

Computing maximum likelihood thresholds using graph rigidity

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Abstract

The maximum likelihood threshold (MLT) of a graph G is the minimum number of samples to almost surely guarantee existence of the maximum likelihood estimate in the corresponding Gaussian graphical model. Recently a new characterization of the MLT in terms of rigidity-theoretic properties of G was proved [3]. This characterization was then used to give new combinatorial lower bounds on the MLT of any graph. We continue this line of research by exploiting combinatorial rigidity results to compute the MLT precisely for several families of graphs. These include graphs with at most 9 vertices, graphs with at most 24 edges, every graph sufficiently close to a complete graph and graphs with bounded degrees.

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1 Introduction

Let G be a graph with n vertices. The Gaussian graphical model associated with G is the set of n -variate normal distributions $\mathcal{N}(0, \Sigma)$ so that if ij is *not* an edge of G , then $(\Sigma^{-1})_{ij} = 0$, i.e. the corresponding random variables are conditionally independent given all of the other random variables.

A question, originally posed by Dempster [9] and popularised by Lauritzen (see [20] for the history), that has gotten a lot of attention after Uhler’s foundational work [20] on the topic is: *for a fixed graph G , how many datapoints¹ are needed for the maximum likelihood estimator of the associated Gaussian graphical model to exist almost surely?* This minimum number of datapoints is called the *maximum likelihood threshold (MLT)* of G , which we denote $\text{mlt}(G)$.

Efficiently computing the maximum likelihood threshold of an arbitrary graph seems out of reach to current techniques. Instead most of the literature focuses on providing combinatorial bounds on $\text{mlt}(G)$. The basic tool is an algebraic graph parameter, introduced by Uhler [17], that Blekherman and Sinn [4] have called the *generic completion rank (GCR)* of G , denoted $\text{gcr}(G)$. There are various equivalent definitions of $\text{gcr}(G)$. For our purposes, the right definition is due to Gross and Sullivan [12]. They showed that $\text{gcr}(G) = d + 1$, where d is the smallest dimension such that a generic geometric structure made of rigid bars connected at freely rotating joints does not support an equilibrium stress. (Full definitions are given below.)

While, for the complete graph K_n , $\text{mlt}(K_n) = n$, much lower numbers can be achieved for sparser graphs. For obtaining lower bounds on $\text{mlt}(G)$, the starting point is the following simple observation.

Lemma 1.1. *Let G be a graph and H a subgraph of G . Then $\text{mlt}(H) \leq \text{mlt}(G)$.*

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¹We assume that the samples are i.i.d. from a distribution whose probability measure is mutually absolutely continuous with respect to Lebesgue measure.

The central role of the generic completion rank in the study of $\text{mlt}(G)$ comes from the following foundational result of Uhler.

Theorem 1.2 ([20]). *Let G be a graph. Then $\text{mlt}(G) \leq \text{gcr}(G)$.*

To get lower bounds, the authors, in [3], introduced a new graph parameter called the *globally rigid subgraph number* of G , denoted $\text{grn}^*(G)$. This is the largest dimension d so that G contains a globally d -rigid subgraph on at least $d + 2$ vertices. They showed that $\text{mlt}(G) \geq \text{grn}^*(G) + 2$.

Rigorous experiments reported in [3] indicate that, for sparse Erdős-Rényi random graphs, in fact, with high probability $\text{grn}^*(G) + 2 = \text{gcr}(G)$ which implies that, with high probability, $\text{gcr}(G) = \text{mlt}(G)$. For sufficiently small MLT and GCR, this relationship is deterministic.

Theorem 1.3 ([3]). *If G is a graph and $\text{mlt}(G) \leq 3$ or $\text{gcr}(G) \leq 4$, then $\text{mlt}(G) = \text{gcr}(G)$.*

Blekherman and Sinn [4] showed that, in general, a “with high probability” equality is the best one can hope for.

Theorem 1.4 ([4]). *For complete bipartite graphs $K_{m,m}$,*

$$\text{mlt}(K_{m,m}) = o(\text{gcr}(K_{m,m})) \quad (\text{as } m \rightarrow \infty)$$

and, in particular

$$4 = \text{mlt}(K_{5,5}) < \text{gcr}(K_{5,5}) = 5.$$

To summarize the current understanding of $\text{mlt}(G)$: experimental evidence from [3] suggests that, for almost all G , $\text{mlt}(G) = \text{gcr}(G)$; the Blekherman–Sinn examples show that for large MLT and GCR, the two quantities can be arbitrarily far apart; and Theorem 1.3 says that, for very small MLT and GCR, the two quantities *always coincide*. The table in [3, Section 1.4] presents the current state of the art in condensed form.

1.1 Contributions

It is then natural to ask how large or how sparse a graph G needs to be in order to have $\text{mlt}(G) < \text{gcr}(G)$. That is the topic of this paper. Our first results imply that the Blekherman–Sinn example of $K_{5,5}$ is the smallest possible graph where the MLT and GCR do not coincide. No graph with fewer vertices can have this property.

Theorem 1.5. *Let G a graph on at most 9 vertices. Then $\text{mlt}(G) = \text{gcr}(G)$.*

Neither can a graph with fewer edges.

Theorem 1.6. *Let G be a graph with at most 24 edges. Then $\text{mlt}(G) = \text{gcr}(G)$.*

Turning to the general case, we prove that any graph which is nearly complete also must have equal MLT and GCR.

Theorem 1.7. *Let G be a graph whose complement has at most 5 edges. Then $\text{mlt}(G) = \text{gcr}(G) \geq n - 2$.*

So must a graph with sufficiently small minimum and maximum degrees.

Theorem 1.8. *Let G be a connected graph with minimum degree at most 4 and maximum degree at most 5. Then $\text{mlt}(G) = \text{gcr}(G) \leq 5$.*

We believe that Theorem 1.7 might not be best possible, in the sense that the same statement may hold with a number larger than 5. The best we could hope for is 19, since the example of $K_{5,5}$ shows that removing 20 edges from K_{10} does give a graph with different MLT and GCR.

Theorem 1.5 is proved in Section 3, with the more technical parts of the proof deferred until Section 6, and Theorem 1.6 is proved in Section 4. Theorems 1.7 and 1.8 are proved in Section 5. First we give some preliminary results from graph rigidity and recap some necessary theory from [3].

2 MLT and rigidity theory

In this section we introduce the necessary background from rigidity theory, including results on equilibrium stresses and MLT. Figure 1 illustrates the following definitions for $d = 2$.

2.1 Rigid graphs

Definition 2.1. Let $d \in \mathbb{N}$ be a dimension. A *framework in \mathbb{R}^d* is an ordered pair (G, p) where G is a graph with n vertices $\{1, \dots, n\}$ and $p = (p(1), \dots, p(n))$ is a configuration of n points in \mathbb{R}^d . Two frameworks (G, p) and (G, q) are *equivalent* if

$$\|p(j) - p(i)\| = \|q(j) - q(i)\| \quad \text{for all edges } ij \text{ of } G$$

and *congruent* if p and q are related by a Euclidean isometry, i.e. if there exists a Euclidean isometry $T : \mathbb{R}^d \rightarrow \mathbb{R}^d$ such that $q(i) = T(p(i))$ for $i = 1, \dots, n$. (G, p) is *globally rigid* in dimension d if all equivalent d -dimensional frameworks are congruent. If this happens only for some neighborhood U around p , i.e. if (G, p) and (G, q) are congruent whenever $q \in U$ and (G, q) and (G, p) are equivalent, then (G, p) is said to be *rigid* in dimension d .

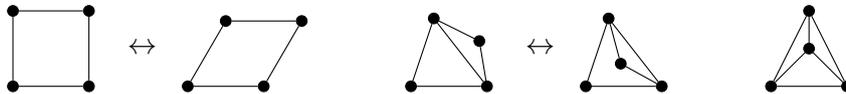


Figure 1: The framework on the left fails to be rigid because there exist arbitrarily close frameworks that are equivalent but not congruent. The frameworks in the middle are rigid but fail to be globally rigid since they are equivalent but not congruent. Finally, the framework on the right is globally rigid and therefore also rigid.

(Global) rigidity of a specific framework is difficult to check [1, 16], but for each dimension d , every graph has a generic behavior. Following [20, 12], we use the following notion of generic. Let p be a configuration of n points in \mathbb{R}^d . We say that p is *generic* if the coordinates of p do not satisfy any non-zero polynomial with rational coefficients. By foundational results of [2, 10], if we let d be a fixed dimension and G a graph, then either every generic d -dimensional framework (G, p) is (globally) rigid or every generic d -dimensional framework (G, p) is not (globally) rigid. Hence, generic (global) rigidity is a graph property, and we call G (*globally*) d -*rigid* if its generic d -dimensional frameworks are (globally) rigid.

