# LEIBNIZ ALGEBRAS IN WHICH ALL CENTRALISERS OF NONZERO ELEMENTS ARE ZERO ALGEBRAS

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Abstract. This paper is concerned with generalising the results for Lie $CT$-algebras to Leibniz algebras. In some cases our results give a generalisation even for the case of a Lie algebra. Results on $A$-algebras are used to show every Leibniz $CT$-algebra over an algebraically closed field of characteristic different from 2,3 is solvable or is isomorphic to $sl_2(F)$. A characterisation is then given for solvable Leibniz $CT$-algebras. It is also shown that the class of solvable Leibniz $CT$-algebras is factor closed.

## 1. Introduction

An algebra $L$ over a field $F$ is called a *Leibniz algebra* if, for every $x, y, z \in L$, we have

$$[x, [y, z]] = [[x, y], z] - [[x, z], y].$$

In other words, the right multiplication operator $R_x : L \to L : y \mapsto [y, x]$ is a derivation of $L$. As a result such algebras are sometimes called *right* Leibniz algebras, and there is a corresponding notion of *left* Leibniz algebras, which satisfy

$$[x, [y, z]] = [[x, y], z] + [y, [x, z]].$$

Clearly, the opposite of a right (left) Leibniz algebra is a left (right) Leibniz algebra, so, in most situations, it does not matter which definition we use.

Every Lie algebra is a Leibniz algebra and every Leibniz algebra satisfying $[x, x] = 0$ for every element is a Lie algebra. They were introduced in 1965 by Bloh ([3]) who called them $D$-algebras, though they attracted more widespread interest, and acquired their current name, through work by Loday and Pirashvili ([8], [9]). They have natural connections to a variety of areas, including algebraic $K$-theory, classical algebraic topology, differential geometry, homological algebra, loop spaces, noncommutative geometry and physics. A number of structural results have been obtained as analogues of corresponding results in Lie algebras.

The *Leibniz kernel* is the set $I = \text{span}\{x^2 : x \in L\}$. Then $I$ is the smallest ideal of $L$ such that $L/I$ is a Lie algebra. Also $[L, I] = 0$.

We define the following series:

$$L^1 = L, \quad L^{k+1} = [L^k, L] \quad (k \geq 1)$$

and

$$L^{(0)} = L, \quad L^{(k+1)} = [L^{(k)}, L^{(k)}] \quad (k \geq 0).$$

Then $L$ is *nilpotent of class n* (resp. *solvable of derived length n*) if $L^{n+1} = 0$ but $L^n \neq 0$ (resp. $L^{(n)} = 0$ but $L^{(n-1)} \neq 0$) for some $n \in \mathbb{N}$. It is straightforward to

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check that $L$ is nilpotent of class $n$ precisely when every product of $n+1$ elements of $L$ is zero, but some product of $n$ elements is non-zero.

The *centraliser* of an element $x$ of an algebra $A$ is $C_A(x) = \{a \in A \mid ax = xa = 0\}$; if the algebra $A$ is clear we will simply write $C(x)$. An algebra $A$ is called *centraliser transitive, or a $CT$-algebra* if $x \in C(y)$ and $y \in C(z)$ imply that $x \in C(z)$. Such algebras, where $A$ is a Lie algebra, have been studied by Klep and Moravec ([7]) in 2010, by Arzhantsev, Makedonskii, and Petravchuk ([7]) in 2011, and by Gorbatsevic ([5]) in 2016. In the last two of these references they are shown to have applications to the classification of finite-dimensional subalgebras in polynomial Lie algebras of rank one and to Lie algebras of vector fields whose orbits are one-dimensional. A similar notion for groups was defined and studied by Weisner [13]) in 1925. Finite nonabelian simple $CT$-groups had been classified by Suzuki ([11]) in 1957. He proved that every finite nonabelian simple $CT$-group is isomorphic to some $PSL(2, 2^f)$, where $f > 1$. Suzuki's result is considered to have been one of the key stones in the proof of the Odd Order Theorem by Feit and Thompson ([4]).

Note that $x \in C(y)$ if and only if $y \in C(x)$. It is clear that $CT$-algebras are subalgebra closed. The *centre* of an algebra $A$ is $Z(A) = \{z \in A \mid za = az = 0 \text{ for all } a \in A\}$. We call $A$ a *zero algebra* if $A = Z(A)$. The following lemma makes clear the title of the paper.

