LINEAR SYSTEMS, SPECTRAL CURVES AND DETERMINANTS

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Abstract. Let $(-A, B, C)$ be a continuous-time linear system with state space a separable complex Hilbert space H, where $-A$ generates a strongly continuous contraction semigroup $(e^{-tA})_{t\geq 0}$ on H, and $\phi(t) = Ce^{-tA}B$ is the impulse response function. Associated with such a system is a Hankel integral operator Γ_ϕ acting on $L^2((0,\infty);\mathbb{C})$ and a Schrödinger operator whose potential is found via a Fredholm determinant by the Faddeev–Dyson formula. Fredholm determinants of products of Hankel operators also play an important role in Tracy and Widom's theory of matrix models and asymptotic eigenvalue distributions of random matrices. This paper provides formulas for the Fredholm determinants that arise thus, and determines consequent properties of the associated differential operators. We prove a spectral theorem for self-adjoint linear systems that have scalar input and output: the entries of Kodaira's characteristic matrix are given explicitly with formulas involving the infinitesimal Darboux addition for $(-A, B, C)$. Under suitable conditions on $(-A, B, C)$ we give an explicit version of Burchnall–Chaundy's theorem, showing that the algebra generated by an associated family of differential operators is isomorphic to an algebra of functions on a particular hyperelliptic curve.

1. Introduction

Fredholm determinants and determinants of multiplicative commutators arise in the theory of continuous-time linear systems $(-A, B, C)$ with state space H, a complex separable Hilbert space. Associated to each such system is its impulse response (or scattering) function $\phi(t) = Ce^{-tA}B$ and a corresponding Hankel operator Γ_ϕ acting on $L^2((0,\infty);\mathbb{C})$, given by $\Gamma_\phi f(x) = \int_0^\infty \phi(x+y)f(y)\,dy$.

We are motivated here by applications from random matrix theory and the spectral theory of Schrödinger operators. The tau function plays a central role for the linear system. Writing $\phi_{(x)}(t) = \phi(t+2x)$, we define the tau function by the Fredholm determinant $\tau(x) = \det(I+\Gamma_{\phi_{(x)}})$. The tau function provides a connection between the scattering function and the potential in Schrödinger's equation. Starting from a Schrödinger equation $-f''+uf = \lambda f$, Faddeev and Dyson showed that under suitable hypotheses, given its scattering function ϕ one could recover the potential u via the formula $u(x) = -2\frac{d^2}{dx^2}\log\tau(x)$.

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$$\begin{tikzcd}
\text{scattering } \phi \arrow[d, "\det"] & (-A,B,C) \arrow[l] \arrow[r] \arrow[d, "\text{Lyapanov eqn}"] & \xi \text{ function} \\
\tau \arrow[d] & R_x \arrow[l, "\text{Gelfand–Levitan}"'] \arrow[r, "\lfloor\cdot\rfloor"'] \arrow[d] & \begin{matrix}\text{diagonal Green's}\\ G(x,x;\lambda)\end{matrix} \arrow[d] \\
\text{potential } u \arrow[d] & T_{GL}(x,y) \arrow[d] & \begin{matrix}\text{Darboux}\\ (d/dx)G(x,x;\lambda)\end{matrix} \\
\text{Schrödinger operator} & \text{Baker–Ahkiezer} &
\end{tikzcd}$$

FIGURE 1.

Figure 1 expresses the connections between various functions and key equations in the theory. The left-hand column involves Hankel operators built from the scattering data. The scattering function ϕ is said to be realised by a linear system if there exists $(-A, B, C)$ such that $\phi(x) = Ce^{-xA}B$; in the terminology of linear systems, this is the impulse response function. Scattering functions can be so realised under hypotheses discussed in [21], [39] and [1], as in Example 6.1.

Under mild hypotheses, we show in Theorem 4.2 that $R_x = \int_x^\infty e^{-tA}BCe^{-tA}\,dt$ gives a family of trace-class operators on H such that $\det(I + R_0) = \det(I + \Gamma_\phi)$. The R_x may be defined as compositions of shifted reachability and controllability operators as in [1, p. 318], although it is more illuminating to observe that they satisfy Lyapunov's equation

$$\begin{aligned}\frac{dR_x}{dx} &= -AR_x - R_xA\\ \Big(\frac{dR_x}{dx}\Big)_{x=0} &= -BC.\end{aligned}\tag{1.1}$$

This opens the route towards a family of algebraic identities linking various operators. The middle column of Figure 1 gives the route from the linear system down to the solution of the corresponding Schrödinger equation via the Gelfand–Levitan equation (5.6), which we solve using the integral kernel T_{GL}. The formula $T_{GL}(x, y) = -Ce^{-xA}(I + R_x)^{-1}e^{-yA}B$ from (5.5) motivates the choice of R_x. For convolution equations on a finite interval, there are analogous formulas [33] and [27].

On the right, we introduce the xi function and diagonal Green's function, which are closely related to the spectral data and determine those equations that are integrable. Formal definitions of these functions are given below in (6.20) and (6.6) .

Tracy and Widom [57] introduced a systematic theory of matrix models involving Fredholm determinants of products of Hankel operators. Unfortunately the space of Hankel integral operators does not have an obvious multiplicative structure, which inhibits calculations. In previous papers [1],[2],[5] and [4], we introduced various tools to reveal the latent algebraic structure of the Hankel operators. In particular we introduced a differential ring of operators $\mathcal{E}$ on H. This differential ring, which in general is not commutative and not self-adjoint, provides a tool for computing the Fredholm determinant of the Hankel integral operators. There is a bracket operator $\lfloor \cdot \rfloor$ taking $\mathcal{E}$ to complex functions on $(0, \infty)$, which is an essential linkage in Figure 1.

In section 2, we recall some basic results concerning continuous-time linear systems and associated Hankel operators. We develop ideas from [4], and obtain Fredholm determinants via linear systems that have input and output space $\mathbb{C}$. In Proposition 2.8, we obtain a necessary condition for an integral kernel to be the product of Hankel operators.

In [57], many formulas involve products of Hankel integral operators that are associated with first order linear ordinary differential equations with 2×2 matrix coefficients. Products of Hankel operators arise as noncommutative differentials, and so we begin section 2 with a discussion of this and the role that the ring of stable rational functions plays in the theory. Some of our computations for products of Hankel operators involve the diagonal derivative of integral kernels. In section 3 we provide a framework for considering these objects which we use in the subsequent parts of the paper.

Let $(-A, B, C)$ be a linear system as above and let $\mathcal{L}^1(H)$ denote the ideal of trace class operators on H. A central role in the theory is played by the subalgebra $\mathcal{E}_0$ of $\mathcal{L}(H)$ that is generated by I, BC and $(\lambda I - A)(\lambda I + A)^{-1}$, and the corresponding quotient algebra $\mathcal{A}_0 = (\mathcal{E}_0 + \mathcal{L}^1(H))/\mathcal{L}^1(H)$. The algebra $\mathcal{A}_0$ is commutative and unital, and hence may be regarded as a space of functions on its maximal ideal space $\mathbb{X}$. For the specific examples we have in mind, it is helpful to regard $\mathbb{X}$ as a spectral curve. In particular cases, we can identify $\mathbb{X}$ with a curve in the sense of algebraic geometry, and introduce suitable tau functions over the curve as in [42] and [43]. These ideas, which lead to a number of determinant formulas, are examined in section 4.

In section 5 we examine a important family of examples of $(-A, B, C)$ which come from Howland operators on $L^2((0, \infty); \mathbb{C})$. Our results explain why the spectral theory of Howland operators is closely linked to that of Hankel integral operators and Schrödinger operators. We show that the systems which come from these operators satisfy the hypotheses of the results in section 4 and hence give specific examples of the determinant formulas obtained.

As above, associated to the linear system $(-A, B, C)$ is a potential $u(x) = -2\frac{d^2}{dx^2} \log \tau(x)$ and corresponding Schrödinger differential operator $L = -\frac{d^2}{dx^2} + u$. In section 6 we show that the operator $(\lambda I - L)^{-1}$ may be expressed via a Green's function $G(x, y; \lambda)$ (see Equation (6.6)), and the diagonal $G(x, x; \lambda)$ may be expressed in terms of the linear system (Equation (6.14)).

For $\arg(-\zeta) \in (-\pi/2, \pi/2)$ we can define the linear systems

$$\Sigma_\zeta = (-A, (\zeta I + A)(\zeta I - A)^{-1}B, C). \tag{1.2}$$

which are related to the original $(-A, B, C)$ by the process of Darboux multiplication; see, for example, [5] and [4, Section 2]. As discussed in [4, Proposition 2.5] and [39], the diagonal Green's function (6.14) describes the infinitesimal Darboux addition via $X(u) = -2\frac{d}{dx}G(x, x; \lambda)$, as discussed in [4, Proposition 2.5] and [39].

For Schrödinger's equation on the real line, Gesztesy and Simon [25] introduced the xi function ξ and established the formula $\xi(x;\lambda) = (1/\pi)\,\mathrm{Im}\log(-G(x, x; \lambda))$. The xi function is a convenient intermediary between the scattering data and the potential for solving the inverse spectral problem. These connections are pursued in section 6.

In section 7 we give a sufficient condition for $(-A, B, C)$ to produce an integrable Schrödinger equation, and we provide an explicit version of Burchnall–Chaundy's theorem, showing how a potential that satisfies the finite stationary KdV hierarchy gives a spectral curve $\mathbb{X}$ of finite genus ℓ. The proof uses identities which originate from Lyapunov's equation (1.1). In [27, (1.1), 5.11)], Gohberg, Kaashoek and Sakhnovic develop a state space approach to the forward and inverse spectral problem for the matrix Schrödinger which involves some aspects that are similar to those of the current paper.

2. Linear systems and Hankel operators

Throughout, let H be a separable complex Hilbert space, regarded as the state space. Let $\mathcal{L}(H)$ be the algebra of bounded linear operators on H with the operator norm and the adjoint $T \mapsto T^\dagger$. We write $H' = \mathcal{L}(H; \mathbb{C})$. We shall let $\mathcal{L}^2(H)$ denote the space of Hilbert–Schmidt operators on H, which is a Hilbert space when equipped with the usual inner product $\langle K, L\rangle = \mathrm{trace}(KL^\dagger)$. Let $\mathcal{L}^1(H)$ denote the space of trace class operators.

Let H_0 be another complex Hilbert space, which serves as the input and output space for our linear system $(-A, B, C)$, so that the input operator is the bounded linear operator $B : H_0 \to H$ and the output operator is the bounded linear operator $C : H \to H_0$. Let $-A$ be the infinitesimal generator of a strongly continuous (C_0) semigroup $(e^{-tA})_{t\geq 0}$ of linear contractions on H. Let $\mathcal{D}(A)$ be the domain of A, which is a dense linear subspace of H and itself a Hilbert space for the graph norm $\|f\|^2_{\mathcal{D}(A)} = \|f\|^2_H + \|Af\|^2_H$.

For bounded input $u : [0, \infty) \to H_0$, output $y : [0, \infty) \to H_0$ and state $x : [0, \infty) \to H$, the continuous time linear system is governed by the ordinary differential equation

$$\begin{aligned} \frac{dx}{dt} &= -Ax + Bu \\ y &= Cx \\ x(0) &= 0. \end{aligned} \tag{2.1}$$

The impulse response function is $\phi(t) = Ce^{-tA}B$, which gives a weakly continuous function $(0,\infty) \to \mathcal{L}(H_0)$, also known as the scattering function. (In many cases we shall take H_0 to be $\mathbb{C}$ and so ϕ is scalar-valued.)

Definition 2.1. Suppose that $\phi \in L^2((0,\infty);\mathcal{L}(H_0))$. The Hankel integral operator with impulse response function ϕ is

$$\Gamma_\phi f(x) = \int_0^\infty \phi(x+t)f(t)\,dt \qquad (f : (0,\infty) \to H_0). \tag{2.2}$$

If Γ_ϕ is trace class on $L^2((0,\infty);H_0)$, then the Fredholm determinant $\det(I+\Gamma_\phi)$ is defined, and we shall be concerned with the problem of computing this determinant in cases of interest.

Given a complex unital algebra $\mathcal{R}$, we say that H is a (left) Hilbert module if there is a unital homomorphism $\Phi : \mathcal{R} \to \mathcal{L}(H)$ which provides a natural pairing $(a,\eta) \mapsto \Phi(a)\eta$ for all $a \in \mathcal{R}$, $\eta \in H$; see [18]. The crucial step is to introduce a differential ring of operators on H. Our results on commutative algebras of ordinary differential operators are also related to those of [20] and [53], except that we use $(-A,B,C)$ as the fundamental datum instead of Grassmannian of subspaces of Hilbert space. Via (1.1) we address the functional identities more directly.

Now we consider the stable rational functions, which are fundamental to linear systems theory as in [58] and [48]. Suppose that $(-A,B,C,D)$ is a linear system with finite-dimensional state space H. Then the linear system is described by the transfer function

$$T(s) = D + C(sI+A)^{-1}B,$$

which is a proper rational function, as in [3]. If $(e^{-tA})_{t\geq 0}$ is of exponential decay, then $T(s)$ is a stable rational function. Here D is the feedthrough operator, included here for completeness. Let $\mathscr{S}$ be the ring of stable rational functions and $\mathcal{R}$ the subring of $\mathscr{S}$ given by $\mathcal{R} = \mathbb{C}[\lambda]$ where $\lambda = 1/(1+s)$. Both $\mathcal{R}$ and $\mathcal{S}$ are principal ideal domains, so every prime ideal is maximal and they have Krull dimension one. The quotient field of $\mathcal{R}$ is equal to $\mathbb{C}(s)$. The state space H is then a finitely-generated torsion module over the principal ideal domain $\mathcal{R}$, and hence has a Jordan decomposition. The Laplace transformation of these variables is simple. Starting from the formula $n!/(1+s)^{n+1} = \int_0^\infty t^n e^{-(s+1)t}dt$, we can conveniently express elements of $\mathcal{R}$ in term of the Laguerre basis of $L^2((0,\infty);\mathbb{C})$; see [3] and [57, (5.36)]. In Lemma 2.5, we obtain conditions for Hankel operators on $L^2((0,\infty);\mathbb{C})$, defined via the Laplace transform, to be bounded or Hilbert–Schmidt. In Proposition 2.8 we obtain a characterization of products of Hankel operators in terms of Pöppe's semi-additive operators; see [45]. These are interesting in their own right, and lead to cocycle formulas as in Theorem 3.13. We now consider the consequences for operators.

Definition 2.2. For $h \in L^2((0,\infty);\mathbb{C})$ we shall let $\mathscr{L}h(s) = \hat{h}(s) = \int_0^\infty e^{-st}h(t)\,dt$, $\operatorname{Re} s > 0$, denote the Laplace transform of h.

Definition 2.3. (i) Let $\mathcal{L}^2(H)$ be the space of Hilbert–Schmidt operators on H with the usual inner product $\langle K, L\rangle = \operatorname{trace}(KL^\dagger)$.

(ii) Let $\mathcal{S}$ be a domain in $\mathbb{C}$ with a C^∞ smooth boundary $\partial\mathcal{S}$, and let ds be the arclength measure on $\partial\mathcal{S}$. Let $\mathcal{D}_{\mathcal{S}}$ be the subalgebra of $L^\infty(\partial\mathcal{S};\mathbb{C})$ such that the norm

$$\|h\|_{\mathcal{D}}^2 = \|h\|_\infty^2 + \iint_{\partial\mathcal{S}\times\partial\mathcal{S}} \frac{|h(x)-h(y)|^2}{|x-y|^2}\frac{ds(x)ds(y)}{\pi} \tag{2.3}$$

is finite. Saitoh [52, Section 5] discussed equivalent conditions for finiteness of this integral.

(iii) For $h \in L^\infty(\partial\mathcal{S};\mathbb{C}))$ let $M_h \in \mathcal{L}(L^2(\partial\mathcal{S};\mathbb{C})$ denote the operator of multiplication by h, namely $f \mapsto hf$.

Definition 2.4. Let $\mathcal{E}, \mathcal{F}$ be a vector subspaces of $\mathcal{L}(H)$.

(i) We write $\mathcal{E}^\dagger = \{X^\dagger : X \in \mathcal{E}\}$ and say that $\mathcal{E}$ is self-adjoint if $\mathcal{E}^\dagger = \mathcal{E}$.

(ii) With $[X,Y] = XY - YX$ for the additive commutator, we write $[\mathcal{E},\mathcal{F}] = \operatorname{span}\{[X,Y] : X \in \mathcal{E}, Y \in \mathcal{F}\}$. The commutator subspace of an algebra $\mathcal{E}$ is $[\mathcal{E},\mathcal{E}]$.

By the Paley–Wiener Theorem, $\mathscr{L} : L^2((0,\infty);\mathbb{C}) \to H^2(RHP)$ gives a unitary operator. In (2.2), Γ_ϕ is a bounded linear operator $L^2((0,\infty);\mathbb{C}) \to L^\infty((0,\infty);\mathbb{C})$. If $t^{1/2}\phi(t) \in L^2((0,\infty);\mathbb{C})$ then Γ_ϕ is a Hilbert–Schmidt operator on $L^2((0,\infty);\mathbb{C})$. The following result links $\mathscr{L}$ with Γ_ϕ.

Lemma 2.5. *Let $h \in L^\infty(0,\infty) \cap L^2((0,\infty);\mathbb{C})$ and let $\phi = \mathscr{L}h$. Then*

(i) $\Gamma_\phi = \mathscr{L}M_h\mathscr{L}$ and so gives a bounded Hankel integral operator on $L^2((0,\infty);\mathbb{C})$.

(ii) Suppose further that $\int_0^\infty t^{-1}|h(t)|^2\,dt$ converges. Then Γ_ϕ is Hilbert–Schmidt.

Proof. (i) Let $P_n(t)$ be the Laguerre polynomial of degree n and let $\ell_n(t) = \sqrt{2}e^{-t}P_n(2t)$. Then $(\ell_n)_{n=0}^\infty$ forms an orthonormal basis for $L^2((0,\infty);\mathbb{C})$ which gives a Hankel matrix

$$\begin{aligned}\langle \Gamma_\phi \ell_n, \ell_m\rangle_{L^2((0,\infty);\mathbb{C})} &= \int_0^\infty h(t)\frac{2(t-1)^{n+m}}{(t+1)^{n+m+2}}\,dt \\ &= \int_{-1}^1 h\Big(\frac{1+u}{1-u}\Big)u^{n+m}\,du \qquad (n,m = 0,1,\dots).\end{aligned} \tag{2.4}$$

For $f \in L^2((0,\infty);\mathbb{C})$,

$$\mathscr{L}M_h\mathscr{L}f(x) = \int_0^\infty e^{-xs}h(s)\int_0^\infty e^{-st}f(t)\,dtds \tag{2.5}$$
$$= \int_0^\infty\int_0^\infty e^{-s(x+t)}h(s)\,ds\,f(t)\,dt$$
$$= \int_0^\infty \phi(x+t)f(t)\,dt. \tag{2.6}$$

Thus $\Gamma_\phi = \mathscr{L}M_h\mathscr{L}$ is a composition of bounded operators and hence is bounded.