Let G be a graph with n vertices and m edges. The *rigidity matrix* $R(G, p)$ of a d -dimensional framework (G, p) is the $m \times dn$ matrix whose rows are indexed by the edges of G , columns indexed by the coordinates of $p(1), \dots, p(n)$, where the entry corresponding to edge e and $p(v)_i$ is $p(v)_i - p(u)_i$ if $e = vu$, and 0 if v is not incident to e . We call G d -*independent* if the rows of $R(G, p)$ are linearly independent for some generic framework (G, p) . In Figure 1, the graphs underlying the frameworks in the middle and on the left are 2-independent, whereas the graph of the framework on the right is not.

A result of Gross and Sullivant [12] implies that the generic completion rank $\text{gcr}(G) = d + 1$, where d is the smallest dimension such that G is d -independent. We take this as the definition of generic completion rank. To define the MLT we go one step further and understand the nature of dependencies.

Given vertices i and j of a graph G , we write $i \sim j$ to indicate that G has an edge between i and j . Let G be a graph with n vertices. Let (G, p) be a framework. An *equilibrium stress* ω of (G, p) is an assignment of weights ω_{ij} to the edges of G so that, for all vertices i

$$\sum_{j \sim i} \omega_{ij} (p(j) - p(i)) = 0 \quad (\text{sum over neighbors of } i).$$

The *equilibrium stress matrix* associated to an equilibrium stress ω is the matrix Ω obtained by setting $\Omega_{ji} = \Omega_{ij} = -\omega_{ij}$ for all edges ij of G , $\Omega_{ii} = \sum_j \omega_{ij}$ and all other entries zero. The *rank* and *signature* of ω are defined to be the rank and signature of Ω , and ω is said to be *PSD* if Ω is positive semi-definite.

The follow result from [3] gives a characterization of the MLT of a graph in terms of equilibrium stresses. The main novelty is to consider the signature of the equilibrium stress, which is essential to get lower bounds on the MLT. For upper bounds on the MLT, one can consider only whether there is any non-zero equilibrium stress for a generic d -dimensional framework. This point of view recovers the upper bound from [12].

Theorem 2.2. *Let G be a graph with n vertices. Then the MLT of G is $d + 1$ if and only if d is the smallest dimension in which no generic d -dimensional framework supports non-zero PSD equilibrium stress.*

We will make repeated use of the following theorem which will allow us to determine the MLT when G is globally d -rigid.

Theorem 2.3 ([8]). *Let G be a graph with $n \geq d + 2$ vertices and d a dimension. If G is globally d -rigid, then there is a generic framework (G, p) with a PSD equilibrium stress of rank $n - d - 1$.*

The theorem will often be applied through the following corollary. The graph G is a d -circuit if it is not d -independent, but every proper subgraph is.

Corollary 2.4 ([3]). *Let G be a d -circuit. Then $\text{gcr}(G) = d + 2$. Further, if G is globally d -rigid then $\text{mlt}(G) = d + 2$.*

2.2 Graph operations

A (d -dimensional) 0-extension of a graph G is the graph obtained from G by adding a new vertex of degree d . A (d -dimensional) 1-extension of G is the graph obtained from G by removing an edge xy , and adding a new vertex of degree $d + 1$ that is adjacent to x , y and $d - 1$ other vertices. The inverse of these operations are called (d -dimensional) 0- and 1-reductions.

The following lemma is relatively straightforward.

Lemma 2.5 ([21, Lemma 11.1.1, Theorem 11.1.7]). *Let G be d -independent and suppose that G' is obtained from G by a 0-extension or a 1-extension. Then G' is d -independent.*

More difficult is the fact that 1-extensions preserve global d -rigidity [5]. In fact one can preserve the existence of a PSD equilibrium stress of full rank [6, Section 9]. That proof can be adapted to obtain the following lemma which we will need. Details are provided in Appendix A.

Lemma 2.6. *Let $G = (V, E)$ be a d -rigid graph that has a generic realization (G, p) in \mathbb{R}^d with a PSD equilibrium stress ω with rank $|V| - d - 2$. Suppose that $G' = (V', E')$ is a d -rigid d -circuit obtained from G by a 1-extension. Then there exists a generic framework (G', p') with a PSD equilibrium stress of rank at least $|V'| - d - 2$.*

Definition 2.7. A graph G is a k -sum of two induced subgraphs G_1 and G_2 each with at least $k + 1$ vertices if G is the union of G_1 and G_2 and $G_1 \cap G_2$ is isomorphic to K_k .

The graphical model literature also refers to k -sums as “weak decompositions” (see, e.g., [15, Definition 2.2]).

Given a graph G with edge ij , we let $G - ij$ denote the graph obtained from G by deleting the edge ij .

Lemma 2.8 ([3]). *Let $1 \leq k \leq d$ and G be a k -sum of subgraphs G_1 and G_2 and ij and edge of $G_1 \cap G_2$. Suppose that there are generic d -dimensional frameworks (G_1, p^1) and (G_2, p^2) that, respectively, support non-zero PSD equilibrium stresses ω^1 and ω^2 , such that $\omega_{ij}^k \neq 0$ for $k = 1, 2$. Let $G' = G - ij$. Then there is a generic d -dimensional framework (G', p) that supports a non-zero PSD equilibrium stress.*

We will also use the following lemma which was implicit in [3]. The cone over a graph $G = (V, E)$ is the graph obtained by adding a new vertex $v_0 \notin V$ to G and the edges v_0v for all $v \in V$.

Lemma 2.9. *Let G' be the cone of G . Then $\text{gcr}(G') = \text{gcr}(G) + 1$ and $\text{mlt}(G') = \text{mlt}(G) + 1$.*

Proof. The first statement follows from a result of Whiteley [22]. The second follows from combining [7, Lemma 4.9] with Theorem 2.2. \square

Lemma 2.9 implies that the cone G over $K_{5,5}$, has $\text{mlt}(G) < \text{gcr}(G)$, so there are non-bipartite examples with this property.

We finish this introductory section with the observation that adding a single edge to a graph can alter the MLT by at most 1. We will not need to use this fact, but it seems interesting in its own right, and is, to our knowledge, new.

Proposition 2.10. *Let e be a non-edge of a graph $G = (V, E)$. Then $\text{mlt}(G) \leq \text{mlt}(G + e) \leq \text{mlt}(G) + 1$.*

Proof. Let $e = vw$ and define the set of non-edges $F := \{vu : u \in V, v \not\sim u\}$. By Lemma 2.9, $\text{mlt}(G + F) = \text{mlt}(G - v) + 1$. By Lemma 1.1 we see that

$$\text{mlt}(G) \leq \text{mlt}(G + e) \leq \text{mlt}(G + F) = \text{mlt}(G - v) + 1 \leq \text{mlt}(G) + 1,$$

which concludes the proof. \square

One may be tempted to conjecture from Proposition 2.10 that MLT is a rank function on the edge set of a graph. It is, however, rather simple to see that it fails the required submodularity condition: if G_1 and G_2 are paths with three vertices such that $G_1 \cup G_2 \cong K_3$ (i.e., they have the same vertex set and share a single edge), then

$$\text{mlt}(G_1 \cup G_2) + \text{mlt}(G_1 \cap G_2) = 3 + 2 = 5 > 4 = 2 + 2 = \text{mlt}(G_1) + \text{mlt}(G_2).$$

3 Graphs with few vertices

We will demonstrate that if G has sufficiently few vertices then $\text{mlt}(G) = \text{gcr}(G)$. This requires a number of technical results which we now derive. The first is computational.

Lemma 3.1. *Every 6-regular graph on 9 vertices is globally 4-rigid.*

Proof. The complement of a 6-regular graph on 9 vertices is 2-regular and there are exactly four isomorphism classes of 2-regular graphs on 9-vertices. In particular, these are the 9-cycle, the disjoint union of a 6-cycle and a 3-cycle, the disjoint union of a 5-cycle and a 4-cycle, and the disjoint union of three 3-cycles.

In all four graphs let the vertex set be $\{v_1, v_2, \dots, v_9\}$. We define G_1 by taking the edge set of K_9 and deleting the 9-cycle with edges $v_1v_2, v_2v_3, \dots, v_8v_9, v_9v_1$. We define G_2 by taking the edge set of K_9 and deleting the 6-cycle with edges $v_1v_2, v_2v_3, \dots, v_5v_6, v_6v_1$ and the 3-cycle with edges v_7v_8, v_8v_9, v_9v_7 . We define G_3 by taking the edge set of K_9 and deleting the 5-cycle with edges $v_1v_2, v_2v_3, \dots, v_4v_5, v_5v_1$ and the 4-cycle with edges $v_6v_7, v_7v_8, v_8v_9, v_9v_6$. Finally, we define G_4 by taking the edge set of K_9 and deleting the 3-cycle with edges v_1v_2, v_2v_3, v_3v_1 , the 3-cycle with edges v_4v_5, v_5v_6, v_6v_4 and the 3-cycle with edges v_7v_8, v_8v_9, v_9v_7 .

