On Extending Scott Modules

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Declaration

Hereby I declare that the present thesis was prepared by me and none of its contents was obtained by means that are against the law.

I also declare that the present thesis is a part of a PhD Programme at Lancaster University. The thesis has never before been a subject of any procedure of obtaining an academic degree.

Alec Gullon
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Abstract

We study a variety of questions related to the Scott modules $\mathcal{S}(G, Q)$ associated to a finite group G, where Q denotes a p-subgroup of G for a given prime p. The main concept we study is that of a p-extendible group, which we define to be a group in which the dimension of $\mathcal{S}(G, Q)$ is minimal for all p-subgroups Q of G. We study those Frobenius groups which are p-extendible and complete a classification of the local subgroups of the sporadic groups which are p-extendible. Furthermore, we study Scott modules associated to finite classical groups which admit (B, N)-pairs that are split at characteristic p. The thesis concludes with some considerations about the relative syzygy $\Omega^2_{P/Q}(k)$ for a certain class of p-groups P.

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Introduction

This thesis is a contribution to the study of Scott modules, a class of modules which can be traced back to work conducted by Alperin, Burry and Scott from the early 1980s; the first explicit mention of these objects in the literature came in 1982, in Burry's paper [6]. If G is a finite group and k is an algebraically closed field of characteristic p, then a Scott kG-module is a certain indecomposable p-permutation kG-module which contains a submodule isomorphic to the trivial kG-module in its socle. This provides one natural perspective to adopt on Scott modules, but others exist. For instance, we may also think of a Scott kG-module as being a certain trivial source kG-module, or as a relatively projective cover of the trivial kG-module; the clearest unifying feature behind these ideas is the requirement that the trivial module be contained in the module as a submodule.

Since their early introduction in the 1980s, Scott modules have featured in the literature, albeit in no consistent pattern. Often, results concerning Scott modules seem to draw inspiration from seemingly disconnected areas of mathematics. Recent examples of this phenomenon include [36], where a method is given to calculate Scott modules in the case where G has a cyclic Sylow p-subgroup, using the Brauer tree of the principal block, and [25] and [20], where it is shown that certain Scott modules remain indecomposable under the Brauer construction. At present, there is no easy method for constructing a given Scott module for an arbitrary finite group and thus plenty of work remains to be done in this area.

If $\mathcal{M}$ is a Scott kG-module, then $\mathcal{M}$ has vertex Q for a p-subgroup $Q \leq G$, and an elementary observation is that the permutation kP-module $k[P/Q]$ is always a direct summand of $\mathcal{M}{\downarrow_P}$ whenever P is a Sylow p-subgroup of G containing Q. The question therefore naturally arises: when is $\mathcal{M}{\downarrow_P} \cong k[P/Q]$, i.e., when is $\mathcal{M}$ an extension of $k[P/Q]$? In fact, one can ask for a stronger property: given a finite group G, when is it the case that every Scott kG-module $\mathcal{M}$ satisfies the above extension property, i.e., when is it the case that every Scott kG-module $\mathcal{M}$ satisfies $\mathcal{M}{\downarrow_P} \cong k[P/Q]$, where Q is a vertex of $\mathcal{M}$ and P is a Sylow p-subgroup of G containing Q? The contents of this thesis mainly arise from an attempt to study and analyse both of these questions. We also concern ourselves with some special cases where Scott modules $\mathcal{M}$ can be described more concretely; specifically, we derive a result which completely describes Scott modules in the case where P is normal in G.

Any given Scott module $\mathcal{M}$ has vertex Q for a p-subgroup $Q \leq G$. Moreover, given any p-subgroup Q contained in a Sylow p-subgroup P of G, there exists a unique Scott module with this vertex, which is denoted by $\mathcal{S}(G, Q)$. Therefore, in order to understand the full range of possible Scott modules that a given finite group G can have, it is necessary to understand the full range of conjugacy classes of p-groups we can have within G. At present, the literature is light on answering this question for finite groups; instead, much more emphasis has been placed on studying the conjugacy classes of p-elements in finite groups. In Chapter 4, we consider the Borel subgroups of some classical groups of small degree and completely describe all of the Scott modules that they can have; part of our efforts involve classifying the conjugacy classes of p-subgroups within these Borel subgroups . Our methods, however, fall far short of providing a general answer to this problem for an arbitrary finite group G.

Main Results

We have mentioned already that there exists no known method which reliably constructs a given Scott module for an arbitrary finite group. In Section 2.4, we derive some theory which helps develop a first step towards answering this question. The following is Theorem 2.4.2 of the thesis.

Theorem. Suppose that G is a finite group and $Q \leq P \in \mathrm{Syl}_p(G)$ with $P \lhd G$. Then:

(i) $\mathcal{S}(PN_G(Q), Q){\downarrow_P} \cong k[P/Q]$;

(ii) $\mathcal{S}(PN_G(Q), Q){\uparrow^G}$ is indecomposable and hence $\mathcal{S}(G, Q) \cong \mathcal{S}(PN_G(Q), Q){\uparrow^G}$.

In particular,

$$\dim \mathcal{S}(G, Q) = \frac{|G : P|}{|N_G(Q)|_{p'}} \cdot |P : Q|.$$

In fact, we have $\mathcal{S}(PN_G(Q), Q) \cong k[PN_G(Q)/QR]$, where R is some subgroup of $PN_G(Q)$ whose order equals $|\,PN_G(Q) : P\,|$. Thus, in the case where P is a normal subgroup of G, $\mathcal{S}(G, Q)$ may be calculated by inducing a known $k[PN_G(Q)]$-module from $PN_G(Q)$ to G and it follows that quite a large amount is known about the representation theory of $\mathcal{S}(G, Q)$ in this case. In the more general context, where P is not normal in G, the above provides full insight into the properties of the $k[N_G(P)]$-module $\mathcal{S}(N_G(P), Q)$, but it is difficult to see how useful it is in understanding the full structure of the kG-module $\mathcal{S}(G, Q)$.

Central to the thesis is the concept of a finite group G being *p-extendible* for a given prime p. We say that a finite group G is p-extendible if for all $P \in \mathrm{Syl}_p(G)$ and $Q \leq P$, we have $\mathcal{S}(G, Q){\downarrow_P} \cong k[P/Q]$. This definition is new and the question that motivates it has not been explored at present in the literature. It is therefore important to know that the definition is not completely trivial; that is to say, that there exist sufficiently "interesting" groups G which are p-extendible. As a starting point, we know that the p-nilpotent groups are themselves p-extendible, a fact which is derived in Proposition 2.3.7. Indeed, in this case, $\mathcal{S}(G, Q)$ is simply equal to the inflation of $k[P/Q]$ along a normal complement for P in G; thus, $\mathcal{S}(G, Q)$ is equal to an obvious extension of $k[P/Q]$, which is not especially interesting. The following result, which is Theorem 3.1.6 of the thesis, gives a richer class of examples to consider.

Theorem. Suppose that $G = K \rtimes H$ is a Frobenius group with Frobenius kernel K and Frobenius complement H and $p \in \pi(H)$. Assume furthermore that H is solvable and any of the following are true:

(i) $p > 3$;

(ii) $p = 3$ and $O_2(H) \ncong Q_8$;

(iii) $p = 2$ and P is cyclic.

Then G is p-extendible.

We note that an analogous statement exists for the case where $G = K \rtimes H$ is a Frobenius group with a nonsolvable Frobenius complement H, however, the details are messier than the case where H is solvable; the reader is directed towards Section 3.1.2 for a full account of the nonsolvable case. We see therefore that the Frobenius groups justify the study of our new definition, since their p-extendibility for the primes under consideration is a rather surprising fact, in comparision to the p-nilpotent groups.

Layout of Thesis

We indicate here how the thesis is laid out, and what the essential contents of each chapter are.

- Chapter 1 starts by covering all of the necessary preliminaries in order to understand the bulk of the thesis; we introduce here basic notions tied into the theories of modular representations and finite groups.

- In Chapter 2, we present a detailed survey into the well-known properties of Scott modules, and include a proof of their existence, which is accredited to Alperin and Scott. We define a *p-extendible* group to be a finite group G in which $\mathcal{S}(G,Q)\downarrow_P \cong k[P/Q]$ for all Sylow p-subgroups P of G and p-subgroups Q of P. We also present some evidence that this definition is not completely trivial; in particular, we show that p-nilpotent groups satisfy this property, and if p is odd, then every abelian p-group is a Sylow p-subgroup of some finite p-extendible group which is not p-nilpotent. We finish the chapter with a new result, which describes the dimension of a given Scott module in the case where G has a normal Sylow p-subgroup P in terms of the order of the local subgroup $N_G(Q)$.

- In Chapter 3, we start by supplementing our collection of examples of finite groups which are p-extendible, by studying the Frobenius complements in Frobenius groups. We then follow this up by studying the necessary structure that a p-extendible group must satisfy, and we demonstrate that the notions of being p-nilpotent and p-extendible are equivalent for p-solvable groups G when p is the smallest prime divisor of $|G|$. The second half of this chapter studies the p-local subgroups of the 26 sporadic groups and answers the following question: if N is a p-local subgroup of a sporadic group and q is a prime divisor of N, when is N q-extendible?

- The main subject in Chapter 4 is the topic of classical groups; using the theory of (B,N)-pairs, we study the Scott modules associated to the subgroup B in a

(B, N)-pair which defines a classical group and classify the possible dimensions for Scott modules in some cases of small degree. We also briefly cover Scott modules associated to the parabolic subgroups.

- In Chapter 5, we break from the main pattern of studying Scott modules, to look at relatively projective covers in more generality. The main topic of this chapter is a look into the properties of the second relative syzygy and a result which gives an upper bound on its dimension.
- Finally, the thesis includes an appendix, which contains MAGMA code that helps to understand the analysis involving sporadic groups in Chapter 3, and also contains any computations that we deem to distract from the flow of the main text.

Chapter 1

Preliminaries

We start with a chapter drawing together the background material we shall need concerning kG-modules and finite groups. Most of the material covered in this chapter is standard, though we include references for less well-known results. We shall take [1] as a standard reference for the topic of representation theory; [17] and [21] will be our main references for group theory.

1.1 Representation Theory

This thesis is ultimately an account of results relating to the modular representation theory of finite groups. We shall adopt a module-theoretic perspective on this theory; thus, the major object of study in this thesis will be the following.

Definition 1.1.1. Let G be a group and k be a field. A (left) **kG-module** $\mathcal{M}$ is a vector space over k together with an action of the group G on $\mathcal{M}$ such that:

(i) $g(m+n) = gm + gn$ for all $g \in G$ and $m, n \in \mathcal{M}$;

(ii) $g(\lambda m) = \lambda(gm)$ for all $g \in G$, $m \in \mathcal{M}$ and $\lambda \in k$;

(iii) $g(hm) = (gh)m$ for all $g, h \in G$ and $m \in \mathcal{M}$.

Thus, a kG-module $\mathcal{M}$ is a vector space equipped with an action of the group G such that each $g \in G$ induces a linear transformation of $\mathcal{M}$ and this correspondence is a homomorphism between the groups G and $GL(\mathcal{M})$; the data of a kG-module can therefore be encoded in a homomorphism $\rho : G \to GL(\mathcal{M})$, and of course, the reverse is true as well.

We now put in place some of the standard notation we shall use throughout the thesis in relation to kG-modules. We fix a finite group G and an algebraically closed field k and assume that k has positive characteristic p; we assume furthermore that any groups we work with have order divisible by p, unless we state otherwise. The **dimension** of a kG-module $\mathcal{M}$ is equal to the dimension of the underlying k-vector space, and we denote this quantity by $\dim \mathcal{M}$. We assume at all times in this thesis that $\dim \mathcal{M}$ is finite; the category of finite-dimensional kG-modules is denoted by $\operatorname{mod} kG$. By a k**-basis** of $\mathcal{M}$, we mean a basis of the underlying vector space.

If $\mathcal{U}, \mathcal{V} \leq \mathcal{M}$ are submodules of $\mathcal{M}$, then we write $\mathcal{M} = \mathcal{U} \oplus \mathcal{V}$ if $\mathcal{U} + \mathcal{V} = \mathcal{M}$ and $\mathcal{U} \cap \mathcal{V} = \{0\}$, and we say that $\mathcal{M}$ is a **direct sum** of its submodules $\mathcal{U}$ and $\mathcal{V}$. We say that a nonzero kG-module $\mathcal{M}$ is **indecomposable** if whenever $\mathcal{M} = \mathcal{U} \oplus \mathcal{V}$ we have either $\mathcal{U} = \{0\}$ or $\mathcal{V} = \{0\}$ and we say that $\mathcal{M}$ is **simple** if $\mathcal{M}$ has no proper, nontrivial submodules. Thus simple modules are clearly indecomposable, but indecomposable modules need not be simple. If $\mathcal{M}$ is a kG-module and $n \in \mathbb{N}$, then we set

$$\mathcal{M}^n := \underbrace{\mathcal{M} \oplus \mathcal{M} \oplus \cdots \oplus \mathcal{M}}_{n \text{ times}}.$$

The symbol $\mathcal{M} \otimes \mathcal{N}$ denotes the **tensor product** of the kG-modules $\mathcal{M}$ and $\mathcal{N}$; thus, the underlying vector space of $\mathcal{M} \otimes \mathcal{N}$ is the tensor product of the vector spaces $\mathcal{M}$ and $\mathcal{N}$, and we define a kG-module structure by setting $g(m \otimes n) = gm \otimes gn$ for all $g \in G$, $m \in \mathcal{M}$ and $n \in \mathcal{N}$.

If $\mathcal{U}_1, \ldots, \mathcal{U}_s \leq \mathcal{M}$ are indecomposable submodules of $\mathcal{M}$, then we write

$$\mathcal{M} = \bigoplus_{i=1}^{s} \mathcal{U}_i$$

if every vector $m \in \mathcal{M}$ can be written uniquely as $m = u_1 + \cdots + u_s$ for some $u_i \in \mathcal{U}_i, 1 \leq i \leq s$, and we refer to the above expression as an **indecomposable decomposition** of the kG-module $\mathcal{M}$. The Krull-Schmidt theorem [1, Theorem 4.3] states that if

$$\mathcal{M} = \bigoplus_{i=1}^{s} \mathcal{U}_i = \bigoplus_{j=1}^{t} \mathcal{V}_j$$

are two indecomposable decompositions for the kG-module $\mathcal{M}$, then $s = t$ and, upon rearranging the submodules if necessary, we have $\mathcal{U}_i \cong \mathcal{V}_i$ for $1 \leq i \leq s$. Thus, an indecomposable decomposition for a kG-module is essentially unique and every finite-dimensional kG-module has such an indecomposable decomposition, so the classification of kG-modules can be reduced to the classification of indecomposable kG-modules. If $\mathcal{U} \leq \mathcal{M}$, we say that $\mathcal{U}$ is a **direct summand** of $\mathcal{M}$ if there exists a submodule $\mathcal{X} \leq \mathcal{M}$ such that $\mathcal{M} = \mathcal{U} \oplus \mathcal{X}$; we refer to the module $\mathcal{X}$ as a **direct sum complement** for $\mathcal{U}$ in $\mathcal{M}$.

For a given kG-module $\mathcal{M}$, we denote by $\mathrm{soc}(\mathcal{M})$ the submodule of $\mathcal{M}$ generated by the simple submodules of $\mathcal{M}$ and refer to this as the **socle** of $\mathcal{M}$; we denote the intersection of all the maximal submodules of $\mathcal{M}$ by $\mathrm{rad}(\mathcal{M})$ and call this the **radical** of $\mathcal{M}$. The trivial kG-module is denoted by k_G.

If $\mathcal{M}$ is a kG-module and $H \leq G$, then we denote by $\mathcal{M}{\downarrow_H}$ the **restriction of** $\mathcal{M}$ **to** H; thus $\mathcal{M}{\downarrow_H}$ is a kH-module with the same underlying vector space as $\mathcal{M}$ and with a kH-module structure defined in terms of how H acts on $\mathcal{M}$. If $\mathcal{N}$ is a kH-module, we say that $\mathcal{N}$ can be **extended to** G if there exists a kG-module $\mathcal{M}$ such that $\mathcal{M}{\downarrow_H} \cong \mathcal{N}$, and we refer to the module $\mathcal{M}$ as an **extension** of $\mathcal{N}$.

We denote by $\mathrm{Hom}_{kG}(\mathcal{M}, \mathcal{N})$ the set of kG-module homomorphisms between $\mathcal{M}$ and $\mathcal{N}$, and $\mathrm{Hom}_k(\mathcal{M}, \mathcal{N})$ the set of linear transformations between $\mathcal{M}$ and $\mathcal{N}$. If $\varphi : \mathcal{M} \to \mathcal{N}$ is a kG-module homomorphism, then $\varphi_H : \mathcal{M}{\downarrow_H} \to \mathcal{N}{\downarrow_H}$ denotes the **restriction** of φ to H and we say that φ is H**-split** if there exists a kH-homomorphism $\gamma : \mathcal{N}{\downarrow_H} \to \mathcal{M}{\downarrow_H}$ such that $\varphi_H\gamma = \mathrm{id}$. If $\mathcal{M}$ is a kG-module, then $\mathcal{M}^*$ denotes the **dual** kG**-module**: thus $\mathcal{M}^* = \mathrm{Hom}_k(\mathcal{M}, k)$ and for each $g \in G$ and $\varphi \in \mathcal{M}^*$, we define $g\varphi : \mathcal{M} \to k$ by $(g\varphi)(m) = \varphi(g^{-1}m)$ for all $m \in \mathcal{M}$.

If $N \lhd G$, then $\mathrm{Inf}_{G/N}^G : \mathrm{mod}\, k[G/N] \to \mathrm{mod}\, kG$ denotes the **inflation** map; thus, if $\mathcal{M}$ is a $k[G/N]$-module, then $\mathrm{Inf}_{G/N}^G(\mathcal{M})$ denotes the kG-module with the same underlying vector space as $\mathcal{M}$ and a G-action given by

$$\underbrace{g \cdot m}_{\text{in } \mathrm{Inf}_{G/N}^G(\mathcal{M})} = \underbrace{[g] \cdot m}_{\text{in } \mathcal{M}}$$

for all $g \in G$ and $m \in \mathcal{M}$, where $[g] \in G/N$ denotes the coset of N in G containing g. If $\varphi : \mathcal{M} \to \mathcal{N}$ is a $k[G/N]$-module homomorphism, then $\mathrm{Inf}_{G/N}^G(\varphi)$ denotes the

kG-module homomorphism between $\mathrm{Inf}_{G/N}^{G}(\mathcal{M})$ and $\mathrm{Inf}_{G/N}^{G}(\mathcal{N})$ induced by inflation. We shall abbreviate $\mathrm{Inf}_{G/N}^{G}$ to just Inf when the context is clear.

If $m_1, \ldots, m_r \in \mathcal{M}$, then $\langle m_1, \ldots, m_r \rangle$ denotes the kG-submodule of $\mathcal{M}$ **generated** by the vectors $m_1, \ldots, m_r$.

We now briefly cover some of the main results and concepts from the field of modular representation theory that we shall find useful in this thesis. The most important concept we will need to cover first is that of *relative projectivity*, which serves as a generalisation of projectivity. Recall that we may regard the group algebra

$$kG = \left\{ \sum_{g \in G} \alpha_g g : \alpha_g \in k \right\}$$

as a kG-module with a G-action given by

$$h \cdot \left(\sum_{g \in G} \alpha_g g \right) = \sum_{g \in G} \alpha_g (hg)$$

for all $h \in G$ and $\alpha_g \in k$; this module is known as the **regular** kG-module. A **free kG-module** is a kG-module isomorphic to $(kG)^n$ for some $n \in \mathbb{N}$.

Definition 1.1.2. [1, Theorem 5.2] We say that a kG-module $\mathcal{P}$ is **projective** if $\mathcal{P}$ satisfies any of the following equivalent properties:

(i) $\mathcal{P}$ is a direct summand of a free kG-module;

(ii) if $\varphi : \mathcal{M} \to \mathcal{P}$ is a surjective homomorphism of kG-modules, then φ is a G-split homomorphism;

(iii) if $\varphi : \mathcal{M} \to \mathcal{N}$ is a surjective kG-homomorphism and $\psi : \mathcal{P} \to \mathcal{N}$ is a kG-homomorphism, then there exists a kG-homomorphism $\rho : \mathcal{P} \to \mathcal{M}$ such that $\varphi\rho = \psi$.

It follows from the Krull-Schmidt theorem that if $\mathcal{P}$ is an indecomposable projective module, $\mathcal{P}$ is a direct summand of the regular kG-module; such modules are known as PIMs (short for **projective indecomposable modules** or **principal indecomposable modules**). It is well-known (see e.g., [1, Theorems 5.3 and 6.6]) that if $\mathcal{P}$ is a PIM, then $\mathrm{soc}(\mathcal{P}) \cong \mathcal{S}$ for some simple kG-module $\mathcal{S}$ and $\mathcal{P}/\,\mathrm{rad}(\mathcal{P}) \cong \mathcal{S}$; furthermore, for each simple kG-module $\mathcal{S}$ there exists precisely one PIM $\mathcal{P}$ with $\mathrm{soc}(\mathcal{P}) \cong \mathcal{S}$.

1.1.1 Induction

The notion of relative projectivity generalises projectivity, but in order to understand it, we need to cover the process of induction. We repeat here for the convenience of the reader the description provided in [1, Section 8]. Suppose that $H \leq G$ and $\mathcal{M}$ is a kH-module. From $\mathcal{M}$ we may construct a kG-module, denoted by $\mathcal{M}{\uparrow}_H^G$ and constructed as follows: First we take the vector space $kG \otimes \mathcal{M}$ and quotient this vector space by the subspace spanned by all vectors of the form $ah \otimes m - a \otimes hm$, where $a \in kG$, $h \in H$ and $m \in \mathcal{M}$. We denote the resultant quotient space by $kG \otimes_{kH} \mathcal{M}$ and take it to be the underlying vector space of $\mathcal{M}{\uparrow}_H^G$; it is common to abuse notation slightly and reappropriate the symbol $a \otimes m$ to represent the coset of this quotient *containing* the elementary tensor $a \otimes m \in kG \otimes \mathcal{M}$, and we shall adopt this convention. We now define $g(a \otimes m) = (ga) \otimes m$ for all $g \in G$, $a \in kG$ and $m \in \mathcal{M}$. The resulting G-action turns $\mathcal{M}{\uparrow}_H^G$ into a well-defined kG-module.

Definition 1.1.3. We refer to the module $\mathcal{M}{\uparrow}_H^G$ as a module **induced** from H to G.

Thus $\mathcal{M}{\uparrow}_H^G$ is a **kG**-module constructed from a **kH**-module. Other notation used in the literature includes $\mathrm{Ind}_H^G(\mathcal{M})$ and $\mathcal{M}^G$. We shall abbreviate the symbol $\mathcal{M}{\uparrow}_H^G$ to just $\mathcal{M}{\uparrow}^G$ if the context is clear.

The following omnibus lemma accounts for many of the basic properties that the above induction process satisfies.

Lemma 1.1.4. Suppose that $H \leq L \leq G$ and let $\mathcal{M}$ and $\mathcal{N}$ be kH-modules. Then:

(i) $\dim \mathcal{M}{\uparrow}^G = |G : H| \dim \mathcal{M}$;

(ii) if $\mathcal{M}$ is a projective kH-module, then $\mathcal{M}{\uparrow}^G$ is a projective kG-module;

(iii) $(\mathcal{M} \oplus \mathcal{N}){\uparrow}^G \cong \mathcal{M}{\uparrow}^G \oplus \mathcal{N}{\uparrow}^G$ and $(\mathcal{M} \otimes \mathcal{N}){\uparrow}^G \cong \mathcal{M}{\uparrow}^G \otimes \mathcal{N}{\uparrow}^G$;

(iv) $(\mathcal{M}^*){\uparrow}^G \cong (\mathcal{M}{\uparrow}^G)^*$;

(v) $(\mathcal{M}{\uparrow}^L){\uparrow}^G \cong \mathcal{M}{\uparrow}^G$.

Proof. Part (i) is Lemma 8.4 in [1]; the remaining parts all appear in Lemma 8.5 of [1]. □

If $\mathcal{M}$ is an indecomposable kH-module, then $\mathcal{M}{\uparrow}_H^G$ might not necessarily be indecomposable; understanding the indecomposable decomposition of $\mathcal{M}{\uparrow}_H^G$ for a general kH-module $\mathcal{M}$ is a hard open question. The following theorem, known as Green's Indecomposability Criterion, gives one special case where the answer is straightforward (see [1, Theorem 8.8]).

Theorem 1.1.5 (Green's Indecomposability Criterion)**.** Suppose that $N \triangleleft G$, $|G : N| = p^r$ for some $r \in \mathbb{N}$ and $\mathcal{M}$ is an indecomposable kN-module. Then $\mathcal{M}{\uparrow}^G$ is indecomposable.

The other result we have available to us that helps us handle the structure of $\mathcal{M}{\uparrow}^G$ is known as Mackey's Theorem. If $H \leq G$, $g \in G$ and $\mathcal{M}$ is a kH-module, then we can define a new $k[gHg^{-1}]$-module $\mathcal{M}^g$ by "transport of structure". That is to say, $\mathcal{M}^g$ has the same underlying vector space as $\mathcal{M}$, and we define an action of gHg^{-1} on $\mathcal{M}^g$ by setting

$$\underbrace{(ghg^{-1}) \cdot m}_{\text{in } \mathcal{M}^g} = \underbrace{h \cdot m}_{\text{in } \mathcal{M}}$$

for all $m \in \mathcal{M}$ and $h \in H$. The following is Lemma 8.7 in [1].

Theorem 1.1.6 (Mackey's Theorem)**.** Suppose that $H, L \leq G$ and $\mathcal{M}$ is a kH-module. Then

$$(\mathcal{M}{\uparrow}_H^G){\downarrow}_L = \bigoplus_{s \in [L \backslash G / H]} ((\mathcal{M}^s){\downarrow}_{L \cap sHs^{-1}}){\uparrow}^L,$$

where $[L \backslash G/H]$ represents a set of double (L, H)-coset representatives in G.

If $N \triangleleft G$, we have $gNg^{-1} = N$ for all $g \in G$, so $\mathcal{M}^g$ is another kN-module, which we refer to as a **conjugate** kN-module of $\mathcal{M}$. We shall denote $\mathcal{M}^g$ by $g \otimes \mathcal{M}$ when $N \triangleleft G$. Note that $g \otimes \mathcal{M}$ need not be isomorphic to $\mathcal{M}$; we say that $\mathcal{M}$ is **G-stable** if $g \otimes \mathcal{M}$ and $\mathcal{M}$ are isomorphic kN-modules for all $g \in G$.

The final important result relating to the process of induction is known as Frobenius Reciprocity, which is given in [1, Theorem 8.6].

Theorem 1.1.7 (Frobenius Reciprocity)**.** Suppose that $H \leq G$, $\mathcal{M}$ is a kG-module and $\mathcal{N}$ is a kH-module. Assume furthermore that G is a finite group and $\mathcal{M}$ and $\mathcal{N}$ are finite-dimensional modules. Then we have the following isomorphisms of vector spaces:

(i) $\mathrm{Hom}_{kG}(\mathcal{M}, \mathcal{N}{\uparrow}^G) \cong \mathrm{Hom}_{kH}(\mathcal{M}{\downarrow}_H, \mathcal{N})$;

(ii) $\mathrm{Hom}_{kG}(\mathcal{N}{\uparrow}^G, \mathcal{M}) \cong \mathrm{Hom}_{kH}(\mathcal{N}, \mathcal{M}{\downarrow}_H)$.

1.1.2 Relative Projectivity

We are now in a position to generalise the notion of a projective module to a relatively projective module. We provide a number of equivalent formulations in the following definition; note the similarity to Definition 1.1.2.

Definition 1.1.8. [1, Proposition 9.1] Let $\mathcal{M}$ be a kG-module and $H \leq G$ be a subgroup of G. Then we say that $\mathcal{M}$ is **relatively** H**-projective** if $\mathcal{M}$ satisfies any of the following equivalent properties:

(i) $\mathcal{M}$ is a direct summand of $(\mathcal{M}{\downarrow}_H){\uparrow}^G$;

(ii) if $\varphi : \mathcal{N} \to \mathcal{M}$ is a surjective kG-homomorphism and φ is an H-split homomorphism, then φ is a G-split homomorphism;

(iii) if $\varphi : \mathcal{N} \to \mathcal{K}$ is a surjective kG-homomorphism and $\psi : \mathcal{M} \to \mathcal{K}$ is a kG-homomorphism, then there is a kG-homomorphism $\rho : \mathcal{M} \to \mathcal{N}$ such that $\varphi\rho = \psi$ provided there is a kH-homomorphism $\sigma : \mathcal{M}{\downarrow}_H \to \mathcal{N}{\downarrow}_H$ such that $\varphi_H\sigma = \psi_H$;

(iv) $\mathcal{M}$ is a direct summand of $\mathcal{N}{\uparrow}^G$ for some kH-module $\mathcal{N}$.

Thus, a projective module is a relatively $\{1\}$-projective module in terms of this new definition. In general, an indecomposable kG-module $\mathcal{M}$ can be relatively H-projective for many subgroups $H \leq G$; the possible subgroups for which $\mathcal{M}$ is relatively H-projective are tightly controlled by a subgroup known as a vertex.

Theorem 1.1.9. [1, Theorem 9.4] Let $\mathcal{M}$ be an indecomposable kG-module.

(i) There exists a p-subgroup Q of G which satisfies the following property: if $H \leq G$, then $\mathcal{M}$ is relatively H-projective if and only if H contains a conjugate of Q. Furthermore, the subgroup Q is unique up to conjugacy in G.

(ii) Given Q as in (i), there exists an indecomposable kQ-module $\mathcal{S}$ such that $\mathcal{M}$ is a direct summand of $\mathcal{S}{\uparrow}^G$.

We refer to the subgroup Q in the above as a **vertex** of $\mathcal{M}$ and the module $\mathcal{S}$ as a **source module** of $\mathcal{M}$. We shall write $\mathrm{vx}(\mathcal{M})$ to refer to a particular vertex of $\mathcal{M}$; if a module $\mathcal{M}$ has a trivial module as its source module, we say that $\mathcal{M}$ is a **trivial source module**.

Remarks 1.1.10. (i) The projective modules are precisely those which have vertex $\{1\}$; thus, relative projectivity can be thought of as a measure of how far an indecomposable module is from being projective.

(ii) If P is a p-group and $Q \leq P$, then the induced module $(k_Q){\uparrow}^P$ is indecomposable and has vertex Q.

(iii) Since vertices are determined up to conjugacy in G, it follows that any indecomposable kG-module is relatively H-projective if H is a subgroup which contains a Sylow p-subgroup of G.

(iv) If a kG-module $\mathcal{M}$ has vertex equal to a Sylow p-subgroup of G, then we say that $\mathcal{M}$ has **maximal vertex**.

The calculation of vertices for a given indecomposable kG-module $\mathcal{M}$ is in general a difficult problem; [42] provides an algorithm for computing the vertex of a given module in MAGMA, and these ideas have been developed further in [12] and applied to find the vertices of simple modules defined over the symmetric group in small degree.

1.1.3 Relatively Projective Resolutions

In this section, we introduce the definition of a relatively projective resolution and put in place the main properties of these objects; we take the details of this section from [38]. If X is a family of subgroups of G, then a kG-homomorphism $\delta : \mathcal{M} \to \mathcal{N}$ is said to be X**-split** if it is H-split for every $H \in X$. Moreover, if $\mathcal{M}$ is a kG-module which has an indecomposable decomposition

$$\mathcal{M} = \bigoplus_{i=1}^{s} \mathcal{U}_i,$$

then we say that $\mathcal{M}$ is **relatively** X**-projective** if each $\mathcal{U}_i$ is relatively H-projective for some $H \in X$.

Definition 1.1.11. Suppose that $\mathcal{M}$ is a kG-module and X is a family of subgroups of G. A **relatively** X**-projective cover** of $\mathcal{M}$ is a pair $(\mathcal{P}, \delta)$ such that:

(i) $\mathcal{P}$ is a relatively X-projective module;

(ii) $\delta : \mathcal{P} \to \mathcal{M}$ is an X-split kG-homomorphism;

(iii) whenever $f \in \mathrm{End}_{kG}(\mathcal{P})$ satisfies $\delta f = \delta$, we have that f is an isomorphism.

Recall that an X-split homomorphism is automatically surjective, which explains the choice of "cover" in the terminology; (iii) in the above means that we cannot replace $\mathcal{P}$ by a proper direct summand and obtain a "smaller" cover, so serves as a minimality condition. When discussing relatively X-projective covers, it is common to refer to the module $\mathcal{P}$ alone and leave δ implicit, although it is an important part of the definition. In the case where X just consists of the trivial subgroup of G, a relatively X-projective cover of $\mathcal{M}$ is a **projective** cover of $\mathcal{M}$; we denote the projective cover of $\mathcal{M}$ by $\mathcal{P}(\mathcal{M})$. In the case where $X = \{H\}$ for a single subgroup $H \leq G$, we shall say that a relatively X-projective cover of $\mathcal{M}$ is a **relatively** H**-projective cover** of $\mathcal{M}$.

We say that two exact sequences

$$\begin{aligned} &0 \to \ker\delta \to \mathcal{P} \xrightarrow{\delta} \mathcal{M} \to 0 \\ \text{and} \quad &0 \to \ker\epsilon \to \mathcal{P}' \xrightarrow{\epsilon} \mathcal{M}' \to 0 \end{aligned}$$

are isomorphic if there exist isomorphisms $\varphi : \ker\delta \to \ker\epsilon$, $\psi : \mathcal{P} \to \mathcal{P}'$ and $\rho : \mathcal{M} \to \mathcal{M}'$ such that the diagram

$$\begin{tikzcd} 0 \arrow[r] & \ker\delta \arrow[r] \arrow[d, "\varphi"] & \mathcal{P} \arrow[r, "\delta"] \arrow[d, "\psi"] & \mathcal{M} \arrow[r] \arrow[d, "\rho"] & 0 \\ 0 \arrow[r] & \ker\epsilon \arrow[r] & \mathcal{P}' \arrow[r, "\epsilon"] & \mathcal{M}' \arrow[r] & 0 \end{tikzcd}$$

commutes.

It is well-known that $\operatorname{mod} kG$ satisfies the following two properties:

(i) every element of $\operatorname{mod} kG$ is a finite direct sum of indecomposable modules which is unique up to order and isomorphism between the indecomposable modules, i.e., the Krull-Schmidt theorem holds;

(ii) if $\mathcal{M} \in \operatorname{mod} kG$ is indecomposable, then $\operatorname{End}_{kG}(\mathcal{M})$ is a local algebra, i.e., every element of $\operatorname{End}_{kG}(\mathcal{M})$ is either invertible or nilpotent (see [1, Theorem 4.2]).

Thus $\operatorname{mod} kG$ is a so-called **Krull-Schmidt category**. On the basis of this, we may deduce the following result.

Proposition 1.1.12. Suppose that $\mathcal{M} \in \operatorname{mod} kG$ and X is a family of subgroups of G. Then:

(i) the module $\mathcal{M}$ has a relatively X-projective cover $(\mathcal{P}, \delta)$;

(ii) if $(\mathcal{P}, \delta)$ and $(\mathcal{P}', \delta')$ are both relatively X-projective covers of $\mathcal{M}$, then

$$0 \to \ker \delta \to \mathcal{P} \xrightarrow{\delta} \mathcal{M} \to 0$$

and

$$0 \to \ker \delta' \to \mathcal{P}' \xrightarrow{\delta'} \mathcal{M} \to 0$$

are isomorphic sequences; in particular, $\mathcal{P} \cong \mathcal{P}'$;

(iii) if $\mathcal{U}$ is a relatively X-projective module, $\epsilon : \mathcal{U} \to \mathcal{M}$ is an X-split kG-homomorphism and $(\mathcal{P}, \delta)$ is a relatively X-projective cover of $\mathcal{M}$, then there exists a relatively X-projective module $\mathcal{Q}$ such that

$$0 \to \ker \delta \to \mathcal{P} \oplus \mathcal{Q} \xrightarrow{\delta} \mathcal{M} \to 0$$

and

$$0 \to \ker \epsilon \to \mathcal{U} \xrightarrow{\epsilon} \mathcal{M} \to 0$$

are isomorphic sequences; in particular, $\mathcal{P}$ is a direct summand of $\mathcal{U}$.

Proof. Since $\operatorname{mod} kG$ is a Krull-Schmidt category, (i) follows from [38, Proposition 1.6]. Parts (ii) and (iii) are proved in [38, Proposition 1.3]. □

Thus relatively X-projective covers of arbitrary kG-modules in $\operatorname{mod} kG$ exist and are essentially unique. Since any finite-dimensional kG-module has a relatively X-projective cover, we may make the following definition.

Definition 1.1.13. A **relatively X-projective resolution** of a module $\mathcal{M} \in \operatorname{mod} kG$ is a positive complex $\mathcal{P}_*$ of relatively X-projective modules and a kG-homomorphism $\delta_1 : \mathcal{P}_1 \to \mathcal{M}$ such that the sequence:

$$\cdots \xrightarrow{\delta_4} \mathcal{P}_3 \xrightarrow{\delta_3} \mathcal{P}_2 \xrightarrow{\delta_2} \mathcal{P}_1 \xrightarrow{\delta_1} \mathcal{M} \to 0$$

is exact, each $(\mathcal{P}_i, \delta_i)$ is a relatively X-projective cover of $\ker \delta_{i-1}$ for all $i \geq 2$ and $(\mathcal{P}_1, \delta_1)$ is a relatively X-projective cover of $\mathcal{M}$.

We set $\Omega^i_X(\mathcal{M}) = \ker \delta_i$ in the above definition, for $i \in \mathbb{N}$, and refer to this kernel as an X**-relative syzygy** of $\mathcal{M}$. We set $\Omega^i_{G/H}(k) = \Omega^i_{\{H\}}(k_G)$ for a subgroup $H \leq G$.

1.1.4 The Green Correspondence

If $L \leq G$, then induction and restriction give a means of relating kL-modules to kG-modules and vice versa, and in certain special cases, this relation satisfies some more precise properties. The Green correspondence is one such instance of this phenomenon. Let Q be a p-subgroup of G and $L \leq G$ with $N_G(Q) \leq L$. If $P, R \leq G$, then $P \leq_G R$ means that $P^x \leq R$ for some $x \in G$.

We define the following three collections of subgroups in G:

$$\begin{aligned} \mathcal{X} &= \{Q^s \cap Q : s \in G, s \notin L\}, \\ \mathcal{N} &= \{Q^s \cap L : s \in G, s \notin L\}, \\ \text{and} \quad \mathcal{Z} &= \{R : R \leq Q, R \nleq_G X \text{ for all } X \in \mathcal{X}\}. \end{aligned}$$

We denote by $\mathcal{Z}(G)$ the set of isomorphism classes of indecomposable kG-modules with vertex in $\mathcal{Z}$ and by $\mathcal{Z}(L)$ the set of isomorphism classes of indecomposable kL-modules with vertex in $\mathcal{Z}$. The following is Theorem 11.1 in [1].

Theorem 1.1.14 (The Green Correspondence)**.** If $\mathcal{M} \in \mathcal{Z}(G)$, then any direct decomposition of $\mathcal{M}{\downarrow_L}$ has a unique direct summand $\mathcal{S}$ such that $\mathrm{vx}(\mathcal{S}) = \mathrm{vx}(\mathcal{M})$. Furthermore, if we set $\mathcal{S} = f(\mathcal{M})$, then $f : \mathcal{Z}(G) \to \mathcal{Z}(L)$ is a one-to-one correspondence and:

(i) $\mathcal{M}{\downarrow_L} = f(\mathcal{M}) \oplus \mathcal{U}$ for some relatively $\mathcal{N}$-projective kL-module $\mathcal{U}$;

(ii) $f(\mathcal{M}){\uparrow^G} = \mathcal{M} \oplus \mathcal{V}$ for some relatively $\mathcal{X}$-projective kG-module $\mathcal{V}$.

We shall denote the map $f : \mathcal{Z}(G) \to \mathcal{Z}(L)$ in the above result by $f(G, L)$ and refer to it as the **Green correspondence between** G **and** L, or simply the **Green correspondence** when G and L are clear from the context. The module $f(\mathcal{M})$ is known as a **Green correspondent** of $\mathcal{M}$. Since restriction and induction commute with dualising, the following property of the Green correspondence is immediate from this theorem.

Proposition 1.1.15. Let $f = f(G, L)$ denote the Green correspondence between G and L. Then $f(\mathcal{M}^*) \cong f(\mathcal{M})^*$ for all $\mathcal{M} \in \mathcal{Z}(G)$.

Proof. See Corollary 7.2.2 of [24]. □

1.1.5 Permutation Modules

Most of the material concerning permutation modules in this section is taken from [24, Chapter 11]. We start with the definition of a permutation module.

Definition 1.1.16. Suppose that $\mathcal{M}$ is a nonzero kG-module. We say that a k-basis X of $\mathcal{M}$ is a **permutation basis** if $gx \in X$ for all $g \in G$ and $x \in X$. Furthermore, we say that $\mathcal{M}$ is a **permutation** kG**-module** if $\mathcal{M}$ contains a permutation basis X.

Given a G-set X, there is an obvious kG-module, denoted kX, which consists of the set
$$kX = \left\{ \sum_{x \in X} \alpha_x x : \alpha_x \in k \right\}$$
together with a kG-module structure induced by the action of G on X. Then kX is clearly a permutation kG-module with permutation basis X; furthermore, if $\mathcal{M}$ is a permutation kG-module with permutation basis X, then $\mathcal{M} \cong kX$.

Remarks 1.1.17. (i) The regular kG-module kG is an example of a permutation kG-module with permutation basis $X = G$.

(ii) If $[G/H]$ denotes a set of left coset representatives of H in G, then $[G/H]$ is a G-set and the kG-module $k[G/H]$ is a permutation kG-module with permutation basis $X = [G/H]$. We have already seen this module in this chapter; indeed, it is well-known that $k[G/H] \cong (k_H)\!\uparrow^G$. If $g \in G$, then we denote by $[g]$ the coset of H in G containing g; thus elements of $k[G/H]$ are of the form

$$\sum_{x \in X} \alpha_x [x],$$

where $\alpha_x \in k$ for all $x \in X$.

(iii) If $\mathcal{M}$ is a permutation kG-module and $O_1, \ldots, O_s$ are the G-orbits of a permutation basis X of $\mathcal{M}$, then

$$\mathcal{M} \cong kX \cong \bigoplus_{i=1}^{s} kO_i.$$

In light of (iii) in the above, the following more refined definition makes sense.

Definition 1.1.18. Suppose that $\mathcal{M}$ is a nonzero kG-module. We say that $\mathcal{M}$ is a **transitive permutation kG-module** if it contains a permutation basis X such that G acts transitively on X, i.e., the action of G on X has one orbit.

The following omnibus result contains all the basic facts we shall require concerning permutation kG-modules.

Lemma 1.1.19. Suppose that G is a finite group. Then the following are true.

(i) Every permutation kG-module is a direct sum of transitive permutation kG-modules.

(ii) A kG-module $\mathcal{M}$ is a transitive permutation kG-module if and only if $\mathcal{M} \cong (k_H)\!\uparrow^G$ for some subgroup $H \leq G$; in particular, if X is a permutation basis for $\mathcal{M}$ and $x \in X$, then we have $\mathcal{M} \cong (k_S)\!\uparrow^G$, where

$$S = \mathrm{Stab}_G(x) := \{g \in G : gx = x\}.$$

(iii) If P is a p-group and $Q \leq P$, then $k[P/Q] \cong (k_Q)\uparrow^P$ is indecomposable and has vertex Q.

(iv) Let $H \leq G$. If $\mathcal{M}$ is a permutation kG-module, then $\mathcal{M}\downarrow_H$ is a permutation kH-module and if $\mathcal{N}$ is a permutation kH-module, then $\mathcal{N}\uparrow^G$ is a permutation kG-module. Furthermore, if $L \leq H$, then $k[H/L]\uparrow^G \cong k[G/L]$.

(v) If G is a group and $H, L \leq G$, then $k[G/H] \cong k[G/L]$ if and only if H and L are conjugate in G.

(vi) If $\mathcal{M}$ and $\mathcal{N}$ are permutation kG-modules, then $\mathcal{M} \oplus \mathcal{N}$, $\mathcal{M} \otimes \mathcal{N}$ and $\mathcal{M}^*$ are all permutation kG-modules.

