$ for each $x \in E$. Hence we have $\|RU\| \leqslant \|R\| < C'$, $\|SRU\| \leqslant \|SR|_F\|\,\|U\| \leqslant \varepsilon$ and

$$\|TRUx\| \geqslant \frac{1}{C}\|Ux\| \geqslant \frac{1-\varepsilon}{C}\|x\| \qquad (x \in E),$$

so that $(S + T)RU$ is bounded below by $(1 - \varepsilon)/C - \varepsilon = 1/C'$ by Lemma 2.1 and the choice of ε. This shows that $S + T$ C'-fixes a copy of E. □

We shall next introduce some notation and terminology related to Banach spaces of the form

$$X = \Big(\bigoplus_{n\in\mathbb{N}} X_n\Big)_{\ell_p} = \Big\{(x_n)_{n\in\mathbb{N}} : x_n \in X_n \ (n \in \mathbb{N}) \text{ and } \sum_{n=1}^{\infty} \|x_n\|^p < \infty\Big\}, \tag{3.1}$$

where $(X_n)_{n\in\mathbb{N}}$ is a sequence of Banach spaces and $p \in [1, \infty)$. For each $n \in \mathbb{N}$, we write $\iota_n\colon X_n \to X$ and $\pi_n\colon X \to X_n$ for the canonical n^{th} coordinate embedding and projection, respectively. Given an operator T on X, we associate with it the $(\mathbb{N}\times\mathbb{N})$-matrix $(T_{j,k})$, where $T_{j,k} = \pi_j T \iota_k\colon X_k \to X_j$ for each pair $j,k\in\mathbb{N}$. We say that T has *finite rows* if, for each $j\in\mathbb{N}$, there exists $k_0\in\mathbb{N}$ such that $T_{j,k}=0$ whenever $k>k_0$, and that T has *finite columns* if, for each $k\in\mathbb{N}$, there exists $j_0\in\mathbb{N}$ such that $T_{j,k}=0$ whenever $j>j_0$.

The following elementary perturbation result is a special case of [10, Lemma 2.7].

Lemma 3.3. *Let T be an operator on a Banach space X of the form* (3.1)*, where X_n is finite-dimensional for each $n\in\mathbb{N}$ and $p\in(1,\infty)$. Then, for each $\varepsilon>0$, there exists an operator $T'\in\mathscr{B}(X)$ with finite rows and finite columns such that the operator $T-T'$ is approximable and has norm at most ε.*

Set $P_0=0$ and $P_n=\sum_{j=1}^n \iota_j\pi_j$ for $n\in\mathbb{N}$. We can then state our final lemma as follows.

Lemma 3.4. *Let X be a Banach space of the form* (3.1)*, let $0\leqslant k_1<k_1'\leqslant k_2<k_2'\leqslant\cdots$ be an increasing sequence of integers, and let $(R_n\colon X_n\to X)_{n\in\mathbb{N}}$ and $(S_n\colon X\to X_n)_{n\in\mathbb{N}}$ be uniformly bounded sequences of operators. Then*

$$Rx=\sum_{n=1}^{\infty}(P_{k_n'}-P_{k_n})R_nx_n \qquad \text{and} \qquad Sx=(S_n(P_{k_n'}-P_{k_n})x)_{n\in\mathbb{N}}, \tag{3.2}$$

where $x=(x_n)_{n\in\mathbb{N}}\in X$, define operators R and S on X of norms at most $\sup_{n\in\mathbb{N}}\|R_n\|$ and $\sup_{n\in\mathbb{N}}\|S_n\|$, respectively.

Proof. Set $C_1=\sup_{n\in\mathbb{N}}\|R_n\|$ and $C_2=\sup_{n\in\mathbb{N}}\|S_n\|$, and let $x=(x_n)_{n\in\mathbb{N}}\in X$ be given. We must show that the elements Rx and Sx defined by (3.2) belong to X and have norms at most $C_1\|x\|$ and $C_2\|x\|$, respectively; the result will then follow because the mappings R and S thus defined are easily seen to be linear.

The required estimate for S is straightforward:

$$\sum_{n=1}^{\infty}\|S_n(P_{k_n'}-P_{k_n})x\|^p\leqslant C_2^p\sum_{n=1}^{\infty}\|(P_{k_n'}-P_{k_n})x\|^p\leqslant C_2^p\|x\|^p.$$

Concerning R, we define $y_j\in X_j$ for each $j\in\mathbb{N}$ as follows: $y_j=\pi_jR_nx_n$ if $k_n<j\leqslant k_n'$ for some (necessarily unique) $n\in\mathbb{N}$, and $y_j=0$ otherwise. Then, for each $m\in\mathbb{N}$, we have

$$\sum_{j=k_m+1}^{\infty}\|y_j\|^p=\sum_{n=m}^{\infty}\sum_{j=k_n+1}^{k_n'}\|\pi_jR_nx_n\|^p=\sum_{n=m}^{\infty}\|(P_{k_n'}-P_{k_n})R_nx_n\|^p\leqslant C_1^p\|(I_X-P_{m-1})x\|^p.$$