By [5, Theorem 1.3], it suffices to show that each G_i has an infinitesimally rigid realization with an equilibrium stress of rank 4. For $1 \leq i \leq 4$ we define the framework (G_i, p) in \mathbb{R}^4 by putting $p(v_1) = (0, 0, 0, 0)$, $p(v_2) = (0, 0, 0, 1)$, $p(v_3) = (0, 0, 4, -1)$, $p(v_4) = (0, 2, 3, 5)$, $p(v_5) = (1, -1, 0, -2)$, $p(v_6) = (1, 3, 7, 0)$, $p(v_7) = (2, -4, -1, 1)$, $p(v_8) = (-9, 0, 2, 11)$ and $p(v_9) = (-3, 3, 1, 6)$. Given these realizations it is simple for the reader to verify that the rigidity matrix has rank $4n - 10 = 26$, that the cokernel of the rigidity matrix is 1-dimensional and that the stress matrix corresponding to any non-zero equilibrium stress of (G_i, p) has rank 4. \square

We will also make use of the following theorem of Jordán. Given a graph $G = (V, E)$ and a subset X of vertices, $i_G(X)$ (or $i(X)$ when the graph G is clear) denotes the number of edges in the subgraph induced by X . G is $(d, \binom{d+1}{2})$ -sparse if $i(X) \leq d|X| - \binom{d+1}{2}$ for all $X \subset V$ with $|X| \geq d$. Then G is $(d, \binom{d+1}{2})$ -tight if it is $(d, \binom{d+1}{2})$ -sparse and $|E| = d|V| - \binom{d+1}{2}$. A graph is *redundantly d -rigid* if it is d -rigid, and remains so after removing any edge.

Theorem 3.2 ([14, Theorems 2.3 and 3.2]). *Let $d \geq 1$, let $k \in \{3, 4\}$ and let G be a graph on $d + k$ vertices. Then G is d -rigid if and only if it contains a spanning $(d, \binom{d+1}{2})$ -tight subgraph. Moreover G is globally d -rigid if and only if G is redundantly d -rigid and $(d + 1)$ -connected.*

The following lemma was implicit in [14]. Let H_d denote the graph obtained by gluing two copies of K_{d+2} along a common K_d subgraph and removing a common edge – see Figure 2 for an illustration when $d = 3$.

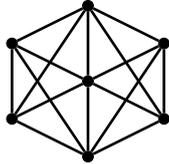


Figure 2: The graph H_3 .

Lemma 3.3 ([14]). *H_d is the unique graph on at most $d + 4$ vertices that is a d -rigid d -circuit and not $(d + 1)$ -connected.*

For disjoint vertex sets A, B of a graph G , we will denote the induced subgraph on the vertex set A by $G[A]$, the non-edges of G by E^c , the non-edges of G induced on the set A by $E^c[A]$, and the set of non-edges of G with one end in A and the other in B by $E^c(A, B)$. The minimum degree of a graph G will be denoted $\delta(G)$. The set of neighbors of a vertex v of G will be denoted $N_G(v)$. Given a set V of vertices, $K(V)$ denotes the complete graph on vertex set V . Given a subset S of edges or vertices of G , we let $G - S$ denote the graph obtained by removing S . Given graphs G and H , $G + H$ denotes the graph whose vertex and edge sets are the unions of the vertex and edge sets of G and H .

Lemma 3.4. *Suppose G is a d -rigid d -circuit with $d + 5$ vertices and minimum degree $d + 1$. Then G has a d -dimensional generic realization with a PSD equilibrium stress of rank at least 3.*

Proof. First suppose G is a $(d + 1)$ -connected d -rigid d -circuit with a vertex v of degree $d + 1$. Then $G - v$ is d -rigid. Since G is $(d + 1)$ -connected and redundantly d -rigid, the graph $G' = G - v_0 + K(N_G(v_0))$ must also be $(d + 1)$ -connected and redundantly d -rigid. By Theorem 3.2, G' is globally d -rigid, and hence by [18, Lemma 4.1], G is globally d -rigid. The result follows from Theorem 2.3.

Now suppose G is not $(d + 1)$ -connected. Since G is d -rigid, there exists a separating set $C \subset V$ of size d . As $\delta(G) = d + 1$, $G - C$ will have exactly two connected components $G[A], G[B]$ where $A = \{a_1, a_2\}$ and $B = \{b_1, b_2, b_3\}$. The complement G^c of G has exactly 9 edges. Since K_{d+2} is not a subgraph of G , $G[C]$ is not complete. Since $|E^c(A, B)| = 6$ it follows that $|E^c[B \cup C]| = 3$ and $|E^c[C]| \in \{1, 2, 3\}$.

If $|E^c[C]| = 1$ and $G[B] \cong K_3$ then, without loss of generality, b_1, b_2 have degree $d + 1$ in G and are incident to edges of $E^c(B, C)$. We can apply a 1-reduction at b_1 and add the missing edge incident to b_2 to result in a smaller d -circuit. A similar argument applies if $G[B]$ has 2 edges. In both cases the resulting d -circuit has $d + 4$ vertices and is not $(d + 1)$ -connected. Hence it is H_d . This case is completed by Lemma 2.6.

Now assume $|E^c[C]| \geq 2$. If three non-edges meet at a vertex $c \in C$, then $G[B \cup C - c]$ is isomorphic to K_{d+2} , contradicting that G is a d -circuit. Hence there are not, and there exists a 1-reduction at a_1 followed by a 0-reduction at a_2 resulting in the graph $K_{d+3} - \{e, f\}$ where e and f do not share a vertex. The result now follows from Theorem 3.2, Theorem 2.3, and Lemma 2.6. \square

The proof of the following lemma is long and technical, so we defer it to the end of the paper (see Section 6).

Lemma 3.5. *Suppose G is a 3-rigid 3-circuit with 9 vertices. Then $\text{mlt}(G) \geq 5$.*

Lemma 3.6. *Let G be a d -circuit on at most $d + 4$ vertices. Then $\text{mlt}(G) \geq d + 2$.*

Proof. By Lemma 3.3, if G is not $(d+1)$ -connected, then $G = H_d$. Since K_{d+2} has a PSD equilibrium stress it follows from Lemma 2.8 that H_d has a PSD equilibrium stress and hence Theorem 2.2 implies that $\text{mlt}(H_d) \geq d+2$. If G is $(d+1)$ -connected, then G is globally rigid by Theorem 3.2 and Corollary 2.4 gives the result. \square

Proof of Theorem 1.5. For any graph H , if we choose d so that $\text{gcr}(H) = d+2$ then H is not d -independent and therefore contains a d -circuit. In light of Lemma 1.1, it therefore suffices to prove any d -circuit G with 9 or fewer vertices has $\text{mlt}(G) \geq d+2$. By Theorem 1.3, we may assume $d \geq 3$ and by Lemma 3.6, we may assume $n \geq d+5$. Thus either $d = 3$ and $n \in \{8, 9\}$, or $d = 4$ and $n = 9$.

Assume that G is d -rigid, i.e. that it has $dn - \binom{d+1}{2} + 1$ edges. If $d = 3$ and $n = 9$ then $\text{mlt}(G) \geq d+2$ by Lemma 3.5. If G has a vertex of degree $d+1$ and $d = 3$ and $n = 8$, or $d = 4$ and $n = 9$, then $\text{mlt}(G) \geq d+2$ by Lemma 3.4 and Theorem 2.2. If G does not have a vertex of degree $d+1$, then $d = 4$, $n = 9$, and G is 6-regular. In this case, Lemma 3.1 implies G is globally rigid and Corollary 2.4 gives the result.

Thus we may assume that G is not d -rigid. Since $n \leq 9$ and $d \geq 3$, [11, Theorem 1] implies that G is obtained from two rigid d -circuits by gluing them together over a common complete subgraph on $(d-1)$ or $(d-2)$ vertices and deleting exactly one edge from the intersection. From above, the two rigid d -circuits that we glue are globally d -rigid. Theorem 2.3 implies that there is a generic framework for each of these with a non-zero PSD equilibrium stress. We may then apply Lemma 2.8 and Theorem 2.2 to conclude that $\text{mlt}(G) = d+2$. \square

4 Graphs with few edges

The purpose of this section is to prove Theorem 1.6. We will use the following results in the proof.

Theorem 4.1 (Jackson and Jordán [13]). *Let $G = (V, E)$ be a graph and suppose $i(X) \leq \frac{1}{2}(5|X| - 7)$ for all $X \subset V$ with $|X| \geq 2$ then G is 3-independent.*

The next lemma generalises [11, Lemma 12] but a careful reading of their proof shows that it works in the generality we state.

