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Facets of the axial three-index assignment polytope

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Abstract

We revisit the facial structure of the axial 3-index assignment polytope. After reviewing known classes of facet-defining inequalities, we present a new class of valid inequalities, and show that they are facets of this polytope. This answers a question posed by Qi and Sun [14]. Moreover, we show that we can separate these inequalities in polynomial time.

Keywords: multi-dimensional assignment; polyhedral methods; facets; separation algorithm;

1. Introduction

The axial 3-index (or 3-dimensional) assignment problem can be described as follows. Given are three disjoint n-sets I, J, K and a weight function $w : I \times J \times K \longrightarrow \mathbb{R}$. The problem is to select a collection of triples $M \subseteq I \times J \times K$ such that each element of each set appears in exactly one triple, and such that total weight of the selected triples is minimized (or maximized). Its formulation as an Integer Linear Program (ILP) is:

$$\min \quad \sum_{i\in I}\sum_{j\in J}\sum_{k\in K} w_{ijk}x_{ijk}$$

$$\text{s.t.} \quad \sum_{j\in J}\sum_{k\in K} x_{ijk} = 1 \qquad \forall i \in I, \tag{1.1}$$

$$\sum_{i\in I}\sum_{k\in K} x_{ijk} = 1 \qquad \forall j \in J, \tag{1.2}$$

$$\sum_{i\in I}\sum_{j\in J} x_{ijk} = 1 \qquad \forall k \in K, \tag{1.3}$$

$$x_{ijk} \in \{0,1\} \qquad \forall i \in I, j \in J, k \in K. \tag{1.4}$$

This problem has many applications; we restrict ourselves here to mentioning that optimization problems in data-association, production, and logistics can often be modeled as 3-index assignment problem; we refer to Spieksma [15] for an overview. In this work

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we contribute to the polyhedral knowledge of the facial structure of the convex hull of the feasible solutions to (1.1)-(1.4). In particular, we give a new class of facet-defining inequalities, and we show that this class can be separated in polynomial time. We also describe known classes of facets by adopting a geometrical point of view, i.e., we organize the variables x_{ijk} in a three-dimensional array (a cube), thereby illustrating the differences between distinct classes of inequalities.

1.1. Literature

It is well-known that, as opposed to the polytope that corresponds to the two-dimensional assignment problem, not all extreme vertices of the polytope corresponding to (1.1)-(1.4) are integral. In fact, different types of fractional vertices exist; work on this topic is reported in Kravtsov [10]. Early work investigating the facial structure of the polytope P_I is described in Balas and Saltzman [4] and Euler [8]. They give different classes of facet-defining inequalities (see Section 2). Subsequently, other classes of facet-defining inequalities are reported in Qi and Balas [12] (see also Qi, Balas and Gwan [13]). Separation algorithms are discussed in Balas and Qi [3]. A nice overview of existing polyhedral results is given in Qi and Sun [14]. This paper also contains the question: "Are there other facet classes such that the right hand sides of their defining inequalities are 2?", to which we provide an (affirmative) answer here. An exact algorithm based on known valid inequalities that are used in conjunction with Lagrangian multipliers is given in Balas and Saltzman [5].

A related polytope is the one that corresponds to the so-called *planar* three-index assignment problem; this is the problem that arises when a collection of triples needs to be selected such that each *pair* of elements from $(I \times J) \cup (I \times K) \cup (J \times K)$ is selected precisely once. The facial structure of this polytope has first been studied in Euler et al. [7]. Also, polytopes that correspond to four-index assignment problems have been studied, see Appa et al. [1]. Recent results that unify these polyhedral results for all multi-index assignment polytopes can be found in Appa et al. [2].

1.2. Preliminaries

To avoid trivialities we assume $n \geq 4$. Let A^n denote the $(0,1)$ matrix corresponding to the constraints (1.1) - (1.3). Thus A^n has n^3 columns (one for each variable) and $3n$ rows (one for each constraint). Then, the 3-index assignment polytope is the following object:

$$P_I^n = \text{conv}\{x \in \{0,1\}^{n^3} : A^n x = \mathbf{1}\},$$

while its linear programming (LP) relaxation is described as:

$$P^n = \{x \in R^{n^3} : A^n x = \mathbf{1}, x \geq 0\}.$$

For reasons of convenience, we will often omit the superscript n, and use A, P_I and P instead. We use $R \equiv (I \cup J \cup K)$; elements of R are called *indices*. We also use $V \equiv I \times J \times K$; elements of V are called *triples*. Given a triple $(i,j,k) \in V$, we refer to i, j and k as *first*, *second*, and *third* indices respectively.

An important object is the so-called *column intersection* graph corresponding to A^n. This graph $G(V,E)$, has a node for each column of A^n (i.e., a node for each triple) and an edge for every pair of columns that have a $+1$ entry in the same row. Notice that each

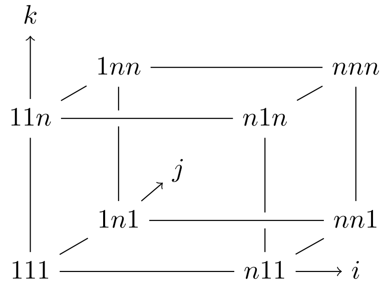

Figure 1: The arrangement of the x_{ijk} variables in a three-dimensional cube.

column of A^n contains three +1's. The intersection of two columns c and d is nothing else but the number of indices that the triples c and d have in common; this number is denoted by $|c \cap d|$. Thus, the edge set E of the column intersection graph is given by $E = \{(c, d) : \{c, d\} \subseteq V, |c \cap d| \geq 1\}$, i.e., two nodes are connected iff the corresponding triples share some index. We call two triples *disjoint* if the corresponding nodes are not connected in G. Clearly, *cliques* (a complete subgraph of G) and *odd cycles* (a cycle consisting of an odd number of vertices in G) are relevant structures. Indeed, it is clear that when given a set of variables that correspond to nodes that form a clique in G, at most one of these variables can equal 1. In other words, a clique in G corresponds to a valid inequality for P^n with righthand side 1, see Balas and Saltzman [4]. Also, a set of variables that correspond to an odd cycle in G gives rise to a valid inequality, see e.g. Euler [8].

In this work, we use well-known concepts from polyhedral theory; for a thorough introduction into this field we refer to Nemhauser and Wolsey [11].

We will adopt a geometrical point of view to illustrate the valid inequalities. To do so, we see the variables x_{ijk} arranged as in a cube, see Figure 1.

We find it convenient to have a symbol for the set of all x-variables that share two indices. More concrete, we define the following sets.

- For a given $(j^*, k^*) \in J \times K$: the set

$$(-, j^*, k^*) \equiv \{(i, j, k) \in V : \; j = j^*, k = k^*\}.$$

We use $x(-, j^*, k^*)$ to denote the total weight of the corresponding variables.

- For a given $(i^*, k^*) \in I \times K$, the set

$$(i^*, -, k^*) \equiv \{(i, j, k) \in V : \; i = i^*, k = k^*\}.$$

We use $x(i^*, -, k^*)$ to denote the total weight of the corresponding variables.

- For a given $(i^*, j^*) \in J \times K$, the set

$$(i^*, j^*, -) \equiv \{(i, j, k) \in V : \; i = i^*, j = j^*\}.$$

We use $x(i^*, j^*, -)$ to denote the total weight of the corresponding variables.

Geometrically, such a set of variables corresponds to an "axis" through the cube depicted in Figure. Further, we write $x(A)$ for $\sum_{q \in A} x_q$.

In the next section we review the known classes of facet-defining inequalities of P_I.

2. A review of known facet classes of P_I

In this section, we review the known facet classes of P_I. There are two classes of facet-defining inequalities with right-hand side (RHS) 1 (Subsection 2.1), and we distinguish four classes of facet-defining inequalities with right-hand side 2 (Subsection 2.2). Subsection 2.3 deals with other facet-defining inequalities.

2.1. Facet-defining inequalities with RHS 1

As described in Subsection 1.2, a clique in the column intersection graph gives rise to a valid inequality. Balas and Saltzman [4] showed that there exist three types of cliques in $G(V, E)$, and two of them give rise to families of valid inequalities that are facet-defining for P_I. It is known that each of these classes can be separated in $O(n^3)$ (see Balas and Qi [3]).

2.1.1. Clique inequalities of type I

Consider a triple $c = (i_c, j_c, k_c) \in V$. For each $c \in V$, define

$$Q(c) = \{(i, j, k) \in V : \ i = i_c, j = j_c \text{ or } i = i_c, k = k_c \text{ or } j = j_c, k = k_c\}.$$

Thus, $Q(c)$ is the set of triples sharing at least two indices with triple c. The corresponding inequalities are clearly valid. For each $c \in V$:

$$x(Q(c)) \leq 1. \tag{2.5}$$

Fact 1. *([4]) Inequalities (2.5) define facets of P_I; these inequalities are called* clique inequalities of type I.

When we organize the variables x_{ijk} in a three-dimensional array (a cube), a clique inequality of type I can be seen as the sum of those x-variables that lie on the three "axes" through a particular cell. Indeed an alternative way of expressing $Q(c)$ is by observing that

$$Q(c) = (-, j_c, k_c) \cup (i_c, -, k_c) \cup (-, j_c, k_c),$$

see Figure 2.

2.1.2. Clique inequalities of type II

Consider two disjoint triples $c = (i_c, j_c, k_c) \in V$ and $d = (i_d, j_d, k_d) \in V$. For each such pair of triples $c, d \in V$, define

$$Q(c, d) = \{(i_c, j_c, k_c), (i_c, j_d, k_d), (i_d, j_c, k_d), (i_d, j_d, k_c)\}.$$

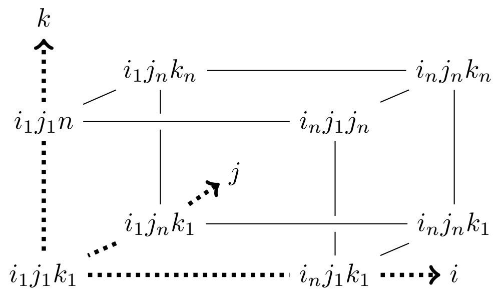

Figure 2: Geometric illustration of a clique inequality of type I; the three dotted axes correspond to the variables in this inequality.

