AN INDUCTIVE CONSTRUCTION OF $(2,1)$-TIGHT GRAPHS

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ABSTRACT. The graphs $G = (V, E)$ with $|E| = 2|V| - \ell$ that satisfy $|E'| \leq 2|V'| - \ell$ for any subgraph $G' = (V', E')$ (and for $\ell = 1, 2, 3$) are the $(2, \ell)$-tight graphs. The Henneberg–Laman theorem characterises $(2, 3)$-tight graphs inductively in terms of two simple moves, known as the Henneberg moves. Recently this has been extended, via the addition of a graph extension move, to the case of $(2, 2)$-tight simple graphs. Here an alternative characterisation is provided by means of vertex-to-K_4 and edge-to-K_3 moves, and this is extended to the $(2, 1)$-tight simple graphs by addition of an edge joining move.

1. INTRODUCTION

The purpose of this paper is to prove an inductive characterisation of simple $(2, 1)$-tight graphs.

Definition 1.1 (Lee and Streinu [14]). *Let $k, \ell \in \mathbb{N}$ and $\ell \leq 2k$. A graph $G = (V, E)$ is (k, ℓ)-sparse if for every subgraph $G' = (V', E')$, $|E'| \leq k|V'| - \ell$ (where if $\ell = 2k$ the inequality only applies if $|V'| \geq k$). G is (k, ℓ)-tight if G is (k, ℓ)-sparse and $|E| = k|V| - \ell$.*

In our notation a graph allows parallel edges and loops, whereas a simple graph allows neither.

The classes of $(2, \ell)$-tight simple graphs play an important role in the theory of 2-dimensional bar-joint frameworks (see, for example, [1, 8] for the general theory). When $l = 3$ these graphs correspond to generic frameworks that are minimally rigid when joints corresponding to the vertices are constrained to lie on a plane (since any framework on a plane has three independent rigid-body motions) [12]. When $l = 2$ these graphs correspond to generic frameworks which are minimally rigid when the joints are constrained to lie on the surface of a cylinder (since this surface allows two independent rigid-body motions) [18]. When $l = 1$ we expect the graphs to correspond to frameworks that are rigid when the joints are constrained to a surface which admits one independent rigid-body motion. These surfaces

include linearly swept surfaces (such as an elliptical cylinder or any ruled surface with parallel rulings) and spun surfaces (such as a circular cone, torus or any surface formed by rotating a smooth curve). These surfaces are important in engineering since they are easily manufactured using the processes of extrusion and turning.

The characterisation of generic framework rigidity typically involves two distinct steps - an inductive construction of the relevant class of graphs and then a proof that the construction steps preserve the required rigidity properties.

The classical result of Henneberg [10] characterises the class of $(2,3)$-tight graphs by recursive operations. Combining this with a result of Lovasz and Yemini [13], extended by Recski [21], leads to:

Theorem 1.2. *[Henneberg [10], Lovasz and Yemini [13], Recski [21]] For a graph $G = (V, E)$ the following are equivalent:*

  (1) *G is $(2,3)$-tight,*
  (2) *G is derivable from K_2 by the Henneberg 1 and Henneberg 2 moves,*
  (3) *for any edge $e \in E(K_{|V(G)|})$, $G \cup \{e\}$ is the edge-disjoint union of two spanning trees.*

Laman [12] then characterised generic minimal rigidity on the plane by showing that the Henneberg 1 and Henneberg 2 moves preserve this property on the plane.

Nixon, Owen and Power [18] obtained the following characterisation of simple $(2,2)$-tight graphs.

Theorem 1.3. *[Nixon, Owen and Power [18], Nash-Williams [16]] For a simple graph $G = (V, E)$ the following are equivalent:*

  (1) *G is $(2,2)$-tight,*
  (2) *G is derivable from K_4 by the Henneberg 1, Henneberg 2 and graph extension moves,*
  (3) *G is the edge-disjoint union of two spanning trees.*

In this characterisation a graph extension move replaces a vertex in the graph by an arbitrary $(2,2)$-tight graph which thereby becomes a $(2,2)$-tight subgraph in the extended graph. [18] also characterised generic minimal rigidity on the cylinder by showing that the Henneberg 1, Henneberg 2 and graph extension moves preserve this property on the cylinder.

Our main result is the following inductive construction of $(2,1)$-tight simple graphs. By $K_5 \backslash e$ we mean the graph formed from the complete graph on 5 vertices by removing an edge, and by $K_4 \sqcup K_4$ we mean the graph formed by taking two copies of K_4 that intersect in a copy of the complete graph K_2. The construction operations are defined at the start of Section 2.

Theorem 1.4. *For a simple graph G the following are equivalent:*

  (1) *G is $(2,1)$-tight,*

(2) G can be derived from $K_5 \setminus e$ or $K_4 \sqcup K_4$ by the Henneberg 1, Henneberg 2, vertex-to-K_4, edge joining and edge-to-K_3 moves,
(3) G is the edge-disjoint union of a spanning tree T and a spanning subgraph P in which every connected component contains exactly one cycle.

We expect that each of the construction moves in (2) of this theorem also preserves minimal generic rigidity on surfaces which admit one rigid body motion. We present this as a conjecture for subsequent investigation.

As a by-product of our arguments we also show the following result giving an alternative inductive construction of $(2,2)$-tight graphs. The construction should be easier to apply since we only insert prescribed small subgraphs rather than an arbitrary graph in the class.