(ii) The Laplace transform $\mathscr{L}$ on $L^2((0,\infty);\mathbb{C})$ is a self-adjoint operator, with square $\mathscr{L}^2$ given by Carleman's Hankel operator [46, Theorem 2.6] with integral kernel $1/(x+y)$ on $L^2((0,\infty);\mathbb{C})$ which is known to have spectrum $[0,\pi]$. We have equality of spectra as operators on $L^2((0,\infty);\mathbb{C})$

$$\sigma(\Gamma_\phi)\setminus\{0\} = \sigma(\mathscr{L}M_h\mathscr{L})\setminus\{0\} = \sigma(M_h\mathscr{L}^2)\setminus\{0\},$$

where $M_h\mathscr{L}^2$ has kernel $h(t)/(x+t)$ as an integral operator on $L^2((0,\infty);\mathbb{C})$, and this kernel is Hilbert–Schmidt. We momentarily assume that ϕ is real-valued; otherwise, we split into real and imaginary parts. Then by [54, Theorem 8.1], the singular numbers satisfy

$$\sum_{n=0}^\infty \mu_n(\mathscr{L}M_h\mathscr{L})^2 \leq \sum_{n=0}^\infty \mu_n(M_h\mathscr{L}^2)^2$$

so $\mathscr{L}M_h\mathscr{L}$ is also Hilbert–Schmidt. □

Definition 2.6. (i) For $t > 0$, let $S_t \in \mathcal{L}(L^2((0,\infty);\mathbb{C})$ denote the shift operator $S_tf(x) = f(x-t)\mathbb{I}_{(0,\infty)}(x-t)$, $x > 0$, and let $S_t^\dagger$ be its adjoint.

(ii) [45] Let $(K_t)_{t>0}$ be a family of integral operators on $L^2((0,\infty);\mathbb{C})$ with kernels $(k_t)_{t>0}$, so $K_tf(x) = \int_0^\infty k_t(x,y)f(y)\,dy$. We say that $(K_t)_{t>0}$ is semi-additive if there exists a function k such that $k_t(x,y) = k(x+t,y+t)$ for all $x,y,t > 0$.

(iii) For a differentiable function of two variables, we write ∂_Δ for the diagonal derivative, so $\partial_\Delta k(x,y) = (\frac{\partial}{\partial x} + \frac{\partial}{\partial y})k(x,y)$.

The families $(S_t)_{t>0}$ and $(S_t^\dagger)_{t>0}$ both give strongly continuous contraction semigroups on $L^2((0,\infty);\mathbb{C})$, and $(S_t)_{t>0}$ is a semigroup of isometries. Note that for $\phi \in L^2((0,\infty);\mathbb{C})$ the space $Y_\phi = \mathrm{cl\,span}\{\phi(x+t) : t > 0\}$ gives a closed linear subspace of $L^2((0,\infty);\mathbb{C})$ that is invariant under the adjoint shift semigroup $(S_t^\dagger)_{t>0}$, so $Y_\phi^\perp = L^2 \ominus Y_\phi$ is invariant under the shift semigroup $(S_t)_{t>0}$. The space $Y_\phi^\perp$ is the null space of $\Gamma_\phi^\dagger$, which has kernel $\bar\phi(x+y)$; indeed, by Plancherel's formula,

$$h \in Y_\phi^\perp \iff \int_0^\infty \phi(x+y)^\dagger h(y)\,dy = 0 \qquad (x>0)$$
$$\iff \int_{-\infty}^\infty \overline{\hat\phi_{(x)}(iy)}\hat h(iy)\,\frac{dy}{\pi i} = 0 \qquad (x>0).$$

Hankel integral operators Γ on $L^2((0,\infty);\mathbb{C})$ are characterized by the intertwining relation $S_t^\dagger \Gamma = \Gamma S_t$ for all $t > 0$. Pöppe [45] called such operators additive in view of $\phi(x+t)$ in their kernels (2.2). The notion of semi-additivity is related to products of Hankel operators, as in Proposition 2.8 below.

For ease of notation we shall write $\mathcal{L}^2$ for $\mathcal{L}^2(L^2((0,\infty);\mathbb{C}))$ in the lemma below. For $t > 0$ define $\sigma_t : \mathcal{L}^2 \to \mathcal{L}^2$ by $\sigma_t(K) = S_t^\dagger K S_t$. A small calculation shows that if $K \in \mathcal{L}^2$ has kernel $k(x,y) \in L^2((0,\infty)\times(0,\infty);\mathbb{C})$ then $\sigma_t(K) = S_t^\dagger K S_t$ is the integral operator with kernel $k(x+t, y+t)$. Let $\tilde\sigma_t(K) = S_t K S_t^\dagger$.

Proposition 2.7. *Let $(\sigma_t)_{t>0}$ and $(\tilde\sigma_t)_{t>0}$ be as above. Then*

(i) $(\sigma_t)_{t>0}$ is a strongly continuous contraction semigroup on $\mathcal{L}^2$.
(ii) For all $K \in \mathcal{L}^2$, $\sigma_t(K) \to 0$ in the strong operator topology as $t \to \infty$.
(iii) the infinitesimal generator of $(\sigma_t)_{t>0}$ is the diagonal derivative ∂_Δ.
(iv) $(\tilde\sigma_t)_{t>0}$ is a strongly continuous contraction semigroup on $\mathcal{L}^2$.
(v) For all $K, L \in \mathcal{L}^2$, $\tilde\sigma_t(KL) = \tilde\sigma_t(K)\tilde\sigma_t(L)$ and $\langle \tilde\sigma_t(K), L\rangle = \langle K, \sigma_t(L)\rangle$.

Proof. (i) This is straightforward. Note that $\|S_t^\dagger K S_t\|_{\mathcal{L}^2} \le \|S_t^\dagger\| \|K\|_{\mathcal{L}^2} \|S_t\| = \|K\|_{\mathcal{L}^2}$.

(ii) If $K \in \mathcal{L}^2$ has kernel k then

$$\|\sigma_t(K)\|_{\mathcal{L}^2}^2 = \iint_{(0,\infty)\times(0,\infty)} |k(x+t,y+t)|^2 \, dxdy$$

which converges to 0 as $t \to \infty$.

(iii) For suitable differentiable k the operator $d\sigma_t(K)/dt$ has kernel $(\frac{\partial}{\partial x} + \frac{\partial}{\partial y})k(x+t,y+t)$, so ∂_Δ is the diagonal derivative.

(iv) As in (ii), we have

$$\|K - \sigma_t(K)\|_{\mathcal{L}^2}^2 = \iint_{(0,\infty)\times(0,\infty)} |k(x,y) - k(x+t,y+t)|^2 \, dxdy$$

which converges to 0 as $t \to 0+$.

(v) For the first identity we have that since $S_t^\dagger S_t = I$,

$$\tilde\sigma_t(KL) = S_t K L S_t^\dagger = S_t K S_t^\dagger S_t L S_t^\dagger = \tilde\sigma_t(K)\tilde\sigma_t(L).$$

For the second,

$$\mathrm{trace}(\tilde\sigma_t(K)L^\dagger) = \mathrm{trace}(S_t K S_t^\dagger L^\dagger) = \mathrm{trace}(K S_t^\dagger L^\dagger S_t) = \mathrm{trace}(K\sigma_t(L)^\dagger). \qquad \square$$

Proposition 2.8. *Suppose that $\phi, \psi \in L^2((0,\infty);\mathbb{C}^n)$ with ϕ and ψ absolutely continuous and let $K = \Gamma_\phi^\top \Gamma_\psi$ have kernel k. Then the kernel $\partial_\Delta k_t(x,y)$ determines an integral operator that has finite rank.*

Proof. We have $k(x,y) = \int_0^\infty \phi(x+z)^\top \psi(z+y)\, dz$. Then by integration by parts, we find

$$\partial_\Delta \int_0^\infty \phi(x+z)^\top \psi(z+y)\, dz = -\phi(x)^\top \psi(y). \qquad \square$$

3. Commutators, cocycles and Hankel products for multiply connected planar domains

We extend the notion of a Hankel operator to Hardy spaces on multiply connected planar domains, and show that Hankel operators and Toeplitz operators together satisfy semi-commutator identities. Products of Hankel operators arise as noncommutative differentials, which are defined as follows.

Definition 3.1. Let $\mathcal{R}$ and $\mathcal{L}$ be unital complex algebras.

(i) We define the space of noncommutative differentials, denoted $\Omega^1\mathcal{R}$, as the subspace of $\mathcal{R}\otimes\mathcal{R}$ that makes the sequence

$$0 \longrightarrow \Omega^1\mathcal{R} \longrightarrow \mathcal{R}\otimes\mathcal{R} \xrightarrow{\mu} \mathcal{R} \longrightarrow 0 \tag{3.1}$$

exact, where $\mu(a\otimes b) = ab$ is the multiplication map. The space $\Omega^1\mathcal{R}$ is spanned by $a\,db = ab\otimes 1 - a\otimes b$ for $a, b \in \mathcal{R}$. Evidently $\Omega^1\mathcal{R}$ is a left $\mathcal{R}$-module.

(ii) Let $\rho : \mathcal{R} \to \mathcal{L}$ be a $\mathbb{C}$-linear map. We define its curvature to be $\varpi : \mathcal{R}\otimes\mathcal{R} \to \mathcal{L}$ where as the bilinear operator determined by $\varpi(f_1, f_2) = \rho(f_1f_2) - \rho(f_1)\rho(f_2)$, so $f_1df_2 \mapsto \varpi(f_1, f_2)$ is a linear map $\Omega^1\mathcal{R} \to \mathcal{L}$. (The curvature is related to the notion of semi-commutator on operator theory. This is not to be confused with the curvature of modules discussed in [18, p. 116]). The dual space of $\Omega^1\mathcal{R}$ is identified in Definition 3.5.

(iii) Suppose further that $\mathcal{R}$ is commutative. Then $\Omega^1\mathcal{R}$ may be regarded as an ideal in $\mathcal{R}\otimes\mathcal{R}$ with square ideal $(\Omega^1\mathcal{R})^2$, so the quotient of ideals $\Omega^1_{\mathcal{R}} = \Omega^1\mathcal{R}/(\Omega^1\mathcal{R})^2$ is a left $\mathcal{R}$-module called the space of first-order Kähler differentials; see [29, p. 173].

(iv) The space of n^{th}-order Kähler differentials $\Omega^n_{\mathcal{R}}$ is the skew-symmetric subspace $\Omega^1_{\mathcal{R}}\wedge\cdots\wedge\Omega^1_{\mathcal{R}}$ of $\otimes_{j=1}^n\Omega^1_{\mathcal{R}}$ with n factors, in which our tensors are formed over the field $\mathbb{C}$.

In Proposition 2.8, the diagonals of kernels emerge in the course of calculations involving products of Hankel operators. In the following section, we express these in a unified framework which will enable us to formulate results in later sections more easily, such as Theorem 4.2.

Suppose that $\mathcal{A}$ is a commutative and unital Banach algebra, so the projective tensor product $\mathcal{A}\hat{\otimes}\mathcal{A}$ is likewise. Let $\mathcal{A}'$ denote the dual space of $\mathcal{A}$, let $\mathbb{X}$ be the maximal ideal space, identified with a subspace of $\mathcal{A}'$, and let $a \mapsto \hat{a}$ be the Gelfand transform $\mathcal{A} \to C(\mathbb{X};\mathbb{C})$. From the formula $\psi(a_0\otimes a_1) = \psi(a_0\otimes 1)\psi(1\otimes a_1)$, Tomiyama observed that the maximal ideal space of $\mathcal{A}\hat{\otimes}\mathcal{A}$ is $\mathbb{X}\times\mathbb{X}$. Let $\mathbb{X}_\Delta = \{(\phi,\phi) : \phi \in \mathbb{X}\}$ be its diagonal.

Proposition 3.2. *Let $\mathcal{A}$ be semisimple.*

(i) Under the Gelfand transform $\mathcal{A}\hat{\otimes}\mathcal{A} \to C(\mathbb{X}\times\mathbb{X};\mathbb{C})$, the space $\Omega^1\mathcal{A}$ is mapped into $\{\hat{a} \in C(\mathbb{X}\times\mathbb{X};\mathbb{C}) : \hat{a}|\mathbb{X}_\Delta = 0\}$.

(ii) For every nonzero derivation $\partial : \mathcal{A} \to \mathcal{A}$, there exists a unique map $a_0 da_1 \mapsto \widehat{(a_0 \partial a_1)}$ which is nonzero.

Proof. (i) Here $\mathbb{X}_\Delta$ is the image of the canonical inclusion $\mathbb{X} \to \mathbb{X} \times \mathbb{X} : \phi \mapsto (\phi, \phi)$, which is compatible with $\phi \mapsto \phi \circ \mu$, and $\Omega^1 \mathcal{A}$ is the nullspace of μ.

(ii) We can regard $\mathcal{A}$ as a natural $\mathcal{A}$-bimodule for multiplication. Then the map $\Omega^1 \mathcal{A} \mapsto \mathcal{A} : a_0 da_1 \mapsto a_0 \partial a_1$ exists by the universal property of $\Omega^1 \mathcal{A}$; see [50, Proposition 8.4.4]. The Gelfand transform is injective on the semisimple Banach algebra $\mathcal{A}$. □

For a complex unital algebra $\mathcal{R}$, there is a unique differential graded algebra $\Omega\mathcal{R} = \oplus_{n=-\infty}^{\infty} \Omega^n \mathcal{R}$ such that $\Omega^0 \mathcal{R} = \mathcal{R}$ with a graded differential $d : \Omega^n \mathcal{R} \to \Omega^{n+1} \mathcal{R}$ with the following universal property. Given any differential graded algebra S and homomorphism $u : \mathcal{R} \to S$, there is a unique homomorphism of differential graded algebras $\Omega\mathcal{R} \to S$ that extends u.

Definition 3.3. Let $b : \Omega\mathcal{R} \to \Omega\mathcal{R}$ be the linear map determined by $b : \Omega^{n+1}\mathcal{R} \to \Omega^n \mathcal{R}$ $b(\omega da) = (-1)^n (\omega a - a\omega)$. Then $b^2 = 0$, so there is a complex

$$0 \longleftarrow \mathcal{R} \longleftarrow \Omega^1 \mathcal{R} \longleftarrow \Omega^2 \mathcal{R} \longleftarrow \cdots$$

with differential b of order (-1). Then the Hochschild homology $HH_*(\mathcal{R})$ is the homology of the complex $(\Omega\mathcal{R}, b)$.

Remark 3.4. Let $\mathcal{R}$ be a complex and unital algebra with the commutator subspace $[\mathcal{R}, \mathcal{R}] = \mathrm{span}\{[r, s] : r, s \in \mathcal{R}\}$. Let

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{R} \longrightarrow \mathcal{A} \longrightarrow 0$$

be an exact sequence of algebras and homomorphisms such that $[\mathcal{R}, \mathcal{R}] \subseteq \mathcal{I}$.

(i) Then $\mathcal{A} = \mathcal{R}/\mathcal{I}$ is a unital and commutative algebra. Indeed, $[\mathcal{A}, \mathcal{A}] = \{0\}$, so $\mathcal{A}$ is commutative. There is an injective map $\Omega^j_{\mathcal{A}} \to HH_j(\mathcal{A})$ from the Kähler differentials to the Hochschild homology of $\mathcal{A}$ by [50, Prop 7.2.3] or[38, Prop 1.1.10] for $j = 1, 2, \dots$. From this we have $HH_0(\mathcal{A}) = \mathcal{A}$, and $\Omega^1_{\mathcal{A}} = HH_1(\mathcal{A})$; also $\Omega^2_{\mathcal{A}}$ is a direct summand of $HH_2(\mathcal{A})$. If $\mathcal{A}$ is smooth, then $\Omega^2_{\mathcal{A}} = HH_2(\mathcal{A})$ by [38, Theorem 3.4.4]. In particular, the polynomial algebra $\mathbb{C}[x_1, \dots, x_n]$ is smooth. Also, if $\mathcal{A}$ is the ring of algebraic functions on a nonsingular algebraic variety over $\mathbb{C}$, then $\mathcal{A}$ is smooth by [38, Prop 3.4.2].

(ii) With $\mathbb{X}$ the maximal ideal space of $\mathcal{A}$, there is a Gelfand map $\mathcal{A} \to C(\mathbb{X}; \mathbb{C})$, so $\Omega^j_{\mathcal{A}}$ may be regarded as a space of Kähler differential forms on $\mathbb{X}$.

(iii) There is a surjective map $HH_n(\mathcal{R}) \to HH_n(\mathcal{A})$ for $n = 0, 1, \dots$ by [38].

In our applications in Proposition 3.8 and 4.8, $\mathcal{R}$ is an algebra of operators and $\mathcal{A}$ may be viewed as a commutative algebra of symbols of these operators. The Kähler differentials give a measure of the complexity of $\mathcal{R}$. We consider the dual space of $\Omega^1 \mathcal{R}$ in Definition 3.5 as in [50, section 7.2].

Definition 3.5. Let $\mathcal{D}$ a complex unital algebra.

(i) Let $\varphi_n : \mathcal{D}^{n+1} \to \mathbb{C}$ a multilinear functional. Then the Hochschild coboundary is $(b\varphi_n) : \mathcal{D}^{n+2} \to \mathbb{C}$ given by

$$(b\varphi_n)(a_0, \dots, a_{n+1}) = \sum_{j=0}^{n} (-i)^j \varphi_n(a_0, \dots, a_j a_{j+1}, \dots, a_n) + (-1)^{n+1} \varphi_n(a_{n+1}a_0, \dots, a_n).$$

If $(b\varphi_n) = 0$, then we call φ_n a Hochschild n-cocycle.

(ii) Suppose further that $\varphi_n(a_1, \dots, a_n, a_0) = (-1)^n \varphi_n(a_0, \dots, a_n)$. Then φ_n is a cyclic n-cocycle. By [13, p. 186], such a φ_n is equivalent to a trace $\tilde{\varphi}_n : \Omega^n \mathcal{D} \to \mathbb{C}$ via $\varphi_n(a_0, \dots, a_n) = \tilde{\varphi}_n(a_0 da_1 \dots da_n)$.

(iii) An even normalized cochain is a $\mathbb{C}$-linear map $\psi_{2m} : \mathcal{D}^{2m+1} \to \mathbb{C}$ such that

$$\psi_{2m}(a_0, a_1, \dots, a_{2m}) = 0$$

if $a_j = 1$ for some $j = 1, \dots, 2m$. Equivalently, ψ_{2m} is a multiplinear map $\mathcal{D} \times (\mathcal{D}/\mathbb{C}) \times \dots \times (\mathcal{D}/\mathbb{C}) \to \mathbb{C}$, where we identify $\mathbb{C}$ with the subspace of $\mathcal{D}$ spanned by the unit.