Proof. Most of these properties are proved in [24]: parts (i) and (ii) appear in Lemma 11.1.1; part (iii) is Lemma 11.1.5; part (iv) is Lemma 11.1.7 (although the proof is clear from the definitions); and part (vi) is Lemma 11.1.6. Part (v) is well-known and a reference can be found in [4, Lemma 2.3.1]. $\square$

1.2 Group Theory

We shift focus now to background material related to finite groups. We start by consolidating some standard notation and terminology in one place. We shall assume throughout that all groups considered are finite; typically, if G denotes such a finite group, then we will assume that p divides $|G|$. The set of Sylow p-subgroups of G will be denoted by $\mathrm{Syl}_p(G)$, and we shall frequently use the symbols P to refer to an arbitrary element of $\mathrm{Syl}_p(G)$ and Q to refer to an arbitrary p-subgroup of G; the symbol P_q will be used to denote a Sylow q-subgroup of G for some prime q. We set $|G|_p = |P|$ where $P \in \mathrm{Syl}_p(G)$, and we set $|G|_{p'} = |G|/|G|_p$. A group G is a **p'-group** if $|G|_p = 1$. The symbol $\pi(G)$ denotes the set of prime divisors of $|G|$ and the order of an element $x \in G$ is denoted by $o(x)$; the **exponent** of G is the lowest common multiple of the orders of elements in G. We set $[x, y] = x^{-1}y^{-1}xy$ for all $x, y \in G$ and refer to this element as a **commutator**.

The **centre** of G is denoted by $Z(G)$, the **commutator** subgroup by G' and the **Frattini** subgroup by $\Phi(G)$; recall that if $N \lhd G$, then G/N is abelian if and only if $G' \leq N$. The symbol $O_p(G)$ denotes the largest normal p-subgroup of G, and $O_{p'}(G)$ denotes the largest normal p'-subgroup of G. A group G is **nilpotent** if $O_p(G) \in \mathrm{Syl}_p(G)$ for all $p \in \pi(G)$.

Given $x, y \in G$ and $H \leq G$, we write $x \sim_H y$ if x **is conjugate to** y in H, i.e., if there exists $h \in H$ such that $x^h = y$. Similarly, if $S, T \subseteq G$ and $H \leq G$, then $S \sim_H T$ means that S **is conjugate to** T in H. Given elements $x_1, \ldots, x_s \in G$ and subsets $X_1, \ldots, X_t \subseteq G$, we write

$$\langle x_1, \ldots, x_s, X_1, \ldots, X_t \rangle$$

to denote the subgroup of G generated by the elements $x_1, \ldots, x_s$ and subsets $X_1, \ldots, X_t$. If $X, Y \subseteq G$, then XY denotes the set product, i.e., $XY = \{xy : x \in X, y \in Y\}$; we recall that if $H, K \leq G$, then

$$|HK| = \frac{|H||K|}{|H \cap K|}.$$

Given a subgroup $H \leq G$, a set of left coset representatives of H in G will be known as a **left transversal** of H in G; similarly, a set of right coset representatives of H in G will be known as a **right transversal** of H in G. We denote a fixed left transversal by $[G/H]$ and a fixed right transversal by $[H\backslash G]$; we shall always assume that $1 \in [G/H]$ and $1 \in [H\backslash G]$. If $g \in G$, then $[g]$ denotes the coset of H in G which contains g.

We denote by $\mathrm{Aut}(G)$ the automorphism group of a finite group G. A subgroup $H \leq G$ is **characteristic** in G if $\varphi(H) = H$ for all $\varphi \in \mathrm{Aut}(G)$, and we write $H \,\mathrm{char}\, G$ in this situation. We shall frequently make use of the following two facts for a chain of groups $H_1 \leq H_2 \leq \cdots \leq H_r$ (see [17, Theorem 2.1.2]):

(i) if $H_1 \,\mathrm{char}\, H_2 \,\mathrm{char}\, H_3 \,\mathrm{char} \cdots \mathrm{char}\, H_{r-1} \,\mathrm{char}\, H_r$, then $H_1 \,\mathrm{char}\, H_r$;

(ii) if $H_1 \,\mathrm{char}\, H_2 \,\mathrm{char}\, H_3 \,\mathrm{char} \cdots \mathrm{char}\, H_{r-1} \lhd H_r$, then $H_1 \lhd H_r$.

Note that if $P \lhd G$ is a Sylow p-subgroup of G, then $P \,\mathrm{char}\, G$.

We write $G = N \times K$ if G is a direct product of two normal subgroups $N, K \lhd G$ and we write $G = N \rtimes K$ if G is a semidirect product of a normal subgroup N and another, not necessarily normal, subgroup K; in both situations, we have $G = NK$ and $N \cap K = \{1\}$, so every $g \in G$ may be written uniquely as $g = nk$ for some $n \in N$ and $k \in K$. If H is a finite group and G is a subgroup of Σ_n, the symmetric group of degree

n, we denote by $H \wr G$ the **wreath product** of H by G; thus $H \wr G$ is isomorphic to the semidirect product $K \rtimes G$, where

$$K = H^{(1)} \times \cdots \times H^{(n)},$$

$H^{(i)} \cong H$ for $1 \leq i \leq n$ and elements of G act on the set $\{H^{(i)} : 1 \leq i \leq n\}$ by permuting the superscripts $\{1, \ldots, n\}$. If H and K are groups which each contain a common central subgroup Z, then $H *_Z K$ denotes the **central product** of H and K with respect to Z; thus $H *_Z K \cong (H \times K)/Z$.

We shall say that H is **subnormal** in G if there exists a chain of subgroups

$$H = H_0 \lhd H_1 \lhd \cdots \lhd H_{r-1} \lhd H_r = G.$$

Furthermore, we shall say that G is **solvable** if there exists a subnormal series

$$\{1\} = N_0 \lhd N_1 \lhd \cdots \lhd N_{r-1} \lhd N_r = G$$

such that for all i, $|N_i/N_{i-1}| = q$ for some $q \in \pi(G)$; we shall say that G is p**-solvable** if there exists a subnormal series of G such that for all i, N_i/N_{i-1} is either a p-group or a p'-group. The Feit-Thompson theorem asserts that any group of odd order is solvable and hence p-solvable for all primes p.

We shall say that G is **metacyclic** if G contains a normal cyclic subgroup A such that $G/A \cong B$ is cyclic, and we say that G is **split metacyclic** if $G = A \rtimes B$ for cyclic subgroups $A, B \leq G$.

If $H \leq G$, then a subgroup X of G is said to be a **complement** of H in G if $G = HX$ and $H \cap X = \{1\}$. If $\pi \subseteq \pi(G)$, then a **Hall** π**-subgroup** of G is a subgroup H such that

$$|H| = \prod_{q \in \pi} |G|_q.$$

We shall find the following well-known facts useful when working with complements.

Proposition 1.2.1. Suppose that G is a finite group and H is a Hall π-subgroup of G for some $\pi \subseteq \pi(G)$.

(i) If G is solvable, then H has a complement in G.

(ii) (Schur-Zassenhaus Theorem). If $H \lhd G$, then H has a complement in G.

Proof. Part (i) is [17, Lemma 6.4.1] and part (ii) is [17, Theorem 6.2.1]. $\square$

We recall that if $N \lhd G$, then there is a one-to-one correspondence between the subgroups of G which contain N and subgroups of G/N, induced by the quotient map $G \to G/N$. We shall express this relation using the bar notation: thus, we set $\overline{G} = G/N$ and denote subgroups of $\overline{G}$ by $\overline{S}$, where S is the corresponding subgroup of G containing N. Recall that $\overline{S} \lhd \overline{G}$ if and only if $S \lhd G$.

We denote the symmetric group on n letters by Σ_n and the alternating group on n letters by A_n. The cyclic group of order n is denoted C_n.

Definition 1.2.2. We shall frequently make use of the following well-known groups.

(i) If P is abelian and the exponent of P is p, then we say that P is **elementary abelian** and write $P = E_{p^n}$, where $|P| = p^n$. Note that

$$E_{p^n} \cong \underbrace{C_p \times C_p \times \cdots \times C_p}_{n \text{ times}}.$$

(ii) We denote by D_{2n} the **dihedral group** of order $2n$, i.e., the rotations of a regular n-gon. Thus

$$D_{2n} = \langle a, b : a^n = b^2 = 1, ab = ba^{-1} \rangle \cong C_n \rtimes C_2.$$

(iii) We denote by SD_{2^n} the **semidihedral group** of order 2^n for $n \geq 4$, which has the following presentation:

$$SD_{2^n} = \langle a, b : a^{2^{n-1}} = b^2 = 1, ab = ba^{2^{n-2}-1} \rangle.$$

(iii) We use Q_{2^n} to denote the **generalised quaternion group** of order 2^n for $n \geq 3$; this group has the presentation:

$$Q_{2^n} = \langle a, b : a^{2^{n-1}} = b^4 = 1, a^{2^{n-2}} = b^2, ab = ba^{-1} \rangle.$$

(iv) The **extraspecial groups** of order p^3 are given by:

$$p_+^{1+2} = \langle a, b, c : a^p = b^p = c^p = 1, [a, c] = [b, c] = 1, [a, b] = c \rangle$$

and

$$p_-^{1+2} = \langle a, b : a^{p^2} = b^p = 1, ab = ba^{p+1} \rangle.$$

These are the two non-abelian groups of order p^3 up to isomorphism; note that $2_+^{1+2} \cong D_8$ and $2_-^{1+2} \cong Q_8$.

1.2.1 Linear and Classical Groups

We will require an understanding of linear and classical groups in Section 3.4 and Chapter 4. In this section, we remind the reader of how these groups are defined and put in place the notation that will be in force for the rest of the thesis; the reader is referred to [41, Chapter 3] for the proofs of any standard facts.

Suppose that V is an n-dimensional vector space over a finite field $F = \mathbb{F}_q$ of order q for some prime power $q = p^r$ with $r \in \mathbb{N}$. The **general linear group**, denoted by $GL_n(q)$, consists of all the invertible linear transformations $\varphi : V \to V$. We denote by $SL_n(q)$ the **special linear group**, which consists of those elements in $GL_n(q)$ with determinant 1, and by $L_n(q)$ the **projective linear group**, which is defined to be the quotient $L_n(q) := SL_n(q)/Z(SL_n(q))$. The general linear groups, special linear groups and projective linear groups are sometimes referred to loosely as **linear groups**.

The other classical groups arise as subgroups of general linear groups which fix certain extra structures on the relevant vector space. Recall that a **bilinear form** is a map $f : V \times V \to F$ such that $f(\lambda u + v, w) = \lambda f(u, w) + f(v, w)$ and $f(u, \lambda v + w) = \lambda f(u, v) + f(u, w)$ for all $\lambda \in F$ and $u, v, w \in V$. We say that f is **symmetric** if $f(u, v) = f(v, u)$ for all $u, v \in V$; **skew-symmetric** if $f(u, v) = -f(v, u)$ for all $u, v \in V$; and **alternating** if $f(v, v) = 0$ for all $v \in V$. If $F = \mathbb{F}_{q^2}$, then we set $\overline{x} = x^q$

for all $x \in F$ and we shall say that a map $f : V \times V \to F$ satisfying

$$\begin{aligned} & f(\lambda u + v, w) = \lambda f(u, w) + f(v, w) \\ \text{and} \quad & f(u, v) = \overline{f(v, u)} \end{aligned}$$

for all $u, v, w \in V$ and $\lambda \in F$ is a **conjugate-symmetric sesquilinear form**.

Given a bilinear form or a sesquilinear form f defined on V, we set $\mathrm{Isom}(V, f)$ to denote the set of $\varphi \in GL_n(q)$ such that $f(\varphi(u), \varphi(v)) = f(u, v)$ for all $u, v \in V$ and call this the **isometry group of V with respect to f**. Furthermore, we set $\mathrm{Sim}(V, f)$ to be equal to the set of $\varphi \in GL_n(q)$ for which there exists $\lambda \in F$ such that $f(\varphi(u), \varphi(v)) = \lambda f(u, v)$ for all $u, v \in V$; this is the **similarity group of V with respect to f**.

We are now ready to define the three main types of classical groups; in the following, we denote the $(n \times n)$-identity matrix by I_n. If $\dim V = 2m$, then $Sp_{2m}(q)$ denotes the **symplectic group** of degree $2m$ over F, which is the isometry group of a nonzero alternating bilinear form f on V. It is well-known that, up to isomorphism, there is one such group in any even dimension and for any finite field F. We define $PSp_{2m}(q)$ to be the quotient $Sp_{2m}(q)/\{\pm I_{2m}\}$ and call this the **projective symplectic group**; we also set $GSp_{2m}(q)$ to equal the similarity group of V with respect to f.

If $\dim V = n$ and $F = \mathbb{F}_{q^2}$, then $GU_n(q)$ denotes the **general unitary group** of degree n over F, which is the isometry group of a conjugate-symmetric sesquilinear form f on V; note that $GU_n(q) \leq GL_n(q^2)$. We denote by $SU_n(q)$ the **special unitary group**, i.e., $SU_n(q) = GU_n(q) \cap SL_n(q^2)$; and by $U_n(q)$ the **projective unitary group**, that is to say, the quotient $SU_n(q)/Z(SU_n(q))$.

If $\dim V = n$ is odd and $F = \mathbb{F}_q$ with $\mathrm{char}\, F > 2$, then $O_n(q)$ denotes the **orthogonal group** of degree n over F, which is the isometry group of a nonzero symmetric bilinear form f on V. In this case, there is up to isomorphism just one orthogonal group. On the other hand, if $\dim V = 2m$ is even, then we get two different orthogonal groups, depending on certain special properties of the bilinear form f. We say that f is of **plus type** if there exists a subspace U in V of dimension m such that f is identically zero on $U \times U$, and it is of **minus type** otherwise. The **orthogonal groups** $O^+_{2m}(q)$ and $O^-_{2m}(q)$ are then the isometry groups corresponding to nonzero symmetric bilinear forms of the obvious type. When we wish to discuss all three of the above cases generically, we shall revert to the symbol $O_n(q)$, with the understanding that for even n, this notation is potentially ambiguous.

For $\epsilon \in \{+, -, \emptyset\}$, we denote by $SO_n^\epsilon(q)$ the **special orthogonal group** corresponding to $O_n^\epsilon(q)$, i.e., $SO_n^\epsilon(q) = O_n^\epsilon(q) \cap SL_n(q)$; by $\Omega_n^\epsilon(q)$, we denote the kernel of the spinor norm map, which is an index 2-subgroup of $SO_n^\epsilon(q)$; and we set $P\Omega_n^\epsilon(q) := \Omega_n^\epsilon(q)/\{\pm I_n\}$ when n is even, and $P\Omega^\epsilon(q) := \Omega_n^\epsilon(q)$ if n is odd. For details concerning the spinor norm map, the reader should refer to [41, 3.7].

Finally, we need to define the orthogonal groups over fields of characteristic 2. If $\dim V = n$ is even and greater than or equal to 6, and $F = \mathbb{F}_q$ with $\operatorname{char} F = 2$, then for each vector $v \in V$ of norm 1, the map defined by

$$t_v : w \mapsto w + f(w, v)v$$

in terms of a nonzero symmetric bilinear form f on V is known as an **orthogonal transvection**. The **orthogonal group** $O_n^+(q)$ is defined to be the subgroup of $GL_n(q)$ generated by these transvections. Moreover, the **quasideterminant** of a given $x \in O_n^+(q)$ is defined to be 1 or -1 according to whether x can be written as a product of either an even number or an odd number of these transvections. The set of elements which are of quasideterminant 1 then form a subgroup of $O_n^+(q)$ which we denote either $\Omega_n^+(q)$ or, for the purposes of being consistent with our terminology for the simple classical groups, $P\Omega_n^+(q)$. A similar construction provides another corresponding orthogonal group of "minus" type, which we denote by $O_{2m}^-(q)$, and this too contains a subgroup generated by certain transvections, which we denote by either $\Omega_n^-(q)$ or $P\Omega_n^-(q)$. We refer the reader to the discussion contained in [41, 3.8] for further details on the constructions for both $O_n^+(q)$ and $O_n^-(q)$ in the case where F has characteristic 2.

We refer to any of the above symplectic, unitary or orthogonal groups as a **classical group**. Of course, the classical groups play an important role in the classification of the finite simple groups; the following result clarifies which of the above groups are simple.

Theorem 1.2.3. The following classical groups are all simple:

(i) $L_n(q)$ if $n > 2$ or $q > 3$;

(ii) $PSp_{2m}(q)$ if $m = 1$ and $q > 3$, or $m = 2$ and $q > 2$, or $m > 2$;

(iii) $U_n(q)$ if $n = 2$ and $q > 3$, or $n = 3$ and $q > 2$, or $n > 3$;

(iv) for $\epsilon \in \{+, -, \emptyset\}$, $P\Omega_n^\epsilon(q)$ if $n = 5$ and q is odd, or $n \geq 6$.

In addition to the above theorem, we have the following well-known isomorphisms between classical groups of low degree:

$$\begin{aligned} L_2(2) &\cong \Sigma_3 & L_2(3) &\cong A_4 \\ L_2(4) \cong L_2(5) &\cong A_5 & L_2(7) &\cong L_3(2) \\ L_2(9) &\cong A_6 & L_4(2) &\cong A_8. \end{aligned}$$

At times, we shall provide examples which involve the classical groups described above, and when we do so, we shall find it helpful to think of them as being groups of matrices. Thus, we may think of $GL_n(q)$ as consisting of the $(n\times n)$-matrices with entries in a finite field of order q, and we denote this field by $\mathbb{F}_q$ and its multiplicative subgroup by $\mathbb{F}_q^\times$. By a **diagonal matrix**, we mean a matrix with diagonal $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ for some $\alpha_i \in \mathbb{F}_q^\times$, and zeroes everywhere else; and by a **scalar matrix**, we mean a matrix with diagonal $(\alpha, \alpha, \ldots, \alpha)$ for some fixed $\alpha \in \mathbb{F}_q^\times$, and zeroes everywhere else. We denote by e_{ij} the matrix with a 1 in its (i, j)-th entry, and zeroes everywhere else. A **permutation matrix** is a matrix which has precisely one entry equal to 1 in every row and column, and zeroes everywhere else; a **monomial matrix** is a matrix which is a product of a diagonal matrix and a permutation matrix.

1.2.2 Fusion

Suppose that G is a finite group and $P \in \mathrm{Syl}_p(G)$. Much of the background material covered in this section is taken from [17, Chapter 7] and [21, Chapter 5]. If $x, y \in P$, then we say that x and y are **fused** in G if $x \sim_G y$, but $x \not\sim_P y$; more generally, we say that two subsets $X, Y \subseteq P$ are **fused** in G if $X \sim_G Y$, but $X \not\sim_P Y$. The question of fusion in a finite group G dates back some time now; the early forming of these ideas started roughly around the mid twentieth century (a good summary of the early state of the theory can be found in [16]). Central to the theory is the notion that conjugation is "locally controlled"; that is to say, conjugation in G can be understood best in terms of the subgroups of G of the form $N_G(Q)$, where Q is a p-subgroup of G and $Q \neq \{1\}$. In particular, Alperin's fusion theorem ([17, Section 7.2]) makes this evident.

We now set up some of the standard terminology we require related to these ideas. A subgroup $H \leq G$ is said to be **p-local** or **local** if $H = N_G(Q)$ for some nontrivial p-subgroup Q. If $P \leq H \leq G$, then we say that H **controls G-fusion** of P in G if, whenever $x \sim_G y$ for $x, y \in P$, we have $x \sim_H y$. Burnside's theorem [17, Theorem 7.1.1] shows that if P is abelian, then $N_G(P)$ controls G-fusion of P in G; however, in general, $N_G(P)$ need not control G-fusion of P in G.

Given a subgroup $K \leq G$ and a right transversal $T = [G/K]$, we define a right action of G on T induced by right multiplication; that is to say, if $t \in T$ and $g \in G$, then we define $t \cdot g$ to be the element in T such that $[t \cdot g] = [tg]$. A direct calculation shows that

$$\prod_{t \in T} tg(t \cdot g)^{-1} \in K$$

for all $g \in G$ and hence the map $v_{G,K} : G \to K/K'$ defined by

$$v_{G,K}(g) = \left[\prod_{t \in T} tg(t \cdot g)^{-1} \right].$$

is well-defined. We refer to this map as the **transfer homomorphism** of K in G; note that it is indeed a homomorphism and is independent of the choice of transversal T (see [21, Theorem 5.1]). We set $A^p(G) = \ker v_{G,P}$. Given subgroups $P \leq H \leq G$, we say that H **controls p-transfer** of P in G if $A^p(H) = H \cap A^p(G)$.

We define the **focal subgroup** of H in G to be

$$\mathrm{Foc}_H(G) = \langle x^{-1}y : x, y \in H, x \sim_G y \rangle.$$

Relating to the focal subgroup and control of p-transfer, we have the following results.

Proposition 1.2.4. Suppose that $P \leq H \leq G$ and $P \in \mathrm{Syl}_p(G)$.

(i) We have $\mathrm{Foc}_P(G) = P \cap G' = P \cap A^p(G)$. Moreover, there exists a normal subgroup $K \lhd G$ such that $\mathrm{Foc}_P(G) \leq K$ and $G/K \cong P/\mathrm{Foc}_P(G)$.

(ii) (Grün's Theorem) We have

$$\mathrm{Foc}_P(G) = \langle P \cap N_G(P)', P \cap Q' : Q \in \mathrm{Syl}_p(G)\rangle.$$

(iii) If H controls G-fusion of P in G, then H controls p-transfer of P in G.

Proof. Part (i) is a combination of Theorem 7.3.1 in [17] and Theorem 5.21 in [21], and part (ii) is Theorem 7.4.2 in [17]. Part (iii) is Corollary 5.22 in [21]. □

We shall say that $Q \leq P$ is **weakly closed in** P **with respect to** G if, whenever $Q^x \leq P$ for some $x \in G$, we have $Q^x = Q$; we shall sometimes simply say that Q is **weakly closed** in G if the context is clear. If $Z(P)$ is weakly closed in P with respect to G, then we say that G is p-**normal**.

Proposition 1.2.5. Suppose that G is a finite group, $P \in \mathrm{Syl}_p(G)$ and $Q \leq P$.

(i) If $Q \lhd P$, then Q is normal in G if and only if Q is subnormal and weakly closed in G.

(ii) If $N \lhd G$ and $\overline{G} = G/N$, then $\overline{Q}$ is weakly closed in $\overline{G}$ if Q is weakly closed in G; furthermore, if N and Q have coprime orders, then $\overline{Q}$ is weakly closed in $\overline{G}$ if and only if Q is weakly closed in G.

Proof. Both these properties are well-known; for a proof of (i), we refer the reader to [14, Lemma 2.3]. Part (ii) is immediate from the definitions. □

1.2.3 p-Nilpotence

Let $P \in \mathrm{Syl}_p(G)$. If $G = O_{p'}(G) \rtimes P$, then we say that G is p-**nilpotent**, and we say that $O_{p'}(G)$ is a **normal** p-**complement** of P in G. The following theorem of Frobenius provides an equivalent characterisation for G to be p-nilpotent in terms of control of G-fusion; in order to get the statement, we combine Theorems 5.25 and 5.26 in [21].

Theorem 1.2.6 (Frobenius' Theorem)**.** Suppose that G is a finite group and $P \in \mathrm{Syl}_p(G)$. Then the following are equivalent:

(i) G is p-nilpotent;

(ii) P controls G-fusion of P in G;

(iii) $N_G(Q)$ is p-nilpotent for every $Q \leq P$, $Q \neq \{1\}$;

(iv) $N_G(Q)/C_G(Q)$ is a p-group for every $Q \leq P$.

In addition to Frobenius's Theorem, the following is worth mentioning.

Theorem 1.2.7 (Burnside's Theorem)**.** Suppose that G is a finite group, $P \in \mathrm{Syl}_p(G)$ and $P \leq Z(N_G(P))$. Then G is p-nilpotent.

Chapter 2

Scott Modules

In this chapter, we put in place the definition and main properties of a class of kG-modules, known as *Scott modules*, which arise as certain direct summands of permutation kG-modules. This thesis is primarily concerned with questions related to these objects, so we take some time to establish their existence and cover some of the well-known properties they satisfy.

To this end, we start the chapter with a detailed explanation of what a Scott module is, including a proof of the existence of such modules, and establish some of the basic properties they satisfy. In Section 2.3, we prove some lemmas that we will find helpful in the sequel; we follow this up in the final section, where we provide a new result which helps describe the structure of Scott modules in the case where G has a normal Sylow p-subgroup.

2.1 Existence and Basic Properties

We have covered what it means for a kG-module to be considered a permutation kG-module. A related concept goes as follows.

Definition 2.1.1. [5, Definition 0.1] A kG-module $\mathcal{M}$ is said to be a p-**permutation** kG-module if whenever Q is a p-subgroup of G, there exists a k-basis X of $\mathcal{M}$ such that $ux \in X$ for all $u \in Q$ and $x \in X$.

Note that this definition is equivalent to requiring that $\mathcal{M}{\downarrow_P}$ is a permutation kP-module for all $P \in \mathrm{Syl}_p(G)$, and this is sometimes how the definition is presented.

It is not necessary for a p-permutation kG-module to be a permutation kG-module, but any permutation kG-module is obviously a p-permutation kG-module. Thus the class of permutation kG-modules is a subset of the class of p-permutation kG-modules. Parts (i) and (ii) of the following result are familiar properties that p-permutation modules share with permutation modules; on the other hand, (iii) describes a key difference.

Proposition 2.1.2. [5, Proposition 0.2] The following are true.

(i) If $\mathcal{M}$ and $\mathcal{M}'$ are p-permutation kG-modules, then so are the kG-modules $\mathcal{M} \oplus \mathcal{M}'$ and $\mathcal{M} \otimes \mathcal{M}'$.

(ii) If $H \leq G$ and $\mathcal{M}$ is a p-permutation kG-module, then $\mathcal{M}{\downarrow_H}$ is a p-permutation kH-module; moreover, if $\mathcal{N}$ is a p-permutation kH-module, then $\mathcal{N}{\uparrow^G}$ is a p-permutation kG-module.

(iii) Any direct summand of a p-permutation kG-module is a p-permutation kG-module.

In fact, the relationship between permutation and p-permutation modules is stronger than a first look might suggest. The following well-known characterisation makes this clear: indeed, the class of p-permutation kG-modules coincides with the class of direct summands of permutation kG-modules.

Theorem 2.1.3. [5, 0.4] Let $\mathcal{M}$ be an indecomposable kG-module. The following are equivalent:

(i) $\mathcal{M}$ is a p-permutation kG-module;

(ii) there exists a subgroup $H \leq G$ such that $\mathcal{M}$ is isomorphic to a direct summand of $k[G/H]$;

(iii) $\mathcal{M}$ is a trivial source kG-module.

Suppose now that $H \leq G$. A key observation concerning the permutation kG-module $k[G/H]$ is that it contains a unique submodule isomorphic to k_G, namely:

$$\mathcal{V} = \left\langle \sum_{x \in [G/H]} [x] \right\rangle.$$

It should be clear that $\mathcal{V}$ is a submodule isomorphic to k_G, since $[G/H]$ is a transitive G-set. The fact that $\mathcal{V}$ is the *unique* submodule isomorphic to k_G can be deduced from Frobenius reciprocity, since we have

$$\begin{aligned} k_H &\cong \mathrm{Hom}_{kH}(k_H, k_H) \cong \mathrm{Hom}_{kH}((k_G){\downarrow_H}, k_H) \\ &\cong \mathrm{Hom}_{kG}(k_G, (k_H){\uparrow^G}) \cong \mathrm{Hom}_{kG}(k_G, k[G/H]), \end{aligned}$$

so that $\dim(\mathrm{Hom}_{kG}(k_G, k[G/H])) = 1$. A consequence of this observation is that $k[G/H]$ must contain a unique direct summand in any indecomposable decomposition, say $\mathcal{S}$, such that k_G is a submodule of $\mathrm{soc}(\mathcal{S})$. We refer to this module as a *Scott module*; the following result, due to Scott and Alperin, describes some key properties it must satisfy.

Theorem 2.1.4 (Scott-Alperin Theorem)**.** [31, Theorem 4.8.4] Let G be a group and $H \leq G$. Suppose that $Q \in \mathrm{Syl}_p(H)$. Then there exists an indecomposable direct summand $\mathcal{S}$ of $k[G/H]$, which is uniquely determined (up to isomorphism) by any of the following three properties:

(i) k_G is a submodule of $\mathrm{soc}(\mathcal{S})$;

(ii) k_G is a submodule of $\mathrm{top}(\mathcal{S}) = \mathcal{S}/\mathrm{rad}(\mathcal{S})$;

(iii) $\mathcal{S}$ has vertex Q, and if $f = f(G, N_G(Q))$ denotes the Green correspondence between G and $N_G(Q)$, then $f(\mathcal{S})$ is the projective cover of $k_{N_G(Q)/Q}$ when viewed as a $k[N_G(Q)/Q]$-module.

Furthermore, in any indecomposable decomposition of $k[G/H]$, there exists a unique indecomposable direct summand isomorphic to $\mathcal{S}$.

Proof. We have already observed that $k[G/H]$ contains a unique (up to isomorphism) indecomposable direct summand $\mathcal{S}$ satisfying (i), so all we need do is show that the three properties listed above are satisfied by $\mathcal{S}$ and $\mathcal{S}$ alone. Let $\mathcal{V}$ denote the unique submodule of $k[G/H]$ which is isomorphic to k_G; we first show that (i) implies (iii), so assume that $\mathcal{V}$ is a submodule of $\mathrm{soc}(\mathcal{S})$. Since $\mathcal{S}$ is a direct summand of $k[G/H]$, it follows that $\mathcal{S}$ is relatively H-projective and hence $\mathrm{vx}(\mathcal{S}) \leq H$, so without loss of generality, we have $\mathrm{vx}(\mathcal{S}) \leq Q$. On the other hand, if we write $[G/H] = \{1\} \cup X$, where $X = [G/H] - \{1\}$, then

$$(k[G/H]){\downarrow_Q} \cong \mathcal{V}{\downarrow_Q} \oplus \mathcal{X},$$

where $\mathcal{X}$ denotes the kQ-module generated by X. Since $\mathcal{V}{\downarrow_Q} \leq \mathcal{S}{\downarrow_Q}$ and $\mathcal{V}{\downarrow_Q}$ is a direct summand of $(k[G/H]){\downarrow_Q}$, it follows that $\mathcal{V}{\downarrow_Q}$ is a direct summand of $\mathcal{S}{\downarrow_Q}$. Thus, if $\mathrm{vx}(\mathcal{S}) = R < Q$, then $\mathcal{S}$ is a direct summand of $(\mathcal{S}{\downarrow_R}){\uparrow^G}$ and we see that k_Q is a direct summand of

$$\bigoplus_{t \in [Q \backslash G/R]} ((\mathcal{S}{\downarrow_R})^t){\downarrow_{Q \cap tRt^{-1}}}){\uparrow^Q},$$

by Mackey's Theorem. But this then implies that k_Q is relatively projective for some proper subgroup of Q, which is impossible, since $\mathrm{vx}(k_Q) = Q$. Thus $\mathrm{vx}(\mathcal{S}) = Q$.

By Frobenius reciprocity, we have

$$\mathrm{Hom}_{kG}(k_G, f(\mathcal{S}){\uparrow^G}) \cong \mathrm{Hom}_{N_G(Q)}(k_{N_G(Q)}, f(\mathcal{S}))$$

and the left hand side is clearly nonzero, since $\mathcal{S}$ is a direct summand of $f(\mathcal{S}){\uparrow^G}$. Thus $k_{N_G(Q)}$ is a submodule of $f(\mathcal{S})$. By the Green correspondence, we know that $f(\mathcal{S})$ has vertex Q; moreover, Mackey's Theorem gives

$$(k[G/H]){\downarrow_{N_G(Q)}} \cong \bigoplus_{t \in [N_G(Q) \backslash G/H]} k[N_G(Q)/N_G(Q) \cap tHt^{-1}].$$

Thus $f(\mathcal{S})$ is a direct summand of $k[N_G(Q)/L]$ for some $L \leq N_G(Q)$ with $Q \leq L$. Since $Q \lhd N_G(Q)$, it follows that Q acts trivially on $k[N_G(Q)/L]$ and hence we may regard $f(\mathcal{S})$ as a $k[N_G(Q)/Q]$-module, which we denote by $\overline{f(\mathcal{S})}$. If $\varphi : \mathcal{M} \to \overline{f(\mathcal{S})}$ is a surjective $k[N_G(Q)/Q]$-homomorphism, then the corresponding homomorphism $\mathrm{Inf}_{N_G(Q)/Q}^{N_G(Q)}(\varphi) : \mathrm{Inf}_{N_G(Q)/Q}^{N_G(Q)}(\mathcal{M}) \to \mathrm{Inf}_{N_G(Q)/Q}^{N_G(Q)}(\overline{f(\mathcal{S})})$ induced by inflation is Q-split; since $\mathrm{Inf}_{N_G(Q)/Q}^{N_G(Q)}(\overline{f(\mathcal{S})}) = f(\mathcal{S})$ and $\mathrm{vx}(\overline{f(\mathcal{S})}) = Q$, it follows that $\mathrm{Inf}_{N_G(Q)/Q}^{N_G(Q)}(\varphi)$ splits as a $k[N_G(Q)]$-homomorphism. Thus φ is a split $k[N_G(Q)/Q]$-homomorphism and we see that $\overline{f(\mathcal{S})}$ is a projective $k[N_G(Q)/Q]$-module which contains $k_{N_G(Q)/Q}$ as a submodule; it is therefore the projective cover of $k_{N_G(Q)/Q}$, as required.

To show that (iii) implies (i), note that f is a bijection, so if $f(\overline{\mathcal{S}})$ satisfies (iii) for another direct summand $\overline{\mathcal{S}}$, we have $f(\overline{\mathcal{S}}) \cong f(\mathcal{S})$ and hence $\overline{\mathcal{S}} \cong \mathcal{S}$.

Finally, we show that (i) and (ii) are equivalent. In order to do so, note that $f(\mathcal{S})$ is self-dual as a $k[N_G(Q)/Q]$-module, since it is a projective $k[N_G(Q)/Q]$-module, so by Proposition 1.1.15 we have

$$f(\mathcal{S}^*) \cong f(\mathcal{S})^* \cong f(\mathcal{S})$$

and hence $\mathcal{S}^* \cong \mathcal{S}$. Since $k_G \leq \mathrm{soc}(\mathcal{S})$ if and only if $k_G \leq \mathrm{soc}(\mathcal{S})^*$ and $\mathrm{top}(\mathcal{S}^*) \cong \mathrm{soc}(\mathcal{S})^*$, it follows that $k_G \leq \mathrm{soc}(\mathcal{S})$ if and only if $k_G \leq \mathrm{top}(\mathcal{S})$, and we conclude that (i) is equivalent to (ii). □

The module $\mathcal{S}$ defined in the above result is a direct summand of $k[G/H]$ and hence is a p-permutation kG-module. As alluded to previously, we refer to it as the **Scott module of H in G**, and we denote it by $\mathcal{S}(G, H)$; note that $\mathcal{S}(G, H)$ is uniquely determined by the subgroup H and not by the particular choice of Sylow p-subgroup Q of H we make. As a first step towards narrowing down which Scott modules are worth studying, we have the following, which allows us to restrict attention to when H itself is a p-group.

Proposition 2.1.5. [31, Corollary 8.5] Let G be a finite group and $H, L \leq G$. Furthermore, let $Q \in \mathrm{Syl}_p(H)$ and $P \in \mathrm{Syl}_p(L)$. Then:

(i) $\mathcal{S}(G, H) \cong \mathcal{S}(G, Q)$;

(ii) $\mathcal{S}(G, H) \cong \mathcal{S}(G, L)$ if and only if $Q \sim_G P$.

Proof. By (iii) of the Scott-Alperin Theorem, it follows that $\mathcal{S}(G, H)$ and $\mathcal{S}(G, Q)$ are the Green correspondents of the same $k[N_G(Q)]$-module and are hence isomorphic, so (i) follows. Thus, all we need do for (ii) is show that $\mathcal{S}(G, Q) \cong \mathcal{S}(G, P)$ if and only if $Q \sim_G P$. If $\mathcal{S}(G, Q) \cong \mathcal{S}(G, P)$, then Q and P are both vertices of $\mathcal{S}(G, Q)$ and hence $Q \sim_G P$, by Theorem 1.1.9. On the other hand, if $Q \sim_G P$, then $k[G/Q] \cong k[G/P]$ by Proposition 1.1.19 (v) and hence $\mathcal{S}(G, Q)$ and $\mathcal{S}(G, P)$ are both direct summands of $k[G/Q]$ containing k_G as a submodule, so they must be isomorphic kG-modules. □

If Q is a p-subgroup of G, then $\mathcal{S}(G, Q)$ is referred to as the **Scott module of G with vertex Q** and is sometimes denoted in the literature by $\mathcal{S}(Q)$. In general, it is not obvious what $\mathcal{S}(G,Q)$ will look like for an arbitrary p-subgroup Q of a finite group G; the following covers a couple of special cases.

Proposition 2.1.6. Suppose that G is a finite group and $P \in \mathrm{Syl}_p(G)$. Then:

(i) $\mathcal{S}(G,P) \cong k_G$;

(ii) $\mathcal{S}(G,\{1\}) \cong \mathcal{P}(k_G)$.

Proof. For (i), note that $\mathcal{S}(G,P) \cong \mathcal{S}(G,G) \cong k_G$, whence the result follows. The second part is immediate from (iii) of the Scott-Alperin Theorem, since $f(G,N_G(Q))$ is trivial when $Q = \{1\}$ and hence $f(\mathcal{S}(G,\{1\})) = \mathcal{S}(G,\{1\})$ is the projective cover of k_G. □

From now on, we shall assume that G is a finite group, and let $P \in \mathrm{Syl}_p(G)$ denote a fixed Sylow p-subgroup of G. If we wish to refer to a general p-subgroup of G, we do so via the symbol Q; as we have shown above, we may assume without loss of generality that $Q \leq P$ when studying $\mathcal{S}(G,Q)$, and we shall adopt this convention.

Remarks 2.1.7. (i) Since $k[P/Q]$ is indecomposable, it follows that $\mathcal{S}(P,Q) \cong k[P/Q]$. In general, it is unpredictable when $\mathcal{S}(G,Q) \cong k[G/Q]$ for more arbitrary finite groups G; we shall provide some examples of this phenomenon in the last section of this chapter (see Corollary 2.4.3).

(ii) It follows from the Scott-Alperin Theorem that $\mathcal{S}(G,Q)$ is a relatively Q-projective cover of k_G, which gives another equivalent way of viewing Scott modules.

We finish the section with a result that gives a well-known alternative formulation of a Scott module.

Theorem 2.1.8. Suppose that G is a finite group and Q is a p-subgroup of G. Then there exists a unique indecomposable p-permutation kG-module $\mathcal{S}$ such that $\mathrm{vx}(\mathcal{S}) = Q$ and k_G is a submodule of $\mathcal{S}$; in particular, $\mathcal{S} \cong \mathcal{S}(G,Q)$.

Proof. If $\mathcal{S}$ is a p-permutation kG-module such that $\mathrm{vx}(\mathcal{S}) = Q$, then by Theorem 2.1.3, it follows that $\mathcal{S}$ is a direct summand of $(k_Q){\uparrow^G} \cong k[G/Q]$. Thus, if $\mathcal{S}$ contains a submodule isomorphic to k_G, by the Scott-Alperin Theorem, we have $\mathcal{S} \cong \mathcal{S}(G,Q)$. □

2.2 Behaviour Under Induction and Restriction

Given a finite group G and a p-subgroup $Q \leq G$, the most direct path to calculating $\mathcal{S}(G, Q)$ would be as follows:

(i) find an indecomposable decomposition of $k[G/Q]$;

(ii) determine which direct summand in this indecomposable decomposition contains a submodule isomorphic to k_G.

Modern day algebraic programming languages are well-equipped to handle both of these steps, but the first involves increasingly lengthy computation times as $|G : Q|$ grows larger. Our first result in this section provides a means of avoiding this inefficiency when calculating $\mathcal{S}(G, Q)$.

Proposition 2.2.1. Suppose that G is a finite group and $Q \leq H \leq G$. Then $\mathcal{S}(G, Q)$ is a direct summand of $\mathcal{S}(H, Q)\uparrow^G$, and there is precisely one direct summand isomorphic to $\mathcal{S}(G, Q)$ in any indecomposable decomposition of $\mathcal{S}(H, Q)\uparrow^G$.

Proof. Since $\mathcal{S}(H, Q)$ is a direct summand of $k[H/Q]$, it follows from Lemmas 1.1.4 and 1.1.19 that $\mathcal{S}(H, Q)\uparrow^G$ is a direct summand of $k[G/Q]$. Furthermore, $\mathcal{S}(H, Q)\uparrow^G$ contains a submodule isomorphic to k_G, since $\mathcal{S}(H, Q)$ contains a submodule isomorphic to k_H. Thus, by the Krull-Schmidt Theorem and the Scott-Alperin Theorem, we know that in any indecomposable decomposition of $\mathcal{S}(H, Q)\uparrow^G$ there is precisely one direct summand which is isomorphic to $\mathcal{S}(G, Q)$, as required. □

Remark 2.2.2. This provides a more efficient means of calculating $\mathcal{S}(G, Q)$, through repeated induction. To be more precise, suppose that we have a chain of subgroups:

$$P = H_0 < H_1 < H_2 < \cdots < H_{r-1} < H_r = G.$$

Then the above shows that $\mathcal{S}(H_{i+1}, Q)$ is a direct summand of $\mathcal{S}(H_i, Q)\uparrow_{H_i}^{H_{i+1}}$ for $0 \leq i \leq r-1$. Using this approach to calculating $\mathcal{S}(H_i, Q)$, we keep the dimension of the modules we are decomposing much smaller than a more direct approach.

A partial converse to the above statement is given by the following, which was proved in [6].

Proposition 2.2.3. Suppose that G is a finite group and $Q \leq H \leq G$. Assume that $\mathcal{V}$ is an indecomposable direct summand of $k[H/L]$ for some subgroup $L \leq H$. If $\mathcal{S}(G,Q)$ is a direct summand of $\mathcal{V}{\uparrow}^G$, then $Q \in \mathrm{Syl}_p(L)$ and $\mathcal{V} \cong \mathcal{S}(H,Q)$.

Proof. By Lemma 1.1.4, we know that $\mathcal{S}(G,Q)$ is a direct summand of $k[G/L]$. Thus, $\mathcal{S}(G,Q)$ is a direct summand of $k[G/L]$ containing k_G as a submodule; by (iii) of the Scott-Alperin theorem, we therefore know that $Q \in \mathrm{Syl}_p(L)$. Thus $\mathcal{S}(H,L) \cong \mathcal{S}(H,Q)$, by Proposition 2.1.5. Furthermore, by the hypothesis and Proposition 2.2.1, we know that $\mathcal{S}(G,Q)$ is a direct summand of both $\mathcal{S}(H,Q){\uparrow}^G$ and $\mathcal{V}{\uparrow}^G$. If $\mathcal{V} \ncong \mathcal{S}(H,Q)$, then we have

$$k[H/L] \cong \mathcal{V} \oplus \mathcal{S}(H,Q) \oplus \mathcal{X}$$

for some direct summand $\mathcal{X}$ and hence

$$k[G/L] \cong \mathcal{V}{\uparrow}^G \oplus \mathcal{S}(H,Q){\uparrow}^G \oplus \mathcal{X}{\uparrow}^G.$$

Thus $k[G/L]$ contains two distinct direct summands in an indecomposable decomposition which are isomorphic to k_G, a contradiction. So $\mathcal{V} \cong \mathcal{S}(H,Q)$, as required. □

We can rephrase the previous two results slightly differently in terms of the concept of *multiplicities of Scott modules*. Given a kG-module $\mathcal{M}$, we say that the **multiplicity** of $\mathcal{S}(G,Q)$ in $\mathcal{M}$ is the number of distinct direct summands appearing in an indecomposable decomposition of $\mathcal{M}$ which are isomorphic to $\mathcal{S}(G,Q)$. Proposition 2.2.1 therefore states that the multiplicity of $\mathcal{S}(G,Q)$ in $\mathcal{S}(H,Q){\uparrow}^G$ is one; on the other hand, Proposition 2.2.3 tells us that if the multiplicity of $\mathcal{S}(G,Q)$ in $\mathcal{V}{\uparrow}^G$ is nonzero and $\mathcal{V}$ is an indecomposable p-permutation kH-module, then $\mathcal{V} \cong \mathcal{S}(H,Q)$. The multiplicity of $\mathcal{S}(G,Q)$ in a general kG-module $\mathcal{M}$ is given by the following result, due to Green (see [15, 4.10]). Before stating the result, we set up some notation: given a kG-module $\mathcal{M}$, we set $I_H(\mathcal{M})$ to be the H-fixed points of $\mathcal{M}$, i.e.,

$$I_H(\mathcal{M}) = \{m \in \mathcal{M} : hm = m \text{ for all } h \in H\};$$

moreover, we let $I_{H,G}(\mathcal{M}) = \mathrm{tr}_H^G(I_H(\mathcal{M}))$, where tr_H^G denotes the relative trace map, i.e., the map $\mathrm{tr}_H^G : I_H(\mathcal{M}) \to \mathcal{M}$ given by

$$\mathrm{tr}_H^G(m) = \sum_{t \in [G/H]} tm$$

for all $m \in I_H(\mathcal{M})$.