Lemma 4.2. *Let G be the 2-sum of two 3-rigid 3-circuits G_1, G_2 . If G' is formed from G by a 1-extension, then G' is either minimally 3-rigid, or G' is the 2-sum of two 3-rigid 3-circuits.*

Lemma 4.3 ([21, Lemma 11.1.9]). *Let G_1, G_2 be subgraphs of a graph G and suppose that $G = G_1 \cup G_2$.*

- (a) *If $G_1 \cap G_2$ is 3-rigid and G_1, G_2 are 3-independent then G is 3-independent.*
- (b) *If $|V(G_1) \cap V(G_2)| \leq 2$, $u \in V(G_1) \setminus V(G_2)$ and $v_2 \in V(G_2) \setminus V(G_1)$ then $\text{rank } R(G + uv, p) = \text{rank } R(G, p) + 1$ for any generic realization p in \mathbb{R}^3 .*

In the proof of Theorem 1.6 we will also use repeatedly the following counting argument. First we need some definitions. Let $G = (V, E)$ be a graph. Take $X, Y \subset V$. We will say that X is 3-critical if $|X| \geq 3$ and $i(X) = 3|X| - 6$. Also, we will use $d_G(X, Y)$ to denote the number of edges in G of the form xy where $x \in X \setminus Y$ and $y \in Y \setminus X$ (again we will simply use $d(X, Y)$ if the graph is clear from the context).

Remark 4.4. Let $G = (V, E)$ be $(3, 6)$ -sparse and suppose that $X, Y \subset V$ are 3-critical. Then

$$\begin{aligned} 3|X| - 6 + 3|Y| - 6 &= i(X) + i(Y) = i(X \cup Y) + i(X \cap Y) - d(X, Y) \\ &\leq 3|X \cup Y| - 6 + i(X \cap Y) - d(X, Y). \end{aligned}$$

If $|X \cap Y| = 2$ and $G[X \cup Y]$ has no edges or $|X \cap Y| \geq 3$ then $i(X \cap Y) = 3|X \cap Y| - 6$ and hence equality holds throughout and $d(X, Y) = 0$. If $|X \cap Y| = 2$ and $G[X \cap Y] \cong K_2$ then $i(X \cap Y) = 1$, $d(X, Y) \leq 1$ and $i(X \cup Y) = 3|X \cup Y| - 6 - (1 - d(X, Y))$. Lastly if $|X \cap Y| = 1$ then $i(X \cap Y) = 0$, $d(X, Y) \leq 3$ and $i(X \cup Y) = 3|X \cup Y| - 6 - (3 - d(X, Y))$.

Lemma 4.5. *Let G be a 3-connected graph formed from two disjoint 3-independent graphs G_1 and G_2 by adding at most 4 edges between them. Then G is 3-independent.*

Proof. Let F be the set of at most 4 edges between G_1 and G_2 . Suppose that F is an independent (in the graph theoretical sense) set in G . Then the result follows from an elementary body-bar type argument (see, for example, [19]). Since G is 3-connected in each remaining possibility there exists an edge $e \in F$ such that $G - e$ has a 2-vertex-separation $\{x, y\}$ where $x, y \in G_1$ (by relabelling G_1, G_2 if necessary) and x, y are incident to edges of F . Since G_1 and $G_2 \cup \{x, y\}$ are 3-independent (by Lemma 2.5) we may now use Lemma 4.3(b) to see that G is 3-independent. \square

Proof of Theorem 1.6. Choose any $G = (V, E)$ with $|E| \leq 24$. If $|V| \leq 9$ then $\text{mlt}(G) = \text{gcr}(G)$ by Theorem 1.5, so we may assume that $|V| \geq 10$. Similarly, if $\text{gcr}(G) \leq 4$ then $\text{mlt}(G) = \text{gcr}(G)$ by Theorem 1.3, so we may suppose that $\text{gcr}(G) \geq 5$. Fix $d = \text{gcr}(G) - 2 \geq 3$. Since G is d -dependent it contains a d -circuit H and since $\text{gcr}(G) = d + 2$ we have $\text{gcr}(H) = d + 2$. Now $\text{mlt}(H) \leq \text{mlt}(G) \leq \text{gcr}(G) = d + 2$ (by Theorem 1.2). Hence it will suffice for us to prove that if G is a d -circuit then $\text{mlt}(G) = \text{gcr}(G)$.

As G is a d -circuit it is easy to deduce from Lemma 2.5 that $\delta(G) \geq d + 1$. Since $|V| \geq 10$, the handshaking lemma implies that

$$24 \geq |E| \geq \frac{d+1}{2}|V| \geq 5d + 5. \quad (1)$$

Hence $d = 3$ and now Equation (1) implies $24 \geq 2|V|$ with equality if and only if G is 4-regular. If $\delta(G) \geq 5$ then $|E| \geq \frac{5}{2}|V|$ and, since $|V| \geq 10$, this contradicts the hypothesis that $|E| \leq 24$. Hence $\delta(G) = 4$. If $\Delta(G) = |V| - 1$ then G has a cone vertex u , and $\text{gcr}(G - u) = \text{gcr}(G) - 1 = 4$ (since the cone of a d -independent graph is $(d + 1)$ -independent [22]). By Theorem 1.3, it follows that $\text{mlt}(G - u) = \text{gcr}(G - u) = 4$. By Lemma 2.9, we have

$$\text{mlt}(G) = \text{mlt}(G - u) + 1 = \text{gcr}(G - u) + 1 = \text{gcr}(G).$$

Hence we may suppose that $\Delta(G) \leq |V| - 2$. If $\Delta(G) \leq 5$ then $\text{mlt}(G) = \text{gcr}(G)$ by Theorem 1.8, hence we may suppose that $\Delta(G) \geq 6$ and so G is not 4-regular. Thus $|V| \in \{10, 11\}$.

If G is not 3-connected (as d -circuits must be 2-connected) then it is the 2-sum of two 3-circuits G_1, G_2 , each with at most 7 vertices. By Theorem 1.5 and Corollary 2.4, $\text{mlt}(G_1) = \text{mlt}(G_2) = 5$. Hence by Theorem 2.2 and Lemma 2.8, $\text{mlt}(G) \geq 5$. Since $\text{gcr}(G) = 5$, $\text{mlt}(G) = 5$ by Theorem 1.3. So we may assume that G is 3-connected.

Case 1: $|V| = 11$. For $X \subset V$ with $|X| \geq 2$, define the value $f(X) := \frac{1}{2}(5|X| - 7)$. For $X \subset V$ with $2 \leq |X| \leq 4$ we have $i_G(X) \leq \binom{|X|}{2} \leq f(X)$, and $i_G(V) \leq 24 = f(V)$. Since G is a 3-circuit, it does not contain a copy of K_5 , hence $i_G(X) \leq 9 \leq f(X)$ whenever $|X| = 5$. When removing $k \leq 4$ vertices, we must remove at least $\sum_{i=1}^k 4 - (i - 1)$ edges (the minimum is achieved by removing a degree 4 vertex and then removing all but one of its neighbours), hence $i_G(X) \leq f(X)$ if $|X| \geq 7$. If $|X| = 6$, then $i_G(X) \leq 3|X| - 6 = 12$ and $f(X) = \frac{1}{2}(5|X| - 7) = \frac{23}{2}$. Hence $i_G(X) > f(X)$ if and only if $G[X]$ is minimally 3-rigid. Suppose that this is the case. As G is a 3-circuit, $G[V \setminus X]$ is 3-independent. Let t denote the number of edges between X and $V \setminus X$. Then, since $\delta(G) \geq 4$ and $i_G(V \setminus X) \leq 12 - t$, we have

$$20 = 4|V \setminus X| \leq \sum_{v \in V \setminus X} d_G(v) = 2i_G(V \setminus X) + t \leq 2(12 - t) + t = 24 - t,$$

and hence $t \leq 4$. Now G is 3-independent by Lemma 4.5, a contradiction. Hence $i_G(X) \leq f(X)$ if $|X| = 6$ and so $i_G(X) \leq f(X)$ for all $X \subset V$ with $|X| \geq 2$. Thus G is 3-independent by Theorem 4.1, a contradiction.

Case 2: $|V| = 10$. As $\delta(G) = 4$ and $\Delta(G) \geq 6$, $|E| \geq 21$. If $|E| = 21$ then $i_G(X) \leq f(X)$ for all $X \subset V$. This follows similarly to the above. In particular it is clear for all $X \subset V$ with $2 \leq |X| \leq 4$, it holds for $|X| = 5$ since K_5 is a 3-circuit, and for all $6 \leq |X| \leq 10$ it follows from repeated removal of minimum degree vertices. Hence $|E| \in \{22, 23, 24\}$. Since $|E| \leq 24 < 3|V| - 5$, G is 3-flexible. Furthermore, since every proper subgraph of G is 3-independent, G is $(3, 6)$ -sparse. Let u be a degree 4 vertex.

Claim. *There exists a pair $x, y \in N(u)$ such that $H := G - u + xy$ is $(3, 6)$ -sparse.*

Proof. Let $N(u) = \{x, y, z, w\}$. Since G does not contain a subgraph isomorphic to K_5 , without loss of generality we may suppose that $xy \notin E$. Now, by [13, Lemma 3.1], $G - v + xy$ is not $(3, 6)$ -sparse if and only if there exists a 3-critical set $X \subset V - u$ with $x, y \in X$. Suppose X is the maximal such set with respect to inclusion. If $N(v) \subset X$ then we contradict the $(3, 6)$ -sparsity of G so without loss of generality we may assume $w \notin X$.