Definition 3.6. Let $\mathcal{D}$ a complex unital algebra and let $\mathcal{J}$ be an ideal in $\mathcal{L}(H)$. Let $\rho : \mathcal{D} \to \mathcal{L}(H)$ be a $\mathbb{C}$-linear map such that the curvature takes values in $\mathcal{J}$. Then ρ is said to be a homomorphism modulo $\mathcal{J}$.

Example 3.7. Let $\mathcal{J} = \mathcal{L}^m(H)$ so the ideal generated by m-fold products of elements of $\mathcal{J}$ is $\mathcal{J}^m = \mathcal{L}^1(H)$, the trace class operators and suppose that ϖ takes values in $\mathcal{J}$, so

$$\psi_{2m}(a_0, a_1, \dots, a_{2m}) = \operatorname{trace}\big(\rho(a_0)\varpi(a_1, a_2) \dots \varpi(a_{2m-1}, a_{2m})\big)$$

gives a normalized even cochain. We can write $a_0 da_1 da_2 \mapsto \rho(a_0)\varpi(a_1, a_2)$.

We consider cocycles over multiply connected domains. Let $\mathcal{S}$ be a bounded planar domain with boundary $\partial\mathcal{S}$ made of C^∞ smooth and positively oriented Jordan arcs C_j $j = 0, \dots, \ell$ that do not intersect. We let D_j be the inside of C_j, and suppose that D_j are disjoint for $j = 1, \dots, \ell$, so $\mathcal{S} = D_0 \setminus \cup_{j=1}^{\ell} \operatorname{cl}(D_j)$. That is, $\mathcal{S}$ may be regarded as a lake surrounding islands D_j, and so may be multiply connected; see [14]. Let s denote arclength measure on $\partial\mathcal{S}$, and let $L^2 = L^2(\partial\mathcal{S}; ds; \mathbb{C})$.

Let $\mathscr{C}_j$ be the Cauchy integral operator

$$\mathscr{C}_j f(z) = \frac{\varepsilon_j}{2\pi i} \int_{C_j} \frac{f(\zeta)}{\zeta - z} \frac{d\zeta}{ds}\, ds \qquad (f \in L^2), \tag{3.2}$$

where $\varepsilon_0 = 1$ and $\varepsilon_j = -1$ for $j = 1, \dots, \ell$. Let H^2 be the Hardy subspace of L^2 consisting of functions that are the boundary values of holomorphic functions on $\mathcal{S}$; then the orthogonal projection $\mathcal{R}_{H^2} : L^2 \to H^2$ is the Riesz–Szegö projection. For this $\mathcal{S}$, let $\mathcal{D}$ be as in (2.3) for $\mathcal{S}$, let $\mathcal{M}_{\mathcal{D}} = \{M_f : f \in \mathcal{D}\}$, and let $\mathcal{A}$ be the subalgebra of $\mathcal{D}$ given by the functions that are holomorphic on $\mathcal{S}$. The following results address the technical issue that Cauchy integrals are convenient to use for computations, and definitions such as (4.8), whereas $\mathcal{R}_{H^2}$ is more convenient as an operator.

Proposition 3.8. *(i) Let $\mathcal{E}_{\mathcal{S}}$ be the subalgebra of $\mathcal{L}(L^2)$ that is generated by $\mathcal{M}_{\mathcal{D}}$ and $\mathscr{C}_j$ for $j = 0, \dots, \ell$. Then $\mathcal{E}_{\mathcal{S}}/(\mathcal{E}_{\mathcal{S}} \cap \mathcal{L}^2(L^2))$ is a commutative complex algebra. Furthermore $M_{f_0}[\mathcal{R}_{H^2}, M_{f_1}][\mathcal{R}_{H^2}, M_{f_2}]$ is trace class for all $f_0, f_1, f_2 \in \mathcal{D}$, and*

$$\psi_2(f_0, f_1, f_2) = \mathrm{trace}(M_{f_0}[\mathcal{R}_{H^2}, M_{f_1}][\mathcal{R}_{H^2}, M_{f_2}]) \tag{3.3}$$

gives a cochain $\mathcal{D}^3 \to \mathbb{C}$.

(ii) Let $\mathcal{E}$ be the subalgebra of $\mathcal{L}(L^2)$ that is generated by $\mathcal{M}_{\mathcal{D}}$ and $\mathcal{R}_{H^2}$. Then $\mathcal{E}/(\mathcal{E} \cap \mathcal{L}^2(L^2))$ is a commutative and self-adjoint complex algebra.

(iii) Let $\mathcal{E}_{H^2} = \{X \in \mathcal{E} : X\mathcal{R}_{H^2} = \mathcal{R}_{H^2} X \mathcal{R}_{H^2}\}$. Then $\mathcal{E}_{H^2}$ is a complex subalgebra of $\mathcal{E}$ and there is homomorphism $\mathcal{E}_{H^2} \to \mathcal{L}(H^2) : X \mapsto X\mathcal{R}_{H^2}$, such that the range contains the Toeplitz operators $\mathcal{R}_{H^2} M_f \mathcal{R}_{H^2}$ on H^2 with symbols $f \in \mathcal{A}$.

(iv) Let $\Gamma_f = \mathcal{R}_{H^2} M_f (I - \mathcal{R}_{H^2})$ for $f \in \mathcal{A}$. Then $[T_f^\dagger, T_f]$ is trace class if and only if Γ_f is Hilbert–Schmidt.

Proof. (i) By repeatedly applying Cauchy's theorem, one shows that $\mathscr{C}_j\mathscr{C}_k = 0$ for all $j \neq k$ and $\mathscr{C}_j\mathscr{C}_j = \mathscr{C}_j$ for $j = 0, \dots, \ell$. Note that $\mathcal{E}_{\mathcal{S}}$ is a unital complex algebra. For all $f \in \mathcal{D}$, the operator $[\mathscr{C}_j, M_f]$ has kernel $(f(z) - f(\zeta))(z - \zeta)^{-1}(dz/ds)$, which is Hilbert–Schmidt on $L^2(\partial\mathcal{S}; ds; \mathbb{C})$. We also have $[M_f, M_g] = 0$ for all $f, g \in \mathcal{D}$.

The next step is to check that the commutators of any pair of elements of the form $M_{f_0}\mathscr{C}_{j_0} \dots \mathscr{C}_{j_m} M_{f_{j_m}}$ is Hilbert–Schmidt. Since $\mathcal{L}^2(L^2)$ is an ideal, and $T \mapsto [T, U]$ is a derivation for all $U \in \mathcal{L}(L^2)$, this reduces to showing that the commutators of pairs of elements of $\{\mathscr{C}_j, M_f : f \in \mathcal{D}, j = 0, \dots, \ell\}$ are Hilbert–Schmidt. Hence $[\mathcal{E}_{\mathcal{S}}, \mathcal{E}_{\mathcal{S}}] \subset \mathcal{L}^2(L^2)$ and all the commutators belong to the Hilbert–Schmidt ideal, so $[\mathcal{E}_{\mathcal{S}}, \mathcal{E}_{\mathcal{S}}] \subseteq \mathcal{L}^2(L^2)$. It follows that the ideal generated by $[\mathcal{E}_{\mathcal{S}}, \mathcal{E}_{\mathcal{S}}]$ is contained in the ideal $\mathcal{E}_{\mathcal{S}} \cap \mathcal{L}^2(L^2)$ of $\mathcal{E}_{\mathcal{S}}$, so $\mathcal{E}_{\mathcal{S}}/(\mathcal{E}_{\mathcal{S}} \cap \mathcal{L}^2(L^2))$ is a commutative algebra.

We observe that $\mathscr{C} = \sum_{j=0}^{\ell} \mathscr{C}_j$ satisfies

$$\mathscr{C} f(z) = \sum_{j=0}^{\ell} \mathscr{C}_j f(z) = \sum_{j=0}^{\ell} \frac{\varepsilon_j}{2\pi i} \int_{C_j} \frac{f(\zeta)}{\zeta - z} \frac{d\zeta}{ds} \, ds = \frac{1}{2\pi i} \int_{\partial\mathcal{S}} \frac{f(\zeta)}{\zeta - z} \, d\zeta.$$

If f is continuous on $\mathrm{cl}(\mathcal{S})$ and holomorphic on $\mathcal{S}$, then $\mathscr{C} f = f$ by the Cauchy integral formula. Also $[\mathscr{C}_j, M_{f_1}][\mathscr{C}, M_{f_2}]$ is trace class for all $f_0, f_1, f_2 \in \mathcal{D}$, so $M_{f_0}[\mathscr{C}, M_{f_1}][\mathscr{C}, M_{f_2}]$ is trace class with

$$\begin{aligned} \psi_2(f_0, f_1, f_2) &= \mathrm{trace}\big(M_{f_0}[\mathscr{C}, M_{f_1}][\mathscr{C}, M_{f_2}]\big) \\ &= \frac{1}{(2\pi i)^2} \iint_{\partial\mathcal{S} \times \partial\mathcal{S}} f_0(z) \frac{f_1(z) - f_1(\zeta)}{z - \zeta} \frac{f_2(\zeta) - f_2(z)}{\zeta - z} \, dz d\zeta, \end{aligned} \tag{3.4}$$

where $\psi_2(f_0, 1, f_2) = \psi_2(f_0, f_1, 1) = 0$. Hence ψ_2 gives a bounded linear functional $\Omega^2\mathcal{D} \to \mathbb{C}$. We pause to note that this formula for the even cochain ψ_2 is not to be confused with the superficially similar expression for the odd cocycle in [13, p. 209]; indeed, $b\psi_2$ is an odd cocycle, possibly nonzero. See [50, 7.10].

Except in special cases such as when $\partial\mathcal{S}$ is a circle, $\mathscr{C}$ does not give a self-adjoint operator and is not equal to $\mathcal{R}_{H^2}$. Hence we need to convert (3.4) into a statement about the Riesz projection. We consider Plemelj's formula

$$Ef(z) = \frac{1}{2}f(z) + \frac{1}{2\pi i}\,\text{p.v.}\int_{\partial\mathcal{S}} \frac{f(\zeta)}{\zeta - z}\frac{d\zeta}{ds}\,ds, \qquad z \in \partial\mathcal{S}, \tag{3.5}$$

which is defined for all $f \in C^\infty(\partial\mathcal{S};\mathbb{C}))$, and gives $E \in \mathcal{L}(L^2)$ such that $E\mathcal{R}_{H^2} = \mathcal{R}_{H^2}$ and $\mathcal{R}_{H^2}E = E$. Then one can easily show that $E^2 = E$, and $H^2 = EL^2$.

While E is not self-adjoint, we proceed to show [34, Theorem 5.2] by calculus that $i(E^\dagger - E)$ is Hilbert–Schmidt and self-adjoint. With a unit speed parametrization of C_j, we have $z = \gamma(t)$ and $\zeta = \gamma(u)$ where $\gamma'(t) = e^{i\theta(t)}$ for some real function θ, hence by the mean-value theorem

$$\begin{aligned}
&\frac{\gamma'(t)}{2\pi i(\gamma(t) - \gamma(u)} - \frac{\bar\gamma'(u)}{2\pi i(\bar\gamma(t) - \bar\gamma(u)}\\
&\quad = \frac{1}{(u-t) + (i/2)(u-t)^2\theta(\bar t)e^{i\theta(\bar t) - i\theta(t)}} - \frac{1}{(u-t) + (i/2)(u-t)^2\theta(\bar u)e^{i\theta(u) - i\theta(\bar u)}}\\
&\quad = \frac{(i/2)(\theta'(\bar u)e^{i\theta(u) - i\theta(\bar u)} - \theta'(\bar t)e^{i\theta(\bar t) - i\theta(t)})}{(1 + (i/2)(u-t)\theta'(\bar t)e^{i\theta(\bar t) - i\theta(t)})(1 + (i/2)(u-t)\theta(\bar u)e^{i\theta(u) - i\theta(\bar u)})}
\end{aligned}$$

for some $\bar t$ and $\bar u$ between u and t, so the kernel of $E^\dagger - E$ is continuous and vanishes on the diagonal. We can therefore form $E^\dagger - E$ and remove the principal value used in the definition (3.5) of E, and deduce that $E^\dagger - E$ is Hilbert–Schmidt; equivalently one can work with the more intuitive expression $\mathscr{C}^\dagger - \mathscr{C}$. Further, one has the Hilbert–Schmidt operator $\mathcal{R}_{H^2}(E^\dagger - E) = \mathcal{R}_{H^2} - E$. As in Definition 2.3 $[M_h, \mathscr{C}]$ is Hilbert–Schmidt for $h \in \mathcal{D}$. Hence we have an identity of Hilbert–Schmidt operators

$$[M_h, \mathcal{R}_{H^2}] = [M_h, E] + [M_h, \mathcal{R}_{H^2}(E^\dagger - E)] = [M_h, \mathscr{C}] + [M_h, \mathcal{R}_{H^2}(\mathscr{C}^\dagger - \mathscr{C})].$$

(ii) We need to show that $[\mathcal{E}, \mathcal{E}] \subseteq \mathcal{L}^2(L^2)$. As above $\mathcal{L}^2(L^2)$ is an ideal, and $T \mapsto [T, U]$ is a derivation for all $U \in \mathcal{L}(L^2)$, so it suffices to note that $[M_f, \mathcal{R}_{H^2}] \in \mathcal{L}^2(L^2)$ for all $f \in \mathcal{D}$. Hence the ideal generated by $[\mathcal{E}_\mathcal{S}, \mathcal{E}_\mathcal{S}]$ is contained in the ideal $\mathcal{L}^2(L^2) \cap \mathcal{E}$ of $\mathcal{E}$, so $\mathcal{E}/(\mathcal{E} \cap \mathcal{L}^2(L^2))$ is a commutative algebra. Note that $M_f^\dagger = M_{\bar f}$ and $\mathcal{R}_{H^2}^\dagger = \mathcal{R}_{H^2}$, so the algebra is self-adjoint.

(iii) Let $\mathcal{A}$ be the commutative Banach algebra given by the $f \in \mathcal{D}$ that are holomorphic on $\mathcal{S}$. Then $\mathcal{S}$ is contained in the maximal ideal space of $\mathcal{A}$. Let $T_f = \mathcal{R}_{H^2}M_f\mathcal{R}_{H^2}$; then $\{T_f : f \in \mathcal{A}\}$ gives a commutative unital subalgebra of $\mathcal{L}(H^2)$, such that the spectrum of T_f is $\mathrm{cl}\{f(z) : z \in \mathcal{S}\}$ for all $f \in \mathcal{A}$. Indeed, for all $z \in \mathcal{S}$, there exists $h_z \in H^2$ such that $\langle h, h_z\rangle = h(z)$ for all $h \in H^2$, then $T_f^\dagger h_z = \overline{f(z)} h_z$, so the spectrum of $T_f^\dagger$ contains $\mathrm{cl}\{\overline{f(z)} : z \in \mathcal{S}\}$. Also $(\lambda - f)^{-1} \in \mathcal{A}$ for all $\lambda \in \mathbb{C} \setminus \mathrm{cl}\{f(z) : z \in \mathcal{S}\}$. For $f \in \mathcal{A}$, we can identify $T_f = M_f \mathcal{R}_{H^2}$ with an element of $\mathcal{E}_{H^2}$, and thereby obtain a homomorphism $\mathcal{A} \to \mathcal{E}_{H^2}$, $f \mapsto T_f$.

(iv) For $f \in \mathcal{D}$, we introduce $T_f \in \mathcal{L}(H^2, H^2)$, $\Gamma_f \in \mathcal{L}(L^2 \ominus H^2, H^2)$, $\tilde{\Gamma}_f \in \mathcal{L}(H^2, L^2 \ominus H^2)$ and $\tilde{\Gamma}_f \in \mathcal{L}(L^2 \ominus H^2, L^2 \ominus H^2)$ by the operator block matrix

$$M_f = \begin{array}{cc} \begin{bmatrix} T_f & \Gamma_f \\ \tilde{\Gamma}_f & \tilde{T}_f \end{bmatrix} & \begin{array}{c} H^2 \\ L^2 \ominus H^2 \end{array} \end{array}$$

From $M_{\bar f} = M_f^\dagger$, we deduce $T_{\bar f} = T_f^\dagger$ and $\Gamma_{\bar f} = \tilde{\Gamma}_f^\dagger$. Then from $M_{fh} = M_f M_h$, we obtain the semi-commutator identity $T_{fh} = T_f T_h + \Gamma_f \tilde{\Gamma}_h$. For $f \in \mathcal{A}$, we have $\tilde{\Gamma}_f = 0$, so $\Gamma_{\bar f} = 0$. Now for $f, h \in \mathcal{A}$, we have $T_{fh} = T_f T_h$. Also,

$$T_{f\bar h} = T_f T_{\bar h} + \Gamma_f \tilde{\Gamma}_{\bar h} = T_f T_h^\dagger + \Gamma_f \Gamma_h^\dagger;$$

and likewise

$$T_{\bar h f} = T_{\bar h} T_f + \Gamma_{\bar h} \tilde{\Gamma}_f = T_h^\dagger T_f,$$

so by subtracting these equations, we deduce that $\Gamma_f \Gamma_h^\dagger = [T_h^\dagger, T_f]$. If Γ_f and Γ_h are Hilbert–Schmidt, then $[T_h^\dagger, T_f]$ is trace class. Conversely, if $[T_f^\dagger, T_f]$ is trace class, then Γ_f is Hilbert–Schmidt. □

Definition 3.9. Let H be a complex Hilbert space and let $\mathcal{E}$ be a complex subalgebra of $\mathcal{L}(H)$ with commutator subspace $[\mathcal{E}, \mathcal{E}]$. If $[\mathcal{E}, \mathcal{E}] \subset \mathcal{L}^1(H)$ and $\mathcal{E}$ is self-adjoint, then $\mathcal{E}$ is said to be crypto-integral of dimension one; see [31]. (Note that all higher order commutators $[[U, V], W]$ have zero trace for $U, V, W \in \mathcal{E}$.)

Corollary 3.10. *Let $\mathcal{E}$ be the complex subalgebra of $\mathcal{L}(L^2)$ that is generated by $\mathcal{L}^1(L^2)$, $\mathscr{C}_j$, $\mathscr{C}_j^\dagger$ and M_f for all $f \in C^\infty(\partial\mathcal{S}; \mathbb{C})$. Then*

(i) $\mathcal{E}$ is crypto-integral of dimension one.

