**Revisiting even and odd square-free numbers**

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An integer is *square-free* if none of its prime factors appears to a power greater than 1 (this includes 1, since it has no prime factors). Denote by $F(x)$ the number of square-free positive integers not greater than $x$. It is well known that $F(x) \sim (6/\pi^2)x$ as $x \to \infty$, where the notation $f(x) \sim g(x)$ means $f(x)/g(x) \to 1$ as $x \to \infty$ (see, for example, [1, Theorem 333]).

In response to a conjecture in [2], the author showed in [3] that, asymptotically, two thirds of the square free numbers are odd and one third even. This was done by modifying the proof of the result for $F(x)$, using concepts like the Möbius function and Dirichlet convolutions. Here, hoping to spare any future readers unnecessary effort, I offer a much more elementary proof.

Let $F_1(x)$ be the number of even square-free integers not greater than $x$, and $F_2(x)$ the number of odd ones. There is a very obvious relationship. An even square-free number $n$ is necessarily of the form $4k+2$: then $\frac{n}{2} = 2k+1$ is odd and square-free. The converse is obviously true as well. Consequently, $F_1(x) = F_2(x/2)$. Since $F(x) = F_1(x) + F_2(x)$, we have

$$F(x) = F_2(x) + F_2\left(\frac{x}{2}\right). \tag{1}$$

Now $F(x) \sim cx$, where $c = 6/\pi^2$. Suppose we know that $F_1(x) \sim ax$ and $F_2(x) \sim bx$. Then (1) implies that $F(x) \sim \frac{3}{2}bx$, hence $\frac{3}{2}b = c$, so $b = \frac{2}{3}c$ and $a = \frac{1}{3}c$. Apparently game over!

Of course, the snag is that we don't know, until we have proved it, that $F_2(x)/x$ tends to any limit as $x \to \infty$. The fact that $F_1(x) + F_2(x) \sim cx$, without further information, certainly does not imply that $F_1(x)/x$ and $F_2(x)/x$ tend to limits, even for positive, increasing functions $F_1$ and $F_2$.

The matter will be resolved by inverting (1) to express $F_2(x)$ in terms of $F(x)$. To start, we have $F(x/2) = F_2(x/2) + F_2(x/4)$, hence, with (1),

$$F(x) - F\left(\frac{x}{2}\right) = F_2(x) - F_2\left(\frac{x}{4}\right).$$

So for each $r \geq 1$,

$$F\left(\frac{x}{2^{2r}}\right) - F\left(\frac{x}{2^{2r+1}}\right) = F_2\left(\frac{x}{4^r}\right) - F_2\left(\frac{x}{4^{r+1}}\right).$$

Add these identities for $0 \leq r \leq k-1$. By cancellation on the right-hand side, we obtain

$$F(x) - F\left(\frac{x}{2}\right) + F\left(\frac{x}{4}\right) - \cdots - F\left(\frac{x}{2^{2k-1}}\right) = F_2(x) - F_2\left(\frac{x}{4^k}\right). \tag{2}$$

Clearly, $F_2(t) = 0$ when $t < 1$, so when $4^k > x$, the right-hand side of (2) is simply $F_2(x)$, and we have indeed expressed $F_2(x)$ in terms of $F(x)$. However, for our purposes, we will apply (2) with another choice of $k$.

We now choose $\varepsilon > 0$ and let the definition of a limit do the work. There exists $x_0$ such that $(c - \varepsilon)x \leq F(x) \leq (c + \varepsilon)x$ for all $x \geq x_0$. We will show that for all large enough $x$, $F_2(x)$ lies between $(\frac{2}{3}c - 4\varepsilon)x$ and $(\frac{2}{3}c + 4\varepsilon)x$. We deal with the upper bound first. Let $k$ be the largest integer such that $x/2^{2k-1} \geq x_0$. Then $x/2^{2k} < 2x_0$, and for $r \leq k - 1$, we have

$$F\left(\frac{x}{2^{2r}}\right) \leq (c + \varepsilon)\frac{x}{2^{2r}},$$

$$F\left(\frac{x}{2^{2r+1}}\right) \geq (c - \varepsilon)\frac{x}{2^{2r+1}}.$$

So the left-hand side of (2) is not greater than

$$cx\left(1 - \frac{1}{2} + \frac{1}{2^2} - \cdots - \frac{1}{2^{2k-1}}\right) + \varepsilon x\left(1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{2k-1}}\right).$$

By the geometric series, the first bracket equals $\frac{2}{3}(1 - 1/2^{2k})$ and the second bracket is less than 2. So

$$F_2(x) \leq (\tfrac{2}{3}c + 2\varepsilon)x + F_2\left(\frac{x}{4^k}\right).$$

Now $x/4^k \leq 2x_0$ and obviously $F_2(t) \leq t$ for all $t$, so for $x > x_0/\varepsilon$, we have

$$F_2(x) \leq (\tfrac{2}{3}c + 2\varepsilon)x + 2x_0 < (\tfrac{2}{3}c + 4\varepsilon)x,$$

as required. With minor modifications, a similar proof establishes the lower bound.

*References*


1. G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers* (5th ed), Oxford Univ. Press (1979).
2. J. A. Scott, Square-free integers once again, *Math. Gaz.* **92** (2008) 70–71.
3. G. J. O. Jameson, Even and odd square-free numbers, *Math. Gaz.* **94** (2010), 123–127.



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