Theorem 1.5. *For a simple graph $G = (V, E)$ the following are equivalent:*
(1) *G is $(2,2)$-tight,*
(2) *G is derivable from K_4 by the Henneberg 1, Henneberg 2, vertex-to-K_4 and edge-to-K_3 moves.*

The main difficulty in proving theorem 1.4 is the requirement that the inductive construction involves only simple graphs. This requirement arises because we are interested in frameworks in which the distance between a pair of joints is the usual distance measured as a straight line in 3-space. Minimal rigidity then clearly requires that two vertices are joined by at most edge. Whitely [24] has considered frameworks embedded on surfaces in which the distance between a pair of joints is a geodesic distance over the surface. In this case a pair of vertices may be separated by more than one distinct geodesic distance and the class of graphs may be extended to include multiple edges between a pair of vertices. Similarly periodic frameworks [2, 15, 22] on the plane may include edges connecting between different cells and result in graphs with multiple edges.

We note that for the case of (k,ℓ)-tight graphs (permitting parallel edges and loops) there are elegant recursive constructions requiring Henneberg type operations only [5, 6].

A further motivation for our work is the hope that understanding the recursive constructions for $(2,\ell)$-tight graphs of the various types will provide insight into $(3,6)$-tight graphs. These are the graphs relevant to major open problems in 3-dimensional rigidity theory [8, 23, 26]. Note that these graphs are necessarily simple and are outside the matroidal range. Indeed for $\ell < 6$ adding *any* $\ell - 3$ edges to a $(3,6)$-tight graph results in a graph with a decomposition into three edge disjoint spanning trees but for $l = 6$ it does not, see [9].

From our main theorems one can quickly derive sparsity variants. That is, characterisations of $(2,\ell)$-sparsity in terms of recursive operations. If Conjecture 4.1 is true then this has applications in computer aided design [20] where the emphasis is on establishing whether a system of constraint equations admits a matrix with linearly independent rows.

The paper is organised as follows. Section 2 defines the recursive moves we will consider. The key difficulty is the construction theory of Section 3, in which we discuss the sufficiency of the moves. The main step is Lemma 3.9. Here we take a seemingly mild requirement that each edge in a copy of K_3 is in at least two copies of K_3 or is in a separate $(2, 1)$-tight subgraph. This leads to the strong conclusion that every copy of K_3 is in a copy of K_4. This convenient property is used to derive the key implication in the proofs of Theorems 1.4 and 1.5. Finally Section 4 discusses rigidity theory and potential applications of our results therein.

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2. Simple $(2, \ell)$-tight Graphs

It will be convenient for us to define $f(H) := 2|V(H)| - |E(H)|$ for a graph H.

Definition 2.1. *Let $\ell = 1, 2, 3$. A simple graph G is $(2, \ell)$-sparse if $f(H) \geq \ell$ for all subgraphs H of G with at least one edge and is $(2, \ell)$-tight if it is $(2, \ell)$-sparse and $f(G) = \ell$.*

We begin by recalling and formally defining the construction moves under consideration. Define the *Henneberg* 0 move to be the addition of a vertex of degree 0 or of degree 1 to a graph. The inverse Henneberg 0 move is the removal of a vertex of degree 0 or degree 1 from a graph.

The *Henneberg* 1 *move* [10], is the addition of a degree 2 vertex to a graph. The inverse Henneberg 1 move is the removal of a degree 2 vertex from a graph.

The *Henneberg* 2 *move* [10], removes an edge uv and adds a vertex x and edges xu, xv, xw for some vertex w. The inverse Henneberg 2 move removes a degree 3 vertex x (and incident edges xu, xv, xw) and adds an edge uv, uw or vw.

Let G be $(2, 1)$-sparse containing a copy of K_4. Write G/K_4 for the (possibly multi)graph formed by contracting this copy of K_4 to a vertex v_*. That is G/K_4 has vertex set $(V(G) \setminus V(K_4)) \cup \{v_*\}$ and edge set $(E(G) \setminus E(K_4)) \cup E_*$ where E_* consists of the edges vv_* associated with edges vw where $v \in G/K_4$ and $w \in K_4$. We call this operation a K_4-*to-vertex move*. The inverse move, the *vertex-to-K_4 move* is illustrated in Figure 1.

The graph extension move mentioned in the introduction refers to the construction of G from G/H where H is a proper induced $(2, 2)$-tight subgraph of G. This move was used in [18] and is similar to vertex expansion moves used in graph theory, [4].

Let G be a graph with an edge uv such that the neighbours of v are $a_1, \ldots, a_n$. The *edge-to-K_3 move*, see Figure 2, (often referred to as vertex splitting in the literature, [25]) removes the edge uv and the vertex v
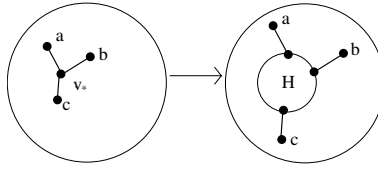
FIGURE 1. With $H = K_4$, an example of the vertex-to-K_4 move and, with H a proper induced $(2,2)$-tight subgraph of G, graph extension.

and all the edges va_i, it replaces them with the vertices v_1, v_2 and edges uv_1, uv_2, v_1v_2, plus some bipartition of the remaining edges v_1a_j and v_2a_k (with one side possibly empty). The inverse move, called the K_3-*to-edge move*, takes a copy of K_3 (with vertices u, v_1, v_2), removes the edges uv_1, uv_2, v_1, v_2, merges two vertices v_1, v_2 into a single vertex v which is adjacent to all the vertices v_1 and v_2 were adjacent to and adds the edge uv.
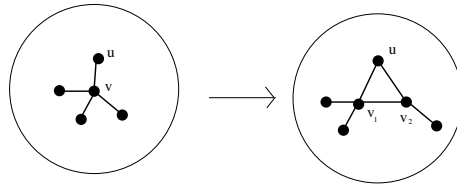

FIGURE 2. The edge-to-K_3 move.

