Non-selfadjoint operator algebras generated by unitary semigroups

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Dedicated to my loving wife,

Anna.

Declaration

Hereby I declare that the present thesis was prepared by me and none of its contents was obtained by means that are against the law.

I also declare that the present thesis is a part of a PhD Programme at Lancaster University. The thesis has never before been a subject of any procedure of obtaining an academic degree.

The thesis contains research carried out jointly: Chapter 3 forms the basis of the paper [37] co-authored with S. C. Power. Hereby I declare that I made a full contribution to all aspects of this research and the writing of this paper.

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Abstract

The parabolic algebra was introduced by Katavolos and Power, in 1997, as the weak*-closed operator algebra acting on $L^2(\mathbb{R})$ that is generated by the translation and multiplication semigroups. In particular, they proved that this algebra is reflexive, in the sense of Halmos, and is equal to the Fourier binest algebra, that is, to the algebra of operators that leave invariant the subspaces in the Volterra nest and its analytic counterpart.

We prove that a similar result holds for the corresponding algebras acting on $L^p(\mathbb{R})$, where $1 < p < \infty$. It is also shown that the reflexive closures of the Fourier binests on $L^p(\mathbb{R})$ are all order isomorphic for $1 < p < \infty$.

The weakly closed operator algebra on $L^2(\mathbb{R})$ generated by the one-parameter semigroups for translation, dilation and multiplication by $e^{i\lambda x}$, $\lambda \geq 0$, is shown to be a reflexive operator algebra with invariant subspace lattice equal to a binest. This triple semigroup algebra, $\mathcal{A}_{ph}$, is antisymmetric in the sense that $\mathcal{A}_{ph} \cap \mathcal{A}_{ph}^* = \mathbb{C}I$, it has a nonzero proper weakly closed ideal generated by the finite-rank operators, and its unitary automorphism group is $\mathbb{R}$. Furthermore, the 8 choices of semigroup triples provide 2 unitary equivalence classes of operator algebras, with $\mathcal{A}_{ph}$ and $\mathcal{A}_{ph}^*$ being chiral representatives.

In chapter 4, we consider analogous operator norm closed semigroup algebras. Namely, we identify the norm closed parabolic algebra A_p with a semicrossed product for the action on analytic almost periodic functions by the semigroup of one-sided translations and we determine its isometric isomorphism group. Moreover, it is shown that the norm closed triple semigroup algebra $A_{ph}^{G_+}$ is the triple semi-crossed product

$A_p \times_v G_+$, where v denotes the action of one-sided dilations. The structure of isometric automorphisms of $A_{ph}^{G_+}$ is determined and $A_{ph}^{G_+}$ is shown to be chiral with respect to isometric isomorphisms.

Finally, we consider further results and state open questions. Namely, we show that the quasicompact algebra $\mathcal{QA}_p$ of the parabolic algebra is strictly larger than the algebra $\mathbb{C}I + K(H)$, and give a new proof of reflexivity of certain operator algebras, generated by the image of the left regular representation of the Heisenberg semigroup $\mathbb{H}_+$.

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Introduction

The operator algebras considered in this thesis are basic examples of Lie semigroup algebras by which we mean a weak operator topology closed algebra generated by the image of a Lie semigroup in a unitary representation of the ambient Lie group. The study of the reflexivity, in the sense of Halmos [62], of non-selfadjoint algebras that are generated by semigroups of operators was begun by Sarason in 1966 [68], where he proved that $H^\infty(\mathbb{R})$, viewed as a multiplication algebra on $H^2(\mathbb{R})$, is reflexive. Since then, several results about 2-parameter Lie semigroup algebras have been obtained.

Let $\{D_\mu, \mu \in \mathbb{R}\}$ and $\{M_\lambda, \lambda \in \mathbb{R}\}$ be the groups of translation and multiplication respectively acting on the Hilbert space $L^2(\mathbb{R})$, given by

$$D_\mu f(x) = f(x-\mu), \quad M_\lambda f(x) = e^{i\lambda x} f(x).$$

It is well-known that these 1-parameter unitary groups are continuous in the strong operator topology (SOT), that they provide an irreducible representation of the Weyl-commutation relations, $M_\lambda D_\mu = e^{i\lambda\mu} D_\mu M_\lambda$, and that the SOT-closed operator algebra they generate is the von Neumann algebra $B(L^2(\mathbb{R}))$ of all bounded operators. (See Taylor [72], for example.) On the other hand it was shown by Katavolos and Power in [38] that the weak*-closed non-selfadjoint operator algebra $\mathcal{A}_p$, known as the parabolic algebra, generated by the semigroups for $\mu \geq 0$ and $\lambda \geq 0$ is a reflexive algebra, containing no selfadjoint operators, other than real multiples of the identity, and containing no nonzero finite rank operators.

The hyperbolic algebra, denoted by $\mathcal{A}_h$ was first considered by Katavolos and Power in [39] and the invariant subspace lattice Lat $\mathcal{A}_h$, viewed as a lattice of projections with the weak operator topology, was identified as a 4-dimensional manifold. Furthermore, Levene and Power have shown ([44]) the reflexivity of an analogous hyperbolic algebra, the algebra generated by the multiplication and dilation semigroups on $L^2(\mathbb{R})$. The latter semigroup is given by the operators V_t, with

$$V_t f(x) = e^{t/2} f(e^t x),$$

for $t \geq 0$. The notation reflects the fact that translation unitaries are induced by the biholomorphic automorphims of the upper half plane which are of parabolic type, and the dilation unitaries are induced by those of hyperbolic type.We also note that Levene [43] has shown the reflexivity of the Lie semigroup operator algebra of $SL_2(\mathbb{R}^+)$ for its standard representation on $L^2(\mathbb{R})$ in terms of the composition operators of biholomorphic automorphisms.

One of the aims in establishing reflexivity and related properties is to understand better the algebraic structure of these somewhat mysterious algebras. Establishing reflexivity can provide a route to constructing operators in the algebra and thereby deriving further algebraic properties.

Although the reflexivity of non-selfadjoint operator algebras has been studied intensively over the last fifty years, the developments have been largely confined within the limits of operator algebras acting on Hilbert spaces. For example, general nest algebras, being the most characteristic class of reflexive noncommutative non-selfadjoint operator algebras since they were introduced by Ringrose in 1965 [64], have a well-developed general theory on Hilbert spaces (Davidson [15]). However, only sporadic results can be found for nest algebras on Banach spaces (see [70], [74], [12]).

In chapter 2, we consider the operator algebras $\mathcal{A}^p_{par}$ on $L^p(\mathbb{R})$ for $1 < p < \infty$, which are similarly generated by the multiplication and translation semigroups, viewed now as bounded operators on $L^p(\mathbb{R})$. Our main result is that $\mathcal{A}^p_{par}$ is also reflexive and, moreover, is equal to $\mathcal{A}^p_{FB}$, the algebra of operators that leave invariant each subspace in the Fourier binest $\mathcal{L}^p_{FB}$ given by

$$\mathcal{L}^p_{FB} = \{0\} \cup \{L^p[t, \infty) \, : \, t \in \mathbb{R}\} \cup \{e^{i\lambda x} H^p(\mathbb{R}) \, : \, \lambda \in \mathbb{R}\} \cup \{L^p(\mathbb{R})\}$$

where $H^p(\mathbb{R})$ is the usual Hardy space for the upper half plane. This lattice of closed subspaces is a binest equal to the union of two complete continuous nests of closed subspaces.

A complication in establishing the reflexivity of the parabolic and hyperbolic algebras on Hilbert space is the absence of an approximate identity of finite rank operators, a key device in the theory of nest algebras ([15], [21], [22]). However, it was shown that the subspace of Hilbert-Schmidt operators is dense for both algebras and that these operators could be used as an alternative. In contrast Annoussis, Katavolos and Todorov [2] have shown that direct integral decomposition arguments provide a route to reflexivity for various discrete noncommutative semigroup algebras. For the L^p theory we need a corresponding substitute. We define a right ideal of what we refer to as (p, q)-integral operators which we show is able to play the role of the (two-sided) ideal of Hilbert-Schmidt operators. As a substitute for the techniques of Hilbert space geometry and tensor product identifications used in [37], [38], [44], we make use of more involved measure theoretic arguments appropriate for the (p, q)-integrable operators.

We also obtain a number of properties of the parabolic algebra on $L^p(\mathbb{R})$, that correspond to the classical case. Namely, $\mathcal{A}^p_{par}$ is antisymmetric (or triangular [33]), in an appropriate sense, and $\mathcal{A}^p_{par}$ contains no non-trivial finite rank operators. Furthermore, the lattice of $\mathcal{A}^p_{par}$ is order isomorphic to the lattice of $\mathcal{A}^2_{par}$ for all $1 < p < \infty$.

In chapter 3, we consider the ultraweakly closed operator algebra acting on $L^2(\mathbb{R})$ which is generated by the parabolic algebra $\mathcal{A}_p$, together with the semigroup of dilation operators $V_t, t \geq 0$. Our main result is that this operator algebra is reflexive and is equal to $\operatorname{Alg}\mathcal{L}$, the WOT-closed algebra of operators that leave invariant each subspace in the lattice $\mathcal{L}$ of closed subspaces given by

$$\mathcal{L} = \{0\} \cup \{L^2(-\alpha, \infty), \alpha \geq 0\} \cup \{e^{i\beta x}H^2(\mathbb{R}), \beta \geq 0\} \cup \{L^2(\mathbb{R})\}.$$

This lattice is a binest, being a sublattice of the Fourier binest $\mathcal{L}_{FB}$. We denote the triple semigroup algebra by $\mathcal{A}_{ph}$ since it is generated by the algebras $\mathcal{A}_p$ and $\mathcal{A}_h$.

As stated above, the lattice $\mathcal{L}_{\mathrm{FB}}$, endowed with the weak operator topology for the orthogonal projections of these spaces, is homeomorphic to the unit circle and forms the topological boundary of a bigger lattice $\operatorname{Lat}\operatorname{Alg}\mathcal{L}_{\mathrm{FB}}$, the so-called reflexive closure of $\mathcal{L}_{\mathrm{FB}}$. This lattice is equal to the full lattice $\operatorname{Lat}\mathcal{A}_p$ of all closed invariant subspaces of $\mathcal{A}_p$ and is homeomorphic to the unit disc. In contrast we see that the binest $\mathcal{L}$ for $\mathcal{A}_{ph}$ is reflexive as a lattice of subspaces; $\mathcal{L} = \operatorname{Lat}\operatorname{Alg}\mathcal{L}$.

As in the analysis of $\mathcal{A}_p$ and $\mathcal{A}_h$ the classical Paley-Wiener theorem (in the form $F(H^2(\mathbb{R})) = L^2(\mathbb{R}^+)$) and the F. and M. Riesz theorem feature repeatedly in our arguments. The analysis of the triple semigroup algebra $\mathcal{A}_{ph}$ turns out to be considerable more challenging than that of the parabolic algebra. For the determination of the subspace $\mathcal{A}_{ph} \cap \mathcal{C}_2$ we obtain a two-variable variant of the Paley-Wiener theorem which is of independent interest. This asserts that if a function $k(x, y)$ in $L^2(\mathbb{R}^2)$ vanishes on a proper cone C with angle less than π, and its two-variable Fourier transform

F_2k vanishes on the (anticlockwise) rotated cone $R_{-\pi/2}C$, then k lies in the closed linear span of a pair of extremal subspaces with this property. These subspaces are rotations of the "quarter subspace" $L^2(\mathbb{R}^+) \otimes H^2(\mathbb{R})$. This is a seemingly classical function theoretic fact but we are unaware of any precedent. We also obtain a number of further interesting properties.

- The triple semigroup algebra $\mathcal{A}_{ph}$ is also antisymmetric. In contrast to $\mathcal{A}_p$ and $\mathcal{A}_h$ the algebra contains non-zero finite rank operators which generate a proper weak operator topology closed ideal.

- The unitary automorphism group is isomorphic to $\mathbb{R}$ and is implemented by the group of dilation unitaries.

- We also see that, unlike the parabolic algebra, $\mathcal{A}_{ph}$ has *chirality* in the sense that $\mathcal{A}_{ph}$ and $\mathcal{A}_{ph}^*$ are the reflexive algebras of *spectrally isomorphic* binests which are not unitarily equivalent. Also the 8 choices of triples of continuous proper semigroups from $\{M_\lambda : \lambda \in \mathbb{R}\}$, $\{D_\mu : \mu \in \mathbb{R}\}$ and $\{V_t : t \in \mathbb{R}\}$ give rise to exactly 2 unitary equivalence classes of operator algebras.

In chapter 4, we turn to the analysis of analogous *norm closed* operator algebras generated by semigroups. In the norm closed case considered here we take advantage of the theory of discrete semicrossed products. In particular, we prove that there are natural identifications

$$A_p = AAP \times_\tau \mathbb{R}_d^+ \ , \ A_{ph}^{\mathbb{Z}^+} = A_p \times_v \mathbb{Z}^+ \ , \ A_{ph}^{\mathbb{R}^+} = A_p \times_v \mathbb{R}_d^+$$

where A_p is the norm closed parabolic algebra, AAP is the algebra of analytic almost periodic functions in $L^\infty(\mathbb{R})$ and $A_{ph}^{G^+}$ is generated by A_p and a semigroup $\{V_t : t \in G^+\}$.

The notion of semicrossed products began with Arveson [6] in 1967, and was developed by the studies developed by Peters [52] and McAsey and Muhly [48] in the early eighties. Since then, several studies of semicrossed products of C^*-algebras have been under investigation by various authors [17, 35, 58]. To avoid categorical issues we shall simply define all the semicrossed products algebras that we consider as subalgebras of their associated C^*-crossed products [51]. Indeed, in the case of the semicrossed product A_p, this algebra coincides with its universal counterpart, defined as usual in terms of all contractive covariant representations of the generator semigroup [60]. However, we do not know if this persists for the triple semicrossed product algebra A_{ph}^{G+}.

Many of the results of isomorphisms of crossed products are concerned with the case of the discrete group $\mathbb{Z}$ (see [58]), whereas we also deal with the group of the real numbers endowed again with the discrete topology. This case is more subtle since the group C^*-algebra of $\mathbb{R}_d$ is the algebra of the almost periodic functions, which brings into play limit characters that arise from the Bohr compactification of $\mathbb{R}_d$. Moreover, the introduction of the triple semigroup semicrossed product makes the identification of the maximal ideal space of the algebra problematic. Nevertheless we obtain the following main results. We determine explicitly the isometric automorphism groups of the norm closed parabolic algebra A_p and the norm closed triple semigroup algebras A_{ph}^{G+}, where $G = \mathbb{Z}$ or $\mathbb{R}_d$. Also we show that the norm closed triple semigroup algebras are chiral with respect to isometric isomorphisms.

In the final chapter, we provide further results and state open questions. We introduce the quasicompact algebra $\mathcal{QA}_p$, that is the C*-algebra that arises from the intersection of the quasitriangular algebra of A_p and its adjoint algebra. The main result here is that $\mathcal{QA}_p$ is strictly larger than the algebra $\mathbb{C}I + K(H)$. The proof gives a novel construction of bounded operators in the algebra that takes advantage of the unbounded triangular truncation. In the second part of the chapter, we provide a new proof of reflexivity of the operator algebra, considered by Anoussis, Katavolos and

Todorov [2], which is generated by the image of the left regular representation of the Heisenberg semigroup $\mathbb{H}^+$, and we also consider related algebras.

Chapter 1

An Introduction to Operator Algebras

1.1 Preliminaries

1.1.1 Fourier series

The theory of Fourier series (and Fourier transform presented in the next subsection) can be found in [40]. Throughout this chapter we are viewing the unit circle $\mathbb{T}$ as the quotient group $\mathbb{R}/2\pi\mathbb{Z}$. Given a function $f \in L^1(\mathbb{T})$ and $n \in \mathbb{Z}$, the nth Fourier coefficient of f is

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-in\theta}d\theta.$$

Proposition 1.1.1. *A function $f \in L^1(\mathbb{T})$ satisfies $\hat{f}(n) = 0$ for all $n \in \mathbb{Z}$, if and only if $f = 0$.*

We focus first our attention on $L^2(\mathbb{T})$, since the set of functions $\{e^{inx}\}_{n\in\mathbb{Z}}$ is an orthonormal basis of the space. Hence it follows by Parseval's identity that for every f

in $L^2(\mathbb{R})$ we have

$$f = \sum_{n=-\infty}^{\infty} \hat{f}(n)e^{inx} \quad \text{and} \quad \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 = \|f\|^2.$$

In the general case, let $f \in L^1(\mathbb{T})$ and write

$$s_n(x) = \sum_{k=-n}^{n} \hat{f}(k)e^{ikx}$$

for the partial sums of the Fourier series of f. One might hope that still s_n converge to f in L^1 norm, but this is not necessarily the case. However, there are other ways to recapture f from its Fourier series. Define

$$\sigma_n = \frac{1}{n+1}(s_0 + s_1 + \cdots + s_n),\ n \in \mathbb{Z}^+$$

the Cesaro means of the Fourier series for f. Then

$$\sigma_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)K_n(x-t)dt$$

where K_n is the Fejer's kernel given by

$$K_n(x) = \frac{1}{n+1}\left(\frac{\sin\frac{n+1}{2}x}{\sin\frac{1}{2}x}\right)^2.$$

Fejer's kernel is an approximate identity on $L^1(\mathbb{T})$ and has the following properties

(i) $K_n \geq 0$;

(ii) $\frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(x)dx = 1$;

(iii) if $0 < \delta < \pi$, then $\lim\limits_{n\to\infty} \sup\limits_{|x|\geq\delta} |K_n(x)| = 0$.