Theorem 2.2.4. Suppose that $\mathcal{M}$ is a kG-module. The multiplicity of $S(G,Q)$ in $\mathcal{M}$ is equal to $\dim(I_{Q,G}(\mathcal{M})/J)$, where

$$J = \{x \in I_{Q,G}(\mathcal{M}) : \alpha(x) = 0 \text{ for all } \alpha \in I_Q(\mathcal{M}^*)\}.$$

We now turn our attention towards what happens when we restrict $\mathcal{S}(G,Q)$ to a subgroup H. The following serves as a complement to Proposition 2.2.1.

Proposition 2.2.5. Suppose that G is a finite group and $Q \leq H \leq G$. Then $\mathcal{S}(H,Q)$ is a direct summand of $\mathcal{S}(G,Q){\downarrow_H}$.

Proof. Since $\mathcal{S}(G,Q)$ is a relatively Q-projective cover of k_G, it follows that there exists a surjective kG-homomorphism $\delta : \mathcal{S}(G,Q) \to k_G$ which is Q-split. In particular, $\delta_H : \mathcal{S}(G,Q){\downarrow_H} \to k_H$ is a surjective kH-homomorphism which is Q-split and $\mathcal{S}(G,Q){\downarrow_H}$ is relatively Q-projective, since $\mathcal{S}(G,Q)$ has vertex Q. It follows from Proposition 1.1.12 that $\mathcal{S}(H,Q)$ is a direct summand of $\mathcal{S}(G,Q){\downarrow_H}$, as required. □

We note one key difference between Propositions 2.2.1 and 2.2.5; $\mathcal{S}(H,Q){\uparrow^G}$ has a *single* direct summand in any given indecomposable decomposition which is isomorphic to $\mathcal{S}(G,Q)$, whereas $\mathcal{S}(G,Q){\downarrow_H}$ may contain multiple distinct direct summands isomorphic to $\mathcal{S}(H,Q)$ in an indecomposable decomposition, by Mackey's Theorem.

The following more general result covers the case where Q is not necessarily a subgroup of H.

Proposition 2.2.6. [22, Theorem 1.7] Let G be a finite group, $H \leq G$ and Q be a p-subgroup of G. Suppose that R is an element which is maximal in the poset $\{Q^g \cap H : g \in G\}$. Then $\mathcal{S}(H,R)$ is a direct summand of $\mathcal{S}(G,Q){\downarrow_H}$.

Proof. By assumption, $R = Q^x \cap H$ for some $x \in G$. We shall prove the result by induction on $|Q|/|R|$. If $|Q|/|R| = 1$, then we have $|Q| = |R|$, and it follows that

$R \leq H$, so the result follows from Proposition 2.2.5. We may therefore assume that $|Q| > |R|$. Set $N = N_G(R)$ and let S be a maximal element in the poset

$$\Omega = \{Q^g \cap N : R < Q^g \cap N, g \in G\}.$$

We note that Ω is nonempty, since $R < Q^x$ and hence $N_{Q^x}(R) > R$, so $Q^x \cap N = N_{Q^x}(R) > R$. Thus S exists and $S = Q^y \cap N$ for some $y \in G$. Note that $R < S$ and hence $|Q|/|S| < |Q|/|R|$, so by the inductive hypothesis, it follows that $\mathcal{S}(G,Q){\downarrow_N}$ has a direct summand $\mathcal{U} \cong \mathcal{S}(N,S)$. In particular, $\mathcal{U}{\downarrow_{N_H(R)}}$ has a direct summand $\overline{\mathcal{U}}$ which contains a submodule isomorphic to $k_{N_H(R)}$; moreover, by Mackey's Theorem, we have

$$(k[N/S]){\downarrow_{N_H(R)}} \cong \bigoplus_{n \in [N_H(R) \backslash N/S]} k[N_H(R)/N_H(R) \cap nSn^{-1}].$$

Note that for all $n \in N$ we have

$$R \leq N_H(R) \cap nSn^{-1} \leq H \cap (Q^{yn^{-1}} \cap N) \leq H \cap Q^{yn^{-1}},$$

so $R = N_H(R) \cap nSn^{-1}$. Thus

$$(k[N/S]){\downarrow_{N_H(R)}} \cong \bigoplus_{n \in [N_H(R) \backslash N/S]} k[N_H(R)/R]$$

and it follows that $\overline{\mathcal{U}} \cong \mathcal{S}(N_H(R), R)$. In particular, there exists a direct summand $\mathcal{V}$ of $\mathcal{S}(G,Q){\downarrow_H}$ such that $\mathcal{V}{\downarrow_{N_H(R)}}$ contains $\overline{\mathcal{U}}$ as a direct summand. By the Green correspondence and [1, Lemma 11.3], it follows that $\mathrm{vx}(\mathcal{V}) = \mathrm{vx}(\overline{\mathcal{U}}) = R$ and hence $\mathcal{V} \cong \mathcal{S}(H,R)$, since $\mathcal{V}$ is the unique direct summand of $\overline{\mathcal{U}}{\uparrow^H}$ with vertex R. Thus, $\mathcal{V} \cong \mathcal{S}(H,R)$ and we are done. □

2.3 Bounding $\dim \mathcal{S}(G,Q)$

In this section, we consider group-theoretic properties that allow us to place bounds on the size of $\dim \mathcal{S}(G,Q)$, along with related questions and proofs. We shall start with the following simple observation, which motivates one of the central definitions of the thesis.

Proposition 2.3.1. Suppose that G is a finite group, $P \in \mathrm{Syl}_p(G)$ and $Q \leq P$. Then $k[P/Q]$ is an indecomposable direct summand of $\mathcal{S}(G,Q){\downarrow_P}$. In particular:

(i) $\dim \mathcal{S}(G,Q) \geq |P:Q|$;

(ii) if $\dim \mathcal{S}(G,Q) = |P:Q|$, then $\mathcal{S}(G,Q){\downarrow_P} \cong k[P/Q]$;

(iii) $|P:Q|$ divides $\dim \mathcal{S}(G,Q)$.

Proof. The first statement follows from Proposition 2.2.5. Parts (i) and (ii) are then immediate corollaries. For part (iii), by Mackey's Theorem we have

$$k[G/Q]{\downarrow_P} \cong ((k_Q){\uparrow^G}){\downarrow_P} \cong \bigoplus_{s\in[P\backslash G/Q]} (k_{P\cap sQs^{-1}}){\uparrow^P} \cong \bigoplus_{s\in[P\backslash G/Q]} k[P/P\cap sQs^{-1}].$$

Now we note that $\dim k[P/P\cap sQs^{-1}]$ is clearly a multiple of $|P:Q|$ for all $s \in G$ and hence any direct summand of $k[G/Q]$ has as its dimension a multiple of $|P:Q|$. Thus part (iii) holds. $\square$

We see therefore that if $\dim \mathcal{S}(G,Q)$ is minimal, then $\mathcal{S}(G,Q)$ is just an extension of $k[P/Q]$; describing the structure of $\mathcal{S}(G,Q)$ then amounts to describing this extension. The above imposes a lower bound on $\dim \mathcal{S}(G,Q)$; the following can be used to determine an upper bound, and also gives a sufficient condition for when $\mathcal{S}(G,Q){\downarrow_P} \cong k[P/Q]$.

Proposition 2.3.2. Suppose that G is a finite group, $P \in \mathrm{Syl}_p(G)$ and $Q \leq P$.

(i) If $H \leq G$ and $Q \in \mathrm{Syl}_p(H)$, then $\dim \mathcal{S}(G,Q) \leq |G:H|$.

(ii) If $|G:H| = |P:Q|$ for a subgroup $H \leq G$ and $Q \leq H$, then $\mathcal{S}(G,Q){\downarrow_P} \cong k[P/Q]$ and $\mathcal{S}(G,Q)$ is a permutation kG-module.

Proof. Since $Q \in \mathrm{Syl}_p(H)$, it follows that $\mathcal{S}(G,Q) \cong \mathcal{S}(G,H)$; but $\mathcal{S}(G,H)$ is a direct summand of $k[G/H]$ and $\dim k[G/H] = |G:H|$, whence the first part follows. For the second part, note that if $|G:H| = |P:Q|$ and $Q \leq H$, then $|H| = |Q||G:P|$, so $Q \in \mathrm{Syl}_p(H)$. Hence $\dim \mathcal{S}(G,Q) \leq |P:Q|$ by (i) and it follows that $\dim \mathcal{S}(G,Q) = |P:Q|$, so $\mathcal{S}(G,Q){\downarrow_P} \cong k[P/Q]$; moreover, $\mathcal{S}(G,Q)$ is a direct summand of $k[G/H]$ and we have shown that $\dim \mathcal{S}(G,Q) = \dim k[G/H]$, so in fact $\mathcal{S}(G,Q) \cong k[G/H]$ and hence $\mathcal{S}(G,Q)$ is a permutation kG-module. $\square$

Example 2.3.3. Let $G = \Sigma_{p^2}$ and P be a Sylow p-subgroup of G for an odd prime p. Then it is well-known that $P \cong C_p \wr C_p$; explicitly, we have $P = E \rtimes Q$, where $E \cong E_{p^p}$ is an elementary abelian group of order p^p generated by the p-cycles

$$\sigma_i = (1 + (i-1)p, 2 + (i-1)p, \ldots, p + (i-1)p)$$

for $1 \leq i \leq p$ and $Q \cong C_p$ is generated by the element

$$w = \prod_{i=1}^{p} (i, i+p, \ldots, i + (p-1)p).$$

In particular, note that ${}^{w}\sigma_i = \sigma_{i+1}$ for $1 \leq i \leq p-1$ and ${}^{w}\sigma_p = \sigma_1$.

For $1 \leq j \leq p-1$, define

$$n_j = \prod_{i=1}^{p} (i, i+p, \ldots, i+jp);$$

Note that $n_{p-1} = w$. We may consider Σ_p as a subgroup of Σ_{p^2}, consisting of those permutations acting trivially on $\{p+1, \ldots, p^2\}$; note that $P' = \langle (1, 2, \ldots, p) \rangle \in \mathrm{Syl}_p(\Sigma_p)$. Since $N_{\Sigma_p}(P') = P' \rtimes C_{p-1}$, there exists an element $\tau \in N_{\Sigma_p}(P')$ of order $p-1$.

We set $T = \langle \tau, {}^{w}\tau, \ldots, {}^{w^{p-1}}\tau \rangle$ and $N = \langle n_j : 1 \leq j \leq p-1 \rangle$. Then each n_j acts on T via conjugation by permuting the p distinct conjugates of τ that generate T, so $H = TN$ is a subgroup of G. Moreover, assume that $n \in N$ and $n = t$ for some $t \in T$; thus

$$n = \prod_{k=0}^{p-1} ({}^{w^k}\tau)^{\alpha_k}$$

with $0 \leq \alpha_k < p-1$. If $1 \leq i \leq p$ and $0 \leq k \leq p-1$, we have $n(i+kp) \in \{i, i+p, \ldots, i+(p-1)p\}$ and $t(i+kp) \in \{1+kp, 2+kp, \ldots, p+kp\}$, so $n(i+kp) = i+kp$ for all i and hence $\alpha_k = 0$ for all k, so $n = 1$. Thus $N \cap T = \{1\}$ and it follows that $H = T \rtimes N$. If

$$\overline{n_j} = \prod_{i=1}^{p} (i + (j-1)p, i + jp)$$

for $1 \leq j \leq p-1$, then a direct calculation shows that $\overline{n_j} = (\overline{n_{j-1}} \cdots \overline{n_1}) n_j$ and since $\overline{n_1} = n_1$, it follows that each $\overline{n_j} \in N$. We may now check the relations in the Coxeter presentation given in [11, 6.2] to verify that $N \cong \Sigma_p$. Thus $|H| = (p-1)^p p!$ and

$Q \leq H$, so H contains Q as a Sylow p-subgroup. It follows from Proposition 2.3.2 (i) that

$$\dim \mathcal{S}(G,Q) \leq |G:H| = \frac{p^2!}{(p-1)^p p!}.$$

Remark 2.3.4. The converse to (ii) of Proposition 2.3.2 is not necessarily true. That is, if $Q \leq P \leq G$ and $\mathcal{S}(G,Q)\downarrow_P \cong k[P/Q]$, then G does not necessarily contain a subgroup H such that $Q \in \mathrm{Syl}_p(H)$ and $|G:H|$ is a prime power.

An example can be found using MAGMA: let $S = \Sigma_8, p = 7$ and $Q = \{1\}$. Then S contains a subgroup G of order $336 = 2^4 \cdot 3 \cdot 7$, namely:

$$G = \langle (1,7)(2,3)(4,5)(6,8), (1,6,2,4,7,8) \rangle.$$

Moreover, if $P \in \mathrm{Syl}_7(G)$, then $\mathcal{S}(G,Q)\downarrow_P \cong kP \cong k[P/Q]$; however, G contains no subgroup of order $336/7 = 48$.

It is possible for $\mathcal{S}(G,Q)\downarrow_P \cong k[P/Q]$ whilst $\mathcal{S}(G,Q)$ is not a permutation kG-module. Indeed, in the previous example, we know that $\mathcal{S}(G,Q)\downarrow_P \cong kP$, but $\mathcal{S}(G,Q)$ cannot be a permutation kG-module, since by Proposition 1.1.19 (ii) we would then have $\mathcal{S}(G,Q) \cong k[G/H]$ for a subgroup $H \leq G$ of order $|G:P|$ and no subgroup of this order exists in G. On the other hand, it might be the case that for every p-subgroup $Q \leq P$, $\mathcal{S}(G,Q)$ is a permutation kG-module such that $\mathcal{S}(G,Q)\downarrow_P \cong k[P/Q]$. The following accounts for one such class of groups.

Proposition 2.3.5. Suppose that G is a finite group, $P \in \mathrm{Syl}_p(G)$ and $N \triangleleft G$ is a p'-group. Let $\overline{G} = G/N$. If $Q \leq P$, then $\mathrm{Inf}_{\overline{G}}^{G}(\mathcal{S}(\overline{G},\overline{Q})) \cong \mathcal{S}(G,Q)$. Furthermore:

(i) if $G = N \rtimes H$ for some subgroup $H \leq G$ with $P \leq H$, then $\mathcal{S}(G,Q)\downarrow_H \cong \mathcal{S}(H,Q)$;

(ii) if G is p-nilpotent, then $\mathcal{S}(G,Q)$ is a permutation kG-module and $\mathcal{S}(G,Q)\downarrow_P \cong k[P/Q]$.

Proof. Let $\mathcal{M} = \mathrm{Inf}_{\overline{G}}^{G}(\mathcal{S}(\overline{G},\overline{Q}))$. Note first that $\mathcal{M}$ is clearly a p-permutation module which contains a submodule isomorphic to k_G, so by Theorem 2.1.8 we are done with the first part if we can show that $\mathrm{vx}(\mathcal{M}) = Q$. Since $\mathcal{S}(\overline{G},\overline{Q})$ is relatively $\overline{Q}$-projective, it follows that $\mathcal{S}(\overline{G},\overline{Q})$ is a direct summand of $\mathcal{N}\uparrow_{\overline{Q}}^{\overline{G}}$ for some $k\overline{Q}$-module $\mathcal{N}$; moreover, by [4, 1.1.3], we have

$$\mathrm{Inf}_{\overline{G}}^{G}(\mathcal{N}\uparrow_{\overline{Q}}^{\overline{G}}) \cong (\mathrm{Inf}_{\overline{Q}}^{QN}(\mathcal{N}))\uparrow_{QN}^{G}.$$

Thus, $\mathcal{M}$ is a direct summand of a relatively QN-projective kG-module and hence $\mathrm{vx}(\mathcal{M}) \leq Q$. On the other hand, suppose that $\mathrm{vx}(\mathcal{M}) = R < Q$ and $\varphi : \mathcal{U} \to \mathcal{S}(\overline{G}, \overline{Q})$ is a surjective $k\overline{G}$-homomorphism which is $\overline{R}$-split. Then the corresponding homomorphism $\mathrm{Inf}_{\overline{G}}^{G}(\varphi) : \mathrm{Inf}_{\overline{G}}^{G}(\mathcal{U}) \to \mathcal{M}$ induced by inflation is R-split and is hence G-split; since N acts trivially on $\mathrm{Inf}_{\overline{G}}^{G}(\mathcal{U})$ and $\mathcal{M}$, it follows that φ is $\overline{G}$-split and hence $\mathcal{S}(\overline{G}, \overline{Q})$ is relatively $\overline{R}$-projective, a contradiction, since $\mathrm{vx}(\mathcal{S}(\overline{G}, \overline{Q})) = \overline{Q}$ and $\overline{R} < \overline{Q}$. So $\mathrm{vx}(\mathcal{M}) = Q$ and it follows that $\mathcal{M} \cong \mathcal{S}(G, Q)$, as required.

If $G = N \rtimes H$, then $H \cong G/N$ and $\mathrm{Inf}_{H}^{G}(\mathcal{S}(H, Q)) \cong \mathcal{S}(G, Q)$, so $\mathcal{S}(G, Q)\downarrow_H \cong \mathcal{S}(H, Q)$ follows. For (ii), note that if G is p-nilpotent, then $\mathcal{S}(G, Q) \cong \mathrm{Inf}_{P}^{G}(k[P/Q])$ and since $k[P/Q]$ is a permutation kP-module, $\mathrm{Inf}_{P}^{G}(k[P/Q])$ is a permutation kG-module. $\square$

Motivated partly by this result, we make the following definition, which will play an important role in this thesis.

Definition 2.3.6. Suppose that G is a finite group. We say that G is ***p*-extendible** if for every $P \in \mathrm{Syl}_p(G)$ and $Q \leq P$, we have $\mathcal{S}(G, Q)\downarrow_P \cong k[P/Q]$.

We record for later reference the observation that p-nilpotent groups are p-extendible.

Corollary 2.3.7. If G is p-nilpotent, then G is p-extendible.

Note that p-nilpotent groups satisfy the stronger condition that $\mathcal{S}(G, Q)$ is a permutation kG-module for all p-subgroups $Q \leq G$, which is not present in the above definition. The following example shows that if p is odd, then any abelian p-group P can be a Sylow p-subgroup of some nontrivial p-extendible group G which is not p-nilpotent.

Example 2.3.8. Suppose that P is an abelian p-group for an odd prime p, let $A = \mathrm{Aut}(P)$ and set $q = p^t$ to be the exponent of P, where $t \in \mathbb{N}$. We let $R = (\mathbb{Z}/q\mathbb{Z})^{\times}$, that is to say, the multiplicative group of the integers modulo q. For each $r \in R$, we may define a map $\varphi_r : P \to P$ by setting $\varphi_r(x) = x^r$ for all $x \in P$; since P is abelian, it follows that $\varphi_r \in A$ for all $r \in R$, and since q is the exponent of P, any two such automorphisms φ_r and φ_s with $1 \leq r < s < q$ are distinct. Furthermore, $|R| = p^t - p^{t-1} = p^{t-1}(p-1)$, so R contains a subgroup of order $p-1$; we call this subgroup H. Let

$$A(H) = \{\varphi_h : h \in H\} \leq A.$$

We now define a semidirect product $G = P \rtimes A(H)$; thus elements of G are of the form (x, φ_r) for some $x \in P$ and $r \in H$, and we have a group product defined by $(x, \varphi_r) \cdot (y, \varphi_s) = (xy^r, \varphi_{rs})$. Note first that G is not p-nilpotent, since if it were then $G = P \times A(H)$ would be abelian; on the other hand, if $x \in P$ and $1 \leq r < s < q$ with $r, s \in H$, we have

$$(x, \varphi_r) \cdot (x, \varphi_s) = (x, \varphi_s) \cdot (x, \varphi_r)$$

if and only if

$$(x^{r+1}, \varphi_{rs}) = (x^{s+1}, \varphi_{sr})$$

if and only if $x^{r+1} = x^{s+1}$, and this clearly cannot hold for all $x \in P$ since $r, s < q$. Furthermore, $P \in \mathrm{Syl}_p(G)$. Finally, G is p-extendible, since if $Q \leq P$, then

$$QH = \{(u, \varphi_h) : u \in Q, h \in H\}$$

is a subgroup of G such that $|QH| = |Q||G : P|$; thus QH is a subgroup of index $|P : Q|$ in G containing Q as a Sylow p-subgroup, so by Proposition 2.3.2 (ii), it follows that $\mathcal{S}(G,Q){\downarrow_P} \cong k[P/Q]$. So G is p-extendible but not p-nilpotent.

Propositions 2.3.2 and 2.3.5 give sufficient conditions for when $\mathcal{S}(G,Q)$ is an extension of $\mathcal{S}(P,Q)$. This can be viewed in the more general context of when $\mathcal{S}(G,Q)$ is an extension of $\mathcal{S}(H,Q)$ for a subgroup H such that $P \leq H \leq G$; Proposition 2.3.5 accounts for one such case, however, this depends on the existence of a normal complement of H in G. In Proposition 2.3.2 (ii), we provided a sufficient condition for when $\mathcal{S}(G,Q){\downarrow_P} \cong k[P/Q]$ which did not require the existence of a normal complement. The following is the best generalisation of this statement we can find

Theorem 2.3.9. Suppose that $Q \leq P \leq H \leq G$, where $P \in \mathrm{Syl}_p(G)$. Suppose that there exists a subgroup $K \leq G$ such that:

(i) $Q \in \mathrm{Syl}_p(K)$;

(ii) $Q = H \cap K$;

(iii) $G = HK$.

Then there exists a direct summand $\mathcal{V}$ of $k[G/Q]$ such that $\mathcal{V}{\downarrow_H} \cong k[H/Q]$.

Proof. Let $T = [H/Q]$ be a left transversal of Q in H. The assumptions on K imply that

$$|G| = \frac{|H||K|}{|H \cap K|} = |H : Q||K|$$

so $G = TK$ and a k-basis for kG can be taken to be TK. If we let $X = [K/Q]$, then it follows that a left transversal of Q in G is given by

$$TX = \{tx : t \in T, x \in X\}$$

and this set is a k-basis of $k[G/Q]$.

For each $t \in T$, define a vector $v(t) \in k[G/Q]$ by

$$v(t) = \sum_{x \in X} [tx]$$

and let $\mathcal{V}$ be the subspace of $k[G/Q]$ spanned by the vectors $v(t)$. For each $g \in G$, we have $g \cdot [tx] = [t'x']$ for some $t' \in T$ and $x' \in X$. Suppose that $g \cdot [tx_1] = [t'x']$ and $g \cdot [tx_2] = [t''x'']$, where $t', t'' \in T$ and $x', x'' \in X$. Then $gtx_1 = t'x'q'$ and $gtx_2 = t''x''q''$ for some $q', q'' \in Q$ and hence we conclude that

$$gt = t'x'q'x_1^{-1} = t''x''q''x_2^{-1}.$$

Since $\langle Q, X \rangle = K$, it follows that $t' = t''$. In particular, $g \cdot v(t) = v(t')$ for some $t' \in T$. Since $\mathcal{V}$ is the subspace spanned by the vectors $v(t)$, we see that $\mathcal{V}$ is a permutation kG-submodule of $k[G/Q]$. Moreover, if $h \in H$ and $ht = t'q'$ for some $t' \in T$ and $q' \in Q$, then since Q acts on X by left multiplication, we see that

$$h \cdot v(t) = h \cdot \left(\sum_{x \in X} [tx] \right) = \sum_{x \in X} [t'q'x] = \sum_{x \in X} [t'x] = v(t'),$$

so $\mathcal{V}{\downarrow_H} \cong k[H/Q]$.

We now show that $\mathcal{V}$ is a direct summand of $k[G/Q]$ by constructing a direct sum complement $\mathcal{W}$. For each $t \in T$ and $x \in X^* := X - \{1\}$, define a vector $w(t, x) \in k[G/Q]$ by

$$w(t, x) = [tx] - [t]$$

and let $\mathcal{W}$ be the subspace generated by the vectors $w(t, x)$. If $t \in T$ and $g \in G$, then $g \cdot [t] = [t'x']$ for some $t' \in T$ and $x' \in X$ and hence for all $x \in X^*$, we have $g \cdot [tx] = [t'x'']$ for some $x'' \in X$ with $x'' \neq x'$. It follows that

$$g \cdot w(t, x) = [t'x''] - [t'x'].$$

If $x' = 1$, then $g \cdot w(t, x) = w(t', x'') \in \mathcal{W}$. On the other hand, if $x'' = 1$, then $g \cdot w(t, x) = -w(t', x') \in \mathcal{W}$. In all other cases, we get

$$g \cdot w(t, x) = [t'x''] - [t'x'] = w(t', x'') - w(t', x') \in \mathcal{W}.$$

It follows that $\mathcal{W}$ is a kG-submodule of $k[G/Q]$.

We claim that $k[G/Q] = \mathcal{V} \oplus \mathcal{W}$. Indeed, if

$$\sum_{t \in T} \alpha_t v(t) + \sum_{t \in T, x \in X^*} \beta_{t,x} w(t, x) = 0$$

for some $\alpha_t, \beta_{t,x} \in k$, then by comparing coefficients of $[t]$, we obtain

$$\alpha_t - \sum_{x \in X^*} \beta_{t,x} = 0$$

for all $t \in T$. On the other hand, by comparing coefficients of $[tx]$ it follows that $\alpha_t + \beta_{t,x} = 0$ for all $t \in T$ and $x \in X^*$. We conclude that $\alpha_t + |X^*|\alpha_t = 0$, i.e., $|X|\alpha_t = 0$. But $|X|$ and p are coprime, so we must have $\alpha_t = 0$ for all $t \in T$ and hence $\beta_{t,x} = 0$ for all $t \in T$ and $x \in X^*$. Thus, the vectors in the set

$$\mathcal{B} = \{v(t) : t \in T\} \cup \{w(t, x) : t \in T, x \in X^*\}$$

are linearly independent and it follows that $\mathcal{V} + \mathcal{W} = \mathcal{V} \oplus \mathcal{W}$. Moreover, the k-basis $\mathcal{B}$ of $\mathcal{V} \oplus \mathcal{W}$ consists of

$$|T| + (|X| - 1)|T| = |X||T| = \dim k[G/Q]$$

vectors, so $\mathcal{V} \oplus \mathcal{W} = k[G/Q]$. Thus $\mathcal{V}$ is a direct summand of $k[G/Q]$ such that $\mathcal{V}{\downarrow_H} \cong k[H/Q]$, as required. □

Remark 2.3.10. If we assume that $P = H$ in the above result, then $k[P/Q] \cong \mathcal{S}(P, Q)$ and K contains Q as a Sylow p-subgroup and has prime-power index in G, so (ii) of Proposition 2.3.2 follows as a special case of this result. Moreover, $\mathcal{V}$ is not necessarily indecomposable and hence it may not be the case that $\mathcal{S}(G, Q) \cong \mathcal{V}$.

We finish the section by proving a simple property that we will find useful when trying to determine if a group G is p-extendible or not.

Proposition 2.3.11. Let G be a finite group, $P \in \mathrm{Syl}_p(G)$ and $N \lhd G$ be a p'-group. Then G is p-extendible if and only if $\overline{G} = G/N$ is p-extendible.

Proof. By Proposition 2.3.5, the inflation map $\mathrm{Inf}_{\overline{G}}^{G} : \mathrm{mod}\, k\overline{G} \to \mathrm{mod}\, kG$ induces a one-to-one correspondence between Scott modules associated to $k\overline{G}$ and Scott modules associated to kG. In particular, if $Q \leq P \in \mathrm{Syl}_p(G)$, then $\dim \mathcal{S}(\overline{G}, \overline{Q}) = \dim \mathcal{S}(G, Q)$; thus $\mathcal{S}(G, Q){\downarrow_P} \cong k[P/Q]$ if and only if $\dim \mathcal{S}(G, Q) = |P : Q|$ if and only if $\dim \mathcal{S}(\overline{G}, \overline{Q}) = |\overline{P} : \overline{Q}|$ if and only if $\mathcal{S}(\overline{G}, \overline{Q}){\downarrow_{\overline{P}}} \cong k[\overline{P}/\overline{Q}]$. So G is p-extendible if and only if $\overline{G}$ is p-extendible. $\square$

2.4 Scott Modules in the Normal Case

In this section, we shall provide a description of the Scott module $\mathcal{S}(G, Q)$ when $Q \leq P \lhd G$ and $P \in \mathrm{Syl}_p(G)$. This description relies upon the associated local subgroup $N_G(Q)$.

Given that $P \lhd G$, the conjugate modules $g \otimes \mathcal{M}$ of any kP-module $\mathcal{M}$ are themselves kP-modules for all $g \in G$. We define

$$I(\mathcal{M}) = \{g \in G : g \otimes \mathcal{M} \cong \mathcal{M} \text{ as } kP\text{-modules}\}$$

and refer to this as the **inertial subgroup** of $\mathcal{M}$ in G. We are interested in the inertial subgroup that corresponds to the module $k[P/Q]$; the following describes this subgroup.

Lemma 2.4.1. Suppose that G is a finite group and $Q \leq P \in \mathrm{Syl}_p(G)$ with $P \lhd G$. Then $I(k[P/Q]) = PN_G(Q)$.

Proof. If $g \in G$ and $T = [P/Q]$, then $g \otimes k[P/Q]$ is a transitive permutation kP-module with permutation basis $X = \{g \otimes [t] : t \in T\}$. Thus $g \otimes k[P/Q] \cong k[P/S]$, where $S = \mathrm{Stab}_P(g \otimes [1])$, by Proposition 1.1.19 (ii). Since $S = {}^gQ$, we therefore have $g \otimes k[P/Q] \cong k[P/{}^gQ]$.

Thus $g \in I(k[P/Q])$ if and only if ${}^gQ \sim_P Q$ if and only if ${}^gQ = {}^uQ$ for some $u \in P$, which happens if and only if $u^{-1}g \in N_G(Q)$ for some $u \in P$. So $g \in I(k[P/Q])$ if and only if $g \in PN_G(Q)$ and the result follows. $\square$

Theorem 2.4.2. Suppose that G is a finite group and $Q \leq P \in \mathrm{Syl}_p(G)$ with $P \trianglelefteq G$. Then:

(i) $\mathcal{S}(PN_G(Q), Q)\downarrow_P \cong k[P/Q]$;

(ii) $\mathcal{S}(PN_G(Q), Q)\uparrow^G$ is indecomposable and hence $\mathcal{S}(G, Q) \cong \mathcal{S}(PN_G(Q), Q)\uparrow^G$.

In particular,
$$\dim \mathcal{S}(G, Q) = \frac{|G : P|}{|N_G(Q)|_{p'}} \cdot |P : Q|.$$

Proof. Note that $N_P(Q) = P \cap N_G(Q)$ is a normal Sylow p-subgroup of $N_G(Q)$ and hence by the Schur-Zassenhaus theorem, there exists a subgroup $R \leq N_G(Q)$ such that $N_G(Q) = N_P(Q) \rtimes R$. Furthermore,

$$|PN_G(Q)| = \frac{|P||N_G(Q)|}{|P \cap N_G(Q)|} = |P||R|$$

and hence QR is a subgroup of $PN_G(Q)$ such that $|PN_G(Q) : QR| = |P : Q|$. By Proposition 2.3.2 (ii), it follows that $\mathcal{S}(PN_G(Q), Q)\downarrow_P \cong k[P/Q]$. The fact that $\mathcal{S}(I(k[P/Q]), Q)\uparrow^G$ is indecomposable is well-known (see, e.g., [26, Proposition 4.1]), and we know that $\mathcal{S}(G, Q)$ is a direct summand of $\mathcal{S}(I(k[P/Q]), Q)\uparrow^G$ from Proposition 2.2.1; by Lemma 2.4.1 we have $I(k[P/Q]) = PN_G(Q)$, so (ii) follows. Thus

$$\dim \mathcal{S}(G, Q) = |G : PN_G(Q)||P : Q| = \frac{|G : P|}{|N_G(Q)|_{p'}} \cdot |P : Q|,$$

as required. $\square$

This result allows us to extend the statement of Proposition 1.1.19 (iii) to the case where P is normal in G. In particular, we have the following corollary.

Corollary 2.4.3. Suppose that $Q \leq P \in \mathrm{Syl}_p(G)$ and $P \lhd G$. Then $(k_Q)\uparrow^G$ is indecomposable if and only if $N_G(Q)$ is a p-group.

Theorem 2.4.2 also gives us the converse of Proposition 2.3.2 (ii) in the case where P is normal in G.

Corollary 2.4.4. Suppose that G is a finite group, P is a normal Sylow p-subgroup of G and $Q \leq P$. Then $\mathcal{S}(G,Q)\downarrow_P \cong k[P/Q]$ if and only if Q is contained as a Sylow p-subgroup of a subgroup $H \leq G$ such that $|G : H| = |P : Q|$.

Proof. If Q is contained as a Sylow p-subgroup of a subgroup $H \leq G$ and $|G : H| = |P : Q|$, then it follows that $\mathcal{S}(G,Q)\downarrow_P \cong k[P/Q]$ by Proposition 2.3.2. Assume on the other hand that $\mathcal{S}(G,Q)\downarrow_P \cong k[P/Q]$. By Theorem 2.4.2, $G = PN_G(Q)$ and since $N_P(Q)$ is a normal Sylow p-subgroup of $N_G(Q)$, by the Schur-Zassenhaus theorem, there exists a subgroup $R \leq N_G(Q)$ such that $|R| = |G : P|$. The result now follows with $H = RQ$. □

Example 2.4.5. Let $G = \Sigma_{p^2}$ for an odd prime p and set P to be equal to the Sylow p-subgroup given in Example 2.3.3; thus, we recall that $P = E \rtimes Q$, where $E = \langle \sigma_1, \dots, \sigma_p \rangle$ with

$$\sigma_i = (1 + (i-1)p, 2 + (i-1)p, \dots, p + (i-1)p)$$

for $1 \leq i \leq p$ and $Q = \langle w \rangle$ with

$$w = \prod_{i=1}^{p} (i, i+p, \dots, i + (p-1)p).$$

We know from [7, Section 4] that $N_G(P) = P \rtimes (C_{p-1})^2$; thus $N_G(P)$ is generated by P and two elements of order $p-1$, which we denote by s_1 and s_2. We construct s_1 and s_2 as follows: First, recall that we may regard Σ_p as a subgroup of Σ_{p^2} by considering those permutations which act trivially on $\{p+1, \dots, p^2\}$, and there exists $\tau \in N_{\Sigma_p}(\sigma_1)$ of order $p-1$ such that $\sigma_1^\tau = \sigma_1^l$ for some $1 \leq l \leq p-1$. Moreover, since $C_{\Sigma_p}(\sigma_1) = \langle \sigma_1 \rangle$, it follows that l has order $p-1$ in $\mathbb{F}_p^\times$, as otherwise we would have $l^k \equiv 1 \bmod p$ for some $k < p-1$ and hence

$$(\sigma_1)^{\tau^k} = \sigma_1^{l^k} = \sigma_1,$$

so $\tau^k \in C_{\Sigma_p}(\sigma_1)$, a contradiction. We set

$$s_1 = \prod_{i=0}^{p-1} {}^{w^i}\tau.$$

Note that by construction, if $n = r + qp$ for some $1 \le r \le q$ and $0 \le q \le p-1$, we have $s_1(n) = s_1(r) + qp$. A direct calculation now shows that $w^{s_1} = w$ and $\sigma_i^{s_1} = \sigma_i^l$ for all i, so $s_1 \in N_G(P)$. For $1 \le r \le p$ and $0 \le q \le p-1$, define $f(r+qp) = (q+1) + (r-1)p$; then f is an element of Σ_{p^2} of order 2. We let $s_2 = s_1^f$ and note that $(\sigma_1 \cdots \sigma_p)^f = w$, so it follows that $w^{s_2} = w^l$. If $n = r + qp$ with $1 \le r \le p$ and $0 \le q \le p-1$, then

$$\begin{aligned} s_2(r+qp) &= (fs_1f)(r+qp) = (fs_1)((q+1)+(r-1)p) \\ &= f(s_1(q+1)+(r-1)p) = r + (s_1(q+1)-1)p \end{aligned}$$

so

$$\begin{aligned} \sigma_i^{s_2} &= (s_2(1+(i-1)p), s_2(2+(i-1)p), \ldots, s_2(p+(i-1)p)) \\ &= (1+(s_1(i)-1)p, 2+(s_1(i)-1)p, \ldots, p+(s_1(i)-1)p) = \sigma_{s_1(i)}. \end{aligned}$$

Thus $w^{s_2} = w^l$ and $\sigma_i^{s_2} = \sigma_{s_1(i)}$ for $1 \le i \le p$, so $s_2 \in N_G(P)$. Finally,

$$\begin{aligned} (s_2^{-1}s_1s_2)(r+qp) &= (s_2^{-1}s_1)(r+(s_1(q+1)-1)p) \\ &= s_2^{-1}(s_1(r)+(s_1(q+1)-1)p) \\ &= s_1(r)+qp = s_1(r+qp), \end{aligned}$$

so $s_2^{-1}s_1s_2 = s_1$. Thus $N_G(P) = P \rtimes \langle s_1, s_2 \rangle$, as required.

Let

$$R = \{\sigma_1^{a_1} \cdots \sigma_p^{a_p} w^k : p \text{ divides } a_1 + \cdots + a_p + k\}.$$

Then a direct calculation shows that R is a maximal, normal and nonabelian subgroup of P of order p^p. Set $N = N_G(P)$; we claim that $N_N(R) = \langle P, s_1s_2 \rangle$. Indeed, if $s = s_1^\alpha s_2^\beta$ with $0 \le \alpha, \beta < p-1$ and $\sigma = \sigma_1^{a_1} \cdots \sigma_p^{a_p} w^k \in R$, then we have $(\sigma_i^{a_i})^{s_1^\alpha} = (\sigma_i^{s_1^\alpha})^{a_i} = \sigma_i^{l^\alpha a_i}$, so

$$\sigma^{s_1^\alpha} = \sigma_1^{l^\alpha a_1} \cdots \sigma_p^{l^\alpha a_p} w^k.$$

Since conjugation by s_2 permutes the elements $\sigma_1, \ldots, \sigma_p$ and $(w^k)^{s_2^\beta} = w^{l^\beta k}$, it follows that $\sigma^s \in R$ if and only if

$$l^\alpha(a_1 + \cdots + a_p) + l^\beta k \equiv 0 \bmod p.$$

As $\sigma \in R$, we know that $a_1 + \cdots + a_p \equiv -k \bmod p$, so $l^\beta k \equiv l^\alpha k \bmod p$; moreover, $0 \leq \alpha, \beta < p - 1$ and l has order equal to $p - 1$, so it follows that $\alpha = \beta$. Thus $N_N(R) = \langle P, s_1 s_2 \rangle$ and we deduce that $|N_N(R)|_{p'} = p - 1$. It follows from Theorem 2.4.2 that

$$\dim \mathcal{S}(N_G(P), R) = \frac{(p-1)^2 p}{p-1} = (p-1)p.$$

Example 2.4.6. Set $G = GL_4(11)$ and take $p = 5$. Then $|G|_5 = 5^4$ and if $u \in GF(11)^\times$ has order 5, we may take

$$P = \left\{ \begin{bmatrix} u^\alpha & 0 & 0 & 0 \\ 0 & u^\beta & 0 & 0 \\ 0 & 0 & u^\gamma & 0 \\ 0 & 0 & 0 & u^\delta \end{bmatrix} : 0 \leq \alpha, \beta, \gamma, \delta \leq 4 \right\} \cong C_5^4$$

as an element of $\mathrm{Syl}_5(G)$. Let M denote the group of monomial matrices in G; then P is a normal subgroup of M and $M = DS$, where D denotes the diagonal matrices in G and S denotes the permutation matrices in G. In particular, $D \leq C_M(P)$, so if $Q \leq P$, determining $N_M(Q)$ amounts to determining $N_S(Q)$. So, for example, if we take

$$Q = \langle u e_{11} + e_{22} + e_{33} + e_{44} \rangle,$$

then $|N_S(Q)|$ is equal to the number of permutation matrices in Σ_4 corresponding to a permutation which fixes 1. Thus $|N_S(Q)| = |\Sigma_3| = 6$ and we get $|N_M(Q)| = |D| \cdot 6 = 10^4 \cdot 6$. So by Theorem 2.4.2, it follows that

$$\dim \mathcal{S}(M, Q) = \frac{|M : Q|}{|N_M(Q)|_5'} = \frac{24 \cdot 10^4}{2^4 \cdot 6} = 4 \cdot |P : Q|.$$

On the other hand, if

$$Q = \langle u e_{11} + u^2 e_{22} + e_{33} + e_{44} \rangle,$$

then the only nontrivial permutation matrix which normalises Q is that corresponding to the permutation $(3, 4)$, so $|N_S(Q)| = 2$. Thus $|N_M(Q)| = |D| \cdot 2$, and again from Theorem 2.4.2, we deduce that

$$\dim \mathcal{S}(M, Q) = \frac{|M : Q|}{|N_M(Q)|_{5'}} = 12 \cdot |P : Q|.$$

Chapter 3

Frobenius Groups and p-Extendibility

In this chapter, we carry out an investigation into the properties of p-extendible groups. We have seen already that p-nilpotent groups are themselves p-extendible and this serves as a natural class of p-extendible groups; we add to this in our first section, where we investigate the Scott modules associated to Frobenius groups and show that for certain primes p, Frobenius groups are p-extendible. In the following two sections, we prove some necessary conditions that a p-extendible group must satisfy and relate this to questions concerned with fusion and transfer. Finally, in the last section, we classify when a p-local subgroup of a sporadic group is q-extendible for a given prime q.

3.1 Frobenius Groups

We start by putting in place the key definitions and structure we shall need in the results to come. We take [17] and [21] as standard references on the subject of Frobenius groups.

Theorem 3.1.1. [21, Theorem 6.4] Suppose that G is a finite group. Then G is said to be a **Frobenius group** if:

(i) $G = K \rtimes H$ for proper, nontrivial subgroups $K, H \leq G$;

(ii) $H \cap H^x = \{1\}$ for all $x \in G - H$.

We assume for the rest of this section that G denotes an arbitrary Frobenius group. We refer to the normal subgroup K in the above as the **Frobenius kernel** of G and the complement H of K in G as the **Frobenius complement** of G. Both the kernel and the complement satisfy certain strong conditions, some of which are dealt with in the following result.

Theorem 3.1.2. [17, Theorem 10.3.1] If $G = K \rtimes H$ is a Frobenius group, then the following conditions hold:

(i) K and H have coprime orders;

(ii) K is nilpotent;

(iii) the Sylow p-subgroups of H are cyclic if p is odd and are cyclic or generalised quaternion if $p = 2$;

(iv) if $|H|$ is odd, then H is a metacyclic group.

A consequence of (i) in the above is that if $Q \leq P \in \mathrm{Syl}_p(G)$, then we may assume without loss of generality that either $P \leq K$ or $P \leq H$, and this will not affect our analysis when studying the module $\mathcal{S}(G, Q)$. Our first result relates to the case when a Sylow p-subgroup P is contained in the Frobenius kernel K.

Theorem 3.1.3. Suppose that $G = K \rtimes H$ is a Frobenius group, and $p \in \pi(K)$. Then:

(i) G contains a normal Sylow p-subgroup P;

(ii) G is p-extendible if and only if $P \rtimes H$ is p-extendible;

(iii) if P is cyclic, then G is p-extendible.

Proof. Since K is nilpotent by Theorem 3.1.2 (ii) and $|K|_p = |G|_p$, it follows that if $P \in \mathrm{Syl}_p(K)$, then $P \,\mathrm{char}\, K \lhd G$ and hence $P \lhd G$ is a Sylow p-subgroup of G. Furthermore, if we let X be equal to the product of the Sylow q-subgroups of K, $q \in \pi(K), q \neq p$, then $X \,\mathrm{char}\, K \lhd G$ and hence $X \lhd G$. Thus $G = X \rtimes (P \rtimes H)$. Part (ii) now follows from Proposition 2.3.11. For the last part, suppose that $Q \leq P$. Since P is cyclic, it follows that $Q \,\mathrm{char}\, P$ and hence $Q \lhd (P \rtimes H)$. Thus, from Theorem 2.4.2, we deduce that $\mathcal{S}(P \rtimes H, Q){\downarrow_P} \cong k[P/Q]$ for all p-subgroups $Q \leq P$, and hence $P \rtimes H$ is p-extendible, so (iii) follows from (ii). $\square$

The study of Frobenius groups splits into the study of those which have solvable Frobenius complements and those which have nonsolvable Frobenius complements. In both cases, the possible structure of the complement is described by theorems which were originally proved by Zassenhaus. We shall study the two cases separately, starting with the case where the complement is solvable.

3.1.1 Solvable Complements

Throughout this section, we assume that $G = K \rtimes H$ is a Frobenius group with a Frobenius complement H which is solvable. The following lemma completely describes the possible structure we can have in this case.