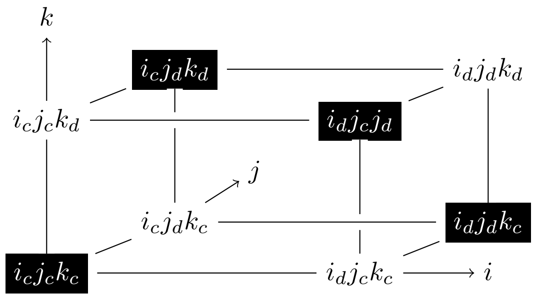

Figure 3: Geometric illustration of a clique inequality of type II; the four highlighted cells correspond to the four variables in this inequality.

Thus, $Q(c, d)$ is the set of triples that has two indices in common with d, and one with c, together with triple c; notice that $Q(c, d)$ contains exactly four triples. The corresponding inequalities are clearly valid. For each disjoint pair $c, d \in V$:

$$x(Q(c, d)) \leq 1. \tag{2.6}$$

Fact 2. *([4]) Inequalities (2.6) define facets of P_I; these inequalities are called* clique inequalities of type II*.*

2.2. Facet-defining inequalities with RHS 2

There are four classes known of facet-defining inequalities with right-hand side 2; these classes are members of larger classes of facet-defining inequalities that have arbitrary right-hand sides (see Qi and Sun [14] for a nice overview). Below we describe each of these classes restricted to right-hand side 2. It is shown in [14] that each of these four classes can be separated in $O(n^3)$ time.

2.2.1. *Lifted 5-hole inequalities*

Balas and Saltzman [4] describe a class of facet-defining inequalities that correspond to cycles of odd length in G; this class can have an arbitrary right-hand side. Here, we restrict ourselves to describing those inequalities that have right-hand side 2, and we will refer to them as *lifted 5-hole* inequalities. Let U consist of two elements of I, two elements of J, and a single element of K, i.e., $U = \{i_1, i_2, j_1, j_2, k_1\} \subset R$. Of course, the roles of I, J, K in the definition of U can be interchanged. For each such $U \subset R$, define

$$S(U) = \{(i,j,k) \in V : \ |(i,j,k) \cap \{i_1, i_2, j_1, j_2, k_1\}| \geq 2\}.$$

Thus, $S(U)$ contains the triples that have at least two indices in common with $U = \{i_1, i_2, j_1, j_2, k_1\}$. The corresponding inequalities are valid. For each $U = \{i_1, i_2, j_1, j_2, k_1\} \subset R$:

$$x(S(U)) \leq 2. \tag{2.7}$$

Fact 3. *([4]) Inequalities (2.7) define facets of P_I; these inequalities are called* lifted 5-hole inequalities.

Informally, we can view the left-hand side of a lifted 5-hole inequality as the union of four (specific) clique inequalities of type I. Indeed, it is easily verified that $S(U) = Q(i_1, j_1, k_1) \cup Q(i_1, j_2, k_1) \cup Q(i_2, j_1, k_1) \cup Q(i_2, j_2, k_1)$, see Figure 4. Thus, informally said, a lifted 5-hole inequality consists of 8 axes. In fact, clique inequalities of type I, as well as the lifted 5-hole inequalities, can be seen as members of a larger class of facet-defining inequalities (called facet class Q in [14], see also [4]).

2.2.2. *$P(2)$ inequalities*

This class of inequalities was introduced by Qi and Balas [12] (see also Qi et al. [13]), and can be seen as a generalization of the clique inequalities of type II. Consider two disjoint sets of indices $U, W \subset R$. We define

$$C_1(U) \equiv \{(i,j,k) \in V : \ i,j,k \in U\}, \text{ and} \tag{2.8}$$

$$C_2(U,W) \equiv \{(i,j,k) \in V : |(i,j,k) \cap U| = 1, |(i,j,k) \cap W| = 2\}. \tag{2.9}$$

Thus, $C_1(U)$ consists of those triples whose indices are contained in U, while $C_2(U,W)$ contains triples that share precisely one index with U, and precisely two indices with W. We now apply definitions (2.8) and (2.9) to the following two choices of U and W. Here is a first choice:

$$U = \{i_1, i_2, j_1, j_2, k_1, k_2\}, W = \{i_3, j_3, k_3\}. \tag{2.10}$$

This leads to

$$\begin{aligned} C_1(U) &= \{(i_1,j_1,k_1), (i_1,j_1,k_2), (i_1,j_2,k_1), (i_1,j_2,k_2), \\ &\quad (i_2,j_1,k_1), (i_2,j_1,k_2), (i_2,j_2,k_1), (i_2,j_2,k_2)\}, \text{ and} \\ C_2(U,W) &= \{(i_1,j_3,k_3), (i_3,j_1,k_3), (i_3,j_3,k_1), (i_2,j_3,k_3), (i_3,j_2,k_3), (i_3,j_3,k_2)\}. \end{aligned}$$

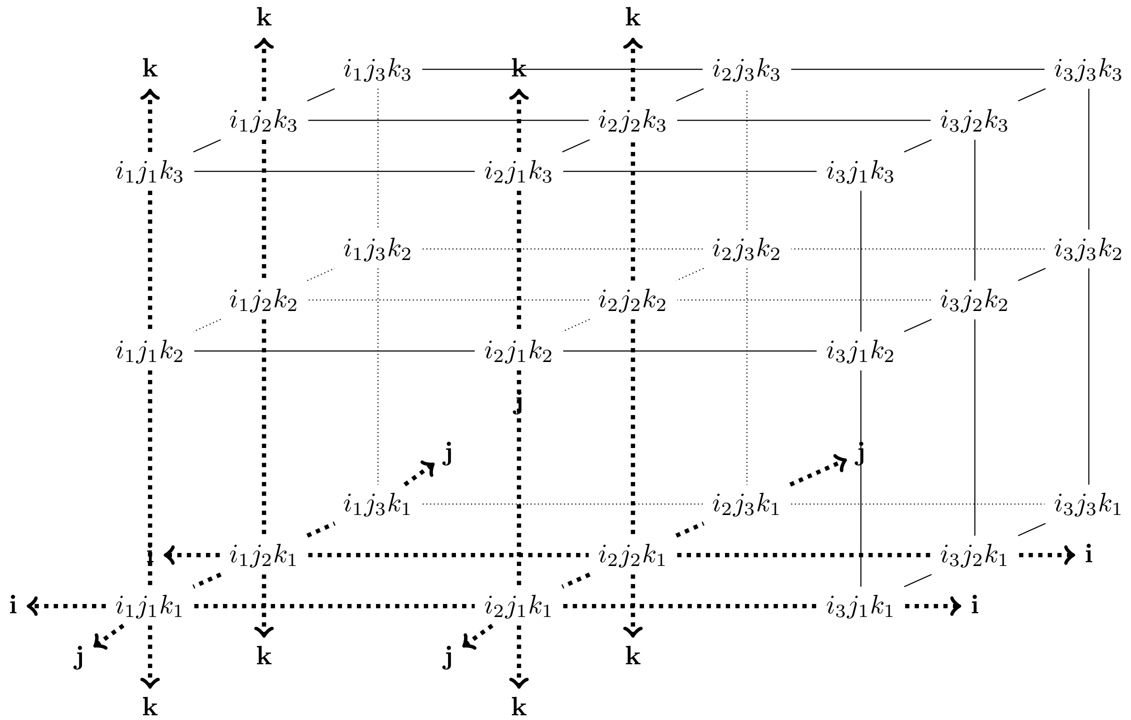

Figure 4: Geometric illustration of a lifted 5-hole inequality; the eight dotted axes correspond to the variables in this inequality.

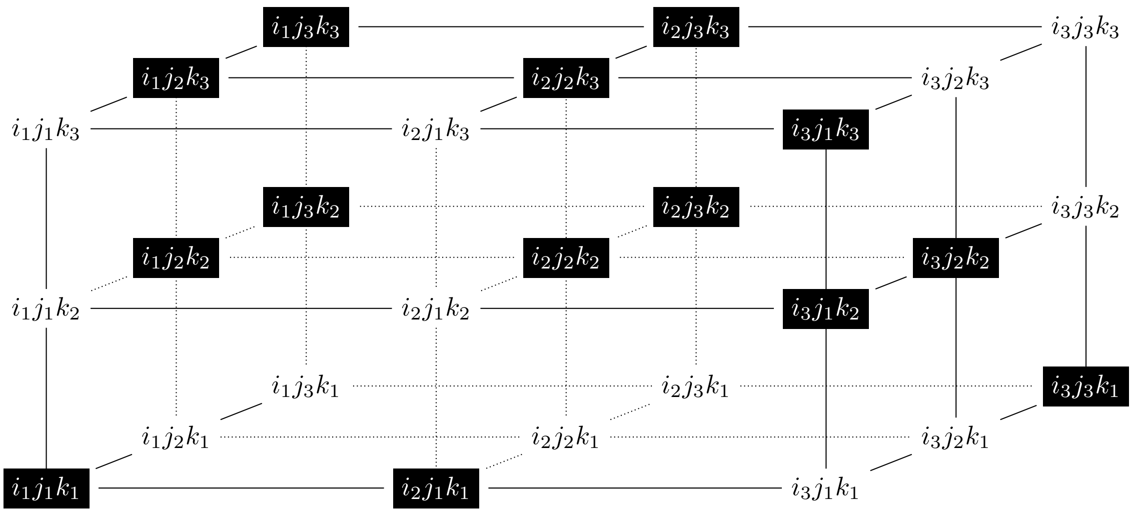

Figure 5: Geometric illustration of a $P(2)$ inequality; the fourteen highlighted cells correspond to the fourteen variables in this inequality.

And here is a second choice for the sets U, W:

$$U = \{i_1, i_2, j_1, k_1\}, W = \{i_3, j_2, j_3, k_2, k_3\}. \tag{2.11}$$

This leads to

$$\begin{aligned} C_1(U) &= \{(i_1, j_1, k_1), (i_2, j_1, k_1))\}, \text{ and} \\ C_2(U, W) &= \{(i_1, j_2, k_2), (i_1, j_2, k_3), (i_1, j_3, k_2), (i_1, j_3, k_3), (i_2, j_2, k_2), (i_2, j_2, k_3), \\ &\quad (i_2, j_3, k_2), (i_2, j_3, k_3), (i_3, j_1, k_2), (i_3, j_1, k_3), (i_3, j_2, k_2), (i_3, j_3, k_1)\}. \end{aligned}$$

The following inequalities are valid. For each disjoint pair of sets $U, W \subset R$ satisfying (2.10) or (2.11):

$$x(C_1(U)) + x(C_2(U, W)) \leq 2. \tag{2.12}$$

Fact 4. *([4]) Inequalities (2.12) define facets of P_I; these inequalities are called $P(2)$* inequalities.