Let K and H be $(2,1)$-tight graphs with vertices $u \in K$ and $v \in H$. We will say that G is formed from K and H by an *edge joining move*, see Figure 3 if $V(G) = V(K) \cup V(H)$ and $E(G) = E(K) \cup E(H) \cup \{uv\}$. Further, if there is a single edge uv joining two $(2,1)$-tight subgraphs G and H, then we will call the inverse move an *edge separation move*.

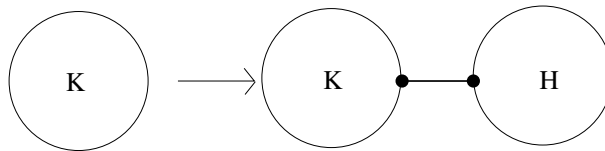
FIGURE 3. The edge joining move.

With respect to Theorem 1.5 note that Figure 4 illustrates the necessity of the K_3-to-edge move when we restrict graph contraction to the K_4-to-vertex move.

We note that $(1) \Leftrightarrow (3)$ in Theorem 1.4 can be proven in an elementary way by showing that the construction operations preserve the spanning

subgraph decomposition. More efficiently, these implications follow from matroidal results; the $(1,1)$-tight graphs are the bases of the cycle matroid and the $(1,0)$-tight graphs are the bases of the bicycle matroid. The union (on the same ground set of edges) of a cycle matroid and a bicycle matroid (with empty intersection) give the results, see [5, 7, 16, 24].
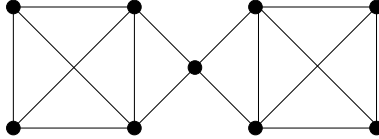

FIGURE 4. A $(2,2)$-tight graph that requires the K_3-to-edge move when we restrict the graph contraction move to the K_4-to-vertex move.

3. CONSTRUCTION THEORY

In this section we consider $(1) \Rightarrow (2)$ in Theorems 1.4 and 1.5. That is, we consider whether an arbitrary $(2,2)$ or $(2,1)$-tight graph can be reduced by applying one of our short list of moves (relevant to each case) to produce a smaller $(2,2)$-tight or $(2,1)$-tight graph.

We begin by showing that in a $(2,1)$-tight or $(2,2)$-tight graph, an inverse Henneberg 2 move is available unless all degree 3 vertices are in copies of K_4.

Lemma 3.1. *Let G be $(2,\ell)$-tight for $\ell = 1,2$ with a vertex $v \in V(G)$ of degree 3 with neighbours v_1, v_2, v_3 in G. Then either v is contained in a copy of K_4 or $G' = (G \setminus v) \cup e$ (for $e = v_1v_2, v_2v_3$ or v_3v_1) is $(2,\ell)$-tight.*

Proof. With suitable labeling of vertices, we distinguish three cases corresponding to the possible edges among the neighbours of v. Either

   (1) $v_1v_2, v_1v_3, v_2v_3 \in E$,
   (2) $v_1v_2 \notin E, v_1v_3, v_2v_3 \in E$, or
   (3) $v_1v_2, v_2v_3 \notin E$.

In case 1, v, v_1, v_2, v_3 induce a copy of K_4 in G.

Figure 5 illustrates the proof in cases 2, 3. Define Y_{12} to be a $(2,\ell)$-tight subgraph of G containing v_1, v_2 but not v_3, v. Similarly define Y_{13} and Y_{23}.

In case 2, $G' = (V \setminus v, (E \setminus \{vv_1, vv_2, vv_3\}) \cup v_1v_2)$ is $(2,\ell)$-tight unless there exists a subgraph Y_{12} of G. But then the addition of v, v_3 and their five incident edges to Y_{12} gives a subgraph Y of G with $f(Y) = \ell - 1$ which contradicts the fact that G is $(2,\ell)$-tight.
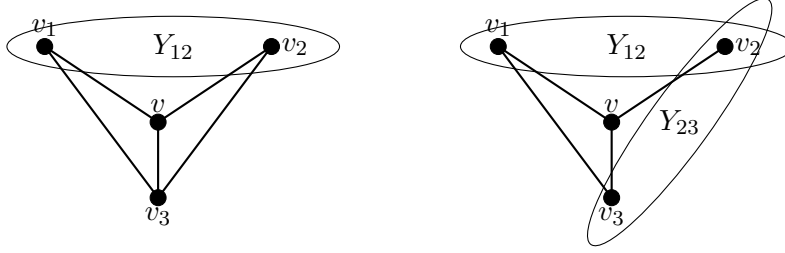
FIGURE 5. The graph on the left illustrates case (2): if there was a subgraph Y_{12} preventing the application on an inverse Henneberg 2 move on v then the graph pictured would be over-counted. Similarly the graph on the right illustrates case (3): if there are subgraphs preventing the addition of $v_1 v_2$ and $v_2 v_3$ then the graph pictured would be over-counted.

In case 3, either $G' = (V \setminus v, (E \setminus \{vv_1, vv_2, vv_3\}) \cup v_1 v_2)$ or $G' = (V \setminus v, (E \setminus \{vv_1, vv_2, vv_3\}) \cup v_2 v_3)$ is $(2, \ell)$-tight unless there exists subgraphs Y_{12} and Y_{23} of G. Then

$$f(Y_{12} \cup Y_{23}) = f(Y_{12}) + f(Y_{23}) - f(Y_{12} \cap Y_{23}) \leq \ell + \ell - \ell = \ell$$

since $Y_{12} \cap Y_{23} \supseteq v_2$ and $Y_{12} \cap Y_{23} \subset G$. But then the addition of v and its three incident edges to $Y_{12} \cup Y_{23}$ gives a subgraph Y of G with $f(Y) = \ell - 1$ which contradicts the fact that G is $(2, \ell)$-tight. $\square$

Lemma 3.2. Let $G = (V, E)$ be a $(2, \ell)$-tight graph for $\ell = 1, 2$. Then G has either an inverse Henneberg 1 move, an inverse Henneberg 2 move or at least $2l$ degree 3 vertices, each of which is in a copy of K_4.