Lemma 3.1.4. [23, Chapter 37, Theorem 4.9] Suppose that $G = K \rtimes H$ is a Frobenius group and H is solvable. Then H contains a normal split metacyclic subgroup $N = A \rtimes B$, where $A, B \leq N$ are cyclic and have coprime orders. Furthermore, the quotient H/N satisfies the following:

(i) H/N is isomorphic to a subgroup of Σ_4;

(ii) if $O_2(H) \ncong Q_8$, then H/N has order dividing 4.

In the result that follows, the groups H and N should be thought of as being the same as the groups H and N described in the above Lemma.

Lemma 3.1.5. Suppose that H is solvable and $N \lhd H$, with $|H : N| = 2^s 3^t$ for some $s, t \geq 0$. Assume further that $N = A \rtimes B$ for cyclic subgroups A and B which have coprime orders, and let p be an odd prime. If $p > 3$ or $t = 0$, then H is p-extendible.

Proof. Let $P \in \mathrm{Syl}_p(H)$. Since $p > 3$ or $t = 0$, we may assume without loss of generality that $P \in \mathrm{Syl}_p(N)$; furthermore, A and B have coprime orders, so we may assume that $P \leq A$ or $P \leq B$. Assume first that $P \leq A$ and $Q \leq P$. Then $P \operatorname{char} A \lhd N$, so $P \operatorname{char} N \lhd H$ and it follows that $P \lhd H$. Since A is cyclic, so too is P and hence $Q \operatorname{char} P \lhd H$, i.e., $Q \lhd H$. Thus, by Theorem 2.4.2, we have $\mathcal{S}(H, Q){\downarrow_P} \cong k[P/Q]$ and hence H is p-extendible, as required.

Suppose instead that $P \leq B$. If P_q denotes a Sylow q-subgroup of A for some $q \in \pi(A)$, then $P_q \operatorname{char} A \lhd N$ and it follows that $P_q \operatorname{char} N \lhd H$, so $P_q \lhd H$. Since A is a product of normal subgroups of H, we see that $A \lhd H$; furthermore, A is a p'-group, so we know from Proposition 2.3.11 that H is p-extendible if and only if $\overline{H} = H/A$ is p-extendible. Note now that $\overline{N} \lhd \overline{H}$ and $\overline{N} \cong B$ is cyclic. Thus, a Sylow p-subgroup $\overline{P}$ of $\overline{H}$ is cyclic and we have $\overline{P} \lhd \overline{H}$; in particular, if $\overline{Q} \leq \overline{P}$, then $\overline{Q}\,\mathrm{char}\,\overline{P} \lhd \overline{H}$ and it follows that $\overline{Q} \lhd \overline{H}$, so from Theorem 2.4.2 we deduce that $\mathcal{S}(\overline{H}, \overline{Q}){\downarrow_{\overline{P}}} \cong k[\overline{P}/\overline{Q}]$. We conclude that $\overline{H}$ is p-extendible and hence H is p-extendible, as required. $\square$

Theorem 3.1.6. Suppose that $G = K \rtimes H$ is a Frobenius group and $p \in \pi(H)$. Assume furthermore that H is solvable and any of the following are true:

(i) $p > 3$;

(ii) $p = 3$ and $O_2(H) \ncong Q_8$;

(iii) $p = 2$ and P is cyclic.

Then G is p-extendible.

Proof. We may assume without loss of generality that $P \leq H$ and by Proposition 2.3.11, the problem reduces to showing that H is p-extendible. By Lemma 3.1.4, we know that if either (i) or (ii) holds, then H contains a normal subgroup N such that $|H : N|$ divides 24 and N is split metacyclic and satisfies the conditions of Lemma 3.1.5. Thus, H is p-extendible and so too is G. If $p = 2$ and P is cyclic, then G is 2-nilpotent (see [17, Theorem 7.6.1]), and hence G is p-extendible by Corollary 2.3.7. $\square$

3.1.2 Nonsolvable Complements

We now turn our attention towards the case where our Frobenius complement, H, is nonsolvable. In this case, we have the following structure theorem.

Lemma 3.1.7. [32, Theorem 18.6] Suppose that $G = K \rtimes H$ is a Frobenius group and H is nonsolvable. Then H contains a normal subgroup N such that $|H : N| \leq 2$ and $N = SL_2(5) \times M$ for some metacyclic group M of order coprime to 30.

Thus, if $G = K \rtimes H$ is a Frobenius group and H is nonsolvable with $p \in \{2, 3, 5\}$, then the above together with (i) of Theorem 3.1.2 tells us that a Sylow p-subgroup of G is isomorphic to a Sylow p-subgroup of $SL_2(5)$. We describe these Sylow p-subgroups in the following, along with some additional properties we will require.

Lemma 3.1.8. Let $G = SL_2(5)$. Then:

(i) if $P_5 \in \mathrm{Syl}_5(G)$, P_5 has a complement in G;

(ii) $SL_2(5) = (SL_2(5))'$, i.e., $SL_2(5)$ is a perfect group;

(iii) $SL_2(5)$ contains a subgroup of index 6.

Proof. From [8], we know that $P_2 \cong Q_8$ and $|N_G(P_2)| = 3 \cdot |P_2|$; thus $N_G(P_2)$ is a complement for P_5 in G. We take (ii) from [29, Theorem 24.17]. As a subgroup of index 6, we may take

$$B = \left\{ \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} \in SL_2(5) : * \in \mathbb{F}_5 \right\} \leq G.$$

□

We shall also need the following two lemmas, the first of which we take from [23, Chapter 37, Lemma 4.5].

Lemma 3.1.9. Let P be a Sylow p-subgroup of a finite group G. If P is cyclic and $P \not\leq G'$, then G is p-nilpotent.

Lemma 3.1.10. If $G = K \rtimes H$ is a Frobenius group, then $G' = KH'$.

Proof. Note that

$$G/KH' = KH/KH' \cong H/H',$$

so G/KH' is abelian and it follows that $G' \leq KH'$. We claim that $KH' \leq G'$; to see this, it is enough to show that $K \leq G'$. Let $h \in H$, $h \neq 1$ and consider the commutators $k^{-1}h^{-1}kh, l^{-1}h^{-1}lh \in K$, where $k, l \in K$. Then if $k^{-1}h^{-1}kh = l^{-1}h^{-1}lh$, we have

$$(lk^{-1})h^{-1}(lk^{-1})^{-1} = h^{-1}.$$

Since G is a Frobenius group, we must have $l = k$, by (ii) of Definition 3.1.1. Thus, for any $h \in H, h \neq 1$, the map $\alpha_h : K \to K$ given by $\alpha_h(k) = [k, h]$ is a bijection, so $K = \alpha_h(K) \leq [K, H] \leq G'$. Thus $K \leq G'$ and it follows that $KH' \leq G'$, so $G' = KH'$, as required. □

We can now state and prove our main result on Frobenius groups with nonsolvable Frobenius complements.

Theorem 3.1.11. Suppose that $G = K \rtimes H$ is a Frobenius group and H is nonsolvable. Let $p \in \pi(H), p > 2$ and $Q \leq P \in \mathrm{Syl}_p(G)$.

(i) If $p > 3$, then $\dim \mathcal{S}(G, Q) \leq 2|P : Q|$.

(ii) If $p = 3$, then $\dim \mathcal{S}(G, Q) \leq 4|P : Q|$.

Proof. By Theorem 3.1.2, we assume without loss of generality that $P \leq H$. Let $N = SL(2, 5) \times M \leq H$ be the subgroup given by Lemma 3.1.7 and note that $H' \leq N$, since H/N is abelian of order less than or equal to 2. Moreover, M is metacyclic, so there exists a normal cyclic subgroup A of M such that M/A is cyclic. Finally, we know that P is cyclic for all $p \geq 3$, by Theorem 3.1.2.

By Lemma 3.1.10, $G' = KH'$, so if $P \not\leq H'$, then $P \not\leq G'$. But then Lemma 3.1.9 implies that G is p-nilpotent, and hence p-extendible, so the bound on the dimension holds in this case.

We may therefore assume that $P \leq H' \leq N$. If $P \not\leq N'$, then N is p-nilpotent by Lemma 3.1.9 and we hence have $N = O_{p'}(N) \rtimes P$. Then $S = K \rtimes (O_{p'}(N) \rtimes Q)$ is a subgroup of G which contains Q as a Sylow p-subgroup. By Proposition 2.3.2 (i), it follows that

$$\dim \mathcal{S}(G, Q) \leq |G : S| = \frac{|K||H|}{|K||O_{p'}(N)||Q|} \leq 2\frac{|N|}{|Q||O_{p'}(N)|} = 2|P : Q|$$

and both (i) and (ii) hold in this case.

We therefore assume that $P \leq N'$. Note that

$$N' = (SL_2(5))' \times M' \leq SL_2(5) \times A,$$

by (ii) of Lemma 3.1.8 and the fact that M/A is abelian. If $p > 5$, then we may assume that $P \leq A$. Since $P \operatorname{char} A$, it follows that $P \lhd A \lhd M \lhd N \lhd H$ and hence $P \operatorname{char} H$. Thus $Q \operatorname{char} P \operatorname{char} H$ and we deduce that $Q \lhd H$. By the Schur-Zassenhaus theorem, P has a complement $R \leq H$ of order $|H : P|$. Then $S = K \rtimes (Q \rtimes R)$ is a subgroup of G such that $|G : S| = |P : Q|$, so it follows from Proposition 2.3.2 (ii) that $\mathcal{S}(G, Q){\downarrow_P} \cong k[P/Q]$ and the bound on the dimension holds in this case.

If $p = 5$, then $P \leq SL_2(5)$ has order 5 and we may assume that $Q = \{1\}$, since if $Q = P$, then $\mathcal{S}(G, P) \cong k_G$ by Proposition 2.1.6. By Lemma 3.1.8, P has a complement in $SL_2(5)$ and hence has one in N. Let the complement of P in N be denoted R; then $S = K \rtimes R$ has index

$$|G : S| = |H : R| = 10 = 2|P : Q|.$$

Thus $\dim \mathcal{S}(G, \{1\}) \leq 2|P : Q|$, by Proposition 2.3.2 (i), and all cases in (i) have been accounted for.

Finally, if $p = 3$, then $P \leq SL_2(5)$ has order 3 and we may again assume that $Q = \{1\}$. Then Lemma 3.1.8 implies that $SL_2(5)$ contains a subgroup of index 6 and hence so does N. Calling this subgroup R, it follows that $S = K \rtimes R$ has index

$$|G : S| = |H : R| \leq 2|N : R| = 12 = 4|P : Q|.$$

Thus $\dim \mathcal{S}(G, \{1\}) \leq 4|P : Q|$, which gives the bound in (ii). $\square$

Remark 3.1.12. Note that the case where H is a nonsolvable Frobenius complement still provides another class of p-extendible groups; in contrast to Theorem 3.1.6, the p-extendible groups in question are of the form $K \rtimes N$, where $N \cong SL_2(5) \times M$ and M is metacyclic of order coprime to 30, and we assume that $p > 3$.

3.2 p-Extendibility

In the next two sections, we shall investigate necessary conditions that are required for a finite group G to be p-extendible. We have seen already that p-nilpotent groups are the most basic example of p-extendible groups; furthermore, a group G is p-nilpotent if and only if a Sylow p-subgroup P of G controls G-fusion of P in G. A similar necessary condition exists for p-extendible groups.

Proposition 3.2.1. Suppose that G is a finite p-extendible group and $P \in \mathrm{Syl}_p(G)$. If $Q, Q' \leq P$ and $Q \sim_G Q'$, then $Q \sim_P Q'$.

Proof. By assumption, we have $\mathcal{S}(G,Q){\downarrow_P} \cong k[P/Q]$ and $\mathcal{S}(G,Q'){\downarrow_P} \cong k[P/Q']$. Furthermore, the fact that $Q \sim_G Q'$ implies that $\mathcal{S}(G,Q) \cong \mathcal{S}(G,Q')$, by Proposition 2.1.5. Thus $k[P/Q] \cong k[P/Q']$ and by Proposition 2.1.5 again, we have $Q \sim_P Q'$. □

Motivated by this observation, we make the following definition.

Definition 3.2.2. Suppose that G is a finite group, $H \leq G$ and $P \in \mathrm{Syl}_p(G)$ with $P \leq H \leq G$. Suppose that G, H and P satisfy the following property:

$$\text{for all } Q, Q' \leq P, \text{ if } Q \sim_G Q', \text{ then } Q \sim_H Q'.$$

Then we say that H **controls the subgroup fusion of** P **in** G.

Remarks 3.2.3. (i) We have shown that in any p-extendible group G, a Sylow p-subgroup necessarily controls its own subgroup fusion in G, but the reverse implication does not necessarily hold. For a counterexample, we may take any finite group G such that $|P| = p$ but $\mathcal{P}(k_G)$ is not an extension of kP; then P clearly controls the subgroup fusion of P in G, but it is not p-extendible (for an example of such a group, we refer to [29]).

(ii) We observe that the above definition bears a resemblance to control of G-fusion in fusion systems, but the two are different; in particular, P controls the G-fusion of P in G if and only if G is p-nilpotent, which underlines one key difference (see [28] for an explanation on the terminology). To understand where this difference comes from, note that in the fusion system $\mathcal{F}_P(G)$, if P controls G-fusion and $Q^g = Q' \leq P$, then there exists a homomorphism $c_u : Q \to Q'$ induced by conjugation for some $u \in P$ such that $c_g = c_u$; on the other hand, what we call control of subgroup fusion does not require that Q be conjugate to Q' via the *same* homomorphism.

Recall that a group G is p-normal if $Z(P)$ is weakly closed in P with respect to G for a given $P \in \mathrm{Syl}_p(G)$. The following imposes a similar condition on the normal subgroups of P in a p-extendible group.

Proposition 3.2.4. Suppose that G is a finite p-extendible group and let $P \in \mathrm{Syl}_p(G)$. Then every normal subgroup of P is weakly closed in P with respect to G. In particular, G is p-normal.

Proof. Let $Q \lhd P$ and suppose that $Q \sim_G Q'$ with $Q' \leq P$. Since G is p-extendible, P controls the subgroup fusion of P in G, so it follows that $Q \sim_P Q'$, i.e., $Q^u = Q'$ for some $u \in P$. But $Q \lhd P$, so $Q = Q'$. Thus Q is weakly closed in P with respect to G. $\square$

We have mentioned that p-nilpotent groups are p-extendible, but the reverse implication does not hold in general, as we saw in Example 2.3.8. On the other hand, under certain circumstances, it might be the case that a group G is p-extendible if and only if it is p-nilpotent. Our next main result accounts for one such situation, and will require the following lemmas.

Lemma 3.2.5. Suppose that $W \leq Z(P)$ and W is weakly closed in P with respect to G. Then $N_G(W)$ controls G-fusion of P in G.

Proof. Suppose that $x, x^g \in P$ for some $g \in G$. Then $W, W^g \leq C_G(x^g)$ and it follows that $W \leq P'$ and $W^g \leq (P')^c$ for some $P' \in \mathrm{Syl}_p(C_G(x^g))$ and $c \in C_G(x^g)$. Thus, we deduce that $W^{gc^{-1}} \leq P'$ and hence $W^{gc^{-1}}$ and W belong to a common Sylow p-subgroup of G; since W is weakly closed in P with respect to G, it follows that $W = W^{gc^{-1}}$. In particular, we have $x^{gc^{-1}} = (x^g)^{c^{-1}} = x^g$ and $gc^{-1} \in N_G(W)$, so the statement follows. $\square$

Lemma 3.2.6. Suppose that G is a finite group and $P \in \mathrm{Syl}_p(G)$.

(i) If G is p-nilpotent and $Q \lhd P$ is normal in G, then $|C_G(Q)|$ is divisible by $|G : P|$.

(ii) If $p = \min \pi(G)$ and every maximal subgroup of P is normal in G, then G is p-nilpotent.

Proof. Part (ii) is given in [10, Lemma 2.6]. For (i), note that if G is p-nilpotent, then $|O_{p'}(G)| = |G : P|$. If $u \in Q$ and $x \in O_{p'}(G)$, then $uxu^{-1}x^{-1} \in Q \cap O_{p'}(G) = \{1\}$, so $ux = xu$ and it follows that $O_{p'}(G) \leq C_G(Q)$, whence the result follows. $\square$

We have seen that in a p-extendible group G, every normal subgroup of a Sylow p-subgroup is weakly closed in G and this allows us to establish that in some cases, being p-extendible is equivalent to being p-nilpotent. Our method of proof in the following is based heavily on that in [10, Theorem 3.1], which was the original inspiration for this result.

Theorem 3.2.7. Suppose that G is a finite group which is p-solvable with $p = \min \pi(G)$. Assume further that $P \in \mathrm{Syl}_p(G)$ and every normal subgroup of P is weakly closed in P with respect to G. Then G is p-nilpotent.

Proof. We assume for a contradiction that G is a counterexample of minimal order. Throughout the proof, we shall make use of the fact that if $P \leq H < G$, then H is p-nilpotent; indeed, it is clear H satisfies the hypothesis of the theorem and since $|H| < |G|$, the minimal assumption on G forces H to be p-nilpotent.

Note that $O_{p'}(G) = \{1\}$: indeed, by Proposition 1.2.5 (ii), if $O_{p'}(G) > \{1\}$, then $G/O_{p'}(G)$ is a group of order smaller than $|G|$ satisfying the hypothesis of the theorem and hence $G/O_{p'}(G)$ is p-nilpotent, so G is p-nilpotent as well, contrary to assumption. Since $O_{p'}(G) = \{1\}$ and G is p-solvable, we have $C_G(O_p(G)) \leq O_p(G)$ by [34, Theorem 9.31].

We next show that P is nonabelian. Assume that P is abelian and let $R \in \mathrm{Syl}_r(N_G(P))$ with $r \neq p$. If M is a maximal subgroup of P, it is subnormal and weakly closed in P with respect to PR and is hence normal in PR, by Proposition 1.2.5 (i); thus PR is p-nilpotent, by Proposition 3.2.6 (ii). In particular, $R \leq C_G(P)$, so it follows that $N_G(P) = C_G(P)$; by Burnside's theorem, we see that G is a p-nilpotent group, contrary to assumption. So P must be nonabelian.

Since G is p-solvable and $P < G$, there exists a prime $r \in \pi(G)$ with $r \neq p$ and a Sylow r-subgroup R of G such that $PR \leq G$, by [17, Theorem 6.3.5]. If $PR < G$, then PR is p-nilpotent and hence $|R|$ divides $|C_G(O_p(G))|$ by Proposition 3.2.6 (i), which contradicts the fact that $C_G(O_p(G)) \leq O_p(G)$. So $G = PR$.

Next we show that $N_G(P) = P$ and P is a maximal subgroup of G. Note that $N_G(P) < G$, as otherwise every maximal subgroup of P would be weakly closed and subnormal in G and hence G would be p-nilpotent. On the other hand, if $P < M < G$, then M is p-nilpotent and hence $|M : P|$ divides $|C_M(O_p(G))|$ by Proposition 3.2.6 (i), which is a contradiction since $C_M(O_p(G)) \leq O_p(G)$. So $P = N_G(P)$ and P is maximal in G.

Since P is maximal in G, we either have $N_G(Z(P)) = P$ or $N_G(Z(P)) = G$. If $N_G(Z(P)) = P$, then it follows from Lemma 3.2.5 that P controls the G-fusion of P in G and G is hence p-nilpotent. So we must have $N_G(Z(P)) = G$ and hence $Z(P) \trianglelefteq G$.

We now claim that $O_p(G) = Z(P)$. Since P is maximal in G, we know that $\Phi(G) \leq O_p(G)$; if $\Phi(G) \neq \{1\}$, then $G/\Phi(G)$ satisfies the hypothesis of the theorem and hence is p-nilpotent, which implies that G is p-nilpotent (see [34, 9.3.4]), a contradiction. So $\Phi(G) = \{1\}$. By [27, Lemma 2.8], it follows that

$$O_p(G) = R_1 \times R_2 \times \cdots \times R_t$$

for some minimal normal subgroups R_i of G. In particular, G/R_i satisfies the hypothesis of the result, so we have

$$K = G/R_1 \times \cdots \times G/R_t$$

is a p-nilpotent group. If $t > 1$, then G is isomorphic to a subgroup of K, and hence G is p-nilpotent; we must therefore have $t = 1$ and $O_p(G)$ is a minimal normal subgroup of G. Since $Z(P) \triangleleft G$, it follows that $Z(P) = O_p(G)$ and hence $C_G(Z(P)) = Z(P)$. But $C_G(Z(P)) \leq P$ and hence $Z(P) = P$, which is a contradiction, since P is nonabelian. Thus no minimal counterexample exists, and we are done. □

Corollary 3.2.8. Suppose that G is a finite group and G is p-solvable with $p = \min \pi(G)$. Then G is p-extendible if and only if G is p-nilpotent.

Proof. If G is p-extendible, then every normal subgroup of P is weakly closed in P with respect to G by Proposition 3.2.4 and hence by Theorem 3.2.7, we have that G is p-nilpotent. On the other hand, if G is p-nilpotent, then G is p-extendible by Corollary 2.3.7. □

3.3 The Focal Subgroup

We prove a few results related to the focal subgroup in this section and finish with a conjecture relating to the control of G-fusion in p-extendible groups. We start with a result which serves as a useful tool for finding p-groups $Q \leq P$ such that $\mathcal{S}(G, Q){\downarrow_P} \cong k[P/Q]$.

Proposition 3.3.1. Suppose that G is a finite group and $P \in \mathrm{Syl}_p(G)$. If $\mathrm{Foc}_P(G) \leq Q \leq P$, then $\mathcal{S}(G, Q){\downarrow_P} \cong k[P/Q]$.

Proof. By Proposition 1.2.4 (i), there exists a normal subgroup N of G such that $G/N \cong P/\operatorname{Foc}_P(G)$ and $\operatorname{Foc}_P(G) \le N$. Thus

$$|N| = |G : P||\operatorname{Foc}_P(G)|$$

and $\operatorname{Foc}_P(G) \in \operatorname{Syl}_p(N)$. Since $\operatorname{Foc}_P(G) \le P$, we have $Q \cap N = \operatorname{Foc}_P(G)$ and hence

$$|QN| = \frac{|Q||N|}{|Q \cap N|} = \frac{|Q|}{|\operatorname{Foc}_P(G)|}|G : P||\operatorname{Foc}_P(G)| = |Q||G : P|.$$

By Proposition 2.3.2 (ii), the result follows. □

We recall that the focal subgroup relates to the control of p-transfer in a group. In particular, for a p-extendible group, the following statement concerning focal subgroups exists.

Theorem 3.3.2. Suppose that $P \in \operatorname{Syl}_p(G)$ and G is p-extendible. Then $\operatorname{Foc}_P(G) = \operatorname{Foc}_P(N_G(P))$.

Proof. Let $N = N_G(P)$. Recall that

$$\begin{aligned} &\operatorname{Foc}_P(G) = \langle P \cap N', P \cap Q' : Q \in \operatorname{Syl}_p(G)\rangle \\ \text{and} \quad &\operatorname{Foc}_P(N_G(P)) = P \cap N', \end{aligned}$$

by Grün's theorem. Thus, in order to prove the result, it is sufficient to show that $P \cap Q' \le P \cap N'$ for all $Q \in \operatorname{Syl}_p(G)$. Suppose that $x \in G$, and set $L = P \cap (P^x)'$. Then

$$L = P \cap (P^x)' = P \cap (P')^x,$$

so $L^{x^{-1}} = P^{x^{-1}} \cap P' \le P$. Since G is p-extendible, it follows from Proposition 3.2.1 that $L^u = L^{x^{-1}}$ for some $u \in P$. In particular, we have

$$L = L^{x^{-1}u^{-1}} = (P^{x^{-1}} \cap P')^{u^{-1}} = P^{x^{-1}u^{-1}} \cap P'^{u^{-1}} \le P',$$

since $P' \lhd P$ and $u \in P$ and it follows that $L \le P' \le N'$. Since $L \le P$, we get that $L \le P \cap N'$, and the result follows. □

3.4 p-Local Subgroups in the Sporadic Groups

In this section, we shall investigate the following question:

> "Suppose that G is a sporadic group, x is an element of order p for some prime $p \in \pi(G)$ and $N = N_G(\langle x \rangle)$. For what primes $q \in \pi(N)$ is N q-extendible?"

Recall that such a subgroup N is an example of a p-**local subgroup** of G; the structure of these normalisers was developed through much of the first half of the 1900s as part of the classification of the finite simple groups and is included in [18] and [19], the latter of which also includes the structure of the local subgroups $N_G(P)$, where $P \in \mathrm{Syl}_p(G)$ and $p \in \pi(G)$. In the sections that follow, we shall provide a complete answer to the above question using the tables in [18], barring a few small exceptions. At times, we will need additional structure concerning N to support our arguments, and for this we will rely on the database of sporadic groups found at [3]; code to support our findings in MAGMA is provided in the appendix if the reader wishes to check our assertions.

Before proceeding to outline our methodology, we note that the question of p-extendibility for the sporadic groups is already settled, by the following result of Malle and Weigel, proved in [30].

Theorem 3.4.1. Let G be a finite nonabelian simple group with $\mathcal{P}(k_G){\downarrow_P} \cong kP$, where $P \in \mathrm{Syl}_q(G)$ for a prime $q \in \pi(G)$. Then one of the following holds:

(i) $G = A_q$ and $q \geq 5$;

(ii) $G = L_2(q)$ and $q \geq 5$;

(iii) $G = L_n(r)$ and $(r^n - 1)/(r - 1) = q^f$ for $f \in \mathbb{N}$;

(iv) $G = M_{11}$ and $q = 11$; or

(v) $G = M_{23}$ and $q = 23$.

In particular, since M_{11} has a Sylow 11-subgroup of order 11 and M_{23} has a Sylow 23-subgroup of order 23, it follows that M_{11} is 11-extendible and M_{23} is 23-extendible and this completely classifies the sporadic groups which are q-extendible for a given prime q. We shall find this result of use in the arguments to come, since many of the above groups appear as subgroups or subquotients of the p-local subgroups of the sporadic groups.

3.4.1 Methodology

We start by outlining some techniques which allow us to check a p-local subgroup N for q-extendibility. Throughout, N will denote a fixed finite group, but should be thought of as representing a particular p-local subgroup of a sporadic group, and P will denote a Sylow q-subgroup of N for some $q \in \pi(N)$. The following omnibus lemma accounts for a number of the standard cases that appear in the tables in [18].

Lemma 3.4.2. Suppose that N is a finite group and $q \in \pi(N)$.

(i) If N is q-nilpotent, then N is q-extendible.

(ii) If $H \triangleleft N$ and $q \notin \pi(H)$, then N is q-extendible if and only if N/H is q-extendible.

(iii) If N is a metacyclic group and there exists $H \triangleleft N$ such that H is cyclic, N/H is cyclic and H is a Hall subgroup of N, then N is q-extendible for all primes $q \in \pi(N)$.

(iv) If $P \in \mathrm{Syl}_q(N)$, $|P| = q$ and N contains a subgroup of index q, then N is q-extendible.

Proof. Parts (i) and (ii) are just restatements of Propositions 2.3.7 and 2.3.11 respectively. For (iii), assume that $q \in \pi(H)$ and $P \in \mathrm{Syl}_q(N)$. Then we must have $P \leq H$ and $P \,\mathrm{char}\, H \triangleleft N$, so $P \triangleleft N$. Thus, if $Q \leq P$, we have $Q \,\mathrm{char}\, P \triangleleft N$ and hence $Q \triangleleft N$. From Theorem 2.4.2, it follows that $\mathcal{S}(N,Q){\downarrow_P} \cong k[P/Q]$. So N is q-extendible if $q \in \pi(H)$; otherwise, $q \in \pi(N/H)$ and we know that N/H is q-extendible, since N/H is abelian and hence q-nilpotent. By (ii), it follows that N is q-extendible, so (iii) follows. For the last part, the assumptions imply that $\mathcal{S}(N,\{1\}){\downarrow_P} \cong kP$ by Proposition 2.3.2 (ii), and $\mathcal{S}(N,P){\downarrow_P} \cong k_P \cong k[P/P]$ by Proposition 2.1.6, so it follows that N is q-extendible. $\square$

The remaining cases where we have q-extendibility use an approach based on the following lemma.

Lemma 3.4.3. Suppose that N is a finite group and $P \in \mathrm{Syl}_q(N)$ for some $q \in \pi(N)$. Suppose that for every $Q \leq P$, $Q \neq P$, there exists a subgroup $H \leq N$ such that $Q \leq H$ and $|H| = |Q||N : P|$. Then N is q-extendible.

Proof. By Proposition 2.3.2 (ii), the assumptions imply that $\mathcal{S}(N,Q){\downarrow_P} \cong k[P/Q]$ for all $Q \leq P$, $Q \neq P$, and $\mathcal{S}(N,P){\downarrow_P} \cong k_P \cong k[P/P]$ by Proposition 2.1.6, so N is q-extendible. $\square$

Thus, the following algorithm may be used to check a sufficient condition for a group N to be q-extendible.

Algorithm 1. Suppose that N is a finite group and $q \in \pi(N)$. The following algorithm returns `true` if N is q-extendible.

Step 1: Fix a Sylow q-subgroup P of N and let X denote the set of proper subgroups of P. Set $r = \log_q(|P|)$.
Step 2: Let Y denote the set of representatives of N-conjugacy classes of subgroups in N; for $0 \leq i \leq r-1$, let $Y(i)$ denote the set of representatives in Y of order $q^i|N : P|$. If $Y(i) = \emptyset$ for any i, then terminate the algorithm and return `false`.
Step 3: Pick $Q \in X$ at random and for each $u \in P$, check that Q^u is a subgroup of some element in $Y(i)$, where $|Q| = q^i$. If there exists $u \in P$ such that this holds, then set X to equal $X - Q^P$, where

$$Q^P = \{Q^u : u \in P\}$$

and then repeat Step 3. Otherwise, return `false`.
Step 4: Continue repeating Step 3 until $X = \emptyset$, at which point return `true`.

Note that if Algorithm 1 returns `false` for a given finite group N and prime $q \in \pi(N)$, it does not necessarily mean that N is not q-extendible. We include a MAGMA implementation of the above algorithm in the appendix of this thesis, and we shall clearly indicate in the text where we have utilised it to verify the q-extendibility of a given p-local subgroup.

This covers the techniques we can employ to show that N is q-extendible, so we now turn our attention to how we might show that N is not q-extendible. The bulk of our arguments in this direction work by showing that $\mathcal{S}(N,\{1\})\!\downarrow_P \not\cong kP$, i.e., the projective cover of k_N is not an extension of kP. Following the notation in [30], for a finite group N and $q \in \pi(N)$, we shall set $c_q(N) = \dim(\mathcal{S}(N,\{1\}))/|N|_q$. By Proposition 2.3.1 (iii) we know that $c_q(N) \in \mathbb{N}$, and if $c_q(N) > 1$, then N is clearly not q-extendible. Moreover, we have the following.

Lemma 3.4.4. [30, Proposition 2.2] Let N be a finite group, $H \triangleleft N$ and $q \in \pi(N)$. Then $c_q(N) \geq c_q(H)c_q(N/H)$. Furthermore, if H is q-solvable, then $c_q(N) = c_q(N/H)$, and if N/H is solvable, then $c_q(N) = c_q(H)$.

Sometimes, studying the projective module $\mathcal{S}(N,\{1\})$ is not sufficient, and in this situation we need to show that $\dim \mathcal{S}(N,Q) > |P:Q|$ for some nontrivial subgroup $Q < P$ to determine that N is not q-extendible. To assist us with this, we make the following observation.

Lemma 3.4.5. Let $N = H \times K$ be a finite group and $q \in \pi(N)$. Suppose that q divides both $|H|$ and $|K|$, $P_H \in \mathrm{Syl}_q(H)$ and $P_K \in \mathrm{Syl}_q(K)$, so that $P = P_H \times P_K \in \mathrm{Syl}_q(N)$. Assume that there exists $x \in Z(P_H)$ such that:

(i) $o(x) = q$;

(ii) $x^h \in P_H - \{x\}$ for some $h \in H$.

Then N is not q-extendible.

Proof. We know that $Z(P_K) > 1$, so let $u \in Z(P_K)$ be an element of order q. Now set $Z = \langle xu \rangle$; then $Z \triangleleft P$, but $Z^h = \langle x^h u \rangle \leq P$ and $Z^h \neq Z$. Thus Z is not weakly closed in P with respect to N and N is hence not q-extendible, by Proposition 3.2.4. □

Examples 3.4.6. (i) If $N = H \times \Sigma_3$ and $3 \in \pi(H)$, then $(1,2,3) \in Z(P)$, where $P = \langle (1,2,3) \rangle \in \mathrm{Syl}_3(\Sigma_3)$. Furthermore, $(1,2,3)^{(1,2)} = (1,3,2) \in P$, so by Lemma 3.4.5, it follows that N is not 3-extendible.

(ii) If $N = H \times A_4$ and $2 \in \pi(H)$, then $P = \langle (1,2)(3,4), (1,3)(2,4) \rangle \in \mathrm{Syl}_2(A_4)$. Furthermore, $(1,2)(3,4)^{(1,3,2)} = (1,3)(2,4) \in P$, so it follows from Lemma 3.4.5 that N is not 2-extendible.

(iii) If $N = H \times D_{2q}$ and $q \in \pi(H)$ with q odd, then N is not q-extendible. Indeed, if we let

$$D_{2q} = \langle a, b : a^q = b^2 = 1, ab = ba^{-1} \rangle,$$

then $a^b = a^{-1}$ and so it follows from Lemma 3.4.5 that N is not q-extendible.

We shall also find the following of use.

Lemma 3.4.7. Let N be a finite group and $H \triangleleft N$ be a q-group for some $q \in \pi(N)$. Set $\overline{N} = N/H$ and suppose that $\overline{P} \in \mathrm{Syl}_q(\overline{N})$. If there exists $\overline{Q} \triangleleft \overline{P}$ such that $\overline{Q}$ is not weakly closed in $\overline{P}$ with respect to $\overline{N}$, then N is not q-extendible.

Proof. We have $\overline{P} = P/H$ for some $P \in \mathrm{Syl}_q(N)$ with $H \leq P$ and $\overline{Q} = Q/H$ for some $Q \triangleleft P$ with $H \leq Q$. Furthermore, there exists $[x] \in \overline{N}$ with $x \in N$ such that $\overline{Q}^{[x]} \neq \overline{Q}$ and $\overline{Q}^{[x]} \leq \overline{P}$. Since $\overline{Q}^{[x]} = \overline{Q^x}$, it follows that $Q^x \neq Q$, and hence Q is not weakly closed in P with respect to N. So N is not q-extendible by Proposition 3.2.4. □

The final tool we have to test for non q-extendibility relates to the following two statements concerned with subgroups of N.

Lemma 3.4.8. Suppose that N is a finite group and N is a q-extendible group for some $q \in \pi(N)$. Let $P \in \mathrm{Syl}_q(N)$.

(i) If $P \leq H \leq N$, then H is q-extendible.

(ii) If $H \triangleleft N$ and $|N : H| = q^a$ for some $a \in \mathbb{N}_0$, then H is q-extendible.

Proof. The first statement follows easily from Proposition 2.2.5. For the second, suppose that $P_H \in \mathrm{Syl}_q(H)$ and $Q \leq P_H$ with $\mathcal{S}(H,Q){\downarrow_{P_H}} \ncong k[P_H/Q]$. Then $\dim \mathcal{S}(H,Q) > |P_H : Q|$, and by Green's indecomposability criterion, $\mathcal{S}(H,Q){\uparrow^N}$ is indecomposable. Thus $\mathcal{S}(N,Q) \cong \mathcal{S}(H,Q){\uparrow^N}$ and has dimension

$$|N : H| \cdot (\dim \mathcal{S}(H,Q)) > |N : H||P_H : Q| = |P : Q|,$$

which contradicts the q-extendibility of N. So H is q-extendible. □

This covers the theoretical means we have to show that a given p-local subgroup N is not q-extendible. At times however, the structure provided in the tables in [18] is too vague for us to argue theoretically, and in these cases we may show that N is not q-extendible using the following algorithm.

Algorithm 2. Suppose that N is a finite group and $q \in \pi(N)$. The following algorithm returns `true` if N is not q-extendible.

Step 1: Fix a Sylow q-subgroup P of N and let X denote the set of normal subgroups in P.
Step 2: For each $Q \in X$, let $C(Q) = \{Q^n : n \in N, Q^n \leq P\}$. If $|C(Q)| > 1$, then return `true`.
Step 3: If $|C(Q)| = 1$ for all $Q \in X$, return `false`.

Note that if this algorithm returns `false` for a given finite group N and prime $q \in \pi(N)$, it does not necessarily mean that N is q-extendible. As with Algorithm 1, we include a MAGMA implementation of this algorithm in the appendix.

3.4.2 Notation

We now explain the notation used in the tables; most of this is taken from [18, Section 5.3], but we repeat it here for the convenience of the reader. Within each table, we list the possible classes of conjugacy classes of elements of order p that lie in the particular sporadic group G; these are labelled following the notation in [18] as $pA, pB, pC, \ldots$ where multiple such classes arise for a specific prime p. Corresponding to the class pX, we describe the structure of the normaliser $N_G(\langle x \rangle)$, where x is an element of type pX.

We represent the cyclic group of order n by simply writing n; this should not be confused with p^{a+b}, which represents a certain group with centre of order p^a and of exponent p if $p > 2$ (e.g., an extraspecial group). Additionally, the symbol $\underline{n}$ represents a group of order n, with no additional structure implied about $\underline{n}$.

The elementary abelian group of order 2^n is represented by E_{2^n}, while $(D_8^*)^n$ and $(Q_8^*)^n$ represent a 2-group which is a central product of n copies of D_8 and Q_8 respectively. If A and B are groups of any type, then AB represents a group X which contains a normal subgroup H such that $H \cong A$ and $X/H \cong B$. Furthermore, $A \cdot B$ and $A\#B$ represent a group X of type AB with the following additional constraints:

(i) $A \cdot B$ implies that $C_X(H) \leq H$ and X splits over H;

(ii) $A\#B$ implies that $H \not\leq Z(X)$ and X does not split over H.

We write N to denote a fixed p-local subgroup of G, and when we are studying a particular p-local subgroup N, we write P_q to represent a Sylow q-subgroup of N.

To avoid unnecessary repetition, we identify a class of possible structures for N where the q-extendibility can be determined by general arguments. Thus, given a p-local subgroup N, we say that it is of type (X) with $X \in \{A, B, C, D, E, F, G\}$ if it satisfies the corresponding structure outlined in the following list.

(A) **$N = n \cdot m$ for coprime integers n and m.**
If this occurs, then N is q-extendible for all primes $q \in \pi(N)$ by Lemma 3.4.2 (iii).

(B) **$N = HK$, where $K = A_q$ or $K = \Sigma_q$, $\pi(N) = \pi(K)$, $q \geq 5$ is prime and $q \notin \pi(H)$.**
By Theorem 3.4.1 and Lemma 3.4.4, we know that if $K = A_q$, then N is q-extendible and not s-extendible for all $s \in \pi(N) - \{q\}$. On the other hand, if $K = \Sigma_q$, then since Σ_q contains Σ_{q-1} as a subgroup of index q, it follows that K is q-extendible by Lemma 3.4.2 (iv) and hence so is N, by Lemma 3.4.2 (ii); moreover, Σ_q contains A_q as a normal subgroup and hence $c_s(K) > 1$ for all $s \in \pi(N) - \{q\}$, since $c_s(A_q) > 1$ for all such s, so $c_s(N) > 1$. We conclude that N is not s-extendible for all $s \in \pi(N) - \{q\}$.

(C) **$N = HL_n(r)$, where $\pi(N) = \pi(L_n(r))$, $(r^n - 1)/(r - 1) = q$ for a prime q, $q \notin \pi(H)$ and $|L_n(r)|_q = q$.**
By Theorem 3.4.1, we know that $c_s(L_n(r)) > 1$ for all $s \in \pi(N) - \{q\}$ and hence $c_s(N) > 1$ by Lemma 3.4.4, so N is not s-extendible if $s \in \pi(N) - \{q\}$. On the other hand, $L_n(r)$ is q-extendible by Theorem 3.4.1 and since $q \notin \pi(H)$ it follows that N is q-extendible as well, by Lemma 3.4.2 (ii).

(D) **$N = H \times K$, where $K = A_q$ or $K = \Sigma_q$, $\pi(N) = \pi(K)$, $q \geq 5$ is a prime number and $q \in \pi(H)$.**
By Theorem 3.4.1, we know that if $K = A_q$, then $c_s(K) > 1$ for all $s \in \pi(N) - \{q\}$, and hence $c_s(N) > 1$ for all such s by Lemma 3.4.4, so N is not s-extendible for all $s \in \pi(N) - \{q\}$. Note that $P = \langle (1, 2, \ldots, q) \rangle \in \mathrm{Syl}_q(K)$ and the conjugacy class of $(1, 2, \ldots, q)$ in A_q is half the size of the conjugacy class of $(1, 2, \ldots, q)$ in Σ_q. Thus, $(1, 2, \ldots, q) \sim_{A_q} (1, 2, \ldots, q)^\alpha$ for some $\alpha \in \{2, \ldots, q - 1\}$, so there exist two distinct nontrivial powers of $(1, 2, \ldots, q)$ which are conjugate in K and by Lemma 3.4.5 it

follows that N is not q-extendible. We conclude that N is not s-extendible for all primes $s \in \pi(N)$.

(E) **$N = H\,\mathrm{Aut}(S)$ for a simple group S such that $c_q(S) > 1$ for all $q \in \pi(S)$ and $\pi(N) = \pi(S)$.**
Since $S \lhd \mathrm{Aut}(S)$, it follows that $c_q(\mathrm{Aut}(S)) > 1$ and hence $c_q(N) > 1$ for all $q \in \pi(N)$, by Lemma 3.4.4. Thus N is not q-extendible for all primes $q \in \pi(N)$.

(F) **$N = HS$ for a simple group S such that $c_q(S) > 1$ for all $q \in \pi(S)$ and $\pi(N) = \pi(S)$.**
By Lemma 3.4.4, we know that $c_q(N) > 1$ for all $q \in \pi(N)$, so N is not q-extendible for all primes $q \in \pi(N)$.

(G) **There exists $H \lhd N$ such that $\pi(N) = \pi(H)$ and $c_q(H) > 1$ for all $q \in \pi(N)$.**
Since $c_q(H) > 1$ for all $q \in \pi(N)$ and $H \lhd N$, it follows from Lemma 3.4.4 that $c_q(N) > 1$ for all $q \in \pi(N)$, so N is not q-extendible for all $q \in \pi(N)$. Note that we have this type if the normal subgroup H is of type (D), (E) or (F).

Our analysis is carried out on a case-by-case basis and for each case we set N to equal $N_G(\langle x \rangle)$, where x is an element of the type being studied in that case. If we wish to refer back to a prior case, we do so using the terminology (X, Y), where X denotes the specific sporadic group and Y refers to the particular conjugacy class of p-element: so, e.g., $(M_{11}, 2A)$ refers to the analysis which was carried out when studying $G = M_{11}$ and $N = N_G(\langle x \rangle)$ for an element $x \in M_{11}$ of type $2A$. We shall omit from our discussion those cases which fall under any of the types $(A) - (G)$, but we clearly indicate when a given p-local subgroup is of such a type in the tables.

3.4.3 Mathieu Groups

There are five sporadic Mathieu groups, which were originally discovered by Émile Mathieu between 1860 and 1873. As the oldest class of sporadic groups, it is unsurprising that they have the simplest structure amongst the four classes. Our results involving the five Mathieu groups are summarised in Tables 3.1 - 3.5.

$G = M_{11}, \quad |G| = 2^4 \cdot 3^2 \cdot 5 \cdot 11$

<u>2A</u>: If $q = 2$, then from [8, Lemma 1], we know that a Sylow 2-subgroup of N is given by:

$$P_2 = \left\langle A := \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}, B := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\rangle \cong SD_{16}.$$

If we let $Q = \langle A^2 \rangle$, then $Q \lhd P_2$, but

$$Q^M = \langle A^6 B \rangle \neq Q, \quad \text{where} \quad M = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Thus N is not a 2-extendible group by Proposition 3.2.4. On the other hand, P_2 is a subgroup of index 3 in $GL_2(3)$ and $|P_3| = 3$, so it follows that N is 3-extendible, by Lemma 3.4.2 (iv).

<u>3A</u>: If $q = 2$, then Σ_3 is 2-nilpotent and hence N is 2-nilpotent, so N is 2-extendible. On the other hand, N is not 3-extendible, by Example 3.4.6 (i).

$G = M_{12}, \quad |G| = 2^6 \cdot 3^3 \cdot 5 \cdot 11$

<u>2B</u>: If $q = 2$, then N is not 2-extendible, by Algorithm 2. On the other hand, Σ_3 is 3-extendible, and hence so is N, by Lemma 3.4.2 (ii).