Thus, an inequality of the class $P(2)$ consists of 14 cells, see Figure 5.

2.2.3. Bull inequalities

This class of inequalities was described in Gwan and Qi [9]. It is a class of inequalities with arbitrary right-hand side; here, we restrict our attention to the case where the right hand side equals 2. Notice that this class of inequalities contains variables whose coefficient has value 2.

Consider a single triple from V, say (i_1, j_1, k_1), and consider a set $U = \{i_2, j_2\}$ (with $i_1 \neq i_2, j_1 \neq j_2$); let us call $W = \{i_1, j_1, k_1\} \cup U$. Define

$$F(U) = \{(i,j,k) \in V : \ |(i,j,k) \cap W| \geq 2, 1 \leq |(i,j,k) \cap \{i_1, j_1, k_1\}| \leq 2\}\}.$$

Thus, $F(U)$ contains those triples that share at least two indices with W, and either one or two indices with $\{i_1, j_1, k_1\}$. The following inequalities are valid. For each $(i_1, j_1, k_1) \in V$ and $U \subset R$:

$$2x_{i_1,j_1,k_1} + x(F(U)) \leq 2. \tag{2.13}$$

Fact 5. *([9]) Inequalities (2.13) define facets of P_I; these inequalities are called* bull inequalities.

Notice that we can write

$$F(U) \cup (i_1, j_1, k_1) = \\ \{(i_1, j_1, -), (i_1, -, k_1), (-, j_1, k_1), (i_1, j_2, -), (i_2, j_1, -), (i_2, -, k_1), (-, j_2, k_1)\}.$$

Thus, a bull inequality consists of 7 axes and a single variable with coefficient 2, see Figure 6 for an illustration.

2.2.4. Comb inequalities

This class of inequalities was also described in Gwan and Qi [9]. Again, it is a class of inequalities with arbitrary right-hand side; here, we restrict our attention to the case where the right hand side equals 2.

Let $i_1, i_2, i_3 \in I$, $j_1, j_2, j_3 \in J$, $k_1, k_2, k_3 \in K$ be pairwise distinct indices in R, and let

$$U = \{(i_1, j_2, k_2), (i_1, j_3, k_3), (i_2, j_2, k_3), (i_2, j_3, k_2), (i_3, j_1, k_1), (i_3, j_2, k_2), (i_3, j_3, k_3)\}. \tag{2.14}$$

The following inequalities are valid. For each $(i_1, j_1, k_1) \in V$ and U satisfying (2.14):

$$x(U) + x[(i_1, j_1, -) \cup (i_1, -, k_1)] \leq 2. \tag{2.15}$$

Fact 6. *([9]) Inequalities (2.15) define facets of P_I; these inequalities are called* comb inequalities.

Thus, a comb inequality consists of 2 axes and 7 cells, see Figure 7 for an illustration.

2.3. Other facet-defining inequalities

Based on odd-cycles present in the column intersection graph G, Euler [8] described a class of facet-defining inequalities. Indeed, an odd cycle in G gives rise to a valid inequality, and, in some circumstances (see [8]), such a valid inequality can be lifted to

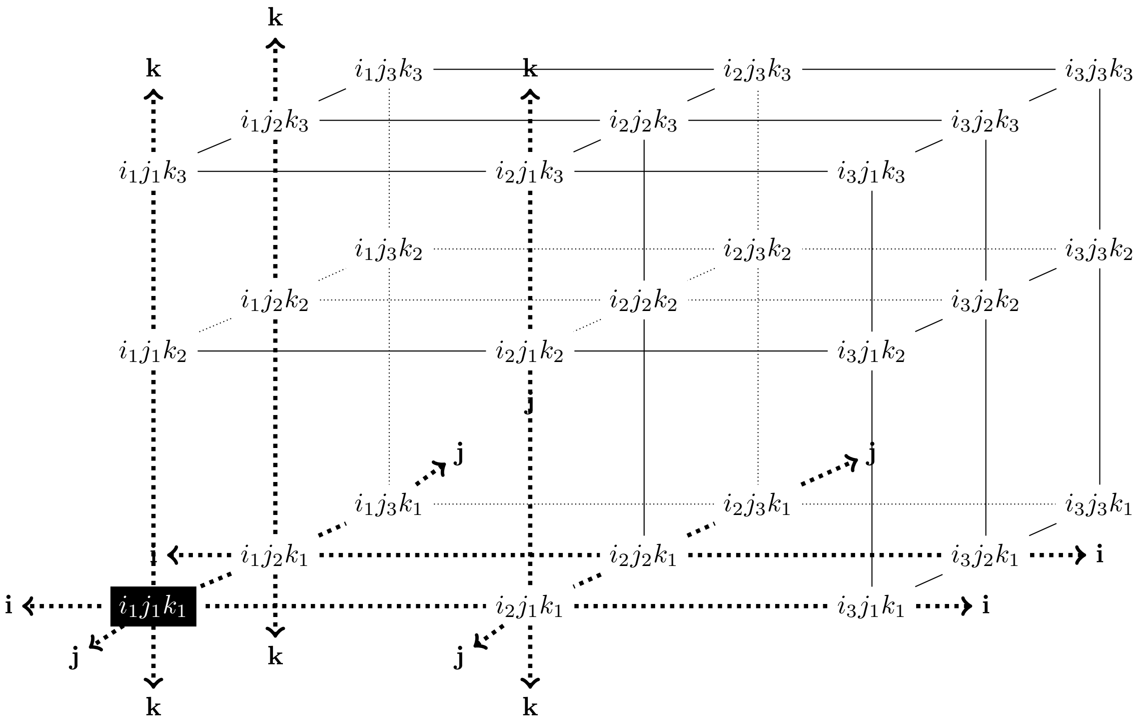

Figure 6: Geometric illustration of a bull inequality; the seven dotted axes correspond to the variables in this inequality, whereas the highlighted cell corresponds to the variable with coefficient 2.

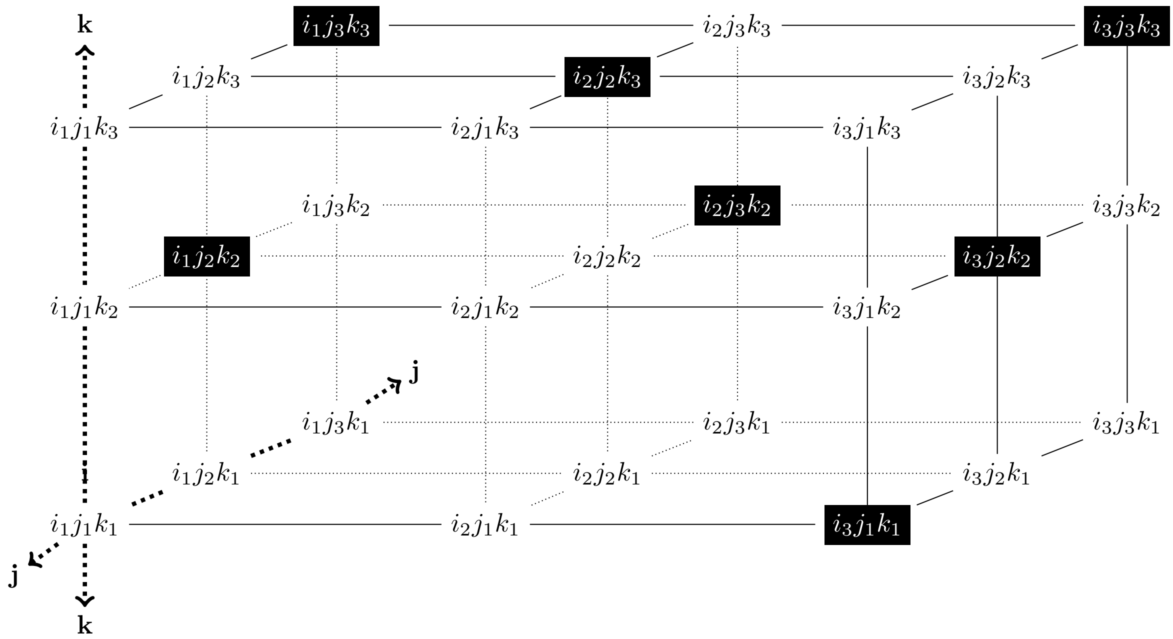

Figure 7: Geometric illustration of a comb inequality; the two dotted axes, and the seven highlighted cells, correspond to the variables in this inequality.

a facet-defining inequality. Although we refrain from giving a precise description of the resulting inequalities, we note here that the right-hand side of this class of inequalities equals $n-1$.

As far as we aware, the classes of inequalities that we covered in this section constitute all known facet-defining inequalities of the polytope P_I.

3. Wall Inequalities

3.1. A new class of valid inequalities

In this section we present a new class of valid inequalities that we call *wall* inequalities. We will prove in Section 3.2 that these inequalities define facets of P_I, thereby answering a question asked by [9].

Let $i_1, i_2, i_3 \in I$, $j_1, j_2, j_3 \in J$, $k_1, k_2 \in K$ be pairwise distinct indices in R. We define the following set of triples:

$$\begin{aligned} B =\{&(i_1,j_1,k_1),(i_1,j_2,k_2),(i_2,j_1,k_2),(i_2,j_2,k_1),\\ &(i_3,j_3,-),(i_3,-,k_1),(i_3,-,k_2),(-,j_3,k_1),(-,j_3,k_2)\}. \end{aligned} \tag{3.16}$$

Consider now the following inequalities. For each B satisfying (3.16):

$$x(B) \leq 2. \tag{3.17}$$

These inequalities are valid, as witnessed by the following lemma.

Lemma 7. *Inequalities (3.17) are valid.*

Proof. Inequalities (3.17) can be obtained by adding equations (1.1) with $i = i_3$, (1.2) with $j = j_3$ and (1.3) with $k = k_1, k_2$, and by adding a clique inequality of type II: $x(Q((i_2, j_2, k_1), (i_1, j_1, k_2))) \leq 1$. Next, integer rounding, i.e., dividing the resulting inequality by 2 and rounding down all coefficients to the nearest integers, gives a wall inequality. ■ □

We note that inequalities (3.17) can be written as

$$\begin{aligned} x(B) = x(Q(i_3,j_3,k_2)) &+ x(Q((i_1,j_1,k_2),(i_2,j_2,k_1)))\\ &+ x[(i_3,-,k_1)\cup(-,j_3,k_1)], \end{aligned} \tag{3.18}$$

where $Q(i_3, j_3, k_2)$ is the set of variables in a clique inequality of type I corresponding to triple (i_3, j_3, k_2) and $Q((i_1, j_1, k_2), (i_2, j_2, k_1))$ is the set of variables in a clique inequality of type II corresponding to triples $(i_1, j_1, k_2), (i_2, j_2, k_1)$. Thus, a wall inequality consists of five axes and four cells, see Figure 8 for an illustration.