Consider the pair $\{y, w\} \subset N(v)$. If there exists a 3-critical set $W \subset V - u$ with $y, w \in W$ then by Remark 4.4 either $i(X \cap W) \leq 3|X \cap W| - 6$, or $|X \cap W| = 1$, or $|X \cap W| = 2$ and $G[X \cap W] \cong K_2$. In the first case $i(X \cup W) = 3|X \cup W| - 6$ contradicting the maximality of X . In the second case we have $V = X \cup W \cup \{u\}$ so we may suppose $z \in X$. We now note that both $G[X \cup \{u\}]$ and $G[W]$ are $(3, 6)$ -tight, and hence minimally 3-rigid as G is a 3-circuit. Hence the graph G' formed from gluing $G[X \cup \{u\}]$ and $G[W \cup \{u\}]$ at the edge uy is 3-independent with 1 degree of freedom by Lemma 4.3(a). If $d(X, W) = 0$, then $G = G'$, contradicting the hypothesis that G is a 3-circuit. If $d(X, W) \geq 1$ then, by Lemma 4.3(b), G is 3-rigid, contradicting the fact that G is 3-flexible. Hence we have $|X \cap W| = 2$ and $G[X \cap W] \cong K_2$ (note that $x \notin W$). If $V = X \cup W \cup u$ then without loss of generality we may assume $z \in X$. Now, since $d(X, W) = 0$, the 3-independent graph $G - uw$ has a 2-vertex-separation $X \cap W := \{y, y'\}$ where $yy' \in E$. It follows from Lemma 4.3(b) that G is 3-rigid, a contradiction. So there exists a vertex in $V \setminus (X \cup W \cup u)$. Since $|E| \leq 24$ and $\delta(G) = 4$, this vertex must be z and $d(z) = 4$. Note that $xw \notin E$ and $G - \{u, z\}$ is 3-independent, so $G - \{u, z\} + xw$ is 3-independent by Lemma 4.3(b). Hence $G - u + xw$ is 3-independent by Lemma 2.5 giving the claim. Hence we may assume that $yw \in E$ and by symmetry that $xw \in E$.

Suppose that $z \in X$. If $zw \in E$ then we contradict the maximality of X , so we may suppose $zw \notin E$. Then there exists a 3-critical subset $Y \subset V - u$ with $w, z \in Y$. By the maximality of X we have $|X \cap Y| \leq 2$. If $|X \cap Y| = 1$ then, since $|E| \leq 24$, $\{w, z\}$ is a 2-vertex-separation of G contradicting the fact that G is 3-connected. So $|X \cap Y| = 2$ but then we contradict the hypothesis that $|E| \leq 24$.

Finally, suppose that $z \notin X$. Then as above we deduce that $xz, yz \in E$. Suppose that $wz \notin E$ and there exists a 3-critical set $Z \subset V - u$ with $w, z \in Z$. If x or y is contained in Z then we may relabel and apply the above argument to obtain a contradiction. Hence we may assume $x, y \notin Z$. Since $|V| = 10$ and X is maximal we have $1 \leq |X \cap Z| \leq 2$. Now Remark 4.4 implies that $d(X, Z) \leq 3$ but we have already shown that $xw, xz, yw, yz \in E$. This is a contradiction and so we must have $wz \in E$. Now $5 \leq |X| \leq 7$. If $|X| \in \{5, 6\}$ then since $G[X \cup \{u, w, z\}]$ is $(3, 6)$ -tight and $|E| \leq 24$ there must exist a vertex in $V \setminus (X \cup \{u, w, z\})$ of degree at most 3, contradicting the fact that $\delta(G) = 4$. Finally, if $|X| = 7$ then, since $|E| \leq 24$, $\{x, y\}$ is a 2-vertex-separation of G . This contradiction completes the proof of the claim. \square

If H is 3-independent then G is also 3-independent as G is formed from H by a 1-extension, hence H is 3-dependent. Thus there exists a set $X \subset V - u$ such that $H[X]$ is a 3-circuit. If $\{x, y\} \not\subset X$ then $H[X] \subset G$, contradicting that G is a 3-circuit, hence $x, y \in X$. As $H[X]$ is a $(3, 6)$ -sparse 3-circuit with at most 9 vertices, $H[X]$ is the 2-sum of two 3-circuits H_1, H_2 where $|V(H_1)| = 5$ and $|V(H_2)| \in \{5, 6\}$. By Theorem 3.2 H_1 and H_2 are 3-rigid. By Lemma 4.2, either G is the 2-sum of two 3-circuits G_1, G_2 (and hence not 3-connected), or G is 3-rigid. However, both cases contradict our assumptions. \square

5 Many edges and bounded degrees

In this short section we will demonstrate that if G is sufficiently close to being complete, or has sufficiently small vertex degrees then $\text{mlt}(G) = \text{gcr}(G)$.

Proof of Theorem 1.7. Let G be a graph on n vertices and fix $d = \text{gcr}(G) - 2$. If $n \leq 4$ the theorem is trivial (for example, from Theorem 1.3) so suppose $n \geq 5$. The graph G has at least $\binom{n}{2} - 5$ edges. We have

$$\binom{n}{2} - 5 > (n - 4)n - \binom{n - 3}{2}.$$

Since any $(n-4)$ -independent graph is $(n-4, \binom{n-3}{2})$ -sparse, G is $(n-4)$ -dependent and therefore G contains an $(n-4)$ -circuit and $d \geq n-4$. It follows from the definition of the GCR that G contains a d -circuit on at most $d+4$ vertices. Lemmas 1.1 and 3.6 and Theorem 1.2 now imply $\text{mlt}(G) = d+2$, completing the proof. \square

The next proof requires the following result of Jackson and Jordán.

Theorem 5.1 ([13]). *Let G be a connected graph with minimum degree at most $d+1$ and maximum degree at most $d+2$. Then G is d -independent if and only if $i(X) \leq d|X| - \binom{d+1}{2}$ for any vertex set $X \subset V$ with $|X| \geq d+2$.*

Proof of Theorem 1.8. The degree hypothesis implies that any set $X \subset V$ satisfies $i(X) \leq \frac{1}{2}(5|X|-1)$. Hence $i(X) \leq 4|X| - 10$ for all $|X| \geq 6$ and $i(X) \leq 3|X| - 6$ for all $|X| \geq 10^2$. Applying these two observations with Theorem 5.1 implies that we have $\text{gcr}(G) \leq 5$, with equality if and only if G contains a 3-circuit on at most 9 vertices (since no 4-circuit exists on at most 5 vertices). If $\text{gcr}(G) \leq 4$ then Theorem 1.3 gives the result. Hence we may suppose $\text{gcr}(G) = 5$ and let H be a 3-circuit contained in G with $|V(H)| \leq 9$. The result follows from applying Theorem 1.5 to H . \square

6 Completing the proof of Theorem 1.5

It remains to prove Lemma 3.5. We first deal with the case when G is 4-connected. In what follows, we make repeated implicit use of the fact that every vertex in a d -circuit has degree at least $d+1$.

Lemma 6.1. *Let G be a 4-connected 3-rigid 3-circuit on 9 vertices. Then G has a generic realization in \mathbb{R}^3 with a PSD equilibrium stress.*

Proof. A counting argument shows that G has a vertex v_0 of degree 4. Since G is a 3-circuit, there exist distinct vertices x, y adjacent to v_0 such that $xy \notin E$. Let G' be the result of the 1-reduction at v_0 that adds xy . Then G' contains a 3-circuit H .

If $|V(H)| = 8$ then $H = G'$. The connectivity of H is at least 3 (as otherwise G would not be 4-connected), hence by [11] H is 3-rigid. The result now follows from Lemmas 2.6 and 3.4.

If $|V(H)| = 7$ then G' is formed from H by a 0-extension that adds a vertex v_1 . Since G is a 4-connected 3-circuit, v_0 and v_1 must be adjacent in G , and $v_1 \notin \{x, y\}$. We now note that G can be formed from H by two 1-extensions; the first will remove the edge xy and connect the vertex v_0 to $N_G(v_0) - v_1 + u$ for some vertex $u \in V(H)$, and the second will remove the edge uv_0 and attach v_1 to all its neighbours in G . Since H_3 has a 3-dimensional generic realization with a PSD equilibrium stress of rank 2 and any globally 3-rigid graph has a PSD equilibrium stress of rank 3 (Theorem 2.3), the result now follows from Lemmas 2.6 and 3.3 and Theorem 3.2.