<u>3A</u>: Since N is 2-nilpotent, it is 2-extendible. On the other hand, N is not 3-extendible, by Algorithm 2.

<u>3B</u>: If $q = 2$ or 3, then we know that N is not q-extendible, by Examples 3.4.6 (i) and (ii).

<u>5A</u>: We know that $P_5 \lhd N$, so $N \cong 5 \cdot 8$ and hence it follows that N is q-extendible for all $q \in \pi(N)$, by Lemma 3.4.2 (iv).

$G = M_{22}, \quad |G| = 2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$

<u>2A</u>: Let $P = \langle (1,2,3,4), (1,2)(3,4) \rangle \in \mathrm{Syl}_2(\Sigma_4)$. Then $Q = \langle (1,3)(2,4) \rangle \lhd P$ and Q is not weakly closed in P with respect to Σ_4, since $Q^{(2,3)} = \langle (1,2)(3,4) \rangle \neq Q$. By Lemma 3.4.7, it follows that N is not 2-extendible. On the other hand, Σ_4 is 3-extendible by

Class	Normaliser in G	Order	q-extendible	Type?
$2A$	$GL_2(3)$	$2^4 \cdot 3$	3	-
$3A$	$\Sigma_3 \times \Sigma_3$	$2^2 \cdot 3^2$	2	-
$5A$	$5 \cdot 4$	$2^2 \cdot 5$	$2, 5$	A
$11A$	$11 \cdot 5$	$5 \cdot 11$	$5, 11$	A

Table 3.1 p-local subgroups: $G = M_{11}$

Class	Normaliser in G	Order	q-extendible	Type?
$2A$	$2 \times \Sigma_5$	$2^4 \cdot 3 \cdot 5$	5	B
$2B$	$(Q_8^*)^2 \cdot \Sigma_3$	$2^6 \cdot 3$	3	-
$3A$	$3^{1+2} \cdot E_{2^2}$	$3^3 \cdot 2^2$	2	-
$3B$	$\Sigma_3 \times A_4$	$2^3 \cdot 3^2$	-	-
$5A$	$10 \cdot 4$	$2^3 \cdot 5$	$2, 5$	-
$11A$	$11 \cdot 5$	$5 \cdot 11$	$5, 11$	A

Table 3.2 p-local subgroups: $G = M_{12}$

Class	Normaliser in G	Order	q-extendible	Type?
$2A$	$E_{2^4} \cdot \Sigma_4$	$2^7 \cdot 3$	3	-
$3A$	$(3 \times A_4) \cdot 2$	$2^3 \cdot 3^2$	3	-
$5A$	$5 \cdot 4$	$2^2 \cdot 5$	$2, 5$	A
$7A$	$7 \cdot 3$	$3 \cdot 7$	$3, 7$	A
$11A$	$11 \cdot 5$	$5 \cdot 11$	$5, 11$	A

Table 3.3 p-local subgroups: $G = M_{22}$

Class	Normaliser in G	Order	q-extendible	Type?
$2A$	$E_{2^4} \cdot L_3(2)$	$2^7 \cdot 3 \cdot 7$	7	C
$3A$	$(3 \times A_5) \cdot 2$	$2^3 \cdot 3^2 \cdot 5$	5	-
$5A$	$15 \cdot 4$	$2^2 \cdot 3 \cdot 5$	$2, 3, 5$	A
$7A$	$14 \cdot 3$	$2 \cdot 3 \cdot 7$	$2, 3, 7$	A
$11A$	$11 \cdot 5$	$5 \cdot 11$	$5, 11$	A
$23A$	$23 \cdot 11$	$11 \cdot 23$	$11, 23$	A

Table 3.4 p-local subgroups: $G = M_{23}$

Lemma 3.4.2 (iv), since the order of a Sylow 3-subgroup of Σ_4 is 3 and P is a subgroup of Σ_4 of index 3. By Lemma 3.4.2 (ii), it follows that N is 3-extendible.

$\underline{3A}$: Note that $P = \langle (1,2)(3,4), (1,3)(2,4) \rangle \in \mathrm{Syl}_2(A_4)$ and $Q = \langle (1,2)(3,4) \rangle$ is a normal subgroup of P which is not weakly closed in P with respect to A_4, since $Q^{(1,3,2)} = \langle (1,3)(2,4) \rangle \neq Q$. Thus A_4 is not 2-extendible, by Proposition 3.2.4. By a combination of Lemmas 3.4.8 (i) and (ii), it follows that N is not 2-extendible. Furthermore, N is 3-extendible, by Algorithm 1.

$$G = M_{23}, \qquad |G| = 2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23$$

$\underline{3A}$: If $q = 2$ or 3, then $c_q(A_5) > 1$ by Theorem 3.4.1 and hence $c_q(3 \times A_5) > 1$ by Lemma 3.4.4, so $c_q(N) > 1$ and hence N is not q-extendible. On the other hand, $c_5(A_5) = 1$ by Theorem 3.4.1, so $c_5(3 \times A_5) = 1$ and hence $c_5(N) = 1$, by Lemma 3.4.4. Since $|P_5| = 5$, it follows that N is 5-extendible.

$$G = M_{24}, \qquad |G| = 2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$$

$\underline{5A}$: We saw in the case $(M_{22}, 3A)$ that A_4 is not 2-extendible and hence neither is N, by a combination of Lemma 3.4.8 (i) and (ii). On the other hand, $c_3(A_4) = 1$, since A_4 contains a subgroup of index 3, so $c_3(5 \times A_4) = 1$ and hence $c_3(N) = 1$, by Lemma 3.4.4. Since $|P_3| = 3$, it follows that N is 3-extendible. Finally, note that $P_5 \lhd N$ and $|P_5| = 5$, so N is 5-extendible by Lemma 3.4.2 (iv).

$\underline{7A}$: Note that $P_7 \lhd N$ and $N/P_7 \cong 3 \times \Sigma_3$. Since Σ_3 is 2-extendible, so too is $3 \times \Sigma_3$ and hence N is 2-extendible, by Lemma 3.4.2 (ii). On the other hand, by Example 3.4.6 (i), we know that $3 \times \Sigma_3$ is not 3-extendible and hence neither is N, by Lemma 3.4.2 (ii). Finally, $P_7 \lhd N$ and $|P_7| = 7$, so N is 7-extendible by Lemma 3.4.2 (iv).

3.4.4 Leech Lattice Groups

The Leech lattice groups are a class of seven sporadic groups which are involved in the group Co_0, the full automorphism group of the Leech lattice. The notation Co_0 is in honour of John Conway, who studied this automorphism group in the 1980s, and the

Class	Normaliser in G	Order	q-extendible	Type?
$2A$	$(D_8^*)^3 \cdot L_3(2)$	$2^{10} \cdot 3 \cdot 7$	7	C
$2B$	$E_{2^6} \cdot \Sigma_5$	$2^9 \cdot 3 \cdot 5$	5	B
$3A$	$3A_6 \cdot 2$	$2^4 \cdot 3^3 \cdot 5$	-	G
$3B$	$\Sigma_3 \times L_3(2)$	$2^4 \cdot 3^2 \cdot 7$	7	C
$5A$	$(5 \times A_4) \cdot 4$	$2^4 \cdot 3 \cdot 5$	$3, 5$	-
$7A$	$(7 \cdot 3) \times \Sigma_3$	$2 \cdot 3^2 \cdot 7$	$2, 7$	-
$11A$	$11 \cdot 10$	$2 \cdot 5 \cdot 11$	$2, 5, 11$	A
$23A$	$23 \cdot 11$	$11 \cdot 23$	$11, 23$	A

Table 3.5 p-local subgroups: $G = M_{24}$

three largest Leech lattice groups are all named after him; the others include HS, the Higman-Sims group, the Janko group J_2, the McLaughlin group McL and the Suzuki group Suz. Our findings for the seven Leech lattice groups are summarised in Tables 3.6 - 3.12.

$G = HS, \quad |G| = 2^9 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11$

5A: Note that N is 2-nilpotent, so it is 2-extendible. On the other hand, P_5 is a Sylow 5-subgroup of G, so P_5 contains an element of type $(5C)$, say x. It follows from Table 3.6 that $|N_N(\langle x \rangle)|_{5'} \leq 2^2$, so by Theorem 2.4.2, we see that

$$\dim \mathcal{S}(N, \langle x \rangle) = \frac{|N : P_5|}{|N_N(\langle x \rangle)|_{5'}} \cdot |P_5 : \langle x \rangle| \geq \frac{2^4}{2^2} \cdot |P_5 : \langle x \rangle| > |P_5 : \langle x \rangle|.$$

Thus N is not 5-extendible.

5C: Since N is 2-nilpotent, it is 2-extendible. On the other hand, N is not 5-extendible, by Algorithm 2.

$G = J_2, \quad |G| = 2^7 \cdot 3^3 \cdot 5^2 \cdot 7$

3B: This case is identical to the case $(M_{12}, 3B)$; we conclude that N is not q-extendible for all $q \in \pi(N)$.

Class	Normaliser in G	Order	q-extendible	Type?
$2A$	$(D_8 * D_8 * 4)\Sigma_5$	$2^9 \cdot 3 \cdot 5$	5	B
$2B$	$2 \times \mathrm{Aut}(A_6)$	$2^6 \cdot 3^2 \cdot 5$	-	E
$3A$	$\Sigma_3 \times \Sigma_5$	$2^4 \cdot 3^2 \cdot 5$	5	B
$5A$	$5^{1+2} \cdot (8 \cdot 2)$	$2^4 \cdot 5^3$	2	-
$5B$	$(5 \cdot 4) \times A_5$	$2^4 \cdot 3 \cdot 5^2$	-	D
$5C$	$E_{5^2} \cdot 4$	$2^2 \cdot 5^2$	2	-
$7A$	$7 \cdot 6$	$2 \cdot 3 \cdot 7$	$2, 3, 7$	A
$11A$	$11 \cdot 5$	$5 \cdot 11$	$5, 11$	A

Table 3.6 p-local subgroups: HS

Class	Normaliser in G	Order	q-extendible	Type?
$2A$	$(Q_8 * D_8) \cdot A_5$	$2^7 \cdot 3 \cdot 5$	5	B
$2B$	$E_{2^2} \times A_5$	$2^4 \cdot 3 \cdot 5$	5	B
$3A$	$3A_6 \cdot 2$	$2^4 \cdot 3^3 \cdot 5$	-	G
$3B$	$\Sigma_3 \times A_4$	$2^3 \cdot 3^2$	-	-
$5A$	$D_{10} \times A_5$	$2^3 \cdot 3 \cdot 5^2$	-	D
$5C$	$D_{10} \times D_{10}$	$2^2 \cdot 5^2$	2	-
$7A$	$7 \cdot 6$	$2 \cdot 3 \cdot 7$	$2, 3, 7$	A

Table 3.7 p-local subgroups: $G = J_2$

Class	Normaliser in G	Order	q-extendible	Type?
$2A$	$2A_8$	$2^7 \cdot 3^2 \cdot 5 \cdot 7$	-	F
$3A$	$3^{1+4} \cdot (SL_2(5)\#2)$	$2^4 \cdot 3^6 \cdot 5$	5	-
$3B$	$E_{3^4} \cdot \Sigma_4$	$2^3 \cdot 3^5$	-	-
$5A$	$(5^{1+2} \cdot 3) \cdot 8$	$2^3 \cdot 3 \cdot 5^3$	$2, 3$	-
$5B$	$E_{5^2} \cdot 4$	$2^2 \cdot 5^2$	2	-
$7A$	$14 \cdot 3$	$2 \cdot 3 \cdot 7$	$2, 3, 7$	A
$11A$	$11 \cdot 5$	$5 \cdot 11$	$5, 11$	A

Table 3.8 p-local subgroups: $G = McL$

<u>$5C$</u>: Note that D_{10} is 2-nilpotent, so N is 2-nilpotent and hence 2-extendible. On the other hand, N is not 5-extendible by Example 3.4.6 (iii).

$$G = McL, \qquad |G| = 2^7 \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11$$

<u>$3A$</u>: If $q = 2$ or 3, then $c_q(L_2(5)) > 1$, by Theorem 3.4.1. Since $L_2(5)$ is isomorphic to a quotient group of $SL_2(5)$, it follows from Lemma 3.4.4 that $c_q(SL_2(5)) > 1$ and hence $c_q(SL_2(5)\#2) > 1$. Thus $c_q(N) > 1$ and we determine that N is neither 2-extendible nor 3-extendible. On the other hand, $c_5(L_2(5)) = 1$ by Theorem 3.4.1 and hence $c_5(SL_2(5)) = 1$ by Lemma 3.4.4, since $Z(SL_2(5))$ is solvable. Since $(SL_2(5)\#2)/SL_2(5) \cong C_2$ is solvable, it follows that $c_5(SL_2(5)\#2) = 1$. Moreover, $|P_5| = 5$, so we see that $SL_2(5)\#2$ is 5-extendible, and hence so is N, by Lemma 3.4.2 (ii).

<u>$3B$</u>: As was seen in the case $(M_{22}, 2A)$, Σ_4 is not 2-extendible, and hence neither is N by Lemma 3.4.2 (ii). On the other hand, N is not 3-extendible, by Algorithm 2.

<u>$5A$</u>: Note that N is 2-nilpotent, so N is 2-extendible. Furthermore, $P_5 \triangleleft N$, so P_2P_5 is a subgroup of index 3 in N; since $|P_3| = 3$, it follows from Lemma 3.4.2 (iv) that N is 3-extendible. Finally, note that $P_5 \in \mathrm{Syl}_5(G)$, so there exists an element of type $(5B)$ in P_5, say x. In particular, $|N_N(\langle x\rangle)|_{5'} \leq 2^2$, so by Theorem 2.4.2, it follows that

$$\dim \mathcal{S}(N, \langle x\rangle) = \frac{|N : P_5|}{|N_N(\langle x\rangle)|_{5'}} \cdot |P_5 : \langle x\rangle| \geq \frac{2^3 \cdot 3}{2^2} \cdot |P_5 : \langle x\rangle| > |P_5 : \langle x\rangle|$$

and hence N is not 5-extendible.

<u>$5B$</u>: Note that N is 2-nilpotent, so it is 2-extendible. On the other hand, N is not 5-extendible, by Algorithm 2.

$$G = Suz, \qquad |G| = 2^{13} \cdot 3^7 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$$

<u>$3B$</u>: From [8, Lemma 1], we know that a Sylow 2-subgroup of $SL_2(3)$ is given by:

$$P = \left\langle A := \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}, B := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\rangle \cong Q_8.$$

Class	Normaliser in G	Order	q-extendible	Type?
$2A$	$(Q_8^*)^2\#\Omega_6^-(2)$	$2^{13}\cdot 3^4\cdot 5$	-	F
$2B$	$(E_{2^2}\times L_3(4))\cdot 2$	$2^9\cdot 3^2\cdot 5\cdot 7$	-	G
$3A$	$3U_4(3)\cdot 2$	$2^8\cdot 3^7\cdot 5\cdot 7$	-	G
$3B$	$3^{2+4}\cdot(SL_2(3)*D_8)$	$2^5\cdot 3^7$	-	-
$3C$	$(E_{3^2}\cdot 2)\times A_6$	$2^4\cdot 3^4\cdot 5$	-	F
$5A$	$(D_{10}\times A_6)\#2$	$2^5\cdot 3^2\cdot 5^2$	-	G
$5B$	$(D_{10}\times A_5)\#2$	$2^4\cdot 3\cdot 5$	-	G
$7A$	$((7\cdot 3)\times A_4)\cdot 2$	$2^3\cdot 3^2\cdot 7$	7	-
$11A$	$11\cdot 10$	$2\cdot 5\cdot 11$	$2,5,11$	A
$13A$	$13\cdot 6$	$2\cdot 3\cdot 13$	$2,3,13$	A

Table 3.9 p-local subgroups: $G=Suz$

Class	Normaliser in G	Order	q-extendible	Type?
$2A$	$(D_8^*)^4\Omega_8^+(2)$	$2^{21}\cdot 3^5\cdot 5^2\cdot 7$	-	F
$2B$	$(E_{2^2}\times G_2(4))\cdot 2$	$2^{12}\cdot 3^3\cdot 5^2\cdot 7\cdot 13$	-	G
$2C$	$E_{2^{11}}\,\mathrm{Aut}(M_{12})$	$2^{18}\cdot 3^3\cdot 5\cdot 11$	-	E
$3A$	$3Suz\cdot 2$	$2^{14}\cdot 3^8\cdot 5^2\cdot 7\cdot 11\cdot 13$	-	G
$3B$	$(3\times 3)U_4(3)\cdot E_{2^2}$	$2^9\cdot 3^8\cdot 5\cdot 7$	-	G
$3C$	$3^{1+4}\cdot GSp_4(3)$	$2^8\cdot 3^9\cdot 5$	-	-
$3D$	$\Sigma_3\times A_9$	$2^7\cdot 3^5\cdot 5\cdot 7$	-	F
$5A$	$(D_{10}\times J_2)\#2$	$2^9\cdot 3^3\cdot 5^3\cdot 7$	-	G
$5B$	$(D_{10}\times(A_5\wr 2))\#2$	$2^7\cdot 3^2\cdot 5^3$	-	-
$5C$	$5^{1+2}\cdot GL_2(5)$	$2^5\cdot 3\cdot 5^4$	-	-
$7A$	$((7\cdot 3)\times A_7)\cdot 2$	$2^4\cdot 3^3\cdot 5\cdot 7^2$	-	-
$7B$	$((7\cdot 3)\times L_2(7))\cdot 2$	$2^4\cdot 3^2\cdot 7^2$	-	-
$11A$	$(11\cdot 10)\times\Sigma_3$	$2^2\cdot 3\cdot 5\cdot 11$	$2,3,5,11$	-
$13A$	$((13\cdot 6)\times A_4)\cdot 2$	$2^4\cdot 3^2\cdot 13$	13	-
$23A$	$23\cdot 11$	$11\cdot 23$	$11,23$	A

Table 3.10 p-local subgroups: $G=Co_1$

The subgroup $Q = \langle B \rangle$ is normal in P and is not weakly closed in P with respect to $SL_2(3)$, since

$$Q^M = \langle A \rangle \neq Q, \quad \text{where } M = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}.$$

There therefore exists a normal subgroup of a Sylow 2-subgroup of $SL_2(3) * D_8$ which is not weakly closed in $SL_2(3) * D_8$, so $SL_2(3) * D_8$ is not 2-extendible, by Proposition 3.2.4. Thus N is not 2-extendible by Lemma 3.4.2 (ii). Furthermore, N is not 3-extendible, by Algorithm 2.

<u>$7A$</u>: We saw in the case $(M_{22}, 3A)$ that A_4 is not 2-extendible, and hence neither is $(7 \cdot 3) \times A_4$ by Lemma 3.4.8 (i). Since $(7 \cdot 3) \times A_4$ is a normal subgroup of N with index 2, it follows that N is not 2-extendible by Lemma 3.4.8 (ii). Furthermore, N is not 3-extendible, by Algorithm 2. Finally, $P_7 \lhd N$ and $|P_7| = 7$, so it follows that N is 7-extendible, by Lemma 3.4.2 (iv).

$$G = Co_1, \quad |G| = 2^{21} \cdot 3^9 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 23$$

<u>$3C$</u>: If $q \in \pi(N)$, then $c_q(PSp_4(3)) > 1$ by Theorem 3.4.1. Since $PSp_4(3)$ is isomorphic to a quotient group of $Sp_4(3)$, it follows from Lemma 3.4.4 that $c_q(Sp_4(3)) > 1$. Furthermore, $Sp_4(3)$ is a subgroup of index 2 in $GSp_4(3)$, so we deduce that $c_q(GSp_4(3)) > 1$ and hence $c_q(N) > 1$ for all $q \in \pi(N)$. Thus N is not q-extendible for all $q \in \pi(N)$.

<u>$5B$</u>: If $q = 2$ or 3, then $c_q(A_5) > 1$ by Theorem 3.4.1 and hence $c_q(A_5 \times A_5) > 1$ by Lemma 3.4.4. Thus $c_q(A_5 \wr 2) > 1$ and hence $c_q(D_{10} \times (A_5 \wr 2)) > 1$. We conclude that $c_q(N) > 1$ and hence N is not q-extendible. If $q = 5$, then $A_5 \wr 2$ contains an element x of order 5 such that $x \in P \in \mathrm{Syl}_5(A_5 \wr 2)$ and there exists $h \in A_5 \wr 2$ with $x^h \neq x$ and $x^h \in P$. By Lemma 3.4.5, it follows that $D_{10} \times (A_5 \wr 2)$ is not 5-extendible and hence neither is N, by Lemma 3.4.8 (i).

<u>$5C$</u>: If $q = 2$ or 3, then since $L_2(5)$ is isomorphic to a quotient group of $SL_2(5)$ and $c_q(L_2(5)) > 1$ by Theorem 3.4.1, it follows from Lemma 3.4.4 that $c_q(SL_2(5)) > 1$. But $SL_2(5) \lhd GL_2(5)$, so $c_q(GL_2(5)) > 1$ and hence $c_q(N) > 1$, so N is not q-extendible. It follows from Algorithm 2 that N is not 5-extendible.

<u>$7A$</u>: Note that $(7 \cdot 3) \times A_7$ is a group of type (D) and hence $c_q((7 \cdot 3) \times A_7) > 1$ for all $q \in \pi(N)$. Thus $c_q(N) > 1$ by Lemma 3.4.4 and hence N is not q-extendible for all $q \in \pi(N)$.

<u>$7B$</u>: By [8], a Sylow 2-subgroup of $SL_2(7)$ is given by:

$$P = \left\langle A := \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, B := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\rangle \cong Q_{16}$$

Furthermore, $Q = \langle A^2 \rangle$ is normal in P and is not weakly closed in P with respect to $SL_2(7)$, since

$$Q^M = \langle A^4 B \rangle \neq Q, \quad \text{where} \quad M = \begin{bmatrix} 0 & 2 \\ 3 & 1 \end{bmatrix}.$$

Thus a Sylow 2-subgroup of $L_2(7)$ contains a normal subgroup which is not weakly closed in $L_2(7)$, so it follows that $L_2(7)$ is not 2-extendible by Proposition 3.2.4. By a combination of Lemmas 3.4.8 (i) and (ii), we conclude that N is not 2-extendible. Furthermore, by Theorem 3.4.1, we know that $c_3(L_2(7)) > 1$ and hence by Lemma 3.4.4, we see that $c_3(N) > 1$, so N is not 3-extendible. Finally, note that a Sylow 7-subgroup of $SL_2(7)$ is given by:

$$P = \left\langle X := \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right\rangle$$

and we have

$$X^M = X^2, \quad \text{where } M = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}.$$

Thus $(7 \cdot 3) \times L_2(7)$ is not 7-extendible by Lemma 3.4.5 and hence N is not 7-extendible, by Lemma 3.4.8 (i).

<u>$11A$</u>: Since $P_{11} \lhd N$ and $N/P_{11} \cong 10 \times \Sigma_3$ is 2-nilpotent, it follows that N is 2-extendible, by Lemma 3.4.2 (ii). Furthermore, Σ_3 is 3-extendible, so N is 3-extendible. If $q = 5$ or 11, then $11 \cdot 10$ is q-extendible by Lemma 3.4.2 (iii) and hence N is q-extendible. So N is q-extendible for all $q \in \pi(N)$.

<u>$13A$</u>: As we saw in the case $(M_{22}, 3A)$, A_4 contains a normal subgroup of a Sylow 2-subgroup of A_4 which is not weakly closed in A_4, so it follows from Lemma 3.4.7 that $(13 \cdot 6) \times A_4$ is not 2-extendible. We therefore see that N is not 2-extendible, by Lemma 3.4.8 (ii). Checking with Algorithm 2, we determine that N is not 3-extendible. Finally, note that $P_{13} \lhd N$ and $|P_{13}| = 13$, so N is 13-extendible, by Lemma 3.4.2 (iv).

$$G = Co_2, \quad |G| = 2^{18} \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$$

<u>$3A$</u>: Note that $(D_8{*}Q_8)\#\Sigma_5$ is a group of type (B), so if $q = 2$ or 3, then $c_q((D_8{*}Q_8)\#\Sigma_5) > 1$ and hence $c_q(N) > 1$, by Lemma 3.4.4, so N is not q-extendible. On the other hand, Σ_5 is 5-extendible and hence N is 5-extendible by Lemma 3.4.2 (ii).

<u>$5A$</u>: The same argument as seen in $(Suz, 3B)$ shows that $4 * SL_2(3)$ is not 2-extendible and hence $(4 * SL_2(3))\#2$ is not 2-extendible by Lemma 3.4.8 (ii). By Lemma 3.4.2 (ii), it follows that N is not 2-extendible. On the other hand, $SL_2(3) \cong Q_8 \rtimes C_3$ is 3-nilpotent, so $4 * SL_2(3)$ is as well and hence $c_3(4 * SL_2(3)) = 1$. It follows that $c_3((4 * SL_2(3))\#2) = 1$ by Lemma 3.4.4 and since $|P_3| = 3$, we deduce that $(4 * SL_2(3))\#2$ is 3-extendible. Thus N is 3-extendible, by Lemma 3.4.2 (ii). Finally, we show using Algorithm 2 that N is not 5-extendible.

<u>$7A$</u>: Note that N is 2-nilpotent, so it is 2-extendible. Furthermore, since $P_7 \lhd N$, P_2P_7 is a subgroup of N of index 3, so N is 3-extendible by Lemma 3.4.2 (iv). Finally, note that $P_7 \lhd N$ and $|P_7| = 7$, so N is 7-extendible, by Lemma 3.4.2 (iv) again.

$$G = Co_3, \quad |G| = 2^{10} \cdot 3^7 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$$

<u>$3A$</u>: If $q \in \pi(N)$, then we know that $c_q(L_2(9)) > 1$ by Theorem 3.4.1. Since $L_2(9)$ is a quotient group of $SL_2(9)$, it follows from Lemma 3.4.4 that $c_q(SL_2(9)) > 1$ and hence $c_q(4 \times SL_2(9)) > 1$. Thus $c_q((4 * SL_2(9)) \cdot 2) > 1$ and hence $c_q(N) > 1$. So N is not q-extendible for all primes $q \in \pi(N)$.

<u>$3B$</u>: If $q = 2$ or 3, then $c_q(\Sigma_5) > 1$ by Theorem 3.4.1 and hence $c_q(N) > 1$ by Lemma 3.4.4, so N is not q-extendible. Note that a Sylow 5-subgroup of Σ_5 has a complement in Σ_5, so it follows that P_5 has a complement in N; thus N is 5-extendible, by Lemma 3.4.2 (iv).

Class	Normaliser in G	Order	q-extendible	Type?
$2A$	$(D_8^*)^4 Sp_6(2)$	$2^{18} \cdot 3^4 \cdot 5 \cdot 7$	-	F
$2B$	$(E_{2^4} \times (D_8^*)^3)A_8$	$2^{17} \cdot 3^2 \cdot 5 \cdot 7$	-	F
$2C$	$E_{2^{10}}\,\mathrm{Aut}(A_6)$	$2^{15} \cdot 3^2 \cdot 5$	-	E
$3A$	$3^{1+4} \cdot ((D_8 * Q_8)\#\Sigma_5)$	$2^8 \cdot 3^6 \cdot 5$	5	-
$3B$	$\Sigma_3 \times \mathrm{Aut}(PSp_4(3))$	$2^8 \cdot 3^5 \cdot 5$	-	E
$5A$	$5^{1+2} \cdot ((4 * SL_2(3))\#2)$	$2^5 \cdot 3 \cdot 5^3$	3	-
$5B$	$(5 \cdot 4) \times \Sigma_5$	$2^5 \cdot 3 \cdot 5^2$	-	D
$7A$	$((7 \cdot 3) \times D_8)2$	$2^4 \cdot 3 \cdot 7$	$2, 3, 7$	-
$11A$	$11 \cdot 10$	$2 \cdot 5 \cdot 11$	$2, 5, 11$	A
$23A$	$23 \cdot 11$	$11 \cdot 23$	$11, 23$	A

Table 3.11 p-local subgroups: $G = Co_2$

Class	Normaliser in G	Order	q-extendible	Type?
$2A$	$2Sp(6, 2)$	$2^{10} \cdot 3^4 \cdot 5 \cdot 7$	-	F
$2B$	$2 \times M_{12}$	$2^7 \cdot 3^3 \cdot 5 \cdot 11$	-	F
$3A$	$3^{1+4} \cdot ((4 * SL_2(9)) \cdot 2)$	$2^6 \cdot 3^7 \cdot 5$	-	-
$3B$	$E_{3^5} \cdot (2 \times \Sigma_5)$	$2^4 \cdot 3^6 \cdot 5$	5	-
$3C$	$\Sigma_3 \times (L_2(8) \cdot 3)$	$2^4 \cdot 3^4 \cdot 7$	-	-
$5A$	$5^{1+2} \cdot (24 \cdot 2)$	$2^4 \cdot 3 \cdot 5^3$	$2, 3$	-
$5B$	$(5 \cdot 4) \times A_5$	$2^4 \cdot 3 \cdot 5^2$	-	D
$7A$	$(7 \cdot 6) \times \Sigma_3$	$2^2 \cdot 3^2 \cdot 7$	$2, 7$	-
$11A$	$(11 \cdot 5) \times 2$	$2 \cdot 5 \cdot 11$	$2, 5, 11$	A
$23A$	$23 \cdot 11$	$11 \cdot 23$	$11, 23$	A

Table 3.12 p-local subgroups: $G = Co_3$

<u>$3C$</u>: If $q = 2$ or 7, then $c_q(L_2(8)) > 1$ by Theorem 3.4.1, so $c_q(L_2(8) \cdot 3) > 1$ by Lemma 3.4.4. Thus $c_q(N) > 1$ and it follows that N is not q-extendible. On the other hand, if $q = 3$, then it follows that N is not 3-extendible, by Example 3.4.6 (i).

<u>$5A$</u>: Note that N is 2-nilpotent and is hence 2-extendible. Furthermore, $24 \cdot 2$ is 3-extendible and hence N is 3-extendible, by Lemma 3.4.2 (ii). We use Algorithm 2 to show that N is not 5-extendible.

<u>$7A$</u>: Note that $P_7 \lhd N$ and $N/P_7 \cong 6 \times \Sigma_3$. Since $6 \times \Sigma_3$ is 2-nilpotent, it follows that N/P_7 is 2-extendible and hence N is 2-extendible, by Lemma 3.4.2 (ii). Furthermore, by Example 3.4.6 (i), $6 \times \Sigma_3$ is not 3-extendible, so neither is N. Finally, $P_7 \lhd N$, so N is 7-extendible, by Lemma 3.4.2 (iv).

3.4.5 Pariahs

The six sporadic groups which are not involved in the monster in some way are known as the *pariahs*. This class of groups includes the three remaining Janko groups, the O'Nan group $O'N$, the Rudvalis group Ru and the Lyons group Ly. The local structure of these six groups is simpler than that of the monster sections, so we study them first. Our results are summarised in Tables 3.13 - 3.18.

$$G = J_1, \quad |G| = 2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19$$

<u>$3A$</u>, <u>$5A$</u>: Since Σ_3 and D_{10} are 2-nilpotent, it follows that N is 2-nilpotent and hence 2-extendible. Furthermore, N is 3-extendible by Lemma 3.4.2 (ii), since Σ_3 is 3-extendible; by the same reasoning, N is 5-extendible, since D_{10} is 5-extendible.

$$G = J_3, \quad |G| = 2^7 \cdot 3^5 \cdot 5 \cdot 17 \cdot 19$$

<u>$3B$</u>: Note that N is 2-nilpotent and hence is 2-extendible. Checking with Algorithm 1, we verify that N is 3-extendible.

<u>$5A$</u>: This case is the same as the case $(J_1, 3A)$; thus, we see that N is q-extendible for all $q \in \pi(N)$.

$G = J_4, \quad |G| = 2^{21} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11^3 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 43$

$\underline{2A}$: Note that $\pi(N) = \pi(M_{22})$ and $3M_{22} \cdot 2$ is a group of type (G) by Theorem 3.4.1. Thus $c_q(3M_{22} \cdot 2) > 1$ and hence $c_q(N) > 1$ for all $q \in \pi(N)$, by Lemma 3.4.4. Thus N is not q-extendible for all $q \in \pi(N)$.

$\underline{2B}$: By Theorem 3.4.1, we have $c_q(M_{22}) > 1$ for all $q \in \pi(N)$. Thus, by Lemma 3.4.4, it follows that $c_q(M_{22} \cdot 2) > 1$ and hence $c_q(N) > 1$. So N is not q-extendible for all $q \in \pi(N)$.

$\underline{5A}$: If $q = 2$ or 3, then $c_q(L_3(2)) > 1$ by Theorem 3.4.1, so $c_q(5 \times (E_{2^3} \# L_3(2)) > 1$ by Lemma 3.4.4 and hence $c_q(N) > 1$, so N is not q-extendible. On the other hand, $P_5 \lhd N$ and $|P_5| = 5$, so N is 5-extendible by Lemma 3.4.2 (iv). Finally, $c_7(L_3(2)) = 1$ by Theorem 3.4.1, so $c_7(5 \times (E_{2^3} \# L_3(2)) = 1$ by Lemma 3.4.4. Thus $c_7(N) = 1$ and since $|P_7| = 7$, it follows that N is 7-extendible.

$\underline{7A}$: If $q = 2$ or 3, then by Theorem 3.4.1, we know that $c_q(\Sigma_5) > 1$ and hence $c_q(N) > 1$ by Lemma 3.4.4. Thus N is not q-extendible. On the other hand, Σ_5 is 5-extendible, so N is 5-extendible by Lemma 3.4.2 (ii). Finally, note that $P_7 \lhd N$ and $|P_7| = 7$, so by Lemma 3.4.2 (iv), we see that N is 7-extendible.

$\underline{11A}$: We saw in the case $(Suz, 3B)$ that $SL_2(3)$ is not 2-extendible and hence $11^{1+2} \cdot SL_2(3)$ is not 2-extendible by Lemma 3.4.2 (ii). Furthermore, $11^{1+2} \cdot SL_2(3)$ is contained as an index 2-subgroup of a subgroup H of N which has index 5 in N. By Lemma 3.4.8 (i) and (ii), it follows that N is not 2-extendible. By Theorem 3.4.1, we know that $c_3(L_2(3)) = 1$, and since $L_2(3)$ is a quotient group of $SL_2(3)$, by Lemma 3.4.4, we see that $c_3(SL_2(3)) = 1$. Thus $c_3(N) = 1$, by Lemma 3.4.4 again; since $|P_3| = 3$, we deduce that N is 3-extendible. By Lemma 3.4.2 (ii), we see that N is 5-extendible, since $11^{1+2} \cdot SL_2(3)$ is a $5'$-group. Finally, note that $P_{11} \lhd N$ and $P_{11} \in \mathrm{Syl}_{11}(G)$. Thus, P_{11} contains an element of type $(11B)$, say x. Since $|N_N(\langle x \rangle)|_{11'} \leq 2^2 \cdot 5$, it follows from Theorem 2.4.2 that

$$\dim \mathcal{S}(N, \langle x \rangle) = \frac{|N : P_{11}|}{|N_N(\langle x \rangle)|_{11'}} \cdot |P_{11} : \langle x \rangle| \geq \frac{2^4 \cdot 3 \cdot 5}{2^2 \cdot 5} \cdot |P_{11} : \langle x \rangle| > |P_{11} : \langle x \rangle|,$$

so N is not 11-extendible.

<u>11B</u>: Note that $P_{11} \lhd N$ and N/P_{11} is a group of order $20 = 2^2 \cdot 5$. By Sylow's theorems, we know that if n_5 denotes the number of Sylow 5-subgroups of N/P_{11}, then $n_5 \equiv 1 \bmod 5$ and n_5 divides 4; thus $n_5 = 1$ and it follows that N/P_{11} is 2-nilpotent and hence 2-extendible. So N is 2-extendible, by Lemma 3.4.2 (ii). Furthermore, N is 5-nilpotent and hence 5-extendible. Finally, $D_{22} \times D_{22}$ is not 11-extendible, by Example 3.4.6 (iii) and thus by Lemma 3.4.8 (i), we deduce that N is not 11-extendible.

$G = O'N, \quad |G| = 2^9 \cdot 3^4 \cdot 5 \cdot 7^3 \cdot 11 \cdot 19 \cdot 31$

<u>5A</u>: Note that $P_3, P_5 \lhd N$, so N is 2-nilpotent and hence 2-extendible. On the other hand, N is not 3-extendible, which we verify using Algorithm 2. Finally, since $P_5 \lhd N$ and $|P_5| = 5$, it follows that N is 5-extendible, by Lemma 3.4.2 (iv).

<u>7A</u>: If $q = 2$ or 3, then $3 \times D_8$ is clearly q-extendible and hence N is q-extendible, by Lemma 3.4.2 (ii). On the other hand, $P_7 \in \mathrm{Syl}_7(G)$ and hence contains an element of type (7B), say x. Since $P_7 \lhd N$ and $|N_N(\langle x\rangle)|_{7'} \leq 2 \cdot 3$, it follows from Theorem 2.4.2 that

$$\dim \mathcal{S}(N, \langle x\rangle) = \frac{|N : P_7|}{|N_N(\langle x\rangle)|_{7'}} \cdot |P_7 : \langle x\rangle| \geq \frac{2^3 \cdot 3}{2 \cdot 3} \cdot |P_7 : \langle x\rangle| > |P_7 : \langle x\rangle|.$$

Thus N is not 7-extendible.

<u>7B</u>: Since $P_7 \lhd N$, it follows that P_3P_7 is a subgroup of index 2 in N, so N is 2-extendible, by Lemma 3.4.2 (iv). Furthermore, N is 3-extendible, since it is 3-nilpotent. Finally, by Example 3.4.6 (iii), we know that $D_{14} \times 7$ is not 7-extendible and hence neither is N, by Lemma 3.4.8 (i).

$G = Ru, \quad |G| = 2^{14} \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 13 \cdot 29$

<u>5A</u>: Since $P_5 \lhd N$, it follows that N is 2-nilpotent and hence 2-extendible. On the other hand, note that $P_5 \in \mathrm{Syl}_5(G)$ and hence P_5 contains an element of type (5B), say x. Since $P_5 \lhd N$ and $|N_N(\langle x\rangle)|_{5'} \leq 2^4$, it follows from Theorem 2.4.2 that

$$\dim \mathcal{S}(N, \langle x\rangle) = \frac{|N : P_5|}{|N_N(\langle x\rangle)|_{5'}} \cdot |P : \langle x\rangle| \geq \frac{2^5}{2^4} \cdot |P : \langle x\rangle| > |P, \langle x\rangle|.$$

Class	Normaliser in G	Order	q-extendible	Type?
$2A$	$2 \times A_5$	$2^3 \cdot 3 \cdot 5$	5	B
$3A$	$\Sigma_3 \times D_{10}$	$2^2 \cdot 3 \cdot 5$	$2, 3, 5$	-
$5A$	$\Sigma_3 \times D_{10}$	$2^2 \cdot 3 \cdot 5$	$2, 3, 5$	-
$7A$	$7 \cdot 6$	$2 \cdot 3 \cdot 7$	$2, 3, 7$	A
$11A$	$11 \cdot 10$	$2 \cdot 5 \cdot 11$	$2, 5, 11$	A
$19A$	$19 \cdot 6$	$2 \cdot 3 \cdot 19$	$2, 3, 19$	A

Table 3.13 p-local subgroups: J_1

Class	Normaliser in G	Order	q-extendible	Type?
$2A$	$(Q_8 * D_8) \cdot A_5$	$2^7 \cdot 3 \cdot 5$	5	B
$3A$	$(3 \times A_6) \cdot 2$	$2^4 \cdot 3^3 \cdot 5$	-	G
$3B$	$(E_{3^2}3^{1+2}) \cdot 2$	$2 \cdot 3^5$	$2, 3$	-
$5A$	$D_{10} \times \Sigma_3$	$2^2 \cdot 3 \cdot 5$	$2, 3, 5$	-
$17A$	$17 \cdot 8$	$2^3 \cdot 17$	$2, 17$	A
$19A$	$19 \cdot 9$	$3^2 \cdot 19$	$3, 19$	A

Table 3.14 p-local subgroups: J_3

Class	Normaliser in G	Order	q-extendible	Type?
$2A$	$(Q_8^*)^6 \#(3M_{22} \cdot 2)$	$2^{21} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11$	-	-
$2B$	$E_{2^{11}}(M_{22} \cdot 2)$	$2^{19} \cdot 3^2 \cdot 5 \cdot 7$	-	-
$3A$	$(6M_{22}) \cdot 2$	$2^9 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11$	-	G
$5A$	$(5 \times (E_{2^3} \# L_3(2))) \cdot 4$	$2^8 \cdot 3 \cdot 5 \cdot 7$	$5, 7$	-
$7A$	$(7 \cdot 3) \times \Sigma_5$	$2^3 \cdot 3^2 \cdot 5 \cdot 7$	$5, 7$	-
$11A$	$(11^{1+2} \cdot SL_2(3))10$	$2^4 \cdot 3 \cdot 5 \cdot 11^3$	$3, 5$	-
$11B$	$(D_{22} \times D_{22}) \cdot 5$	$2^2 \cdot 5 \cdot 11^2$	$2, 5$	-
$23A$	$23 \cdot 22$	$2 \cdot 11 \cdot 23$	$2, 11, 23$	A
$29A$	$29 \cdot 14$	$2 \cdot 7 \cdot 29$	$2, 7, 29$	A
$31A$	$31 \cdot 10$	$2 \cdot 5 \cdot 31$	$2, 5, 31$	A
$37A$	$37 \cdot 12$	$2^2 \cdot 3 \cdot 37$	$2, 3, 37$	A
$43A$	$43 \cdot 14$	$2 \cdot 7 \cdot 43$	$2, 7, 43$	A

Table 3.15 p-local subgroups: J_4

Class	Normaliser in G	Order	q-extendible	Type?
$2A$	$(4L_3(4)) \cdot 2$	$2^9 \cdot 3^2 \cdot 5 \cdot 7$	-	G
$3A$	$(E_{3^2} \cdot 2) \times A_6$	$2^4 \cdot 3^4 \cdot 5$	-	F
$5A$	$(D_{10} \times (E_{3^2} \cdot 4))\#2$	$2^4 \cdot 3^2 \cdot 5$	$2, 5$	-
$7A$	$7^{1+2} \cdot (3 \times D_8)$	$2^3 \cdot 3 \cdot 7^3$	$2, 3$	-
$7B$	$(D_{14} \times 7) \cdot 3$	$2 \cdot 3 \cdot 7^2$	$2, 3$	-
$11A$	$11 \cdot 10$	$2 \cdot 5 \cdot 11$	$2, 5, 11$	A
$19A$	$19 \cdot 6$	$2 \cdot 3 \cdot 19$	$2, 3, 19$	A
$31A$	$31 \cdot 15$	$3 \cdot 5 \cdot 31$	$3, 5, 31$	A

Table 3.16 p-local subgroups: $O'N$

Class	Normaliser in G	Order	q-extendible	Type?
$2A$	$2^{11}\Sigma_5$	$2^{14} \cdot 3 \cdot 5$	5	B
$2B$	$E_{2^2} \times {}^2B_2(2^{3/2})$	$2^8 \cdot 5 \cdot 7 \cdot 13$	-	F
$3A$	$3\#\,\mathrm{Aut}(A_6)$	$2^5 \cdot 3^3 \cdot 5$	-	E
$5A$	$(5^{1+2} \cdot Q_8) \cdot 4$	$2^5 \cdot 5^3$	2	-
$5B$	$(5 \cdot 4) \times A_5$	$2^4 \cdot 3 \cdot 5^2$	-	D
$7A$	$(D_{14} \times E_{2^2}) \cdot 3$	$2^3 \cdot 3 \cdot 7$	$3, 7$	-
$13A$	$(13 \cdot 4 \times E_{2^2}) \cdot 3$	$2^4 \cdot 3 \cdot 13$	$3, 13$	-
$29A$	$29 \cdot 14$	$2 \cdot 7 \cdot 29$	$2, 7, 29$	A

Table 3.17 p-local subgroups: Ru

Class	Normaliser in G	Order	q-extendible	Type?
$2A$	$2A_{11}$	$2^7 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11$	11	D
$3A$	$(3McL) \cdot 2$	$2^8 \cdot 3^7 \cdot 5^3 \cdot 7 \cdot 11$	-	G
$3B$	$3^{2+4} \cdot ((4 * SL_2(5)) \cdot 2)$	$2^5 \cdot 3^7 \cdot 5$	5	-
$5A$	$5^{1+4} \cdot ((4 * SL_2(9))\#2)$	$2^6 \cdot 3^2 \cdot 5^6$	-	-
$5B$[1]	$(5 \times (5^{1+2} \cdot \Sigma_3)) \cdot 4$	$2^3 \cdot 3 \cdot 5^4$	$2, 3$	-
$7A$[2]	$((7 \cdot 3) \times SL_2(3)) \cdot 2$	$2^4 \cdot 3^2 \cdot 7$	7	-
$11A$	$(11 \cdot 5) \times \Sigma_3$	$2 \cdot 3 \cdot 5 \cdot 11$	$2, 3, 5, 11$	A
$31A$	$31 \cdot 6$	$2 \cdot 3 \cdot 31$	$2, 3, 31$	A
$37A$	$37 \cdot 18$	$2 \cdot 3^2 \cdot 37$	$2, 3, 37$	A
$67A$	$67 \cdot 22$	$2 \cdot 11 \cdot 67$	$2, 11, 67$	A

Table 3.18 p-local subgroups: Ly

Thus N is not 5-extendible.