3.2. Wall inequalities define facets of P_I

Here we prove the main theorem.

Theorem 8. *Inequalities (3.17) define facets of P_I.*

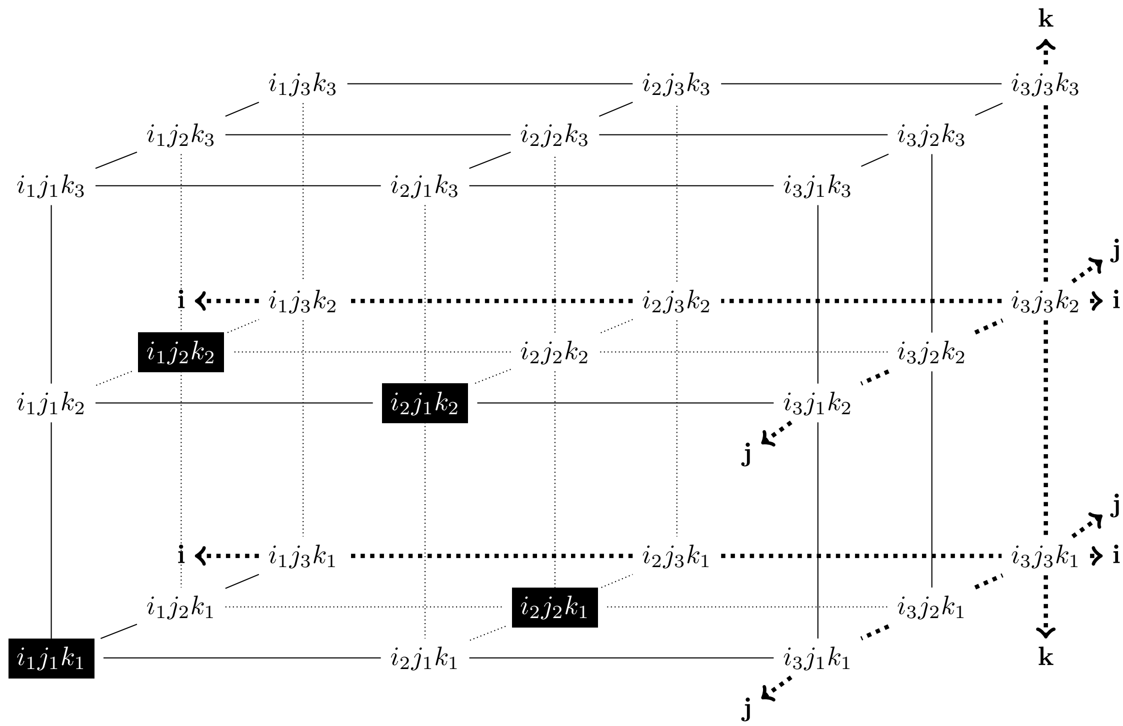

Figure 8: Geometric illustration of a wall inequality; the five dotted axes, and the four highlighted cells, correspond to the variables in this inequality.

Proof. Let us first explain the plan we follow in order to prove that $x(B) \leq 2$ defines a facet of P_I. An inequality defines a facet of P_I when it is satisfied by every $x \in P_I$ and the dimension of the polyhedron $P^B \equiv \{x \in P_I : x(B) = 2\}$ is equal to the dimension of $P_I - 1$ (see [11]). To prove that this is the case we will show that

- an inequality from (3.17) does not define an improper face, and
- adding $x(B) = 2$ to the constraints defining P_I increases the rank of the equality system of P_I by exactly one.

The latter statement means that any equation that is satisfied by all $x \in P^B$, is a linear combination of the equations in the system defining P^B. Since the dimension of the polyhedron P is equal to the number of variables in the system defining P minus rank of the equality system of P, proving the second point above implies $\dim(P^B) = \dim(P_I) - 1$.

To prove that an inequality from (3.17) does not induce an improper face, we need to exhibit a feasible solution with $x(B) \leq 1$. Here is such a feasible solution: $x_{i_3+\ell, j_3+\ell, k_2+\ell} = 1$ for $\ell = 0, \ldots, n-1$ (indices should be read modulo n; the values of the indices i_3, j_3, k_2 follow from the specific wall inequality under consideration).

To show that an inequality from (3.17) defines a facet of P_I i.e., that $\dim(P^B) = \dim(P_I) - 1$, we use the same approach as used in [4] and [9]. Namely, we exhibit scalars $\lambda_i, i \in I, \mu_j, j \in J, \nu_k, k \in K$ and a scalar π such that if $\alpha x = \alpha_0$ for all $x \in P^B$, then the scalars λ_i, μ_j, ν_k, and π satisfy:

$$\alpha_{ijk} = \lambda_i + \mu_j + \nu_k \quad \text{if} \quad (i,j,k) \in V \backslash B, \tag{3.19}$$
$$\alpha_{ijk} = \lambda_i + \mu_j + \nu_k + \pi \quad \text{if} \quad (i,j,k) \in B, \text{ and} \tag{3.20}$$
$$\alpha_0 = \sum_{i \in I} \lambda_i + \sum_{j \in J} \mu_j + \sum_{k \in K} \nu_k + 2\pi. \tag{3.21}$$

To prove (3.19) and (3.20), we repeatedly apply the following *interchange procedure.*

1. Consider a solution $x \in P_I$ containing two disjoint triples (i,j,k) and (a,b,c), i.e., we have $x_{ijk} = x_{abc} = 1$.
2. Construct a solution $\bar{x}$ from x by interchanging the first index in the two selected triples (i,j,k) and (a,b,c): $\bar{x}_{ajk} = \bar{x}_{ibc} = 1$. Observe that $\bar{x} \in P_I$.
3. Deduce the value of α_{ijk} from (i) and (ii) using $\alpha x = \alpha \bar{x}$, which now implies $\alpha_{ijk} = \alpha_{ajk} + \alpha_{ibc} - \alpha_{abc}$.

The above procedure describes a *first index* interchange; clearly, a similar procedure exists involving a second and third index interchange. Without of loss of generality let us assume that $i_1 = 1, i_2 = 2, i_3 = 3, j_1 = 1, j_2 = 2, j_3 = 3, k_2 = 2, k_3 = 3$.

We define for all $i \in I$, $j \in J$ and $k \in K$:

$$\lambda_i = \alpha_{inn} - \alpha_{nnn}, \tag{3.22}$$
$$\mu_j = \alpha_{njn} - \alpha_{nnn}, \text{ and} \tag{3.23}$$
$$\nu_k = \alpha_{nnk}. \tag{3.24}$$

Then, in order to prove (3.19), we need to prove for $(i, j, k) \in V \backslash B$

$$\alpha_{ijk} = \lambda_i + \mu_j + \nu_k = \alpha_{inn} + \alpha_{njn} + \alpha_{nnk} - 2\alpha_{nnn} \tag{3.25}$$

In the following, when we illustrate a solution $x \in P_I$, we only write those variables in the set B that take positive values.

We first deduce four equations which we will use in proving (3.25) for each $(i, j, k) \notin B$. Consider a solution $x \in P^B$ such that $x_{nnn} = x_{i33} = 1$. Using a first index interchange, we obtain $\bar{x} \in P^B$ with $\bar{x}_{inn} = \bar{x}_{n33} = 1$. Using $\alpha x = \alpha \bar{x}$ we have

$$\alpha_{nnn} + \alpha_{i33} = \alpha_{inn} + \alpha_{n33}. \tag{3.26}$$

Note that (3.26) is true for every $i \in I$.

Consider a solution $x \in P^B$ such that $x_{nnn} = x_{3j3} = 1$. Using a second index interchange, we obtain $\bar{x} \in P^B$ with $\bar{x}_{njn} = \bar{x}_{3n3} = 1$. Therefore,

$$\alpha_{3n3} = \alpha_{nnn} + \alpha_{3j3} - \alpha_{njn}. \tag{3.27}$$

Note that this is true for every $j \in J$.

Again, consider a solution $x \in P^B$ such that $x_{nnn} = x_{3j2} = 1$. Using a second index interchange, we obtain $\bar{x} \in P^B$ with $\bar{x}_{njn} = \bar{x}_{3n2} = 1$. Therefore,

$$\alpha_{3n2} = \alpha_{nnn} + \alpha_{3j2} - \alpha_{njn}. \tag{3.28}$$

Note that this is true for every $j \in J$.

Now, consider a solution $x \in P^B$ such that $x_{nnn} = x_{33k} = 1$. Using a third index interchange, we obtain $\bar{x} \in P^B$ with $\bar{x}_{nnk} = \bar{x}_{33n} = 1$. Therefore,

$$\alpha_{33n} = \alpha_{nnn} + \alpha_{33k} - \alpha_{nnk}. \tag{3.29}$$

Observe that (3.29) is true for all $k \in K$.

3.2.1. Proving (3.19)

If at least two indices of i, j, k are equal to n then it is easy to see that (3.25) holds, and hence (3.19) follows. Below we consider the cases when at least two indices of i, j, k are not equal to n.

Case 1: when $i = n$, $j \neq n$ and $k \neq n$
Substituting $i = n$ in (3.25), implies that we need to show the following:

$$\alpha_{njk} = \alpha_{njn} + \alpha_{nnk} - \alpha_{nnn}. \tag{3.30}$$

We consider all possible cases of j and k as follows. We explain in detail the three steps in the interchange procedure mentioned above for the case when $j = 1$, $k \neq 2$. For other possible values of j and k such that $(n, j, k) \notin B$ we omit the complete details in proving

(3.25); instead we give the start solution, the type of index interchange, and the new solution in Table 1.