Proof. G has no degree 1 vertices since this would imply that there is an edge $ab \in E(G)$ and $G = Y \cup ab$ with $b \notin V(Y)$ and $f(Y) = l - 1$.

Assume G has no inverse Henneberg 1 move. Then every vertex has degree at least three.

Label the vertices $1, \ldots, |V|$ and let $d(i)$ denote the degree of vertex i. The summation over the degree of all vertices in G gives $2|E|$. Hence the condition that G is $(2, \ell)$-tight gives

$$\text{(1)} \qquad \sum_{i=1}^{|V|} (4 - d(i)) = 2l.$$

Since $d(i) \geq 3$ this implies G has at least 2ℓ degree 3 vertices. By Lemma 3.1 G has an inverse Henneberg 2 move or each of these degree 3 vertices is in a copy of K_4. $\square$

We will say that a K_3-to-edge or a K_4-to-vertex move is *allowable* if it results in a graph which is simple and $(2, \ell)$-tight.

The following lemma shows that a K_4-to-vertex move is allowable provided that the copy of K_4 does not have two vertices in a single copy of K_3.

We use the notation $K_n(v_1, \ldots, v_n)$ for a subgraph of G which is a copy of the complete graph K_n on the vertices $v_1, \ldots, v_n$.

Lemma 3.3. *Let G be a $(2, \ell)$-tight graph with $|V(G)| > 4$ and let $G \to G/K_4$ be a K_4-to-vertex move. Then G/K_4 is simple and $(2, \ell)$-tight unless there is a K_3 in G with $|V(K_3 \cap K_4)| = 2$.*

Proof. G/K_4 is simple unless there is a vertex $v \in V(G) \setminus V(K_4)$ and edges $a, b \in E(G)$ with $a, b \in V(K_4)$. In this case $|V(K_3(v, a, b) \cap K_4)| = 2$.

$f(G/K_4) = f(G)$ so G/K_4 is $(2, \ell)$-tight unless there is a $Y' \subset G/K_4$ with $f(Y') < l$. There is a corresponding $Y \subset G$ such that $Y' = Y/K_4$. But then $f(Y) < l$ because $f(Y) = f(Y')$ which contradicts the $(2, \ell)$-sparsity of G. $\qquad\square$

The following lemma describes when a K_3-to-edge move is allowable. Note that a $(2, \ell)$-tight graph containing no copy of K_3 admits an inverse Henneberg move by Lemmas 3.2 and 3.1.

Lemma 3.4. *Let G be a $(2, \ell)$-tight graph and $G \to G'$ a K_3-to-edge move in which the vertices $a, b \in K_3(a, b, c)$ are the vertices in G which are merged. Then G' is simple unless there is a $K_3(a, b, d)$ in G with $d \neq c$ and G' is $(2, \ell)$-sparse unless there is a $Y \subset G$ with $ab \in E(Y)$, $c \notin V(Y)$ and Y is $(2, \ell)$-tight.*

Proof. G' is simple provided there is no vertex d different from c and two edges da, db. This gives the first condition.

G' is $(2, \ell)$-sparse provided it has no subgraph Y' with $f(Y') < l$. Y' is also a subgraph of G unless it derives from a subgraph $Y \subset G$ with $ab \in E(Y)$ and $f(Y') < f(Y)$ only if $c \notin Y$. $\qquad\square$

There are three possible K_3-to-edge moves which can be applied to a copy of K_3 in G. If none of these results in a simple graph then there are three further copies of K_3 in G and, if these are distinct, there are six further K_3-to-edge moves which might result in a simple graph. We will use this growth in the number of copies of K_3 to show that if G contains a copy of K_3 then either G has an allowable K_3-to-edge move or every copy of K_3 is in a copy of K_4 (Lemma 3.9 below). This K_4 gives an allowable K_4-to-vertex move unless it is adjacent to a copy of K_3 which, by this argument, must also be in another copy of K_4. This allows us to put a strong constraint on the possible graphs which contain a copy of K_3 but no allowable K_3-to-edge or K_4-to-vertex move (Lemma 3.10 below).

In order to keep track of the way in which copies of K_3 may share edges in a $(2, \ell)$-tight graph we first define a triangle sequence which is a set of nested subgraphs of G and derive some of its properties.

Definition 3.5. *Let G be a simple graph. A triangle sequence in G is a nested set of subgraphs*

$$M_3 \subset M_4 \subset \cdots \subset M_i \cdots \subset M_n \subseteq G$$

where M_3 is a copy of K_3, $E(M_i)$ and $V(M_i)$ are respectively the sets of edges and vertices of M_i, $|V(M_i)| = |V(M_{i-1})| + 1$ and if $V(M_i) \setminus V(M_{i-1}) = v_i$ then $E(M_i) \setminus E(M_{i-1}) = v_i a_i, v_i b_i$ where $a_i b_i \in E(M_{i-1})$ and $a_i b_i$ is in exactly one copy of K_3 in M_{i-1}. We use $S(M_i)$ to denote the set of edges in $E(M_i)$ which are in exactly one copy of K_3 in M_i (so $a_i b_i \in S(M_{i-1})$).