Theorem 1.1.2. *Let f be a function in $L^p(\mathbb{T})$, where $1 \leq p < \infty$. Then the Cesaro means σ_n of the Fourier series for f converge to f in the L^p-norm. If f is in $L^\infty(\mathbb{T})$ then $\{\sigma_n\}$ converges to f in the weak*-topology on $L^\infty(\mathbb{T})$. In addition, if f is continuous, then $\{\sigma_n\}$ converges uniformly to f.*

1.1.2 Fourier transform

Let F be the Fourier transform on $L^1(\mathbb{R})$, which is given by the formula

$$Ff(x) := \hat{f}(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(y) e^{-ixy} dy.$$

By the Riemann-Lebesgue lemma the Fourier transform of an L^1 function is a continuous function vanishing at infinity. In addition f has zero Fourier transform if and only if $f = 0$. This comes out readily from the inverse formula, which in the special case that $\hat{f}$ is integrable takes the form

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{f}(y) e^{ixy} dy.$$

Given now a function in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, it follows from the Plancherel theorem that its Fourier transform is in $L^2(\mathbb{R})$ and the Fourier transform map is an isometry with respect to the L^2 norm. This implies that the Fourier transform restricted to $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ can be extended uniquely to an isometric map $L^2(\mathbb{R}) \to L^2(\mathbb{R})$. This isometry is actually a unitary and it is called the Fourier-Plancherel Transform. In addition, by the inverse formula on $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, we obtain that $F^2 f(x) = f(-x), F^3 = F^*$ and $F^4 = I$.

We know $F : L^1(\mathbb{R}) \to L^\infty(\mathbb{R})$ contractively, and we have seen that it also extends to a contraction $L^2(\mathbb{R}) \to L^2(\mathbb{R})$. Therefore, by the Riesz interpolation theorem, F

defines for every $p \in [1,2]$ a contractive linear map $L^p(\mathbb{R}) \to L^q(\mathbb{R})$, where q is the conjugate exponent of p.

1.1.3 The Hardy space H^p, $p \in [1, \infty]$

The details of the theory of Hardy spaces can be found in [25, 31] and [41].

Given an open subset $\mathbb{S}$ of the complex plane, we denote by $Hol(\mathbb{S})$ the set of holomorphic functions on $\mathbb{S}$. For any $p \geq 1$ we define the Hardy space of the open unit disk $\mathbb{D}$ as follows

$$
\begin{aligned}
H^p(\mathbb{D}) &= \left\{ f \in Hol(\mathbb{D}) \,:\, \sup_{0 \leq r \leq 1} \int_{-\pi}^{\pi} |f(re^{i\theta})|^p \frac{d\theta}{2\pi} < \infty \right\},\ 1 \leq p < \infty, \\
H^\infty(\mathbb{D}) &= \left\{ f \in Hol(\mathbb{D}) \,:\, \sup_{z \in \mathbb{D}} |f(z)| < \infty \right\}.
\end{aligned}
$$

If we consider the boundary behavior of holomorphic functions, we can identify $H^p(\mathbb{D})$ with a closed subspace of $L^p(\mathbb{T})$. A key tool in this theory is the Poisson kernel, that is the family of functions P_r, for $0 \leq r < 1$ given by

$$
P_r(\theta) = \frac{1 - r^2}{1 - 2r\cos\theta + r^2}.
$$

Check that $\{P_r\}_{r \in [0,1)}$ is an approximate identity for $L^1(\mathbb{T})$, since it satisfies the properties of a kernel:

(i) $P_r(\theta) \geq 0$;

(ii) $\frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\theta) d\theta = 1$, $0 \leq r < 1$;

(iii) if $0 < \delta < \pi$, then $\lim_{r \to 1} \sup_{|\theta| \geq \delta} |P_r(\theta)| = 0$.

Define the Poisson integral of a function $\tilde{f} \in L^p(\mathbb{T})$ by the formula

$$f(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \tilde{f}(t)P_r(\theta - t)dt.$$

Theorem 1.1.3. (Fatou) *A function f lies in $H^p(\mathbb{D})$ for $1 \leq p \leq +\infty$ if and only if f is the Poisson integral of a function $\tilde{f} \in L^p(\mathbb{T})$, such that*

$$\int_{-\pi}^{\pi} \tilde{f}(\theta)e^{in\theta}d\theta = 0,\ n = 1, 2, 3, \dots. \tag{1.1}$$

Then f has non-tangential limits which exist and agree with $\tilde{f}$ at almost every point of the unit circle. In addition, we obtain that $\|f\|_p = \|\tilde{f}\|_p$.

Hence we define the Hardy space $H^p(\mathbb{T})$ of the unit circle as the space of all functions in $L^p(\mathbb{T})$ that have zero nth Fourier coefficient, for every $n < 0$. When $p = 1$, this identification is not obvious but it follows from the F. and M. Riesz theorem ([31]).

Theorem 1.1.4. (Szegö) *Let $1 \leq p \leq \infty$ and $f \in L^p(\mathbb{T})$ be a function with $\log|f| \in L^1(\mathbb{T})$. Define the function*

$$[f] : \mathbb{D} \to \mathbb{C} : z \mapsto \exp\left\{\frac{1}{2\pi}\int_{\mathbb{T}} \frac{e^{i\theta} + z}{e^{i\theta} - z}\log|f(\theta)|d\theta\right\}. \tag{1.2}$$

Then $[f]$ lies in $H^p(\mathbb{D})$ and the boundary value function $B[f]$ satisfies

$$|B[f](\theta)| = |f(\theta)|,\ \textit{for almost every}\ \theta \in \mathbb{T}.$$

Moerover, for every nonzero function $f \in H^1(\mathbb{T})$, the function $\log|f(\theta)|$ lies in $L^1(\mathbb{T})$.

Let now ω be the homeomorphism of $\overline{\mathbb{D}}\backslash\{1\}$ onto the closed upper half plane $\overline{\mathbb{C}^+}$

$$\omega(z) = i\frac{1+z}{1-z}.$$

Then ω induces for every $p \in [1, \infty)$ the isometric map

$$\Phi_p : L^p(\mathbb{D}) \to L^p(\mathbb{C}^+) : (\Phi_p f)(z) = \left(\frac{1}{\sqrt{\pi}(z+i)}\right)^{2/p} f(\omega^{-1}(z)),\ z \in \mathbb{C}^+, \tag{1.3}$$

and

$$\Phi_\infty : L^\infty(\mathbb{D}) \to L^\infty(\mathbb{C}^+) : (\Phi_\infty f)(z) = f(\omega^{-1}(z)),\ z \in \mathbb{C}^+. \tag{1.4}$$

Therefore, we identify the Hardy spaces of the disc with the corresponding spaces

$$H^p(\mathbb{C}^+) = \left\{ f \in Hol(\mathbb{C}^+) \ : \ \sup_{y>0} \int_{\mathbb{R}} |f(x+iy)|^p dx < \infty \right\},\ 1 \leq p < \infty,$$
$$H^\infty(\mathbb{C}^+) = \left\{ f \in L^\infty(\mathbb{C}^+) \ : \ \sup_{z \in \mathbb{C}^+} |f(z)| < \infty \right\}.$$

Similarly, we can lift the Poisson formula to the half plane and get the respective identifications with closed subspaces of $L^p(\mathbb{R})$

$$H^p(\mathbb{R}) = \left\{ f \in L^p(\mathbb{R}) \ : \ \int_{\mathbb{R}} \frac{f(t)}{t+z} dt = 0,\ z \in \mathbb{C}^+ \right\},\ 1 \leq p < \infty,$$
$$H^\infty(\mathbb{R}) = \left\{ f \in L^\infty(\mathbb{R}) \ : \ \int_{\mathbb{R}} f(t) \left(\frac{1}{t+z} - \frac{1}{t+i}\right) dt = 0,\ z \in \mathbb{C}^+ \right\}.$$

The following corollary about the set of zeros of a function f in H^p is immediate from Theorem 1.1.4 and echos the fact that f is a non-tangential limit of holomorphic functions.

Corollary 1.1.5. *Every function $f \in H^p(\mathbb{T})$, with $1 \leq p \leq \infty$, cannot vanish on a set of strictly positive Lebesgue measure unless f is identically zero. The same also holds for $f \in H^p(\mathbb{R})$.*

The next theorem gives a characterization of $H^2(\mathbb{R})$, using the Fourier-Plancherel transform.

Theorem 1.1.6. (Paley-Wiener theorem) $FH^2(\mathbb{R}) = L^2(\mathbb{R}^+)$.

A trivial application of the above theorem shows that the set $\overline{H^2(\mathbb{R})}$ of complex conjugates $\overline{f}$ of functions $f \in H^2(\mathbb{R})$ is the orthogonal complement of $H^2(\mathbb{R})$. Hence the subspaces $H^2(\mathbb{R})$ and $L^2(\mathbb{R}^+)$ are in generic position[1] in the sence of Halmos ([26]).

Theorem 1.1.7. (Riesz factorization theorem) *A function f is in $H^1(\mathbb{R})$ if and only if there exist $g, h \in H^2(\mathbb{R})$ with $f = g \cdot h$ and $\|f\|_1 = \|g\|_2\|h\|_2$.*

Combining the two above theorems we get that the integral of a function $f \in H^1(\mathbb{R})$ is zero. Indeed, let g, h be in $H^2(\mathbb{R})$, such that $f = h \cdot g$. Then

$$\int_{\mathbb{R}} f(x)dx = \langle g, \overline{h} \rangle = 0.$$

We continue with two elementary density lemmas for the Hardy spaces $H^p(\mathbb{R})$ on the line, for $p \in (1, \infty)$. For each u in the open upper half plane $\mathbb{C}^+$ of $\mathbb{C}$ let

$$b_u(x) = \frac{1}{x+u},\ x \in \mathbb{R}.$$

Since b_u extends to a holomorphic function in the upper half plane, given by the formula $z \mapsto \frac{1}{z+u}$, it is a routine calculation to show that it lies in $H^p(\mathbb{R})$, for every $p \in (1, \infty)$.

Lemma 1.1.8. *The linear spans of the sets $D_1 = \{b_u | u \in \mathbb{C}^+\}$, $D_2 = \{b_u b_w | u, w \in \mathbb{C}^+\}$ are both dense in $H^p(\mathbb{R})$, for $1 < p < \infty$.*

Proof. Fix some $p \in (1, \infty)$ and suppose that there exists some $f \in H^p(\mathbb{R})$ that does not lie in the closed linear span of D_1. Then by the Hahn - Banach theorem, there exists a function g in $L^q(\mathbb{R})$, where q is the conjugate exponent of p, such that $\int_{\mathbb{R}} b_u g = 0$, for all $u \in \mathbb{C}^+$, and $\int_{\mathbb{R}} fg \neq 0$. But

$$\int_{\mathbb{R}} b_u g = 0,\ \forall u \in \mathbb{C}^+ \Leftrightarrow \int_{\mathbb{R}} \frac{g(x)}{x+u} dx = 0,\ \forall u \in \mathbb{C}^+ \Leftrightarrow g \in H^q(\mathbb{R}).$$

[1]We say that two subspaces M, N of a Hilbert space H are in generic position when $M \cap N = M \cap N^\perp = M^\perp \cap N = M^\perp \cap N^\perp = \{0\}$.

Consider now the respective Poisson extensions of the functions f and g on the upper half plane. Hölder's inequality yields that the product of the respective Poisson extensions of f and g on the upper half plane lies in $H^1(\mathbb{C}^+)$. Hence the boundary value function fg is in $H^1(\mathbb{R})$, so $\int_{\mathbb{R}} fg = 0$, which gives a contradiction.

Now, for any distinct $u, w \in \mathbb{C}^+$, observe that

$$b_u(x)b_w(x) = \frac{b_u(x) - b_w(x)}{w - u}.$$

Define

$$h_n = (ni - u)b_u b_{ni} = b_u - b_{ni}.$$

Since $h_n \to b_u$ pointwise, as $n \to \infty$, and $|h_n(x)| \leq |b_u(x)|$ for sufficiently large n, for all $x \in \mathbb{R}$, it follows from dominated convergence that $h_n \overset{\|\cdot\|_p}{\to} b_u$. Therefore, given $u \in \mathbb{C}^+$, the function b_u lies in the closed linear span of D_2, so by the first part of the lemma, the proof is complete. $\square$

Lemma 1.1.9. *Let* $\mathbb{C}^+_{\mathbb{Q}} = \{u \in \mathbb{C}^+ : u = x + iy, \text{ where } x, y \in \mathbb{Q}\}$. *For every* $t \in \mathbb{R}$ *the countable set*

$$\Lambda_t = \{b_u D_t b_w | u, w \in \mathbb{C}^+_{\mathbb{Q}}\}$$

is dense in $H^p(\mathbb{R})$, *for every* $p \in (1, \infty)$.

Proof. Observe first that

$$D_t b_w(x) = \frac{1}{(x - t) + w} = \frac{1}{x + (w - t)} = b_{w-t}(x), \; x \in \mathbb{R}.$$

Since $\mathbb{Q}$ is dense in $\mathbb{R}$, the rest of the proof is a simple application of dominated convergence. $\square$

Definition 1.1.10. A function $f \in L^\infty(\mathbb{T})$ is called **unimodular** if $|f(\theta)| = 1$ for almost every $\theta \in \mathbb{T}$. The identification (1.4) allows us to extend the definition of unimodular functions in $L^\infty(\mathbb{R})$.

Beurling's theorem ([29], [63]) gives a characterization of the closed subspaces of an $L^p(\mathbb{T})$ space that are shift invariant. We recall that a subspace $K \subseteq L^p(\mathbb{T})$ is shift invariant if $e^{ix}K$ is contained in K. Here we give the analogue of the result on the real line.

Theorem 1.1.11. (Beurling) *Given $p \in (1, \infty)$, let M be a closed subspace of $L^p(\mathbb{R})$ such that $e^{i\lambda x}M \subseteq M$ for all $\lambda \in \mathbb{Z}^+$. Then M is either of the form $L^p(E)$ for some Borel subset $E \subseteq \mathbb{R}$ or M is equal to $\varphi H^p(\mathbb{R})$ for some unimodular function φ.*

Proposition 1.1.12. *Let $\varphi \in L^\infty(\mathbb{R})$ be a unimodular function such that $\varphi H^2(\mathbb{R}) = H^2(\mathbb{R})$. Then φ is constant a.e..*

1.2 Fundamental algebras

In this section we review briefly the theory of C*-algebras, von Neumann and nest algebras, which will be necessary to read this thesis. Most of this theory can be found in the monographs of Davidson [15, 16] and in [49].

1.2.1 C*-algebras

Definition 1.2.1. A **Banach algebra** $\mathcal{A}$ is a complex algebra equipped with a complete submultiplicative norm:

$$\|ab\| \leq \|a\| \, \|b\|, \forall a, b \in \mathcal{A}.$$

If $\mathcal{A}$ has a unit $\mathbf{1}$ then it is called **unital** and we may assume that $\|\mathbf{1}\| = 1$.

Definition 1.2.2. If $\mathcal{A}$ is a unital Banach algebra, the **spectrum** of an element $a \in \mathcal{A}$ is

$$\sigma(a) = \{\lambda \in \mathbb{C} : \lambda \mathbf{1} - a \text{ is not invertible}\}.$$

The spectrum of an element a is always a non-empty compact set. Therefore, the **spectral radius** $\rho(a) = \max\{|\lambda| : \lambda \in \sigma(a)\}$ is a non-negative real number.

Proposition 1.2.3. *For each a in a Banach algebra $\mathcal{A}$, the spectral radius is determined by $\rho(a) = \lim_{n\to\infty} \|a^n\|^{1/n}$.*

Definition 1.2.4. A **C*-algebra** A is a Banach algebra equipped with an involution $a \mapsto a^*$ satisfying the C^*-condition

$$\|a^*a\| = \|a\|^2, \, \forall a \in \mathcal{A}.$$

An element a in $\mathcal{A}$ is called **normal** when $a^*a = aa^*$. Also, a is **unitary**, if we have $a^*a = aa^* = \mathbf{1}$. A normal element a is **selfadjoint**, when it satisfies the property $a = a^*$. We will call a selfadjoint element a **positive** if $\sigma(a) \subset \mathbb{R}^+$. Finally, a is a **projection** if it satisfies $a = a^* = a^2$; that is, a is a selfadjoint idempotent.

Proposition 1.2.5. *If a is a selfadjoint element of a C^*-algebra $\mathcal{A}$, then the C^*-property implies $\rho(a) = \|a\|$.*

Corollary 1.2.6. *There is at most one norm on a Banach $*$-algebra making it to a C^*-algebra.*

Let now H be a Hilbert space. The collection of bounded linear operators on H, denoted by $B(H)$, is a C*-algebra. The linear structure is clear. The product is by composition of operators and the involution is given by the adjoint operator. The

C*-norm is the operator norm given by

$$\|T\| = \sup\{\|Th\| \,:\, h \in H, \|h\| \leq 1\},$$

for any T in $B(H)$.

Definition 1.2.7. A **representation** of a C*-algebra $\mathcal{A}$, or a norm-closed subalgebra, is a pair (H, φ), where H is a Hilbert space and $\varphi : \mathcal{A} \to B(H)$ is a contractive homomorphism. Also, φ is called **non-degenerate** if for every nonzero $h \in H$ there exists $a \in \mathcal{A}$ such that $\varphi(a)h \neq 0$.

Every homomorphism between two C*-algebras is contractive if and only if it is a $*$-homomorphism. Furthermore, when these morphisms are also injective, then they are isometric. A representation is called **faithful**, when it is injective. Moreover, the image of a representation of a C^*-algebra is always closed ([32]).

An important class of representations of a C*-algebra $\mathcal{A}$, or a norm-closed subalgebra $\mathcal{A}$, are the characters on $\mathcal{A}$. A **character** acting on $\mathcal{A}$ is a bounded nonzero multiplicative linear functional $\varphi : \mathcal{A} \to \mathbb{C}$. The set of all the characters on $\mathcal{A}$ is denoted by $\mathfrak{M}(\mathcal{A})$ and it is called the **character space** of $\mathcal{A}$.

Proposition 1.2.8. *If $\mathcal{A}$ is a commutative norm closed algebra, then the set $\mathfrak{M}(\mathcal{A})$ is in one-to-one correspondence with the set of maximal ideals in $\mathcal{A}$.*

The character space $\mathfrak{M}(\mathcal{A})$ of a norm closed algebra $\mathcal{A}$, equipped with the weak*-topology

$$\varphi_i \xrightarrow{w^*} \varphi \Leftrightarrow \varphi_i(a) \to \varphi(a),\, \forall a \in \mathcal{A}$$

is a weak*-closed subset of the unit ball of the dual space of $\mathcal{A}$. Hence, by Alaoglu's theorem ([14]), it is weak*-compact.