Let $x \in P^B$ be such that $x_{n1k} = x_{33n} = x_{222} = 1$. Using a third index interchange we obtain $\bar{x} \in P^B$ such that $\bar{x}_{n1n} = \bar{x}_{33k} = \bar{x}_{222} = 1$. By $\alpha x = \alpha \bar{x}$ we have:

$$\alpha_{n1k} + \alpha_{33n} = \alpha_{n1n} + \alpha_{33k}.$$

Substituting the value of α_{33n} from (3.29) we get the required equality:

$$\alpha_{n1k} = \alpha_{nnk} + \alpha_{n1n} - \alpha_{nnn}.$$

In the column 'remarks' of Table 1, we mention the equality used (e.g., (3.29) in the above case) in deducing the expression for α_{ijk}. Notice that when $i = n$, $j = 3$ and $k \in \{2, 3\}$, $(i, j, k) \in B$, and we need to prove (3.20).

case	start sol.	interchange type	new sol.	remarks
$j \in \{1, 2, 3\}$				
$j = 1$, $k \neq 2$	$x_{n1k}, x_{33n}, x_{222}$	3	$\bar{x}_{n1n}, \bar{x}_{33k}, \bar{x}_{222}$	(3.29)
$j = 1, k = 2$	$x_{n12}, x_{33n}, x_{123}$	3	$\bar{x}_{n1n}, \bar{x}_{332}, \bar{x}_{123}$	(3.29)
$j = 2$, $k \neq 2$	$x_{n2k}, x_{33n}, x_{112}$	3	$\bar{x}_{n2n}, \bar{x}_{33k}, \bar{x}_{112}$	(3.29)
$j = 2$, $k = 2$	$x_{n22}, x_{33n}, x_{213}$	3	$\bar{x}_{n2n}, \bar{x}_{332}, \bar{x}_{213}$	(3.29)
$j = 3$, $k \in \{2, 3\}$				$(i, j, k) \in B$
$j = 3$, $k \notin \{2, 3\}$	$x_{n3k}, x_{3n3}, x_{112}$	2	$\bar{x}_{nnk}, \bar{x}_{333}, \bar{x}_{112}$	(3.27)
$j \notin \{1, 2, 3\}$				
$k = 2$	$x_{nj2}, x_{33n}, x_{123}$	3	$\bar{x}_{njn}, \bar{x}_{332}, \bar{x}_{123}$	(3.29)
$k \neq 2$	$x_{njk}, x_{33n}, x_{112}$	3	$\bar{x}_{n2n}, \bar{x}_{33k}, \bar{x}_{112}$	(3.29)

Table 1: Proving (3.19) when $i = n$, $j \neq n$, $k \neq n$

Case 2: when $i \neq n$, $j = n$ and $k \neq n$
We consider all possible values of i and k such that $(i, n, j) \notin B$ in Table 2. Straight forward calculations prove the corresponding version of (3.25):

$$\alpha_{ink} = \alpha_{inn} + \alpha_{nnk} - \alpha_{nnn}. \tag{3.31}$$

Case 3: when $i \neq n$, $j \neq n$ and $k = n$
Similar to the above two cases we prove the following version of (3.25)

$$\alpha_{ijn} = \alpha_{inn} + \alpha_{njn} - \alpha_{nnn} \tag{3.32}$$

for all possible cases of the values of i and j in Table 3.

Case 4: when $i \neq n$, $j \neq n$ and $k \neq n$
We now prove (3.25) for the case when $i \neq n$,$j \neq n$, $k \neq n$. Let $x \in P^B$ such that $x_{nnn} = x_{ijk} = 1$ with $(i, j, k) \in B$. Note that such a solution always exists. We define $\bar{x}$ by doing a first index interchange; we get $\bar{x}_{inn} = \bar{x}_{njk} = 1$. By $\alpha x = \alpha \bar{x}$, we have:

$$\alpha_{nnn} + \alpha_{ijk} = \alpha_{inn} + \alpha_{njk}. \tag{3.33}$$

case	start sol.	interchange type	new sol.	remarks
$k \in \{2,3\}$				
$i = 3$				$(i,j,k) \in B$
$k = 2,\ i = 2$	x_{2n2},x_{33n},x_{123}	3	$\bar{x}_{2nn}$,$\bar{x}_{332}$,$\bar{x}_{123}$	(3.29)
$k = 2,\ i \notin 2,3$	x_{in2},x_{33n},x_{213}	3	$\bar{x}_{inn}$,$\bar{x}_{332}$,$\bar{x}_{213}$	(3.29)
$k = 3,\ i = 2$	x_{2n3},x_{33n},x_{112}	3	$\bar{x}_{2nn}$,$\bar{x}_{333}$,$\bar{x}_{112}$	(3.29)
$k = 3,\ i \notin 2,3$	x_{in3},x_{33n},x_{222}	3	$\bar{x}_{inn}$,$\bar{x}_{333}$,$\bar{x}_{222}$	(3.29)
$k \notin \{2,3\}$				
$k \notin \{2,3\},\ i = 1$	x_{1nk},x_{n33},x_{222}	1	$\bar{x}_{nnk}$,$\bar{x}_{133}$,$\bar{x}_{222}$	(3.26)
$k \notin \{2,3\},\ i \neq 1$	x_{ink},x_{n33},x_{112}	1	$\bar{x}_{nnk}$,$\bar{x}_{133}$,$\bar{x}_{112}$	(3.26)

Table 2: Proving (3.19) when $i \neq n$, $j = n$, $k \neq n$

case	start sol.	interchange type	new sol.	remarks
$i \notin \{1,2,3\}$				
$i \notin \{1,2,3\},\ j = 2$	x_{i2n},x_{n33},x_{112}	1	$\bar{x}_{n2n}$,$\bar{x}_{i33}$,$\bar{x}_{112}$	(3.26)
$i \notin \{1,2,3\},\ j \neq 2$	x_{ijn},x_{n33},x_{222}	1	$\bar{x}_{njn}$,$\bar{x}_{i33}$,$\bar{x}_{222}$	(3.26)
$i \in \{1,2,3\}$				
$i = 1,\ j \neq 2,3$	x_{1jn},x_{n33},x_{222}	1	$\bar{x}_{njn}$,$\bar{x}_{133}$,$\bar{x}_{222}$	(3.26)
$i = 1,\ j = 3$	x_{13n},x_{3n3},x_{222}	2	$\bar{x}_{1nn}$,$\bar{x}_{333}$,$\bar{x}_{222}$	(3.27)
$i = 1,\ j = 2$	x_{12n},x_{3n2},x_{213}	2	$\bar{x}_{1nn}$,$\bar{x}_{322}$,$\bar{x}_{213}$	(3.28)
$i = 2,\ j \neq 1,3$	x_{2jn},x_{n33},x_{112}	1	$\bar{x}_{njn}$,$\bar{x}_{233}$,$\bar{x}_{112}$	(3.26)
$i = 2,\ j = 3$	x_{23n},x_{3n3},x_{112}	2	$\bar{x}_{2nn}$,$\bar{x}_{333}$,$\bar{x}_{112}$	(3.27)
$i = 2,\ j = 1$	x_{21n},x_{3n2},x_{123}	2	$\bar{x}_{2nn}$,$\bar{x}_{312}$,$\bar{x}_{123}$	(3.28)
$i = 3,\ j \neq 1,3$	x_{3jn},x_{n33},x_{112}	1	$\bar{x}_{njn}$,$\bar{x}_{333}$,$\bar{x}_{112}$	(3.26)
$i = 3,\ j = 1$	x_{31n},x_{n33},x_{222}	1	$\bar{x}_{n1n}$,$\bar{x}_{333}$,$\bar{x}_{222}$	(3.26)
$i = 3,\ j = 3$				$(i,j,k) \in B$

Table 3: Proving (3.19) when $i \neq n$, $j \neq n$, $k = n$

Using equation (3.30) we get

$$\alpha_{ijk} = \alpha_{inn} + \alpha_{nnk} + \alpha_{njn} - 2 \cdot \alpha_{nnn}. \tag{3.34}$$

This completes the proof of equation (3.25), and hence (3.19) is true.

3.2.2. Proving (3.20)

For $(i,j,k) \in B$ we define

$$\pi_{ijk} = \alpha_{ijk} - \lambda_i - \mu_j - \nu_k. \tag{3.35}$$

Next, to prove (3.20), we show that all π_{ijk} are equal. To do this, we first prove that $\pi_{222} = \pi_{213} = \pi_{123} = \pi_{112}$ and then derive the rest of the relations from these equalities.

Consider $x \in P^B$ such that $x_u = x_t = x_r = 1$, where $r = (3,3,1)$, $u = (1,1,3)$ and $t = (2,2,2)$. Define $\bar{x}$ from x by a first index interchange with $\bar{u} = (2,1,3)$ and

$\bar{t} = (1, 2, 2)$. Note that $\bar{u}, t \in B$; $u, \bar{t} \notin B$ and $\bar{x} \in P^B$. Since $\alpha x = \alpha \bar{x}$, we have:

$$\alpha_u + \alpha_t = \alpha_{\bar{u}} + \alpha_{\bar{t}}. \tag{3.36}$$

Substituting the values of α_u and $\alpha_{\bar{t}}$ from equation (3.19) and the values of α_t and $\alpha_{\bar{u}}$ from equation (3.35) we obtain

$$\pi_t + \lambda_2 + \mu_2 + \nu_2 + \lambda_1 + \mu_1 + \nu_3 = \pi_{\bar{u}} + \lambda_2 + \mu_1 + \nu_3 + \lambda_1 + \mu_2 + \nu_2 \tag{3.37}$$

or $\pi_{222} = \pi_{213}$.

Again, consider $x \in P^B$ such that $x_u = x_t = x_r = 1$, where $r = (3, 3, 1)$, $u = (1, 1, 3)$ and $t = (2, 2, 2)$. A third index interchange will give $\bar{u} = (1, 1, 2)$ and $\bar{t} = (2, 2, 3)$. Using $\alpha x = \alpha \bar{x}$, we have:

$$\pi_t + \lambda_1 + \mu_1 + \nu_3 + \lambda_2 + \mu_2 + \nu_2 = \pi_{\bar{u}} + \lambda_1 + \mu_1 + \nu_2 + \lambda_2 + \mu_2 + \nu_3$$

which implies $\pi_{222} = \pi_{112}$.

Similarly, consider $x \in P^B$ such that $x_r = x_u = x_t = 1$, where $r = (3, 3, 1)$, $u = (5, 1, 2)$ and $t = (1, 2, 3)$. Define $\bar{x}$ from x by a first index interchange with $\bar{u} = (1, 1, 2)$ and $\bar{t} = (5, 2, 3)$. Note that $\bar{u}, t \in B$; $u, \bar{t} \notin B$ and $\bar{x} \in P^B$. Again by $\alpha x = \alpha \bar{x}$, we have:

$$\pi_t + \lambda_5 + \mu_1 + \nu_2 + \lambda_1 + \mu_2 + \nu_3 = \pi_{\bar{u}} + \lambda_1 + \mu_1 + \nu_2 + \lambda_5 + \mu_2 + \nu_3$$

or $\pi_t = \pi_{\bar{u}}$ i.e., $\pi_{123} = \pi_{112}$.