(ii) The map $\rho : C^\infty(\partial\mathcal{S}; \mathbb{C}) \to \mathcal{E}$, $f \mapsto E M_f E$, is a homomorphism modulo $\mathcal{L}^1(L^2)$;

(iii) The cyclic 1-cocycle $\varphi : C^\infty(\partial\mathcal{S}; \mathbb{C}) \times C^\infty(\partial\mathcal{S}; \mathbb{C}) \to \mathbb{C}$, $\varphi(f_0, f_1) = \mathrm{trace}(M_{f_0}[E, M_{f_1}])$, satisfies

$$\varphi(f_0, f_1) = \frac{1}{2\pi i} \int_{\partial\mathcal{S}} f_0(z(s)) \frac{df_1}{ds}(z(s))\, ds \tag{3.6}$$

and

$$\mathrm{trace}(M_{f_0}[E, M_{f_1}]) = \mathrm{trace}(M_{f_0}[\mathcal{R}_{H^2}, M_{f_1}]) \qquad (f_0, f_1 \in C^\infty(\partial\mathcal{S}; \mathbb{C})). \tag{3.7}$$

Proof. (i) Each C_j is diffeomorphic to the unit circle $C(0, 1)$, so one can use Fourier series to show that $(f(z) - f(\zeta))/(z - \zeta)$ gives a trace-class integral kernel for all $f \in C^\infty(C_j; \mathbb{C})$. The proof of Proposition 3.8 shows that $\mathscr{C}_j - \mathscr{C}_j^\dagger \in \mathcal{L}^1(L^2)$.

(ii) The operator $M_{f_0}[E, M_{f_1}]$ may be expressed as an integral operator on $L^2(\partial\mathcal{S}; ds; \mathbb{C})$ with kernel

$$f_0(z)\frac{f_1(\zeta) - f_1(z)}{2\pi i(\zeta - z)}\frac{dz}{ds}$$

by (3.5), so trace $M_{f_0}[E, M_{f_1}]$ is given via the chain rule by the integral along the diagonal, as stated in (3.6).

We have curvature

$$\varpi(f_0, f_1) = \rho(f_0)\rho(f_1) - \rho(f_0 f_1) = EM_{f_0}[E, M_{f_1}]E$$

where by a computation written in detail in [50, page 76]

$$\mathrm{trace}_{EL^2}(\rho(f_1)\rho(f_0) - \rho(f_1 f_0)) = \mathrm{trace}_{L^2}((E - I)M_{f_0}[E, M_{f_1}])$$

and

$$\mathrm{trace}_{EL^2}(\rho(f_0)\rho(f_1) - \rho(f_1 f_0)) = \mathrm{trace}_{L^2}(EM_{f_0}[E, M_{f_1}]);$$

hence the corresponding bilinear map is

$$\varphi(f_0, f_1) = \mathrm{trace}_{EL^2}\big(\varpi(f_0, f_1) - \varpi(f_1, f_0)\big) = \mathrm{trace}_{L^2}\big(M_{f_0}[E, M_{f_1}]\big). \tag{3.8}$$

The key point is that $E \in \mathcal{L}(L^2)$ is an idempotent, $E^2 = E$; self-adjointness is not required. From (3.8) it is clear that $\varphi(f_1, f_0) = -\varphi(f_1, f_0)$; the proof that $b\varphi = 0$ is known as the noncommutative Bianchi identity [50, Proposition 2.8.2]. Hence φ is indeed a cyclic 1-cocycle.

(iii) As in (ii), $\varphi_{(1)}(f_0, f_1) = \mathrm{trace}(M_{f_0}[\mathcal{R}_{H^2}, M_{f_1}])$ gives a cyclic 1-cocycle on $C^\infty(\partial\mathcal{S}; \mathbb{C})$. We proceed to show that ψ_1 coincides with φ. We introduce the one-parameter family $E_u = (1 - u)E + u\mathcal{R}_{H^2}$ for $u \in [0, 1]$, and we find that $E_u^2 = E_u$ and $E_u\mathcal{R}_{H^2} = \mathcal{R}_{H^2}$ and $\mathcal{R}_{H^2}E_u = E_u$. Then $\mathcal{R}_{H^2} - E_u = (1 - u)\mathcal{R}_{H^2}(E^\dagger - E)$, so $\frac{dE_u}{du} = \mathcal{R}_{H^2}(E^\dagger - E)$ is trace class. Then

$$\varphi_{(u)}(f_0, f_1) = \mathrm{trace}\big(M_{f_0}[E_u, M_{f_1}]\big)$$

satisfies

$$\begin{aligned}\frac{d}{du}\varphi_u(f_0, f_1) &= \mathrm{trace}\Big(M_{f_0}\big[\frac{dE_u}{du}, M_{f_1}\big]\Big)\\ &= \mathrm{trace}\Big(\frac{dE_u}{du}[M_{f_1}, M_{f_0}]\Big) = 0,\end{aligned} \tag{3.9}$$

so $\varphi_{(0)}(f_0, f_1) = \varphi_{(1)}(f_0, f_1)$, as required. □

Remark 3.11. We now introduce the pairing of cyclic 1-cocycles with $K_1(C^\infty(\partial\mathcal{S}; \mathbb{C}))$. Let $\mathcal{A} = C^\infty(\partial\mathcal{S}; \mathbb{C})$, $\mathcal{B} = M_{k\times k}(\mathbb{C})$, and $\mathcal{A} \otimes M_{k\times k}(\mathbb{C}) = M_{k\times k}(\mathcal{A})$, which are complex unital algebras; let $GL_k(\mathcal{A}) = \{X \in M_{k\times k}(\mathcal{A}) : \exists Y \in M_{k\times k}(\mathcal{A}); XY = YX = I\}$. There is a natural homomorphism of differential graded algebras $\pi : \Omega(\mathcal{A} \otimes \mathcal{B}) \to \Omega(\mathcal{A}) \otimes \Omega(\mathcal{B})$, and the dual operation is the cup product of cochains, as in [12, Theorem 9]. We let φ be an n-linear cochain on $\mathcal{A}$, and ψ be a m-linear cochain on $\mathcal{B}$, so $\varphi\sharp\psi = (\varphi \otimes \psi) \circ \pi$ is an

$n+m$-linear cochain on $\mathcal{A}\otimes\mathcal{B}$. Then by [12, Proposition 15], there is a pairing between cyclic 1-cocycles and elements of $GL_k(\mathcal{A})$

$$\langle [u],[\varphi]\rangle = \frac{1}{8\pi i}\varphi\sharp\,\mathrm{trace}\big(u^{-1}-I, u-I\big) \qquad (\varphi\in Z^1_\lambda, u\in GL_k(\mathcal{A})).$$

The Chern character maps the group $K_1(C(\partial\mathcal{S};\mathbb{C}))$ to the odd-dimensional Čech cohomology groups $H^*(\partial\mathcal{S};\mathbb{Q})$ with rational coefficients as in [56, page 174], so $K_1(C(\partial\mathcal{S};\mathbb{C}))\otimes\mathbb{Q}$ is isomorphic to $H^{odd}(\partial\mathcal{S};\mathbb{C})$. This discussion may be compared with [20, Theorem 5.2].

Given $X,Y\in\mathcal{L}(H)$, and a complex polynomial $f(x,y)=\sum_{j,k=0}^N a_{jk}x^jy^k$, we write $f(X,Y)=\sum_{j,k=0}^N a_{jk}X^jY^k$.

Proposition 3.12. *Let $\mathcal{E}$ be as in Corollary 3.10 and let $Z\in\mathcal{E}$, and let $X=\frac{1}{2}(Z+Z^\dagger)$ and $Y=\frac{1}{2i}(Z-Z^\dagger)$ so that $Z=X+iY$.*

(i) Then (i) X, Y and $i[X,Y]$ are self-adjoint elements of $\mathcal{E}$ and $i[X,Y]\in\mathcal{L}^1(L^2)$. $Z=X+iY$ where $X,Y\in\mathcal{E}$ are self-adjoint and $i[X,Y]\in\mathcal{E}$ is a self-adjoint and trace-class operator.

(ii) There exists a compactly supported function $P_Z\in L^1(\mathbb{R}^2,dxdy;\mathbb{R})$ such that

$$\mathrm{trace}\big([f(X,Y),h(X,Y)]\big)=\frac{1}{2\pi i}\iint_{\mathbb{R}^2}P_Z(x,y)\frac{\partial(f,h)}{\partial(x,y)}\,dxdy \tag{3.10}$$

for all polynomials $f,h\in\mathbb{C}[x,y]$.

(iii) Let P_Z be as in (ii) and let

$$\varphi(f,h)=\iint_{\mathcal{S}}P_Z(x,y)\frac{\partial(f,h)}{\partial(x,y)}\,dxdy. \tag{3.11}$$

Then $\varphi: C^\infty(\mathbb{C};\mathbb{C})\times C^\infty(\mathbb{C};\mathbb{C})\to\mathbb{C}$ defines a cyclic 1-cocycle.

Proof. (i) This follows immediately, since $\mathcal{E}$ is self-adjoint and $[Z,Z^\dagger]\in[\mathcal{E},\mathcal{E}]\subseteq\mathcal{L}^1(L^2)$.

(ii) By (i), the map $\mathbb{C}[x,y]\to\mathcal{L}(L^2): f(x,y)\mapsto f(X,Y)$ is a homomorphism modulo $\mathcal{L}^1(H)$. The existence of P_Z follows from Pincus's theorem [44] and uniqueness follows from [9, Proposition 5.2]; see also Lemma 4.7.

(iii) Clearly $\varphi(f,h)=-\varphi(h,f)$ and by elementary calculus,

$$\frac{\partial(f_0f_1,f_2)}{\partial(x,y)}-\frac{\partial(f_0,f_1f_2)}{\partial(x,y)}+\frac{\partial(f_2f_0,f_1)}{\partial(x,y)}=0, \tag{3.12}$$

hence $\varphi(f_0f_1,f_2)-\varphi(f_0,f_1f_2)+\varphi(f_2f_0,f_1)=0$, so $b\varphi=0$. The Kähler differential $d:\Omega^1_{\mathbb{C}[x,y]}\to\Omega^2_{\mathbb{C}[x,y]}$ takes $fdh\mapsto\frac{\partial(f,h)}{\partial(x,y)}dxdy$. Since P_Z has compact support, we can assume that f,h are C^∞ and of compact support. Then we can define $e^{itX}\in\mathcal{L}(L^2)$ via the usual power series and use the Fourier inversion theorem to introduce

$$f(X,Y)=\frac{1}{(2\pi)^2}\iint_{\mathbb{R}^2}e^{itX}e^{iuY}\hat{f}(t,x)\,dtdu \qquad (f\in C_c^\infty(\mathbb{R}^2;\mathbb{C})).$$

where the integral converges as a Bochner-Lebesgue integral in $\mathcal{L}(L^2)$.

One can regard (3.11) as determining $\tilde{\varphi}:\Omega^1 C^\infty(\mathcal{S};\mathbb{C})\to\mathbb{C}$, $\tilde{\varphi}(fdh)=\varphi(f,h)$, or as a linear functional $\Omega^2_{C^\infty(\mathcal{S};\mathbb{C})}\to\mathbb{C}$ such that $df\wedge dh\mapsto\varphi(f,h)$. $\square$

We encounter a variant of the formula (3.10) in Proposition 4.8. One can deduce the following result from [13, Theorem 5, p. 75/291] or (1.5) of [49], and we give a more explicit formula (3.14) for the cocycle.

Proposition 3.13. *Let $\mathcal{D}=\mathcal{D}_{\partial\mathbb{D}}$ be the subalgebra of $L^\infty(\partial\mathbb{D};\mathbb{C})$ as in Definition 2.3.*

(i) Let $T_f=\mathcal{R}_{H^2}M_f\mathcal{R}_{H^2}$. Then the Toeplitz map $\rho:\mathcal{D}\to\mathcal{L}(L^2(0,\infty);\mathbb{C})$, $f\mapsto T_f$ gives a homomorphism modulo $\mathcal{L}^1(L^2(0,\infty);\mathbb{C})$, and the associated curvature is a product of Hankel integral operators

$$\varpi(f_1,f_2)=\rho(f_1f_2)-\rho(f_1)\rho(f_2)=\Gamma^\dagger_{\bar{f}_1}\Gamma_{f_2}. \tag{3.13}$$

(ii) The range of $\varpi:\Omega^1\mathcal{D}\to\mathcal{L}^1(L^2(0,\infty))$ is the set of trace-class integral kernels $K\in L^2((0,\infty)\times(0,\infty);\mathbb{C})$ such that $\frac{d}{dt}K(x+t,y+t)$ is a finite rank kernel, namely a product of vectorial Hankel integral operators.

(iii) The map

$$\varphi_1(f_1,f_2)=\frac{1}{2\pi i}\iint_{\mathbb{D}}\frac{\partial(f_1,f_2)}{\partial(x,y)}\,dxdy\qquad(f_1,f_2\in\mathcal{D}). \tag{3.14}$$

defines a cyclic 1-cocycle.

Proof. (i) There is a natural unitary isomorphism $H^2(\mathbb{D})\to H^2(RHP)$ given by $f(z)\mapsto(\zeta+1)^{-1}f((\zeta-1)/(\zeta+1))$ for $\operatorname{Re}\zeta>0$. Then there is a unitary isomorphism $H^2(RHP)\to L^2((0,\infty);\mathbb{C})$, $f\mapsto(2\pi)^{-1}\int_{-\infty}^{\infty}e^{i\omega t}f(i\omega)\,d\omega$ as in the Paley–Wiener Theorem. The composition of conjugation by these unitary maps takes $\Gamma_h\in\mathcal{L}(H^2(\mathbb{D}))$ to a Hankel integral operator in $\mathcal{L}(L^2((0,\infty);\mathbb{C}))$, which belongs to $\mathcal{L}^2((L^2(0,\infty);\mathbb{C}))$ whenever $\Gamma_h\in\mathcal{L}^2(H^2(\mathbb{D}))$.

Each $h\in\mathcal{D}_{\mathbb{D}}$ determines a harmonic function $h:\mathbb{D}\to\mathbb{C}$ via the Poisson integral; then we can introduce a harmonic function $h((\zeta-1)/(\zeta+1))$ on $\{\zeta:\operatorname{Re}\zeta>0\}$; note that

$$\iint_{\mathbb{R}\times\mathbb{R}}\frac{|h(\frac{iw-1}{iw+1})-h(\frac{iv-1}{iv+1})|^2}{|w-v|^2}\,dvdw=\iint_{[0,2\pi]\times[0,2\pi]}\frac{|h(e^{i\theta})-h(e^{i\phi})|^2}{|e^{i\theta}-e^{i\phi}|^2}\,d\theta d\phi. \tag{3.15}$$

(ii) As in Proposition 2.8 have

$$(f_1,f_2)\mapsto\operatorname{trace}\big(\Gamma^\dagger_{\bar{f}_1}\Gamma_{f_2}\big)=\sum_{\ell=0}^{\infty}(\ell+1)\hat{f}_1(\ell)\hat{f}_2(-\ell).$$

In particular, each $h\in\mathcal{D}_{\mathbb{D}}$ produces a Hilbert–Schmidt Hankel operator via this route, so the semi-commutator is trace class.

(iii) By (i), we can introduce

$$\varphi_1(f_1,f_2)=\operatorname{trace}(\varpi(f_1,f_2)-\varpi(f_2,f_1)), \tag{3.16}$$

so φ is well defined and we clearly have $\varphi_1(f_2,f_1)=-\varphi_1(f_1,f_2)$. We can extend $f,h\in\mathcal{D}_{\mathbb{D}}$ to harmonic functions on $\mathbb{D}$ via the Poisson integral so that $\nabla f,\nabla h$ are square integrable with respect to area measure, so the Jacobian $\partial(f,h)/\partial(x,y)$ is integrable by the Cauchy–Schwarz inequality. Note that by [31, p. 151] or (3.13),

$$\begin{aligned}\operatorname{trace}(\varpi(f,h)-\varpi(h,f)) &= \operatorname{trace}\big([\rho(f),\rho(h)]\big)\\ &= \frac{1}{2\pi i}\iint_{\mathbb{D}}\frac{\partial(f,h)}{\partial(x,y)}\,dxdy\\ &= \frac{1}{2\pi i}\int_{\partial\mathbb{D}} f\,dh \qquad (3.17)\end{aligned}$$

and the identity (3.12). Hence the Hochschild coboundary operator is

$$\begin{aligned}(b\varphi_1)(f_1,f_2,f_3) &= \operatorname{trace}\big(\varpi(f_1f_2,f_3)-\varpi(f_3,f_1,f_2)+\varpi(f_1,f_2f_3)\\ &\qquad -\varpi(f_2f_3,f_1)+\varpi(f_3f_1,f_2)-\varpi(f_2,f_3f_1)\big)\\ &= \operatorname{trace}\big(\rho(f_3)\rho(f_1f_2)-\rho(f_1f_2)\rho(f_3)+\rho(f_1)\rho(f_2f_3)\\ &\qquad -\rho(f_2f_3)\rho(f_1)+\rho(f_2)\rho(f_1f_3)-\rho(f_1f_3)\rho(f_2)\big)\\ &= 0. \qquad (3.18)\end{aligned}$$

This result justifies introducing $\mathcal{D}$, since the integral in the cocycle formula converges absolutely only for $f_1, f_2 \in \mathcal{D}$. □

In the next section we introduce cocycle formulas for linear systems and determinants of multiplicative commutators.

4. Linear systems and Fredholm determinant formulas

Let $(-A,B,C)$ be a continuous time linear system with state space H and input and output space $\mathbb{C}$. In Theorem 4.2 below we introduce an operator algebra on the state space that captures most of the essential information about the linear system and shows how the ODE (2.1) gives rise to a differential equation in this operator algebra. Under further hypotheses, we use Theorem 4.2 in Corollary 4.4 to define an essential spectrum $\mathbb{S}_e$ for a linear system. In Lemma 6.8, we introduce a special differential ring which relates the algebras in Theorem 4.2 to differential equations on the real line.

Definition 4.1. (i) Let $(R_x)_{x>0}$ be a family of operators such that $\langle R_x f, h\rangle$ is differentiable for all $f \in \mathcal{D}(A)$ and $h \in \mathcal{D}(A^\dagger)$. Then Lyapunov's equation for $(-A,B,C)$ is

$$\frac{dR_x}{dx} = -AR_x - R_xA, \quad x>0; \qquad \Big(\frac{dR_x}{dx}\Big)_{x=0} = -BC.$$

(ii) Suppose that $[(\lambda I + A)^{-1}, (\mu I + A^\dagger)^{-1}] \in \mathcal{L}^1(H)$. Then we say that A and $A^\dagger$ are resolvent commuting modulo $\mathcal{L}^1(H)$, or almost resolvent commuting.