Theorem 1.2.9. *If $\mathcal{A}$ is a commutative C^*-algebra, then the **Gelfand transform***

$$\mathcal{A} \to \mathcal{C}_0(\mathfrak{M}(\mathcal{A})) : a \mapsto \hat{a}, \quad where \;\; \hat{a}(\varphi) = \varphi(a), (\varphi \in \hat{\mathcal{A}}).$$

is an isometric $*$-isomorphism between C^*-algebras.

In the noncommutative case, a C*-algebra may have no multiplicative linear functionals (e.g. the C*-algebra of complex $n \times n$ matrices $M_n(\mathbb{C})$). Nonetheless, we have similar results using positive linear functionals [2] of norm 1, which are called **states**.

Theorem 1.2.10. (GNS construction) *For every state f on a C^*-algebra $\mathcal{A}$ there is a triple (π_f, H_f, ξ_f), where π_f is a representation of $\mathcal{A}$ on some Hilbert space H_f and $\xi_f \in H_f$ is a cyclic (i.e. $\overline{\pi_f(\mathcal{A})\xi_f} = H_f$) unit vector such that*

$$f(a) = \langle \pi_f(a)\xi_f, \xi_f \rangle, \forall a \in \mathcal{A}.$$

Theorem 1.2.11. (Gelfand - Naimark) *For every C^*-algebra $\mathcal{A}$ there exists a faithful representation (π, H).*

Therefore, every abstract C^*- algebra can be thought as a closed subalgebra of bounded linear operators acting on a Hilbert space H.

1.2.2 Norm closed algebras of analytic functions

The disc algebra

The disc algebra $A(\mathbb{D})$ is the algebra of holomorphic functions $f : \mathbb{D} \to \mathbb{C}$, where f extends to a continuous function on the closed unit disc. Given the supremum norm,

[2] A linear functional φ acting on a C*-algebra $\mathcal{A}$ is called positive if $\varphi(a) \geq 0$, for every positive element $a \in \mathcal{A}$.

$A(\mathbb{D})$ becomes a norm closed subalgebra of $H^\infty(\mathbb{D})$. One can check that the function

$$f(z) = e^{\frac{z+1}{z-1}} \tag{1.5}$$

lies in $H^\infty(\mathbb{D})$, but not in $A(\mathbb{D})$, so $A(\mathbb{D})$ is a proper subalgebra of $H^\infty(\mathbb{D})$ (see Chapter 6 in [31] for further details). Hence, if we identify each $f \in A(\mathbb{D})$ with its boundary values, $A(\mathbb{D})$ consists of the continuous functions on the unit circle, whose Fourier coefficients vanish on the negative integers. Since each f is the norm limit of its Cesaro polynomials, we have

$$A(\mathbb{D}) = \|\cdot\|\text{-alg}\{e^{inx} : n \in \mathbb{Z}^+\}.$$

The disc algebra has been studied extensively over the last century. We shall focus on the maximal ideal space of $A(\mathbb{D})$, since it will play an important role in chapter 4. One can check that given $\lambda \in \overline{\mathbb{D}}$, the set

$$I_\lambda = \{f \in A(\mathbb{D}) : f(\lambda) = 0\}$$

is a maximal ideal in $A(\mathbb{D})$, since it is the kernel of the point evaluation character $f \mapsto f(\lambda)$. The following result can be found in [31].

Theorem 1.2.12. *Every maximal ideal of $A(\mathbb{D})$ is the kernel of a point evaluation character, for some point λ in the closed unit disc.*

The algebra of analytic almost periodic functions

The theory of almost periodic functions was mainly created in 1925 by Bohr [10] and was substantially developed during the 1930s by Bochner, Besicovich, Stepanov and others. The reader can refer to [1, 9, 46, 69] for more details.

Definition 1.2.13. A set $E \subseteq \mathbb{R}$ is called relatively dense if there exists $\lambda > 0$ such that any interval of length λ contains at least one element of E. Given a function $f : \mathbb{R} \to \mathbb{C}$ and $\epsilon > 0$, a real number τ is called an ϵ-translation number of f, if

$$\sup_{t\in\mathbb{R}} |f(t+\tau) - f(t)| \leq \epsilon.$$

A continuous function f is **almost periodic** if and only if for any $\epsilon > 0$ the set of ϵ-translation numbers is relatively dense in $\mathbb{R}$.

Example 1.2.14. Every trigonometric polynomial of the form

$$p(x) = \sum_{k=1}^{n} c_k e^{i\lambda_k x}, \text{ with } c_k \in \mathbb{C}, \lambda_k \in \mathbb{R},$$

is an almost periodic function.

We denote by $AP(\mathbb{R})$ the algebra of almost periodic functions and we equip it with the supremum norm. Using a standard approximation argument, one can check that $AP(\mathbb{R})$ is a norm closed selfadjoint algebra of $C_b(\mathbb{R})$, hence it is a C*-algebra. In addition, we have the following result (see [9], Chapter 1).

Proposition 1.2.15. *If $f \in AP(\mathbb{R})$ and $\inf\{|f(x)| : x \in \mathbb{R}\} > 0$, then $1/f \in AP(\mathbb{R})$.*

It can be shown that $AP(\mathbb{R})$ is isometrically isomorphic to the C*-algebra $C(\mathbb{R}_B)$ of continuous functions on the Bohr compactification of the real numbers ([69]). Recall that the Bohr compactification of the real line, denoted by $\mathbb{R}_B$, can be identified by Pontryagin's duality theorem with the dual topological group of the discrete real line ([65]).

Theorem 1.2.16. (Approximation theorem) *For every $f \in AP(\mathbb{R})$ and $\epsilon > 0$, there exists a trigonometric polynomial p_ϵ, such that $\|f - p_\epsilon\| < \epsilon$.*

Theorem 1.2.17. (Mean value theorem) *Let f be an almost periodic function. The mean value of f, given by the formula*

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(t)dt,$$

exists.

Applying the mean value theorem we can define Fourier coefficients for almost periodic functions. Given $\lambda \in \mathbb{R}$ define the contractive linear map ϵ_λ by

$$\epsilon_\lambda(f) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(t)e^{-i\lambda t}dt. \tag{1.6}$$

Bochner introduced an important class of approximation polynomials, known as the Bochner - Fejer polynomials, which is suggested by Fejer's classical theorem on the Cesaro summability of the Fourier series of a periodic function.

Theorem 1.2.18. *Given f almost periodic function, the set of nonzero coefficients of f is at most countable.*

Let now $B = \{b_1, b_2, \dots, b_n, \dots\}$ be a countable set of real numbers. The set B is called **rationally independent** if for every $r_1, r_2, \dots, r_n \in \mathbb{Q}$, $n \in \mathbb{N}$ arbitrary, the equality

$$r_1b_1 + r_2b_2 + \cdots + r_nb_n = 0$$

implies that all of $r_1, r_2, \dots, r_n$ are zero. A rationally independent set B is a **rational basis** of a countable set $\Lambda = \{\lambda_1, \lambda_2, \dots, \lambda_n, \dots\}$, if every λ_n is representable as a finite linear combination of the b_j's with rational coefficients, that is

$$\lambda_n = r_1^{(n)}b_1 + r_2^{(n)}b_2 + \cdots + r_{m_k}^{(n)}b_{m_k},\ (n = 1, 2, \dots)$$

where $r_j^{(n)} \in \mathbb{Q}$. It is clear that every countable set Λ of real numbers has a basis B contained in the set. If Λ is rationally independent, then take $B = \Lambda$; otherwise the basis can be obtained by eliminating successively those λ_n's that are linear combinations of the preceding ones.

Suppose now f is an almost periodic function and $B = \{b_1, b_2, \ldots, b_n, \ldots\}$ is a rational basis for the nonzero Fourier coefficients of f. Denote by $K_{b_1,\ldots,b_m}$ the Bochner - Fejer kernel, that is the function given by

$$K_{b_1,\ldots,b_m}(t) = \sum_{\substack{|\nu_1|<(m!)^2 \\ \cdots\cdots\cdots \\ |\nu_m|<(m!)^2}} \left(1 - \frac{|\nu_1|}{(m!)^2}\right) \cdots \left(1 - \frac{|\nu_m|}{(m!)^2}\right) e^{-it\left(\frac{\nu_1}{m!}b_1 + \cdots + \frac{\nu_m}{m!}b_m\right)}. \tag{1.7}$$

One can check that this composite kernel shares the same properties with the Fejer kernel. Namely

(i) $K_{b_1,\ldots,b_m} \geq 0$;

(ii) $\lim\limits_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} K_{b_1,\ldots,b_m}(t)dt = 1$;

For more details, the reader can look at [46]. Define the Bochner - Fejer trigonometric polynomials of f by

$$\begin{aligned} \sigma_m(f)(t) &= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(t+s) K_{b_1,\ldots,b_m}(s)ds = \\ &= \sum_{\substack{|\nu_1|<(m!)^2 \\ \cdots\cdots\cdots \\ |\nu_m|<(m!)^2}} \left(1 - \frac{|\nu_1|}{(m!)^2}\right) \cdots \left(1 - \frac{|\nu_m|}{(m!)^2}\right) \epsilon_{\frac{\nu_1}{m!}b_1 + \cdots + \frac{\nu_m}{m!}b_m}(f) e^{-it\left(\frac{\nu_1}{m!}b_1 + \cdots + \frac{\nu_m}{m!}b_m\right)}. \end{aligned}$$

Check that the terms of these polynomials differ from zero if and only if the respective Fourier coefficients of f are nonzero.

Theorem 1.2.19. *For every almost periodic function f*

$$\sigma_m(f) \overset{\|\cdot\|}{\to} f, \quad \textit{as } m \to \infty.$$

We focus now on the non-selfadjoint algebra of analytic almost periodic functions, that is the norm closed algebra generated by the functions $\{e^{i\lambda x} : \lambda \geq 0\}$. It is evident that this algebra, denoted by $AAP(\mathbb{R})$, is contained in $H^\infty(\mathbb{R})$, so it is an integral domain.

Proposition 1.2.20. *$AAP(\mathbb{R})$ is properly contained in $H^\infty(\mathbb{R})$.*

Proof. Let f be the step function on $\mathbb{T}$

$$f(x) = \begin{cases} 1, & \text{if } x \in [-\pi, 0] \\ 2, & \text{if } x \in (0, \pi) \end{cases}.$$

One can check that $f \in L^\infty(\mathbb{T})$ and $\log|f| \in L^1(\mathbb{T})$, so it follows by Theorem 1.1.4 that the function $[f]$ given by formula (1.2) lies in $H^\infty(\mathbb{D})$. Moreover, the boundary function of $[f]$, which we call $B[f]$, satisfies $|B[f]| = |f|$ almost everywhere on $\mathbb{T}$. We apply now the isometric map Φ_∞ given in (1.4) to transfer the function $[f]$ to the upper half plane. Define

$$g(z) := (\Phi_\infty[f])(z) = [f]\left(\frac{z-i}{z+i}\right), \ (z \in \mathbb{C}^+).$$

Since $g \in H^\infty(\mathbb{C}^+)$, its boundary function, denoted by Bg, lies in $H^\infty(\mathbb{R})$. In addition

$$|Bg(x)| = \begin{cases} 1, & \text{for almost every } x > 0 \\ 2, & \text{for almost every } x < 0 \end{cases}.$$

Hence the function Bg is not in $AAP(\mathbb{R})$. □

These types of algebras have been studied by Besicovich ([9], Chapter III) and the theory has been considerably extended by the work of Arens and Singer in the 1960s ([4, 5, 11]). We exhibit two results of this theory that we shall use extensively in Chapter 4. First we describe the continuous automorphisms of $AAP(\mathbb{R})$. Define the multiplicative linear map $\varphi_{c,k}$ on the non-closed algebra of analytic trigonometric polynomials by

$$\varphi_{c,k}(e^{i\lambda x}) = c(\lambda)e^{ik\lambda x},$$

where $k > 0$ and $c : \mathbb{R} \to \mathbb{T}$ homomorphism (so $c \in \mathbb{R}_B$).

Theorem 1.2.21. (Arens) *The map $\varphi_{c,k}$ extends to a continuous automorphism of $AAP(\mathbb{R})$. Moreover, for every continuous automorphism φ of $AAP(\mathbb{R})$, there exist $k > 0$ and $c \in \mathbb{R}_B$, such that $\varphi = \varphi_{c,k}$.*

Consider now the maximal ideal space of $AAP(\mathbb{R})$. If we identify $AAP(\mathbb{R})$ with the closed subalgebra of $H^\infty(\mathbb{C}^+)$, we obtain the point evaluation characters

$$\chi_z(f) = f(z),$$

where $z \in \mathbb{C}^+$. It is evident that there are others. For instance, if I denotes the set of functions f in $AAP(\mathbb{R})$, such that $f(n)$ converges to zero as n goes to infinity, it is clear that I is a proper closed ideal in $AAP(\mathbb{R})$. Hence I is contained in a maximal ideal in $AAP(\mathbb{R})$, so there exists a character χ in $\mathfrak{M}(AAP(\mathbb{R}))$, such that $\chi(f) = 0$, for every $f \in I$. It is obvious now that χ is not one of the characters χ_z. The natural question, as in the case of H^∞, is to ask if the set of the point evaluation characters χ_z is dense in $\mathfrak{M}(AAP(\mathbb{R}))$. The following theorem gives an affirmative answer to this question.

Theorem 1.2.22. (Arens - Singer) *The maximal ideal space of $AAP(\mathbb{R})$ can be identified with the compact topological space*

$$\mathbb{R}_B \times [0, \infty) \cup \{\infty\}.$$

1.2.3 Operator topologies and compact operators

There are several Hausdorff locally convex topologies that can be defined on $B(H)$ besides the operator norm [32]. The **strong operator topology** or SOT-topology is defined as the topology of pointwise convergence on the Hilbert space H. So given a net $(T_i)_i$ of operators in $B(H)$, it follows

$$T_i \overset{\text{SOT}}{\to} T, \text{ when } T_i x \overset{i}{\to} Tx, \forall x \in H.$$

The topology on $B(H)$ generated by the separating family of seminorms

$$B(H) \to \mathbb{R}^+ : T \mapsto |\langle Tx, y\rangle|, (x, y \in \mathbb{C})$$

is called the **weak operator topology** or WOT- topology on $B(H)$. Hence we write

$$T_i \overset{\text{WOT}}{\to} T \Leftrightarrow \langle T_i x, y\rangle \overset{i}{\to} \langle Tx, y\rangle, \forall x, y \in H.$$

Recall now that a **finite rank operator** is a bounded operator such that its range is finite dimensional. In particular, an operator F of finite rank n takes the form

$$F : H \to H \ : \ x \mapsto \sum_{k=1}^{n} \langle x, h_k\rangle g_k,$$

with $\{h_k\}$ and $\{g_k\}$ linearly independent. In addition, an operator K is **compact** if the image of the unit ball under K is precompact. Equivalently, K is the norm limit of finite rank operators. The set of compact operators acting on H is a closed ideal in $B(H)$ and it will be denoted by $K(H)$.

Let now K be a compact operator. It follows from the spectral theorem that the positive operator $|K| = (K^*K)^{1/2}$ has eigenvalues $s_1 \geq s_2 \geq \ldots$ with $\lim_n s_n = 0$.

The numbers $s_n = s_n(K)$ are called the **singular values** of K. For $1 \leq p \leq \infty$, let $\mathcal{C}_p$ denote the **von Neumann - Schatten classes** of compact operators, such that $\{s_n(K)\}$ belongs to ℓ^p.

Define a norm on $\mathcal{C}_p$ by

$$\|K\|_p = \left(\sum_{n \geq 1} s_n(K)^p \right)^{1/p}.$$

Then $\| \cdot \|_p$ is indeed a norm (see Corollary 1.9 in [15]) and the space $(\mathcal{C}_p, \| \cdot \|_p)$ is a Banach space. Of particular interest are the **trace class** operators $\mathcal{C}_1$, and the **Hilbert-Schmidt** operators $\mathcal{C}_2$. Recall that a bounded operator acting on $L^2(\mathbb{R})$ is Hilbert-Schmidt if and only if it can be represented as an integral operator $\operatorname{Int} k$ with kernel $k \in L^2(\mathbb{R}^2)$ ([27]), given by

$$(\operatorname{Int} k\, f)(x) = \int_{\mathbb{R}} k(x,y) f(y) dy.$$

Proposition 1.2.23. *Let E be a orthonormal basis of a Hilbert space H. Then the map*

$$\mathcal{C}_1 \to K(H)^* : u \mapsto tr(u \cdot), \;\; \textit{where } tr(v) = \sum_{x \in E} \langle v(x), x \rangle$$

is an isometric linear isomorphism, that is independent of the choice of the basis. The same also holds for the map

$$B(H) \to \mathcal{C}_1^* : v \mapsto tr(\cdot v).$$

Thus, $B(H)$ can be viewed as a dual space. The **ultraweak** or **weak*-topology** on $B(H)$ is the weak*-topology on $B(H)$, generated by the seminorms

$$B(H) \to \mathbb{R}^+ : u \mapsto |tr(uv)|, \ (v \in \mathcal{C}_1).$$

Remark 1.2.24.

1. In general, the WOT topology is coarser than both SOT and weak*-topology, while the norm topology is the finest. However, the WOT and weak* topologies coincide on the norm bounded sets of $B(H)$, and hence the closed unit ball of $B(H)$ is WOT and weak*-compact, by Alaoglu's theorem ([14]).

2. A linear functional acting on $B(H)$ is WOT-continuous if and only if it is SOT-continuous. Hence it follows by the Hahn - Banach separation theorem ([14]) that a convex subset of $B(H)$ is SOT-closed if and only if it is WOT-closed.

3. The linear operations are continuous for all of these topologies, while the ring multiplication is separately continuous. Also, it is easy to check that the involution operator $T \mapsto T^*$ is WOT and weak*-continuous, while it is not SOT-continuous.

4. Since the WOT topology is coarser than both SOT and weak*-topology, a SOT-convergent (or weak*-convergent) net of operators is automatically WOT-convergent. On the other hand, a norm-bounded net is WOT-convergent if and only if it is weak*-convergent.

1.2.4 von Neumann algebras

Let H be a Hilbert space and $\mathcal{S} \subseteq \mathbf{B}(H)$. We define its **commutant** to be the set

$$\mathcal{S}' = \{T \in B(H) : TS = ST, \forall S \in \mathcal{S}\}.$$

The set $\mathcal{S}'$ is a unital algebra and it is SOT-closed.