Thus, at this point we have shown that:

$$\zeta \equiv \pi_{222} = \pi_{112} = \pi_{123} = \pi_{213}.$$

It still remains to show that for all i, j, k, the following is true:

$$\pi_{3j2} = \pi_{i32} = \pi_{i33} = \pi_{33k} = \pi_{3j3} = \zeta.$$

We prove this by exhibiting pairs of feasible solutions in the following way. Consider $x \in P^B$ such that $x_r = x_u = x_t = 1$, where $r = (3, 1, 3)$, $u = (2, 2, 1)$ and $t = (i, 3, 2)$ with $i \notin \{2, 3\}$. Construct $\bar{x}$ from x by a first index interchange yielding $\bar{u} = (2, 2, 2)$ and $\bar{t} = (i, 3, 1)$. Note that $\bar{u}, t \in B$; $u, \bar{t} \notin B$ and $\bar{x} \in P^B$. Again by $\alpha x = \alpha \bar{x}$, we have:

$$\pi_t + \lambda_2 + \mu_2 + \nu_1 + \lambda_i + \mu_3 + \nu_2 = \pi_{\bar{u}} + \lambda_2 + \mu_2 + \nu_2 + \lambda_i + \mu_3 + \nu_1$$

or $\pi_t = \pi_{\bar{u}}$ i.e.,

$$\pi_{i32} = \pi_{222} = \zeta \text{ for } i \notin \{2, 3\}.$$

Next, consider $x \in P^B$ such that $x_r = x_u = x_t = 1$, such that $r = (2, 3, 3)$, $u = (3, j, k)$ and $t = (1, 1, 2)$, with $j \notin \{3\}$ and $k \notin \{2, 3\}$, a third index interchange will give $\bar{u} = (3, j, 2)$ and $\bar{t} = (1, 1, k)$. Again using $\alpha x = \alpha \bar{x}$ implies

$$\pi_{3j2} = \pi_{112} = \zeta \text{ for } j \neq 3.$$

For simplicity, in rest of the cases we avoid complete working of details and we simply illustrate start and new solutions, and type of interchange used in each case (as before).

Therefore, for the rest of the cases consider $x \in P^B$ such that $x_r = x_u = x_t = 1$ with following cases:

- $r = (1, 2, 3)$, $u = (6, 3, k)$ and $t = (3, j, 2)$, with $j \neq 3$ and $k \notin \{2, 3\}$, a first index interchange will give $\bar{u} = (3, 3, k)$ and $\bar{t} = (6, j, 2)$. Applying $\alpha x = \alpha \bar{x}$ will give $\pi_t = \pi_{\bar{u}}$ i.e.,
$$\pi_{33k} = \pi_{3j2} = \zeta \text{ for } k \notin \{2, 3\}.$$
- $r = (1, 1, 2)$, $u = (3, 2, k)$ and $t = (i, 3, 3)$, with $i \notin \{1, 3\}$ and $k \notin \{2, 3\}$, a second index interchange will give $\bar{u} = (3, 3, k)$ and $\bar{t} = (i, 2, 3)$ which implies
$$\pi_{i33} = \pi_{33k} = \zeta \text{ for } i \notin \{1, 3\}.$$
- $r = (1, 1, 2)$, $u = (i, 3, 7)$ and $t = (3, j, 3)$, with $i \notin \{1, 3\}$ and $j \notin \{1, 3\}$, a third index interchange will give $\bar{u} = (i, 3, 3)$ and $\bar{t} = (3, j, 7)$, which gives us
$$\pi_{3j3} = \pi_{i33} \text{ for } j \notin \{1, 3\}.$$
- $r = (1, 2, 3)$, $u = (4, 4, 1)$ and $t = (3, 3, 2)$, a third index interchange will give $\bar{t} = (4, 4, 2)$ and $\bar{u} = (3, 3, 1)$ which implies
$$\pi_{332} = \pi_{331} = \zeta.$$
- $r = (3, 7, 3)$, $u = (1, 1, 1)$ and $t = (2, 3, 2)$, a third index interchange will give $\bar{t} = (2, 3, 1)$ and $\bar{u} = (1, 1, 2)$ which implies
$$\pi_{232} = \pi_{112} = \zeta.$$
- $r = (2, 2, 2)$, $u = (4, 4, 4)$ and $t = (3, 3, 3)$, a first index interchange will give $\bar{u} = (3, 4, 4)$ and $\bar{t} = (4, 3, 3)$. Notice that here we have $t, \bar{t} \in B$ and $u, \bar{u} \notin B$. Hence we have $\pi_t = \pi_{\bar{t}}$ and this implies
$$\pi_{333} = \pi_{433} = \zeta.$$
- $r = (3, 4, 2)$, $u = (2, 1, 1)$ and $t = (1, 3, 3)$, a third index interchange will give $\bar{u} = (2, 1, 3)$ and $\bar{t} = (1, 3, 1)$ which implies
$$\pi_{133} = \pi_{213} = \zeta.$$
- $r = (2, 2, 2)$, $u = (4, 4, 4)$ and $t = (3, 1, 3)$, a second index interchange will give $\bar{u} = (4, 1, 4)$ and $\bar{t} = (3, 4, 3)$. Again, observe that here we have $t, \bar{t} \in B$ and $u, \bar{u} \notin B$. Hence we have $\pi_t = \pi_{\bar{t}}$ and this implies
$$\pi_{313} = \pi_{343} = \zeta.$$

Therefore we get, for all i, j, and k, the following:

$$\pi_{3j2} = \pi_{33k} = \pi_{i33} = \pi_{3j3} = \pi_{112} = \pi_{123}$$
$$= \pi_{222} = \pi_{213} = \pi_{i32}.$$

3.2.3. Proving (3.21)

Let $\tilde{x}$ be defined by

$$\begin{aligned} \tilde{x}_{ijk} &= 1, \text{ if } \ i = j = k \\ &= 0, \text{ otherwise} \end{aligned}$$

Then $\tilde{x} \in P^B$, hence $\alpha\tilde{x} = \alpha_0$. Substituting the values of α from (3.19), (3.20) will give us (3.21).

■

Finally, we remark the following. Since our polytope P_I is not full-dimensional, there is no unique representation of a facet-defining inequality. Indeed, by adding or subtracting an equality from (1.1)- (1.3), another, equivalent representation of a facet-defining inequality can appear. Hence, it is conceivable that a wall inequality is nothing else but another representation of some already known inequality. That, however, is not the case. For each class of known facet-defining inequalities that we covered in Section 2, we can exhibit a fractional point satisfying equalities (1.1)- (1.3), such that it is not cut away by the known class, but is cut away by the wall inequality. We refer to Dokka [6] for the precise details.

4. Separation

In this section we address the separation problem corresponding to the wall facets. More specifically, we give an $O(n^4)$ separation algorithm to decide whether a given $x \in P$ that satisfies the clique inequalities of type I and type II, violates a wall inequality.

For convenience, let us define the concept of a *large* triple, and a *large* axis. These concepts are defined with respect to a given (fractional) solution $x \in P$. We call a triple $c \in V$ *large* if $x_c > \frac{1}{7}$. Similarly, we call an axis $(i, j, -)$ large (respectively $(i, -, k), (-, j, k)$) when $x(i, j, -) > \frac{1}{7}$ (respectively when $x(i, -, k) > \frac{1}{7}, x(-, j, k) > \frac{1}{7}$). We assume the following sets of large triples are pre-computed in a preprocessing step:

$$LT(i) \equiv \{(j, k) \in J \times K : (i, j, k) \text{ is large}\},$$
$$LT(j) \equiv \{(i, k) \in I \times K : (i, j, k) \text{ is large}\},$$
$$LT(k) \equiv \{(i, j) \in I \times J : (i, j, k) \text{ is large}\}.$$

Further, we will use LT to denote the set of all large triples, i.e.,

$$LT \equiv \{(i, j, k) \in I \times J \times K : (i, j, k) \text{ is large}\}.$$

Also, the following sets of large axes are pre-computed:

$$\begin{aligned}
LAJ(i) &\equiv \{j \in J : (i,j,-) \text{ is large}\},\\
LAK(i) &\equiv \{k \in K : (i,-,k) \text{ is large}\},\\
LAI(j) &\equiv \{i \in I : (i,j,-) \text{ is large}\},\\
LAK(j) &\equiv \{k \in K : (-,j,k) \text{ is large}\},\\
LAI(k) &\equiv \{i \in I : (i,-,k) \text{ is large}\},\\
LAJ(k) &\equiv \{j \in J : (-,j,k) \text{ is large}\}.
\end{aligned}$$

Notice that all these sets can be computed in $O(n^3)$ time. Large triples (axes) play a vital role in our separation algorithm, because of the fact that for a fixed $r \in R$ there are at most a constant number of large triples, and large axes that contain r. We record the following straightforward observations in a lemma.

Lemma 9. *Given is some $x \in P$. The following statements are true:*

(i) For each $i \in I$ the number of pairs $(j,k) \in J \times K$ such that triple (i,j,k) is large is at most 6.

(ii) For each $i \in I$ the number of $j \in J$ such that the axis $(i,j,-)$ is large is at most 6.

(iii) The number of large triples in x equals at most $7n$.

Proof. We argue by contradiction.

Ad (i) Suppose statement (i) is not true, then at least 7 pairs $(j,k) \in J \times K$ exist with $x(i,j,k) > \frac{1}{7}$. This implies:

$$\sum_{j \in J} \sum_{k \in K} x(i,j,k) > 7 \times \frac{1}{7} = 1,$$

which contradicts $x \in P$.

Ad (ii) Similar to (i).

Ad (iii) Suppose statement (iii) is not true, then the number of large triples exceeds $7n$. But then, total value of all x-variables exceeds $7n \times \frac{1}{7} = n$, which contradicts $x \in P$.

□ ■

In the following subsections we will prove the following theorem:

Theorem 10. *The separation problem for wall inequalities (3.16) can be solved in $O(n^4)$ time.*

Recall that B stands for the set of triples present in some wall inequality, see (3.16). We use $B_1 \subset B$ to denote four of these triples, i.e., we set

$$B_1 \equiv \{(i_1, j_1, k_1), (i_1, j_2, k_2), (i_2, j_1, k_2), (i_2, j_2, k_1)\}.$$

Notice that wall inequalities (3.16) are *symmetric* in the following sense: the values of indices k_1 and k_2, as well as i_1 and i_2 (or j_1 and j_2) can be interchanged without changing the inequality. We use this symmetry later on.