**Lemma 1.1.** *The algebra $A$ is a $CT$-algebra if and only if $C(x)$ is a zero algebra for all $0 \neq x \in L$.*

*Proof.* Let $A$ be a $CT$-algebra and let $y, z \in C(x)$. Then $y \in C(x)$ and $x \in C(z)$, so $y \in C(z)$; that is $yz = zy = 0$.

Conversely, suppose that $C(x)$ is a zero algebra for all $x \neq 0$, and let $x \in C(y)$ and $y \in C(z)$. Then $x, z \in C(y)$, so $xz = zx = 0$, whence $x \in C(z)$. □

**Lemma 1.2.** *If $A$ is a $CT$-algebra and $Z(A) \neq 0$, then $A$ is a zero algebra.*

*Proof.* Let $0 \neq x \in Z(A)$. Then $A = C(x)$ is a zero algebra. □

We call an algebra $A$ an *$A$-algebra* if every nilpotent subalgebra of $A$ is a zero algebra. Then $CT$-algebras are a subclass of the class of $A$-algebras.

**Lemma 1.3.** *Every $CT$-algebra is an $A$-algebra.*

*Proof.* Let $N$ be a nilpotent $CT$-algebra. Then $Z(N) \neq 0$, so the result follows from Lemma 1.2.. □

The next section is concerned with generalising the results for Lie $CT$-algebras to Leibniz algebras. In some cases our results give a generalisation even for the case of a Lie algebra. Leibniz $A$-algebras were studied by Towers in [12]. The results proved there are used to show every Leibniz $CT$-algebra over an algebraically closed field of characteristic different from 2,3 is solvable or is isomorphic to $sl_2(F)$; the previously recorded result for Lie algebras asumed characteristic zero. A characterisation is then given for solvable Leibniz $CT$-algebras. It is also shown that the class of solvable Leibniz $CT$-algebras is factor closed.

From now on, $L$ will denote a finite-dimensional Leibniz algebra over a general field $F$ (unless specified otherwise). The notation $\oplus$ will denote an algebra direct sum, whereas $\dot{+}$ will denote a direct sum of the underlying vector space structure alone.

## 2. Main results

The following lemma will prove useful.

**Lemma 2.1.** *Let $L = A\dot{+}Fx$ be a Leibniz $CT$-algebra over a field $F$, where $A$ is a zero ideal of $L$ and $x^2 \in A$. Then, either $L$ is a zero algebra or $R_x|_A$ has no zero eigenvalue in $F$ and $Ax = A$.*

*Proof.* Suppose that $R_x|_A$ has a zero eigenvalue in $F$. Let $0 \neq y \in A$ be a corresponding eigenvector, so $yx = 0$. If $xy = 0$, we have that $L = C(y)$ is a zero algebra. So suppose that $xy \neq 0$. Then

$$(xy)x = x(yx) + x^2y = 0 \text{ and } x(xy) = x^2y - (xy)x = 0.$$

Thus, $L = C(xy)$ is a zero algebra again.

Suppose next that $R_x|_A$ has no zero eigenvalue in $F$. Let $L = L_0\dot{+}L_1$ be the Fitting decomposition of $L$ relative to $R_x$. Then $L_1 \subseteq A$. Suppose that $B = L_0 \cap A \neq 0$ and that $R_x^n(B) = 0$, $R_x^{n-1}(B) \neq 0$. Let $0 \neq y \in R_x^{n-1}(B)$. Then $yx = 0$ and $R_x$ has a zero eigenvalue in $F$, a contradiction. It follows that $B = 0$ and so $A = L_1$ and $Ax = A$. □

*Cyclic* Leibniz algebras, $L$, are generated by a single element. In this case $L$ has a basis $a, a^2, \dots, a^n (n > 1)$ and product $a^n a = \alpha_2 a^2 + \dots + \alpha_n a^n$. Let $T$ be the matrix for $R_a$ with respect to the above basis. Then $T$ is the companion matrix for $p(x) = x^n - \alpha_n x^{n-1} - \dots - \alpha_2 x = p_1(x)^{n_1} \dots p_r(x)^{n_r}$, where the $p_j$ are the distinct irreducible factors of $p(x)$. Then we have the following result.