Definition 1.2.25. A **von Neumann algebra** $\mathcal{M}$ acting on a Hilbert space H is a selfadjoint subset of $B(H)$, that satisfies the property

$$\mathcal{M} = \mathcal{M}''.$$

Theorem 1.2.26. (Double commutant theorem) *If $\mathcal{A} \subseteq B(H)$ is a unital selfadjoint algebra, then the following are equivalent :*

(α) $\mathcal{A}$ is a von Neumann algebra

(β) $\mathcal{A}$ is SOT-closed

In general, a C*-algebra may have no non-trivial projections. For example, if X is a connected locally compact Hausdorff space, then the C*-algebra $C_0(X)$ of continuous functions on X contains no non-trivial projections . However, in the subcategory of von Neumann algebras, we always have sufficiently many projections, in order to form a generating set in the sense of the following proposition.

Proposition 1.2.27. *Let $\mathcal{M}$ be a von Neumann algebra acting on Hilbert space H. Then $\mathcal{M}$ is the norm closed linear span of its projections. Furthermore, if H is separable, there is a countable set $\mathcal{E} \subseteq \mathcal{M}$ of projections, such that $\mathcal{E}'' = \mathcal{M}$.*

1.2.5 Reflexive algebras

Given a set $\mathcal{S}$ of operators, the lattice of all closed subspaces, that are left invariant by every element of $\mathcal{S}$, is denoted $\operatorname{Lat}\mathcal{S}$. Similarly, if $\mathcal{L}$ is a lattice of subspaces, then $\operatorname{Alg}\mathcal{L}$ denotes the algebra of all bounded operators leaving each element of $\mathcal{L}$ invariant.

Definition 1.2.28. An algebra $\mathcal{A}$ is **reflexive**, if $\mathcal{A} = \operatorname{Alg}\operatorname{Lat}\mathcal{A}$; in the same spirit, a lattice $\mathcal{L}$ is called **reflexive** if $\mathcal{L} = \operatorname{Lat}\operatorname{Alg}\mathcal{L}$.

Remark 1.2.29. 1. For any lattice of subspaces $\mathcal{L}$, the algebra $\operatorname{Alg}\mathcal{L}$ is WOT-closed;

2. for any algebra $\mathcal{A}$, we have

$$\mathcal{A} \subseteq \overline{\mathcal{A}}^{w^*} \subseteq \overline{\mathcal{A}}^{\mathrm{WOT}} \subseteq \operatorname{Alg}\operatorname{Lat}\mathcal{A};$$

3. therefore, if an algebra is reflexive, it coincides with its weak*-closure and its WOT-closure.

Thus a reflexive operator algebra is determined by its invariant subspaces. The origins of studying reflexive algebras go back in 1966, when Donald Sarason proved that $H^\infty(\mathbb{R})$ viewed as a multiplication algebra on $H^2(\mathbb{R})$, is reflexive ([68]). The class of reflexive algebras can be considered as a non-selfadjoint generalization of von Neumann algebras, since if $\mathcal{M}$ is a von Neumann algebra, then $\operatorname{Lat}\mathcal{M}$ consists of all the subspaces, whose projections lie in the commutant of $\mathcal{M}$. Hence $\operatorname{Alg}\operatorname{Lat}\mathcal{M} = \mathcal{M}''$. On the other hand, the analogue of double commutant theorem does not hold for non-selfadjoint algebras. A counterexample of a SOT-closed algebra which is not reflexive is the subalgebra of 2 by 2 matrices of the form

$$\left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \ : \ a, b \in \mathbb{C} \right\}.$$

This algebra is smaller than the algebra of upper triangular matrices

$$\left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \ : \ a, b, c \in \mathbb{C} \right\}$$

but has the same invariant subspaces, so it is not reflexive. The next definition generalizes reflexivity for operator spaces.

Definition 1.2.30. Let $\mathcal{S}$ be a subspace of $B(H)$ for some Hilbert space H. The **reflexive hull** of $\mathcal{S}$ is the set

$$Ref(\mathcal{S}) = \{T \in B(H) : T\xi \in \overline{\mathcal{S}\xi}, \text{ for all } \xi \in H\},$$

where $\mathcal{S}\xi$ is the linear subspace $\{S\xi \,:\, S \in \mathcal{S}\}$.

Proposition 1.2.31. *Let $\mathcal{A}$ be a unital subalgebra of $B(H)$. Then $Ref(\mathcal{A}) = \operatorname{Alg}\operatorname{Lat}\mathcal{A}$.*

Proof. Let $A \in Ref(\mathcal{A})$ and $K \in \operatorname{Lat}\mathcal{A}$. Then for every $\xi \in K$, we get that $A\xi \in \overline{\mathcal{A}\xi}$. Since $\mathcal{A}\xi \subseteq K$, it follows that A lies in $\operatorname{Alg}\operatorname{Lat}\mathcal{A}$. On the other hand, let A be in $\operatorname{Alg}\operatorname{Lat}\mathcal{A}$ and $\xi \in H$. Since $\mathcal{A}$ is unital, $\xi \in \overline{\mathcal{A}\xi}$. Then $\overline{\mathcal{A}\xi} \in \operatorname{Lat}\mathcal{A}$, so $A\xi \in \overline{\mathcal{A}\xi}$, hence $A \in Ref(\mathcal{A})$. $\square$

In this section, we will focus on nest algebras, which have been studied intensely in the last 50 years ([15, 57]), since their consideration by Ringrose in [64]. Their importance, even in finite-dimensions, lies in the fact that they provide the most fundamental class of noncommutative non-selfadjoint operator algebras.

Definition 1.2.32. A **nest** is a totally ordered set $\mathcal{N}$ of closed subspaces of a Hilbert space H containing $\{0\}$ and H, which is complete with respect to the natural lattice operations, namely the intersection and the closed span.

Given a subspace N belonging to a nest $\mathcal{N}$, define

$$N_- = \vee\{N' \in \mathcal{N} : N' < N\}$$

and

$$N_+ = \wedge\{N' \in \mathcal{N} : N' > N\}.$$

The non-trivial subspaces $N \ominus N_-$ are called **atoms** of $\mathcal{N}$. If the atoms of $\mathcal{N}$ span H, then the nest is **atomic**. If there are no atoms, it is called **continuous**.

Definition 1.2.33. Given a nest $\mathcal{N}$, the **nest algebra** $\mathcal{T}(\mathcal{N})$ is defined as the set of all operators T, such that $TN \subseteq N$ for every $N \in \mathcal{N}$.

Every nest algebra is SOT-closed proper non-selfadjoint subalgebra of $B(H)$ if and only if the nest is not trivial.

Example 1.2.34.

- Let P_n be an increasing sequence of finite dimensional subspaces, such that their union is dense in H. Then $\mathcal{P} = \{P_n : n \geq 1\} \cup \{\{0\}, H\}$ is an atomic nest. $\mathcal{T}(\mathcal{N})$ consists of all the operators which have a block upper triangular matrix with respect to $\mathcal{P}$.

- Let $H = L^2(\mathbb{R})$. For each $t \in \mathbb{R}$, let N_t consist of all functions f in $L^2(\mathbb{R})$ such that $f(x) = 0$ a.e. on $(-\infty, t]$. Then $\mathcal{N}_v = \{N_t : t \in \mathbb{R}\} \cup \{\{0\}, L^2(\mathbb{R})\}$ is a continuous nest, which is known as the **Volterra nest**.

- Given a nest $\mathcal{N}$ on a Hilbert space H and a unitary operator $U : H \to \mathcal{K}$, then the set $U\mathcal{N}$ is nest on the Hilbert space $\mathcal{K}$. Thus, if we denote by F the Fourier-Plancherel transform on $L^2(\mathbb{R})$, the nest $\mathcal{N}_a = F^*\mathcal{N}_v$ is called the **analytic nest**.

Clearly, a nest algebra is reflexive. It turns out that every nest is also reflexive ([64]). Actually, a nest algebra contains an abundance of operators. Note that a von Neumann algebra may have no nonzero finite rank operators. For example, the **multiplication algebra** $\mathcal{M}_m = \{M_f : f \in L^\infty(\mathbb{R})\}$, that is the algebra of operators $M_f \in B(L^2(\mathbb{R}))$ given by

$$M_f g = fg, \ (g \in L^2(\mathbb{R})), \tag{1.8}$$

contains no compact operators. In contrast, for nest algebras we have the following result (see 3.11 in[15]) :

Theorem 1.2.35. (Erdos density theorem) *The finite rank contractions in a nest algebra $\mathcal{A}$ are dense in the unit ball of $\mathcal{A}$ in the SOT-topology. Hence it follows by Remark 1.2.24 that they are dense in the unit ball as well in the WOT and the weak*-topology.*

Apart from reflexivity of a nest algebra $\mathcal{A}$, that enables us to examine if a given element of $B(H)$ lies in $\mathcal{A}$, we have the following theorem about its distance induced by the operator norm ([7, 55]).

Theorem 1.2.36. (Arveson's distance formula) *Let $\mathcal{A}$ be a nest algebra on $B(H)$, corresponding to a nest $\mathcal{N}$. Then, for every operator $T \in B(H)$, we have :*

$$d(T, \mathcal{A}) := \inf_{A \in \mathcal{A}} \|T - A\| = \sup_{N \in \mathcal{N}} \|(I - P_N)TP_N\|.$$

Given a nest algebra $\mathcal{A}$, its compact pertubation $\mathcal{A} + K(H)$ is called the **quasitriangular algebra** associated with $\mathcal{A}$. Since $K(H)$ is an ideal in $B(H)$, it is evident that $\mathcal{A} + K(H)$ is an algebra. The fact that it is also norm closed follows from the weak*-density of $\mathcal{A} \cap K(H)$ in $\mathcal{A}$ ([23]). In fact, this is a corollary of an elementary theorem of Rudin [66].

Theorem 1.2.37. (Rudin) *Suppose that Y and Z are closed subspaces of a Banach space X and let Φ be a collection of linear transformations in $B(X)$, such that*

1. $\Lambda Z \subseteq Z, \forall \Lambda \in \Phi$;
2. $\Lambda X \subseteq Y, \forall \Lambda \in \Phi$;
3. $\sup\{\|\Lambda\| : \Lambda \in \Phi\} = M < +\infty$;
4. *Given $y \in Y$ and $\epsilon > 0$ there exists a $\Lambda \in \Phi$ such that $\|y - \Lambda y\| < \epsilon$.*

Then $Z + Y$ is also a closed subspace of X.

Corollary 1.2.38. (Fall, Arveson, Muhly) *Let $\mathcal{A}$ be a nest algebra acting on a Hilbert space H. Then the **quasitriangular algebra** $\mathcal{A} + K(H)$ is norm closed.*

Proof. It suffices to show that $\mathcal{A} + K(H)$ satisfies the conditions of Theorem 1.2.37. Take $Z = \mathcal{A}$, $Y = K(H)$ and define Φ be the collection of transformations

$$\Lambda_F : B(H) \to B(H) : T \mapsto FT$$

with F finite rank operator in the unit ball of $\mathcal{A}$. The three first conditions are evidently satisfied. The fourth condition holds, since by Theorem 1.2.35 there exists a net of finite rank operators that converge to the identity operator in the weak*-topology. □

1.3 The parabolic and the hyperbolic algebra

1.3.1 Translation - multiplication algebras

Before introducing the parabolic and the hyperbolic algebras, it is helpful to consider the weak*-closed algebras on $L^2(\mathbb{R})$ generated by the multiplication and translation operators, as these algebras have the same set of invariant subspaces as their generators. Define the operator groups :

$$\lambda \to M_\lambda : (M_\lambda f)(x) = e^{i\lambda x} f(x)$$

and

$$\mu \to D_\mu : (D_\mu f)(x) = f(x - \mu).$$

Note that these two groups satisfy the so-called **Weyl relations** (see[47])

$$M_\lambda D_\mu = e^{i\lambda\mu} D_\mu M_\lambda, \; \forall \lambda, \mu \in \mathbb{R}.$$

By elementary functional analysis, the weak*-closed algebra generated by the unitary group $\{M_\lambda\}_{\lambda\in\mathbb{R}}$ consists of all the multiplication operators :

$$\text{weak}^*\text{-alg}\{M_\lambda \,:\, \lambda \in \mathbb{R}\} = \{M_\varphi \,:\, \varphi \in L^\infty(\mathbb{R})\}.$$

Note that the algebra on the right hand side is the multiplication algebra $\mathcal{M}_m$, defined in the previous section. One can check that $\mathcal{M}_m$ is a maximal selfadjoint abelian algebra of $B(L^2(\mathbb{R}))$, so it is WOT-closed (see for example Theorem 7.8 in [15]). Since it contains the generators M_λ for all λ, it suffices to show the opposite inclusion. If that is not true, then by the Hahn - Banach separation theorem ([14]), there is a function $\varphi \in L^\infty$ and a weak*-continuous linear functional on $B(L^2(\mathbb{R}))$, say

$$\omega : \mathcal{B}(L^2(\mathbb{R})) \to \mathbb{C},$$

that annihilates the algebra on the left hand side and $\omega(M_\varphi) = 1$. On the other hand, the restriction of ω on the multiplication algebra $\mathcal{M}_m$ induces a weak*-continuous functional on $L^\infty(\mathbb{R})$, which we denote by ω again. Hence there exist $h \in L^1(\mathbb{R})$, such that

$$\omega(f) \to \int_{\mathbb{R}} f(x)h(x)dx.$$

Now since $\omega(M_\lambda) = 0$, for every $\lambda \in \mathbb{R}$, the Fourier transform of h is the zero function, which implies that $h = 0$. Therefore $\omega(M_\varphi) = 0$, for every $\varphi \in L^\infty$, so our claim is true.

Similarly we have

$$\text{weak}^*\text{-alg}\{M_\lambda : \lambda \geq 0\} = \{M_\varphi : \varphi \in H^\infty(\mathbb{R})\} =: \mathcal{M}_{H^\infty}.$$

Since the translation group is unitarily equivalent to the multiplication group via the Fourier transform, there are similar identifications for the **translation algebra** $\mathcal{D}_m$, that is generated by the operators D_μ. The next result can be found in [72].

Proposition 1.3.1. *The algebra $\mathcal{M} =$ weak*-alg$\{M_\lambda, D_\mu : \lambda, \mu \in \mathbb{R}\}$ coincides with the algebra of all bounded operators on $L^2(\mathbb{R})$.*

Finally, the identification, that ties these ideas with nest algebras,

$$\text{weak}^*\text{-alg}\{M_\lambda, D_\mu : \lambda \in \mathbb{R}, \mu \geq 0\} = \operatorname{Alg}\mathcal{N}_v$$

is given in [61].

1.3.2 The parabolic algebra

Define on the Hilbert space $L^2(\mathbb{R})$, the (doubly) non-selfadjoint **Fourier binest algebra**

$$\mathcal{A}_{FB} = \operatorname{Alg}(\mathcal{N}_a \cup \mathcal{N}_v),$$

where $\mathcal{N}_a$ and $\mathcal{N}_v$ are the analytic and Volterra nest, respectively. It's trivial to check that $\mathcal{A}_{FB}$ is a reflexive algebra, being the intersection of two reflexive algebras, and that $\mathcal{A}_{FB}$ contains no non-zero finite rank operators and no non-trivial selfadjoint operators, i.e. $\mathcal{A}_{FB} \cap \mathcal{A}_{FB}^* = \mathbb{C}I$.

Define now the **parabolic algebra** $\mathcal{A}_p$ as the weak*-closed operator algebra that is generated by the two SOT-continuous unitary semigroups of operators $\{M_\lambda, \lambda \geq 0\}$

and $\{D_\mu,\ \mu \geq 0\}$ acting on $L^2(\mathbb{R})$. Since the generators of $\mathcal{A}_p$ leave the binest invariant, we have that $\mathcal{A}_p \subseteq \mathcal{A}_{FB}$. Katavolos and Power showed in [38] that these two algebras are equal, but we will present this result as it was proved in [42] :

Proposition 1.3.2. *Given $k \in L^2(\mathbb{R}^2)$, define $\Theta_p(k) : (x,t) \mapsto k(x, x-t)$. Then*

$$\mathcal{A}_{FB} \cap \mathcal{C}_2 \subseteq \{\operatorname{Int} k \mid \Theta_p(k) \in H^2(\mathbb{R}) \otimes L^2(\mathbb{R}^+)\}$$

where $\operatorname{Int} k$ *denotes the Hilbert-Schmidt operator acting on $L^2(\mathbb{R})$ given by*

$$(\operatorname{Int} k\, f)(x) = \int_{\mathbb{R}} k(x,y) f(y) dy.$$

Now, given $h \in H^2 \cap H^\infty(\mathbb{R}), \varphi \in L^1 \cap L^2(\mathbb{R}^+)$, let $h \otimes \varphi$ denote the function $(x,y) \mapsto h(x)\varphi(y)$. The integral operator $\operatorname{Int} k$, that is induced by the function $k = \Theta_p^{-1}(h \otimes \varphi)$, lies in the parabolic algebra. In particular, we have $\operatorname{Int} k = M_h \Delta_\varphi$, where Δ_φ is the bounded operator that is defined by the sesquilinear form

$$\langle \Delta_\varphi f, g\rangle = \int_{\mathbb{R}} \int_{\mathbb{R}} \varphi(t) D_t f(x) \overline{g(x)} dx dt, \text{ where } f, g \in L^2(\mathbb{R}).$$

Since the linear span of such functions k of separate variables is dense in the space $H^2(\mathbb{R}) \otimes L^2(\mathbb{R}^+)$, it follows by the proposition above that

$$\{\operatorname{Int} k \mid \Theta_p(k) \in H^2(\mathbb{R}) \otimes L^2(\mathbb{R}^+)\} \subseteq \mathcal{A}_p \cap \mathcal{C}_2$$

and this implies

$$\mathcal{A}_{FB} \cap \mathcal{C}_2 \subseteq \mathcal{A}_p \cap \mathcal{C}_2.$$

The opposite inclusion is evident, so the Fourier binest algebra and the parabolic algebra contain the same Hilbert-Schmidt operators.