Theorem 4.2. *Suppose that A has dense domain $\mathcal{D}(A)$ and is maximally accretive. Suppose also that $C(sI+A)^{-1} \in H^2(RHP; H')$ and $(sI+A)^{-1}B \in H^2(RHP; H)$. Let $\mathcal{E}_{\mathcal{S}}$ be* *the unital complex subalgebra of $\mathcal{L}(H)$ that is generated by I, BC, and $(A-\lambda I)(A+\lambda I)^{-1}$ for all $\lambda > 0$, and let $\mathcal{E}$ be the closure of $\mathcal{E}_{\mathcal{S}}$ in the weak operator topology.*

(i) Let $\mathcal{A}_0 = \mathcal{E}_0/(\mathcal{E}_0 \cap \mathcal{L}^1(H))$. Then $\mathcal{A}_0$ is a commutative unital complex algebra, and there is an exact sequence of algebras

$$0 \longrightarrow \mathcal{L}^1(H) \longrightarrow \mathcal{E}_0 + \mathcal{L}^1(H) \longrightarrow \mathcal{A}_0 \longrightarrow 0.$$

(ii) Let $\varphi_1 : \mathcal{E}_0 \times \mathcal{E}_0 \to \mathbb{C}$ be the bilinear form $\varphi_1(V,W) = \mathrm{trace}([V,W])$. Then $V \mapsto \varphi_1(V,W)$ gives a trace on $\mathcal{E}_0$ for all $W \in \mathcal{E}_0$, and φ_1 is a cyclic 1-cocycle such that

$$\varphi_1(V,W) = \log\det\bigl(e^V e^W e^{-V} e^{-W}\bigr) \qquad (V,W \in \mathcal{E}_0). \tag{4.1}$$

(iii) Also $\mathcal{E}$ contains the family of trace class operators

$$R_x = \int_x^\infty e^{-tA} BC e^{-tA}\, dt \qquad x \geq 0, \tag{4.2}$$

and $(R_x)_{x\geq 0}$ satisfies Lyapunov's equation.

Proof. (i) Since $\mathcal{L}^1(H)$ is an ideal in $\mathcal{L}(H)$, one can check that all commutators of elements of $\mathcal{E}_0$ belong to $\mathcal{L}^1(H)$, so the commutator subspace of $\mathcal{E}_0 + \mathcal{L}^1(H)$ is contained in $\mathcal{L}^1(H)$.

(ii) A trace on $\mathcal{E}_0$ is equivalent to a linear functional on the commutator quotient space $\mathcal{E}_0/[\mathcal{E}_0,\mathcal{E}_0]$. Let $\mathcal{M}$ be an $\mathcal{E}_0$-bimodule, and $\delta : \mathcal{E}_0 \to \mathrm{Der}(\mathcal{M})$, $V \mapsto \delta_V$ where δ_V is the inner derivation $\delta_V(M) = VM - MV$. Then the inner derivations give a Lie algebra since $[\delta_V, \delta_W] = \delta_{[V,W]}$. By (i), $\varphi_1(V,W) = \mathrm{trace}([V,W])$ is well defined. Let $\mathrm{Rad}(\varphi_1) = \{V \in \mathcal{E}_0 : \varphi_1(V,M) = 0 \quad \forall M \in \mathcal{E}_0\}$. Then for $V, W \in \mathcal{E}_0$, we have $[V,W] \in \mathcal{L}^1(H)$, hence $\mathrm{trace}([V,W]M - M[V,W]) = 0$ for all $M \in \mathcal{L}(H)$; that is $[\mathcal{E}_0,\mathcal{E}_0] \subseteq \mathrm{Rad}(\varphi_1)$. We have $\varphi_1(V,W) = -\varphi_1(W,V)$ and

$$\begin{aligned}&\varphi_1(UV,W) - \varphi_1(U,VW) + \varphi_1(WU,V)\\ &\qquad\qquad = \mathrm{trace}\bigl(UVW - WUV - UVW + VWU + WUV - VWU\bigr) = 0.\end{aligned}$$

The formula (4.1) is valid by Pincus's identity [30, p. 182] since $[V,W] \in \mathcal{L}^1(H)$.

(iii) By the Lumer–Phillips theorem [28], $-A$ generates a strongly continuous contraction semigroup $(e^{-tA})_{t\geq 0}$ on H, and $(sI+A)^{-1} \in \mathcal{L}(H)$ for all $\mathrm{Re}\, s > 0$. For $\lambda > 0$, let $(-A)_\lambda = -(\lambda/2) - (\lambda/2)(A-\lambda I)(A+\lambda I)^{-1}$. We note that each $(-A)_\lambda$ belongs to $\mathcal{E}_0$ and $(-A)_\lambda f \to -Af$ in H as $\lambda \to \infty$ for all $f \in \mathcal{D}(A)$. Also $(\exp((-A)_\lambda t))_{t\geq 0}$ gives a uniformly continuous contraction semigroup on H, since $(-A)_\lambda \in \mathcal{L}(H)$. Hence $e^{-tA}BCe^{-tA} \in \mathcal{E}$ is an operator of rank one, and hence trace class, with $\|e^{-tA}BCe^{-tA}\|_{\mathcal{L}^1(H)} \leq \|e^{-tA}B\|_H \|Ce^{-tA}\|_{H'}$.

As in [1, Proposition 2.1], we have $C(sI+A)^{-1}=\int_0^\infty e^{-st}Ce^{-tA}\,dt$ and $(sI+A)^{-1}B=\int_0^\infty e^{-st}e^{-tA}B\,dt$, where by Plancherel's formula for Hilbert-space functions

$$\int_0^\infty \|Ce^{-tA}\|^2_{H'}\,dt=\lim_{\varepsilon\to 0+}\int_{-\infty}^\infty \|C((\varepsilon+i\omega)I+A)^{-1}\|^2_{H'}\,\frac{d\omega}{2\pi},$$
$$\int_0^\infty \|e^{-tA}B\|^2_{H}\,dt=\lim_{\varepsilon\to 0+}\int_{-\infty}^\infty \|((\varepsilon+i\omega)I+A)^{-1}B\|^2_{H}\,\frac{d\omega}{2\pi},$$

and the right-hand sides are finite by hypothesis. Hence by the Cauchy–Schwarz inequality

$$\int_0^\infty \|e^{-tA}BCe^{-tA}\|_{\mathcal{L}^1(H)}\,dt\le\Big(\int_0^\infty \|e^{-tA}B\|^2_H\,dt\Big)^{1/2}\Big(\int_0^\infty \|Ce^{-tA}\|^2_{H'}\,dt\Big)^{1/2};\qquad(4.3)$$

so R_0, and likewise R_x for $x>0$, define trace class operators in $\mathcal{E}_0$. The operator $-A^\dagger$ generates the strongly continuous semigroup $(e^{-tA^\dagger})_{t\ge0}$ of contractions on H by [28, Theorem 4.3]. For $f\in\mathcal{D}(A^\dagger)$ and $h\in\mathcal{D}(A)$, the expression $\langle R_xh,f\rangle=\int_x^\infty\langle BCe^{-tA}h,e^{-tA^\dagger}f\rangle\,dt$ is differentiable, and one easily checks using the fundamental theorem of calculus that Lyapunov's equation holds. $\square$

Remark 4.3. (1) We have used the ideal $\mathcal{L}^1(H)$ since the Fredholm determinant is defined on $\{I+\Gamma:\Gamma\in\mathcal{L}^1(H)\}$ as we use in Lemma 5.3 below. A similar result to Theorem 4.2(i) holds with the ideal of compact operators $\mathcal{K}$ in place of $\mathcal{L}^1(H)$, so $(\mathcal{E}_0+\mathcal{K})/\mathcal{K}$ is a commutative subalgebra of the Calkin algebra $\mathcal{L}(H)/\mathcal{K}$. This is of interest when A has essential spectrum, and brings us within the scope of Proposition 3.2.

(2) Let $(e^{-tA})_{t\ge0}$ be the shift semigroup $(S_t)_{t\ge0}$ as in Definition 2.6. Then for suitable B and C, the hypotheses and hence conclusions of Theorem 4.2(i),(ii),(iii) hold.

(3) To deal with self-adjoint Schrödinger operators, as in section 7, we use the following variant, and refer to [1] for examples relating to scattering theory. Corollary 4.4 introduces an essential spectrum for linear systems that satisfy some conditions relating to self-adjointness. Examples 5.1(ii) and 6.1 relate to Corollary 4.4 (i) and (ii), while in Corollary 4.8 we obtain a determinant formula for multiplicative commutators relating to Corollary 4.4(iii).

Corollary 4.4. *Suppose that both the linear systems $(-A,B,C)$ and $(-A^\dagger,C^\dagger,B^\dagger)$ satisfy the hypotheses of Theorem 4.2, and that either*

(i) $C=B^\dagger$ and A is self-adjoint and non-negative; or
(ii) $C=B^\dagger$ and iA is self-adjoint; or more generally
(iii) A and $A^\dagger$ are resolvent commuting modulo $\mathcal{L}^1(H)$.

Let $\mathcal{E}_*$ be the unital complex subalgebra of $\mathcal{L}(H)$ that is generated by $I, BC, C^\dagger B^\dagger$ and $(A-\lambda I)(A+\lambda I)^{-1}$ and $(A^\dagger-\mu I)(A^\dagger+\mu I)^{-1}$ for all $\lambda,\mu>0$. Then

- *$\mathcal{E}_*$ is crypto-integral of dimension one;*
- *the weak closure of $\mathcal{E}_*$ is a von Neumann algebra;*

- *the quotient of the weak closure of $\mathcal{E}_*$ by the compact operators gives a commutative C^* algebra with spectrum $\mathbb{S}_e$, so there exists a compact set $\mathbb{S}_e$ and a 1unital $*$-homomorphism $\mathcal{E}_* \to C(\mathbb{S}_e;\mathbb{C})$ with dense range. In case (i), $\mathbb{S}_e$ is a subset of $[0,\infty)$; whereas in case (ii), $\mathbb{S}_e$ is a subset of the unit circle $\mathbb{S}^1$.*

Proof. (i) This is a special case of Theorem 4.2, in which $\phi(t) = Ce^{-tA}B$ is real. Hence $\mathcal{E}_0 = \mathcal{E}_*$ is a self-adjoint subalgebra of $\mathcal{L}(H)$ such that the commutator subspace is contained in $\mathcal{L}^1(H)$. The canonical quotient map $\pi : \mathcal{L}(H) \to \mathcal{L}(H)/\mathcal{K}(H)$ to the Calkin algebra restricts to a map on $\mathcal{E}_*$ which is zero on $[\mathcal{E}_*, \mathcal{E}_*]$. Hence there is commutative $C*$-algebra $\mathcal{C}$ such that $\pi(\mathcal{E}_*)$ is dense in $\mathcal{C}$. By the Gelfand-Naimark theorem, $\mathcal{C}$ is $*$-isomorphic to $C(\mathbb{S}_e;\mathbb{C})$ for some compact set $\mathbb{S}_e$. Note that BC is compact, hence $\pi(BC) = 0$; also the spectrum of $(\lambda I - A)(\lambda I + A)^{-1}$ is contained in $[0,\infty)$ for all $\lambda > 0$; hence the essential spectrum of $(\lambda I - A)(\lambda I + A)^{-1}$ is contained in $[0,\infty)$ for all $\lambda > 0$; hence $\mathbb{S}_e \subset [0,\infty)$.

(ii) This is a special case of Theorem 4.2, in which $(e^{itA})_{t\in\mathbb{R}}$ is a unitary group and $\phi(t) = Ce^{-tA}C^\dagger$ has $\bar{\phi}(t) = \phi(-t)$. Here $\mathcal{E}_*$ is a self-adjoint subalgebra of $\mathcal{L}(H)$ such that the commutator subspace is contained in $\mathcal{L}^1(H)$. Here the spectrum of $(\lambda I - A)(\lambda I + A)^{-1}$ is contained in $\mathbb{S}^1$, hence the essential spectrum of $(\lambda I - A)(\lambda I + A)^{-1}$ is contained in $\mathbb{S}^1$, so $\mathbb{S}_e \subseteq \mathbb{S}^1$.

(iii) The proof is as for Theorem 4.2. The key identity is

$$\big[(A - \lambda I)(A + \lambda I)^{-1}, (A^\dagger - \mu I)(A + \mu I)\big] = 4\lambda\mu\big[(A + \lambda I)^{-1}, (A^\dagger + \mu I)^{-1}\big], \tag{4.4}$$

where the right-hand side is trace class. □

Example 4.5. For $\kappa > 0$, let $\Delta_\kappa = -\frac{d^2}{dx^2} + \kappa^2$ in $L^2(\mathbb{R};\mathbb{C})$, so by Fourier analysis one shows that Δ_κ has inverse operator G, where G is the integral operator on $L^2(\mathbb{R};\mathbb{C})$ with kernel $g(x,y;\kappa) = e^{-\kappa|x-y|}/(2\kappa)$. Let $q \in L^1 \cap L^2(\mathbb{R};\mathbb{R})$, and write $q = q_1 q_2$ where $q_1, q_2 \in L^2(\mathbb{R};\mathbb{R})$. Then $M_{q_1}G$ has kernel $q_1(x)e^{-\kappa|x-y|}/(2\kappa)$, which is Hilbert–Schmidt; likewise $M_{q_2}G$ is Hilbert–Schmidt. There exists $\kappa_0 > 0$ such that $\Delta_\kappa + M_q$ is invertible for all $\kappa > \kappa_0$, with inverse $(I + \Delta_\kappa^{-1}M_q)^{-1}\Delta_\kappa^{-1}$. We have

$$\big[(\Delta_\kappa + M_q)^{-1}, \Delta_\kappa^{-1}\big] = -(\Delta_\kappa + M_q)^{-1}\big[\Delta_\kappa + M_q, \Delta_\kappa^{-1}\big](\Delta_\kappa + M_q)^{-1} \tag{4.5}$$

$$= -(I + GM_q)^{-1}G\big(M_{q_1}\big[M_{q_2}, G\big] + \big[M_{q_1}, G\big]M_{q_2}\big)G(I + M_qG)^{-1} \tag{4.6}$$

is the sum of terms which involve two Hilbert–Schmidt factors, hence is trace class. Thus, the operators Δ_κ and $\Delta_\kappa + M_q$ are almost resolvent commuting, although their difference M_q can be an unbounded operator, and Δ_κ^{-1} is not compact. Likewise, one can show that $A = \Delta_\kappa + iM_q$ and $A^\dagger = \Delta_\kappa - iM_q$ are almost resolvent commuting.

In section 5, we obtain various results in the context of Corollary 4.4(i); Example 6.1 relates to Corollary 4.4(ii). We proceed towards Proposition 4.8, which applies in the context of Corollary 4.4(iii). The determining function, and related determinants, are more conveniently introduced via discrete-time linear systems. We shall discuss this next, before we return to continuous-time systems at the end of the section.

Definition 4.6. (i) Let $U \in \mathcal{L}(H)$, $B \in \mathcal{L}(H_0, H)$, $C \in \mathcal{L}(H, H_0)$, $D \in \mathcal{L}(H_0)$. We denote by $\begin{bmatrix} U & B \\ C & D \end{bmatrix}$ the discrete-time linear system

$$\begin{aligned} x_{n+1} &= Ux_n + Bu_n \\ y_n &= Cx_n + Du_n \qquad (n = 0, 1, \dots) \end{aligned}$$

with input $(u_n)_{n=0}^{\infty}$ with $u_n \in H_0$, state $(x_n)_{n=0}^{\infty}$ with $x_n \in H$ and output $(y_n)_{n=0}^{\infty}$ with $y_n \in H_0$. The transfer function for this system is $\Phi(\lambda) = D + C(\lambda I - U)^{-1}B$.

(ii) Let $V \in \mathcal{L}(H)$. For the family of linear systems

$$\begin{bmatrix} U & (\mu I - V)^{-1}B \\ iC & I \end{bmatrix} \qquad (\mu \in \mathbb{C} \setminus \sigma(V)), \tag{4.7}$$

Pincus [44, page 223] defined the determining function

$$E(\lambda, \mu) = I + iC(\lambda I - U)^{-1}(\mu I - V)^{-1}B, \qquad (\lambda \in \mathbb{C} \setminus \sigma(U),\ \mu \in \mathbb{C} \setminus \sigma(V)). \tag{4.8}$$

The main application of the determining function is to compute determinants, starting with the identity

$$\det E(\lambda, \mu) = \det(I + i(\mu I - V)^{-1}BC(\lambda I - U)^{-1}) \qquad 1(\lambda \in \mathbb{C} \setminus \sigma(U),\ \mu \in \mathbb{C} \setminus \sigma(V)).$$

In particular, let $U, V \in \mathcal{L}(H)$ be self-adjoint and let $B \in \mathcal{L}^2(H_0, H)$ and $C \in \mathcal{L}^2(H, H_0)$ satisfy $[U, V] = iBC$. Then the product iBC is trace class and skew self-adjoint, and by some straightforward manipulations, one can show that

$$\det\Big((\lambda I - U)(\mu I - V)(\lambda I - U)^{-1}(\mu I - V)^{-1}\Big) = \det\Big(I + iC(\lambda I - U)^{-1}(\mu I - V)^{-1}B\Big),$$

where the left-hand side is the determinant of a multiplicative commutator. The scope of this formula is extended via Pincus's principal function, namely the function P introduced in the following lemma.

Lemma 4.7. *(Pincus) Let $Z = U + iV$ be an almost normal operator.*

(i) There exists a unique and compactly supported $P \in L^1(\mathbb{R}^2; dxdy; \mathbb{R})$ that satisfies

$$\det\Big((\lambda I - U)(\mu I - V)(\lambda I - U)^{-1}(\mu I - V)^{-1}\Big) = \exp\Big(\frac{1}{2\pi i}\iint \frac{P(x, y)}{(x - \lambda)(y - \mu)}\, dxdy\Big).$$

(ii) In particular suppose that $B = C^\dagger$ *where* $C : H \to \mathbb{C}$ *has rank one with* $[U, V] = iBC$*. Then the function* P *in (i) is supported on* $\sigma(U) \times \sigma(V)$*, takes values in* $[0, 1]$ *and satisfies*

$$\det E(\lambda, \mu) = \exp\Big(\frac{1}{2\pi i}\iint_{\sigma(U)\times\sigma(V)} \frac{P(x, y)}{(\lambda - x)(\mu - y)}\, dxdy\Big).$$

Proof. (i) See [9, page 153]. Uniqueness is addressed in [9, Proposition 5.2].

(ii) See [44, Theorem 7.1]. □

Proposition 4.8. *Suppose as in Corollary 4.4(iii) that* A *and* $A^\dagger$ *are almost resolvent commuting, and let* $Z = (I - A)(I + A)^{-1}$*.*

(i) Then Z *is almost normal and* $Z = U + iV$ *where* $U, V \in \mathcal{L}(H)$ *are self-adjoint with a principal function as in Lemma 4.7(i) such that*

$$\operatorname{trace}\big([f(U, V), h(U, V)]\big) = \frac{1}{2\pi i}\iint_{\sigma(U)\times\sigma(V)} \frac{\partial(f, h)}{\partial(x, y)} P(x, y)\, dxdy \tag{4.9}$$

for all polynomials $f(x, y), h(x, y) \in \mathbb{C}[x, y]$*.*

(ii) There exists H_0 *and a continuous time linear system* $(-A, B, C)$ *with input and output space* H_0 *such that* $\mathcal{E} = alg\{I, Z, Z^\dagger\}$ *is crypto-integral of dimension one with* $2BC = [Z^\dagger, Z]$*.*

(iii) In particular, if $[Z^\dagger, Z]$ *has rank one, then one can choose* $(-A, B, C)$ *to have input and output space* $\mathbb{C}$*.*

Proof. (i) Under the canonical quotient map $\mathcal{L}(H) \to \mathcal{L}(H)/\mathcal{L}^1(H)$, the images of $Z = (I - A)(I + A)^{-1}$ and $Z^\dagger = (I - A^\dagger)(I + A^\dagger)^{-1}$ give commuting elements of the algebra $\mathcal{L}(H)/\mathcal{L}^1(H)$, so Z is almost normal and as such has an essential spectrum in $\mathbb{C}$. Now $[Z^\dagger, Z] = 2i[U, V] \in \mathcal{L}^1(H)$ is self-adjoint, so (i) of Lemma 4.7 applies.