**Theorem 2.2.** *Let $L$ be a cyclic Leibniz algebra. Then the following are equivalent:*

(i) *$L$ is a $CT$-algebra;*
(ii) *$L$ is an $A$-algebra; and*
(iii) *$\alpha_2 \neq 0$, and then $L = L^2\dot{+}F(a^n - \alpha_n a^{n-1} - \dots - \alpha_2 a)$ and we can take $p_1(x)^{n_1} = x$.*

*Proof.* The equivalence of (ii) and (iii) is given by [12, Theorem 12]. Lemma 1.3 gives that (i) implies (ii), so it simply remains to show that (iii) implies (i).

So suppose that $L$ is as described in (iii). Put $b = a^n - \alpha_n a^{n-1} - \dots - \alpha_2 a$. Then it is easy to check that $b^2 = 0$ and $R_b|_{L^2}$ has no zero eigenvalue in $F$. Let $x = n + \lambda b$ where $n \in L^2$ and $\lambda \in F$. Now $bL^2 = bI = 0$, so straightforward calculations show that

$$C(x) = \left\{ \begin{array}{lll} L^2 & \text{if} & \lambda = 0, n \neq 0 \\ Fb & \text{if} & \lambda \neq 0, n = 0 \\ 0 & \text{if} & \lambda \neq 0, n \neq 0 \end{array} \right\}$$

Hence $L$ is a $CT$-algebra □

**Note** that the above result shows that we may not have $xA = A$ in Lemma 2.1. For, if $L$ is a cyclic Leibniz $A$-algebra, then $Ix = I$, but $xI = 0$.

**Lemma 2.3.** *If $L$ is a Leibniz algebra and $N$ is a zero ideal of $L$, we can consider $N$ as a right $L/N$-module under the action $n(x + N) = nx$ for all $x \in L$, $n \in N$. Then, each element $R_{x+N}$ with $x \notin N$ has no zero eigenvalue under this action if and only if $C_L(n) \subseteq N$ for all $n \in N$.*

*Proof.* Suppose $R_{x+N}$ has a zero eigenvalue under the action, where $x \notin N$. Then, there exists $0 \neq n \in N$ such that $nx = 0$. If $xn = 0$, then $x \in C_L(n) \setminus N$. If $xn \neq 0$, then $x \in C_L(xn) \setminus N$, as in Lemma 2.1 above.

Conversely, if $x \in C_L(n) \setminus N$ for some $n \in N$, we have $nx = 0$ and $R_{x+N}$ has a zero eigenvalue. □

**Proposition 2.4.** *Let $L$ be a Leibniz algebra and $N$ be a zero ideal of $L$ such that $L/N$ is a $CT$-algebra and $C_L(n) \subseteq N$ for all $0 \neq n \in N$. Then $L$ is a $CT$-algebra.*

*Proof.* Let $0 \neq x \in L$ and take $y, z \in C_L(x)$, so $xy = yx = xz = zx = 0$. If $y \in N$, then $x \in C_L(y) \subseteq N$. Hence $y, z \in C_L(x) \subseteq N$, giving $yz = 0$.

So assume that $y \notin N$. Then $yz \in N$, by the hypothesis. Now

$$x(yz) = (xy)z - (xz)y = 0 \text{ and } (yz)x = y(zx) + (yx)z = 0,$$

so $x \in C_L(yz) \subseteq N$. Thus $C_L(x) \subseteq N$ and $yz = 0$. The result follows. □

**Note** that If $L$ is a nilpotent cyclic Leibniz algebra of dimension greater than 1, then $L/I$ is a Lie $CT$ algebra, but $L$ is not a $CT$-algebra. Next we have the two main classification results.

**Theorem 2.5.** *Let $L$ be a Leibniz $CT$-algebra over an algebraically closed field $F$ of characteristic $\neq 2, 3$. Then $L$ is solvable or is isomorphic to $sl_2(F)$.*