If $|V(H)| = 6$ then H is globally 3-rigid by Theorem 3.2. Since $|E(G')| = 19$ and $|E(H)| = 13$, G' has 6 edges not in H . Given a, b are the two vertices in $G' - V(H)$ with a having equal or higher degree than b in G' , one of the two possibilities must hold: (i) both a and b have degree 3 in G' , or (ii) a has degree 4 in G' , b has degree 3 in G' , and there exists an edge between a and b . In both cases we have that $v_0b \in E$, and in (i) we have $v_0a \in E$ also. In case (i) we must have distinct vertices $s, t \in V \setminus \{a, b, v_0, x, y\}$ adjacent to a and b respectively as otherwise G would not be 4-connected. Hence in case (i) we can obtain G from H by three 1-extensions; the first to add v_0 attached to x, y, s, t and the next to split two of the edges v_0s and v_0t and add the vertices a, b . If case (ii) holds then G can be formed from H by a 0-extension to add b adjacent to its neighbours plus a vertex w in the neighbourhood of a in G , a 1-extension at wb to add a , and a 1-extension at xy to add v_0 . In either case, the result will hold by Lemma 2.6 and Theorem 2.3.

Finally, suppose $|V(H)| = 5$, i.e. G' has a 5-clique. Let a, b, c be the three vertices in $G' - V(H)$. We will show that there exists a 1-reduction of G at a, b or c resulting in a graph that does not contain a 5-clique, hence reducing the problem to one of the previous cases. We first note that v_0 can be adjacent to at most two of a, b, c as $x, y \in V(H)$. If any of a, b, c are adjacent to four vertices in H then G must contain either $K_6 - \{e, f\}$ (e, f independent) or K_5 which contradicts that G is a 3-circuit.

²The cases when $|X| = 6$ and respectively $|X| = 10$ follow since $i(X)$ is an integer.

The 4-connectivity of G implies that each of a, b, c has a neighbour in H . If a has exactly 1 neighbour in H then a is adjacent to v_0 and has degree 4. By a quick case analysis we can see that there is a 1-reduction at a (in G) creating a graph with no 5-clique. Hence we may assume each of a, b, c has either 2 or 3 neighbours in H . If all three have 3 neighbours in H then G would have a vertex of degree 3, hence we may assume a has only two neighbours in H . If a, b, c all have two neighbours in H then we may assume that $av_0 \notin E$ and hence a has degree 4. As above, we can apply a 1-reduction at a (in G) to create a graph with no 5-clique. Hence we may assume that b has 3 neighbours in H . If c has 2 neighbours in H , then a quick case analysis shows that one of a, b, c has degree 4 in G and we again reduce that vertex instead of v_0 . Lastly if c has 3 neighbours in H , then a certainly has degree 4 in G (otherwise G' would have too many edges) and we finish in the same manner. \square

We lastly deal with the case when G is not 4-connected. It will be convenient to define a *node* of G to be a vertex of degree 4 and to use N to denote the set of nodes of G . We also define a *deleted k -sum* of two graphs G_1, G_2 to be the graph obtained by gluing G_1 and G_2 along a common k -clique, then removing one edge from this common clique.

Lemma 6.2. *Let G be a 3-rigid 3-circuit on 9 vertices with a separating set $C = \{c_1, c_2, c_3\}$. Then G has a generic realization in \mathbb{R}^3 with a PSD equilibrium stress.*

Proof. Let $A, B \subset V$ be chosen such that $|A| \leq |B|$, $A \cup B \cup C = V$, $A \cap B = A \cap C = B \cap C = \emptyset$, and there exist no edge joining A and B . As G is a 3-circuit on 9 vertices, either $|A| = |B| = 3$, or $|A| = 2$ and $|B| = 4$. Note that C cannot be a clique, as this would imply either $G[A \cup C]$ or $G[B \cup C]$ is a 3-circuit.

Suppose that $G[B]$ is disconnected. Then $|B| = 4$ and so $|A| = 2$. Since 3-circuits have minimum degree at least 4, the only 3-circuit that satisfies the above conditions is the graph described in Figure 3(a). This graph can be formed by the union of three copies of K_5 glued at three vertices $\{c_1, c_2, c_3\}$ with two edges c_1c_2, c_1c_3 removed. It can be shown that G has a 3-dimensional generic realization with a PSD equilibrium stress³. Hence we may suppose both $G[A]$ and $G[B]$ are connected.

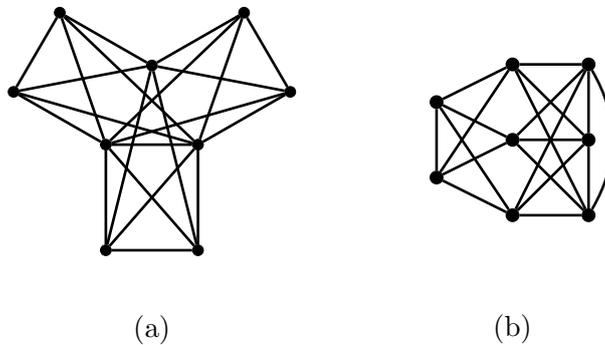


Figure 3: (a) A graph formed from gluing three copies of K_5 at three vertices and then deleting two edges in their intersection. (b) A 3-rigid 3-circuit.

Claim. *If $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3\}$, then G has a 3-dimensional generic realization with a PSD equilibrium stress.*

Proof. Note that $|E^c| = 14$ and $|E^c(A, B)| = 9$. If $|E^c[C]| = 1$ then G is a deleted 3-sum of two smaller graphs. The result now follows from [11, Lemma 17(a)], Lemma 2.8 and the fact that all 3-circuits on 7 or fewer vertices support a PSD equilibrium stress (see Lemma 3.6).

³Since equilibrium stresses are invariant under affine transformations, we can find three generic frameworks $(K_5, p^1), (K_5, p^2), (K_5, p^3)$ with PSD equilibrium stresses $\omega^1, \omega^2, \omega^3$ respectively so that: (i) $p_{c_i}^1 = p_{c_i}^2 = p_{c_i}^3$ for each $i \in \{1, 2, 3\}$, (ii) $\omega_{c_1c_2}^1 + \omega_{c_1c_2}^2 + \omega_{c_1c_2}^3 = 0$, (iii) $\omega_{c_1c_3}^1 + \omega_{c_1c_3}^2 + \omega_{c_1c_3}^3 = 0$, and (iv) the framework (G, p) formed by gluing all three frameworks at the vertices c_1, c_2, c_3 and deleting the edges c_1c_2, c_1c_3 is regular. The obtained framework will have a PSD equilibrium stress $\omega = \bar{\omega}^1 + \bar{\omega}^2 + \bar{\omega}^3$, where $\bar{\omega}^i$ is the extension of ω^i to the edges of $G + c_1c_2 + c_1c_3$.

Now suppose $|E^c[C]| \in \{2, 3\}$. Since $|A| = |B|$, we may assume, without loss of generality, that the number of non-edges with an end in A is less than the number of non-edges with an end in B ; we shall define these sets as $E_A^c := E^c[A \cup C] \setminus E^c[C]$ and $E_B^c := E^c[B \cup C] \setminus E^c[C]$. We now have three possible cases; $|E_A^c| = 0$, $|E_A^c| = |E_B^c| = 1$, or $|E_A^c| = 1$ and $|E_B^c| = 2$.

Suppose $|E_A^c| = 0$. Since $G[A \cup C]$ cannot be 3-dependent, we must have $|E_B^c| = 2$ and $|E^c[C]| = 3$. By checking the possible non-edge combinations, we note that either no vertex of C is a node and G can be formed from the 3-rigid 3-circuit described in Figure 3(b) by a 1-extension (and hence we are done by Lemmas 2.6 and 3.4), or C contains a node adjacent to only one vertex in B . As the only graph that satisfies the latter condition contains a double-banana subgraph (i.e. the flexible 3-circuit formed by the deleted 2-sum of two copies of K_5), which can be found by deleting the node in C , we are done by Lemma 2.8.

Now suppose $|E_A^c| = |E_B^c| = 1$ (and hence $|E^c[C]| = 3$). If the non-edges in E_A^c and E_B^c share an end then G will contain a double-banana subgraph, so we may assume otherwise. By checking all the remaining non-edge combinations, we see that we can always 1-reduce to a node in A that adds an edge between vertices in C , then apply another 1-reduction to a node in B that adds an edge between vertices in C , and end up with the graph H_3 . By Lemma 2.8, H_3 has a PSD equilibrium stress, and by observation of the corresponding stress matrix we note it must have rank at least 2. The result now follows from Lemma 2.6.