$\underline{7A}$: Note first that N is not 2-extendible, which we verify using Algorithm 2. On the other hand, N is 3-nilpotent and hence 3-extendible. Furthermore, $P_7 \lhd N$ and $|P_7| = 7$, so N is 7-extendible by Lemma 3.4.2 (iv).

$\underline{13A}$: We start by showing that N is not 2-extendible, using Algorithm 2. Furthermore, N is 3-nilpotent and hence 3-extendible. Finally, $P_{13} \lhd N$ and $|P_{13}| = 13$, so N is 13-extendible by Lemma 3.4.2 (iv).

$G = Ly, \quad |G| = 2^8 \cdot 3^7 \cdot 5^6 \cdot 7 \cdot 11 \cdot 31 \cdot 37 \cdot 67$

$\underline{3B}$: If $q = 2$ or 3, then $c_q(L_2(5)) > 1$ by Theorem 3.4.1 and hence $c_q(SL_2(5)) > 1$ by Lemma 3.4.4. It hence follows that $c_q(4 * SL_2(5)) > 1$ and hence $c_q(N) > 1$, so N is not q-extendible. On the other hand, note that $c_5(L_2(5)) = 1$ by Theorem 3.4.1, so by Lemma 3.4.4, we see that $c_5(4 * SL_2(5)) = 1$ and $c_5(N) = 1$. Since $|P_5| = 5$, it follows from Lemma 3.4.2 (iv) that N is 5-extendible.

$\underline{5A}$: Repeating the steps carried out in the case $(Ly, 3B)$, we see that if $q \in \pi(N)$, then $c_q(L_2(9)) > 1$ and hence $c_q(N) > 1$, so N is not q-extendible.

$\underline{5B}$: Note that $P_5 \lhd N$ and $N/P_5 \cong \Sigma_3 \cdot 4$. Since Σ_3 is 2-nilpotent, it follows that $\Sigma_3 \cdot 4$ is 2-extendible and hence so is N by Lemma 3.4.2 (ii). Furthermore, a Sylow 3-subgroup of $\Sigma_3 \cdot 4$ is normal in $\Sigma_3 \cdot 4$ and has order 3, so by Lemma 3.4.2 (iv), it follows that $\Sigma_3 \cdot 4$ is 3-extendible and hence N is 3-extendible by Lemma 3.4.2 (ii). We are unable to determine whether or not N is 5-extendible using Algorithms 1 and 2; this is mainly due to the fact that there is no permutation representation of Ly which is defined on a small enough number of points (Lyons original construction was on $8,835,156$ points and a review of the literature reveals no known permutation representations of smaller degree).

$\underline{7A}$: From [8], we know that a Sylow 2-subgroup of $SL_2(3)$ is given by:

$$P = \left\langle A := \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\rangle \cong Q_8.$$

Then $Q = \langle A^2, B\rangle$ is normal in P but is not weakly closed in P with respect to $SL_2(3)$, since

$$Q^M = \langle A^2, A\rangle \neq Q, \quad \text{where} \quad M = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Thus $SL_2(3)$ is not 2-extendible by Proposition 3.2.4, and by Lemma 3.4.8 (i) and (ii), it follows that N is not 2-extendible. Note that $P_7 \lhd N$ and $|P_7| = 7$, so N is 7-extendible by Lemma 3.4.2 (iv). When it comes to determining if N is 3-extendible or not, we encounter a similar problem to that discussed in the case $(Ly, 5B)$; since there is no reasonably sized permutation representation of Ly, we are unable to use Algorithms 1 and 2 to conclude that N is or is not 3-extendible.

3.4.6 Monster Sections

The last remaining class of sporadic groups we shall consider is the class of *monster sections*. The largest one of these is the Monster itself (or the *Fischer-Griess* Monster). The remaining groups all appear as subgroups or subquotients of the Monster, and include the Held group He, the Harada-Norton group HN, the Thompson group Th, three groups due to Fischer and the Baby Monster (a name suggested by Conway). Our findings involving this class of eight groups are summarised in Tables 3.19 - 3.26.

$$G = He, \quad |G| = 2^{10} \cdot 3^3 \cdot 5^2 \cdot 7^3 \cdot 17$$

7A: If $q = 2$ or 3, then $c_q(L_3(2)) > 1$ by Theorem 3.4.1 and hence $c_q(N) > 1$ by Lemma 3.4.4, so N is not q-extendible. On the other hand, a Sylow 7-subgroup P of $SL_3(2)$ is given by:

$$P = \left\langle X := \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \right\rangle$$

and we have

$$X^M = X^2, \quad \text{where} \quad M = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

[1] It is unknown if $(5 \times (5^{1+2} \cdot \Sigma_3)) \cdot 4$ is 5-extendible or not.
[2] It is unknown if $((7 \cdot 3) \times SL_2(3)) \cdot 2$ is 3-extendible or not.

It hence follows from Lemma 3.4.5 that N is not 7-extendible.

7C: Note that $P_7 \lhd N$ and hence by Lemma 3.4.2 (ii), N is 2-extendible if and only if $3 \times \Sigma_3$ is 2-extendible if and only if Σ_3 is 2-extendible, and we have seen that Σ_3 is 2-extendible, so it follows that N is 2-extendible. On the other hand, by Example 3.4.6 (i), we know that $3 \times \Sigma_3$ is not 3-extendible, so N is not 3-extendible either. Finally, note that $P_7 \in \mathrm{Syl}_7(He)$ and therefore P_7 contains an element of type $(7D)$, say x. Then $|N_N(\langle x \rangle)|_{7'} \leq 2 \cdot 3$, so

$$\dim \mathcal{S}(N, \langle x \rangle) = \frac{|N : P_7|}{|N_N(\langle x \rangle)|_{7'}} \cdot |P_7 : \langle x \rangle| \geq \frac{2 \cdot 3^2}{2 \cdot 3} \cdot |P_7 : \langle x \rangle| > |P_7 : \langle x \rangle|.$$

Thus N is not 7-extendible by Theorem 2.4.2.

7D: Since $P_7 \lhd N$, it follows that $P_3 P_7$ is a subgroup of index 2 in N and hence N is 2-extendible, by Lemma 3.4.2 (iv). Furthermore, N is 3-nilpotent and hence 3-extendible. Finally, note that $7 \times D_{14}$ is not 7-extendible, by Example 3.4.6 (iii) and hence N is not 7-extendible by Lemma 3.4.8 (i).

$G = HN, \quad |G| = 2^{14} \cdot 3^6 \cdot 5^6 \cdot 7 \cdot 11 \cdot 19$

2B: If $q = 2$ or 3, then $c_q(A_5) > 1$ by Theorem 3.4.1, so $c_q(A_5 \wr 2) > 1$ by Lemma 3.4.4 and hence $c_q(N) > 1$, so N is not q-extendible. On the other hand, as we saw in the case $(Co_1, 5B)$, $A_5 \wr 2$ is not 5-extendible, so N is not 5-extendible by Lemma 3.4.2 (ii).

3B: If $q = 2$ or 3, then $c_q(L_2(5)) > 1$ by Theorem 3.4.1, so $c_q(4 * SL_2(5)) > 1$ by Lemma 3.4.4 and hence $c_q(N) > 1$, so N is not q-extendible. On the other hand, note that $c_5(L_2(5)) = 1$ by Theorem 3.4.1 and hence $c_5(SL_2(5) * 4) = 1$ by Lemma 3.4.4. Since the order of a Sylow 5-subgroup of $SL_2(5) * 4$ is 5, it follows that $SL_2(5) * 4$ is 5-extendible. Thus N is 5-extendible by Lemma 3.4.2 (ii).

5B: At [3], generators for N are provided in the table listing the maximal subgroups of G; specifically, the generators found under "$5^{1+4}.2^{1+4}.5.4$" generate a group isomorphic to N. Since $|N| = 2^7 \cdot 5^6$, we know that N is solvable, so it follows from Theorem 3.2.7 that N is 2-extendible if and only if it is 2-nilpotent; we verify using MAGMA that P_5 is not normal in N, so it follows that N is not 2-extendible. Furthermore,

Class	Normaliser in G	Order	q-extendible	Type?
$2A$	$(E_{2^2}L_3(4)) \cdot 2$	$2^9 \cdot 3^2 \cdot 5 \cdot 7$	-	G
$2B$	$(D_8^*)^3 \cdot L_3(2)$	$2^{10} \cdot 3 \cdot 7$	7	C
$3A$	$3\#\Sigma_7$	$2^4 \cdot 3^3 \cdot 5 \cdot 7$	7	B
$3B$	$\Sigma_3 \times L_3(2)$	$2^4 \cdot 3^2 \cdot 7$	7	C
$5A$	$(D_{10} \times A_5)\#2$	$2^5 \cdot 3 \cdot 5^2$	-	G
$7A$	$(7 \cdot 3) \times L_3(2)$	$2^3 \cdot 3^2 \cdot 7^2$	-	-
$7C$	$7^{1+2} \cdot (3 \times \Sigma_3)$	$2 \cdot 3^2 \cdot 7^3$	2	-
$7D$	$(7 \times D_{14}) \cdot 3$	$2 \cdot 3 \cdot 7^2$	$2, 3$	-
$17A$	$17 \cdot 8$	$2^3 \cdot 17$	$2, 17$	A

Table 3.19 p-local subgroups: He

Class	Normaliser in G	Order	q-extendible	Type?
$2A$	$(2HS) \cdot 2$	$2^{11} \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11$	-	G
$2B$	$(D_8^*)^4(A_5 \wr 2)$	$2^{14} \cdot 3^2 \cdot 5^2$	-	-
$3A$	$(3 \times A_9) \cdot 2$	$2^7 \cdot 3^5 \cdot 5 \cdot 7$	-	G
$3B$	$3^{1+4} \cdot (4 * SL_2(5))$	$2^4 \cdot 3^6 \cdot 5$	5	-
$5A$	$(D_{10} \times U_3(5))\#2$	$2^4 \cdot 3^2 \cdot 5^3 \cdot 7$	-	G
$5B$	$(5^{1+4} \cdot (2^{1+4} \cdot (5 \cdot 4)))$	$2^7 \cdot 5^6$	-	-
$5C$	$E_{5^3} \cdot (4 * SL_2(5))$	$2^4 \cdot 3 \cdot 5^4$	-	-
$5E$	$((5 \times 5^{1+2}) \cdot E_{2^2})4$	$2^4 \cdot 5^4$	2	-
$7A$	$((7 \cdot 3) \times A_5) \cdot 2$	$2^3 \cdot 3^2 \cdot 5 \cdot 7$	$5, 7$	-
$11A$	$(11 \cdot 10) \times 2$	$2^2 \cdot 5 \cdot 11$	$2, 5, 11$	-
$19A$	$19 \cdot 9$	$3^2 \cdot 19$	$3, 19$	A

Table 3.20 p-local subgroups: HN

checking with MAGMA, we verify that P_5 contains a unique maximal subgroup Q such that $Q \lhd N$ and Q contains an element of type $(5E)$. Let $H = Q \cdot (2^{1+4})$; then $|N_H(\langle x \rangle)|_{5'} \leq 2^4$, so

$$\dim \mathcal{S}(H, \langle x \rangle) = \frac{|H : P_5|}{|N_H(\langle x \rangle)|_{5'}} \cdot |P_5 : \langle x \rangle| \geq \frac{2^5}{2^4} \cdot |P_5 : \langle x \rangle| > |P_5 : \langle x \rangle|$$

and it follows that H is not 5-extendible by Theorem 2.4.2. Furthermore, H is a normal subgroup of index 5 in $K = Q \cdot (2^{1+4} \cdot 5)$, so N is not 5-extendible by a combination of Lemma 3.4.8 (i) and (ii).

$\underline{5C}$: We saw in the case $(HN, 3B)$ that $4 * SL_2(5)$ is not 2-extendible, so neither is N, by Lemma 3.4.2 (ii); furthermore, we saw that $c_3(4 * SL_2(5)) > 1$, so $c_3(N) > 1$ by Lemma 3.4.4 and hence N is not 3-extendible. Note that $N \cong (5 \times (E_{5^2} \cdot SL_2(5))2$ (see [18, Table 5.3w]) and a Sylow 5-subgroup P of $SL_2(5)$ contains an element X such that there exists $M \in SL_2(5)$ with $X^M \in P$ and $X^M \neq X$; indeed, let

$$X = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad M = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}.$$

Regard elements of $5 \times (E_{5^2} \cdot SL_2(5))$ as pairs and set $5 \cong C_5 = \langle w \rangle$. Then

$$Q = \{(w^i, eX^i) : 0 \leq i \leq 4, e \in E_{5^2}\}$$

is a normal subgroup of P_5. Furthermore, it is not weakly closed closed in P_5 with respect to $5 \times (E_{5^2} \cdot SL_2(5))$, since $(w, X)^M = (w, X^2) \notin Q$. So $(5 \times (E_{5^2} \cdot SL_2(5))$ is not 5-extendible by Proposition 3.2.4 and hence N is not 5-extendible by Lemma 3.4.8 (i).

$\underline{5E}$: We know that N is 2-nilpotent and hence it is 2-extendible. Moreover, following the steps covered in the case $(HN, 5B)$, we obtain a MAGMA implementation of N, and we verify that N is not 5-extendible using this.

$\underline{7A}$: If $q = 2$ or 3, then $c_q(A_5) > 1$ by Theorem 3.4.1 and hence $c_q(N) > 1$ by Lemma 3.4.4, so it follows that N is not q-extendible. On the other hand, $c_5(A_5) = 1$, so $c_5(N) = 1$

Class	Normaliser in G	Order	q-extendible	Type?
$2A$	$(D_8^*)^4 A_9$	$2^{15} \cdot 3^4 \cdot 5 \cdot 7$	-	F
$3A$	$(3 \times G_2(3)) \cdot 2$	$2^7 \cdot 3^7 \cdot 7 \cdot 13$	-	G
$3B$	$\underline{3^9} \cdot GL_2(3)$	$2^4 \cdot 3^{10}$	-	-
$3C$	$E_{3^5} \cdot (2\Sigma_6)$	$2^4 \cdot 3^6 \cdot 5$	-	-
$5A$	$5^{1+2} \cdot ((4 * SL_2(3))\#2)$	$2^5 \cdot 3 \cdot 5^3$	3	-
$7A$	$((7 \cdot 3) \times L_2(7)) \cdot 2$	$2^4 \cdot 3^2 \cdot 7^2$	-	-
$13A$[3]	$((13 \cdot 6) \times 3)2$	$2^2 \cdot 3^2 \cdot 13$	$2, 13$	-
$19A$	$19 \cdot 18$	$2 \cdot 3^2 \cdot 19$	$2, 3, 19$	A
$31A$	$31 \cdot 15$	$3 \cdot 5 \cdot 31$	$3, 5, 31$	A

Table 3.21 p-local subgroups: Th

Class	Normaliser in G	Order	q-extendible	Type?
$2A$	$2U_6(2)$	$2^{16} \cdot 3^6 \cdot 5 \cdot 7 \cdot 11$	-	F
$2B$	$(2 \times (D_8^*)^4)O_6^-(2)$	$2^{17} \cdot 3^4 \cdot 5$	-	-
$2C$	$\underline{2^{13}} \cdot (\underline{3^3} \cdot 8)$	$2^{16} \cdot 3^3$	-	-
$3A$	$\Sigma_3 \times (U_4(3) \cdot 2)$	$2^9 \cdot 3^7 \cdot 5 \cdot 7$	-	-
$3B$	$3^{1+6} \cdot (\underline{2^7} \cdot E_{3^2} \cdot 2)$	$2^8 \cdot 3^9$	-	-
$3C$	$E_{3^5} \cdot ((Q_8 * Q_8) \cdot (\Sigma_3 \times \Sigma_3))$	$2^7 \cdot 3^7$	-	-
$3D$	$\underline{3^6} \cdot GL_2(3)$	$2^4 \cdot 3^7$	-	-
$5A$	$(5 \cdot 4) \times \Sigma_5$	$2^6 \cdot 3 \cdot 5^2$	-	D
$7A$	$(7 \cdot 6) \times \Sigma_3$	$2^2 \cdot 3 \cdot 7$	$2, 3, 7$	-
$11A$	$(11 \cdot 5) \times 2$	$2 \cdot 5 \cdot 11$	$2, 5, 11$	A
$13A$	$13 \cdot 6$	$2 \cdot 3 \cdot 13$	$2, 3, 13$	A

Table 3.22 p-local subgroups: Fi_{22}

by Lemma 3.4.4 and it follows that N is 5-extendible, since $|P_5| = 5$. Finally, $P_7 \lhd N$ and $|P_7| = 7$, so N is a 7-extendible group, by Lemma 3.4.2 (iv).

$\underline{11A}$: Note that $P_{11} \lhd N$ and $N/P_{11} \cong 10 \times 2$. Moreover, 10×2 is 2-extendible, since it is 2-nilpotent, and it is clearly 5-extendible. By Lemma 3.4.2 (ii), it follows that N is 2-extendible and 5-extendible. Furthermore, $P_{11} \lhd N$ and $|P_{11}| = 11$, so N is 11-extendible, by Lemma 3.4.2 (iv).

$$G = Th, \quad |G| = 2^{15} \cdot 3^{10} \cdot 5^3 \cdot 7^2 \cdot 13 \cdot 19 \cdot 31$$

<u>$3B$</u>: As we saw in the case $(M_{11}, 2A)$, $GL_2(3)$ is not 2-extendible and hence neither is N, by Lemma 3.4.2 (ii). Furthermore, [3] provides generators for a certain maximal subgroup of G labelled by "3.32.3.(3 × 32).3^2 : $2S_4$" which is isomorphic to N, and using this, we show N is not 3-extendible, by Algorithm 2.

<u>$3C$</u>: For all $q \in \pi(N)$, we have $c_q(A_6) > 1$ by Theorem 3.4.1 and hence $c_q(2\Sigma_6) > 1$ by Lemma 3.4.4, so $c_q(N) > 1$ and hence N is not q-extendible.

<u>$5A$</u>: The argument used in the case $(Suz, 3B)$ shows that $4 * SL_2(3)$ is not 2-extendible and hence $(4 * SL_2(3))\#2$ is not 2-extendible by Lemma 3.4.8 (ii), so N is not 2-extendible by Lemma 3.4.2 (ii). On the other hand, $c_3(L_2(3)) = 1$ by Theorem 3.4.1, so $c_3(SL_2(3) * 4) = 1$ by Lemma 3.4.4 and it follows that $c_3(N) = 1$; since $|P_3| = 3$, we deduce that N is 3-extendible. Finally, [3] provides generators for a certain maximal subgroup of G labelled by "5^{1+2} : $4S_4$" which is isomorphic to N; using Algorithm 2 on this group, we show that N is not 5-extendible.

<u>$7A$</u>: This case is identical to the case $(Co_1, 7B)$; we conclude immediately that N is not q-extendible for all $q \in \pi(N)$.

<u>$13A$</u>: Note that $(13 \cdot 6) \times 3$ contains a normal Hall subgroup of order $3^2 \cdot 13$, so N is 2-nilpotent and hence 2-extendible. On the other hand, $P_{13} \lhd N$ and $|P_{13}| = 13$, so it follows that N is 13-extendible by Lemma 3.4.2 (iv). There is no reasonably sized permutation representation of Th (the smallest is on 141,127,000 points) and at present, no way of determining if N is 3-extendible or not using Algorithms 1 or 2.

$$G = Fi_{22}, \quad |G| = 2^{17} \cdot 3^9 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$$

<u>$2B$</u>: If $q \in \pi(N)$, then $\Omega_6^-(2) \lhd O_6^-(2)$ and $c_q(\Omega_6^-(2)) > 1$ by Theorem 3.4.1 since $\Omega_6^-(2)$ is simple, so it follows that $c_q(O_6^-(2)) > 1$ and hence $c_q(N) > 1$ by Lemma 3.4.4. Thus N is not q-extendible for all $q \in \pi(N)$.

<u>$2C$</u>: If $q \in \pi(N)$, then we show that N is not q-extendible using Algorithm 2.

[3] It is unknown if $((13 \cdot 6) \times 3) \cdot 2$ is 3-extendible or not.

<u>$3A$</u>: If $q \in \pi(N)$, then since $U_4(3)$ is simple, it follows from Theorem 3.4.1 that $c_q(U_4(3)) > 1$, so by Lemma 3.4.4, we deduce that $c_q(U_4(3) \cdot 2) > 1$ and $c_q(N) > 1$. So N is not q-extendible for all $q \in \pi(N)$.

<u>$3B$</u>: If $q \in \pi(N)$, then we show that N is not q-extendible using Algorithm 2.

<u>$3C$</u>: If $q = 2$, then we show that N is not 2-extendible using Algorithm 2. On the other hand, if $q = 3$, then by Example 3.4.6 (i), there exists $Q \lhd P \in \mathrm{Syl}_3(\Sigma_3 \times \Sigma_3)$ which is not weakly closed in $\Sigma_3 \times \Sigma_3$ and hence there exists $\overline{Q} \lhd \overline{P} \in \mathrm{Syl}_3((Q_8 * Q_8) \cdot (\Sigma_3 \times \Sigma_3))$ which is not weakly closed in $(Q_8 * Q_8) \cdot (\Sigma_3 \times \Sigma_3)$. It follows from Lemma 3.4.7 that N is not 3-extendible.

<u>$3D$</u>: As we saw in the case $(M_{11}, 2A)$, $GL_2(3)$ is not 2-extendible and hence neither is N, by Lemma 3.4.2 (ii). Furthermore, N is not 3-extendible, by Algorithm 2.

<u>$7A$</u>: This case has already been analysed in the case $(Co_3, 7A)$ and we may therefore conclude that N is not q-extendible for all $q \in \pi(N)$.

$$G = Fi_{23}, \qquad |G| = 2^{18} \cdot 3^{13} \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 23$$

<u>$2C$</u>: If $q \in \pi(N)$, then since $\Omega_6^-(2)$ is simple, it follows that $(3 \times \Omega_6^-(2)) \cdot 2$ is a group of type (G), so $c_q((3 \times \Omega_6^-(2)) \cdot 2) > 1$ and hence $c_q(N) > 1$ by Lemma 3.4.4. So N is not q-extendible for all $q \in \pi(N)$.

<u>$3B$</u>: Since $|N| = 2^{11} \cdot 3^{13}$, we know that N is solvable and hence 2-solvable, so N is 2-extendible if and only if it is 2-nilpotent by Corollary 3.2.8. Checking with MAGMA, we verify that N is not 2-nilpotent, so it is not 2-extendible. On the other hand, we may use Algorithm 2 to verify that the quotient $K \cong (Q_8^*)^3 3^{1+2} GL_2(3)$ contains a subgroup $Q \lhd P \in \mathrm{Syl}_3((Q_8^*)^3 3^{1+2} GL_2(3))$ which is not weakly closed in P with respect to K, so it follows from Lemma 3.4.7 that N is not 3-extendible.

<u>$3C$</u>: If $q \in \pi(N)$, then since $\Omega_5(3)$ is simple, it follows from Theorem 3.4.1 that $c_q(\Omega_5(3)) > 1$ and we deduce that $c_q(2 \times (\Omega_5(3) \cdot 2)) > 1$ and hence $c_q(N) > 1$, by Lemma 3.4.4. Thus N is not q-extendible for all $q \in \pi(N)$.

<u>$3D$</u>: Since $|N| = 2^5 \cdot 3^{10}$, we know that N is solvable and hence 2-solvable, so N is 2-extendible if and only if it is 2-nilpotent by Corollary 3.2.8. Checking with MAGMA,

we verify that N is not 2-nilpotent, so it is not 2-extendible. On the other hand, we may show that N is not 3-extendible using Algorithm 2.

$\underline{7A}$: If $q = 2$ or 3, then it follows from Theorem 3.4.1 that $c_q(\Sigma_5) > 1$, so $c_q(N) > 1$ by Lemma 3.4.4 and hence N is not q-extendible. On the other hand, we know that Σ_5 is 5-extendible, so N is 5-extendible by Lemma 3.4.2 (ii). Finally, note that $P_7 \lhd N$ and $|P_7| = 7$, so N is 7-extendible, by Lemma 3.4.2 (iv).

$\underline{11A}$: Since N is 2-nilpotent it is 2-extendible. Furthermore, $P_{11} \lhd N$, so we know that P_2P_{11} is a subgroup of index 5 in N, so it follows that N is 5-extendible by Lemma 3.4.2 (iv). Finally, $P_{11} \lhd N$ and $|P_{11}| = 11$, so N is 11-extendible, by Lemma 3.4.2 (iv).

$\underline{13A}$: Note that $P_{13} \lhd N$ and $N/P_{13} \cong 6 \times \Sigma_3$. Since $6 \times \Sigma_3$ is 2-nilpotent, it is 2-extendible, and hence N is 2-extendible by Lemma 3.4.2 (ii). On the other hand, $6 \times \Sigma_3$ is not 3-extendible by Example 3.4.6 (i), so N is not 3-extendible by Lemma 3.4.7 (ii). Finally, note that $P_{13} \lhd N$ and $|P_{13}| = 13$, so N is 13-extendible, by Lemma 3.4.2 (iv).

$$G = Fi'_{24}, \qquad |G| = 2^{21} \cdot 3^{16} \cdot 5^2 \cdot 7^3 \cdot 11 \cdot 13 \cdot 17 \cdot 23 \cdot 29$$

$\underline{2B}$: Since $U_4(3)$ is a simple group, by Theorem 3.4.1 it follows that $3U_4(3) \cdot 2$ is a group of type (G) and hence $c_q(3U_4(3) \cdot 2) > 1$ for all $q \in \pi(N)$. Thus $c_q(N) > 1$ for all $q \in \pi(N)$ by Lemma 3.4.4, and we deduce that N is not q-extendible for all $q \in \pi(N)$.

$\underline{3A}$: Since $P\Omega_8^+(3)$ is a simple group, by Theorem 3.4.1 we know that $c_q(P\Omega_8^+(3)) > 1$ for all $q \in \pi(N)$ and hence $c_q(P\Omega_8^+(3) \cdot 3) > 1$ by Lemma 3.4.4. Thus N is a group of type (G) and it follows that N is not q-extendible for all $q \in \pi(N)$.

$\underline{3B}$: Since $U_5(2)$ is a simple group, Theorem 3.4.1 shows that $c_q(U_5(2)) > 1$ for all $q \in \pi(N)$, so $c_q(U_5(2) \cdot 2) > 1$ and hence $c_q(N) > 1$ for all $q \in \pi(N)$, by Lemma 3.4.4. So N is not q-extendible for all $q \in \pi(N)$.

$\underline{3C}$: We know that $P\Omega_6^-(3)$ is a quotient group of $\Omega_6^-(3)$ and $c_q(P\Omega_6^-(3)) > 1$ for all $q \in \pi(N)$ by Theorem 3.4.1, so $c_q(\Omega_6^-(3)) > 1$ for all $q \in \pi(N)$ by Lemma 3.4.4. Thus $c_q(N) > 1$ for all $q \in \pi(N)$ and we see that N is not q-extendible for all $q \in \pi(N)$.

Class	Normaliser in G	Order	q-extendible	Type?
$2A$	$2Fi_{22}$	$2^{18} \cdot 3^9 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$	-	F
$2B$	$((2 \times 2)U_6(2)) \cdot 2$	$2^{18} \cdot 3^6 \cdot 5 \cdot 7 \cdot 11$	-	G
$2C$	$(E_{2^2} \times (D_8^*)^4)((3 \times \Omega_6^-(2)) \cdot 2)$	$2^{18} \cdot 3^5 \cdot 5$	-	-
$3A$	$\Sigma_3 \times \Omega_7(3)$	$2^{10} \cdot 3^{10} \cdot 5 \cdot 7 \cdot 13$	-	F
$3B$	$3^{1+8}(Q_8^*)^3 3^{1+2} GL_2(3)$	$2^{11} \cdot 3^{13}$	-	-
$3C$	$E_{3^6} \cdot (2 \times (\Omega_5(3) \cdot 2))$	$2^8 \cdot 3^{10} \cdot 5$	-	-
$3D$	$\underline{2^5 3^{10}}$	$2^5 \cdot 3^{10}$	-	-
$5A$	$(5 \cdot 4) \times \Sigma_7$	$2^6 \cdot 3^2 \cdot 5^2 \cdot 7$	7	B
$7A$	$(7 \cdot 6) \times \Sigma_5$	$2^4 \cdot 3^2 \cdot 5 \cdot 7$	$5, 7$	-
$11A$	$((11 \cdot 5) \times E_{2^2}) \cdot 2$	$2^3 \cdot 5 \cdot 11$	$2, 5, 11$	-
$13A$	$(13 \cdot 6) \times \Sigma_3$	$2^2 \cdot 3^2 \cdot 13$	$2, 13$	-
$17A$	$17 \cdot 16$	$2^4 \cdot 17$	$2, 17$	A
$23A$	$23 \cdot 11$	$11 \cdot 23$	$11, 23$	A

Table 3.23 p-local subgroups: Fi_{23}

Class	Normaliser in G	Order	q-extendible	Type?
$2A$	$2Fi_{22}2$	$2^{18} \cdot 3^9 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$	-	G
$2B$	$(D_8^*)^6(3U_4(3) \cdot 2)$	$2^{21} \cdot 3^7 \cdot 5 \cdot 7$	-	-
$3A$	$(3 \times (P\Omega_8^+(3) \cdot 3)) \cdot 2$	$2^{13} \cdot 3^{14} \cdot 5 \cdot 7 \cdot 13$	-	-
$3B$	$3^{1+10}(U_5(2) \cdot 2)$	$2^{11} \cdot 3^{16} \cdot 5 \cdot 11$	-	-
$3C$	$E_{3^7}\Omega_6^-(3)2$	$2^8 \cdot 3^{13} \cdot 5 \cdot 7$	-	-
$3D$	$\underline{2^6 3^{14}}$	$2^6 \cdot 3^{14}$	-	-
$3E$	$((3 \times 3) \cdot 2) \times G_2(3)$	$2^7 \cdot 3^8 \cdot 7 \cdot 13$	-	F
$5A$	$(D_{10} \times A_9)\#2$	$2^8 \cdot 3^4 \cdot 5^2 \cdot 7$	-	G
$7A$	$(7 \cdot 6) \times A_7$	$2^4 \cdot 3^3 \cdot 5 \cdot 7^2$	-	D
$7B$	$7^{1+2} \cdot (6 \times \Sigma_3)$	$2^2 \cdot 3^2 \cdot 7^3$	2	-
$11A$	$((11 \cdot 5) \times A_4) \cdot 2$	$2^3 \cdot 3 \cdot 5 \cdot 11$	$3, 5, 11$	-
$13A$	$((13 \cdot 6) \times (E_{3^2} \cdot 2))2$	$2^3 \cdot 3^3 \cdot 13$	$2, 13$	-
$17A$	$17 \cdot 16$	$2^4 \cdot 17$	$2, 17$	A
$23A$	$23 \cdot 11$	$11 \cdot 23$	$11, 23$	A
$29A$	$29 \cdot 14$	$14 \cdot 29$	$14, 29$	A

Table 3.24 p-local subgroups: Fi_{24}'

<u>$3D$</u>: We may obtain a permutation representation of N in MAGMA using some ad hoc methods, which are described in Appendix A.1.1. Since $|N| = 2^6 \cdot 3^{14}$, we know that N is 2-solvable, and hence N is 2-extendible if and only if N is 2-nilpotent by Corollary 3.2.8; we verify using MAGMA that N is not 2-nilpotent, and hence it is not 2-extendible. Moreover, N contains a normal subgroup H of order 3^6 and Algorithm 2 shows that N/H contains a 3-group which is normal in a Sylow 3-subgroup of N/H and is not weakly closed in N/H. By Lemma 3.4.7 and Proposition 3.2.4, it follows that N is not 3-extendible, as required.

<u>$7B$</u>: Since 6 and Σ_3 are both 2-nilpotent, it follows that $6 \times \Sigma_3$ is 2-nilpotent and hence 2-extendible, so N is 2-extendible by Lemma 3.4.2 (ii); on the other, $6 \times \Sigma_3$ is not 3-extendible by Example 4.2.5 (i), and hence N is not 3-extendible either. Note that $P_7 \in \mathrm{Syl}_7(G)$; then $Z(P_7)$ is cyclic of order 7 and $Z(P_7) \lhd P_7$, so comparing the orders of $(7A)$ and $(7B)$ in Table 3.24, we know that $N \cong N_G(Z(P))$. This provides a computationally feasible way of constructing N in MAGMA, and using Algorithm 2, we confirm that N is not 7-extendible.

<u>$11A$</u>: We saw in the case $(M_{22}, 3A)$ that A_4 is not 2-extendible and hence it follows that N is not 2-extendible by a combination of Lemma 3.4.8 (i) and (ii). On the other hand, $c_3(A_4) = 1$ and hence $c_3(N) = 1$ by Lemma 3.4.4; since $|P_3| = 3$, it follows that N is 3-extendible; similar reasoning shows that $c_5(N) = 1$ and hence N is 5-extendible. Finally, note that $P_{11} \lhd N$ and $|P_{11}| = 11$, so N is 11-extendible.

<u>$13A$</u>: Note that N contains a normal Hall subgroup of order $3^3 \cdot 13$, so N is 2-nilpotent and hence 2-extendible. Furthermore, we verify using Algorithm 2 that N is not 3-extendible. Finally, $P_{13} \lhd N$ and $|P_{13}| = 13$, so N is 13-extendible.

$$G = B, \quad |G| = 2^{41} \cdot 3^{13} \cdot 5^6 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 32 \cdot 47$$

<u>$2A$</u>: We note that ${}^2E_6(2)$ is a finite simple group of Lie type, and hence by Theorem 3.4.1, we have $c_q({}^2E_6(2)) > 1$ for all $q \in \pi(N)$. Thus N is a group of type (G), so $c_q(N) > 1$ for all $q \in \pi(N)$ and hence N is not q-extendible for all $q \in \pi(N)$.

<u>$2D$</u>: Note that $\Omega_8^+(2)$ is simple and hence $c_q(\Omega_8^+(2)) > 1$ for all $q \in \pi(N)$ by Theorem 3.4.1. Furthermore, $\Omega_8^+(2)$ is isomorphic to a normal subgroup of $SO_8^+(2)$, which is

a normal subgroup of $O_8^+(2)$, so by Lemma 3.4.4, it follows that $c_q(O_8^+(2)) > 1$ and hence $c_q(N) > 1$ for all $q \in \pi(N)$. Thus N is not q-extendible for all $q \in \pi(N)$.

$\underline{3A}$: Since Fi_{22} is simple, it follows from Theorem 3.4.1 that $c_q(Fi_{22}) > 1$ for all $q \in \pi(N)$, so $c_q(Fi_{22} \cdot 2) > 1$ and hence $c_q(N) > 1$ by Lemma 3.4.4. Thus N is not q-extendible for all $q \in \pi(N)$.

$\underline{3B}$: Note that $\Omega_6^-(2)$ is simple and hence $c_q(\Omega_6^-(2)) > 1$ for all $q \in \pi(N)$ by Theorem 3.4.1. Furthermore, $\Omega_6^-(2)$ is isomorphic to a normal subgroup of $SO_6^-(2)$, which is a normal subgroup of $O_6^-(2)$, so by Lemma 3.4.4, it follows that $c_q(\Omega_6^-(2)) > 1$ and hence $c_q(N) > 1$ for all $q \in \pi(N)$. Thus N is not q-extendible for all $q \in \pi(N)$.

$\underline{5A}$: Since HS is simple, we know from Theorem 3.4.1 that $c_q(HS) > 1$ for all $q \in \pi(N)$, so $c_q(HS \cdot 2) > 1$ and hence $c_q(N) > 1$ for all $q \in \pi(N)$. Thus N is not q-extendible for all $q \in \pi(N)$.

$\underline{5B}$: If $q = 2$ or 3, then $c_q(A_5) > 1$ by Theorem 3.4.1 and hence by Lemma 3.4.4, it follows that $c_q((Q_8 * D_8)A_5) > 1$. Hence $c_q(5^{1+4} \cdot ((Q_8 * D_8)A_5)) > 1$ and $c_q(N) > 1$. Thus N is not 2-extendible or 3-extendible. We are unable to determine if N is 5-extendible or not using Algorithms 1 or 2, due to the fact that no known permutation representation for B exists on a reasonable number of points (indeed, the original construction for B, due to Leon and Sims, was on $13,571,955,000$ points).

$\underline{7A}$: Since $(4^3 - 1)/(4 - 1) = 21$ is not a prime power, it follows from Theorem 3.4.1 that $c_q(L_3(4)) > 1$ for all $q \in \pi(N)$. Thus, $(2L_3(4)) \cdot 2$ is a group of type (G) and hence so is N; thus, $c_q(N) > 1$ for all $q \in \pi(N)$ and it follows that N is not q-extendible for all $q \in \pi(N)$.

$\underline{11A}$: Note that $P_{11} \lhd N$ and $N/P_{11} \cong 10 \times \Sigma_5$, which is a group of type (D). Thus, we know that if $q = 2, 3$ or 5, then $10 \times \Sigma_5$ is not q-extendible and hence neither is N by Lemma 3.4.2 (ii). On the other hand, $P_{11} \lhd N$ and $|P_{11}| = 11$, so N is 11-extendible, by Lemma 3.4.2 (iv).

$\underline{13A}$: Note that $P_{13} \lhd N$ and $N/P_{13} \cong 12 \times \Sigma_4$. We saw in the case $(M_{22}, 2A)$ that Σ_4 contains a 2-group which is normal in a Sylow 2-subgroup of Σ_4 but is not weakly closed in Σ_4. Thus, by Lemma 3.4.7, it follows that N is not 2-extendible. On the other hand,

$12 \times \Sigma_4$ contains a subgroup isomorphic to $12 \times \Sigma_3$, and by Example 3.4.6 (i), we see that $12 \times \Sigma_4$ is not 3-extendible, so neither is N, by Lemma 3.4.2 (ii). Finally, $P_{13} \lhd N$ and $|P_{13}| = 13$, so N is 13-extendible.

<u>17*A*</u>: Since $P_{17} \lhd N$, it follows that N is 2-nilpotent and hence 2-extendible. Furthermore, $|P_{17}| = 17$, so we can deduce that N is 17-extendible, by Lemma 3.4.2 (iv).

<u>19*A*</u>: Note that $P_{19} \lhd N$ and $N/P_{19} \cong 18 \times 2$ is an abelian group, so N/P_{19} is 2-extendible and 3-extendible; by Lemma 3.4.2 (ii), it follows that N is 2-extendible and 3-extendible. Furthermore, $P_{19} \lhd N$ and $|P_{19}| = 19$, so we see that N is 19-extendible, by Lemma 3.4.2 (iv).

$$G = M, \quad |G| = 2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71$$

<u>3*B*</u>: If $q \in \pi(N)$, then $c_q(Suz) > 1$ by Theorem 3.4.1 and thus $c_q(2Suz) > 1$ by Lemma 3.4.4; it hence follows that $c_q(3^{1+12}(2Suz)) > 1$ and hence $c_q(N) > 1$. Thus N is not q-extendible for all $q \in \pi(N)$.

<u>5*B*</u>: If $q \in \pi(N)$, then $c_q(J_2) > 1$ by Theorem 3.4.1 and thus $c_q(2J_2) > 1$ by Lemma 3.4.4. It therefore follows that $c_q((4 * 2J_2) \cdot 2) > 1$ and thus $c_q(N) > 1$; hence N is not q-extendible for all $q \in \pi(N)$.

<u>7*B*</u>: If $q = 2, 3$ or 5, then $c_q(A_7) > 1$ by Theorem 3.4.1 and hence $c_q((2A_7 \times 3) \cdot 2) > 1$ by Lemma 3.4.4. Thus $c_q(N) > 1$ and it follows that N is not q-extendible. Furthermore, [3] has a permutation representation of N on $16,807$ points, which is the maximal subgroup labelled "$7^{1+4} : (3 \times 2S_7)$". Using this permutation representation, we verify that N is not 7-extendible, using Algorithm 2.

<u>13*A*</u>: If $q = 2$ or 3, then $c_q(L_3(2)) > 1$ and hence by Lemma 3.4.4, we deduce that $c_q(N) > 1$, so N is not q-extendible. On the other hand, $c_7(L_3(2)) = 1$, so by Lemma 3.4.4, we see that $c_7(N) = 1$; since $|P_7| = 7$, it follows that N is 7-extendible. Finally, $P_{13} \lhd N$ and $|P_{13}| = 13$, so N is 13-extendible.

[4] It is unknown if $(5^{1+4} \cdot ((Q_8 * D_8)A_5)) \cdot 4$ is 5-extendible or not.

Class	Normaliser in G	Order	q-extendible	Type?
$2A$	$(2\,{}^2E_6(2))\cdot 2$	$2^{38}\cdot 3^9\cdot 5^2\cdot 7^2\cdot 11\cdot 13\cdot 17\cdot 19$	-	-
$2B$	$(D_8^*)^{11}(Co_2)$	$2^{41}\cdot 3^6\cdot 5^3\cdot 7\cdot 11\cdot 23$	-	F
$2C$	$(E_{2^2}\times F_4(2))2$	$2^{27}\cdot 3^6\cdot 5^2\cdot 7^2\cdot 13\cdot 17$	-	G
$2D$	$2^9 2^{16} O_8^+(2)$	$2^{38}\cdot 3^5\cdot 5\cdot 7$	-	-
$3A$	$\Sigma_3\times(Fi_{22}\cdot 2)$	$2^{17}\cdot 3^9\cdot 5^2\cdot 7\cdot 11\cdot 13$	-	-
$3B$	$3^{1+8}(Q_8^*)^3 O_6^-(2)$	$2^{14}\cdot 3^{13}\cdot 5$	-	-
$5A$	$(5\cdot 4)\times(HS\cdot 2)$	$2^{12}\cdot 3^2\cdot 5^4\cdot 7\cdot 11$	-	-
$5B$[4]	$(5^{1+4}\cdot((Q_8 * D_8)A_5))\cdot 4$	$2^9\cdot 3\cdot 5^6$	-	-
$7A$	$((7\cdot 3)\times(2L_3(4)\cdot 2))2$	$2^9\cdot 3^3\cdot 5\cdot 7^2$	-	-
$11A$	$(11\cdot 10)\times\Sigma_5$	$2^4\cdot 3\cdot 5^2\cdot 11$	11	-
$13A$	$(13\cdot 12)\times\Sigma_4$	$2^5\cdot 3^2\cdot 13$	13	-
$17A$	$((17\cdot 8)\times E_{2^2})2$	$2^6\cdot 17$	$2, 17$	-
$19A$	$(19\cdot 18)\times 2$	$2^2\cdot 3^2\cdot 19$	$2, 3, 19$	-
$23A$	$(23\cdot 11)\times 2$	$2, 11, 23$	$2, 11, 23$	A
$31A$	$31\cdot 15$	$3\cdot 5\cdot 31$	$3, 5, 31$	A
$47A$	$47\cdot 23$	$23\cdot 47$	$23, 47$	A

Table 3.25 p-local subgroups: B

Class	Normaliser in G	Order	q-extendible	Type?
$2A$	$2B$	$2^{42}\cdot 3^{13}\cdot 5^6\cdot 7^2\cdot 11\cdot 13\cdot 17\cdot 19\cdot 23\cdot 32\cdot 47$	-	F
$2B$	$(D_8^*)^{12}(Co_1)$	$2^{43}\cdot 3^9\cdot 5^4\cdot 7^2\cdot 11\cdot 13\cdot 23$	-	F
$3A$	$(3Fi_{24}')\cdot 2$	$2^{22}\cdot 3^{17}\cdot 5^2\cdot 7^3\cdot 11\cdot 13\cdot 17\cdot 23\cdot 29$	-	G
$3B$	$3^{1+12}(2Suz)\cdot 2$	$2^{15}\cdot 3^{20}\cdot 5^2\cdot 7\cdot 11\cdot 13$	-	-
$3C$	$\Sigma_3\times Th$	$2^{16}\cdot 3^{11}\cdot 5^3\cdot 7^2\cdot 13\cdot 19\cdot 31$	-	F
$5A$	$(D_{10}\times HN)2$	$2^{16}\cdot 3^6\cdot 5^7\cdot 7\cdot 11\cdot 19$	-	G
$5B$	$5^{1+6}\cdot((4 * 2J_2)\cdot 2)$	$2^{10}\cdot 3^3\cdot 5^9\cdot 7$	-	-
$7A$	$((7\cdot 3)\times He)\cdot 2$	$2^{11}\cdot 3^4\cdot 5^2\cdot 7^4\cdot 17$	-	G
$7B$	$7^{1+4}\cdot(2A_7\times 3)\cdot 2$	$2^5\cdot 3^3\cdot 5\cdot 7^6$	-	-
$11A$	$((11\cdot 5)\times M_{12})\cdot 2$	$2^7\cdot 3^3\cdot 5^2\cdot 11^2$	-	G
$13A$	$((13\cdot 6)\times L_3(2))\cdot 2$	$2^5\cdot 3^2\cdot 7\cdot 13$	$7, 13$	-
$13B$	$13^{1+2}\cdot((SL_2(3)\times 3)4)$	$2^5\cdot 3^2\cdot 13^3$	-	-
$17A$	$((17\cdot 8)\times L_2(7))2$	$2^7\cdot 3\cdot 7\cdot 17$	$7, 17$	-
$19A$	$((19\cdot 9)\times A_5)\cdot 2$	$2^3\cdot 3^3\cdot 5\cdot 19$	$5, 19$	-
$23A$	$(23\cdot 11)\times\Sigma_4$	$2^3\cdot 3\cdot 11\cdot 23$	$3, 11, 23$	-
$29A$	$((29\cdot 14)\times 3)2$	$2^2\cdot 3\cdot 7\cdot 29$	$2, 3, 7, 29$	-
$31A$	$(31\times 15)\times\Sigma_3$	$2\cdot 3^2\cdot 5\cdot 31$	$2, 5, 31$	-
$41A$	$41\cdot 40$	$2^3\cdot 5\cdot 41$	$2, 5, 41$	A
$47A$	$93\cdot 23$	$23\cdot 93$	$23, 93$	A
$59A$	$59\cdot 29$	$29\cdot 59$	$29, 59$	A
$71A$	$71\cdot 35$	$5\cdot 7\cdot 71$	$5, 7, 71$	A

Table 3.26 p-local subgroups: M

<u>13B</u>: We saw in the case $(Ly, 7A)$ that $SL_2(3)$ is not 2-extendible and hence neither is $SL_2(3) \times 3$ by Lemma 3.4.2 (ii); thus $(SL_2(3) \times 3)4$ is not 2-extendible by Lemma 3.4.8 (ii) and we conclude that N is not 2-extendible. Moreover, by Theorem 3.4.1, we have $c_3(L_3(2)) > 1$ and hence it follows from Lemma 3.4.4 that $c_3(N) > 1$, so N is not 3-extendible. Moreover, [3] has a permutation representation of N on 2,197 points under the maximal group labelled "$13^{1+2} : (3 \times S_4)$" and using Algorithm 2 and this representation, we verify that N is not 13-extendible.