*Proof.* Since $F$ has cohomological dimension 0, $L = R \dot{+} S$, where $R$ is the radical of $L$ and $S$ is a direct sum of ideals isomorphic to $sl_2(F)$, by Lemma 1.3 and [12, Theorem 2]. Suppose that $R \neq 0$. If $S \neq 0$, it is clear from Lemma 1.1 that $S \cong sl_2(F)$, and there is an element $x \in S$ such that $R_x$ is nilpotent. Then $R_x$ has a zero eigenvalue and $A = N \dot{+} Fx$, where $N$ is the nilradical of $L$, must be a zero algebra, by Lemma 2.1. Thus, $x \in C_L(N)$. But $C_L(N)$ is an ideal of $L$, and so $S \subseteq C_L(N)$. Pick $0 \neq y \in N$. Then $S \subset C(y)$, which is a zero algebra. Hence $S = 0$. □

**Theorem 2.6.** *Let $L$ be a solvable Leibniz $CT$-algebra of derived length $n + 1$. Then*

(i) *$L = A_n \dot{+} A_{n-1} \dot{+} \ldots \dot{+} A_0$, where $A_i$ is an a zero subalgebra of $L$ and $L^{(i)} = A_n \dot{+} A_{n-1} \dot{+} \ldots \dot{+} A_i$ for $0 \leq i \leq n$;*

(ii) *$L$ splits over the nilradical $N$, which equals $L^{(n)}$; and*

(iii) *for every element $x \in L \setminus N$, $R_x|_N$ has no zero eigenvalue in $F$, and $Nx = N$.*

*Proof.* (i) We have that $L = A_n \dot{+} A_{n-1} \dot{+} \ldots \dot{+} A_0$ and $L^{(i)} = A_n \dot{+} A_{n-1} \dot{+} \ldots \dot{+} A_i$, by [12, Corollary 1].

(ii) Also, $N = A_n \oplus (N \cap A_{n-1} \oplus \ldots \oplus N \cap A_0)$ and $Z(L^{(i)}) = N \cap A_i$ for each $0 \leq i \leq n$, by [12, Theorem 5]. Suppose that $N \cap A_i \neq 0$ for some $0 \leq i \leq n-1$. Let $0 \neq x \in N \cap A_i$. Then $C_L(x) \supseteq L^{(i)}$. But $L^{(i)}$ is not a zero algebra for $0 \leq i \leq n-1$, so we have a contradiction.

(iii) Let $x \in L \setminus N$. Then $L(x) = N \dot{+} Fx$ satisfies the hypothesis of Lemma 2.1, since $x^2 \in I \subseteq N$. If $L(x)$ is a zero algebra, then $x \in C_L(N) = N$, by [12, Lemma 7], a contradiction. Hence $R_x|_N$ has no zero eigenvalue in $F$, and $Nx = N$, by Lemma 2.1. □

**Definition 1.** *A Leibniz algebra $L$ is called* **completely solvable** *if $L^2$ is nilpotent.*

Over a field of characteristic zero, every solvable Leibniz algebra is completely solvable, and so every solvable Leibniz $CT$-algebra has derived length at most 2. However, this is not the case over fields of positive characteristic, even for Lie algebras, as the following example, which is taken from [6, pages 52, 53], shows.

**Example 2.7.** *Let*

$$e = \begin{bmatrix} 0 & 1 & 0 & . & . & . & 0 \\ 0 & 0 & 1 & 0 & . & . & 0 \\ \vdots & & & & & & \vdots \\ 0 & . & . & . & . & 0 & 1 \\ 1 & 0 & . & . & . & . & 0 \end{bmatrix}, \quad f = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 2 & \dots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \dots & p-1 \end{bmatrix},$$

*let $F$ be a field of prime characteristic $p$ and put $L = Fe + Ff + F^p$ with product $[a + \mathbf{x}, b + \mathbf{y}] = [a, b] + (\mathbf{x}b - \mathbf{y}a)$ for all $a, b \in Fe + Ff$, $\mathbf{x}, \mathbf{y} \in F^p$.*

Then, straightforward calculations show that

$$C(\alpha e + \beta f + \mathbf{x}) = \left\{ \begin{array}{lll} Ff + F\mathbf{x_1} & \text{if} & \alpha = \beta = 0, \mathbf{x} = \mathbf{x_1} \\ F^p & \text{if} & \alpha = \beta = 0, \mathbf{x} \neq \mathbf{x_1} \\ F(\alpha e + \beta f + \mathbf{x}) & \text{if} & \alpha \neq 0 \\ F(\beta f + \mathbf{x}) + F\mathbf{x_1} & \text{if} & \alpha = 0, \beta \neq 0, \end{array} \right\}$$

where $\mathbf{x_1} = (1, 0, \dots, 0)$, and all of these are zero algebras.