Theorem 10 relies on the following lemma.

Lemma 11. *Any violated wall inequality falls into at least one of the following three cases:*

Case 1: No triple in B_1 is large.

Case 2: A triple from B_1, as well as the axis $(i_3, j_3, -)$, are large.

Case 3: A triple from B_1 with a third index k from $\{k_1, k_2\}$, as well as an axis with third index k' from $\{k_1, k_2\}, k' \neq k$, are large.

Proof. Imagine a violated wall inequality where none of these cases apply. Then it must be the case that all large triples from B_1, as well as all large axes, share an index from $\{k_1, k_2\}$, say k_1. However, since $x \in P$, we have $x[(i_3, -, k_1) \cup (-, j_3, k_1)] + x(i_1, j_1, k_1) + x(i_2, j_2, k_1) \leq 1$. Thus, the sum of the remaining variables in the wall inequality, being $x[(i_3, j_3, -) \cup (i_3, -, k_2) \cup (-, j_3, k_2)] + x(i_2, j_1, k_2) + x(i_1, j_2, k_2)$ must exceed 1; this is impossible since each of these terms is not large. □ ■

We will now show how to detect a violated wall inequality in each of the three cases given in Lemma 11.

4.1. Case 1: when no triple in B_1 is large

As mentioned before, we assume that the given (fractional) solution $x \in P$ satisfies the clique inequalities of type I and type II. We now give some properties of a violated wall inequality when no triple in B_1 is large.

Lemma 12. *For a violated wall inequality with no large triple in B_1, the following statements are true:*

(i) at least one of the axes $(-, j_3, k_1)$ and $(-, j_3, k_2)$ is large,

(ii) at least one of the axes $(i_3, -, k_1)$ and $(i_3, -, k_2)$ is large,

(iii) at least one of the axes $(i_3, -, k_1)$ and $(-, j_3, k_1)$ is large,

(iv) at least one of the axes $(i_3, -, k_2)$ and $(-, j_3, k_2)$ is large.

Proof.

Ad (i) Since $x \in P$, we know that

$$x[(i_3, j_3, -) \cup (i_3, -, k_1) \cup (i_3, -, k_2)] \leq 1.$$

Together with $x(B_1) \leq \frac{4}{7}$, it follows that, for a wall inequality to be violated, at least one of the axes $(-, j_3, k_1), (-, j_3, k_2)$ must be large.

Ad (ii) A similar argument as above using $x[(i_3, j_3, -) \cup (-, j_3, k_1) \cup (-, j_3, k_2)] \leq 1$ applies.

Ad (iii) Since x satisfies the clique inequalities of type I, and in particular: $x(Q(i_3, j_3, k_2)) \leq 1$, statement (iii) follows from $x(B_1) \leq \frac{4}{7}$.

Ad (iv) A similar argument as above using $x(Q(i_3, j_3, k_1)) \leq 1$ applies.

□ ■

Correctness of Algorithm 1 Consider a violated wall inequality. It follows from Lemma 12, and from symmetry, that it is enough to consider the case when $(i_3, -, k_1)$ and $(-, j_3, k_2)$ are large. We now assume that $x(i_1, j_1, k_1) \geq \max\{(i_1, j_2, k_2), (i_2, j_1, k_2), (i_2, j_2, k_1)\}$; we come back to this assumption later. Algorithm 1 starts by enumerating over $K \times K$ to consider all pairs k_1 and k_2. For each fixed k_1 and k_2, each $i_3 \in LAI(k_1)$ and $j_3 \in LAJ(k_2)$ are considered to identify a violated inequality. Clearly, since $(i_3, -, k_1)$ and $(-, j_3, k_2)$ are large, it follows that $i_3 \in LAI(k_1)$ and $j_3 \in LAJ(k_2)$; no other i_3, j_3 need to be considered.

In addition, we claim that for a violated wall inequality to exist, it must be true that there exist $i_1, j_1 \in I \times J$ such that:

$$x(i_1, j_1, k_1) > \frac{1 - x[(i_3, -, k_1) \cup (-, j_3, k_1)]}{4}. \tag{4.38}$$

Indeed, suppose this were not true, then

$$x(i_1, j_1, k_1) \leq \frac{1 - x[(i_3, -, k_1) \cup (-, j_3, k_1)]}{4},$$

which is equivalent with:

$$4x(i_1, j_1, k_1) \leq 1 - x[(i_3, -, k_1) \cup (-, j_3, k_1)],$$

which by our earlier assumption, implies:

$$x(i_1, j_1, k_1) + x(i_1, j_2, k_2) + x(i_2, j_1, k_2) + x(i_2, j_2, k_1) + x[(i_3, -, k_1) \cup (-, j_3, k_1)] \leq 1. \tag{4.39}$$

However, since clique inequalities of type I are satisfied, we have:

$$x[(i_3, j_3, -) \cup (i_3, -, k_2) \cup (-, j_3, k_2)] \leq 1. \tag{4.40}$$

Inequalities (4.39) and (4.40) would imply that no violated wall inequality exists, and hence it is true that for a violated wall inequality to exist, (4.38) must hold. Thus, we can use (4.38) to build a list of all $(i_1, j_1) \in I \times J$. Then the inequality is checked for each $(i_2, j_2) \in I \times J$ for fixed i_3, j_3, k_2, k_1 and for each $(i_1, j_1) \in S$. Hence, in this case of no large triple in B_1, a violated wall inequality is found if one exists. We point out that that the assumption $x(i_1, j_1, k_1) \geq \max\{(i_1, j_2, k_2), (i_2, j_1, k_2), (i_2, j_2, k_1)\}$ is indeed without loss of generality: one of these four elements has the largest weight among them, and the arguments used above go through for each choice of maximum-weight element.

Algorithm 1 Separation algorithm for Wall Facets - Case 1

```
{No large triple in B_1}
S := ∅
for all k_1, k_2 ∈ K × K do
  for all i_3 ∈ LAI(k_1) do
    for all j_3 ∈ LAJ(k_2) do
      for all (i_1, j_1) ∈ I × J do
        if (4.38) is satisfied then
          S := S ∪ {(i_1, j_1)}
        end if
      end for
      for all (i_1, j_1) ∈ S do
        for all (i_2, j_2) ∈ I × J do
          if x(B) > 2 then
            Output x(B) ≤ 2 as violated wall inequality
          end if
        end for
      end for
    end for
  end for
end for
```

Complexity of Algorithm 1 The first 'for' loop runs $O(n^2)$ times. By Lemma 9 and by the definition of the sets LA, the second and third 'for' loops each run $O(1)$ times. The loop to find the set of (i_1, j_1)'s satisfying (4.38) runs $O(n^2)$ times. However, the cardinality of this set S is 3. To see this, suppose there exist 4 pairs (i_1^h, j_1^h), $h = 1, \ldots, 4$, satisfying (4.38). This implies:

$$\sum_{h=1}^{4} x(i_1^h, j_1^h, k_1) + x[(-, j_3, k_1) \cup (i_3, -, k_1) > 1,$$

which contradicts $x \in P$. Thus, the cardinality of the set S is at most 3. Therefore, the sixth 'for' loop runs $O(1)$ times, while the last loop runs in $O(n^2)$. This gives a the total complexity of Algorithm 1 of $O(n^4)$.

4.2. Case 2: A triple from B_1, as well as the axis $(i_3, j_3, -)$, are large

In this case, the algorithm looks for a violated inequality when there is a large triple in B_1, and when the axis $(i_3, j_3, -)$ is large. Without loss of generality we assume that the large triple is (i_1, j_2, k_2). As in case 1, we assume that the given solution $x \in P$ satisfies the clique inequalities of type I and II. The algorithm to identify a violated wall inequality in this case is given in Algorithm 2.

Correctness of Algorithm 2 Algorithm 2 starts by choosing a candidate for i_3 in I. Then the set $LAJ(i_3)$ is enumerated for j_3 making use of the fact that $(i_3, j_3, -)$ is large.

Since $x \in P$ satisfies all clique inequalities of type II, it follows that

$$x[(i_3, j_3, -) \cup x(i_3, -, k_1) \cup x(i_3, -, k_2) \cup x(-, j_3, k_1) \cup x(-, j_3, k_2)] > 1, \tag{4.41}$$

for a wall inequality to be violated.

Let us assume that the following is true:

$$x(i_3, -, k_1) \quad \geq \quad \max\{x(i_3, -, k_2), x(-, j_3, k_1), x(-, j_3, k_2)\}. \tag{4.42}$$

Algorithm 2 Separation algorithm for Wall Facets - case 2

```
{triple (i1, j2, k2) and axis (i3, j3, −) are large}
S := ∅
for all i3 ∈ I do
  for all j3 ∈ LAJ(i3) do
    for all k1 ∈ K do
      if (4.43) is satisfied then
        S := S ∪ {k1}
      end if
    end for
    for all k1 ∈ S1 do
      for all k2 ∈ K do
        for all (i2, j1) ∈ I × J do
          for all (i1, j2) ∈ LT(k2) do
            if x(B) > 2 then
              Output x(B) ≤ 2 as violated wall inequality
            end if
          end for
        end for
      end for
    end for
  end for
end for
```

Then it follows that a wall inequality can only be violated when

$$x(i_3, -, k_1) > \frac{1 - x(i_3, j_3, -)}{4}. \tag{4.43}$$

Indeed, if this were not true then we have:

$$x(i_3, -, k_1) \leq \frac{1 - x(i_3, j_3, -)}{4},$$

which is equivalent with:

$$4x(i_3, -, k_1) \leq 1 - x(i_3, j_3, -),$$

leading to (using (4.42)):

$$x[(i_3,-,k_1)\cup(i_3,-,k_2)\cup(-,j_3,k_1)\cup(-,j_3,k_2)]\le 1-x(i_3,j_3,-),$$

contradicting (4.41). Now, Algorithm 2 enumerates over all $k_1\in K$ to make a list S of all k_1 satisfying (4.43) for a fixed i_3 and j_3. Then for each choice of $k_1\in S$, and fixed i_3, j_3, the algorithm enumerates over all $k_2\in K$. Next, for a fixed choice of i_3,j_3,k_2,k_1, the algorithm enumerates over all (i,j) pairs for i_2, j_1. Finally, for a fixed i_2,i_3,j_1,j_3,k_1,k_2, the algorithm checks the inequality for all candidates of i_1 and j_2 such that $(i_1,j_2)\in LT(k_2)$. Since we assumed triple (i_1,j_2,k_2) to be large, it is enough to consider the (i_2,j_1) pairs in $LT(k_2)$ to identify a violated wall inequality in this case. Notice that assumption (4.42) is indeed without loss of generality: one of the four axes in (4.42) has the largest weight among them, and straightforward modifications of (4.43) can then be used.