<u>17A</u>: If $q = 2$ or 3, then by Theorem 3.4.1, we know that $c_q(L_2(7)) > 1$ and hence $c_q(N) > 1$ by Lemma 3.4.4, so N is not q-extendible. On the other hand, $c_7(L_2(7)) = 1$, so $c_7(N) = 1$, and since $|P_7| = 7$, we conclude that N is 7-extendible. Finally, $P_{17} \lhd N$ and $|P_{17}| = 17$, so N is 17-extendible, by Lemma 3.4.2 (iv).

<u>19A</u>: If $q = 2$ or 3, then $c_q(A_5) > 1$ by Theorem 3.4.1 and hence $c_q(N) > 1$ by Lemma 3.4.4, so N is not q-extendible. On the other hand, $c_5(A_5) = 1$, so $c_5(N) = 1$ and since $|P_5| = 5$, we conclude that N is 5-extendible. Finally, $P_{19} \lhd N$ and $|P_{19}| = 19$, so N is 19-extendible, by Lemma 3.4.2 (iv).

<u>23A</u>: Note that $P_{23} \lhd N$ and $N/P_{23} \cong 11 \times \Sigma_4$. We saw in the case $(M_{22}, 2A)$ that Σ_4 is not 2-extendible, so neither is N by Lemma 3.4.2 (iv). On the other hand, $11 \times \Sigma_4$ is clearly 11-extendible, so N is 11-extendible. Finally, $P_{23} \lhd N$ and $|P_{23}| = 23$, so N is 23-extendible, by Lemma 3.4.2 (iv).

<u>29A</u>: Note that $((29 \cdot 14) \times 3)$ contains a normal Hall subgroup of order $3 \cdot 7 \cdot 29$ and hence N is 2-nilpotent, so N is 2-extendible. Furthermore, since P_7P_{29} is normal in N, it follows that $P_2P_7P_{29}$ is a subgroup of index 3 in N, and since $|P_3| = 3$, we deduce that N is 3-extendible, by Lemma 3.4.2 (iv). Moreover, $c_7(29 \cdot 14) = 1$, so $c_7((29 \cdot 14) \times 3) = 1$ by Lemma 3.4.4 and hence $c_7(N) = 1$; since $|P_7| = 7$, it follows that N is 7-extendible. Finally, $P_{29} \lhd N$ and $|P_{29}| = 29$, so N is 29-extendible, by Lemma 3.4.2 (iv).

<u>31A</u>: Note that $P_{31} \lhd N$ and $N/P_{31} \cong 15 \times \Sigma_3$. Since Σ_3 is 2-extendible, it follows that N is 2-extendible, by Lemma 3.4.2 (iv); on the other hand, Σ_3 is not 3-extendible by Example 3.4.6 (i), so N is not 3-extendible. Furthermore, $15 \times \Sigma_3$ is clearly 5-extendible, so N is 5-extendible. Finally, since $P_{31} \lhd N$ and $|P_{31}| = 31$, we conclude that N is 31-extendible, by Lemma 3.4.2 (iv).

Chapter 4

Classical Groups

In this chapter, we apply the results developed in Sections 2.3 and 2.4 to some of the classical groups described in Section 1.2.1. We start by recalling the details of (B, N)-pairs, which serve as a useful tool for unifying the structure of the finite classical groups; in particular, we study the Scott modules associated to the subgroup B in such (B, N)-pairs. We finish the chapter by looking at the Levi decomposition and how certain p-subgroups of a classical group define Scott modules which extend to a corresponding parabolic subgroup.

4.1 (B, N)-Pair Preliminaries

Much of the material in this section is standard, and is taken from [13, Sections 1.6, 1.7]. Throughout this section, we let G be an abstract finite group, but we will mainly think of G as being a particular classical group in examples. We start with the definition of a (B, N)-pair.

Definition 4.1.1. We say that G admits a (B, N)**-pair** if there exist subgroups $B, N \leq G$ such that the following conditions hold:

(i) $G = \langle B, N \rangle$;

(ii) $H = B \cap N \lhd N$ and the quotient $W := N/H$ is generated by a set of elements $S = \{s_i : i \in I\}$ such that $s_i^2 = 1$;

(iii) if $s_i = n_i H$ and $n \in N$, then:

(a) $n_i B n_i \neq B$ and;

(b) $n_i B n \subseteq B(n_i n)B \cup BnB$.

If G admits a (B,N)-pair, we refer to the quotient W as the **Weyl** group of the (B,N)-pair. The order $|I|$ is referred to as the **rank** of the (B,N)-pair; note that it may be infinite, but we shall always assume that $|I|$ is finite. We shall refer to the elements of S as the **simple reflections** of W; note that the simple reflections are **involutions**, i.e., elements of order 2. If $w \in W$ and $w = nH$, then without loss of generality we set $BwB := BnB$.

Remarks 4.1.2. Suppose that G is a linear algebraic group defined over an algebraically closed field of characteristic p. A maximal, closed, connected, solvable subgroup of G is known as a **Borel** subgroup of G. It is well-known (see [13, Section 3.4]) that any linear algebraic group has a Borel subgroup, and any two Borel subgroups of G are conjugate in G. Furthermore, in the situation where G is affine and G admits a reductive (B,N)-pair ([13, Definition 3.4.5]), the subgroup B of the (B,N)-pair is in fact a Borel subgroup. Classical groups (defined over algebraically closed fields) and groups of Lie type are both standard examples of groups which admit a reductive (B,N)-pair: see [13, Theorems 1.7.4, 1.7.8] for the former and [35, p. 58, Theorem 6] for the latter.

Now let G be a (infinite order) classical group defined over an algebraically closed field of characteristic p and set $F_q : G \to G$ to be a Frobenius map, where $q = p^e$ for some $e \geq 1$ and $F_q(x) = (x_{ij}^q)$ for all $x \in G$. Then the fixed point set G^{F_q} is a (finite order) classical group defined over a finite field, and if (B,N) is a reductive (B,N)-pair for G with B an F_q-stable subgroup of G, then (B^{F_q}, N^{F_q}) is a (B,N)-pair for G^{F_q} (see Sections 1.17 and 1.18 of [9]). In Carter's book [9], a Borel subgroup of G^{F_q} is defined to be any subgroup of the form B^{F_q}, where B is an F_q-stable Borel subgroup of the classical group G. In both cases, a "Borel" subgroup can be viewed as a subgroup B in a certain (B,N)-pair for the particular classical group; the connection between classical groups and (B,N)-pairs is therefore quite apparent.

Theorem 4.1.3 (Bruhat Decomposition)**.** [13, Proposition 1.6.3] If G admits a (B, N)-pair with Weyl group W, then

$$G = BWB = \coprod_{w \in W} BwB$$

is a disjoint union of double cosets.

Thus the Weyl group W of the (B, N)-pair for G is a set of double (B, B)-coset representatives in G, and it follows that $G = BNB$.

If G admits a (B, N)-pair and $J \subseteq I$, we let N_J denote the inverse image of $\langle s_j : j \in J \rangle$ in N and set $P_J = BN_JB$. This set product is a subgroup of G, known as the **parabolic subgroup** of G associated to J (the fact that it is a subgroup follows from the (B, N)-pair axioms - see [13, 1.6.2] for the details). Furthermore, any subgroup $L \leq G$ satisfying $B \leq L \leq G$ must be of the form $L = P_J$ for some $J \subseteq I$, so the parabolic subgroups exhaust the interim subgroups between B and G (see [37, 9.15] for this fact).

For each subset $J \subseteq I$, we set $S_J = \{s_j : j \in J\}$ and we let $W_J = \langle s : s \in S_J \rangle \leq W$. For all $w \in W_J$, we define $l(w)$ to be the smallest $k \in \mathbb{N}$ such that there exists an expression of the form

$$w = s_1 s_2 \cdots s_k$$

with $s_j \in S$ for $1 \leq j \leq k$ and we refer to $l(w)$ as the **length** of w in W. It is well-known that within any (finite rank) Weyl group W and for any subset $J \subseteq I$, there exists a unique element $w_J \in W_J$ such that $l(w_J) = \max_{w \in W_J} l(w)$, and this element is an involution. We set $w_0 = w_I$, and refer to this as the **longest element** of W. Note that although w_J is an involution, it is never an element of S, unless $|J| = 1$.

We shall mainly be interested in (B, N)-pairs which satisfy some additional structure. The following more refined definition was originally formulated in [33, Definition 3.1].

Definition 4.1.4. If G admits a (B, N)-pair, we say that the (B, N)-pair is **split at characteristic** p if:

(i) $B = U \rtimes H$, where U is a p-group and H is an abelian p'-group;

(ii) $H = \cap_{n \in N} nBn^{-1}$.

Note that the subgroup U is unique in any given subgroup B, but the subgroup H is only unique up to conjugacy. By the Bruhat decomposition, it follows that U is a Sylow p-subgroup of G. Given any $x \in G$ and $w \in W$, we define p-subgroups of U by setting $U_x = U \cap U^x$ and $U_w = U_n$, where $w = nH$ for some $n \in N$. The following is Theorem 3.4 in [33].

Proposition 4.1.5. Suppose that G admits a (B, N)-pair which is split at characteristic p. Then for all $w \in W$, we have

$$U = U_{w_0 w} U_w \quad \text{and} \quad U_{w_0 w} \cap U_w = \{1\}.$$

4.2 The Subgroup B

In this section, we consider some finite classical groups which admit a (B, N)-pair that is split at characteristic p. Our aim is to study $\mathcal{S}(B, Q)$, where Q is a p-subgroup of U. For finite classical groups, it is well-known that a (B, N)-pair is determined up to conjugacy; moreover, we have seen that Scott modules are also determined up to conjugacy. Thus, when studying a particular classical group, we may, without loss of generality, fix a choice of subgroup B. We start with a lemma, which holds for abstract (B, N)-pairs, accounting for some of the subgroups $Q \leq U$.

Lemma 4.2.1. Suppose that G admits a (B, N)-pair which is split at characteristic p, so that $B = U \rtimes H$ with $U \in \mathrm{Syl}_p(G)$. Then $\mathcal{S}(B, U_w){\downarrow_U} \cong k[U/U_w]$ for all $w \in W$.

Proof. By definition, we have $U_w = U \cap U^n$, where $w = nH = Hn$. If $h \in H$, then $nh = h'n$ for some $h' \in H$ and hence

$$(U_w)^h = U^h \cap U^{nh} = U \cap U^{h'n} = U \cap U^n = U_w.$$

Thus $H \leq N_B(U_w)$ and it follows that $|N_B(U_w)|_{p'} = |H| = |B : U|$; we therefore have $\mathcal{S}(B, U_w){\downarrow_U} \cong k[U/U_w]$ by Theorem 2.4.2. □

4.2.1 $GL_n(q)$

Suppose that $G = GL_n(q)$ for a prime power $q = p^e$. The order of $GL_n(q)$ is given by:

$$|GL_n(q)| = q^{\frac{1}{2}n(n-1)}(q-1)(q^2-1)\cdots(q^n-1).$$

Thus, the order of a Sylow p-subgroup of G is $p^{\frac{e}{2}n(n-1)}$; such a Sylow p-subgroup of G is given by the **unitriangular** matrices:

$$U_n(q) = \left\{ \begin{bmatrix} 1 & * & * & \cdots & * & * \\ 0 & 1 & * & \cdots & * & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix} : * \in \mathbb{F}_q \right\}.$$

This subgroup is part of a (B, N)-pair of G which is split at characteristic p; the corresponding subgroup H is the diagonal matrices, and the subgroup $B = U \rtimes H$ is equal to the set of upper triangular matrices. Moreover, N consists of the monomial matrices in $GL_n(q)$ and the Weyl group $W := N/H \cong \Sigma_n$; we may identify W with the group of permutation matrices, since the group of permutation matrices in N forms a left transversal of H in N. We set $B_n(q)$ to be the set of upper triangular matrices in $GL_n(q)$, $H_n(q)$ to be the set of diagonal matrices and $N_n(q)$ to be the set of monomial matrices.

This particular Sylow p-subgroup $U_n(q)$ and its subgroups have been studied now for some time; in particular, in [40], Weir analyses a certain class of subgroups in $U_n(q)$ known as *partition* subgroups. If $i < j$, then we let $U_{i,j} = \{I_n + \alpha e_{ij} : \alpha \in \mathbb{F}_q\}$; note that $U_{i,j}$ is a subgroup of $U_n(q)$ isomorphic to the additive group of $\mathbb{F}_q$ and hence $U_{i,j} \cong E_q$ for all $i < j$. A subgroup $Q \leq U_n(q)$ is said to be a **partition subgroup** of $U_n(q)$ if it is generated by a selection of the subgroups $U_{i,j}$. For example, if $n = 4$, then a direct calculation verifies that

$$Q = \left\{ \begin{bmatrix} 1 & * & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} : * \in \mathbb{F}_q \right\}$$

is a subgroup of $U_4(q)$ and hence is a partition subgroup of $U_4(q)$; in this case, $Q = \langle U_{1,2}, U_{1,4}, U_{2,4}\rangle$.

Proposition 4.2.2. Suppose that $U = U_n(q)$ and $B = B_n(q)$, and let $Q \leq U$ be a partition subgroup of U. Then $\mathcal{S}(B,Q)\!\downarrow_U \cong k[U/Q]$.

Proof. Recall that $B = U \rtimes H$, where $H = H_n(q)$ denotes the set of diagonal matrices. If $h \in H$ and $i < j$, then it is clear that $U_{i,j}^h = U_{i,j}$, since conjugating a matrix in $GL_n(q)$ by h multiplies every entry of the matrix by an element of $\mathbb{F}_q$. Thus, if $Q \leq U$ is a partition subgroup, then $h \in N_B(Q)$ and it follows that $H \leq N_B(Q)$. The result now follows from Theorem 2.4.2, since $U \lhd B$. $\square$

In fact, more can be said in the case where Q is a normal subgroup of U; in this case, the normal partition subgroups are the only subgroups of U for which $\mathcal{S}(B,Q)\!\downarrow_U \cong k[U/Q]$.

Proposition 4.2.3. Suppose that $U = U_n(q)$ and $B = B_n(q)$. If $Q \lhd U$, then $\mathcal{S}(B,Q)\!\downarrow_U \cong k[U/Q]$ if and only if Q is a partition subgroup of U.

Proof. If $\mathcal{S}(B,Q)\!\downarrow_U \cong k[U/Q]$, then by Theorem 2.4.2, it follows that $Q \lhd B$. By [40, Theorem 5], we therefore have that Q is partition subgroup of U. On the other hand, if Q is a partition subgroup of U, then $\mathcal{S}(B,Q)\!\downarrow_U \cong k[U/Q]$ by Proposition 4.2.2. $\square$

Example 4.2.4. Suppose that $G = GL_3(p)$ for an odd prime p. Then $U_3(p) \cong p_+^{1+2}$; explicitly, we may verify the generators and relations given in Definition 1.2.2 using the following matrices:

$$a = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad c = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Let $U = U_3(p), B = B_3(p)$ and $H = H_3(p)$. Then:

(i) a complete list of representatives of the conjugacy classes of subgroups of order p in U is given by: $\langle b\rangle$, $\langle c\rangle$ and $\langle ab^i\rangle$ with $0 \leq i \leq p-1$;

(ii) the subgroups of order p^2 in U are $\langle b, c\rangle$ and $\langle ab^i, c\rangle$ with $0 \leq i \leq p-1$.

We include a proof of this fact in the appendix; see Appendix A.2. It is clear that $\langle a\rangle$, $\langle b\rangle, \langle c\rangle, \langle b, c\rangle$ and $\langle a, c\rangle$ are all partition subgroups and hence $\mathcal{S}(B, Q)\downarrow_U \cong k[U/Q]$ if Q is any of these subgroups, by Proposition 4.2.2.

Let $1 \le i \le p-1$ and set $Q = \langle ab^i\rangle$. Let

$$u = \begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix} \in U \quad \text{and} \quad h = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{bmatrix} \in H$$

where $x, y, z \in \mathbb{F}_p$ and $\alpha, \beta, \gamma \in \mathbb{F}_p^\times$. Then

$$u^{-1} = \begin{bmatrix} 1 & -x & xz - y \\ 0 & 1 & -z \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad ab^i = \begin{bmatrix} 1 & 1 & i \\ 0 & 1 & i \\ 0 & 0 & 1 \end{bmatrix}.$$

A direct calculation now shows that

$$(ab^i)^u = \begin{bmatrix} 1 & 1 & z + i - xi \\ 0 & 1 & i \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{so} \quad (ab^i)^{uh} = \begin{bmatrix} 1 & \alpha^{-1}\beta & \alpha^{-1}\gamma(z + i - xi) \\ 0 & 1 & \beta^{-1}\gamma i \\ 0 & 0 & 1 \end{bmatrix}.$$

Assume that $2 \le m \le p-1$ and

$$(ab^i)^{m-1} = \begin{bmatrix} 1 & m-1 & f(m-1)i \\ 0 & 1 & (m-1)i \\ 0 & 0 & 1 \end{bmatrix},$$

where $f(m-1) = \frac{(m-1)m}{2}$ denotes the $(m-1)$-th triangular number. Then

$$(ab^i)^m = \begin{bmatrix} 1 & m-1 & f(m-1)i \\ 0 & 1 & (m-1)i \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & i \\ 0 & 1 & i \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & m & f(m)i \\ 0 & 1 & mi \\ 0 & 0 & 1 \end{bmatrix}, \qquad (*)$$

so by induction, $(*)$ holds for $1 \le m \le p-1$. Thus, $(ab^i)^{uh} \in Q$ if and only if $\alpha^{-1}\beta = \beta^{-1}\gamma$ and

$$\alpha^{-1}\gamma(z + i - xi) = f(\alpha^{-1}\beta) = \frac{\alpha^{-1}\beta(\alpha^{-1}\beta + 1)}{2},$$

i.e., if and only if $\gamma = \alpha^{-1}\beta^2$ and

$$z = \frac{\beta(\alpha^{-1}\beta + 1)}{2\gamma} + xi - i.$$

In particular, we conclude that $|N_B(Q)| = (p-1)^2p^2$ and it follows from Theorem 2.4.2 that $\dim \mathcal{S}(B,Q) = (p-1)|U:Q|$.

Now let $1 \le i \le p-1$ and $Q = \langle ab^i, c\rangle$. Note that $Q \triangleleft U$ but Q is not a partition subgroup, so we know from Proposition 4.2.3 that $\dim \mathcal{S}(B,Q) > |U:Q|$. Furthermore, since $Q \triangleleft U$, $|N_B(Q)|_{p'} = |N_H(Q)|$. By $(*)$, a general element $u \in Q$ looks like

$$u = (ab^i)^m c^n = \begin{bmatrix} 1 & m & f(m)i - n \\ 0 & 1 & mi \\ 0 & 0 & 1 \end{bmatrix}$$

with $0 \le m, n \le p-1$. Moreover,

$$u^h = \begin{bmatrix} 1 & \alpha^{-1}\beta m & \alpha^{-1}\gamma(f(m)i - n) \\ 0 & 1 & \beta^{-1}\gamma m i \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus, if $m \ne 0$, then $u^h \in Q$ if and only if $\gamma = \alpha^{-1}\beta^2$, so $|N_B(Q)| = (p-1)^2p^3$. By Theorem 2.4.2, it follows that $\dim \mathcal{S}(B,Q) = (p-1)|U:Q|$.

We summarise our findings in Table 4.1. Note that there are no examples of a nonpartition subgroup $Q \le U$ for which $\mathcal{S}(B,Q){\downarrow_U} \cong k[U/Q]$.

4.2.2 Other Classical Groups

The symplectic group $Sp_{2m}(q)$ and the orthogonal groups $O_{2m+1}(q)$ and $O^+_{2m}(q)$ are both examples of classical groups which admit (B,N)-pairs that are split at characteristic

Q	$\lvert N_{B_3(p)}(Q)\rvert_{p'}$	$\dim \mathcal{S}(B_3(p), Q)/\lvert U_3(q) : Q\rvert$
$\{1\}$	$(p-1)^3$	1
$\langle a\rangle$	$(p-1)^3$	1
$\langle b\rangle$	$(p-1)^3$	1
$\langle c\rangle$	$(p-1)^3$	1
$\langle ab^i\rangle$, $1 \leq i \leq p-1$	$(p-1)^2$	$(p-1)$
$\langle a, c\rangle$	$(p-1)^3$	1
$\langle b, c\rangle$	$(p-1)^3$	1
$\langle ab^i, c\rangle, 1 \leq i \leq p-1$	$(p-1)^2$	$(p-1)$

Table 4.1 Scott Modules for $B_3(p) = U_3(p) \rtimes H_3(p)$

p. Moreover, as we shall see, these (B, N)-pairs can be described in terms of $U_n(q)$, $H_n(q)$ and $B_n(q)$, where n denotes the degree of the particular classical group.

In order to make this clear, we cover some generalities, which are taken from [13, Section 1.7]. Suppose that G is a finite group which admits a (B, N)-pair that is split at characteristic p, and $B = U \rtimes H$. Let $\varphi : G \to G$ be a bijective homomorphism such that:

(i) $\varphi(U) = U$, $\varphi(H) = H$ and $\varphi(N) = N$;

(ii) every coset Hn such that $\varphi(Hn) \subseteq Hn$ contains an element which is fixed by φ.

The fixed point set $G^\varphi = \{g \in G : \varphi(g) = g\}$ then admits a (B, N)-pair which is split at characteristic p, namely the pair (B^φ, H^φ). Moreover, we have $B^\varphi = U^\varphi \rtimes H^\varphi$ and U^φ is a Sylow p-subgroup of G^φ. The Weyl group of the new (B, N)-pair is N^φ/H^φ.

For each classical group $Sp_{2m}(q), O_{2m+1}(q)$ and $O^+_{2m}(q)$ contained in a general linear group $GL_n(q)$, there exists a corresponding homomorphism $\varphi : GL_n(q) \to GL_n(q)$ satisfying (i) and (ii) in the above and such that $(GL_n(q))^\varphi$ equals the original classical group. In this section, we shall carry out some computations involving the classical group $Sp_4(p)$, looking at the Scott modules associated to the subgroup $(B_4(p))^\varphi$. For

$n \in \mathbb{N}$, we define the $(n \times n)$-matrix

$$Q(n) = \begin{bmatrix} 0 & 0 & \cdots & 1 \\ \vdots & \vdots & \iddots & \vdots \\ 0 & 1 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \end{bmatrix}.$$

Example 4.2.5. Let $G = Sp_4(p)$ and p be an odd prime. Then $Sp_4(p)$ is equal to the fixed point set of the bijective homomorphism $\varphi : GL_4(p) \to GL_4(p)$ given by $\varphi(g) = n_0^{-1}(g^T)^{-1}n_0$, where

$$n_0 = \begin{bmatrix} 0 & Q(2) \\ -Q(2) & 0 \end{bmatrix}.$$

Thus, $Sp_4(p)$ admits a (B, N)-pair, where

$$B = B_4(p)^{\varphi} \quad \text{and} \quad N = N_4(p)^{\varphi}.$$

Furthermore, a Sylow p-subgroup of $Sp_4(p)$ is given by $U := U_4(p)^{\varphi}$; a direct calculation shows that

$$(U_4(p))^{\varphi} = \left\{ \begin{bmatrix} 1 & x & y & w \\ 0 & 1 & z & y - xz \\ 0 & 0 & 1 & -x \\ 0 & 0 & 0 & 1 \end{bmatrix} : x, y, w, z \in \mathbb{F}_p \right\}.$$

We let

$$a = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad c = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad d = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then $U \cong E \rtimes D$, where $E = \langle a, b, c\rangle \cong E_{p^3}$ and $D = \langle d \rangle \cong C_p$. Furthermore, we have $a^d = ab^{-2}$, $b^d = b$ and $c^d = a^{-1}bc$. The possible subgroups of U are described by the following:

Q	$\lvert N_B(Q)\rvert_{p'}$	$\dim \mathcal{S}(B,Q)/\lvert U:Q\rvert$
$\{1\}$	$(p-1)^2$	1
$\langle b\rangle$	$(p-1)^2$	1
$\langle d\rangle$	$(p-1)^2$	1
$\langle ab\rangle$	$(p-1)^2$	1
$\langle c\rangle$	$(p-1)^2$	1
$\langle b^ic\rangle$, $1 \le i \le p-1$	$2(p-1)$	$(p-1)/2$
$\langle cd^i\rangle$, $1 \le i \le p-1$	$(p-1)$	$(p-1)$
$\langle a,b\rangle$	$(p-1)^2$	1
$\langle b,d\rangle$	$(p-1)^2$	1
$\langle ab, b^ic\rangle$, $4i \equiv -1 \bmod p$	$(p-1)^2$	1
$\langle ab, b^ic\rangle$, $0 \le i \le p-1$, $4i \not\equiv -1 \bmod p$	$2(p-1)$	$(p-1)/2$
$\langle b, cd^i\rangle$, $0 \le i \le p-1$	$(p-1)$	$(p-1)$
$\langle a,b,d\rangle$	$(p-1)^2$	1
$\langle a,b,c\rangle$	$(p-1)^2$	1
$\langle a,b,cd^i\rangle$, $1 \le i \le p-1$	$(p-1)$	$(p-1)$

Table 4.2 Scott Modules for the subgroup $B = B_4(p)^\varphi$ of $Sp_4(p)$

(i) a complete list of representatives of the conjugacy classes of subgroups of order p in U is given by $\langle b\rangle, \langle d\rangle, \langle ab\rangle, \langle b^ic\rangle$ for $0 \le i \le p-1$ and $\langle cd^i\rangle$ for $1 \le i \le p-1$;

(ii) a complete list of representatives of the conjugacy classes of subgroups of order p^2 in U is given by $\langle a,b\rangle$, $\langle b,d\rangle$, $\langle ab, b^ic\rangle$ for $0 \le i \le p-1$ and $\langle b, cd^i\rangle$ for $0 \le i \le p-1$; and

(iii) the subgroups of order p^3 in P are $\langle a,b,d\rangle$ and $\langle a,b,cd^i\rangle$ for $0 \le i \le p-1$.

Again, we shall not prove this here, but we do give a proof in the appendix; see Appendix A.3. Recall that we are interested in the Scott modules $\mathcal{S}(B,Q)$, where Q is one of the subgroups in (i), (ii) or (iii). The Weyl group of our (B,N)-pair, $W = N/(H_4(p))^\varphi \cong C_2 \wr C_2 \cong D_8$. More explicitly, we have

$$W \cong \langle (1,4),(2,3)\rangle \rtimes \langle (1,2)(3,4)\rangle \le \Sigma_4.$$

We may represent elements of the Weyl group by their corresponding permutation matrices in N. There are therefore 8 subgroups of U of the form $U_w = U \cap U^w$ for some $w \in W$, and these are:

$$\begin{array}{rrr} U_{1_{\Sigma_4}} = U & U_{(1,4)} = \langle c \rangle & U_{(1,3)(2,4)} = \langle d \rangle \\ U_{(1,3,4,2)} = \langle b, d \rangle & U_{(1,2,4,3)} = \langle a, c \rangle & U_{2,3} = \langle a, b, d \rangle \\ U_{(1,2)(3,4)} = E & U_{(1,4)(2,3)} = \{1\}. & \end{array}$$

Moreover, $\langle a, c \rangle$ is U-conjugate to $\langle ab, b^i c \rangle$, where $4i \equiv -1 \bmod p$. Indeed, let $\alpha = (p-1)/2$ and set

$$u = \begin{bmatrix} 1 & \alpha & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \alpha+1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

A direct calculation verifies that $a^u = ab$ and $c^u = a^{\alpha+1} b^{(\alpha+1)^2} c$. Thus $c^u \in \langle ab, b^i c \rangle$ if and only if $c^u = (ab)^{\alpha+1}(b^i c)$, which occurs if and only if

$$\alpha + 1 + i \equiv (\alpha+1)^2 \bmod p$$

if and only if

$$2p + 2 + 4i \equiv p^2 + 2p + 1 \bmod p$$

if and only if $4i \equiv -1 \bmod p$. If Q is any of the above subgroups, we have $\mathcal{S}(B, Q)\downarrow_U \cong k[U/Q]$, by Lemma 4.2.1. In addition, $\langle b \rangle = Z(U)$ and hence $\langle b \rangle \triangleleft B$, so $\mathcal{S}(B, \langle b \rangle)\downarrow_U \cong k[U/\langle b \rangle]$, by Theorem 2.4.2.

The dimensions of the remaining cases may be determined using analysis similar to that seen in Example 4.2.4. We have summarised our findings in Table 4.2; details of the computations we used to find these are included in Appendix A.3.

4.3 Parabolic Subgroups

Suppose that G is a finite group that admits a (B, N)-pair which is split at characteristic p. Then $B = U \rtimes H$ for some p-group U and abelian p'-group H, and $U \in \mathrm{Syl}_p(G)$. Furthermore, any subgroup $\Gamma \leq G$ satisfying $B \leq \Gamma \leq G$ is a parabolic subgroup, and is hence of the form P_J for some $J \subseteq I$, where I is the index set for the set of simple roots of the Weyl group of G. In this section, we study some of the Scott modules which are associated to these parabolic subgroups.

These statements can be understood using something called the *Levi decomposition*. In order to describe this topic, some notation is necessary. For a given subset $J \subseteq I$, we define $U_J = U \cap U^{w_J}$, where w_J denotes the longest word in $S_J = \langle s_j : j \in J \rangle$. We define

$$L_J = \langle H, (U_{w_0 w_i})^w : w \in W_J, i \in J \rangle$$

and refer to this as the **Levi** subgroup corresponding to J. The following properties of the subgroups U_J and L_j are reasonably well-known (see [39, Section 3]).

Theorem 4.3.1. (The Levi Decomposition) Let G be a finite group which admits a (B, N)-pair that is split at characteristic p, with index set I for its Weyl group W. Then for each $J \subseteq I$, we have:

(i) $N_G(U_J) = P_J$;

(ii) $P_J = U_J \rtimes L_J$, where L_J denotes the Levi subgroup corresponding to J;

(iii) L_J admits a split (B, N)-pair.

As a consequence of this result, we have the following.

Corollary 4.3.2. Suppose that $J \subseteq I$. Then:

(i) $\mathcal{S}(P_J, U_{w_0 w_J})\downarrow_U \cong k[U/U_{w_0 w_J}]$;

(ii) $\mathcal{S}(P_J, U_J) \cong \mathcal{P}(k_{P_J/U_J})$.

Proof. It is shown in [39, Lemma 2.A] that L_J admits a split (B, N)-pair with Sylow p-subgroup $U_{w_0 w_J}$. By the Levi decomposition, $|P_J : L_J| = U_J = p^a$ for some $a \in \mathbb{N}$, so by Proposition 2.3.2 (ii), the first part follows. The second part follows from the fact that $U_J \lhd P_J$ and (iii) of the Scott-Alperin theorem. $\square$

Example 4.3.3. Let $G = GL(4, q)$, so that we have a set of simple reflections $S = \{(1, 2), (2, 3), (3, 4)\}$, and take $S_J = \{(1, 2), (3, 4)\}$. In this case, the longest word in the subgroup W_J is $(1, 2)(3, 4)$, and the longest word in the Weyl group W of G is

$w_0 = (1,4)(2,3)$, so $w_0 w_J = (1,3)(2,4)$. A direct calculation now shows that

$$U_{w_0 w_J} = \left\{ \begin{bmatrix} 1 & * & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{bmatrix} : * \in \mathbb{F}_q \right\}$$

and

$$L_J = \left\{ \begin{bmatrix} * & * & 0 & 0 \\ * & * & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{bmatrix} \in GL(4,q) : * \in \mathbb{F}_q \right\}.$$

On the other hand

$$U_J = U \cap U^{w_J} = \left\{ \begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} : * \in \mathbb{F}_q \right\},$$

whilst

$$P_J = \left\{ \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{bmatrix} \in GL(4,q) : * \in \mathbb{F}_q \right\}.$$

We can then easily see from the above that $P_J = U_J \rtimes L_J$ and $U_{w_0 w_J} \in \mathrm{Syl}_p(L_J)$.

Chapter 5

Relatively Projective Covers

So far in this thesis, we have been primarily interested in questions related to the Scott module $\mathcal{S}(G, Q)$, where G is a finite group and Q is a p-subgroup of G. We recall that $\mathcal{S}(G, Q)$ is a relatively Q-projective cover of the trivial module k_G, and thus $\mathcal{S}(G, Q)$ may be studied in the wider context of relatively Q-projective resolutions. In this chapter, we take a first step beyond the Scott module by studying the module $\Omega^2_{P/Q}(k)$ for a p-group $P = H \rtimes Q$ which can be written as a semidirect product of a normal subgroup H and a cyclic subgroup Q of order p; our main aim is to study and bound the quantity $\dim(\Omega^2_{P/Q}(k))$.

5.1 Basic Constructions

We start by recalling some basics. Suppose that G is a finite group and $Q \leq G$ is a p-subgroup. By the **augmentation map**, we mean the kG-homomorphism $\sigma : k[G/Q] \to k_G$ given by

$$\sigma\left(\sum_{x \in [G/Q]} \alpha_x [x]\right) = \sum_{x \in [G/Q]} \alpha_x$$

for all elements of $k[G/Q]$. Note that σ is clearly surjective and $k[G/Q]$ is relatively Q-projective; moreover, as we saw in the proof of the Scott-Alperin theorem, k_Q is a direct summand of $(k[G/Q])\downarrow_Q$ and hence σ is a Q-split kG-homomorphism. It follows from Proposition 1.1.12 that there exists an indecomposable direct summand $\mathcal{S}$ of $k[G/Q]$

which is a relatively Q-projective cover of k_G. By (ii) of the Scott-Alperin theorem, this module must be isomorphic to $\mathcal{S}(G,Q)$, since if $\mathcal{M}$ is any other indecomposable direct summand of $k[G/Q]$, we know that k_G is not isomorphic to any submodule of $M/\operatorname{rad}(\mathcal{M})$.

We have provided some results which give necessary conditions for when $\mathcal{S}(G,Q){\downarrow_P} \cong \mathcal{S}(P,Q)$, where $P \in \mathrm{Syl}_p(G)$. In some senses, this may be viewed as the "simplest" behaviour that we might expect from Scott modules, and the question can naturally be extended to relative syzygies: if Q is a p-subgroup of G, when do we have $\Omega^i_{G/Q}(k){\downarrow_P} \cong \Omega^i_{P/Q}(k)$? The following generalises Corollary 2.3.7.

Proposition 5.1.1. Suppose that G is a p-nilpotent group and $Q \leq P \in \mathrm{Syl}_p(G)$. Then $\Omega^i_{G/Q}(k){\downarrow_P} \cong \Omega^i_{P/Q}(k)$ for all $i \geq 1$.

Proof. We have $G = O_{p'}(G) \rtimes P$, so $P \cong G/O_{p'}(G)$, and if

$$\cdots \xrightarrow{\delta_4} \mathcal{P}_3 \xrightarrow{\delta_3} \mathcal{P}_2 \xrightarrow{\delta_2} \mathcal{P}_1 \xrightarrow{\delta_1} k_P \to 0$$

is a relatively Q-projective resolution of k_P, then

$$\cdots \xrightarrow{\overline{\delta_4}} \mathrm{Inf}_P^G(\mathcal{P}_3) \xrightarrow{\overline{\delta_3}} \mathrm{Inf}_P^G(\mathcal{P}_2) \xrightarrow{\overline{\delta_2}} \mathrm{Inf}_P^G(\mathcal{P}_1) \xrightarrow{\overline{\delta_1}} k_G \to 0 \qquad (*)$$

is an exact sequence, where $\overline{\delta_i} = \mathrm{Inf}_P^G(\delta_i)$. Each $\mathcal{P}_i$ is a relatively Q-projective module, so $\mathcal{P}_i$ is a direct summand of $((\mathcal{P}_i){\downarrow_Q}){\uparrow^P}$. By [4, 1.1.3], we have that $\mathrm{Inf}_P^G(\mathcal{P}_i)$ is a direct summand of

$$(\mathrm{Inf}_Q^{QO_{p'}(G)}((\mathcal{P}_i){\downarrow_Q})){\uparrow^G}$$

and hence $\mathrm{Inf}_P^G(\mathcal{P}_i)$ is relatively $QO_{p'}(G)$-projective, so it is Q-projective. Furthermore, if $f \in \mathrm{End}_{kG}(\mathcal{P}_i)$ satisfies $\overline{\delta_i} f = \overline{\delta_i}$, then clearly $\delta_i f_P = \delta_i$ and hence f_P is a bijection, so f is a bijection as well. Thus, $(*)$ is a relatively Q-projective resolution of k_G. Since $(\mathrm{Inf}_P^G(\mathcal{P}_i)){\downarrow_P} \cong \mathcal{P}_i$ and $\overline{(\delta_i)}_P = \delta_i$, it follows that $\Omega^i_{G/Q}(k){\downarrow_P} \cong \Omega^i_{P/Q}(k)$, as required. $\square$

Suppose now that $Q \leq P$ is cyclic of order p and $P = H \rtimes Q$ for some $H \leq P$. Throughout what follows, we let $Q = \langle w \rangle$ for some element w of order p. We start with a couple of preliminary results, the first of which describes the possible homomorphisms we may have involving the kP-modules kP and $k[P/Q]$, and the second of which describes how these homomorphisms split.

Lemma 5.1.2. Suppose that $\mathcal{M}$ is a kP-module and $Q = \langle w \rangle$ for some element w of order p. Then:

(i) for each $m \in \mathcal{M}$, there exists precisely one kP-homomorphism $\delta : kP \to \mathcal{M}$ such that $\delta(1_P) = m$;

(ii) for each $m \in \mathcal{M}$ satisfying $wm = m$, there exists precisely one kP-homomorphism $\delta : k[P/Q] \to \mathcal{M}$ such that $\delta([1_P]) = m$, and any kP-homomorphism between $k[P/Q]$ and $\mathcal{M}$ is of this form for some $m \in \mathcal{M}$.

Proof. Suppose that $\delta : kP \to \mathcal{M}$ is a kP-homomorphism and $m \in \mathcal{M}$. Then

$$\delta\left(\sum_{u \in P} \alpha_u u\right) = \sum_{u \in P} \alpha_u (u\delta(1_P))$$

and hence δ is completely determined by the value $\delta(1_P)$, so there is at most one kP-homomorphism such that $\delta(1_P) = m$. If we define $\delta : kP \to \mathcal{M}$ by

$$\delta\left(\sum_{u \in P} \alpha_u u\right) = \sum_{u \in P} \alpha_u (um),$$

then a routine check confirms that δ is a kP-homomorphism satisfying $\delta(1_P) = m$, so (i) follows.

Now suppose that $\delta : k[P/Q] \to \mathcal{M}$ is a kP-homomorphism. Then

$$\delta([1_P]) = \delta(w[1_P]) = w\delta([1_P]).$$

Thus, if $m \in \mathcal{M}$, then there exists a kP-homomorphism $\delta : k[P/Q] \to \mathcal{M}$ such that $\delta([1_P]) = m$ only if $wm = m$. Moreover, if $wm = m$, then

$$\delta\left(\sum_{x \in [P/Q]} \alpha_x [x]\right) = \sum_{x \in X} \alpha_x (x\delta([1_P]))$$

and hence δ is completely determined by the value $\delta([1_P])$, so there is at most one kP-homomorphism such that $\delta([1_P]) = m$. A routine check verifies that the map $\delta : k[P/Q] \to \mathcal{M}$ given by

$$\delta\left(\sum_{x\in[P/Q]} \alpha_x[x]\right) = \sum_{x\in[P/Q]} \alpha_x(xm)$$

is a well-defined kP-homomorphism, so (ii) follows. □

Following on from this result, we establish some notation connected to the module $\mathcal{M} = k[P/Q]^r \oplus kP^s$, with $r, s \in \mathbb{N}_0$. We set $1_{P/Q}$ to equal the element $[1] \in k[P/Q]$ and let 1_P denote the identity of P. We view elements of the direct sum $\mathcal{M}$ as being $(r+s)$-tuples and for a given $m \in \mathcal{M}$, we let $m[i]$ denote the component of m in the i-th module in the direct sum. For $1 \le i \le r$, we define $\mathbf{m}_i \in \mathcal{M}$ to be the element of $\mathcal{M}$ with $(\mathbf{m}_i)[i] = [1_P]$ and $(\mathbf{m}_i)[j] = 0$ for $j \ne i$; and for $r+1 \le i \le s$, we define $\mathbf{m}_i \in \mathcal{M}$ by the prescription $(\mathbf{m}_i)[i] = 1_P$ and $(\mathbf{m}_i)[j] = 0$ for $j \ne i$. By Lemma 5.1.2, a given kP-homomorphism $\delta : \mathcal{M} \to \mathcal{N}$ is determined completely by the values $\delta(\mathbf{m}_i)$ for $1 \le i \le r+s$, and is well-defined if and only if $w \cdot \delta(\mathbf{m}_i) = \delta(\mathbf{m}_i)$ for $1 \le i \le r$.

Lemma 5.1.3. Suppose that $\mathcal{N}$ is a kP-module and $\mathcal{N}{\downarrow_Q}$ is a permutation kQ-module with permutation basis X. Suppose that $\delta : \mathcal{M} \to \mathcal{N}$ is a surjective homomorphism of kP-modules such that whenever $wx = x$ for some $x \in X$, there exists $m \in \mathcal{M}$ such that $\delta(m) = x$ and $wm = m$. Then δ is Q-split.

Proof. The action of Q on X splits up X into orbits of size 1 and orbits of size p. Let $x_1, \dots, x_s$ be those elements of X which are fixed by Q and $z_1, \dots, z_t$ be representatives of the orbits of size p; then

$$X = \{x_1, \dots, x_s\} \cup \{w^l z_j : 0 \le l \le p-1, 1 \le j \le t\}.$$

By assumption, for $1 \le i \le s$, there exist elements $m_i \in \mathcal{M}$ such that $\delta(m_i) = x_i$ and $wm_i = m_i$. For $1 \le j \le t$, let $m_j^* \in \mathcal{M}$ be such that $\delta(m_j^*) = z_j$ and define a linear transformation $\gamma : \mathcal{N} \to \mathcal{M}$ by letting $\gamma(x_i) = m_i$ for $1 \le i \le s$ and $\gamma(w^l z_j) = w^l m_j^*$ for $0 \le l \le p-1$ and $1 \le j \le t$. Then $\delta\gamma = \mathrm{id}$ and γ is a kQ-homomorphism by construction, so it follows that δ is Q-split, as required. □

5.2 Bounding $\dim \Omega^2_{P/Q}(k)$

In this section, we shall assume that $P = H \rtimes Q$ is a p-group, with $Q = \langle w \rangle$ for an element w of order p.