**Theorem 2.8.** *Let $L$ be a solvable Leibniz $CT$-algebra over an arbitrary algebraically closed field $F$. Then the nilradical of $L$ has codimension at most 2 in $L$.*

*Proof.* We have that $L = N \dot{+} A_1 \dot{+} A_0$, by Theorem 2.6 and [12, Theorem 14]. Suppose $A_1 \neq 0$. Let $A$ be a minimal ideal of $B = N \dot{+} A_1$, inside $N$. Then $A$ is an irreducible $B$-bimodule, and so $A_1 A = 0$ or $ab = -ba$ for every $a \in A$, $b \in B$, by [2, Lemma 1.9]. In either case, $A$ is a minimal right $A_1$-module and $A = Fn$ is one-dimensional, by [10, Lemma 5]. But $C_B(A) \subseteq N$ has dimension at least $\dim B - 1$, so $\dim A_1 \leq 1$. The same argument shows that $\dim A_0 \leq 1$, whence the result. □

If the field $F$ in the above result has characteristic zero, then the codimension is at most one. However, over any field of characteristic $p > 0$, the codimension can be two, even if $L$ is a Lie algebra, as Example 2.7 shows.

If $L$ is a Leibniz algebra and $y \in L$, the *left centraliser of $y$ in $L$*, is $C_L^\ell(y) = \{x \in L \mid xy = 0\}$. It is easy to check that this a subalgebra of $L$.

**Theorem 2.9.** *Let $L$ be a completely solvable Leibniz $CT$-algebra. Then, either $L$ is a zero algebra, or $L = N \dot{+} A_0$ where $N$ is the nilradical, $N^2 = A_0^2 = 0$, $R_x|_N$ has no zero eigenvalue and $Nx = N$ for all $x \in A_0$. If $A_0$ and $A_0'$ are two complements to $N$ in $L$, then there exists $n \in N$ such that $(1 + L_n)(A_0) = A_0'$.*

*Proof.* We have that $L = N \dot{+} A_0$ where $N$ is the nilradical and $N^2 = A_0^2 = 0$, by Theorem 2.6. Suppose that $L$ is not a zero algebra. Then $A_0 \neq 0$, $R_x|_N$ has no zero eigenvalue and $Nx = N$ for all $x \in A_0$, by Lemma 2.1.

For every $y \in L \setminus N$ we have $N = Ny \subseteq Ly \subseteq L^2 \subseteq N$, so $N = Ly$. Pick any $x \in L$. Then $xy \in N = Ny$, so there is an $n \in N$ such that $xy = ny$. Thus $(x - n)y = 0$ and $x - n \in C_L^\ell(y)$. It follows that $L = N + C_L^\ell(y)$. Moreover, $N \cap C_L^\ell(y) = 0$, since $y$ has no zero eigenvalue on $N$, so $C_L^\ell(y)$ is a complement to $N$ in $L$.

Let $A_0$ be any complement to $N$ in $L$. Then $y = n' + a$ for some $n' \in N = Ny$ and $a \in A_0$. Hence, there is an $n \in N$ such that $n' = -ny$ and $a = (1 + L_n)(y)$.

Put $\theta = 1 + L_n$, so $\theta(y) = a$ and $\theta(C_L^\ell(y)) \subseteq C_L^\ell(a)$. But $A_0 = C_L^\ell(a)$ from which the final claim follows. □

**Note** that the above result mirrors the corresponding result in Lie algebras (see [7, Theorem 3]). However, in that result, the complements are conjugate under the inner automorphism $1 + ad\, n$. In our result, $1 + L_a$ is not an automorphism, in general, as the following example shows.

**Example 2.10.** *Let $L$ be the two-dimensional solvable cyclic Leibniz algebra with basis $a, a^2$ and $a^2a = a^2$. It is easy to see that this is a $CT$-algebra. Put $\theta = 1 + L_{a^2}$. Then $\theta(a^2) = a^2$, whereas $\theta(a)\theta(a) = (a + a^2)(a + a^2) = a^2 + a^2$.*

It would have been better if we could have taken $\theta = 1 + R_n$, which is an automorphism. However, if $N = I$ (as in Example 2.10 above), $\theta(A_0) = A_0$.