Taking $m=1$, we see that $y=(y_j)_{j\in\mathbb{N}}$ belongs to X with norm at most $C_1\|x\|$. Moreover, we deduce that the series $\sum_{n=1}^{\infty}(P_{k_n'}-P_{k_n})R_nx_n$ is convergent with sum y because

$$\left\|y-\sum_{n=1}^{m}(P_{k_n'}-P_{k_n})R_nx_n\right\|^p=\sum_{j=k_{m+1}+1}^{\infty}\|y_j\|^p\leqslant C_1^p\|(I_X-P_m)x\|^p\to 0 \quad \text{as} \quad m\to\infty,$$

so that $Rx=y$, and the conclusion follows. □

Proof of Theorem 1.3. The implication $\Leftarrow$ is easy to verify. Suppose that $I_{W_p}=STR$ for some operators $R,S\in\mathscr{B}(W_p)$, and let $C=\sqrt{\|R\|\,\|S\|}$. By replacing R and S with $CR/\|R\|$ and $CS/\|S\|$, respectively, we may suppose that $\|R\|=\|S\|=C$. Then, for each $n\in\mathbb{N}$, the composite operator $TR\iota_n\colon \ell_\infty^n\to W_p$ is bounded below by $1/\|S\|=1/C$ and $\|R\iota_n\|\leqslant\|R\|=C$, so that T C-fixes ℓ_∞^n.

Conversely, suppose that T fixes the family $\{\ell_\infty^n : n \in \mathbb{N}\}$ uniformly. We may without loss of generality suppose that $\|T\| = 1$. Take a free ultrafilter $\mathcal{U}$ on $\mathbb{N}$. Corollary 2.10 shows that the ultrapower $T_\mathcal{U}$ C-fixes a copy of c_0 for some $C \geqslant 1$. Choose constants $C_1 > C_2 > C_3 > C_4 > C$, and set $\varepsilon = \min\{(C_4 - C)/C^2 C_4, 1/C_1^2\} \in (0,1)$. By Lemma 3.3, we can find an operator $T' \in \mathscr{B}(W_p)$ with finite rows and columns such that $\|T - T'\| < \varepsilon/2$. Set $T'' = T'/\|T'\|$. Since

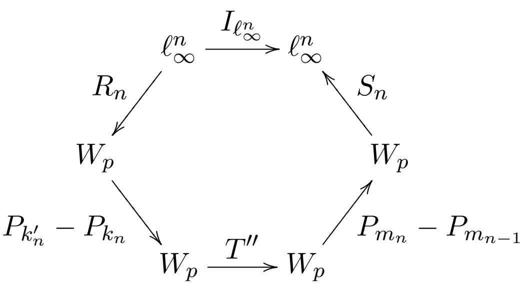
$$\|T_\mathcal{U} - T''_\mathcal{U}\| = \|T - T''\| \leqslant \|T - T'\| + \left\| \left(1 - \frac{1}{\|T'\|}\right) T' \right\| = \|T - T'\| + \big| \|T'\| - \|T\| \big| < \varepsilon \leqslant \frac{C_4 - C}{C^2 C_4},$$

Corollary 2.3 implies that $T''_\mathcal{U}$ C_4-fixes a copy of c_0.

By induction, we shall construct sequences $0 = k_0 = k_0' \leqslant k_1 < k_1' \leqslant k_2 < k_2' \leqslant \cdots$ and $0 = m_0 < m_1 < m_2 < \cdots$ of integers and sequences $(R_n \colon \ell_\infty^n \to W_p)_{n\in\mathbb{N}_0}$ and $(S_n \colon W_p \to \ell_\infty^n)_{n\in\mathbb{N}_0}$ of operators, each having norm at most C_1, such that

$$(I_{W_p} - P_{m_n}) T'' P_{k_n'} = 0 = P_{m_{n-1}} T'' (I_{W_p} - P_{k_n}) \tag{3.3}$$

and the diagram



(3.4)

is commutative for each $n \in \mathbb{N}$.

The only reason that we have included the case $n = 0$ is that it makes the start of the induction trivial (whereas if we began with $n = 1$, we would need to carry out a small amount of checking, which would duplicate parts of the induction step). Indeed, we can simply take $R_0 = S_0 = 0$ (as well as $k_0 = k_0' = m_0 = 0$, as already stated).