Proposition 1.3.3. *$\mathcal{A}_p$ has a bounded approximate identity of Hilbert-Schmidt operators. In other words, there exists a norm bounded sequence $(T_n)_{n\in\mathbb{N}}$ of operators in $\mathcal{A}_p \cap \mathcal{C}_2$ such that $T_n \overset{SOT}{\to} I$.*

Therefore, by a density argument which also features in Chapters 2 and 4 (see [42], Corollary 3.11), we get the following theorem.

Theorem 1.3.4. (Katavolos - Power) *The parabolic algebra coincides with the Fourier binest algebra. Since $\mathcal{A}_{FB}$ is plainly reflexive, the same holds for $\mathcal{A}_p$.*

Finally, we note that the binest $\mathcal{N}_a \cup \mathcal{N}_v$ is not reflexive. In [38], a cocycle argument is used to show that the invariant subspace lattice of the parabolic algebra is

$$\operatorname{Lat} \mathcal{A}_p = \{K_{\lambda,s} | \lambda \in \mathbb{R}, s \geq 0\} \cup \mathcal{N}_v$$

where $K_{\lambda,s} = M_\lambda M_{\varphi_s} H^2(\mathbb{R})$ and $\varphi_s(x) = e^{-isx^2/2}$. Thus, given $s \geq 0$, we have the nest $\mathcal{N}_s = M_{\varphi_s}\mathcal{N}_a$. Any pair of distinct nests in $\operatorname{Lat} \mathcal{A}_p$ intersects only in the trivial subspaces. If we view $\operatorname{Lat} \mathcal{A}_p$ as a set of projections endowed with the SOT-topology, then it is homeomorphic to the closed unit disc and the topological boundary is the binest.

1.3.3 The hyperbolic algebra

In this subsection, we consider the algebra that is generated by the multiplication and dilation semigroups. Particularly, let $\{V_t : t \in \mathbb{R}\}$ be the one parameter SOT-continuous unitary group of dilation operators, acting on $L^2(\mathbb{R})$ by

$$(V_t f)(x) = e^{t/2} f(e^t x).$$

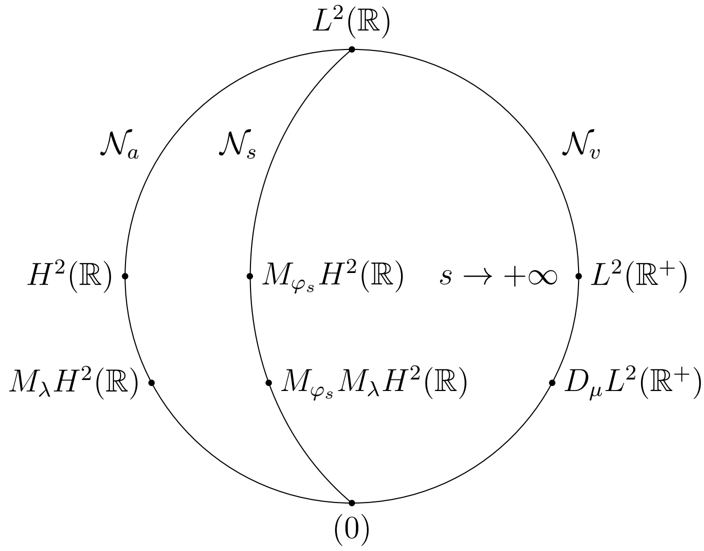

Fig. 1.1 The invariant subspace lattice of the parabolic algebra

One can check that the two groups satisfy the commutation relations

$$V_t M_\lambda = M_{\lambda e^t} V_t.$$

The **hyperbolic algebra**, introduced by Katavolos and Power ([39]), is defined as the weak*-closed operator algebra

$$\mathcal{A}_h = \text{weak}^*\text{-alg}\{M_\lambda, V_t | \lambda, t \geq 0\}.$$

On the other hand, take the following two subspace lattices

$$\mathcal{L}_D = \{L^2[-\alpha, \beta] : \alpha, \beta \in [0, +\infty]\} \cup \{\{0\}, L^2(\mathbb{R})\}$$
$$\mathcal{L}_L = \{d_s H^2(\mathbb{R}) : s \in \mathbb{R}\} \cup \{\{0\}, L^2(\mathbb{R})\}$$

where $d_s : \mathbb{R} \to \mathbb{C}$ is the unimodular function $d_s(x) = |x|^{is}$. Note that neither of these lattices is a nest, since they are not totally ordered. Define now the **dilation lattice**

algebra

$$\mathcal{A}_{DL} = \mathrm{Alg}(\mathcal{L}_D \cup \mathcal{L}_L).$$

In a similar way to how we presented in the parabolic case, Levene and Power ([44]) proved that $\mathcal{A}_h$ is equal to $\mathcal{A}_{DL}$. Since the generators of the hyperbolic algebra leave the bilattice invariant, we have $\mathcal{A}_h \subseteq \mathcal{A}_{DL}$. The key idea again is to identify the Hilbert-Schmidt operators. Let $Q = \{(x, y) \in \mathbb{R}^2 \,|\, xy \geq 0\}$ and p be the almost everywhere defined function

$$p(x) = \sqrt{x}\,\chi_{(0,+\infty)}(x) + i\sqrt{-x}\,\chi_{(-\infty,0)}(x).$$

Proposition 1.3.5. *Given $k \in L^2(Q)$, define $\Theta_h(k) : (x, t) \mapsto \overline{p(x)}e^{t/2}k(x, e^t x)$. Then*

$$\mathcal{A}_{DL} \cap \mathcal{C}_2 \subseteq \{\mathrm{Int}\, k \,|\, \Theta_h(k) \in H^2(\mathbb{R}) \otimes L^2(\mathbb{R}^+)\}.$$

Let $h \in H^2(\mathbb{R})$, $\varphi \in L^1 \cap L^2(\mathbb{R}^+)$ and let $k = \Theta_h^{-1}(h \otimes \varphi)$. Then the integral operator, that is induced by k, lies in the hyperbolic algebra and more specifically $\mathrm{Int}\, k = M_{ph}V_\varphi$, where V_φ is defined by the bounded sesquilinear form

$$\langle V_\varphi f, g\rangle = \int_{\mathbb{R}}\int_{\mathbb{R}} \varphi(t) V_t f(x)\overline{g(x)}dxdt,\ f, g \in L^2(\mathbb{R}).$$

Thus, we have that the Hilbert Schmidt operators in $\mathcal{A}_h$ and $\mathcal{A}_{DL}$ coincide, so by proving that the hyperbolic algebra contains a bounded approximate identity of Hilbert Schmidt operators, we obtain the following result.

Theorem 1.3.6. (Levene - Power) *The dilation lattice algebra and the hyperbolic algebra are equal. Thus $\mathcal{A}_h$ is reflexive.*

The bilattice $\mathcal{L}_D \cup \mathcal{L}_L$ is not reflexive, since the invariant subspace lattice of the hyperbolic algebra is the set

$$\operatorname{Lat} \mathcal{A}_h = \{K_{z,\lambda,\mu} \,|\, z \in \mathbb{C}^*, \lambda, \mu \geq 0\} \cup \mathcal{L}_D,$$

where $K_{z,\lambda,\mu} = M_{\upsilon_z} M_{e_{\lambda,\mu}} H^2(\mathbb{R}),$ and the functions υ_z and $e_{\lambda,\mu}$ are given by the formulas $\upsilon_z(x) = \chi_{(0,+\infty)}(x) + z\,\chi_{(-\infty,0)}(x)$ and $e_{\lambda,\mu}(x) = e^{i(\lambda x + \mu x^{-1})}$, respectively. Once again, this lattice, viewed as a lattice of projections with the SOT-topology, is homeomorphic to a compact and connected manifold, but in this case it is 4-dimensional.

Chapter 2

The parabolic algebra on L^p spaces

In this chapter we introduce the corresponding parabolic algebras on $L^p(\mathbb{R})$. To avoid any confusion we denote the parabolic algebra acting on $L^p(\mathbb{R})$ by $\mathcal{A}^p_{par}$. Therefore the algebra $\mathcal{A}^2_{par}$ is the algebra introduced in the previous chapter. We show that for any $p \in (1, +\infty)$ the parabolic algebra $\mathcal{A}^p_{par}$ is reflexive and is equal to the Fourier binest algebra. To prove this, we define a right ideal of integral operators with Bochner integrable kernel functions, which we show is able to play the role of the (two-sided) ideal of Hilbert-Schmidt operators. Most of the results are contained in [36].

2.1 The space $L^p(\mathbb{R}; L^q(\mathbb{R}))$, for $1 < p, q < \infty$

We now introduce some notation and terminology associated with the classical space $L^p(\mathbb{R}; L^q(\mathbb{R}))$. This space is a space of kernel functions for what we refer to as the (p, q)-integral operators. For more details, we refer the reader to [53],[54].

Let $p, q \in [1, +\infty]$. Define $\mathbb{S}(\mathbb{R}; L^q(\mathbb{R}))$ to be the space of measurable simple functions; i.e. the functions $f : \mathbb{R} \to L^q(\mathbb{R})$ taking only finitely many values :

$$f(x) = \sum_{k=1}^{n} \chi_{A_k}(x) g_k,$$

where $\{A_k\}_{k=1,\dots,n}$ is a finite family of Borel measurable pairwise disjoint sets and where $g_k \in L^q(\mathbb{R})$.

Definition 2.1.1. A function $f : \mathbb{R} \to L^q(\mathbb{R})$ is said to be strongly measurable if there is a sequence (f_n) in $\mathbb{S}(\mathbb{R}; L^q(\mathbb{R}))$, tending to f pointwise a.e.. Also, f is weakly measurable, if given $\omega \in (L^q(\mathbb{R}))^*$ the function $t \mapsto \omega(f(t))$ is Borel measurable.

The relationship between strong and weak measurability is given by the following theorem of Pettis [53], who introduced the notion of almost separably valued functions.

Definition 2.1.2. Let $1 \le q \le \infty$. A function $f : \mathbb{R} \to L^q(\mathbb{R})$ is almost separably valued, if there is a conull Borel set $A \subseteq \mathbb{R}$, such that $f(A)$ is separable.

Theorem 2.1.3. *A function $f : \mathbb{R} \to L^q(\mathbb{R})$ is strongly measurable if and only if it is weakly measurable and almost separably valued.*

Example 2.1.4. Define $f : \mathbb{R} \to L^\infty(\mathbb{R})$ by $f(x) = \chi_{(-\infty,x]}$. Then f is not almost separably valued, and hence not strongly measurable, since $\|f(x) - f(t)\|_\infty = 1$ for $x \ne t$. However, for $q \in (1,\infty)$, the function $g : \mathbb{R} \to L^q(\mathbb{R})$, given by $g(x) = \chi_{(-\infty,x]} f$, where $f \in L^q(\mathbb{R})$, is strongly measurable. To see this, note that $L^q(\mathbb{R})$ is separable and given $\omega \in L^p(\mathbb{R})$, where p is the conjugate exponent of q, we have

$$\omega(g(x)) = \int_{\mathbb{R}} \omega(y)\chi_{(-\infty,x]}(y) f(y)dy = \int_{-\infty}^{x} \omega(y) f(y)dy,$$

which is measurable, being the limit of absolutely continuous functions.

The definition of L^p spaces of L^q-valued functions is analogous to the case of scalar valued functions. One can check first that strong measurability of a function $f : \mathbb{R} \to L^q(\mathbb{R})$ ensures measurability in the usual sense of the scalar-valued function $x \mapsto \|f(x)\|_q$ (see for example [53]). Define $L^p(\mathbb{R}; L^q(\mathbb{R}))$ as the set of equivalence classes (modulo equality for almost every $x \in \mathbb{R}$) of strongly measurable functions

f that satisfy $\left(\int_{\mathbb{R}} \|f(x)\|_q^p dx\right)^{1/p} < \infty$ for $1 \leq p < \infty$, and $\mathrm{esssup}\|f(\cdot)\|_q$ for $p = \infty$. Each of the above spaces endowed with the respective norm

$$\|f\|_{p,q} = \left(\int_{\mathbb{R}} \|f(x)\|_q^p dx\right)^{1/p}, \text{ for } p \in [1, \infty),$$

$$\|f\|_{\infty,q} = \mathrm{esssup}\|f(\cdot)\|_q, \text{ for } p = \infty,$$

becomes a Banach space.

Remark 2.1.5. In the case $p = q = 2$ we have the natural isomorphisms

$$L^2(\mathbb{R}; L^2(\mathbb{R})) \cong L^2(\mathbb{R}) \otimes L^2(\mathbb{R}) \cong L^2(\mathbb{R}^2).$$

For the rest of the subsection, the exponents p, q lie on the open interval $(1, \infty)$. Given $f_1, f_2, \ldots, f_n \in L^p(\mathbb{R})$ and $g_1, g_2, \ldots, g_n \in L^q(\mathbb{R})$, define

$$f : \mathbb{R} \to L^q(\mathbb{R}) : f(x) \mapsto \sum_{k=1}^{n} f_k(x) g_k.$$

We denote this function by $\sum_{k=1}^{n} f_k \otimes g_k$ and we write $\mathbb{F}(\mathbb{R}; L^q(\mathbb{R}))$ for the subspace of $L^p(\mathbb{R}; L^q(\mathbb{R}))$ formed by such functions. Finally, we write $\mathbb{F}(\mathbb{R}; \mathbb{S}(\mathbb{R}))$ for the set of functions $\sum_{k=1}^{n} f_k \otimes \chi_{A_k}$, where $\{A_k\}_{k=1,\ldots,n}$ is a partition of the real line.

Proposition 2.1.6. *The following sets are dense in* $L^p(\mathbb{R}; L^q(\mathbb{R}))$.

1. $\mathbb{S}(\mathbb{R}; L^q(\mathbb{R})) \cap L^p(\mathbb{R}; L^q(\mathbb{R}))$;
2. $\mathbb{F}(\mathbb{R}; L^q(\mathbb{R})) \cap L^p(\mathbb{R}; L^q(\mathbb{R}))$;
3. $\mathbb{F}(\mathbb{R}; \mathbb{S}(\mathbb{R})) \cap L^p(\mathbb{R}; L^q(\mathbb{R}))$.

Proof. The argument for the density of the first two sets can be found in [54]. For the last set it suffices to prove that given $f \in L^p(\mathbb{R}), g \in L^q(\mathbb{R})$, we can find a sequence

(f_n) of elements in $\mathbb{F}(\mathbb{R};\mathbb{S}(\mathbb{R}))\cap L^p(\mathbb{R};L^q(\mathbb{R}))$, that converges to $f\otimes g$ with respect to the $\|\cdot\|_{p,q}$ norm. By the classical theory of L^q spaces, there is a sequence of simple functions

$$g_n=\sum_{k=1}^{n}a_k\chi_{A_k},\ a_k\in\mathbb{C},$$

such that $g_n\to g$ in $L^q(\mathbb{R})$. Then the functions $f\otimes g_n$ lie in $\mathbb{F}(\mathbb{R};\mathbb{S}(\mathbb{R}))\cap L^p(\mathbb{R};L^q(\mathbb{R}))$ and

$$\|f\otimes g_n-f\otimes g\|_{p,q}=\|f\|_p\|g_n-g\|_q\to 0.$$

□

To simplify the notation, we drop $L^p(\mathbb{R};L^q(\mathbb{R}))$ for each of the above sets. So when it causes no confusion, we write $\mathbb{S}(\mathbb{R};L^q(\mathbb{R}))$ for the respective dense subspace of $L^p(\mathbb{R};L^q(\mathbb{R}))$.

The characterization of the dual space of $L^p(\mathbb{R};L^q(\mathbb{R}))$ is again analogous to the scalar valued case, after we take account of duality in the range space $L^q(\mathbb{R})$ (see [54]).

Proposition 2.1.7. *Let $p,q\in(1,\infty)$ be conjugate exponents. The dual space of $L^p(\mathbb{R};L^q(\mathbb{R}))$ is isometrically isomorphic to $L^q(\mathbb{R};L^p(\mathbb{R}))$ by the map*

$$\alpha:L^q(\mathbb{R};L^p(\mathbb{R}))\to(L^p(\mathbb{R};L^q(\mathbb{R})))^*:\alpha(\tilde{k})(k)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}k(x)(y)\tilde{k}(x)(y)dy\,dx.$$

where $k\in L^p(\mathbb{R};L^q(\mathbb{R}))$ and $\tilde{k}\in L^q(\mathbb{R};L^p(\mathbb{R}))$.

Lemma 2.1.8. *Given an operator $T\in B(L^q(\mathbb{R}))$, there is a unique bounded linear operator*

$$\widetilde{T}:L^p(\mathbb{R};L^q(\mathbb{R}))\to L^p(\mathbb{R};L^q(\mathbb{R}))$$

such that given $f\otimes g\in\mathbb{F}(\mathbb{R};L^q(\mathbb{R}))$

$$\widetilde{T}(f\otimes g)=f\otimes Tg.$$

Moreover, the map $T \mapsto \widetilde{T}$ is isometric.