(ii) We can choose H_0 and $B \in \mathcal{L}^2(H_0, H)$ and $C \in \mathcal{L}^2(H, H_0)$ such that $2BC = [Z^\dagger, Z]$. Observe that $\mathrm{alg}\{I, Z, Z^\dagger\}$ is crypto-integral of dimension one, and contains BC.

(iii) If $[Z^\dagger, Z]$ has rank one, then we can choose $B \in \mathcal{L}(\mathbb{C}, H) = H$ and $C \in \mathcal{L}(H, \mathbb{C}) = H'$ such that $2BC = [Z^\dagger, Z]$. □

Proposition 4.8(i) gives a formula for the cocycle φ_1 of Theorem 4.2(ii) in a special case, where the principal function is defined on $\mathbb{C}$. As discussed in [10, Theorem 5.10], this is not the typical situation.

5. Howland operators and Fredholm determinant formulas

A Howland operator in an integral operator of the form

$$Kf(x) = \int_0^\infty \frac{b(x)c(y)}{x + y} f(y)\, dy, \qquad f \in L^2((0, \infty); \mathbb{C})$$

where $b, c \in L^\infty((0,\infty);\mathbb{C})$. Howland observed [32, page 410] that some tools from the theory of Schrödinger operators could be adapted to study such integral operators and that Lyapunov's equation is fundamental. In our context, Howland operators provide an important situation where the conditions of Corollary 4.4(i) hold, and such operators arise from linear systems of a special form which we now describe. In Proposition 5.3, we compute Fredholm determinants via the Gelfand-Levitan equation.

Example 5.1. (i) Let $b, c \in L^\infty((0,\infty);\mathbb{C}) \cap L^2((0,\infty);\mathbb{C})$. Consider the linear system $(-A, B, C)$ with state space $H = L^2((0,\infty);\mathbb{C})$ where $\mathcal{D}(A) = \{f \in L^2((0,\infty);\mathbb{C}) : xf(x) \in L^2((0,\infty);\mathbb{C})\}$. The operators are

$$\begin{aligned} A &: \mathcal{D}(A) \to L^2((0,\infty);\mathbb{C}), \qquad f(u) \mapsto uf(u) \qquad (u>0);\\ B &: \mathbb{C} \to L^2((0,\infty);\mathbb{C}), \qquad x \mapsto bx \qquad (x \in \mathbb{C});\\ C &: L^2((0,\infty);\mathbb{C}) \to \mathbb{C}, \qquad f \mapsto \int_0^\infty c(u)f(u)\,du. \end{aligned} \tag{5.1}$$

The impulse response function is given by the Laplace transform of bc,

$$\phi(t) = Ce^{-tA}B = \int_0^\infty b(u)c(u)e^{-tu}\,du,$$

while $R_x = \int_x^\infty e^{-tA}BCe^{-tA}\,dt$ is the integral operator on $L^2((0,\infty);\mathbb{C})$ that has kernel

$$\frac{b(u)c(v)e^{-(u+v)x}}{u+v} \qquad (u, v > 0).$$

Thus R_x is a Howland operator for all $x > 0$. If $c(u), b(u), c(u)/\sqrt{u}, b(u)/\sqrt{u}$ all belong to $L^2((0,\infty);\mathbb{C})$, then the hypotheses of Theorem 4.2 are satisfied, and $R_0 \in \mathcal{L}^1(H)$. This was stated in [2, Proposition 2.1]; here we give the crucial estimate

$$\|C((\varepsilon + i\omega)I + A)^{-1}\|^2_{H'} = \int_0^\infty \frac{|c(u)|^2}{|\varepsilon + i\omega + u|^2}du,$$

which gives

$$\begin{aligned} \int_{-\infty}^\infty \|C((\varepsilon + i\omega)I + A)^{-1}\|^2_{H'} \frac{d\omega}{2\pi} &= \int_0^\infty |c(u)|^2 \int_{-\infty}^\infty \frac{1}{|\varepsilon + u + i\omega|^2}\frac{d\omega}{2\pi}du\\ &= \frac{1}{2}\int_0^\infty \frac{|c(u)|^2 du}{u+\varepsilon}. \end{aligned} \tag{5.2}$$

Thus $C(sI + A)^{-1}$ belongs to $H^2(RHP; H')$. A similar calculation shows that $(sI + A)^{-1}B \in H^2(RHP; H)$.

(ii) Suppose that $b = \bar{c}$; then $C = B^\dagger$, and Corollary 4.4(i) applies.

(iii) The following Proposition 5.2 introduces a Volterra type group and applies to the corresponding T_{GL} from (5.1).

For $a > 0$, we have an orthogonal decomposition $L^2((0,\infty);\mathbb{C}) = L^2((0,a);\mathbb{C}) \oplus L^2((a,\infty);\mathbb{C})$ where $L^2((a,\infty);\mathbb{C})$ is the image of the shift S_a and $L^2((0,a);\mathbb{C})$ is the nullspace of $S_a^\dagger$. Consider the space $\mathcal{T}$ of all measurable $T : (0,\infty)\times(0,\infty) \to \mathbb{C}$ such that $T(x,y) = 0$ for all $0 < y < x$ and $\int_0^\infty \int_x^\infty |T(x,y)|^2\, dydx$ is finite, so T is upper triangular. Given $T \in \mathcal{T}$ define the bounded linear operator $V_T : L^2((0,\infty);\mathbb{C}) \to L^2((0,\infty);\mathbb{C})$ by

$$V_T f(x) = f(x) + \int_x^\infty T(x,y)f(y)\, dy \qquad (f \in L^2((0,\infty);\mathbb{C})). \tag{5.3}$$

Proposition 5.2. *Let*

$$\mathcal{G} = \{V_T\, :\, T \in \mathcal{T} \text{ and } V_T \text{ is invertible in } \mathcal{L}(L^2(0,\infty);\mathbb{C})\}.$$

Then

- *(i) for all $V \in \mathcal{G}$ and $a > 0$, the operator V restricted to $L^2((0,a);\mathbb{C})$ has image in $L^2((0,a);\mathbb{C})$;*
- *(ii) $\mathcal{G}$ forms a group under operator composition;*
- *(iii) If $V \in \mathcal{G}$ is unitary, then $V = I$.*
- *(iv) Suppose that $T \in \mathcal{T}$, that $T(x,y)$ is continuous on $\{(x,y) \in (0,\infty)^2 : 0 < x < y\}$, and there exists $\varepsilon > 0$ such that $e^{\varepsilon(x+y)}T(x,y)$ is bounded. Then with such a T the formula (5.3) gives an operator $V_T \in \mathcal{G}$.*

Proof. (i) For all $0 < a < x$, we have $\int_0^a T(x,y)\, dy = 0$, which is equivalent to the condition that $\int_0^a T(x,y)f(y)\, dy = 0$ for all $0 < a < x$ and $f \in L^2((0,a);\mathbb{C})$. Recall the shift operator S_a and note that $S_aS_a^\dagger$ is the orthogonal projection $L^2((0,\infty);\mathbb{C}) \to L^2((a,\infty);\mathbb{C})$ and by the preceding calculation $S_aS_a^\dagger V(I - S_aS_a^\dagger) = 0$.

(ii) One considers the equation $V(I+H) = I$, which is equivalent to $H = -V^{-1}(V-I)$, which is upper triangular.

(iii) By (ii), the condition $VV^\dagger = I$ requires the adjoint kernel $\overline{T(y,x)}$ also to be upper triangular, whereas we have $\overline{T(y,x)} = 0$ for all $0 < x < y$ which implies the adjoint kernel is lower triangular; so $T(x,y) = 0$ for all $x, y > 0$.

(iv) Suppose that $|T(x,t)| \le Me^{-\varepsilon(x+y)}$; then by iterated integration over a simplex, one shows that

$$\begin{aligned}\int_{\{x<y_1<y_2<\dots<y_n<y\}} &|T(x,y_1)T(y_1,y_2)\dots T(y_n,y)|dy_1\dots dy_n\\ &\le M^{n+1}e^{-\varepsilon(x+y)}\int_{\{x<y_1<y_2<\dots<y_n<y\}} e^{-2\varepsilon(y_1+\dots+y_n)}dy_1\dots dy_n\\ &\le \frac{M^{n+1}e^{-\varepsilon(x+y)}}{\varepsilon^n 2^n n!}.\end{aligned} \tag{5.4}$$

Thus one shows that the series $\sum_{j=1}^\infty (-1)^j T^j$ converges in $\mathcal{L}^2$, so the spectrum of V_T is $\{1\}$ and V_T is invertible with $V_T^{-1} - I \in \mathcal{T}$. ☐

Proposition 5.3. *Let $(-A, B, C)$ be a continuous time linear system as in Theorem 4.2, with impulse response function ϕ, and let*

$$T_{GL}(x,y) = -Ce^{-xA}(I+R_x)^{-1}e^{-yA}B, \qquad x,y \geq 0. \tag{5.5}$$

Then T_{GL} satisfies the Gelfand–Levitan equation

$$\phi(x+y) + T_{GL}(x,y) + \int_x^\infty T_{GL}(x,z)\phi(z+y)\,dz = 0, \tag{5.6}$$

and $(d/dx)\log\det(I+R_x) = \operatorname{trace} T_{GL}(x,x)$.

Proof. We recall from [5, Theorem 2.5] the basic identity

$$\begin{aligned} Ce^{-(x+y)A}B - Ce^{-xA}(I+R_x)^{-1}e^{-yA}B - Ce^{-xA}(I+R_x)^{-1}\int_x^\infty e^{-zA}BCe^{-zA}e^{-yA}B\,dz \\ = Ce^{-xA}\Big(I - (I+R_x)^{-1} - (I+R_x)^{-1}R_x\Big)e^{-yA}B = 0. \end{aligned}$$

Also, the determinant satisfies

$$\begin{aligned} \frac{d}{dx}\log\det(I+R_x) &= \frac{d}{dx}\operatorname{trace}\log(I+R_x) \\ &= \operatorname{trace}\Big((I+R_x)^{-1}\frac{dR_x}{dx}\Big) \\ &= -\operatorname{trace}\big((I+R_x)^{-1}e^{-xA}BCe^{-xA}\big) \\ &= -Ce^{-xA}(I+R_x)^{-1}e^{-xA}B, \end{aligned} \tag{5.7}$$

as required. □

We can apply Proposition 5.3 directly in the context of Corollary 4.4(i). To deal with Hankel products, we follow the approach of [4] and obtain a formula for the Fredholm determinant and its derivative from Proposition 2.8(vi). Note that if ϕ is the impulse response function of $(-A, B, C)$, then $\phi^\dagger$ is the impulse response function of $(-A^\dagger, C^\dagger B^\dagger)$. Let $(-A, B_1, C_1)$ and $(-A, C_2, B_2)$ be linear systems with state space H an impulse response functions $\phi(t)^\top = C_1e^{-tA}B_1$ and $\psi(t) = C_2e^{-tA}B_2$. To combine these, we let $\lambda \in \mathbb{C}$ be a complex parameter and let $(-\hat{A}, \hat{B}, \hat{C})$ be the linear system

$$(-\hat{A}, \hat{B}, \hat{C}) = \left(\begin{bmatrix} -A & 0 \\ 0 & -A \end{bmatrix}, \begin{bmatrix} 0 & B_1 \\ B_2 & 0 \end{bmatrix}, \begin{bmatrix} \lambda C_1 & 0 \\ 0 & -C_2 \end{bmatrix}\right) \tag{5.8}$$

which has impulse response function

$$\Phi(x) = \hat{C}e^{-x\hat{A}}\hat{B} = \begin{bmatrix} 0 & \lambda C_1 e^{-xA}B_1 \\ -C_2e^{-xA}B_2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \lambda\phi(t)^\top \\ -\psi(x) & 0 \end{bmatrix};$$

we momentarily defer the proof of existence. We introduce the operator function

$$\hat{R}_x = \int_x^\infty e^{-s\hat{A}}\hat{B}\hat{C}e^{-s\hat{A}}\,ds = \int_x^\infty \begin{bmatrix} 0 & -e^{-sA}B_1C_2e^{-sA} \\ \lambda e^{-sA}B_2C_1e^{-sA} & 0 \end{bmatrix} ds. \tag{5.9}$$

Remark 5.4. It is straightforward to produce a suitable linear system that has impulse response function equal to any given rational function. For $a \in LHP$ and $r \in \mathbb{N}$, we take the state space to be $L^2((0,\infty);\mathbb{C})$ and input and output spaces $\mathbb{C}$. Let

$$\begin{aligned} e^{-tA} : L^2((0,\infty);\mathbb{C}) &\to L^2((0,\infty);\mathbb{C}), \qquad f(u) \mapsto e^{-tu} f(u) \qquad (u,t>0); \\ B : \mathbb{C} &\to L^2((0,\infty);\mathbb{C}), \qquad x \mapsto u^{(r-1)/2} e^{au/2} x \qquad (x \in \mathbb{C}); \\ C : L^2((0,\infty);\mathbb{C}) &\to \mathbb{C}, \qquad f \mapsto \frac{1}{(r-1)!}\int_0^\infty u^{(r-1)/2} e^{au/2} f(u)\, du. \end{aligned} \tag{5.10}$$

Then the impulse response function is

$$\phi(t) = Ce^{-tA}B = \frac{1}{(r-1)!}\int_0^\infty u^{r-1} e^{-tu+au}\, du = \frac{1}{(t-a)^r},$$

while $R_x = \int_x^\infty e^{-tA}BCe^{-tA}\, dt$ is the integral operator on $L^2((0,\infty);\mathbb{C})$ that has kernel

$$R_x \leftrightarrow \frac{(uv)^{(r-1)/2)} e^{(a/2-x)(u+v)}}{u+v} \qquad (u,v>0),$$

which has the form of a Howland operator as in Example 5.1 and [32]; here $1/(u+v)$ is the Carleman kernel as in the proof of Lemma 2.5, while the other factors arise from multiplication operators.

Elaborating this example, one can introduce (5.8) to realize the impulse response functions as in ([4, Proposition 3.2]). Given linear systems $(-A, B_1, C_1)$ with impulse response function ϕ_1 and state space H, and $(-A, B_2, C_2)$ with impulse response function ϕ_2, then

$$\left(\begin{bmatrix} -A & 0 \\ 0 & -A \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \begin{bmatrix} C_1 & C_2 \end{bmatrix} \right) \tag{5.11}$$

has impulse response function $\phi_1 + \phi_2$ and state space $H \otimes \mathbb{C}^2$. Thus one can realise rational functions via partial fractions by adding the impulse response functions from systems as in (5.10).

Proposition 5.5. *Let $K = \Gamma_\phi^\top \Gamma_\psi \in \mathcal{L}^2$ be a product of vector-valued Hankel operators and let $K_t = S_t^\dagger K S_t$ be as in Proposition 2.8(vi). Suppose that A has dense domain $\mathcal{D}(A)$ and is maximally accretive. Suppose also that $C_j(sI+A)^{-1} \in H^2(RHP;H')$ and $(sI+A)^{-1}B_j \in H^2(RHP;H)$ for $j=1,2$ and that $(-\hat A, \hat B, \hat C)$ is defined as in (5.8). Then*

$$\hat T_{GL}(x,y) = -\hat C e^{-x\hat A}(I+\hat R_x)^{-1} e^{-y\hat A}\hat B. \tag{5.12}$$

satisfies the Gelfand–Levitan equation

$$\hat\Phi(x+y) + \hat T_{GL}(x,y) + \int_x^\infty \hat T_{GL}(x,z)\hat\Phi(z+y)\, dz = 0, \tag{5.13}$$

and satisfies

$$\operatorname{trace} \hat T_{GL}(x,x) = \frac{d}{dx}\log\det(I+\lambda K_x). \tag{5.14}$$

Proof. See [1, Lemma 5.1] for the Gelfand–Levitan equation. With $\phi_{(x)}(y) = \phi(y+2x)$, the Fredholm determinant satisfies

$$\begin{aligned}\det(I+\lambda K_x) &= \det(I+\lambda\Gamma_{\phi^\top_{(x)}}\Gamma_{\psi_{(x)}})\\ &= \det(I+\Gamma_{\hat\Phi_{(x)}})\\ &= \det(I+\hat R_x). \end{aligned}\tag{5.15}$$

Now $\hat R_x$ satisfies $-\frac{d\hat R_x}{dx} = e^{-x\hat A}\hat B\hat C e^{-x\hat A}$, so by Proposition 2.4 of [1], we have

$$\begin{aligned}\frac{d}{dx}\log\det(I+\hat R_x) &= \operatorname{trace}\Big((I+\hat R_x)^{-1}\frac{d\hat R_x}{dx}\Big)\\ &= -\operatorname{trace}\big((I+\hat R_x)^{-1}e^{-x\hat A}\hat B\hat C e^{-x\hat A}\big)\\ &= -\operatorname{trace}\big(\hat C e^{-x\hat A}(I+\hat R_x)^{-1}e^{-x\hat A}\hat B\big),\end{aligned}\tag{5.16}$$

as required. □

6. The xi function and Darboux addition

Corollary 4.4(ii) applies to several problems about scattering for Schrödinger operators on the real line, as in the following.