Finally, suppose that $|E_A^c| = 1$ and $|E_B^c| = 2$ (and hence $|E^c[C]| = 2$). By relabelling we may assume $c_1c_2, c_2c_3 \in E^c[C]$ (i.e. $c_1c_3 \in E$) and a_1 is a node. If a_1c_2 is a non-edge then $G[A \cup C]$ contains a copy of K_5 , hence we may choose a non-edge $e \in \{c_1c_2, c_2c_3\}$ so that both end points of e are neighbours of a_1 in G . First suppose G has two non-edges b_ic_x, b_jc_x for distinct b_i, b_j . We must have $c_x \neq c_2$, as otherwise $G[B \cup C]$ will contain a copy of K_5 . If c_x is not a node, then G is a deleted 3-sum of the globally 3-rigid graphs $K_6 - \{e, f\}$ (see Theorem 3.2) and K_5 ; hence G will have a PSD equilibrium stress by Lemma 2.8. If c_x is a node then $G - c_x$ is the 2-sum of two copies of K_5 ; hence G will have a PSD equilibrium stress by Lemma 2.8. Now suppose G does not have two vertices in B that are adjacent to the same vertex in the complement of G . Define $G' := G - a_1 + e$. If $E_B^c = \{b_ib_j, b_kc_\ell\}$ for distinct i, j, k , then the 1-reduction $G' - b_i + b_kc_\ell$ of G' is H_3 ; hence the results follows from our previous observations of H_3 and Lemma 2.6. If $E_B^c = \{b_ic_x, b_jc_y\}$ for distinct i, j and distinct x, y , then G' is the 3-sum of a copy of K_5 and the globally 3-rigid 3-circuit $K_{B \cup C} - E_B^c$ (see Theorem 3.2). Hence by Lemma 2.8, G has a PSD equilibrium stress. \square

Claim. *If $A = \{a_1, a_2\}$ and $B = \{b_1, b_2, b_3, b_4\}$ then G has a 3-dimensional generic realization with a PSD equilibrium stress.*

Proof. We note that $a_1a_2 \in E$ and $a_1c_i, a_2c_i \in E$ for each $i \in \{1, 2, 3\}$, as otherwise a_1 and a_2 would have a degree of 3 or less in G . If $G[C]$ has 3 edges, then $G[A \cup C]$ would be K_5 and so G would not be a 3-circuit. If $G[C]$ has 2 edges then G is the 3-sum of K_5 and another 3-circuit with 7 vertices by [11, Lemma 17(a)], and hence G will have a 3-dimensional generic realization with a PSD equilibrium stress by Lemmas 2.8 and 3.4. Suppose that $G[C]$ has either no edges or 1 edge. Applying a 1-reduction at either a_1 or a_2 and then applying a 0-reduction to the remaining vertex in A is equivalent to deleting both a_1 and a_2 and adding an edge between two vertices in C . For brevity we refer to this process as an *A-move*.

Suppose that there exists an *A-move* that gives a graph with minimum degree 2. We can check all the possible cases where this happens by observing that G has 6 non-edges with both ends in $B \cup C$, and at least two non-edges must have both ends in C . In every case we see that G would contain either K_5 or $K_6 - \{e, f\}$ as a subgraph, contradicting that G is a 3-circuit. Hence we may assume that any *A-move* produces a graph with minimum degree 3.

Now suppose that every *A-move* produces a graph with minimal degree 3. As G is a 3-circuit, any vertex of degree 3 of G' must lie in C . By checking the various assignments of non-edges between vertices in B and C we see that $G[C]$ must contain no edges; any possible graph where every *A-move* gives a graph with minimum degree 3 and $G[C]$ contains an edge would force G to contain either K_5 or the 3-circuit $K_6 - \{e, f\}$. This leaves the two possible 3-rigid 3-circuits given in Figure 4. The graph on the left has a vertex that we can apply a 1-reduction to so as to obtain the graph in Figure

3(b). We can verify that the claim holds for the remaining graph on the right in a similar manner as in Lemma 3.1.⁴

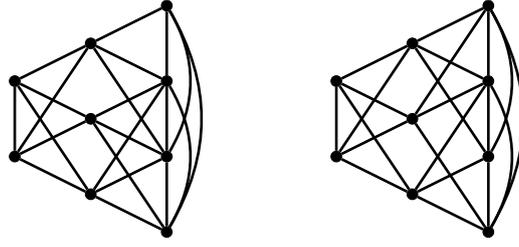


Figure 4: The only two possible graphs that G can be if every A -move gives a graph with minimal degree 3.

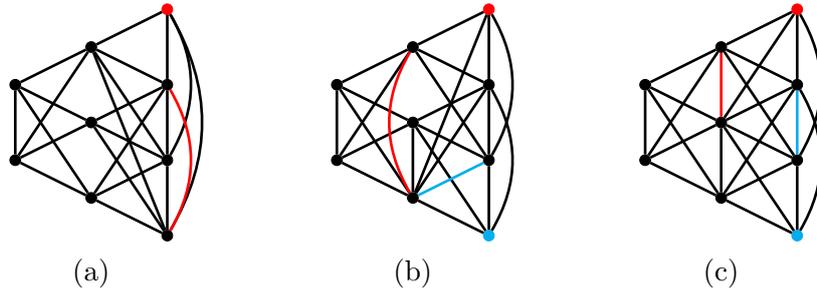


Figure 5: (a) Ignoring the red edge, a 3-rigid 3-circuit that can be formed from the 3-rigid 3-circuit in Figure 3(b) by a 1-extension. (b)–(c) Ignoring the red and blue edges, two graphs that can be formed from H_3 by two consecutive 1-extensions.

Hence, using the handshaking lemma, we may assume that G has an A -move that produces a graph G' with minimal degree 4. If G' is globally 3-rigid the result follows from Theorem 2.3 and Lemma 2.6, so we may suppose G' is not globally 3-rigid. By an easy case-by-case check of graphs on 7 vertices with minimum degree 4 and $16(= 3 \cdot 7 - 5)$ edges, we see that we must have $G' = H_3$. As we are assuming G has no separating set of size 3 with more than 1 edge, it follows that G must be one of the three graphs depicted in Figure 5; we can see this by systematically applying reverse A -moves to H_3 . We can obtain the 3-rigid 3-circuit in Figure 3(b) from the graph in Figure 5(a) by applying a 1-reduction at the red vertex and adding the red edge, hence it will also have a 3-dimensional PSD equilibrium stress by Lemmas 2.6 and 3.4. For the other two graphs in Figure 5, we can apply a 1-reduction at the red vertex to add the red edge and then a 1-reduction at the blue vertex that adds the blue edge to obtain H_3 . Hence Lemmas 2.6 and 3.4 complete the proof. \square

The above claims cover all possibilities and hence complete the proof. \square

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⁴Let $G = (V, E)$ be the graph defined as follows. Put $V = \{a_1, a_2, b_1, b_2, b_3, b_4, c_1, c_2, c_3\}$ and $E = \{a_1a_2, a_1c_1, a_1c_2, a_1c_3, a_2c_1, a_2c_2, a_2c_3, b_1b_2, b_1b_3, b_1b_4, b_2b_3, b_2b_4, b_3b_4, b_1c_1, b_1c_2, b_2c_1, b_2c_2, b_2c_3, b_3c_1, b_3c_3, b_4c_2, b_4c_3\}$. Define (G, p) in \mathbb{R}^3 by putting $p(a_1) = (-42, -45, -40)$, $p(a_2) = (44, 48, 44)$, $p(b_1) = (9, -1, -7)$, $p(b_2) = (-8, -8, 3)$, $p(b_3) = (-1, -4, -5)$, $p(b_4) = (3, -7, 3)$, $p(c_1) = (1, -1, 9)$, $p(c_2) = (-3, -3, -4)$ and $p(c_3) = (-5, -10, -6)$. Given this realization it is simple for the reader to verify that (G_i, p) is infinitesimally rigid in \mathbb{R}^3 and that the unique equilibrium stress of (G, p) has a PSD stress matrix of rank 4. Since G is not globally 3-rigid, it follows that a sufficiently nearby generic framework (G, q) has a rank 4 PSD equilibrium stress.

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A 1-extensions that preserve PSD equilibrium stresses

We prove Lemma 2.6. We first require the following technical result from [7, Lemma 4.9] and the discussion around it.

Lemma A.1. *Let (G, p) be a d -dimensional framework and ω a PSD equilibrium stress of (G, p) and let xy be an edge of G so that $\omega_{xy} > 0$. Then there is a non-singular projective transformation T on \mathbb{R}^d so that $(G, T(p))$ has a PSD equilibrium stress ψ so that $\psi_{xy} < 0$.*

Proof of Lemma 2.6. Let xy be the edge deleted during the 1-extension and z be the vertex added to form G' . Denote the neighbours of z that are neither x nor y by v_1, \dots, v_{d-1} . For each $t \in \mathbb{R}$, define p^t to be the realization of G' in \mathbb{R}^d where $p_v^t = p_v$ for all $v \in V$ and $p_z^t = tp_x + (1-t)p_y$. Also for each $t \in \mathbb{R}$ define the map $\omega^t : E' \rightarrow \mathbb{R}$ by setting $\omega_e^t = \omega_e$ for all $e \in E$, $\omega_{xz}^t = (1-t)^{-1}\omega_{xy}$, $\omega_{yz}^t = t^{-1}\omega_{xy}$ and $\omega_{zv_i}^t = 0$ for all $i \in \{1, \dots, d-1\}$. Note that for each $t \in \mathbb{R}$, ω^t is an equilibrium stress of (G', p^t) . Since G is a d -circuit, $\omega_{xy} \neq 0$. Hence by Lemma A.1, we may suppose that $\omega_{xy} = -1$.