We will often refer to a triangle sequence by the largest graph in the sequence. A maximal length triangle sequence is one which cannot be extended by a single vertex in G. We note that even for a maximal length triangle sequence with largest graph M_n the graph G may contain copies $K_3(a, b, c)$ which are not subgraphs of M_n even though $ab \in E(M_n)$. This may occur if $c \in V(M_n)$ or if $c \notin V(M_n)$ and the edge ab is in more than one copy of K_3 in M_n. Since M_n is itself a graph we may form different triangle sequences within M_n for example by starting with different copies of K_3 in M_n, see Figure 6.
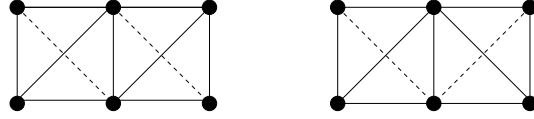

FIGURE 6. Two possible maximal length triangle sequences in $K_4 \sqcup K_4$. In each case the graph shown is the final graph M_6 in the sequence. The dashed lines represent edges which are in $K_4 \sqcup K_4$ but not in M_6. The intermediate graphs in the sequence are obtained by starting with any copy of K_3 and sequentially adding one vertex and two edges from an adjacent copy of K_3

Lemma 3.6. *A triangle sequence in G has the following properties.*

   (1) *$|V(M_i)| = i$ and $|E(M_i)| = 2i - 3$.*
   (2) *The edges in $S(M_i)$ form a spanning cycle of M_i.*
   (3) *For every edge $ab \in E(M_i) \setminus S(M_i)$ the vertex pair a, b separates M_i (as a graph) with the property that if $aa_l, aa_r \in S(M_i)$ then a_l, a_r are in different separation components.*
   (4) *If $K_3(a, b, c)$ is any copy of K_3 in M_n then there is a triangle sequence $M_3' \subset \cdots \subset M_m'$ in M_n such that $M_3' = K_3(a, b, c)$ and $M_m' = M_n$.*

Proof. Property (1) follows by induction since $|V(M_i)| = |V(M_{i-1})| + 1$ and $|E(M_i)| = |E(M_{i-1})| + 2$.

Property (2) follows by induction. The edges of $S(M_3)$ form a spanning cycle of M_3. Assume property (2) is true for M_{i-1}. Let $V(M_i) =$

$V(M_{i-1}), v_i$ and let $E(M_i) = E(M_{i-1}), v_i a_i, v_i b_i$. The edge $a_i b_i \in S(M_{i-1})$ is in $K_3(a_i, b_i, v_i) \subset M_i$ in addition to a copy of K_3 in M_{i-1} so is not in $S(M_i)$. The two edges $a_i v_i$ and $b_i v_i$ are both in $K_3(a_i, b_i, v_i)$ (and in no other copy of K_3 in M_i) so these are in $S(M_i)$. If the edges in $S(M_{i-1})$ form a spanning cycle C_{i-1} of M_{i-1} then the cycle

$$C_i = (C_{i-1} \setminus a_i b_i) \cup a_i v_i \cup b_i v_i$$

forms a spanning cycle of M_i.

Property (3) is also proved by induction. It is trivially true for M_3. Assume it is true for any M_{i-1}. Let $V(M_i) = V(M_{i-1}), v_i$ and let $E(M_i) = E(M_{i-1}), v_i a_i, v_i b_i$. Every vertex pair which separates M_{i-1} also separates M_i with the same components because the vertices a, b are adjacent and so are in the same component of any vertex separation of M_{i-1}. Putting vertex v_i in this component gives a corresponding vertex separation of M_i. The edge ab is the only edge which is in $E(M_i) \setminus S(M_i)$ and not in $E(M_{i-1}) \setminus S(M_{i-1})$. The vertex pair a, b separates the vertex v_i from the vertices $V(M_{i-1}) \setminus a, b$. The neighbours of a in $S(M_i)$ are v_i and a vertex $a_l \neq b \in V(M_{i-1}) \setminus a, b$ and these are separated by a, b.

For property (4) we will show there is a triangle sequence in M_n starting with $K_3(a, b, c)$ and terminating with M'_m for which $m = n$. This implies $|E(M'_m)| = |E(M_n)|$ and that $M'_m = M_n$. Let $M'_3, M'_4 \ldots M'_m$ be a maximal length triangle sequence in M_n starting with $K_3(a, b, c)$. Suppose for a contradiction that $m < n$. The edges in $S(M'_m)$ form a spanning cycle of M'_m and there is a edge $a_m b_m$ in $S(M'_m)$ which is not in $S(M_n)$ (since $m < n$ and a cycle contains no proper subcycles). Since $a_m b_m$ is in $E(M_n)$ but not in $S(M_n)$ there is a vertex v_m in $V(M_n) \setminus V(M'_m)$ such that there is $K_3(a_m, b_m, v_m)$ which in is in M_n and not in M'_m. The edge $a_m b_m$ is therefore in a subgraph $K_3(a_m, b_m, v_m)$ of M_n but is not in M'_m. This implies that $v_m \in V(M_n) \setminus V(M'_m)$ and $a_m v_m, b_m v_m \in E(M_n) \setminus E(M'_m)$ with $a_m b_m \in S(M'_m)$. This contradicts the maximality of the triangle sequence in M_n. $\square$

The following lemma uses a maximum length triangle sequence to show that if G has a copy of K_3 but does not have a K_3-to-edge move then every edge in a copy of K_3 in G is in at least two copies of K_3 in G.

Lemma 3.7. *Let G be a $(2, \ell)$-tight graph for $l = 1, 2$ containing a copy of K_3. Then either*

    (i) there is a K_3-to-edge move in G which gives a $(2, l)$-tight graph or
    (ii) every edge in a copy of K_3 in G is in at least two copies of K_3 in G.

Proof. Suppose that an edge $e = ab \in E(G)$ is in exactly one copy $K_3(a, b, c) \subset G$. By Lemma 3.4, the K_3-to-edge move which merges vertices a and b gives a simple graph G' and G' is $(2, \ell)$-tight unless ab and c have the following property (P): there is a $(2, \ell)$-tight subgraph $Y \subset G$ with $ab \in E(Y)$ and $c \notin V(Y)$.