Now assume that, for some $N \in \mathbb{N}_0$, integers $0 = k_0 = k_0' \leqslant k_1 < k_1' \leqslant \cdots \leqslant k_N < k_N'$ and $0 = m_0 < m_1 < \cdots < m_N$ and operators $(R_n \colon \ell_\infty^n \to W_p)_{n=0}^N$ and $(S_n \colon W_p \to \ell_\infty^n)_{n=0}^N$ of norms at most C_1 have been chosen in accordance with (3.3)–(3.4). Since T'' has finite rows, we can choose $k_{N+1} \geqslant k_N'$ such that $T''_{r,s} = 0$ whenever $1 \leqslant r \leqslant m_N$ and $s > k_{N+1}$. Then we have $P_{m_N} T'' (I_{W_p} - P_{k_{N+1}}) = 0$. For convenience, set $T''_{N+1} = (I_{W_p} - P_{m_N}) T'' (I_{W_p} - P_{k_{N+1}})$. This is a finite-rank perturbation of T'', and consequently $(T''_{N+1})_\mathcal{U}$ is a finite-rank perturbation of $T''_\mathcal{U}$ because ultrapowers of finite-rank operators have finite rank. Hence Lemma 3.2 implies that $(T''_{N+1})_\mathcal{U}$ C_3-fixes a copy of c_0, and thus of ℓ_∞^{N+1}. This, in turn, means that T''_{N+1} C_2-fixes a copy of ℓ_∞^{N+1} by Lemma 2.7; that is, we can find an operator $R_{N+1} \colon \ell_\infty^{N+1} \to W_p$ of norm at most C_2 such that $T''_{N+1} R_{N+1}$ is bounded below by $1/C_2$. The fact that R_{N+1} has finite rank means that we can take $k_{N+1}' > k_{N+1}$ such that $\|(I_{W_p} - P_{k_{N+1}'}) R_{N+1}\| \leqslant 1/C_2 - 1/C_1$. Lemma 2.1 then shows that $(I_{W_p} - P_{m_N}) T'' (P_{k_{N+1}'} - P_{k_{N+1}}) R_{N+1}$ is bounded below by $1/C_1$. Since T'' has finite columns, we can choose $m_{N+1} > m_N$ such that $T''_{r,s} = 0$ whenever $r > m_{N+1}$ and $1 \leqslant s \leqslant k_{N+1}'$. This implies that $(I_{W_p} - P_{m_{N+1}}) T'' P_{k_{N+1}'} = 0$, and consequently

$$(P_{m_{N+1}} - P_{m_N}) T'' (P_{k_{N+1}'} - P_{k_{N+1}}) R_{N+1} = (I_{W_p} - P_{m_N}) T'' (P_{k_{N+1}'} - P_{k_{N+1}}) R_{N+1},$$

which is bounded below by $1/C_1$, so Lemma 3.1 gives an operator $S_{N+1}\colon W_p \to \ell_\infty^{N+1}$ of norm at most C_1 such that the diagram (3.4) commutes for $n = N+1$. Hence the induction continues.

As in Lemma 3.4, we can now define operators R and S on W_p of norms at most C_1 by

$$Rx = \sum_{n=1}^{\infty}(P_{k_n'} - P_{k_n})R_n x_n \quad \text{and} \quad Sx = (S_n(P_{m_n} - P_{m_{n-1}})x)_{n\in\mathbb{N}} \quad (x = (x_n)_{n\in\mathbb{N}} \in W_p).$$

Then, for each $r, s \in \mathbb{N}$, we have

$$\pi_r(ST''R)\iota_s(x) = S_r(P_{m_r} - P_{m_{r-1}})T''(P_{k_s'} - P_{k_s})R_s x = \begin{cases} x & \text{if } r = s \\ 0 & \text{otherwise} \end{cases} \qquad (x \in \ell_\infty^s)$$

by (3.3)–(3.4), and therefore $ST''R = I_{W_p}$. Since

$$\|STR - I_{W_p}\| \leqslant \|S\|\,\|T - T''\|\,\|R\| < C_1^2\varepsilon \leqslant 1$$

by the choice of ε, we conclude that the operator STR is invertible, and the result follows. $\square$

Proof of Theorem 1.1. Theorem 1.3 shows that $\mathscr{M}_{W_p} = \mathscr{S}_{\{\ell_\infty^n\,:\,n\in\mathbb{N}\}}(W_p)$, which is an ideal by Theorem 1.2, and it is therefore the unique maximal ideal of $\mathscr{B}(W_p)$ by the observation of Dosev and Johnson that was stated in the Introduction.

The Banach space W_p is reflexive because $p \in (1, \infty)$. Hence the mapping $T \mapsto T^*$, which maps an operator T to its adjoint T^*, is a linear, anti-multiplicative, isometric bijection of the Banach algebra $\mathscr{B}(W_p)$ onto $\mathscr{B}(W_p^*)$, and so it induces an order isomorphism between the lattices of ideals of these two Banach algebras. In particular, the image under this mapping of the unique maximal ideal $\mathscr{M}_{W_p}$ of $\mathscr{B}(W_p)$ is the unique maximal ideal of $\mathscr{B}(W_p^*)$, and this ideal is given by

$$\{T^* : I_{W_p} \neq STR\ (R, S \in \mathscr{B}(W_p))\} = \{T^* : I_{W_p^*} \neq R^*T^*S^*\ (R, S \in \mathscr{B}(W_p))\} = \mathscr{M}_{W_p^*}. \quad \square$$

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