Lemma 5.2.1. The module $\Omega_{P/Q}(k){\downarrow_Q}$ is a permutation kQ-module with permutation basis equal to the Q-set

$$X = \{[h] - [1] : h \in H, h \neq 1\}.$$

Proof. Since $k[P/Q]$ is indecomposable, the module $k[P/Q]$ together with the augmentation map $\sigma : k[P/Q] \to k_P$ is a relatively Q-projective cover of k_P. Thus $\Omega_{P/Q}(k) \cong \ker \sigma$. Furthermore, $\dim(\ker \sigma) = \dim k[P/Q] - 1 = |P : Q| - 1$ and $|X| = |P : Q| - 1$. Finally, it is clear that the elements of X are linearly independent and X is a Q-set, since $H \triangleleft P$. Thus X is a permutation basis for $\Omega_{P/Q}(k){\downarrow_Q}$, as required. $\square$

Theorem 5.2.2. Let $C = C_H(Q)$ and let Z and T be subsets of P such that:

(i) $Z \subseteq T$;

(ii) $C = \langle Z \rangle$;

(iii) $H = \langle {}^{w^l}t : 0 \leq l \leq p-1, t \in T \rangle$.

Then, if $r = |Z|$ and $s = |T| - |Z|$, we have

$$\dim \Omega^2_{P/Q}(k) \leq |H|^r + |P|^s - (|H| - 1).$$

Proof. Suppose that $T = \{z_1, \ldots, z_r, t_1, \ldots, t_s\}$, where $z_1, \ldots, z_r \in Z$ and $t_1, \ldots, t_s \notin Z$. Let $\mathcal{M} = k[P/Q]^r \oplus kP^s$ and define a kP-homomorphism $\delta : \mathcal{M} \to \Omega_{P/Q}(k)$ by setting

$$\delta(\mathbf{m}_i) = \begin{cases} [z_i] - 1_{P/Q} & \text{if } 1 \leq i \leq r, \\ [t_{i-r}] - 1_{P/Q} & \text{if } r+1 \leq i \leq r+s \end{cases}$$

and extending kP-linearly; by Lemma 5.1.2 and the fact that each $z_i \in C_H(Q)$, it follows that δ is a well-defined kP-homomorphism.

We first show that δ is surjective, so suppose that $h \in H$. Since $H = \langle {}^{w^l}t : 0 \le l \le p-1, t \in T \rangle$, we have $h = u_1 \cdots u_k$, where each u_j is in the set:

$$\mathcal{T} = \{{}^{w^m}(t^n) : 0 \le m \le p-1, n \in \mathbb{N}_0, t \in T\}.$$

Note that $[u] - 1_{P/Q} \in \delta(\mathcal{M})$ for all $u \in \mathcal{T}$: indeed, if $u = {}^{w^m}(t^n)$, then we have

$$\begin{aligned} [u] - 1_{P/Q} &= [{}^{w^m}(t^n)] - 1_{P/Q} = ([{}^{w^m}(t^n)] - [{}^{w^m}(t^{n-1})]) + ([{}^{w^m}(t^{n-1})] - 1_{P/Q}) \\ &= w^m t^{n-1}([t] - 1_{P/Q}) + ([{}^{w^m}(t^{n-1})] - 1_{P/Q}). \end{aligned}$$

Thus, if we assume that for a given m we have $[{}^{w^m}(t^{n-1})] - 1_{P/Q} \in \delta(\mathcal{M})$, it follows that $[{}^{w^m}(t^n)] - 1_{P/Q} \in \delta(\mathcal{M})$, and hence $[u] - 1_{P/Q} \in \delta(\mathcal{M})$, by induction. Thus,

$$\begin{aligned} [h] - 1_{P/Q} &= [u_1 \cdots u_k] - 1_{P/Q} = ([u_1 \cdots u_k] - [u_1 \cdots u_{k-1}]) + ([u_1 \cdots u_{k-1}] - 1_{P/Q}) \\ &= u_1 \cdots u_{k-1}([u_k] - 1_{P/Q}) + ([u_1 \cdots u_{k-1}] - 1_{P/Q}) \end{aligned}$$

and it follows by induction on k that $[h] - 1_{P/Q} \in \delta(\mathcal{M})$. Since

$$X = \{[h] - 1_{P/Q} : h \in H, h \neq 1\}$$

is a permutation kQ-basis for $\Omega_{P/Q}(k){\downarrow_Q}$, we deduce that δ is surjective.

We now show that δ is Q-split using Lemma 5.1.3. Suppose that $[h] - 1 \in X$ satisfies $w \cdot ([h] - 1) = [h] - 1$. Then $h \in C_H(Q)$ and hence $h = u_1 \cdots u_k$, where each u_j is in the set:

$$\mathcal{Z} = \{z^n : n \in \mathbb{N}_0, z \in Z\}.$$

If $z \in Z$ and $z = z_i$ for $1 \le i \le r$, then

$$\delta\left(\sum_{s=0}^{n-1} z_i^s \mathbf{m}_i\right) = \sum_{s=0}^{n-1} z_i^s([z_i] - 1_{P/Q}) = [z_i^n] - 1_{P/Q}.$$

Thus, for each u_j, there exists $m_j \in \mathcal{M}$ such that $\delta(m_j) = [u_j] - 1_{P/Q}$ and $wm_j = m_j$. Assume inductively that there exists m^*_{k-1} such that $\delta(m^*_{k-1}) = [u_1 \cdots u_{k-1}] - 1_{P/Q}$

and $wm^*_{k-1} = m^*_{k-1}$. Then we have

$$\begin{aligned}\delta((u_1 \cdots u_{k-1})m_k + m^*_{k-1}) &= [u_1 \cdots u_{k-1}u_k] - [u_1 \cdots u_{k-1}] + [u_1 \cdots u_{k-1}] - 1_{P/Q} \\ &= [u_1 \cdots u_k] - 1_{P/Q}.\end{aligned}$$

and $w \cdot ((u_1 \cdots u_{k-1})m_k + m^*_{k-1}) = (u_1 \cdots u_{k-1})m_k + m^*_{k-1}$, so it follows by induction on k that there exists $m^*_k \in \mathcal{M}$ such that $\delta(m^*_k) = [u_1 \cdots u_k] - 1_{P/Q} = [h] - 1_{P/Q}$ and $wm^*_k = m^*_k$. By Lemma 5.1.3, we see that δ is Q-split.

Thus, $(\mathcal{M}, \delta)$ consists of a relatively Q-projective module and a surjective kP-homomorphism which is Q-split. By Proposition 1.1.12, it follows that a relatively Q-projective cover of $\Omega_{P/Q}(k)$ is isomorphic to a direct summand of $\mathcal{M}$. Thus

$$\dim(\Omega^2_{P/Q}(k)) \leq \dim \mathcal{M} - \dim \Omega_{P/Q}(k) = |H|^r + |P|^s - (|H| - 1).$$

$\square$

Example 5.2.3. Let $r \geq 1$ and $P = C_{2^r} \wr C_2$. This 2-group may be realised explicitly as a Sylow 2-subgroup of $SL_3(q)$, where q is an odd prime power such that 2^r divides $q-1$, but 2^{r+1} does not (see [8]). In particular, P may be written as a semidirect product $P = H \rtimes Q$, where $H = \langle x, y : x^{2^r} = y^{2^r} = 1 \rangle$ is abelian and $Q = \langle w : w^2 = 1 \rangle$ with $x^w = y$. Thus, $C_H(Q) = \langle xy \rangle$ and $H = \langle x, x^w, xy \rangle$. If we let $Z = \{xy\}$ and $T = \langle xy, x \rangle$, then it follows from Theorem 5.2.2 that

$$\dim \Omega^2_{P/Q}(k) \leq |H| + |P| - (|H| - 1) = |P| + 1 = 2^{2r+1} + 1.$$

Appendix A

Loose Ends

A.1 MAGMA Code

In this section of the appendix, we present MAGMA implementations of the two algorithms discussed in Section 3.4, along with some further additional functions which are helpful when working with the p-local subgroups of sporadic groups.

At [3], MAGMA implementations can be found for all 26 sporadic groups, along with their various maximal subgroups. The information available ranges from complete data for the smaller groups, to incomplete data for the larger monster sections and pariahs. The most helpful data that suits our purposes are the permutation representations, since MAGMA performs well when handling such a representation of a group. All but a few of the sporadic groups have permutation representations on a small (less than $500,000$) number of points; the other means of representating a group, which involves generating the group via a pair of matrices, is less useful for our purposes.

In Figure A.1, we have provided a function which can be used to calculate the p-local subgroups of a given sporadic group G. The function `FindSporadicNormalisers` accepts a sporadic group G, a prime $p \in \pi(G)$ and an optional parameter `limit`, which is set by default to 1; the parameter `limit` should be set to be equal to the number of distinct p-local subgroups in G for the given prime p, which can be read off the corresponding table in Section 3.4. It returns an array of the p-local subgroups it computes. If the `limit` parameter is set to a value less than the actual number of

```
FindSporadicNormalisers := function(G,p:limit:=1)
	P := Sylow(G,p);
	Normalisers := [];
	Con := ConjugacyClasses(P);
	for C in Con do
		// We only care for normalisers N_G(x) where x
		// is an element of order p, so we skip the rest
		if C[1] ne p then
			continue;
		end if;
		include := true;
		Nor := Normaliser(G,sub<G|C[3]>);
		for M in Normalisers do
			// There are no cases where two distinct p-local
			// subgroups have the same order, so we only
			// add M if its order is distinct from all cases so far
			if #M eq #Nor then
				include:= false;
				break;
			end if;
		end for;
		if include then
			Append(~Normalisers,Nor);
			// As a measure of progress, we output the
			// current number of normalisers computed
			#Normalisers;
		end if;
		if #Normalisers eq limit then
			break;
		end if;
	end for;
	return Normalisers;
end function;
```

Fig. A.1 Function used to calculate p-local subgroups of a sporadic group

p-local subgroups in G, then the function will return a number of p-local subgroups equal to `limit`, but there is no determining precisely which ones it returns.

Once these p-local subgroups have been calculated, a user can use the function in Figure A.2 to test a sufficient condition for a given group G to be p-extendible. The function `DoSubgroupsForpExtendibilityExist` accepts a finite group G and a prime p; if it returns `true`, then G is p-extendible. However, in many cases, it will return `false`, and nothing may be determined in this case; all that this means is that the group has failed to pass the sufficient condition tested by the function. Note that this function is a MAGMA implementation of Algorithm 1, an algorithm which was discussed in Section 3.4.

The other option available to the user is the function given in Figure A.3, which tests a sufficient condition for a finite group G to *not* be p-extendible. The function `DoesaNonWeaklyClosedNormalSubgroupExist` accepts a finite group G and a prime p. If it returns `true`, then the group G is definitely not p-extendible; more precisely, the function has determined that if $P \in \mathrm{Syl}_p(G)$, then there exists a subgroup $Q \leq P$

```
DoSubgroupsForpExtendibilityExist := function(G,p)
   P := Sylow(G,p);
      Index := #G div #P;
      r := Floor(Log(p,#P));
      for i in [0..r-1] do
            YSubs := Subgroups(G:OrderEqual := Index*p^i);
            YSubs := [x`subgroup : x in YSubs];
            if #YSubs eq 0 then
                  return false;
            end if;
            QSubs := Subgroups(P:OrderEqual := p^i);
            QSubs := [x`subgroup : x in QSubs];
            for Q in QSubs do
                  Conjs := Conjugates(P,Q);
                  Qpasses := false;
                  for C in Conjs do
                        for Y in YSubs do
                              if C subset Y then
                                    Qpasses := true;
                                    break;
                              end if;
                        end for;
                        if Qpasses then;
                              break;
                        end if;
                  end for;
                  if not Qpasses then
                        return false;
                  end if;
            end for;
      end for;
      return true;
end function;
```

Fig. A.2 MAGMA implementation of Algorithm 1

```
DoesaNonWeaklyClosedNormalSubgroupExist := function(G,p)
      P := Sylow(G,p);
      QSubs := NormalSubgroups(P);
      QSubs := [x`subgroup : x in QSubs];
      for Q in QSubs do
            Conjs := Conjugates(G,Q);
            for C in Conjs do
                  if C subset P and C ne Q then
                        return true;
                  end if;
            end for;
      end for;
      return false;
end function;
```

Fig. A.3 MAGMA implementation of Algorithm 2

```
IspNilpotent := function(G,p)
	P := Sylow(G,p);
	NOrder := #G div #P;
	// Ideally, it would be best to request those normal
	// subgroups which have order equal to NOrder, but
	// there are contexts in which this will not work, so we
	// utilise a slightly less efficient but more general
	// approach
	Subs := NormalSubgroups(G);
	for N in Subs do
		if Order(N`subgroup) eq NOrder then
			return true;
		end if;
	end for;
	return false;
end function;
```

Fig. A.4 Function that tests if a group is p-nilpotent

which is normal in P and not weakly closed in P with respect to G. On the other hand, nothing can be concluded if the function returns `false`; the group G may or may not be p-extendible. This function is a MAGMA implementation of Algorithm 2 from Section 3.4.

These three functions serve as the major tools needed to understand and verify much of the analysis involving MAGMA given in Section 3.4. There is one other convenience function that we provide here: a simple function which tests if a given finite group G is p-nilpotent, which is given in Figure A.4.

A.1.1 The Case $(Fi'_{24}, 3D)$

In this small subsection, we detail how one can approach obtaining a permutation representation of the 3-local subgroup N, where $G = Fi'_{24}$ and $N = N_G(\langle x \rangle)$ for an element $x \in G$ of order 3 and type $(3D)$: see Figure 3.24 and the corresponding analysis for more details. A naive approach to the problem involves using the function `FindSporadicNormalisers` and inputting the permutation representation for G obtained from [3], the prime 3 and setting `limit` to equal 4; however, this approach fails to compute the permutation representation in a reasonable timeframe. In figure A.5, we provide a method which computes a permutation representation for N from the permutation representation of G found at [3], and this may be used to verify the analysis we provided when studying the case $(Fi'_{24}, 3D)$.

```
// Input the permutation representation that is provided
// for Fi_{24} at the online ATLAS
Fi24AdHocMethod := function(G)
       P := Sylow(G,3);
       // We get a set of representatives of the conjugacy
       // classes of elements of order p
       Con := ConjugacyClasses(P);
       Conp := [];
       for C in Con do
              if C[1] eq 3 then
                     Append(~Conp,C[3]);
              end if;
       end for;
       // The MAGMA function ConjugacyClasses calculates its
       // conjugacy classes in descending order of N_P(x),
       // where x is the representative: thus, we know that
       // the entries 5-12 of the array Conp contain elements
       // x for which the order of N_P(x) is 3^14, and
       // we know that one of these elements is of type (3D)
       for i in [5..12] do
              N := Normaliser(G,sub<G|Conp[i]>);
              // We know that x is of type (3D) if and only
              // #N equals 2^6 * 3^14, so we test for this
              if #N eq 2^6 * 3^14 then
                     return N;
              end if;
       end for;
       return 0;
end function;
```

Fig. A.5 Function to calculate a permutation representation for the case $(Fi'_{24}, 3D)$

A.2 Subgroups of p_+^{1+2}

In this section of the appendix, we provide a description of the conjugacy classes of subgroups in p_+^{1+2}, the extraspecial group of order p^3 and exponent p. Recall that we have defined $[x, y] = x^{-1}y^{-1}xy$ for a pair of element x, y in a finite group G.

Proposition A.2.1. Suppose that p is an odd prime and

$$P = p_+^{1+2} = \langle a, b, c : a^p = b^p = c^p = 1, [a, c] = [b, c] = 1, [a, b] = c\rangle.$$

Then:

(i) a complete list of representatives of the conjugacy classes of subgroups of order p in P is given by: $\langle b\rangle, \langle c\rangle$ and $\langle ab^i\rangle$ with $0 \leq i \leq p-1$;

(ii) the subgroups of order p^2 in P are $\langle b, c\rangle$ and $\langle ab^i, c\rangle$ with $0 \leq i \leq p-1$.

Proof. Since the exponent of P is p and any subgroup of order p contains $p-1$ elements which no other subgroup of order p can contain, it follows that there are $(p^3-1)/(p-1) = p^2+p+1$ subgroups of P of order p. We have $a^b = ac$, so $(a^r)^b = a^r c^r$

and $(a^r)^{b^n} = a^r c^{nr}$. Similarly, $(b^{-1})^a = b^{-1}c$, so $(b^s)^a = b^s c^{-s}$ and $(b^s)^{a^m} = b^s c^{-ms}$. Thus, if $a^r b^s c^t, a^m b^n c^l \in P$ with $0 \leq r, s, t, m, n, l \leq p-1$, then

$$(a^r b^s c^t)^{a^m b^n c^l} = (a^r)^{b^n} (b^s)^{a^m b^n} c^t = a^r c^{nr} b^s c^{-ms} c^t = a^r b^s c^{nr-ms+t}. \qquad (*)$$

It thus follows that $|\mathrm{Stab}_P(\langle b \rangle)| = p^2$ and hence $|\mathcal{C}(\langle b \rangle)| = p$. On the other hand, if $1 \leq k \leq p$, then $(ab^i)^k = a^k b^{ki} c^{-f(k-1)i}$, where $f(k-1)$ denotes the $(k-1)$-th triangular number; moreover, from $(*)$, it follows that

$$(ab^i)^{a^m b^n c^l} = ab^i c^{n-im}. \qquad (**)$$

Thus, $a^m b^n c^l \in \mathrm{Stab}_P(\langle ab^i \rangle)$ if and only if $n \equiv im \bmod p$. Thus $|\mathrm{Stab}_P(\langle ab^i \rangle)| = p^2$ and $|\mathcal{C}(\langle ab^i \rangle)| = p$. From $(*)$ and $(**)$, we see that the conjugacy classes $\mathcal{C}(\langle b \rangle)$ and $\mathcal{C}(\langle ab^i \rangle)$ for $0 \leq i \leq p-1$ are all distinct; furthermore, the total number of subgroups in these conjugacy classes is given by

$$\underbrace{1}_{\langle c \rangle} + \underbrace{p}_{\langle b \rangle} + \underbrace{p \cdot p}_{\langle ab^i \rangle} = 1 + p + p^2$$

so we see that we have accounted for all possible subgroups.

Since $Z(P) = \langle c \rangle$, any subgroup of P of order p^2 must contain c, as any normal subgroup of a p-group intersects nontrivially with $Z(P)$. Thus, the subgroups of order p^2 of P are in one-to-one correspondence with subgroups of order p in $P/Z(P) \cong C_p^2$ and there are $(p^2-1)/(p-1) = p+1$ such subgroups. If $\langle ab^i, c \rangle = \langle ab^j, c \rangle$ with $0 \leq i < j \leq p-1$, then there must exist m and n with $0 \leq m, n \leq p-1$ such that

$$ab^i = (ab^j)^m c^n = a^m b^{jm} c^{-f(m-1)j} c^n.$$

But then $m = 1$ and hence $i = j$, a contradiction; thus $\langle ab^i, c \rangle \neq \langle ab^j, c \rangle$ for all $0 \leq i < j \leq p-1$. Furthermore, $b \notin \langle ab^i, c \rangle$, so $\langle b, c \rangle \neq \langle ab^i, c \rangle$ for $0 \leq i \leq p-1$, and we see that the list of subgroups given in (ii) is a complete list, as required. $\square$

A.3 Subgroups of $Sp_4(p)$

In this section of the appendix, we include the computations and analysis which support our findings in Example 4.2.5. We start with a result which describes the possible subgroups in a particular group of order p^4, where p is an odd prime.

Proposition A.3.1. Let $E \cong E_{p^3} = \langle a, b, c : a^p = b^p = c^p = 1\rangle$ and $D = \langle d\rangle \cong C_p$, and suppose that $P = E \rtimes D$ is the semidirect product given by $a^d = ab^{-2}, b^d = b$ and $c^d = a^{-1}bc$. Then:

(i) a complete list of representatives of the conjugacy classes of subgroups of order p in P is given by: $\langle b\rangle$, $\langle d\rangle$, $\langle ab\rangle$, $\langle b^i c\rangle$ for $0 \leq i \leq p-1$ and $\langle cd^i\rangle$ for $1 \leq i \leq p-1$;

(ii) a complete list of representatives of the conjugacy classes of subgroups of order p^2 in P is given by: $\langle a, b\rangle$, $\langle b, d\rangle$, $\langle ab, b^i c\rangle$ for $0 \leq i \leq p-1$ and $\langle b, cd^i\rangle$ for $0 \leq i \leq p-1$;

(iii) the subgroups of order p^3 in P are $\langle a, b, d\rangle$ and $\langle a, b, cd^i\rangle$ for $0 \leq i \leq p-1$.

Proof. Arguing inductively, we find that

$$a^{d^\sigma} = ab^{-2\sigma} \quad \text{and} \quad c^{d^\sigma} = a^{-\sigma} b^{\sigma^2} c$$

for $0 \leq \sigma \leq p-1$. Thus, if $0 \leq \alpha, \beta, \gamma, \varphi, \psi, \sigma \leq p-1$, then

$$\begin{aligned}
(a^\alpha c^\beta d^\gamma)^{d^\sigma(a^\varphi c^\psi)} &= (a^\alpha)^{d^\sigma} (c^\beta)^{d^\sigma} (d^\gamma)^{a^\varphi c^\psi} \\
&= a^\alpha b^{-2\alpha\sigma} a^{-\beta\sigma} b^{\beta\sigma^2} c^\beta (a^{-\varphi} c^{-\psi} d^\gamma (a^\varphi c^\psi) d^{-\gamma} d^\gamma) \\
&= a^{\alpha-\beta\sigma-\varphi} b^{\beta\sigma^2-2\alpha\sigma} c^{\beta-\psi} a^\varphi b^{2\gamma\varphi} a^{\gamma\psi} b^{\gamma^2\psi} c^\psi d^\gamma \\
&= a^{\alpha-\beta\sigma+\gamma\psi} b^{\beta\sigma^2-2\alpha\sigma+2\gamma\varphi+\gamma^2\psi} c^\beta d^\gamma. \qquad (*)
\end{aligned}$$

Furthermore,

$$\begin{aligned}
(a^\alpha c^\beta d^\gamma)(a^\varphi c^\psi d^\sigma) &= a^\alpha c^\beta (d^\gamma a^\varphi c^\psi d^{-\gamma}) d^\gamma d^\sigma \\
&= a^\alpha c^\beta a^\varphi b^{2\gamma\varphi} a^{\gamma\psi} b^{\gamma^2\psi} c^\psi d^{\gamma+\sigma} \\
&= a^{\alpha+\varphi+\gamma\psi} b^{2\gamma\varphi+\gamma^2\psi} c^{\beta+\psi} d^{\gamma+\sigma}. \qquad (**)
\end{aligned}$$

We start by studying the subgroups of P of order p. Note first that $\langle b\rangle = Z(P) \lhd P$ and hence $|\mathcal{C}(\langle b\rangle) = 1|$. Furthermore, by $(*)$, if $0 \leq m, n, t \leq p-1$, then $d^t(a^m c^n) \in \mathrm{Stab}_P(\langle d\rangle)$ if and only if $n \equiv 0 \bmod p$ and $2m + n \equiv 0 \bmod p$, i.e., if and only if $n, m = 0$. Thus, $|\mathrm{Stab}_P(\langle d\rangle)| = p^2$ and $|\mathcal{C}(\langle d\rangle)| = p^2$. If $1 \leq i \leq p-1$, then $d^t(a^m c^n) \in \mathrm{Stab}_P(\langle cd^i\rangle)$ if and only if $-t + in \equiv 0 \bmod p$ and $t^2 + 2im + i^2 n \equiv 0 \bmod p$ if and only if $t \equiv in \bmod p$ and $2im \equiv (-t^2 - i^2 n) \bmod p$. Thus $|\mathrm{Stab}_P(\langle cd^i\rangle)| = p^2$ and $|\mathcal{C}(\langle cd^i\rangle)| = p^2$. Since $\mathrm{Stab}_P(\langle ab\rangle), \mathrm{Stab}_P(\langle b^i c\rangle) \leq E$ and d is not an element of either of these stabilisers, we know that $\mathrm{Stab}_P(\langle ab\rangle) = \mathrm{Stab}_P(\langle b^i c\rangle) = E$. Thus, we have the following orders for each conjugacy class:

$$\begin{aligned} |\mathcal{C}(\langle b\rangle)| = 1, \quad |\mathcal{C}(\langle d\rangle)| = p^2, \quad |\mathcal{C}(\langle ab\rangle)| = p, \quad |\mathcal{C}(\langle b^i c\rangle)| = p \quad \text{for } 0 \leq i \leq p-1, \\ |\mathcal{C}(\langle cd^i\rangle)| = p^2 \quad \text{for } 1 \leq i \leq p-1. \end{aligned}$$

By $(*)$ and $(**)$, it follows that the conjugacy classes $\mathcal{C}(\langle cd^i\rangle)$ for $1 \leq i \leq p-1$ and $\mathcal{C}(\langle d\rangle)$ are all distinct, and the conjugacy classes $\mathcal{C}(\langle ab\rangle)$ and $\mathcal{C}(\langle b^i c\rangle)$ with $0 \leq i \leq p-1$ are all distinct as well. Counting subgroups, we determine that P contains

$$\underbrace{1}_{\langle b\rangle} + \underbrace{p^2}_{\langle d\rangle} + \underbrace{p}_{\langle ab\rangle} + \underbrace{p^2}_{\langle b^i c\rangle} + \underbrace{p^2(p-1)}_{\langle cd^i\rangle} = p^3 + p^2 + p + 1 = \frac{p^4 - 1}{p - 1}$$

distinct subgroups of order p, and hence we have accounted for all of them. Note that this also shows that P has exponent p.

There are three possibilities to consider when looking at the subgroups of P of order p^2: subgroups of the form $Q \leq E$, subgroups of the form $Q = \langle x, yd\rangle$ with $x, y \in E$ and subgroups of the form $Q = \langle xd, yd\rangle$ with $x, y \in E$. If $Q = \langle xd, yd\rangle$ with $x, y \in E$, then $(yd)(d^{-1}x^{-1}) = yx^{-1} \in E$ and $yx^{-1} \neq 1$, and hence we only need consider the first two cases. We have $Q = \langle a^m b^n c^s, a^{\overline{m}} b^{\overline{n}} c^{\overline{s}} d\rangle$ for $0 \leq m, n, s, \overline{m}, \overline{n}, \overline{s} \leq p-1$ is a subgroup of order p^2 in P if and only if

$$(a^m b^n c^s)(a^{\overline{m}} b^{\overline{n}} c^{\overline{s}} d) = (a^{\overline{m}} b^{\overline{n}} c^{\overline{s}} d)(a^m b^n c^s)$$

which by $(**)$, holds if and only if

$$a^{m+\overline{m}} b^{n+\overline{n}} c^{s+\overline{s}} d = a^{\overline{m}+m+s} b^{2m+s+\overline{n}+n} c^{s+\overline{s}} d.$$

This occurs if and only if $m = 0$ and $s = 0$. Thus, the subgroups of the form $Q = \langle x, yd\rangle$ with $x, y \in E$ are of the form $Q = \langle b, a^m c^s d\rangle$ for some m, s such that $0 \le m, s \le p-1$. Moreover, by $(**)$, these subgroups are all distinct, so there are p^2 such subgroups; additionally, there are p^2+p+1 subgroups of E of order p^2. By $(*)$, we have

$$d^{d^t(a^m c^n)} = a^n b^{2m+n} d, \tag{$\dagger$}$$

so $w^t(a^m c^n) \in \mathrm{Stab}_P(\langle b, d\rangle)$ if and only if $n = 0$, so $|\mathrm{Stab}_P(\langle b, d\rangle)| = p^3$. If $0 \le i \le p-1$, then by $(*)$, we have

$$(cd^i)^{d^t(a^m c^n)} = a^{-t+in} b^{t^2+2im+i^2n} cd^i \tag{$\dagger\dagger$}$$

so $d^t(a^m c^n) \in \mathrm{Stab}_P(\langle b, cd^i\rangle)$ if and only if $-t + in \equiv 0 \bmod p$, and it follows that $|\mathrm{Stab}_P(\langle b, cd^i\rangle)| = p^3$. Finally, if $0 \le i \le p-1$, then $\mathrm{Stab}_P(\langle ab, b^i c\rangle) \le E$ and $d \notin \mathrm{Stab}_P(\langle ab, b^i c\rangle)$, so we see that $|\mathrm{Stab}_P(\langle ab, b^i c\rangle)| = p^3$. Thus, since $\langle a, b\rangle \lhd P$, we have the following orders for each conjugacy class:

$$|\mathcal{C}(\langle a, b\rangle)| = 1, \quad |\mathcal{C}(\langle b, d\rangle)| = p, \quad |\mathcal{C}(\langle ab, b^i c\rangle)| = p \quad \text{for } 0 \le i \le p-1,$$
$$|\mathcal{C}(\langle b, cd^i\rangle)| = p \quad \text{for } 0 \le i \le p-1.$$

We claim that the $2(p+1)$ conjugacy classes in the above are disjoint. Note first that if $0 \le i, j \le p-1$, then $\langle ab, b^i c\rangle \notin \mathcal{C}(\langle b, d\rangle)$ and $\langle ab, b^i c\rangle \notin \mathcal{C}(\langle b, cd^j\rangle)$, since $b \notin \langle ab, b^i c\rangle$. Furthermore, if $\mathcal{C}(\langle ab, b^i c\rangle) = \mathcal{C}(\langle ab, b^j c\rangle)$ with $0 \le i < j \le p-1$, then $\langle ab, b^i c\rangle^{d^t} = \langle ab, b^j c\rangle$ with $1 \le t \le p-1$. We therefore have

$$(ab)^{d^t} = ab^{-2t+1} \in \langle ab, b^j c\rangle$$

and hence $ab^{-2t+1} = (ab)^\alpha (b^i c)^\beta$, which by $(**)$ implies that $\beta = 0$ since $i \ne j$ and hence $\alpha = 1$; thus $-2t + 1 \equiv 1 \bmod p$ and $t = 0$, which is a contradiction. Moreover, if $\langle b, d\rangle^{d^t(a^m b^n)} = \langle b, cd^i\rangle$ with $0 \le i \le p-1$, then by $(\dagger)$, $a^n b^{2m+n} d = b^\alpha (cd^i)^\beta$ and by $(**)$, we would have $\beta = 0$ and hence a contradiction. Finally, if $\langle b, cd^i\rangle^{d^t(a^m b^n)} = \langle b, cd^j\rangle$ with $0 \le i < j \le p-1$, then by $(\dagger\dagger)$

$$a^{-t+in} b^{t^2+2im+i^2n} cd^i = b^\alpha (cd^j)^\beta.$$

By $(**)$, we must have $\beta = 1$ and hence $i = j$, a contradiction. Thus, the conjugacy classes are distinct, and we have accounted for p^2 subgroups of the form $\langle x, yd \rangle$ with $x, y \in E$, and $p^2 + p + 1$ subgroups of order p^2 in E.

Finally, we study the subgroups of order p^3 in P. Since $Z(P) = \langle b \rangle$, it follows that any such subgroup must contain b, as otherwise P splits over its centre, which is not possible. Thus, the subgroups of order p^3 in P are in one-to-one correspondence with the subgroups of order p^2 in $P/Z(P) \cong p_+^{1+2}$. By Proposition A.2.1, there are $p+1$ such subgroups, so all we need do is show that the $p+1$ groups described in (iii) are distinct. If $\langle a, b, d \rangle = \langle a, b, cd^i \rangle$ with $1 \leq i \leq p-1$, then $d = a^m b^n (cd^i)^s$. By $(**)$, we must have $is \equiv 1 \bmod p$ and $s \equiv 0 \bmod p$, a contradiction. Clearly, $\langle a, b, d \rangle \neq \langle a, b, c \rangle$. Finally, if $\langle a, b, cd^i \rangle = \langle a, b, cd^j \rangle$ with $1 \leq i < j \leq p-1$, then $cd^i = a^m b^n (cd^j)^s$ which implies that $s \equiv 1 \bmod p$ and $js \equiv i \bmod p$. Thus $i = j$, which is a contradiction. Thus, the subgroups are distinct. $\square$

Recall that $Sp_4(p)$ contains a subgroup B equal to the semidirect product $U \rtimes H$, where

$$U = \left\{ \begin{bmatrix} 1 & x & y & w \\ 0 & 1 & z & y - xz \\ 0 & 0 & 1 & -x \\ 0 & 0 & 0 & 1 \end{bmatrix} : x, y, w, z \in \mathbb{F}_p \right\}$$

and $H = H_4(p) \cap Sp_4(p)$, i.e.,

$$H = \left\{ \begin{bmatrix} \alpha & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & \beta^{-1} & 0 \\ 0 & 0 & 0 & \alpha^{-1} \end{bmatrix} : \alpha, \beta \in \mathbb{F}_p^\times \right\}.$$

Moreover, U is isomorphic to the p-group described in Proposition A.3.1, with explicit generators given by:

$$a = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad c = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad d = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

We present here analysis which covers the cases in Table 4.2 which were not accounted for in Example 4.2.5. Before we do so, we set up some notation. We let

$$u = \begin{bmatrix} 1 & x & y & w \\ 0 & 1 & z & y - xz \\ 0 & 0 & 1 & -x \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad h = \begin{bmatrix} \alpha & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & \beta^{-1} & 0 \\ 0 & 0 & 0 & \alpha^{-1} \end{bmatrix}$$

denote arbitrary elements of U and H respectively. Note that

$$u^{-1} = \begin{bmatrix} 1 & -x & xz - y & -w \\ 0 & 1 & -z & -y \\ 0 & 0 & 1 & x \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

If

$$v = \begin{bmatrix} 1 & m & n & r \\ 0 & 1 & s & n - ms \\ 0 & 0 & 1 & -m \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

then a direct calculation shows that

$$v^{uh} = \begin{bmatrix} 1 & \alpha^{-1}\beta m & \alpha^{-1}\beta^{-1}(mz + n - sx) & \alpha^{-2}\delta(m,n,r,s) \\ 0 & 1 & \beta^{-2}s & \alpha^{-1}\beta^{-1}(n + mz - sx - ms) \\ 0 & 0 & 1 & -\alpha^{-1}\beta m \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{A.1}$$

where $\delta(m,n,r,s)$ is the following polynomial in the variables x, y, w, z:

$$\delta(m,n,r,s) = m(2y - 2xz + sx) - 2nx + r + sx^2. \tag{A.2}$$

Start by letting $Q = \langle ab \rangle$. Then

$$ab = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{implies} \quad (ab)^j = \begin{bmatrix} 1 & 0 & j & j \\ 0 & 1 & 0 & j \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{A.3}$$

for $0 \leq j \leq p-1$. Moreover, $\delta(0,1,1,0) = 1-2x$, so by (A.1), we have

$$(ab)^{uh} = \begin{bmatrix} 1 & 0 & \alpha^{-1}\beta^{-1} & \alpha^{-2}(1-2x) \\ 0 & 1 & 0 & \alpha^{-1}\beta^{-1} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

So $uh \in N_B(Q)$ if and only if $\alpha^{-2}(1-2x) = \alpha^{-1}\beta^{-1}$ if and only if $x = (1-\alpha\beta^{-1})/2$. Thus $|N_B(Q)| = (p-1)^2 p^3$ and hence $|N_B(Q)|_{p'} = (p-1)^2$.

Next, we let $Q = \langle b^i c \rangle$ with $1 \leq i \leq p-1$. Then

$$b^i c = \begin{bmatrix} 1 & 0 & 0 & i \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{implies} \quad (b^i c)^j = \begin{bmatrix} 1 & 0 & 0 & ji \\ 0 & 1 & j & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and $\delta(0,0,i,1) = i + x^2$, so

$$(b^i c)^{uh} = \begin{bmatrix} 1 & 0 & -\alpha^{-1}\beta^{-1}x & \alpha^{-2}(i+x^2) \\ 0 & 1 & \beta^{-2} & -\alpha^{-1}\beta^{-1}x \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{A.4}$$

Thus $(b^i c)^{uh} \in Q$ if and only if $x = 0$ and $\alpha^{-2} = \beta^{-2}$, i.e., if and only if $x = 0$ and $\beta = \pm\alpha$. It follows that $|N_B(Q)| = 2p^3(p-1)$, so $|N_B(Q)|_{p'} = 2(p-1)$.

The final case of a subgroup of order p to consider is when $Q = \langle cd^i \rangle$ with $1 \leq i \leq p-1$. In what follows, $f(j)$ denotes the j-th triangular number and $g(j)$ denotes the sum of the first j triangular numbers; thus $f(j) = j(j+1)/2$ and $g(j) = j(j+1)(j+2)/6$.

Note that

$$cd^i = \begin{bmatrix} 1 & i & 0 & 0 \\ 0 & 1 & 1 & -i \\ 0 & 0 & 1 & -i \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and if} \quad (cd^i)^{j-1} = \begin{bmatrix} 1 & (j-1)i & f(j-2)i & -g(j-2)i^2 \\ 0 & 1 & j-1 & -f(j-1)i \\ 0 & 0 & 1 & -(j-1)i \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

then

$$\begin{aligned}(cd^i)^j &= \begin{bmatrix} 1 & i & 0 & 0 \\ 0 & 1 & 1 & -i \\ 0 & 0 & 1 & -i \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & (j-1)i & f(j-2)i & -g(j-2)i^2 \\ 0 & 1 & j-1 & -f(j-1)i \\ 0 & 0 & 1 & -(j-1)i \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & ji & f(j-2)i + (j-1)i & -g(j-2)i^2 - f(j-1)i^2 \\ 0 & 1 & j & -f(j-1)i - (j-1)i - i \\ 0 & 0 & 1 & -ji \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & ji & f(j-1)i & -g(j-1)i^2 \\ 0 & 1 & j & -f(j)i \\ 0 & 0 & 0 & -ji \\ 0 & 0 & 0 & 1 \end{bmatrix}. \end{aligned} \tag{A.5}$$

Thus (A.5) holds by induction for $1 \leq j \leq p-1$. Furthermore, $\delta(i,0,0,1) = 2iy - 2ixz + ix + x^2$ and hence by (A.1) we have

$$(cd^i)^{uh} = \begin{bmatrix} 1 & \alpha^{-1}\beta i & \alpha^{-1}\beta^{-1}(iz-x) & \alpha^{-2}(2iy - 2ixz + ix + x^2) \\ 0 & 1 & \beta^{-2} & \alpha^{-1}\beta^{-1}(iz - x - i) \\ 0 & 0 & 1 & -\alpha^{-1}\beta i \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{A.6}$$

Thus $(cd^i)^{uh} \in Q$ if and only if $\alpha^{-1}\beta = \beta^{-2}$,

$$\begin{aligned} \alpha^{-1}\beta^{-1}(iz - x) &= f(\alpha^{-1}\beta - 1)i \\ \alpha^{-1}\beta^{-1}(iz - x - i) &= -f(\alpha^{-1}\beta)i, \end{aligned}$$

$$\text{and}\quad \alpha^{-2}(2iy - 2ixz + ix + x^2) = -g(\alpha^{-1}\beta - 1)i^2.$$

The first two conditions hold if and only if $\alpha = \beta^3$ and $x = \alpha\beta f(\alpha^{-1}\beta - 1)i - iz$. We may then check that

$$\begin{aligned}\alpha^{-1}\beta^{-1}(iz - x - i) &= f(\alpha^{-1}\beta - 1)i - \alpha^{-1}\beta^{-1}i\\ &= i\left(\frac{(\alpha^{-1}\beta-1)\alpha^{-1}\beta}{2} - \alpha^{-1}\beta^{-1}\right)\\ &= -i\frac{\beta^{-4}+\beta^{-2}}{2} = -f(\alpha^{-1}\beta)i.\end{aligned}$$

Thus, we get a solution in terms of β if and only if α and x are of the prescribed form and

$$y = \frac{-\alpha^2 g(\alpha^{-1}\beta - 1)i^2 + 2ixz - ix - x^2}{2i}.$$

Thus $|N_B(Q)| = (p-1)p^2$ and hence $|N_B(Q)|_{p'} = (p-1)$.

We move on now to subgroups of order p^2. Firstly, if $Q = \langle a, b\rangle$, then $Q \lhd U$. In fact, we have

$$(ab)^h = \begin{bmatrix} 1 & 0 & \alpha^{-1}\beta^{-1} & \alpha^{-2}\\ 0 & 1 & 0 & \alpha^{-1}\beta^{-1}\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix} \in \langle a, b\rangle$$

so $\langle a, b\rangle \lhd B$ and hence $|N_B(Q)|_{p'} = (p-1)^2$.

If $Q = \langle ab, b^ic\rangle$ with $0 \le i \le p-1$, then

$$(ab)^j(b^ic)^k = \begin{bmatrix} 1 & 0 & j & j+ik\\ 0 & 1 & k & j\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}.$$

By (A.3), we know that $uh \in N_B(Q)$ implies that $uh \in N_B(\langle ab\rangle)$, which holds if and only if $x = (1-\alpha\beta^{-1})/2$. Moreover, if x is of this form, then by (A.4) we know that $(b^ic)^{uh} \in Q$ if and only if

$$\alpha^{-2}(x^2 + i) = -\alpha^{-1}\beta^{-1}x + \beta^{-2}i$$

if and only if

$$\frac{1-2\alpha\beta^{-1}+\alpha^2\beta^{-2}}{4}+i=-\alpha\beta^{-1}\frac{1-\alpha\beta^{-1}}{2}+\alpha^2\beta^{-2}i.$$

Multiplying through both sides by 4 and rearranging, the above holds if and only if

$$1-\alpha^2\beta^{-2}=-4i(1-\alpha^2\beta^{-2}).$$

Thus, if $1+4i\equiv 0 \bmod p$, then $N_B(Q)=N_B(\langle ab\rangle)$ and we have $|N_B(Q)|_{p'}=(p-1)^2$. Otherwise, we must have $1-\alpha^2\beta^{-2}=0$ and hence $\alpha^2=\beta^2$, so $|N_B(Q)|=2(p-1)p^3$. Hence $|N_B(Q)|_{p'}=2(p-1)$.

The final subgroup of order p^2 to consider is the case where $Q=\langle b, cd^i\rangle$ with $0\leq i\leq p-1$. Since $\langle b\rangle \lhd B$, we have that $N_B(\langle cd^i\rangle)\leq N_B(Q)$ and hence $(p-1)\leq |N_B(Q)|_{p'}$. On the other hand, by (A.6), we know that $(cd^i)^{uh}\in Q$ if and only if $\alpha^{-1}\beta=\beta^{-2}$ and $x=\alpha\beta f(\alpha^{-1}\beta-1)i-iz$, so in fact $|N_B(Q)|_{p'}=(p-1)$.

Now let $0\leq i\leq p-1$ and let $Q=\langle a,b,cd^i\rangle$. Since $\langle a,b\rangle\lhd B$, we can determine those $h\in H$ which normalise Q by determining those $h\in H$ for which $(cd^i)^h\in Q$; let X denote this subgroup of B. We have just shown that $p-1\leq |X|$. Furthermore,

$$(cd^i)^h=\begin{bmatrix}1 & \alpha^{-1}\beta i & 0 & 0\\ 0 & 1 & \beta^{-2} & -\alpha^{-1}\beta^{-1}i\\ 0 & 0 & 1 & -\alpha^{-1}\beta i\\ 0 & 0 & 0 & 1\end{bmatrix}.$$

A typical element of Q looks like

$$\begin{aligned}a^jb^k(cd^i)^l &= \begin{bmatrix}1 & 0 & j & k\\ 0 & 1 & 0 & j\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & li & f(l-1)i & -g(l-1)i^2\\ 0 & 1 & l & -f(l)i\\ 0 & 0 & 1 & -li\\ 0 & 0 & 0 & 1\end{bmatrix}\\ &= \begin{bmatrix}1 & li & f(l-1)i+j & *\\ 0 & 1 & l & -f(l)i+j\\ 0 & 0 & 1 & -li\\ 0 & 0 & 0 & 1\end{bmatrix}\end{aligned}$$

for some $* \in \mathbb{F}_p$. Thus, if $i \neq 0$, then $\alpha^{-1}\beta = \beta^{-2}$ and it follows that $|X| \leq p-1$, so $|X| = p-1$. Thus $|N_B(Q)|_{p'} = p-1$. On the other hand, if $i = 0$, then $\langle a, b, c \rangle \lhd U$ and we have that $|N_B(Q)|_{p'} = (p-1)^2$. This is the last subgroup of order p^3 to consider, and we are finally done.

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