Suppose for a contradiction to the lemma that every edge in G which is in exactly one copy of K_3 satisfies this property.

Let $M_3 \subset M_4 \cdots \subset M_i \cdots \subset M_n \subset G$ be a maximal length triangle sequence in G. Every edge in $E(M_n) \setminus S(M_n)$ is in two copies of K_3.

Suppose there is exactly one edge ab in $S(M_n)$ which is in exactly one copy of K_3 in G and therefore satisfies property (P) with corresponding subgraph Y. We will show by induction that $V(M_n) \cap V(Y) = \{a, b\}$ and $f(Y \cup M_n) = \ell$. Since $ab \in S(M_n)$ there is a vertex c in $V(M_n)$ such that $K_3(a, b, c)$ is in M_n. By property (4) of Lemma 3.6 there is a triangle sequence $M'_3 \subset \ldots M'_i \cdots \subset M_n$, starting with $M'_3 = K_3(a, b, c)$ and ending with M_n. $V(M'_3) \cap V(Y) = \{a, b\}$ and $f(Y \cup M'_3) = \ell$. Assume for the induction that $V(M_{i-1}) \cap V(Y) = \{a, b\}$ and that $f(Y \cup M_{i-1}) = \ell$. Let $V(M_i) \setminus V(M_{i-1}) = v_i$. If $v_i \in V(Y)$ then

$$f(Y \cup v_i a_i \cup v_i b_i) = \ell - 2$$

which contradicts the $(2, \ell)$-sparsity of G. Thus $V(M_i) \cap V(Y) = \{a, b\}$ and $f(Y \cup M_i) = \ell$.

Every edge cd in $S(M_n) \setminus ab$ is in a subgraph $K_3(c, d, v)$ of G where $K_3(c, d, v)$ is not a subgraph of M_n. Since M_n is the largest graph in a maximal length triangle sequence we must have $v \in V(M_n)$ else M_n could be extended to include v. But then $f(Y \cup M_n \cup cv) = \ell - 1$ and since $Y \cup M_n \cup cv$ is a subgraph of G this contradicts the $(2, \ell)$-sparsity of G.

Suppose there is more than one edge in $S(M_n)$ which is in exactly one copy of K_3 in G. There are subgraphs Y_1 and Y_2 and edges $a_1 b_1 \in Y_1 \cap M_n$ and $a_2 b_2 \in Y_2 \cap M_n$. If the vertices a_1, b_1, a_2, b_2 are distinct then

$$f(Y_1 \cup Y_2 \cup M_n) \leq 2\ell - 3$$

because $f(Y_1 \cup Y_2) \leq 2l$ and there are $n - 4$ vertices and $2n - 5$ edges in M_n which are not in $Y_1 \cup Y_2$. If two of the vertices a_1, b_1, a_2, b_2 are the same then

$$f(Y_1 \cup Y_2 \cup M_n) = \ell - 1$$

since $f(Y_1 \cup Y_2) = \ell$ and there are $n - 3$ vertices and $2n - 5$ edges in M_n which are not in $Y_1 \cup Y_2$. In either case this contradicts the $(2, \ell)$-sparsity of G for $l = 1, 2$. $\square$

We say that an edge $ab \in E(G)$ is a *chord* of M_n if $a, b \in V(M_n)$ and $e \notin E(M_n)$. Let $[M_n]$ denote the graph induced in G by $V(M_n)$. Then $E([M_n]) \setminus E(M_n)$ is the set of chords of M_n. The set C defined in the next lemma is the set of edges in $S(M_n)$ which are in two or more copies of K_3 in $[M_n]$. We will show that when M_n is the largest subgraph in a maximal length triangle sequence this is the same as a set of edges in $S(M_n)$ which are in two or more copies of K_3 in G. This lemma can then be used to limit the length of a triangle sequence because the number of chords of M_n is limited to one for $\ell = 2$ and to two for $\ell = 1$ by the $(2, \ell)$-sparsity of G.

We use the notation $\cup_{i=1}^m A_i$ to denote $A_1 \cup A_2, \ldots, \cup A_m$ where A_i are sets or graphs.

Lemma 3.8. *Let G be graph and let M_n be a subgraph in a triangle sequence in G with $n > 4$. Let $e_1, \ldots, e_m$ for $m > 0$ be chords of M_n, let $C_i = \{f \in S(M_n) : \exists g \in E([M_n]) \text{ such that } K_3(e_i, f, g) \subset G\}$. Then $|C| \leq 3m$ where $C = \cup_{i=1}^m C_i$.*

Proof. Assume for induction that the lemma is true for all possible choices of $m - 1$ chords of M_n and suppose that $e_1, \ldots, e_m$ are a set of m chords of M_n.

Suppose the chords $e_1, \ldots, e_m$ determine a graph with t distinct vertices and c connected components. Then $t \leq m + c$. Since the edges in $S(M_n)$ form a spanning cycle of M_n each vertex of a chord e_i is incident to two edges in $S(M_n)$. This implies $|C| \leq 2(m + c)$ which implies $|C| \leq 3m$ unless $c > m/2$. We may assume therefore that there is at least one component with exactly one edge which we label as the edge e_m where e_m has no vertices in common with $e_i, i = 1, \ldots, m - 1$.