Given Ω is the equilibrium stress matrix associated with ω , the equilibrium stress matrix associated with ω^t is the matrix

$$\Omega^t = \begin{bmatrix} M_t & \mathbf{0}_{3 \times (|V|-2)} \\ \mathbf{0}_{(|V|-2) \times 3} & \mathbf{0}_{(|V|-2) \times (|V|-2)} \end{bmatrix} + \begin{bmatrix} 0 & \mathbf{0}_{1 \times |V|} \\ \mathbf{0}_{|V| \times 1} & \Omega \end{bmatrix},$$

where

$$M_t = \begin{bmatrix} \frac{-1}{t(1-t)} & \frac{1}{1-t} & \frac{1}{t} \\ \frac{1}{1-t} & \frac{-t}{1-t} & -1 \\ \frac{1}{t} & -1 & \frac{t-1}{t} \end{bmatrix}.$$

Suppose $a \in \mathbb{R}^E$ is an element of the kernel of Ω^t . By observing the z -coordinate of $\Omega^t a$ we see that $a_z = ta_x + (1-t)a_y$. By observing the x - and y -coordinates of $\Omega^t a$ with this substitution we see that $\sum_{v \in N_G(x)} \omega^t(a_x - a_v) = 0$ and $\sum_{v \in N_G(y)} \omega^t(a_y - a_v) = 0$. Thus the nullity of Ω^t is equal to the nullity of Ω , and so $\text{rank } \Omega^t = \text{rank } \Omega + 1 \geq |V'| - d - 2$ for each $t \neq 0, 1$. As $\text{rank } M_t = 1$ for all $t \neq 0, 1$ and

$$M_t \rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \quad \text{as } t \rightarrow \infty,$$

the matrix M_t is PSD for sufficiently large t . Hence we can now fix some $T > 1$ such that Ω^T is PSD.

By Lemma 2.5, (G', p^t) is infinitesimally rigid, hence there exists an open neighbourhood U of p^T where for each $p' \in U$, the framework (G', p') has exactly one (up to scalar multiplication) equilibrium stress and the rigidity matrix $R(G', p')$ has maximal rank over all realizations. It follows that we can define a continuous map $\lambda : U \rightarrow \mathbb{R}^E$ such that $\lambda(q)$ is an equilibrium stress of (G', q) and $\lambda(p^T) = \omega^T$. Suppose that an equilibrium stress $\lambda(q)$ has rank $|V'| - d - 1$ for some $q \in U$. By [5, Theorem 1.3], G' is globally d -rigid. Hence there exists a generic framework (G', p') with a PSD equilibrium stress of rank $|V'| - d - 1$ by Theorem 2.3. Suppose instead that the equilibrium stress $\lambda(q)$ has rank at most $|V'| - d - 2$ for all $q \in U$. Then the rank of the equilibrium stress $\lambda(p^T) = \omega^T$ is maximal over U . As the rank function is lower semi-continuous, there exists an open neighbourhood $U' \subset U$ of p^T where each equilibrium stress $\lambda(q)$ with $q \in U'$ has rank $|V'| - d - 2$.

Define, for each $i \in \{1, \dots, |V'|\}$, the continuous map $\mu_i : U' \rightarrow \mathbb{R}^{|V'|-d-2}$ which maps a realization $q \in U'$ to its i -th highest eigenvalue of the equilibrium stress matrix associated to $\lambda(q)$. Since ω^T is PSD with rank $|V'| - d - 2$, $\mu_i(p^T) > 0$ for all $i \leq |V'| - d - 2$ and $\mu_i(p^T) = 0$ for all $i > |V'| - d - 2$. By the continuity of the μ_i maps, there exists a sufficiently close generic realization $p' \in U'$ where $\mu_i(p') > 0$ for all $i \leq |V'| - d - 2$. Since the rank of $\lambda(p')$ is $|V'| - d - 2$, $\mu_i(q) = 0$ for all $i > |V'| - d - 2$ also. Hence the generic framework (G', p') has a PSD equilibrium stress of rank $|V'| - d - 2$. \square

References

- [1] Timothy G. Abbot. Generalizations of Kempe’s universality theorem. Master’s thesis, Massachusetts Institute of Technology, Cambridge, MA, 2008. URL: <https://dspace.mit.edu/handle/1721.1/44375>.
- [2] Leonard Asimow and Ben Roth. The rigidity of graphs. *Transactions of the American Mathematical Society*, 245:279–289, 1978. doi:10.1090/S0002-9947-1978-0511410-9.
- [3] Daniel Bernstein, Sean Dewar, Steven Gortler, Anthony Nixon, Meera Sitharam, and Louis Theran. Maximum likelihood thresholds via graph rigidity. Preprint, arXiv:2108.02185, 2021. URL: <https://arxiv.org/abs/2108.02185>.
- [4] Grigoriy Blekherman and Rainer Sinn. Maximum likelihood threshold and generic completion rank of graphs. *Discrete & Computational Geometry*, 61(2):303–324, 2019. doi:10.1007/s00454-018-9990-3.
- [5] Robert Connelly. Generic global rigidity. *Discrete Comput. Geom.*, 33(4):549–563, 2005. doi:10.1007/s00454-004-1124-4.
- [6] Robert Connelly. Questions, conjectures and remarks on globally rigid tensegrities, 2009. URL: <http://pi.math.cornell.edu/~connelly/09-Thoughts.pdf>.
- [7] Robert Connelly, Steven J. Gortler, and Louis Theran. Affine rigidity and conics at infinity. *Int. Math. Res. Not. IMRN*, 2018(13):4084–4102, 2018. doi:10.1093/imrn/rnx014.
- [8] Robert Connelly, Steven J. Gortler, and Louis Theran. Generically globally rigid graphs have generic universally rigid frameworks. *Combinatorica*, 40(1):1–37, 2020. doi:10.1007/s00493-018-3694-4.
- [9] Arthur P. Dempster. Covariance selection. *Biometrics*, 28(1):157–175, 1972. doi:10.2307/2528966.
- [10] Steven J Gortler, Alexander D Healy, and Dylan P Thurston. Characterizing generic global rigidity. *American Journal of Mathematics*, 132(4):897–939, 2010. doi:10.1353/ajm.0.0132.
- [11] Georg Grasegger, Hakan Guler, Bill Jackson, and Anthony Nixon. Flexible circuits in the d -dimensional rigidity matroid. *Journal of Graph Theory*, 100(2):315–330, 2022. doi:10.1002/jgt.22780.
- [12] Elizabeth Gross and Seth Sullivant. The maximum likelihood threshold of a graph. *Bernoulli*, 24(1):386–407, 2018. doi:10.3150/16-BEJ881.
- [13] Bill Jackson and Tibor Jordán. The d -dimensional rigidity matroid of sparse graphs. *Journal of Combinatorial Theory, Series B*, 95(1):118–133, 2005. doi:10.1016/j.jctb.2005.03.004.
- [14] Tibor Jordán. A note on generic rigidity of graphs in higher dimension. *Discrete Applied Mathematics*, 297:97–101, 2021. doi:10.1016/j.dam.2021.03.003.
- [15] Steffen L. Lauritzen. *Graphical Models*. Oxford University Press, 1996. doi:10.1002/(SICI)1097-0258(19991115)18:21<2983::AID-SIM198>3.0.CO;2-A.
- [16] J.B. Saxe. *Embeddability of Weighted Graphs in K -space is Strongly NP-hard*. CMU-CS-80-102. Carnegie-Mellon University, Department of Computer Science, 1980. URL: <https://books.google.co.uk/books?id=vCLAGwAACAAJ>.
- [17] Bernd Schulze, Hattie Serocold, and Louis Theran. Frameworks with coordinated edge motions. *arXiv preprint arXiv:1807.05376*, 2018.

- [18] Shin-ichi Tanigawa. Sufficient conditions for the global rigidity of graphs. *J. Combin. Theory Ser. B*, 113:123–140, 2015. doi:[10.1016/j.jctb.2015.01.003](https://doi.org/10.1016/j.jctb.2015.01.003).
- [19] Tiong-Seng Tay. Rigidity of multi-graphs. i. linking rigid bodies in n-space. *Journal of Combinatorial Theory, Series B*, 36(1):95–112, 1984. doi:[10.1016/0095-8956\(84\)90016-9](https://doi.org/10.1016/0095-8956(84)90016-9).
- [20] Caroline Uhler. Geometry of maximum likelihood estimation in gaussian graphical models. *Annals of Statistics*, 40(1):238–261, 2012. doi:[10.1214/11-AOS957](https://doi.org/10.1214/11-AOS957).
- [21] W. Whiteley. Some matroids from discrete applied geometry. In J. E. Bonin, J. G. Oxley, and B. Servatius, editors, *Matroid theory*, number 197 in Contemporary Mathematics, pages 171–311. American Mathematical Society, Providence, RI, 1996. doi:[10.1090/conm/197/02540](https://doi.org/10.1090/conm/197/02540).
- [22] Walter Whiteley. Cones, infinity and 1-story buildings. *Structural Topology*, 8:53–70, 1983. URL: <https://upcommons.upc.edu/handle/2099/1003>.