Finally, we have that solvable Leibniz $CT$-algebras are factor-closed.

**Theorem 2.11.** *Let $L$ be a solvable Leibniz $CT$-algebra and let $J$ be an ideal of $L$. Then $L/J$ is a $CT$-algebra.*

*Proof.* Suppose that $J \not\subseteq N$ and let $x \in J \setminus N$. Then $N = Nx \subseteq J$, by Lemma 2.1, so $J \subset N$ or $N \subseteq J$. We use induction on the derived length $k$ of $L$. Suppose first that $k = 2$, so $L$ is completely solvable. If $N \subseteq J$, we have that $L/N$ is a zero algebra and so $L/J$ is a $CT$-algebra.

So suppose now that $J \subset N$. A similar argument to that used in Lemma 2.1 also shows that $J = Jx$ for all $x \notin N$. Let $x + J \in C_{L/J}(n + J)$ where $n \in N \setminus J$ and suppose that $x \notin N$. Then $nx \in J = Jx$, so $nx = jx$ for some $j \in J$. Thus $(n - j)x = 0$. But now $n = j$, since $L$ is a $CT$-algebra, and this is a contradiction. It follows that $C_{L/J}(n + J) \subseteq N/J$. Now $L/N \cong (L/J)/(N/J)$ is a $CT$-algebra and hence so is $L/J$, by Proposition 2.4.

So suppose the result holds whenever $k \leq m$ and let $L$ have derived length $m+1$. Then $L = N \dot{+} B$ for some subalgebra $B$ of derived length $m$ of $L$, by Theorem 2.6 (ii). Now $L/N \cong B$ is a $CT$-algebra, by the inductive hypothesis. If $N \subseteq J$, then $L/J \cong (L/N)/(J/N) \cong B/B \cap J$, which is a $CT$-algebra, by the inductive hypothesis. If $J \subset N$, then $L/J$ is a $CT$-algebra as in paragraph two above. □

**Corollary 2.12.** *Let $L$ be any Leibniz $CT$-algebra over an algebraically closed field of characteristic $\neq 2, 3$, and let $J$ be an ideal of $L$. Then $L/J$ is a $CT$-algebra.*

*Proof.* This is immediate from Theorems 2.5 and 2.11. □

## References


[1] I.V.Arzhantsev, E. A. Makedonskii, and A. P. Petravchuk, 'Finite-dimensional subalgebras in polynomial Lie algebras of rank one', *Ukrainian Math. J.* **63 (5)** (2011), 708712.

[2] D. Barnes, 'Some theorems on Leibniz algebras', *Comm. Alg.* **39** (2011), 2463-2472.

[3] A. Bloh, 'On a generalization of the concept of Lie algebra'. *Dokl. Akad. Nauk SSSR.* **165** (1965), 471–473.

[4] W. Feit, J. G. Thompson, 'Solvability of groups of odd order'. *Pacific J. Math.* **13** (1963), 7751029.

[5] V.V. Gorbatsevich, 'Lie algebras with abelian centralizers', *Math. Notes* **101 (5)** (2017), 795-801; translated from *Mat. Zametki* **101 (5)** (2017), 690-699.

[6] N. Jacobson, Lie algebras. New York: Dover Publ. (1979).

[7] I. Klep and P. Moravec, 'Lie algebras with abelian centralizers', *Algebra Colloq.* **17(4)** (2010), 629–636.

[8] J.-L. Loday, 'Une version non commutative des algèbres de Lie: les algbres de Leibniz'. *Enseign. Math. (2)* **39 (34)** (1993), 269–293.

[9] J.-L. Loday and T. Pirashvili, 'Universal enveloping algebras of Leibniz algebras and (co)homology', *Math. Annalen* **296 (1)** (1993) 139–158.

[10] H. Strade and R. Farnsteiner, 'Modular Lie algebras and their representations', New York: Marcel Dekker Inc. (1988).

[11] M. Suzuki, 'The nonexistence of a certain type of simple groups of odd order', *Proc. Amer. Math. Soc.* **8** (1957), 686695.

[12] D.A.Towers, 'Leibniz $A$-algebras', *Comm. Math.* **38** (2020), 103-121.

[13] L. Weisner, 'Groups in which the normaliser of every element except identity is abelian', *Bull. Amer. Math. Soc.* **31** (1925), 413416.

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