Proof. Let $f = \sum_{k=1}^{n} \chi_{A_k} \otimes g_k$, such that $g_k \in L^q(\mathbb{R})$ and $\{A_k\}_{k=1,\ldots,n}$ are pairwise disjoint Borel sets. By linearity, calculate

$$\begin{aligned}
\|\widetilde{T} f\|_{p,q}^p &= \left\| \sum_{k=1}^{n} \chi_{A_k} \otimes T g_k \right\|_{p,q}^p = \int_{\mathbb{R}} \left\| \sum_{k=1}^{n} \chi_{A_k}(x) T g_k \right\|_q^p dx = \\
&= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \left| \sum_{k=1}^{n} \chi_{A_k}(x) T g_k(y) \right|^q dy \right)^{p/q} dx = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \sum_{k=1}^{n} \chi_{A_k}(x) |T g_k(y)|^q dy \right)^{p/q} dx = \\
&= \int_{\mathbb{R}} \left(\sum_{k=1}^{n} \chi_{A_k}(x) \|T g_k\|_q^q \right)^{p/q} dx \leq \int_{\mathbb{R}} \left(\sum_{k=1}^{n} \chi_{A_k}(x) \|T\|^q \|g_k\|_q^q \right)^{p/q} dx = \\
&= \|T\|^p \int_{\mathbb{R}} \left(\sum_{k=1}^{n} \chi_{A_k}(x) \|g_k\|_q^q \right)^{p/q} dx = \|T\|^p \int_{\mathbb{R}} \left(\sum_{k=1}^{n} \chi_{A_k}(x) \int_{\mathbb{R}} |g_k(y)|^q dy \right)^{p/q} dx = \\
&= \|T\|^p \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \left| \sum_{k=1}^{n} \chi_{A_k}(x) g_k(y) \right|^q dy \right)^{p/q} dx = \|T\|^p \|f\|_{p,q}^p.
\end{aligned}$$

Since the set $\mathbb{S}(\mathbb{R}; L^q(\mathbb{R}))$ is dense in $L^p(\mathbb{R}; L^q(\mathbb{R}))$, the operator $\widetilde{T}$ is bounded. To show that the mapping $T \mapsto \widetilde{T}$ is isometric, check that given $g \in L^q(\mathbb{R})$

$$\|\chi_{[0,1]} \otimes g\|_{p,q}^p = \int_{\mathbb{R}} \|\chi_{[0,1]}(x) g\|_q^p dx = \int_0^1 dx \, \|g\|_q^p = \|g\|_q^p.$$

This yields an upper bound for the norm of the operator T

$$\|Tg\|_q^p = \|\chi_{[0,1]} \otimes Tg\|_{p,q}^p = \|\widetilde{T}(\chi_{[0,1]} \otimes g)\|_{p,q}^p \leq \|\widetilde{T}\| \, \|\chi_{[0,1]} \otimes g\|_{p,q}^p = \|\widetilde{T}\|^p \, \|g\|_q^p,$$

so the proof is complete. □

Lemma 2.1.9. *Let $p, q \in (1, \infty)$. The linear map*

$$\Theta : L^p(\mathbb{R}; L^q(\mathbb{R})) \to L^p(\mathbb{R}; L^q(\mathbb{R})) : \Theta(f)(x)(y) \mapsto f(x)(x-y)$$

is a bijective isometry onto $L^p(\mathbb{R}; L^q(\mathbb{R}))$.

Proof. It suffices again to consider $f \in \mathbb{S}(\mathbb{R}; L^q(\mathbb{R}))$. Let $f(x) = \sum_{k=1}^{n} \chi_{A_k} \otimes g_k$ as before. First, in order to obtain that Θf is strongly measurable, it suffices to show that given $\omega \in L^p(\mathbb{R})$, the function

$$\omega(\Theta f(\cdot)) = \mathbb{R} \to \mathbb{C} : x \mapsto \omega(\Theta f(x)) = \int_{\mathbb{R}} \omega(y)(\Theta f)(x)(y)dy$$

is measurable. This is trivial to prove, since

$$\begin{aligned}\omega(\Theta f(x)) &= \int_{\mathbb{R}} \omega(y) \sum_{k=1}^{n} \chi_{A_k}(x) g_k(x-y)dy = \\ &= \sum_{k=1}^{n} \chi_{A_k}(x) \int_{\mathbb{R}} \omega(y) g_k(x-y)dy = \sum_{k=1}^{n} \chi_{A_k}(x)(\omega * g_k)(x),\end{aligned}$$

and applying Young's inequality, the function $\omega * g_k$ lies in $L^\infty(\mathbb{R})$. Now

$$\begin{aligned}\|\Theta f\|_{p,q}^p &= \int_{\mathbb{R}} \|\Theta(f)(x)\|_q^p dx = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \left| \sum_{k=1}^{n} \chi_{A_k}(x) g_k(x-y) \right|^q dy \right)^{p/q} dx = \\ &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \left| \sum_{k=1}^{n} \chi_{A_k}(x) g_k(y) \right|^q dy \right)^{p/q} dx = \int_{\mathbb{R}} \|f(x)\|_q^p dx = \|f\|_{p,q}^p.\end{aligned}$$

Since $\Theta^{-1} = \Theta$, the map is bijective. □

2.1.1 The Fourier binest algebra $\mathcal{A}_{FB}^p$ and the parabolic algebra

In this subsection, we give the natural generalization of the Fourier binest algebra on L^p spaces. The Volterra nest $\mathcal{N}_v^p$ is the continuous nest consisting of the subspaces $L^p([t, +\infty))$, for $t \in \mathbb{R}$, together with the trivial subspaces $\{0\}, L^p(\mathbb{R})$. The analytic nest $\mathcal{N}_a^p$ is defined to be the chain of subspaces

$$e^{i\lambda x} H^p(\mathbb{R}), \quad \lambda \in \mathbb{R},$$

together with the trivial subspaces. We claim that the nest $\mathcal{N}_a^p$ is totally ordered; the chain of subspaces $e^{i\lambda x}H^2(\mathbb{R})$ is evidently totally ordered, since it is unitarily equivalent with the Volterra nest. By Lemma 1.1.8 the space $e^{i\lambda x}H^2(\mathbb{R}) \cap L^p(\mathbb{R})$ is dense in $e^{i\lambda x}H^p(\mathbb{R})$, for every $p \in (1, \infty)$, so our claim follows trivially. Note that in the special case where $p = \infty$, the above nests are not complete with respect to the norm closed span, but with the weak*-closed linear span. These nests determine the Volterra nest algebra $\mathcal{A}_v^p = \operatorname{Alg} \mathcal{N}_v^p$ and the analytic nest algebra $\mathcal{A}_a^p = \operatorname{Alg} \mathcal{N}_a^p$, both of which are reflexive operator algebras.

The Fourier binest is the subspace lattice

$$\mathcal{L}_{FB}^p = \mathcal{N}_v^p \cup \mathcal{N}_a^p$$

and the Fourier binest algebra $\mathcal{A}_{FB}^p$ is the non-selfadjoint algebra $\operatorname{Alg} \mathcal{L}_{FB}^p$ of operators which leave invariant each subspace of $\mathcal{L}_{FB}^p$. The reflexivity of $\mathcal{A}_{FB}^p$ is immediate from its definition.

Given $p \in (1, +\infty)$, let J be the flip operator given by $(Jf)(x) = f(-x)$. Note that J is the isometric operator that takes the Volterra nest to its counterpart

$$(\mathcal{N}_v^p)^\perp := \{0\} \cup \{L^p(-\infty, t] \,:\, t \in \mathbb{R}\} \cup \{L^p(\mathbb{R})\}$$

and the analytic nest to

$$(\mathcal{N}_a^p)^\perp := \{0\} \cup \{e^{-i\lambda x}\overline{H^p}(\mathbb{R}) \,:\, \lambda \in \mathbb{R}\} \cup \{L^p(\mathbb{R})\}.$$

Hence $J\mathcal{A}_{FB}^p J$ is the binest algebra generated by the lattice $J\mathcal{L}_{FB}^p = (\mathcal{N}_v^p)^\perp \cup (\mathcal{N}_a^p)^\perp$. Since the spaces $e^{i\lambda x}H^p(\mathbb{R})$ and $L^p[t, \infty)$ are naturally complemented and have trivial subspaces it is straightforward to adjust the Hilbert space arguments [38] to see that

$\mathcal{A}^p_{FB}$ is an antisymmetric operator algebra, meaning that $\mathcal{A}^p_{FB} \cap J\mathcal{A}^p_{FB}J = \mathbb{C}I$, and also that the algebra contains no non-zero finite rank operators.

To define the parabolic algebra $\mathcal{A}^p_{par}$, we recall the definition of the strong operator topology (SOT). Given a net $(T_i)_{i\in I}$ of bounded operators on a Banach space X, we say that $T_i \overset{\text{SOT}}{\to} T$, where $T \in B(X)$, if and only if $T_i x \to Tx$, for every $x \in X$. In other words, the SOT-topology on $B(X)$ is defined as the topology of pointwise convergence on X. Check that in the case that X is a Hilbert space the above definition coincides with the definition of SOT-topology given in subsection 1.2.3.

The parabolic algebra $\mathcal{A}^p_{par}$ is defined as the SOT-closed operator algebra on $L^p(\mathbb{R})$ that is generated by the two isometric semigroups $\{M_\lambda, \lambda \geq 0\}$, $\{D_\mu,\, \mu \geq 0\}$. As we stated in subsection 1.3.2, Katavolos and Power defined the parabolic algebra, in the case $p = 2$, to be the weak*-closed algebra that is generated by the translation and multiplication semigroups and they proved that this algebra is equal to the SOT-closed algebra $\mathcal{A}^2_{FB}$. Hence the two definitions of the parabolic algebra on $L^2(\mathbb{R})$ coincide.

2.1.2 Integral Operators on $L^p(\mathbb{R})$

Let $p \in (1,\infty)$ and q be its conjugate exponent. Given $k \in L^p(\mathbb{R}; L^q(\mathbb{R}))$, the linear map

$$(\operatorname{Int} k\, f)(x) = \int_{\mathbb{R}} k(x)(y) f(y) dy$$

defines a bounded operator on $L^p(\mathbb{R})$. Indeed, given $f \in L^p(\mathbb{R})$, applying the Hölder inequality we obtain

$$\int_{\mathbb{R}} \left| \int_{\mathbb{R}} k(x)(y) f(y) dy \right|^p dx \leq \|k\|^p_{p,q} \|f\|^p_p.$$

We will refer to such an operator as $(\mathbf{p},\mathbf{q})$**-integral operator** and denote the set of (p, q)-integral operators by

$$\mathcal{G}^p = \{\operatorname{Int} k \,:\, k \in L^p(\mathbb{R}; L^q(\mathbb{R}))\}.$$

Remark 2.1.10.

1. The above calculation also proves that the norm $\|\cdot\|_{p,q}$ dominates the operator norm, and so given $(k_n)_{n\geq 1}, k \in L^p(\mathbb{R}; L^q(\mathbb{R}))$, such that $k_n \overset{\|\cdot\|_{p,q}}{\to} k$, then $\operatorname{Int} k_n \to \operatorname{Int} k$.

2. In the special case $p = 2$, then $\mathcal{G}^2 = \mathcal{C}_2$, where $\mathcal{C}_2$ is the ideal of the Hilbert-Schmidt operators on $L^2(\mathbb{R})$.

Lemma 2.1.11. *$\mathcal{G}^p$ is a right ideal in $B(L^p(\mathbb{R}))$.*

Proof. Let $T \in B(L^p(\mathbb{R}))$. Given $f \in L^p(\mathbb{R})$ and $k \in L^p(\mathbb{R}; L^q(\mathbb{R}))$, such that $k = \sum_{\kappa=1}^{n} f_\kappa \otimes g_\kappa$, we have

$$\begin{aligned}(\operatorname{Int} kTf)(x) = \int_{\mathbb{R}} k(x,y)(Tf)(y)dy &= \sum_{\kappa=1}^{n} f_\kappa(x) \int_{\mathbb{R}} g_\kappa(y)(Tf)(y)dy = \\ &= \sum_{\kappa=1}^{n} f_\kappa(x) \int_{\mathbb{R}} T^* g_\kappa(y) f(y)dy,\end{aligned}$$

where T^* is the adjoint operator of T. Therefore $\operatorname{Int} kT = \operatorname{Int} \widetilde{k}$, where $\widetilde{k} = \sum_{\kappa=1}^{n} f_\kappa \otimes T^* g_\kappa$. In the general case, let $k \in L^p(\mathbb{R}; L^q(\mathbb{R}))$ and $k_m = \sum_{\kappa=1}^{n} f_\kappa^{(m)} \otimes g_\kappa^{(m)}$, such that $k_m \overset{\|\cdot\|_{p,q}}{\to} k$. Applying the above argument, we have $\operatorname{Int} k_m T = \operatorname{Int} \widetilde{k_m}$, where $\widetilde{k_m} = \sum_{\kappa=1}^{n} f_\kappa^{(m)} \otimes T^* g_\kappa^{(m)}$. Then, by Lemma 2.1.8, there is a unique operator $\widetilde{T^*} \in B(L^p(\mathbb{R}; L^q(\mathbb{R}))$, such that

$$\begin{aligned}\|\widetilde{k_m} - \widetilde{k_l}\|_{p,q} &= \left\| \sum_{\kappa=1}^{n} f_\kappa^{(m)} \otimes T^* g_\kappa^{(m)} - \sum_{\kappa=1}^{n} f_\kappa^{(l)} \otimes T^* g_\kappa^{(l)} \right\|_{p,q} = \\ &= \left\| \widetilde{T^*} \left(\sum_{\kappa=1}^{n} f_\kappa^{(m)} \otimes g_\kappa^{(m)} - \sum_{\kappa=1}^{n} f_\kappa^{(l)} \otimes g_\kappa^{(l)} \right) \right\|_{p,q} \leq\end{aligned}$$

$$
\begin{aligned}
&\leq \left\|\widetilde{T^*}\right\| \left\| \left(\sum_{\kappa=1}^{n} f_\kappa^{(m)} \otimes g_\kappa^{(m)} - \sum_{\kappa=1}^{n} f_\kappa^{(l)} \otimes g_\kappa^{(l)} \right) \right\|_{p,q} = \\
&= \left\|\widetilde{T^*}\right\| \|k_m - k_l\|_{p,q}.
\end{aligned}
$$

It follows that the sequence $(\widetilde{k_m})_m$ is a Cauchy sequence, so by the completeness of $L^p(\mathbb{R}; L^q(\mathbb{R}))$, it converges to some $\widetilde{k} \in L^p(\mathbb{R}; L^q(\mathbb{R}))$. Since the $\|\cdot\|_{p,q}$ norm dominates the operator norm, the sequence $(\operatorname{Int}\widetilde{k_n})_n$ of (p,q)-integral operators converges to $\operatorname{Int}\widetilde{k}$. Thus, by the uniqueness of the limit, we obtain $\operatorname{Int} kT = \operatorname{Int}\widetilde{k}$. □

2.2 Reflexivity

In this section, we prove that the parabolic algebra $\mathcal{A}^p_{par}$ is reflexive, given $p \in (1, \infty)$. In particular, we will show that $\mathcal{A}^p_{par} = \mathcal{A}^p_{FB}$. Since the generators of $\mathcal{A}^p_{par}$ leave the subspaces of the binest $\mathcal{L}^p_{FB}$ invariant, we have $\mathcal{A}^p_{par} \subseteq \mathcal{A}^p_{FB}$. Hence it suffices to prove that $\mathcal{A}^p_{FB} \subseteq \mathcal{A}^p_{par}$. In the following proposition, we make use of the linear transformation Θ defined in Lemma 2.1.9.

Proposition 2.2.1. *Let* $\operatorname{Int} k \in \mathcal{G}^p \cap \mathcal{A}^p_{FB}$. *Then k satisfies the following properties:*

1. $\Theta k \in L^p(\mathbb{R}; L^q(\mathbb{R}^+))$;

2. *For every Borel set A of finite measure,* $\operatorname{Int}(\Theta k)\chi_A$ *lies in* $H^p(\mathbb{R})$.

Proof. Let $\operatorname{Int} k \in \mathcal{G}^p \cap \mathcal{A}^p_{FB}$.

1. Since $\operatorname{Int} kL^p[t, \infty) \subseteq L^p[t, \infty)$, for every $t \in \mathbb{R}$, it follows that $k(x)(y) = 0$, for almost every $(x, y) \in \mathbb{R}^2$, such that $y > x$. Therefore, $\Theta k(x) \in L^q(\mathbb{R}^+)$ for almost every $x \in \mathbb{R}$.

2. Since $M_{-\lambda}\operatorname{Int} kD_\mu M_\lambda H^p(\mathbb{R}) \subseteq H^p(\mathbb{R})$, for every $\lambda, \mu \in \mathbb{R}$, given functions $f \in H^p(\mathbb{R}), g \in H^q(\mathbb{R})$, we have that

$$\begin{aligned}
&\int_{\mathbb{R}} (M_{-\lambda}\operatorname{Int} kD_\mu M_\lambda f)(x)g(x)dx = 0 \Leftrightarrow \\
&\int_{\mathbb{R}} \left(\int_{\mathbb{R}} e^{-i\lambda x} k(x)(y) e^{i\lambda(y-\mu)} f(y-\mu)g(x)dy\right) dx = 0 \overset{y \to x-y}{\Leftrightarrow} \\
&\int_{\mathbb{R}} \left(\int_{\mathbb{R}} \Theta k(x)(y) e^{-i\lambda(y+\mu)} f(x-y-\mu)g(x)dy\right) dx = 0.
\end{aligned}$$

Therefore, for every $q \in L^1(\mathbb{R})$, we obtain

$$\int_{\mathbb{R}} \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} \Theta k(x)(y) e^{-i\lambda(y+\mu)} f(x-y-\mu)g(x)dy\right) dx\right) q(\mu)d\mu = 0.$$

Take $q(\mu) = \chi_A(\mu)$, where A is a Borel set of finite measure. Then, by Fubini's theorem

$$\begin{aligned}
&\int_{\mathbb{R}} \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} \Theta k(x)(y) e^{-i\lambda(y+\mu)} f(x-y-\mu)g(x)dy\right) dx\right) \chi_A(\mu)d\mu = 0 \Leftrightarrow \\
&\int_{\mathbb{R}} \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} \Theta k(x)(y) e^{-i\lambda(y+\mu)} f(x-y-\mu)g(x)\chi_A(\mu)d\mu\right) dy\right) dx = 0 \overset{\mu \to \mu-y}{\Leftrightarrow} \\
&\int_{\mathbb{R}} \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} \Theta k(x)(y) e^{-i\lambda\mu} f(x-\mu)g(x)\chi_A(\mu-y)d\mu\right) dy\right) dx = 0.
\end{aligned}$$

Thus

$$\int_{\mathbb{R}} \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} \Theta k(x)(y) f(x-\mu)g(x)\chi_A(\mu-y)dy\right) dx\right) e^{-i\lambda\mu}d\mu = 0. \tag{2.1}$$

We claim that the function

$$\Phi : \mathbb{R} \to \mathbb{C} : \mu \mapsto \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \Theta k(x)(y) f(x-\mu)g(x)\chi_A(\mu-y)dy\right) dx$$

is a well defined L^1 function. By Tonelli's theorem, it suffices to show that

$$\int_{\mathbb{R}} \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} \left| \Theta k(x)(y) f(x-\mu) g(x) \chi_A(\mu - y) \right| d\mu \right) dy \right) dx < \infty.$$

We have

$$\begin{aligned} &\int_{\mathbb{R}} \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} \left| \Theta k(x)(y) f(x-\mu) g(x) \chi_A(\mu - y) \right| d\mu \right) dy \right) dx = \\ &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \left(\left| \Theta k(x)(y) \right| \int_{\mathbb{R}} \left| f(x-\mu) \chi_A(\mu - y) \right| d\mu \right) dy \right) \left| g(x) \right| dx = \\ &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \left(\left| \Theta k(x)(y) \right| \int_{\mathbb{R}} \left| f(x-y-\mu) \chi_A(\mu) \right| d\mu \right) dy \right) \left| g(x) \right| dx. \end{aligned}$$

By Young's inequality the function $c := |f| * \chi_A$ lies in $L^p(\mathbb{R})$, so the expression above is equal to

$$\int_{\mathbb{R}} \left(\int_{\mathbb{R}} \left| \Theta k(x)(y) c(x-y) \right| dy \right) \left| g(x) \right| dx = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \left| k(x)(y) c(y) \right| dy \right) \left| g(x) \right| dx$$

which by Hölder's inequality is bounded by $\|c\|_p \|k\|_{p,q} \|g\|_q$, so our claim is proven. Hence it follows by the equation (2.1) that the Fourier transform of the function Φ is the zero function, so we obtain that

$$\int_{\mathbb{R}} \left(\int_{\mathbb{R}} \Theta k(x)(y) f(x-\mu) g(x) \chi_A(\mu - y) dy \right) dx = 0 \tag{2.2}$$

for almost every $\mu \in \mathbb{R}$. Fix some $\mu \in \mathbb{R}$, such that equation (2.2) holds. Hence by Lemma 1.1.9

$$\int_{\mathbb{R}} \left(\int_{\mathbb{R}} \Theta k(x)(y) h(x) \chi_A(\mu - y) dy \right) dx = 0$$

for every h in a dense subset of $H^q(\mathbb{R})$. Moreover, since the set A was freely chosen and $\chi_A(\mu - y) = D_\mu J \chi_A(y) = \chi_B(y)$, where $B = \{\mu - a : a \in A\}$ and J

is again the flip operator, it follows that

$$\begin{aligned}\int_{\mathbb{R}}\left(\int_{\mathbb{R}}\Theta k(x)(y)h(x)\chi_A(y)dy\right)dx &= 0\\ \Rightarrow \int_{\mathbb{R}}(\mathrm{Int}(\Theta k)\chi_A)(x)h(x)dx &= 0\end{aligned}$$

for every Borel set A of finite measure. Hence $\mathrm{Int}(\Theta k)\chi_A$ annihilates a dense subspace of $H^q(\mathbb{R})$, so by Hölder's inequality lies in the annihilator of $H^q(\mathbb{R})$, which is $H^p(\mathbb{R})$.