Example 6.1. We consider a skew symmetric A as in Corollary 4.4(ii), which arises in the scattering theory of Schrödinger's operator. Let $b_1, b_1 \in C_0(\mathbb{R};\mathbb{C})$ such that $b_1(-k) = \overline{b_1(k)}$, $b_2(-k) = \overline{b_2(k)}$, $|b_1(k)| = |b_2(k)|$ and let $b(k) = b_1(k)b_2(k)$, so $b(-k) = \overline{b(k)}$. As in [1, Theorem 4.2] we consider the linear system $(-A, B, C)$ given by

$$\begin{aligned} e^{-xA} : L^2(\mathbb{R};\mathbb{C}) \to L^2(\mathbb{R};\mathbb{C}), &\qquad f(k) \mapsto e^{-ikx}f(k);\\ B : \mathbb{C} \to L^2(\mathbb{R};\mathbb{C}), &\qquad a \mapsto b_1(k)a;\\ C : L^2(\mathbb{R};\mathbb{C}) \to \mathbb{C}, &\qquad f \mapsto \frac{1}{2\pi}\int_{-\infty}^{\infty} b_2(k)f(k)\,dk.\end{aligned}$$

Then $(e^{-xA})_{x\in\mathbb{R}}$ is a unitary group of operators, and $\sigma(A) = i\mathbb{R}$. For $\zeta > 0$, the operator $(\zeta I - A)(\zeta I + A)^{-1}$ is unitary, being given by multiplication by a unimodular function. This example occurs in scattering problems [1, Theorem 1.3 (3) and Theorem 4.2] and [21, Section 5]. To convert our notation to that of [21, p. 486], let the left-hand scattering (reflection) coefficient be $s_{21}(k) = b(k)$ and $\phi(x) = (2\pi)^{-1}\int_{-\infty}^{\infty} s_{21}(k)e^{-ixk}dk$ for $x > 0$; the condition $s_{21}(k) = \overline{s_{21}(-k)}$ ensures that $\phi(x) = \overline{\phi(x)}$. Then the Fredholm determinant is

$$\vartheta(s_{21}) = \det(I+\Gamma_\phi) = \det(I+R_0).\tag{6.1}$$

Note that with $\phi_{(y)}(x) = \phi(x+2y)$, the backward shift $\phi(x) \mapsto \phi_{(y)}(x)$ arises from multiplication $b(k) \mapsto e^{2iky}b(k)$. Using the argument of [22, p. 158] and [17, p 217], one shows that Γ_ϕ is a self-adjoint Hilbert-Schmidt operator with $\|\Gamma_\phi\|_{\mathcal{L}(L^2(0,\infty);\mathbb{C})} < 1$. If there are no bound states, then b extends to define a bounded and holomorphic function on the upper half-plane with $\|b\|_\infty \leq 1$, so $\phi(x) = 0$ for $x < 0$.

We have

$$\begin{aligned} u(x) &= -2\frac{d}{dx}T_{GL}(x,x) \\ &= 2\frac{d}{dx}\Big(Ce^{-xA}\big(I-(I+R_x)^{-1}R_x\big)e^{-xA}B\Big) \\ &= 2\frac{d}{dx}\Big(\phi(2x) + \int_x^\infty T_{GL}(x,z)\phi(z+x)\,dz\Big) \end{aligned} \tag{6.2}$$

which is a variant of Rybkin's formula [51, (5.8)]; see also the trace formula [17, $(1)_R$]. Let H^2 be the usual Hardy space on the upper half plane and $\mathcal{D}$ be the Dirichlet algebra $\mathcal{D}$ of Definition 2.3 for the upper half plane.

Proposition 6.2. *(i) For $u \in L^2$, the impulse response function ϕ has derivative $\phi' \in L^2((-\infty,\infty);\mathbb{C})$.*

(ii) Suppose that

$$\iint_{\mathbb{R}\times\mathbb{R}} \frac{|s_{21}(k) - s_{21}(\kappa)|^2}{|k-\kappa|^2}dk d\kappa \tag{6.3}$$

converges. Then Γ_ϕ is a Hilbert–Schmidt operator on $L^2((0,\infty);\mathbb{C})$, and each s_{21} determines an element of $\mathcal{D}$

(iii) Let T_f be the Toeplitz operator $T_f h = \mathcal{R}_{H^2}(fh)$ for $f \in \mathcal{D}$ and $h \in H^2$. Then the Toeplitz map $\rho : \mathcal{D} \to \mathcal{L}(H^2)$, $f \mapsto T_f$ gives a homomorphism modulo $\mathcal{L}^1(H^2)$.

Proof. (i) The impulse response function has an L^2 derivative since

$$\begin{aligned} \int_0^\infty |\phi'(x)|^2\,dx &\leq \frac{1}{2\pi}\int_{-\infty}^\infty k^2|s_{21}(k)|^2\,dk \\ &\leq \frac{1}{2\pi}\int_{-\infty}^\infty k^2\log\frac{1}{1-|s_{21}(k)|^2}\,dk \\ &\leq \frac{1}{6}\int_{-\infty}^\infty |u(x)|^2\,dx \end{aligned} \tag{6.4}$$

where the final step uses the Zakharov–Faddeev trace formula [60] [51, (4.5)]. If $-d^2/dx^2 + u(x)$ has no bound states, then there is equality in the final line of (6.4).

(ii) By Plancherel's formula, We have

$$\begin{aligned} \|\Gamma_\phi\|^2_{\mathcal{L}^2} &= \int_{\mathbb{R}} |t||\phi(t)|^2\,dt \\ &= \frac{1}{2\pi}\iint_{\mathbb{R}\times\mathbb{R}} \frac{|e^{iht}-1|^2}{h^2}|\phi(t)|^2\,dhdt \\ &= \frac{1}{4\pi^2}\iint_{\mathbb{R}\times\mathbb{R}} \frac{|s_{21}(k+h) - s_{21}(k)|^2}{h^2}\,dkdh. \end{aligned}$$

The coefficients s_{21} satisfy $|s_{21}(k)| \leq 1$ for all $k \in \mathbb{R}$, so when (6.3) is finite, s_{21} determines an element of the Dirichlet algebra $\mathcal{D}$ of definition 2.3.

(iii) Each s_{21} extends to a bounded and harmonic function on the upper half plane $\{k + i\kappa : k \in \mathbb{R}, \kappa > 0\}$; this Poisson extension may not be holomorphic. The algebraic properties are given by Proposition 3.13. With H^2 the usual Hardy space on the upper half plane, and $\mathcal{R}_{H^2} : L^2 \to H^2$ the Riesz projection, each $f \in \mathcal{D}$ gives a Toeplitz operator $T_f \in \mathcal{L}(H^2)$ by $T_f h = \mathcal{R}_{H^2}(fh)$ which is bounded since $f \in L^\infty$. By Proposition 3.13, the semi-commutator $T_{fg} - T_f T_g$ is trace class on H^2 for all $f, g \in \mathcal{D}$. $\square$

Remark 6.3. (i) In particular, $Z = T_f$ has $[Z^\dagger, Z] = T_{\bar f} T_f - T_f T_{\bar f} \in \mathcal{L}^1(H^2)$. In Proposition 3.12 , we obtained trace formulas for functions of $(Z, Z^\dagger)$.

(ii) Suppose that $\phi(t) \to 0$ as $t \to \infty$, that ϕ is absolutely continuous and $\int_0^\infty t|\phi'(t)|dt$ converges. Then by simple estimates, one shows that $\int_0^\infty t|\phi(t)|^2 dt$ converges, so Γ_ϕ is Hilbert-Schmidt. Compare with the conditions in [22].

Following Kreĭn [36], Gesztesy and Simon [25] introduced the *xi* or spectral displacement function associated with a one-dimensional Schrödinger operator, and described how this function connects the Green's function (6.20), the potential and other information in the study of spectral and inverse spectral problems.

Let L denote the densely defined self-adjoint operator $L = -\frac{d^2}{dx^2} + u$ in $L^2(\mathbb{R}; \mathbb{C})$ associated to a potential $u \in C_b(\mathbb{R}, \mathbb{R})$. Suppose that ψ_+, ψ_- are solutions of $L\psi = \lambda\psi$ such that ψ_+ restricts to function in $L^2((0,\infty); \mathbb{C})$ and ψ_- restricts to a function in $L^2((-\infty, 0); \mathbb{C})$.

Definition 6.4. Kodaira's [35] characteristic matrix is

$$\Xi(x,\lambda) = \begin{bmatrix} 2\psi_+\psi_- & \psi_+'\psi_- + \psi_+\psi_-' \\ \psi_+'\psi_- + \psi_+\psi_-' & 2\psi_+'\psi_-' \end{bmatrix} \tag{6.5}$$

all evaluated at (x, λ).

Then $\Xi(x, \lambda)$ is an entire function of λ and $\Xi(x, \lambda) = \Xi(x, \lambda)^\top$. Let $\mathrm{Wr}(f, h) = fh' - f'h$ be the Wronskian. Then Green's function G for $(\lambda I - L)^{-1}$ is

$$G(x, y; \lambda) = \begin{cases} \psi_-(x; \lambda)\psi_+(y, \lambda)/\,\mathrm{Wr}(\psi_-, \psi_+) & (x < y) \\ \psi_-(y; \lambda)\psi_+(x; \lambda)/\,\mathrm{Wr}(\psi_-, \psi_+) & (x > y). \end{cases} \tag{6.6}$$

For any $f \in L^2(\mathbb{R}, \mathbb{C})$, the function $h(x) = \int_{-\infty}^\infty G(x, y; \lambda) f(y)\, dy$ satisfies $f = \lambda h + h'' - uh$.

In their work on the KdV equation, Ercolani and McKean [21] introduced a geometrical multiplier curve. This spectral curve, which could be of an exotic and singular kind, encodes the spectral data of the integrable system. Its geometry, via divisors, line bundles and so forth, can be used to construct explicit solutions to KdV. There is a correspondence between potentials and divisors (in the sense of algebraic geometry; see [29, p 136]). The set of divisors forms an abelian group and the addition rule is associated with the Darboux addition of potentials, as in [39].

We begin by considering the Darboux addition (6.7). Then we consider the infinitesimal Darboux addition (6.13), which is directly related to the diagonal of the Green's function. Suppose that $(e^{-tA})_{t\geq 0}$ is a (C_0) contraction semigroup, so A has dense domain $\mathcal{D}(A)$; that A is accretive, so $\mathrm{Re}\langle Af, f\rangle \geq 0$ for all $f\in\mathcal{D}(A)$; and that $(\lambda I+A)\mathcal{D}(A)$ is dense in H for some $\lambda>0$. Then for all $\zeta\in\mathbb{R}$, the operator $(\zeta I-A)(\zeta I+A)^{-1}$ is a contraction on H. In previous papers [4] and [5, p. 79], we have considered the Darboux addition rule (6.7) on a continuous time linear system, namely

$$(-A,B,C)\mapsto \Sigma_\zeta=(-A,(\zeta I+A)(\zeta I-A)^{-1}B,C) \qquad (|\arg(-\zeta)|<\pi/2). \tag{6.7}$$

As in [4], we are particularly interested in the pair $\Sigma_\infty=(-A,B,C)$ and $\Sigma_0=(-A,-B,C)$, and regard the involution $(-A,B,C)\leftrightarrow(-A,-B,C)$ as analogous to the involution on a hyperelliptic curve, as in [23] or [42]. We introduce the notion of a Darboux curve for a linear system, where the definition is motivated by [21, 3.4].

Definition 6.5. Let $\mathcal{A}_0$ be the unital complex subalgebra of $\mathcal{L}(H)$ that is generated by the operators $(\lambda I-A)(\lambda I+A)^{-1}$ for $\lambda\in\mathbb{C}\setminus\sigma(-A)$, and let $\mathcal{A}$ be the closure of $\mathcal{A}_0$ in $\mathcal{L}(H)$ for the operator norm topology. The Darboux curve of $(-A,B,C)$, denoted $\mathbb{S}$, is the maximal ideal space of the unital commutative Banach algebra $\mathcal{A}$.

The Gelfand transform gives an algebra homomorphism $\mathcal{A}\to C(\mathbb{S};\mathbb{C})$, so elements of $\mathcal{A}$ give functions on $\mathbb{S}$; clearly $\mathbb{S}$ is unchanged by Darboux addition. Note that H is a Hilbert module over $\mathcal{A}$, and H is a Hilbert module over the complex rational functions with poles off $\sigma(A)$. One regards $(\zeta I+A)(\zeta I-A)^{-1}$ as a multiplier on $\mathcal{A}$ or a rational function on $\mathbb{S}$, with inverse $(\zeta I+A)^{-1}(\zeta I-A)$. Darboux addition then corresponds to addition of divisors on $\mathbb{S}$. In the special cases covered by Corollary 4.4 we introduced $\mathbb{S}_e$ which involves the essential spectrum of A, hence $\mathbb{S}_e$ is a closed subset of $\mathbb{S}$. The meaning of ζ will be revealed in Theorem 7.1 as a spectral parameter for a spectral curve $\mathbb{X}$.

Definition 6.6. For $0<\theta\leq\pi$, we introduce the sector $S_\theta=\{z\in\mathbb{C}\setminus\{0\}:|\arg z|<\theta\}$. A closed and densely defined linear operator $-A$ is a generator of type $\mathcal{GA}_b(\theta,M)$ [28, Theorem 5.3] if there exists $\pi/2<\theta<\pi$ and $M\geq 1$ such that S_θ is contained in the resolvent set of $-A$ and $|\lambda|\|(\lambda I+A)^{-1}\|_{\mathcal{L}(H)}\leq M$ for all $\lambda\in S_\theta$. Let $\mathcal{D}(A)$ be the domain of A and $\mathcal{D}(A^\infty)=\cap_{n=0}^\infty\mathcal{D}(A^n)$. See [28, p. 37].

Lemma 6.7. *Let $(-A,B,C)$ be a linear system such that $\|e^{-t_0A}\|_{\mathcal{L}(H)}<1$ for some $t_0>0$, and that B and C are Hilbert–Schmidt operators such that $\|B\|_{\mathcal{L}^2(H_0;H)}\|C\|_{\mathcal{L}^2(H;H_0)}\leq 1$. Suppose further that $-A$ is of type $\mathcal{GA}_b(\theta,M)$. Then the trace class operators $(R_x)_{x>0}$ give the unique solution to Lyapunov's equation (1.1) for $x>0$ that satisfies the initial condition.*

Proof. By [28, Theorem 5.3], there is a (C_0) uniformly bounded semigroup $(e^{-tA})_{t>0}$ which has an analytic extension to the sector $\{t \in \mathbb{C} : \mathrm{Re}\, t > 0, |\arg t| < \theta - \pi/2\}$ so that for all $\varepsilon > 0$ there exists $M_\varepsilon < \infty$ such that $\|e^{-tA}\| \leq M_\varepsilon$ for all $t \in \mathbb{C}$ such that $\mathrm{Re}\, t > 0$ and $|\arg t| < \theta - \pi/2 - \varepsilon$. Then we can follow the proof of [5, Theorem 2.3]. □

Suppose that $(-A, B, C)$ is a continuous time linear system with state space H and input and output space $\mathbb{C}$ which satisfies the conditions of Lemma 6.7, so C is scalar-valued. Let $\mathcal{E} \subseteq L(H)$ be as in Theorem 4.2 [5, Theorem 4.4]. Suppose that $I + R_x$ is invertible in $\mathcal{L}(H)$ and define $F_x = (I + R_x)^{-1}$; then define the bracket operation $\lfloor \cdot \rfloor_x : C((0,\infty); \mathcal{E}) \to C^\infty((0,\infty); \mathbb{C})$

$$\lfloor X \rfloor_x = Ce^{-xA} F_x X F_x e^{-xA} B \qquad (X \in C((0,\infty); \mathcal{E}),\ x > 0), \tag{6.8}$$

so that $\lfloor X \rfloor_x$ is a scalar-valued function of $x > 0$. Here $(F_x)_{x>0}$ is a family of elements of $\mathcal{E}$. Equipped with the special associative multiplication

$$X * Y = X\big(AF_x + F_x A - 2F_x A F_x\big)Y \qquad (X, Y \in C((0,\infty); \mathcal{E})) \tag{6.9}$$

and the formal first-order differential operator

$$\partial X = A(I - 2(I + R_x)^{-1})X + \frac{dX}{dx} + X(I - 2(I + R_x))^{-1}A \qquad (X \in C^\infty(0,\infty); \mathcal{E})); \tag{6.10}$$

then $C^\infty((0,\infty); \mathcal{E})$ becomes a differential ring in sense of [48].

Lemma 6.8. *The bracket operation $\lfloor \cdot \rfloor_x : \mathcal{E} \to C^\infty((0,\infty); \mathbb{C})$ gives a homomorphism of differential rings, so that*

$$\lfloor X * Y \rfloor_x = \lfloor X \rfloor_x \lfloor Y \rfloor_x, \qquad \lfloor \partial X \rfloor_x = \frac{d}{dx} \lfloor X \rfloor_x. \tag{6.11}$$

Proof. This follows from Lyapunov's equation (1.1) as in [5, Theorem 4.4]. □

This is a counterpart to Pöppe's identities for Hankel integral operators, as in [40, section 3.5]. Using Lemma 6.8, one can move between operator identities in the algebra $\mathcal{E}_0$ on H to differential equations for $\mathbb{C}$-valued functions on $(0,\infty)$; see also [4, Section 2].

To associate a Schrödinger operator with a linear system $(-A, B, C)$, we introduce the potential $u(x) = -4\lfloor A \rfloor_x$, which also depends on the B and C via the bracket (6.8). This is equivalent to

$$u(x) = -2\frac{d^2}{dx^2} \log \det(I + R_x) \tag{6.12}$$

as in Dyson's determinant formula; see [5, (1.12)] and (6.1). By Proposition 5.3, the corresponding Schrödinger operator is $L = -\frac{d^2}{dx^2} + u$. The Darboux addition rule (6.7), $\zeta \mapsto \Sigma_\zeta$, produces a family of linear systems for which we can similarly associate bracket operations, potentials u_ζ and Schrödinger operators $L_\zeta = \frac{d^2}{dx^2} + u_\zeta$. The evolution of the potentials under this rule is of the form $u_\zeta = u - \zeta X(u) + o(\zeta)$ where $X(u)$ is a vector field on the space of potentials. Following McKean [39] we showed in [4, Proposition 2.5] that under certain conditions, the infinitesimal Darboux addition $X(u)$ is completely determined by the diagonal Green's function. Let $F_x = (I + R_x)^{-1}$ and introduce

$$X(u) = \frac{-2}{\sqrt{-\lambda}} \lfloor A(I - 2F_x)A(\lambda I + A^2)^{-1} + A(\lambda I + A^2)^{-1}(I - 2F_x)A \rfloor_x. \tag{6.13}$$

as in [4, Proposition 2.5].

Proposition 6.9. *(i) The entries of the characteristic matrix $\Xi(x, \lambda)$ for $x > 0$ belong to $\mathcal{E}$.*

(ii) Let L be the Schrödinger operator in $L^2(\mathbb{R}; \mathbb{C})$, and let $\Xi(x, \lambda)$ be defined for $x \in \mathbb{R}$. Then $\operatorname{Im} \Xi(x, \lambda)$ determines the spectral measure of L.

Proof. (i) We observe that $\det \Xi(x, \lambda) = -\mathrm{Wr}(\psi_+, \psi_-)^2$, which is a non-zero constant, and the top-left entry of Ξ is $\psi_+\psi_-$ as in the numerator of $-G(x, x; \lambda)$ given by (6.6).