Let $e_m = ab$ with $a, b \in V(M_n)$. Each of the vertices a, b is incident to exactly two edges in $S(M_n)$ which we label $aa_l, aa_r, bb_l, bb_r \in S(M_n)$. These edges are all distinct because $ab \notin S(M_n)$. Since the edges in $S(M_n)$ form a cycle we may label the vertices so that there is a (possibly trivial) path $P(a_l, b_l) \in S(M_n)$ which connects a_l, b_l and avoids a, b, a_r, b_r and then $a_r \neq b_l$ and $a_l \neq b_r$, see Figure 7. We may also label so that $a_r \neq b_r$ since if $a_l = b_l$ and $a_r = b_r$ the edges aa_l, a_lb, ba_r, a_ra form a 4-cycle in $S(M_n)$ which contradicts $n > 4$.
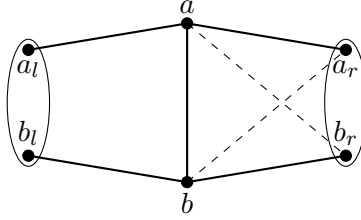

FIGURE 7. A chord ab of M_n with $ab \in V(M_n)$ and adjacent to edges $aa_l, aa_r, bb_l, bb_r \in S(M_n)$. Edges ab_r and ba_r cannot both be in $E(M_n)$ because the vertex pair b, a_r must then separate M_n.

Any edge $f \in S(M_n)$ which is in a 3-cycle with ab has a vertex in common with ab. Given edges e, f there is at most one 3-cycle in G which includes e, f. Thus we have shown $|C_m| \leq 4$. Furthermore, if $|C_m| = 4$ the vertex triples a, b, a_r and a, b, b_r must both induce 3-cycles in $[M_n]$. This implies $ba_r, ab_r \in E(M_n)$ because edges in $e_1, \ldots, e_{m-1}$ have no vertices in common with ab. This contradicts Lemma 3.6 part (3) for the vertex pair b, a_r because the neighbours of b in $S(M_n)$ are b_l, b_r and there would be a path

$b_r a, aa_l, P(a_l, b_l)$ which connects b_r and b_l and excludes b, a_r. Thus $|C_m| \leq 3$ which combines with the induction hypothesis $\cup_{i=1}^{m-1} C_i \leq 3(m-1)$ to give $\cup_{i=1}^{m} C_i \leq 3m$. $\qquad \square$

Lemma 3.9. *Let G be a $(2, \ell)$-tight graph for $\ell = 1, 2$ with the property that every edge ab in a $K_3(a, b, c) \subset G$ is in at least two copies of K_3 in G. Then every copy of K_3 in G is in a copy of K_4.*

Proof. We will show first that every maximal length triangle sequence in G with largest graph M_n satisfies $n \leq 6$ for $\ell = 1$ and $n \leq 4$ for $\ell = 2$.

Since every edge $ab \in S(M_n)$ is in exactly one copy of K_3 in M_n there is a vertex $c \in V(G)$ such that $K_3(a, b, c) \subset G$ and $K_3(a, b, c) \not\subset M_n$. This implies that $c \in V(M_n)$ because otherwise the triangle sequence can be extended with vertex c. Since $K_3(a, b, c) \not\subset M_n$ either ac or bc is a chord of M_n. Every edge in $S(M_n)$ must therefore be in the set C defined in Lemma 3.8 and if $n > 4$ by Lemma 3.8 we have $n = |C| \leq 3m$ where m is the number of chords of M_n in G. $f(M_n \cup_{i=1}^{m} e_i) = 3 - m$ because $f(M_n) = 3$ so $f(M_n \cup_{i=1}^{m} e_i) \geq \ell$ implies $m \leq 3 - \ell$ and $n \leq 3(3 - \ell)$. These imply $n \leq 4$ for $\ell = 2$ and $n \leq 6$ for $\ell = 1$.

For $n = 4$ there is a unique largest graph M_4 and a unique edge from $E([M_n]) \setminus E(M_n)$ which can be added to the graph M_4 so that every edge of $S(M_4)$ is in two copies of K_3. This creates a copy of K_4.

An analysis of the subgraphs induced by the vertices of maximal length triangle sequences M_n with $n \leq 6$ and with the property that every edge in $E(M_n)$ is in two copies of K_3 in G shows that for $l = 1$, $[M_5] = K_5 \setminus e$ or $[M_6] = K_4 \sqcup K_4$. Since every K_3 is in a maximal length triangle sequence and every K_3 in K_4, $K_5 \setminus e$ or $K_4 \cup K_4$ is in a copy of K_4 the proof is complete. $\qquad \square$

Lemma 3.10. *Let G be a $(2, \ell)$-tight graph for $l = 1, 2$ which contains a copy of K_3. Then either $G = K_4$, G has an allowable K_3-to-edge move, an allowable K_4-to-vertex move or every copy of K_3 is in a copy of $K_4 \sqcup K_4$ or $K_5 \setminus e$.*

Proof. Let the copy of K_3 be $K_3(a, b, c)$ and assume G has no allowable K_3-to-edge move or K_4-to-vertex move. By Lemmas 3.7 and 3.9 $K_3(a, b, c)$ is in a $K_4(a, b, c, d)$. Since this does not give an allowable K_4-to-vertex move, by Lemma 3.3 there is a $K_3(c, d, e)$ (say) with a, b, c, d, e all distinct and again by Lemma 3.9 there is a $K_4(c, d, e, g)$. If a, b, c, d, e, g are distinct then $K_3(a, b, c)$ is in a copy of $K_4 \sqcup K_4$ and if $g = a$ or b then $K_3(a, b, c)$ is in a copy of $K_5 \setminus e$. $\qquad \square$

We combine the lemmas in this section to show that all suitable $(2, \ell)$-tight graphs have an allowable reduction move.