□

Our next goal is to determine a dense set of $\mathcal{G}^p \cap \mathcal{A}^p_{FB}$. We start with an approximation lemma.

Lemma 2.2.2. *Let $\varphi \in L^1(\mathbb{R})$. Then, given $p \in [1,\infty)$, the convolution operator*

$$\Delta_\varphi : L^p(\mathbb{R}) \to L^p(\mathbb{R}) : f \mapsto \varphi * f,$$

is bounded. Furthermore, if φ has essential support in $\mathbb{R}^+$, then Δ_φ belongs to the SOT-closed algebra generated by $\{D_t \,|\, t \in \mathbb{R}^+\}$.

Proof. The continuity of Δ_φ is immediate by Young's inequality, which also gives $\|\Delta_\varphi\| \leq \|\varphi\|_1$. The argument of the second claim is similar to that for $p = 2$ [42]. Suppose first that φ has compact support $[a,b]$, for some $b > a \geq 0$. Given $n \in \mathbb{N}$ and $m \in \{0,1,\dots,n-1\}$, define $\alpha_{m,n} = \int_{\tau(m,n)}^{\tau(m+1,n)} \varphi(s)ds$, where $\tau(m,n) = a + \frac{m}{n}(b-a)$. We claim that the sequence $(T_n)_n$ given by

$$T_n = \sum_{m=0}^{n-1} \alpha_{m,n} D_{\tau(m,n)}$$

converges in the SOT-topology to Δ_φ. Consider $f \in L^p$. Then by Hahn - Banach theorem

$$\begin{aligned}\|(\Delta_\varphi - T_n)f\|_p &= \sup\left\{\left|\int_{\mathbb{R}}(\Delta_\varphi - T_n)f(x)g(x)dx\right| \,:\, \|g\|_q = 1\right\} = \\ &= \sup\left\{\left|\int_{\mathbb{R}}\sum_{m=0}^{n-1}\int_{\tau(m,n)}^{\tau(m+1,n)}\varphi(t)\left((D_t - D_{\tau(m,n)})f(x)\right)dtg(x)dx\right| \,:\, \|g\|_q = 1\right\} = \\ &= \sup\left\{\left|\int_{\mathbb{R}}\int_{\mathbb{R}}\varphi(t)\left((D_t - D_{\rho_n(t)})f(x)\right)dtg(x)dx\right| \,:\, \|g\|_q = 1\right\} \leq \\ &\leq \sup\left\{\int_{\mathbb{R}}\left(\int_{\mathbb{R}}\left|\varphi(t)\left((D_t - D_{\rho_n(t)})f(x)\right)g(x)\right|dx\right)dt \,:\, \|g\|_q = 1\right\}\end{aligned}$$

where $\rho_n(t) = a + \frac{b-a}{n}\left\lfloor\frac{(t-a)n}{b-a}\right\rfloor$, $t \in [a,b]$. Now

$$\int_{\mathbb{R}}\left|\varphi(t)\left((D_t - D_{\rho_n(t)})f(x)\right)g(x)\right|dx \leq |\varphi(t)|\,\|(D_t - D_{\rho_n(t)})f\|_p\|g\|_q,$$

so it follows that

$$\|(\Delta_\varphi - T_n)f\|_p \leq \int_{\mathbb{R}}|\varphi(t)|\|(D_t - D_{\rho_n(t)})f\|_p dt.$$

Since $\|(D_t - D_{\rho_n(t)})f\|_p \to 0$ as $n \to \infty$ and $|\varphi(t)|\,\|(D_t - D_{\rho_n(t)})f\|_p \leq 2|\varphi(t)|\,\|f\|_p$, we get that $\|(\Delta_\varphi - T_n)f\|_p \to 0$, by dominated convergence theorem. This proves the second claim of the theorem , in the case where φ has compact support. The general case, is a simple application of Young's inequality. □

Remark 2.2.3. In the $L^2(\mathbb{R})$ case, there is a simpler proof, using the unitary Fourier-Plancherel transform F. Note that

$$\Delta_\varphi = F^* M_{\hat{\varphi}} F.$$

Since $\varphi \in L^1(\mathbb{R}^+)$, it follows that $\hat{\varphi} \in \overline{H^\infty(\mathbb{R})}$. Therefore, the multiplication operator $M_{\hat{\varphi}}$ lies in the SOT-closed algebra generated by $\{M_{-\lambda} : \lambda \in \mathbb{R}^+\}$. Hence, using the fact that $D_\lambda = F^* M_{-\lambda} F$, the proof is complete.

Lemma 2.2.4. *Let $h \in H^p(\mathbb{R})$, $\varphi \in L^q(\mathbb{R}^+)$, where $p \in (1, \infty)$ and q is its conjugate exponent. Define $k = \Theta^{-1}(h \otimes \varphi)$. Then, the operator* $\operatorname{Int} k$ *lies in $\mathcal{G}^p \cap \mathcal{A}^p_{par}$.*

Proof. First, consider $h \in H^\infty(\mathbb{R}), \varphi \in L^1(\mathbb{R}^+)$. Then

$$\begin{aligned}(\operatorname{Int} k f)(x) &= \int_{\mathbb{R}} \Theta^{-1}(h \otimes \varphi)(x)(y) f(y) dy = \\ &= \int_{\mathbb{R}} h(x)\varphi(x-y) f(y) dy = (M_h \Delta_\varphi f)(x),\end{aligned}$$

so $\|\operatorname{Int} k\| \leq \|h\|_\infty \|\varphi\|_1$. By the previous lemma $\Delta_\varphi \in$ SOT-alg$\{D_t : t \in \mathbb{R}^+\}$, hence $\operatorname{Int} k \in \mathcal{A}^p_{par}$. Take now $h \in H^p(\mathbb{R})$ and $\varphi \in L^q(\mathbb{R}^+)$. Then there exist $h_m \in (H^\infty \cap H^p)(\mathbb{R})$, $\varphi_m \in (L^1 \cap L^q)(\mathbb{R}^+)$, such that $h_m \overset{\|\cdot\|_p}{\to} h$ and $\varphi_m \overset{\|\cdot\|_q}{\to} \varphi$. Now it is straightforward to show that $h_m \otimes \varphi_m \overset{\|\cdot\|_{p,q}}{\to} h \otimes \varphi$. Since the norm $\|\cdot\|_{p,q}$ dominates the operator norm and $\mathcal{A}^p_{par}$ is norm closed,

$$\operatorname{Int} k = \operatorname{Int}(\Theta^{-1}(h \otimes \varphi)) \in \mathcal{A}^p_{par}.$$

Moreover, the fact that h and φ lie in $H^p(\mathbb{R})$ and $L^q(\mathbb{R}^+)$ respectively implies that $\operatorname{Int} k \in \mathcal{G}^p$. □

Proposition 2.2.5. $\mathcal{G}^p \cap \mathcal{A}^p_{FB} = \mathcal{G}^p \cap \mathcal{A}^p_{par}$, *for every $p \in (1, \infty)$.*

Proof. Let $\mathcal{G}^p \cap \mathcal{A}^p_{FB}$ be strictly larger than $\mathcal{G}^p \cap \mathcal{A}^p_{par}$. Since the subspace

$$\{k \in L^p(\mathbb{R}; L^q(\mathbb{R})) : \operatorname{Int} k \in \mathcal{G}^p \cap \mathcal{A}^p_{par}\}$$

is closed in $L^p(\mathbb{R}; L^q(\mathbb{R}))$, by Riesz's lemma ([67]) there exists an element $\operatorname{Int} k_0 \in \mathcal{G}^p \cap \mathcal{A}^p_{FB}$, where k_0 lies in the unit sphere of $L^p(\mathbb{R}; L^q(\mathbb{R}))$, such that

$$\inf\{\|k_0 - k\|_{p,q} \,:\, \operatorname{Int} k \in \mathcal{G}^p \cap \mathcal{A}^p_{par}\} > \frac{9}{10}.$$

Since $\mathbb{F}(\mathbb{R}; L^q(\mathbb{R}))$ is dense in $L^p(\mathbb{R}; L^q(\mathbb{R}))$ and Θ is a bijective isometry, there exists $a \in \Theta^{-1}(\mathbb{F}(\mathbb{R}; L^q(\mathbb{R})))$, such that

$$\|k_0 - a\|_{p,q} < \frac{1}{10}.$$

Also, by the boundedness of the Riesz projection from $L^p(\mathbb{R})$ to $H^p(\mathbb{R})$ [31], we can write the element a, as $a = b + c + d + e$ where

$$\begin{aligned}
b &= \Theta^{-1}(\sum_{k=1}^{N_1} h_k^1 \otimes g_k^1), \text{ with } h_k^1 \in H^p(\mathbb{R}), g_k^1 \in L^q(\mathbb{R}^+),\\
c &= \Theta^{-1}(\sum_{k=1}^{N_2} h_k^2 \otimes g_k^2), \text{ with } h_k^2 \in H^p(\mathbb{R}), g_k^2 \in L^q(\mathbb{R}^-),\\
d &= \Theta^{-1}(\sum_{k=1}^{N_3} h_k^3 \otimes g_k^3), \text{ with } h_k^3 \in \overline{H^p(\mathbb{R})}, g_k^3 \in L^q(\mathbb{R}^+),\\
e &= \Theta^{-1}(\sum_{k=1}^{N_1} h_k^4 \otimes g_k^4), \text{ with } h_k^4 \in \overline{H^p(\mathbb{R})}, g_k^4 \in L^q(\mathbb{R}^-).
\end{aligned}$$

Note that at least one of the elements c, d, e has norm bigger than $\frac{1}{4}$. For otherwise, since $\|k_0 - b\|_{p,q} > \frac{9}{10}$, we have

$$\|k_0 - a\|_{p,q} = \|k_0 - b - (c + d + e)\|_{p,q} \geq \|k_0 - b\|_{p,q} - \|c + d + e\|_{p,q} > \frac{3}{20}.$$

Without loss of generality, let $\|c\|_{p,q} > \frac{1}{4}$. By the Hahn - Banach theorem and Proposition 2.1.7, there exists $\omega \in L^q(\mathbb{R}; L^p(\mathbb{R}))$, such that $|\omega(c)| > \frac{1}{4}$ and $\|\omega\| = 1$. Hence, by Proposition 2.1.6 and the definition of the element c, we may assume that

$\|\omega\| \leq \frac{3}{2}$ and ω is given by the formula

$$\begin{aligned}\omega(k) &= \sum_{m=1}^{n} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} k(x)(y) f_m(x) \chi_{A_m}(x-y) dy \right) dx = \\ &= \sum_{m=1}^{n} \int_{\mathbb{R}} (\mathrm{Int}(\Theta k)\chi_{A_m})(x) f_m(x) dx\end{aligned}$$

where $f_m \in \overline{H^q(\mathbb{R})}$ and $\{A_m\}_{m=1,\ldots,n}$ is a family of Borel subsets of $\mathbb{R}^-$. So it follows from Proposition 2.2.1 that

$$\frac{3}{2}\|k_0 - a\|_{p,q} \geq |\omega(k_0 - a)| = |\omega(k_0) - \omega(b) - \omega(c) - \omega(d) - \omega(e)| = |\omega(c)| > \frac{1}{4},$$

which is a contradiction. □

The following proposition and proof follow the pattern for the case $p = 2$, given in [42].

Proposition 2.2.6. *For every $p \in (1, \infty)$, the algebra $\mathcal{A}^p_{par}$ contains a bounded approximate identity of elements in $\mathcal{G}^p$.*

Proof. Take $h_n(x) = \frac{ni}{x+ni}$ and $\varphi_n(y) = n\chi_{[0,1/n]}(y)$. It is trivial to see that $h_n \in H^r(\mathbb{R})$ and $\varphi_n \in L^r(\mathbb{R}^+)$, for every $r \in (1, \infty)$. Moreover, h_n and φ_n lie in the respective unit balls of $H^\infty(\mathbb{R})$ and $L^1(\mathbb{R})$. Let $k_n = \Theta^{-1}(h_n \otimes \varphi_n)$. As in the proof of Lemma 2.2.4, we have $\mathrm{Int}\, k_n = M_{h_n}\Delta_{\varphi_n}$ and $\|\mathrm{Int}\, k_n\| \leq \|h_n\|_\infty \|\varphi_n\|_1 \leq 1$. Since $h_n \to 1$ uniformly on compact sets of the real line, it follows that $M_{h_n} \overset{\mathrm{SOT}}{\to} I$. Now given $f \in C_{\mathrm{C}}(\mathbb{R})$, note that

$$\begin{aligned}\|\Delta_{\varphi_n} f - f\|_p^p &= \int_{\mathbb{R}} \left| \int_{\mathbb{R}} n\chi_{[0,1/n]}(y) f(x-y) dy - f(x) \right|^p dx = \\ &= \int_{\mathbb{R}} \left| \int_0^{1/n} n f(x-y) dy - f(x) \right|^p dx.\end{aligned}$$

Check that $\left|\int_0^{1/n} nf(x-y)dy - f(x)\right|^p \leq 2^p\|f\|_\infty^p \chi_s(x)$, where S is the compact set

$$S = \{x + \tau \,|\, x \in \operatorname{supp} f,\, \tau \in [0,1]\}.$$

Hence by dominated convergence $\Delta_{\varphi_n} f \to f$. Since $C_{\mathrm{C}}(\mathbb{R})$ is dense in $L^p(\mathbb{R})$ it follows that $\Delta_{\varphi_n} \overset{\mathrm{SOT}}{\to} I$. Multiplication is SOT-continuous on the closed unit ball of bounded operators, so $M_{h_n}\Delta_{\varphi_n} \overset{\mathrm{SOT}}{\to} I$. □

Theorem 2.2.7. *For every $p \in (1,\infty)$, the parabolic algebra A^p_{par} is equal to the Fourier binest algebra $\mathcal{A}^p_{FB}$.*

Proof. As we have noted before, it suffices to prove that $\mathcal{A}^p_{FB} \subseteq A^p_{par}$. Let $T \in \mathcal{A}^p_{FB}$ and $(X_n)_{n\geq 1}$ be the bounded approximate identity of the previous proposition. By Lemma 2.1.11 and Proposition 2.2.5, the operators X_nT lie in $\mathcal{G}^p \cap \mathcal{A}^p_{FB} = \mathcal{G}^p \cap \mathcal{A}^p_{par}$. Since $\mathcal{A}^p_{par}$ is SOT-closed, the given operator $T = \mathrm{SOT} - \lim_n X_nT$ lies in $\mathcal{A}^p_{par}$. □

Proposition 2.2.8. *The Fourier binest algebra $\mathcal{A}^\infty_{FB}$ is strictly larger than the parabolic algebra $\mathcal{A}^\infty_{par}$.*

Proof. Recall first that by Proposition 1.2.20 the algebra $AAP(\mathbb{R})$ of analytic almost periodic functions is strictly smaller than $H^\infty(\mathbb{R})$. Choose a function φ that lies in $H^\infty(\mathbb{R})$ and it is not an element of $AAP(\mathbb{R})$. It suffices to show that $M_\varphi \notin \mathcal{A}^\infty_{par}$. If this is not the case, there is some sequence $p_n(M_\lambda, D_\mu)$ in the non-closed algebra generated by $\{M_\lambda, D_\mu \,:\, \lambda, \mu \geq 0\}$ which converges strongly to M_φ. Thus for any $f \in L^\infty(\mathbb{R})$, we have

$$\left\|p_n(M_\lambda, D_\mu)f - M_\varphi f\right\|_\infty \to 0, \text{ as } n \to \infty.$$

Choosing $f \equiv 1$, it follows that

$$\left\|p_n(M_\lambda, I)f - M_\varphi f\right\|_\infty \to 0, \text{ as } n \to \infty,$$

and so $\varphi \in AAP(\mathbb{R})$, a contradiction. $\square$

Remark 2.2.9. It remains unclear to the author whether the parabolic operator algebras $\mathcal{A}^1_{par}$ and $\mathcal{A}^\infty_{par}$ acting on the respective Banach spaces $L^1(\mathbb{R})$ and $L^\infty(\mathbb{R})$ are reflexive operator algebras.