Complexity of Algorithm 2 We will now prove the complexity part. First, notice that the cardinality of S is at most 3. Suppose this were not true, then we have k_1^h, $h=1,2,3,4$, each satisfying (4.43) for a fixed i_3, implying

$$x(i_3,j_3,-)+\sum_{h=1}^{4}x(i_3,-,k_1^h)>1,$$

which is impossible, since $x\in P$. Notice that this argument applies for each possible axis in (4.42) having the largest weight.

The first 'for' loop runs $O(n)$ times, second 'for' loop runs $O(1)$ times, the third 'for' loop runs $O(n)$ times, the fourth 'for' loop runs $O(1)$ times, the fifth loop runs $O(n)$ times, the sixth 'for' loop runs $O(n^2)$ times, and the last 'for' loop runs $O(1)$ times. Hence the overall complexity is $O(n^4)$.

4.3. Case 3: A triple from B_1, as well as an axis with a different third index, are large

In this case, the algorithm looks for a violated inequality when there is a large triple in B_1, and when an axis with a different third index is large. Without loss of generality we assume that the large triple is (i_1,j_2,k_2). As before, we assume that the given solution $x\in P$ satisfies the clique inequalities of type I and II.

It follows that either axis $(i_3,-,k_1)$ or axis $(-,j_3,k_1)$ is large. Symmetry implies that we can assume, without loss of generality, the large axis to be $(i_3,-,k_1)$. Further, we need to distinguish three subcases depending upon which of the remaining four axes has the largest weight.

Subcase A: $\max\{x(i_3,-,k_2),x(-,j_3,k_2)\}\ge\max\{x(i_3,j_3,-),x(-,j_3,k_1)\}$,

Subcase B: $x(i_3,j_3,-)\ge\max\{x(i_3,-,k_2),x(-,j_3,k_1),x(-,j_3,k_2)\}$,

Subcase C: $x(-,j_3,k_1)\ge\max\{x(i_3,j_3,-),x(i_3,-,k_2),x(-,j_3,k_2)\}$.

4.3.1. Subcase A

In this subsection, we assume that one of the two axes containing third index k_2 is heaviest; let us say axis $(-, j_3, k_2)$ is heaviest. The algorithm to identify a violated wall inequality in this case is given in Algorithm 3.

Correctness and Complexity of Algorithm 3 Algorithm 3 starts by considering each possible $(i_1, j_2, k_2) \in LT$. Then, it enumerates over all pairs $i_2, j_1 \in I \times J$, and next for each $j_3 \in J$. Algorithm 3 then makes a list S of j_3's such that

$$x(-, j_3, k_2) > \frac{1 - [x(i_2, j_1, k_2) + x(i_1, j_2, k_2)]}{3}. \tag{4.44}$$

Indeed, notice that otherwise no violated wall inequality exists: using

$$x(-, j_3, k_2) \leq \frac{1 - [x(i_2, j_1, k_2) + x(i_1, j_2, k_2)]}{3},$$

we can arrive at:

$$x(i_3, j_3, -) + x(i_3, -, k_2) + x(-, j_3, k_2) + x(i_2, j_1, k_2) + x(i_1, j_2, k_2) \leq 1,$$

which, since $x \in P$, implies no violated wall inequality exists.

Algorithm 3 Separation algorithm for Wall Facets - subcase A

{triple (i_1, j_2, k_2) and axis $(i_3, -, k_1)$ are large; an axis containing as third index k_2 is heaviest}
$S := \emptyset$
for all $(i_1, j_2, k_2) \in LT$ **do**
  for all $(i_2, j_1) \in I \times J$ **do**
    for all $j_3 \in J$ **do**
      if (4.44) is satisfied **then**
        $S := S \cup \{j_3\}$
      end if
    end for
    for all $j_3 \in S_1$ **do**
      for all $k_1 \in K$ **do**
        for all $i_3 \in LAI(k_1)$ **do**
          if $x(B) > 2$ **then**
            Output $x(B) \leq 2$ as violated wall inequality
          end if
        end for
      end for
    end for
  end for
end for

Let us now argue that the number of such j_3's is at most 2. Indeed, suppose this is not the case and let there be j_3^1, j_3^2, j_3^3 which satisfy (4.44). We have:

$$\sum_{h=1}^{3}(-,j_3^h,k_2)+x(i_2,j_1,k_2)+x(i_1,j_2,k_2) > \quad 1-[x(i_2,j_1,k_2)+x(i_1,j_2,k_2)]$$
$$+ \quad x(i_2,j_1,k_2)+x(i_1,j_2,k_2)=1.$$

This is a contradiction and hence there are at most 2 j_3's. For a fixed i_2, j_1, i_1, j_2, k_2 and for each $j_3 \in S$ the inequality is checked for all $k_1 \in K$ and $i_3 \in LAI(k_2)$. Again this is enough as $(i_3, -, k_1)$ is large.

With respect to complexity: the first 'for' loop runs $O(n)$ times (since by Lemma 9 we have $O(n)$ large triples), the second 'for' loop runs $O(n^2)$ times, third 'for' loop runs $O(n)$ times, fourth 'for' loop runs $O(n)$ times. All other 'for' loops only run $O(1)$ times. Since the third and fourth 'for' loops are parallel, the total complexity is $O(n^4)$.

4.3.2. Subcase B

Let us now consider the case when axis $(i_3, j_3, -)$ is heaviest among the four remaining axes, i.e., when $x(i_3, j_3, -) \geq \max\{x(i_3, -, k_2), x(-, j_3, k_1), x(-, j_3, k_2)\}$. The corresponding algorithm is given as Algorithm 4.

Correctness and Complexity of Algorithm 4 Suppose that we know the values of k_2, k_1 and i_3 of a violated wall inequality. Then, for a violated wall inequality to exist, j_3 should satisfy

$$x(i_3, j_3, -) > \frac{1-[x(i_3,-,k_2)\cup x(i_3,-,k_1)]}{3}. \tag{4.45}$$

Otherwise, it follows that total weight on all five axes does not exceed 1, which is not compatible with the existence of a violated wall inequality, and x satisfying clique inequalities of type II.

Algorithm 4 Separation algorithm for Wall Facets - subcase B

{triple (i_1, j_2, k_2) and axis $(i_3, -, k_1)$ are large; axis $(i_3, j_3, -)$ is heaviest}
$S := \emptyset$
for all $k_2 \in K$ **do**
  for all $k_1 \in K$ **do**
    for all $i_3 \in LAj(k_1)$ **do**
      for all $j_3 \in J$ **do**
        if (4.45) is satisfied **then**
          $S := S \cup \{j_3\}$
        end if
      end for
    end for
    for all $j_3 \in S$ **do**
      for all $(i_2, j_1) \in I \times J$ **do**
        for all $(i_1, j_2) \in LT(k_2)$ **do**
          if $x(B) > 2$ **then**
            Output $x(B) \leq 2$ as violated wall inequality
          end if
        end for
      end for
    end for
  end for
end for

Using a similar reasoning as in Subsection 4.3.1, it can be argued that there are at most 3 j_3's such that (4.45) is satisfied. Algorithm 4 starts by first enumerating over K for k_2. In the second 'for' loop, the algorithm enumerates over K for k_1, and the third 'for' loop enumerates over $LAi(k_1)$ for i_3. The fourth 'for' loop makes a list of j_3's satisfying (4.45), and the fifth 'for' loop enumerates over this list. The sixth 'for' loop runs over all $(i, j) \in I \times J$, while the seventh 'for' loop enumerates over $LT(k_2)$ for (i_1, j_2) and checks the inequality for violation.
The complexity of Algorithm 4 is determined by the first, second, fifth, sixth, and seventh 'for' loops that run respectively in $O(n)$, $O(n)$, $O(1)$, $O(n^2)$, and $O(1)$, which gives a total complexity of $O(n^4)$.

4.3.3. Subcase C

Let us finally consider the case when axis $(-, j_3, k_1)$ is the heaviest, i.e., when $x(-, j_3, k_1) \geq \max\{x(i_3, j_3, -), x(i_3, -, k_2), x(-, j_3, k_2)\}$. The corresponding algorithm is given as Algorithm 5.
Correctness and Complexity of Algorithm 5 In the first 'for' loop, Algorithm 5 enumerates over K for k_1, in the second 'for' loop it enumerates over $LAI(k_1)$ to find i_3. In the third 'for' loop the algorithm makes a list of all j_3's such that

$$x(-, j_3, k_1) > \frac{1 - x(i_3, -, k_1)}{4}. \tag{4.46}$$

Notice that, similarly to Subcase B, if this inequality is not true, it follows that total weight on all five axes does not exceed 1. Thus (4.46) must be true for a violated wall inequality to exist.

Algorithm 5 Separation algorithm for Wall Facets - subcase C

{triple (i_1, j_2, k_2) and axis $(i_3, -, k_1)$ are large; axis $(-, j_3, k_1)$ is heaviest}
$S := \emptyset$
for all $k_1 \in K$ **do**
  for all $i_3 \in LAi(k_1)$ **do**
    for all $j_3 \in J$ **do**
      if (4.46) is satisfied **then**
        $S := S \cup \{j_3\}$
      end if
    end for
    for all $j_3 \in S$ **do**
      for all $(i_2, j_1, k_2) \in V$ **do**
        for all $(i_1, j_2) \in LT(k_2)$ **do**
          if $x(B) > 2$ **then**
            Output $x(B) \leq 2$ as violated wall inequality
          end if
        end for
      end for
    end for
  end for
end for

Again, using a similar reasoning as in Subsection 4.3.1, it follows that there are at most 3 j_3's such that (4.46) is satisfied. In the fourth 'for' loop the algorithm enumerates over all $(i_2, j_1, k_2) \in V$ and in the fifth 'for' loop it enumerates over all $(i_1, j_2, k_2) \in LT(k_2)$.

The complexity of Algorithm 5 is determined by the first, second, fourth, fifth, and sixth 'for' loops that run respectively in $O(n)$, $O(1)$, $O(1)$, $O(n^3)$, and $O(1)$, which gives a total complexity of $O(n^4)$.

5. Conclusion

We have exhibited a new class of valid inequalities for the axial 3-index assignment polytope. This class of valid inequalities, called *wall* inequalities define facets of this polytope, and can be separated in $O(n^4)$ time.

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