Lemma 3.11. *Let G be $(2, 2)$-tight. Then $G = K_4$ or G has an inverse Henneberg 1 move, an inverse Henneberg 2 move, an allowable K_3-to-edge move or an allowable K_4-to-vertex move.*

Proof. Assume G has no inverse Henneberg 1 move and no inverse Henneberg 2. By Lemma 3.2 G has a copy of K_4 and thus a copy of K_3. The proof is completed by Lemma 3.10 since neither $K_4 \sqcup K_4$ nor $K_5 \setminus e$ is $(2, 2)$-sparse. $\square$

Lemma 3.12. *Let G be $(2, 1)$-tight. Then $G = K_4 \sqcup K_4$ or $G = K_5 \setminus e$ or G has an inverse Henneberg 1 move, an inverse Henneberg 2 move, an allowable K_3-to-edge move, an allowable K_4-to-vertex move or an edge separation move.*

Proof. Assume G has no inverse Henneberg 1 move, no inverse Henneberg 2 move, no allowable K_3-to-edge move and no allowable K_4-to-vertex move. By Lemma 3.2 each of the degree-3 vertices in G is in a copy of K_4 and thus in a copy of K_3. By Lemma 3.10 each of these K_3 is in a copy of $K_4 \sqcup K_4$ or $K_5 \setminus e$.

Let $Y = \{Y_1, \ldots, Y_n\}$ be the set of subgraphs of G which are each copies of $K_4 \sqcup K_4$ or $K_5 \setminus e$.

The subgraphs $Y_i \in Y$ are vertex disjoint since

$$f(Y_i \cup Y_j) = f(Y_i) + f(Y_j) - f(Y_i \cap Y_j) = 2 - f(Y_i \cap Y_j)$$

and $(2, 1)$-sparsity requires $f(Y_i \cap Y_j) \leq 1$. Every proper subgraph X of $K_4 \sqcup K_4$ or $K_5 \setminus e$ has $f(X) \geq 2$ so this requires Y_i and Y_j to be vertex disjoint.

Let V_0 and E_0 be the sets of vertices and edges in G which are in none of the $Y_i \in Y$. Then

$$f(G) = \sum_{i=1}^{n} f(Y_i) + 2|V_0| - |E_0|$$

so $|E_0| = 2|V_0| + n - 1$. Each of the vertices in V_0 is incident to at least 4 edges in E_0. If each Y_i is incident to at least 2 edges in E_0 then $|E_0| \geq (4|V_0| + 2n)/2$ for a contradiction.

At least one of the Y_i is incident to at most one edge in E_0. If this Y_i is incident to no edges in E_0 then $G = K_4 \sqcup K_4$ or $G = K_5 \setminus e$ since G is connected. Otherwise Y_i is incident to one edge $e \in E_0$ and e provides an edge separation move. $\square$

Using the above lemmas we reach the stated goal of this section.

Proof of (1) $\Rightarrow$ (2) in Theorem 1.4 or Theorem 1.5. By induction using Lemma 3.11 or Lemma 3.12. $\square$

4. Further Work

We expect to be able to use Theorem 1.4 to prove the following conjecture discussed in the introduction.[1]

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[1]Since submitting this paper we, in collaboration with Stephen Power, have solved the conjecture in the case where $\mathcal{M}$ is irreducible [19].

Conjecture 4.1. *Let $\mathcal{M}$ be a cone, a torus, a union of concentric cones or a union of concentric tori and let p be generic. Then (G, p) is generically minimally rigid on $\mathcal{M}$ if and only if $G = K_2, K_3, K_4$ or G is $(2, 1)$-tight.*

It would also be interesting to consider surfaces that do not admit any rigid-body motions. For such surfaces there are immediate additional problems. For example Equation 1 with $\ell = 0$ shows that the minimum degree in a $(2, 0)$-tight graph may be 4 so additional Henneberg type operations are required. This actually provides additional motivation for studying these graphs since the obvious choices to take are X and V-replacement as studied by Tay and Whiteley [23] in the 3-dimensional setting. Indeed they conjecture that these operations (with additional conditions for V-replacement) preserve rigidity in 3-dimensions.

It is also interesting to note that the d-dimensional version of the edge-to-K_3 move, known in the literature as vertex splitting [25], is one of a very short list of operations known to preserve rigidity in arbitrary dimension. Nevertheless there is no conjectured inductive construction, even in 3-dimensions, that makes use of this. We hope that our methods for dealing with the edge-to-K_3 move for $(2, \ell)$-tight graphs may be useful in finding such a construction.

There are more exotic settings in which the class of $(2, 1)$-tight graphs are the appropriate combinatorial tool needed to classify generic minimal rigidity. For example we could take $\mathcal{M}$ to be two parallel (but not concentric) cylinders. Here there is only one rigid-body motion of $\mathcal{M}$ in $\mathbb{R}^3$, or we may take $\mathcal{N}$ to be a cylinder coaxial to a cone. Again there is only one rigid-body motion (this time a rotation about the central axis). In such reducible settings there is a little more work to do to in considering which surface each framework point lies on. This extra requirement is particularly evident for $\mathcal{N}$, but in either case a $(2, 1)$-tight subgraph realised purely on one cylinder would be overconstrained.

A similar but deeper topic is the problem of when a framework realisation is unique (this is the topic of global rigidity, see for example [3, 11]). To characterise the global rigidity of frameworks supported on an algebraic surface one of the key steps is to analyse the circuits of the rigidity matroid $\mathcal{R}_{\mathcal{M}}$ (this is the linear matroid induced by the linear independence of the rows of the surface rigidity matrix). Since the independent sets in $\mathcal{R}_{\mathcal{M}}$, for $\mathcal{M}$ a cylinder, may be identified with the $(2, 2)$-tight graphs ([18, Theorem 5.4]), the circuits may be identified with a sub-class of the $(2, 1)$-tight graphs. Such a recursive construction is given in [17] and finding a similar construction for circuits in the $(2, 1)$-tight matroid is open.

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