Let λ_0 be the bottom of the spectrum of L. Then by [4, (3.2)], the diagonal Green's function for $(\lambda I - L)^{-1}$ is

$$G(x, x; \lambda) = \frac{1}{\sqrt{-\lambda}} \Big(\frac{1}{2} - \frac{\lfloor A \rfloor_x}{\lambda} + \frac{\lfloor A^3 \rfloor_x}{\lambda^2} - \frac{\lfloor A^5 \rfloor_x}{\lambda^3} + \dots \Big) \qquad (\lambda < -\lambda_0), \tag{6.14}$$

where λ is a spectral parameter for L. Also $X(u) = -2\frac{d}{dx} G(x, x; \lambda)$ gives an element of $\mathcal{E}$, as does $\frac{d^2}{dx^2} G(x, x; \lambda)$. Correspondingly, the off-diagonal entry is $(\psi_+\psi_-)' = \psi_+'\psi_- + \psi_+\psi_-'$. Finally, the remaining entry is

$$2\psi_+'\psi_-' = (\psi_+\psi_-)'' + 2(\lambda - u)\psi_+\psi_-.$$

By Kodaira's theorem [35], for Schrödinger's operator in $L^2(\mathbb{R}; \mathbb{C})$, the spectral measure matrix is

$$\mu(0, \lambda] = \lim_{\varepsilon \to 0+} \frac{1}{\pi} \int_0^\lambda \operatorname{Im} \Xi(x, \nu + i\varepsilon)\, d\nu. \tag{6.15}$$

Here $\mu(\nu, \lambda]$ is a positive semi-definite matrix for $\lambda \geq \nu$, so one can form a positive matrix measure $d\mu(\lambda)$ as in Stieltjes integration.

(ii) Let $m_\pm(x, \lambda)$ be the Weyl functions for $Lf = -\frac{d^2 f}{dx^2} + uf$ in $L^2(\mathbb{R}; \mathbb{C})$ with boundary condition $f(x) = 0$, so that $f + m_+ h \in L^2((0, \infty); \mathbb{C})$ and $f - m_- h \in L^2((-\infty, 0); \mathbb{C})$. Then $m_\pm(x, \lambda)$ are Herglotz functions that have boundary limits

$$m_\pm(x, \lambda) = \lim_{\varepsilon \to 0+} m_\pm(x, \lambda + i\varepsilon).$$

From [26, (6.1) and (6.2)], we have the identities

$$-G(x, x; \lambda) = \frac{-1}{m_+(x, \lambda) + m_-(x, \lambda)}, \tag{6.16}$$

$$-\frac{d}{dx} G(x, x; \lambda) = -\frac{m_+(x, \lambda) - m_-(x, \lambda)}{m_+(x, \lambda) + m_-(x, \lambda)}, \tag{6.17}$$

and so

$$\frac{d}{dx} \log\big(-G(x, x; \lambda)\big) = m_+(x, \lambda) - m_-(x, \lambda). \tag{6.18}$$

Suppose further that the spectrum is purely absolutely continuous. Then the diagonal Green's function determines the spectral measure as

$$\frac{d}{dx}\log\bigl(-G(x,x;\lambda)\bigr) = 2\pi i\frac{d\mu}{d\lambda}, \tag{6.19}$$

at Lebesgue points λ of the spectral density. □

Example 6.10. In Example 6.1 and [1, Theorem 4.2 and Section 6], we have shown that the hypotheses of Proposition 6.9(ii) hold for the scattering case of Schrödinger's equation on the real line. The linear systems $(-A, B, C)$ is associates with a unitary group $(e^{-xA})_{x\in\mathbb{R}}$ that is determined by the scattering data in [1, (4.9)].

Proposition 6.9 amounts to a forward spectral theorem, which determines the spectral measure. The inverse spectral problem is also solved likewise, in the sense that the potential u is determined by partial information about $G(x, x; \lambda)$.

Definition 6.11. Let $u \in C_b(\mathbb{R}; \mathbb{R})$ and let L be Schrödinger's operator $L\psi = -\frac{d^2\psi}{dx^2} + u\psi$ with Green's function as in 6.6.

(i) The *xi* function of Kreĭn and Gesztesy–Simon [25] is defined to be

$$\xi(x,\lambda) = \frac{1}{\pi}\arg\lim_{\varepsilon\to 0+}\bigl(-G(x,x;\lambda+i\varepsilon)\bigr) \qquad (\lambda\in\mathbb{R}). \tag{6.20}$$

(ii) The potential u is discretely dominated if $\xi(x, \lambda) = 1/2$ holds for all $x \in \mathbb{R}$ and almost all λ in the essential spectrum of L with respect to Lebesgue measure.

Now $\log(-G(x, x; \lambda))$ is a Herglotz function, hence has boundary values as in (6.20). Observe that $\xi(x, \lambda) = 1/2$ if and only if $G(x, x; \lambda)$ is purely imaginary. Suppose that u is discretely dominated, that L has spectrum $\sigma(L)$ with $\lambda_0 = \inf\sigma(L)$ and

$$[\lambda_0,\infty)\setminus\sigma(L) = \cup_{j=1}^{\ell}(\alpha_j,\beta_j), \tag{6.21}$$

so the spectrum has ℓ consecutive gaps (α_j, β_j). When ℓ is finite, the potential is finite-gap, or algebro-geometric; Theorem 7.1 provides examples. The notion of a *xi* function can further be generalized via Pincus's determining function to a bilinear trace formula [11, Theorem 3.6].

7. Spectral curves

Let $(-A, B, C)$ be a linear system as in Lemma 6.7, and consider the potential $u(x) = -4\lfloor A\rfloor_x$. As before, let $L = -\frac{d^2}{dx^2} + u$ be the corresponding Schrödinger operator. The main result in this section, Theorem 7.1, gives conditions on $(-A, B, C)$ such that one can identify the algebra generated by a certain family of differential operators with an algebra of functions on a particular hyperelliptic curve. This is consistent with the notion of a multiplier curve, as in [39].

The diagonal Green's function contains much more information than simply the potential. Indeed, it acts as a generating function with coefficients that are differential polynomials in u with universal coefficients which we will identify in the following KdV recursion. Doubling the coefficients of the term in parentheses in (6.14), we consider the functions $f_0 = 1$, $f_m = (-1)^m 2\lfloor A^{2m-1}\rfloor_x$ for $m = 1, 2, \dots$. The differential algebra generated by u and its derivatives with respect to x is $\mathcal{R} = \mathbb{C}[u, \frac{du}{dx}, \frac{d^2u}{dx^2}, \dots]$. Following [24], we regard λ, ζ as variables, and define $F_n(\lambda) \in \mathcal{R}[\lambda]$ by

$$F_n(x;\lambda) = \sum_{j=0}^{n} f_{n-j}(x)\lambda^j.$$

These $\mathcal{R}$-valued polynomials satisfy several identities consequent on the stationary KdV hierarchy. We let $Q_{2n+1}(\lambda) \in \mathcal{R}[\lambda]$ of degree $2n+1$ be the polynomial

$$Q_{2n+1}(\lambda) = \frac{1}{2}\Big(\frac{\partial^2 F_n(x;\lambda)}{\partial x^2}\Big)F_n(x;\lambda) - \frac{1}{4}\Big(\frac{\partial F_n(x;\lambda)}{\partial x}\Big)^2 - \big(u(x)-\lambda\big)F_n(x;\lambda)^2, \tag{7.1}$$

and P_{2n+1} be the differential operator of order $2n+1$ in $\partial/\partial x$ with coefficients in $\mathcal{R}$,

$$P_{2n+1} = \sum_{j=1}^{n}\Big(f_{n-j}(x)\frac{\partial}{\partial x} - \frac{1}{2}\frac{\partial f_{n-j}(x)}{\partial x}\Big)L^j. \tag{7.2}$$

We now give the promised version of Burchnall–Chaundy's theorem which gives conditions for a pair of commuting differential operators to satisfy a polynomial equation and thereby determine an algebraic curve.

Theorem 7.1. *For $(-A, B, C)$ as in Lemma 6.7 and let (f_j) be as in (7.2) and suppose that there exists an integer $\ell \geq 1$ such that $\frac{\partial f_{\ell+1}}{\partial x} = 0$ but $\frac{\partial f_\ell}{\partial x} \neq 0$. Then*

(i) the differential operators satisfy $Q_{2\ell+1}(L) = -P_{2\ell+1}^2$;

(ii) The function $f(\lambda,\zeta) = \zeta^2 - Q_{2\ell+1}(\lambda)$ is independent of x and so $f(\lambda,\zeta) = 0$ determines a hyperelliptic curve $\mathbb{X}$ of genus ℓ;

(iii) the commutative algebra of ordinary differential operators that is generated by L and $P_{2\ell+1}$ is isomorphic to the algebra of regular functions on $\mathbb{X}$.

Proof. (i) In [5, Proposition 5.2], it is shown that the sequence $(f_m)_{m=0}^{\infty}$ lies in $\mathcal{R}$ and satisfies the recursion relation of the stationary KdV hierarchy. By [24, Remark 1.5], there exists a solution of the stationary KdV hierarchy that terminates in the sense there is a solution $(f_m)_{m=0}^{\infty}$ with only finitely many nonzero f_m, $m = 0, 1, \dots, \ell$; hence L commutes with $P_{2\ell+1}$. Note that $\mathcal{R} = \mathbb{C}[u, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \dots, \frac{\partial^{2\ell} u}{\partial x^{2\ell}}]$, giving a Noetherian differential ring. On substituting L for λ in $Q_{2\ell+1}(\lambda)$, one obtains a differential operator $Q_{2\ell+1}(L)$ of order $4\ell+2$. The result now follows as in the proof of [24, Theorem 1.3].

(ii) Let $f(\lambda,\zeta) = \zeta^2 - Q_{2\ell+1}(\lambda)$, which independent of x by the hypothesis on $f_{\ell+1}$ and [24, p. 7]. Also, $f(\lambda,\zeta)$ is irreducible in $\mathbb{C}[\lambda,\zeta]$.

(iii) Let $\mathcal{A}_L$ be the set of differential operators that commute with L, so $\mathcal{A}_L$ is a commutative algebra by a result of Schur. We have seen that $P_{2\ell+1} \in \mathcal{A}_L$. This implies that the greatest common divisor of the orders of the operators in $\mathcal{A}_L$ is one, and so $\mathcal{A}_L$ has rank one [59, p. 182]. Let $\mathrm{Specm}(\mathcal{A}_L)$ be the space of maximal ideals of $\mathcal{A}_L$ which is a subset of the space $\mathrm{Spec}(\mathcal{A}_L)$ of prime ideals. Burchnall and Chaundy [7] showed that $\mathrm{Specm}(\mathcal{A}_L)$ is an irreducible algebraic curve that can be completed by adding one smooth point at infinity; see [59, p. 182] and [43, p. 132] for a modern presentation of this result.

Then $\mathbb{C}[L, P_{2\ell+1}]$ is isomorphic to $\mathbb{C}[\lambda, \zeta]/(f(\lambda, \zeta))$, namely the algebra generated by the coordinate functions on the hyperelliptic curve $\mathbb{X}$; see [47, Remark 2.8]. One can show that $\mathbb{C}[L, P_{2\ell+1}]$ is a complex subalgebra of $\mathcal{A}_L$ such that the dimension of the quotient vector space $\mathcal{A}_L/\mathbb{C}[L, P_{2\ell+1}]$ is finite; see [41, section 2]. We interpret $\mathbb{X}$ as sheets covering $\mathbb{C}$. There is an inclusion $\mathbb{C}[L] \to \mathcal{A}_L$, hence a surjective map of prime ideals $\mathrm{Spec}(\mathcal{A}_L) \to \mathrm{Spec}(\mathbb{C}[L])$ given by $P \mapsto P \cap \mathbb{C}[L]$; see [55, p. 223]. Thus a spectral point $\lambda \in \mathbb{C}$ gives the prime ideal generated by $L - \lambda I$ in $\mathbb{C}[L]$ which is covered by a prime ideal of $\mathcal{A}_L$. □

Remark 7.2. In [6, Theorem 3.1], Brezhnev shows that if the system $(-A, B, C)$ satisfies Theorem 7.1 then the corresponding L is integrable, in the sense that $L\psi = \lambda\psi$ can be solved explicitly by Liouville integration.

The curve $\mathbb{X}$ is hyperelliptic, so there exists a meromorphic $\wp : \mathbb{X} \to \mathbb{C}_\infty$ that is typically $2:1$, with branch points at $\{p \in \mathbb{X} : \wp'(p) = 0\}$; see [42, 3.155]. We introduce the real points on $\mathbb{X}$ by $\mathbb{X}_r = \{(\lambda, \zeta) \in \mathbb{X} : \lambda, \zeta \in \mathbb{R}\}$, which typically gives a disconnected subset which projects to $\{\lambda \in \mathbb{R} : Q_{2\ell+1}(\lambda) > 0\}$ via $(\lambda, \zeta) \mapsto \lambda$.

Suppose that u is real, and consider how $\mathbb{X}_r$ and $\sigma(L)$ are related. In [5, Proposition 3.2] we provide the Baker–Akhiezer function ψ_λ such that $L\psi_\lambda = \lambda\psi_\lambda$, so $\{\lambda \in \mathbb{R} : \psi_\lambda \in L^\infty((0, \infty); \mathbb{C})\}$ is the set of approximate eigenvalues, contained in the L^2 spectrum of L. Then $\mathrm{int}\{\lambda \in \mathbb{R} : \psi_\lambda \in L^\infty((0, \infty) : \mathbb{C})\}$ is contained in the essential spectrum $\sigma_{ess}(L)$. The function $-G(x, x; \lambda)$ is holomorphic on $\mathbb{C} \setminus \sigma(L)$. For real potentials that also satisfy Theorem 7.1, we have $\wp : \mathbb{X} \to \mathbb{C}_\infty$ restricting to $\wp : \mathbb{X}_r \to \sigma_{ess}(L)$. One now has the geometrical situation addressed in Example 7.3.

Example 7.3. Douglas and Yan [19, p. 212] considered cycles on algebraic curves, and postulated the existence of associated 1-cocycles. Corollary 3.10 on multiply connected domains leads to such a result. By Riemann's mapping theorem, we can suppose that $C_0 = C(0, 1)$ and $D_0 = \mathbb{D}$. Then we use the inversion in C_0 via $z \mapsto z/|z|^2$ to take $C_j \mapsto C_{-j}$ and $\mathcal{S} \mapsto \mathcal{S}_1$ to produce a domain $\mathcal{S} \cup C_0 \cup \mathcal{S}_1$ that has boundary $\cup_{j=1}^{\ell}(C_j \cup C_{-j})$; here $C_)$ is not a boundary, but the circle of reflection joinint $\mathcal{S}$ to $\mathcal{S}_0$. By joining C_j to C_{-j} for $j = 1, \dots, \ell$, we can introduce a compact Riemann surface $\mathbb{X}$ called the Schottky double of $\mathcal{S}$. Let $\mathbb{X}_1$ be the open subset of $\mathbb{X}$ that corresponds to $\mathcal{S}_1$ in this identification.

Then any meromorphic function on $\mathbb{X}$ that has all its poles in $\mathbb{X}_1$ gives an element of $\mathcal{A}$. As in [55, p 156], one can introduce $\Omega^1_{\mathbb{X}}$ as the space of differential one-forms that have holomorphic coefficients, classically called the Abelian differentials of the first kind. Let V be an open subset of $\mathbb{X}$ such that $\gamma = \partial V$ is a C^1 contour; let f, g be holomorphic functions on $\mathrm{cl}(V)$. Then fdg is a holomorphic 1-form on V such that $\int_\gamma fdg = 0$ by Stokes's theorem.

As in [16, Figure 1], and [23, p. 103], $\mathbb{X}$ has a homology basis $\{\alpha_1, \dots, \alpha_\ell; \beta_1, \dots, \beta_\ell\}$, so that α_j in $\mathbb{X}_0$ corresponds to C_j and β_j is a contour that arises from a cross-cut linking C_j to C_{-j} for $j = 1, \dots, \ell$. Then one can introduce the group of closed contours on $\mathbb{X}$ modulo homotopy equivalence, to obtain the abelian homology group $H_1(\mathbb{X}; \mathbb{Z})$. Let Z^1 be the space of closed differential 1-forms on $\mathbb{X}$ for the de Rham differential $\{\sigma : d\sigma = 0\}$, so $\Omega^1_{\mathbb{X}}$ is a complex linear subspace of Z^1 as in [23, p. 25]. There is a well-defined pairing $Z^1 \times H_1(\mathbb{X}; \mathbb{Z}) \to \mathbb{C}$ given by $(\omega, \gamma) = \int_\gamma \omega$ [23, p. 44].

Let ϑ be the sheaf of holomorphic functions on $\mathbb{X}$ and ϑ^* the sheaf of nowhere-zero holomorphic functions on $\mathbb{X}$. As in [29, page 446], we have the multiplicative group of holomorphic line bundles on $\mathbb{X}$ denoted $\mathrm{Pic}(\mathbb{X})$, which is canonically isomorphic to $H^1(\mathbb{X}; \vartheta^*)$. From the exponential map $\vartheta \to \vartheta^*$ $f \mapsto e^{2\pi i f}$ we obtain an exact sequence

$$0 \longrightarrow H^1(\mathbb{X}; \mathbb{Z}) \longrightarrow H^1(\mathbb{X}; \vartheta) \longrightarrow \mathrm{Pic}(\mathbb{X}) \longrightarrow H^2(\mathbb{X}; \mathbb{Z}) \longrightarrow \dots$$

where $H^2(\mathbb{X}; \mathbb{Z}) \cong \mathbb{Z}$. Then the Jacobian variety of $\mathbb{X}$ is given by the subgroup $\mathrm{Pic}(\mathbb{X})_0 = \{p \in \mathrm{Pic}(\mathbb{X}) : p \mapsto 0\}$ of $\mathrm{Pic}(\mathbb{X})$, as in [29, page 325]. Mumford gives an algebraic construction of the Jacobian for hyperelliptic curves [42, 3.28].

For spectral theory of self-adjoint operators, as in Theorem 7.1, we introduce $\mathbb{X}$ via the following construction. Consider consecutive and disjoint bounded real intervals $[a_j, b_j]$ with union $S = \cup_{j=0}^{\ell}[a_j, b_j]$, so $\mathcal{S}_0 = \mathbb{C}_\infty \setminus S$ is a $(\ell + 1)$-connected subset of the Riemann sphere. Consider a domain $\mathcal{S}$ with boundary $\partial\mathcal{S}$ made of disjoint circles C_j for $j = 0, \dots, \ell$, let D_j be the inside of C_j and suppose that $D_j \subset D_0$ for $j = 1, \dots, \ell$ and $D_j \cap D_k$ is empty for $j, k = 1, \dots, \ell$; this $\mathcal{S}$ is known as a circular domain. By [14, page 7] there exists a circular domain $\mathcal{S}$ and a conformal bijection $\psi : \mathcal{S} \to \mathcal{S}_0$. We can form the Schottky double $\mathbb{X}$ of $\mathcal{S}$ and thus study $\mathcal{S}_0$ as an open subset of the Riemann surface $\mathbb{X}$ of genus ℓ. In [14], [15] and [16], Crowdy and Marshall give an effective computation for ψ via the Schottky–Klein function.

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