2.3 The lattice of the parabolic algebra

Let $K^p_{\lambda,s} = M_\lambda M_{\varphi_s} H^p(\mathbb{R})$ where $\varphi_s(x) = e^{-isx^2/2}$. This is evidently an invariant subspace for the multiplication semigroup and for $s \geq 0$ one can check that it is invariant for the translation semigroup. Thus for $s \geq 0$ the nest $\mathcal{N}^p_s = M_{\varphi_s}\mathcal{N}^p_a$ is contained in $\operatorname{Lat} \mathcal{A}^p_{par}$ and these nests are distinct. Suppose now that $p = 2$. With the strong operator topology for the associated orthogonal subspace projections it can be shown ([38]) that the set of these nests for $s \geq 0$, together with the Volterra nest $\mathcal{N}^2_v$, is homeomorphic to the closed unit disc. A cocycle argument given in [38] leads to the fact that every invariant subspace for $\mathcal{A}^2_{par}$ is of this form for $p = 2$. That is

$$\operatorname{Lat} \mathcal{A}^2_{par} = \{K^2_{\lambda,s} | \lambda \in \mathbb{R}, s \geq 0\} \cup \mathcal{N}^2_v. \tag{2.3}$$

We prove now the corresponding result for the general case of $\mathcal{A}^p_{par}$, where $1 < p < \infty$.

Let K be a non-trivial element of $\operatorname{Lat} \mathcal{A}^p_{par}$. Then the subspace $K \cap L^2(\mathbb{R})$ is invariant under the generators of the parabolic algebra. Therefore, the $\|\cdot\|_2$-closure of $K \cap L^2(\mathbb{R})$ lies in $\operatorname{Lat} \mathcal{A}^2_{par}$. On the other hand, by Theorem 1.1.11, either $K = L^p(E)$ for some Borel set $E \subseteq \mathbb{R}$ or $K = M_\varphi H^p(\mathbb{R})$ for some unimodular function φ. In the first case, where $K = L^p(E)$, then

$$\overline{L^p(E) \cap L^2(\mathbb{R})}^{\|\cdot\|_2} \in \operatorname{Lat} \mathcal{A}^2_{par} \Rightarrow L^2(E) \in \operatorname{Lat} \mathcal{A}^2_{par} \Rightarrow E = [t, \infty),$$

for some $t \in \mathbb{R}$. In the second case, $K = M_\varphi H^p(\mathbb{R})$, which implies

$$\overline{M_\varphi H^p(\mathbb{R}) \cap L^2(\mathbb{R})}^{\|\cdot\|_2} \in \operatorname{Lat} \mathcal{A}^2_{par} \Rightarrow M_\varphi H^2(\mathbb{R}) \in \operatorname{Lat} \mathcal{A}^2_{par} \Rightarrow M_\varphi = M_{\varphi_s} M_\lambda,$$

for some $s \in [0, +\infty), \lambda \in \mathbb{R}$. Hence, we have the following result.

Theorem 2.3.1. *Given $p \in (1, \infty)$, the invariant subspace lattice of the algebra $\mathcal{A}^p_{par}$ is*

$$\operatorname{Lat} \mathcal{A}^p_{par} = \{K^p_{\lambda,s} | \lambda \in \mathbb{R}, s \geq 0\} \cup \mathcal{N}^p_v.$$

Recall that the reflexive closure of a set of closed subspaces $\mathcal{L}$ is the subspace lattice $\operatorname{Lat} \operatorname{Alg} \mathcal{L}$. Thus the theorem identifies the reflexive closure of the binest $\mathcal{L}^p_{FB}$.

Remark 2.3.2. In [38], Katavolos and Power proved that $\operatorname{Lat} \mathcal{A}^2_{par}$, viewed as a topological space of projections on $L^2(\mathbb{R})$, endowed with the strong operator topology, is homeomorphic to the closed unit disc. In particular, they obtained the so-called strange limit

$$P_{K^2_{\lambda,s}} \overset{\text{SOT}}{\to} P_{L^2[\lambda,+\infty)}, \text{ as } s \to \infty,$$

which relies on the Paley - Wiener theorem and the fact that the Fourier transform is unitary on $L^2(\mathbb{R})$. Even though the Riesz projection from $L^p(\mathbb{R})$ onto $H^p(\mathbb{R})$ remains bounded, it is unknown to the author if the above convergence still holds, for $p \in (1, +\infty) \backslash \{2\}$.

We expect that the operator algebras $\mathcal{A}^p_{par}$, for $1 < p < \infty$, are pairwise non isomorphic, even as rings of linear operators. However, the standard methods for such a demonstration (which go back to Eidelheit [20]) rely on exploiting the presence of rank one operators to deduce an isomorphism between the underlying Banach spaces. Possibly the (p, q)-integral operators could once again play a substitute role in this demonstration.

Chapter 3

The triple semigroup algebra

3.1 Introduction

In this chapter, we consider the weak*-closed operator algebra $\mathcal{A}_{ph}$, that is generated by the semigroups of multiplication, translation and dilation operators, that is the sets of operators M_λ, D_μ, V_t, for $\lambda, \mu, t \geq 0$, respectively. Our main result is that this operator algebra, viewed as a subalgebra of $B(L^2(\mathbb{R}))$, is reflexive and, moreover, is equal to $\operatorname{Alg}\mathcal{L}$, the algebra of operators that leave invariant each subspace in the lattice $\mathcal{L}$ of closed subspaces given by

$$\mathcal{L} = \{0\} \cup \{L^2(-\alpha, \infty), \alpha \geq 0\} \cup \{e^{i\beta x} H^2(\mathbb{R}), \beta \geq 0\} \cup \{L^2(\mathbb{R})\}.$$

This lattice is a binest, being the union of two complete nests of closed subspaces.

We also obtain the following further properties. The triple semigroup algebra $\mathcal{A}_{ph}$ is antisymmetric in the sense that $\mathcal{A}_{ph} \cap \mathcal{A}_{ph}^* = \mathbb{C}I$. In contrast to $\mathcal{A}_p$ and $\mathcal{A}_h$ the algebra $\mathcal{A}_{ph}$ contains non-zero finite rank operators and these generate a proper weak*-closed ideal. Also, $\mathcal{A}_{ph}$ has the rigidity property that its unitary automorphism group is isomorphic to $\mathbb{R}$ and implemented by the group of dilation unitaries.

We also see that, unlike the parabolic algebra, $\mathcal{A}_{ph}$ has *chirality* in the sense that $\mathcal{A}_{ph}$ and $\mathcal{A}_{ph}^*$ are not unitarily equivalent despite being the reflexive algebras of *spectrally isomorphic* binests. Furthermore the 8 choices of triples of continuous proper semigroups from $\{M_\lambda, \lambda \in \mathbb{R}\}$, $\{D_\mu : \mu \in \mathbb{R}\}$ and $\{V_t : t \in \mathbb{R}\}$ give rise to exactly 2 unitary equivalence classes of operator algebras. These results can be found in [37].

3.2 Antisymmetry

We now show that $\mathcal{A}_{ph}$, like its subalgebras $\mathcal{A}_p$ and $\mathcal{A}_h$, is an antisymmetric operator algebra. In fact we shall prove that the containing algebra $\operatorname{Alg}\mathcal{L}$ is antisymmetric. A key step of the proof is the next lemma which will also be useful in the analysis of unitary automorphisms. We write $\mathbb{C}^+$ for the set of complex numbers with positive imaginary part.

Lemma 3.2.1. *Let $h, g \in H^2(\mathbb{R})$, $c, d \in \mathbb{C}^+$ and let $(x+c)h(x) = (x+d)g(x)$ for almost every x in a Borel set A of positive Lebesgue measure. Then $(x+c)h(x) = (x+d)g(x)$ almost everywhere in $\mathbb{R}$.*

Proof. We have

$$\begin{aligned}
(x+c)h(x) = (x+d)g(x) &\Leftrightarrow x(h(x)-g(x)) + c(h(x)-g(x)) + (c-d)g(x) = 0 \\
&\Leftrightarrow (x+c)(h(x)-g(x)) + (x+c)\frac{(c-d)g(x)}{x+c} = 0 \\
&\Leftrightarrow (x+c)\left(h(x)-g(x)+\frac{(c-d)g(x)}{x+c}\right) = 0.
\end{aligned}$$

Since $\frac{1}{x+c} \in H^\infty(\mathbb{R})$ we have $h(x) - g(x) + \frac{(c-d)g(x)}{x+c} \in H^2(\mathbb{R})$ and so it suffices to prove the following. Given $h \in H^2(\mathbb{R})$ and $c \in \mathbb{C}^+$, with $(x+c)h(x) = 0$ almost everywhere in A, then $(x+c)h(x) = 0$ almost everywhere. This is evident from Corollary 1.1.5. $\square$

In the next proof we write D_g for the operator FM_gF^* with $g \in H^\infty(\mathbb{R})$. This lies in the weak*-closed algebra generated by the operators $D_\mu = FM_\mu F^*$, for $\mu \geq 0$, and so belongs to $\mathcal{A}_p$ and to $\operatorname{Alg}\mathcal{L}$.

Theorem 3.2.2. *The selfadjoint elements of* $\operatorname{Alg}\mathcal{L}$ *are real multiples of the identity.*

Proof. Let $A \in \operatorname{Alg}\mathcal{L} \cap (\operatorname{Alg}\mathcal{L})^*$. Then A is reduced by subspaces $L^2(-\mu, +\infty)$, for $\mu \geq 0$, and $M_\lambda H^2(\mathbb{R})$, for $\lambda \geq 0$.

In particular, since A reduces the subspace $L^2(\mathbb{R}^+)$, it commutes with the projections $P_{L^2(\mathbb{R}^+)}$ and $P_{L^2(\mathbb{R}^-)}$. Hence A can be considered as a "block diagonal" operator with respect to the decomposition $L^2(\mathbb{R}^-) \oplus L^2(\mathbb{R}^+)$. Moreover, it follows by elementary measure theory that the compression of A in $L^2(\mathbb{R}^-)$ commutes with every projection M_{χ_B}, where χ_B is the characteristic function of a Borel set B in $L^2(\mathbb{R}^-)$. Since the commutant of a set is always a WOT-closed algebra, it follows that $A\Big|_{L^2(\mathbb{R}^-)}$ commutes with M_f for every $f \in L^\infty(\mathbb{R}^-)$. Since the algebra $\mathcal{M}_m$ of multiplication operators is a maximal abelian von Neumann algebra, we conclude that $A\Big|_{L^2(\mathbb{R}^-)} = M_f$, for some $f \in L^\infty(\mathbb{R}^-)$.

Since A reduces the subspaces $M_\lambda H^2(\mathbb{R})$, for all $\lambda \geq 0$, applying similar arguments for the operator FAF^* where F is the Fourier-Plancherel transform, we get a similar decomposition.

We conclude that A admits two direct sum decompositions

$$A = P_{L^2(\mathbb{R}^-)}M_fP_{L^2(\mathbb{R}^-)} + P_{L^2(\mathbb{R}^+)}XP_{L^2(\mathbb{R}^+)} = P_{H^2(\mathbb{R})}D_gP_{H^2(\mathbb{R})} + P_{\overline{H^2(\mathbb{R})}}YP_{\overline{H^2(\mathbb{R})}},$$

where $f \in L^\infty(\mathbb{R}^-)$, $g \in H^\infty(\mathbb{R})$ and X (resp. Y) is an uniquely determined operator on $L^2(\mathbb{R}^+)$ (resp. $\overline{H^2(\mathbb{R})}$).

Let $h(x) = \frac{1}{x+c}$ with $c \in \mathbb{C}^+$. Then, by the first decomposition,

$$Ah = M_fh + P_{L^2(\mathbb{R}^+)}XP_{L^2(\mathbb{R}^+)}h,$$

$$h^{-1}Ah = f + h^{-1}P_{L^2(\mathbb{R}^+)}XP_{L^2(\mathbb{R}^+)}h$$

and so for x in $\mathbb{R}^-$ we have $h^{-1}(x)(Ah)(x) = f(x)$. Also Ah is in $H^2(\mathbb{R})$ and so by the previous lemma, $h^{-1}Ah$ is determined by f and there is a function φ independent of c which extends f. Thus $h^{-1}Ah = \varphi$ and $Ah = \varphi h$. Since the linear span of the family $\{h : \mathbb{R} \to \mathbb{C} \big| h(x) = \frac{1}{x+c}, c \in \mathbb{C}^+\}$ is dense in $H^2(\mathbb{R})$, we have $A\big|_{H^2(\mathbb{R})} = M_\varphi\big|_{H^2(\mathbb{R})}$. However, by the second decomposition arguing similarly, we have $A\big|_{H^2(\mathbb{R})} = D_g\big|_{H^2(\mathbb{R})}$. Thus, given $h_1 \in H^2(\mathbb{R})\backslash\{0\}$, we have for every $\mu \in \mathbb{R}$,

$$M_\varphi D_\mu h_1 = D_g D_\mu h_1 = D_\mu D_g h_1 = D_\mu M_\varphi h_1.$$

Thus $\varphi(x)h_1(x-\mu) = \varphi(x-\mu)h_1(x-\mu)$ for almost every $x \in \mathbb{R}$ and so $\varphi(x) = c$ almost everywhere for some $c \in \mathbb{C}$. Now we have $A\big|_{H^2(\mathbb{R})} = A\big|_{L^2(\mathbb{R}^-)} = cI$ and it follows from the density of $H^2(\mathbb{R}) + L^2(\mathbb{R}^-)$ in $L^2(\mathbb{R})$ that $A = cI$, as required. $\square$

3.3 Finite rank operators in $\operatorname{Alg}\mathcal{L}$

It follows immediately from the definition of the binest $\mathcal{L}$ that the weak*-closed space

$$\mathcal{I} = P_+B(L^2(\mathbb{R}))(I - Q_+)$$

is contained in $\operatorname{Alg}\mathcal{L}$, where P_+ and Q_+ are the orthogonal projections for $L^2(\mathbb{R}^+)$ and $H^2(\mathbb{R})$. From this and Lemma 3.4.2 it follows that, in contrast to the subalgebras $\mathcal{A}_p$ and $\mathcal{A}_h$, the algebra $\mathcal{A}_{ph}$ contains finite rank operators. Also, it is straightforward to construct a pair of nonzero operators in $\mathcal{I}$ whose product is zero, and so, unlike the semigroup algebra $H^\infty(\mathbb{R})$, it follows also that the triple semigroup algebra $\mathcal{A}_{ph}$ is not an integral domain.

We now show that in fact the space $\mathcal{I}$ contains all the finite rank operators in $\mathrm{Alg}\,\mathcal{L}$. Let $\mathcal{N}_v^-$ and $\mathcal{N}_a^+$ be the subnests of $\mathcal{N}_v$ and $\mathcal{N}_a$ whose union is $\mathcal{L}$.

Proposition 3.3.1. *The weak*-closed ideal generated by the finite rank operators in* $\mathrm{Alg}\,\mathcal{L}$ *is the space* $\mathcal{I}$*. Moreover, each operator of rank* n *is decomposable as a sum of* n *rank one operators in* $\mathrm{Alg}\,\mathcal{L}$*.*

Proof. Let

$$\mathrm{Int}\,k : f \to \sum_{j=1}^{n} \langle f, h_j \rangle g_j$$

be a nonzero finite rank operator in $\mathrm{Alg}\,\mathcal{L}$, with $\{h_j\}$ and $\{g_j\}$ linearly independent functions in $L^2(\mathbb{R})$. There is some $\lambda_0 \geq 0$, such that $M_{\lambda_0} H^2(\mathbb{R}) \cap span\{g_i : i = 1, \ldots, n\} = \{0\}$. Since $M_{\lambda_0} H^2(\mathbb{R}) \in \mathcal{L}$ it follows that if $f \in M_{\lambda_0} H^2(\mathbb{R})$ then $\langle h_i, f \rangle = 0$, for every $i = 1, \ldots, n$. This in turn implies that $h_i \in M_{\lambda_0} \overline{H^2(\mathbb{R})}$.

We see now that the functions h_i have full support and, moreover, their set of restrictions to $\mathbb{R}^+$ is a linearly independent set of functions. Thus there are functions $f_1, \ldots, f_n$ in $L^2(\mathbb{R}^+)$ with $\langle f_i, h_j \rangle = \delta_{ij}$. Since $\mathrm{Int}\,k$ is in $\mathrm{Alg}\,\mathcal{N}_v^-$ it follows that each function g_i lies in $L^2(\mathbb{R}^+)$.

Since $\mathrm{Int}\,k \in \mathrm{Alg}\,\mathcal{N}_a^+$ it now follows that if $f \in H^2(\mathbb{R})$ then $\langle f, h_j \rangle = 0$ for each j. This holds for all such f and so $h_j \in H^2(\mathbb{R})^\perp$ for each j. Since $\mathcal{I} \subseteq \mathrm{Alg}\,\mathcal{L}$ the rank one operators determined by the h_j and g_j lie in $\mathrm{Alg}\,\mathcal{L}$ and the second assertion of the proposition follows. The first assertion follows from this. $\square$

As we will see in the next section, the ideal $\mathcal{I}$ plays a key role in the proof of reflexivity of the triple semigroup algebra.

3.4 Reflexivity

We now show that the algebra $\mathcal{A}_{ph}$ is reflexive, that is $\mathcal{A}_{ph} = \operatorname{Alg}\operatorname{Lat}\mathcal{A}_{ph}$, and for this it will be sufficient to show that $\mathcal{A}_{ph}$ is the binest algebra $\operatorname{Alg}\mathcal{L}$. Figure 3.1 depicts the inclusion of $\operatorname{Lat}\mathcal{A}_{ph}$ in $\operatorname{Lat}\mathcal{A}_p$ implied by the following lemma.

Lemma 3.4.1. *$\operatorname{Lat}\mathcal{A}_{ph} = \mathcal{L}$.*

Proof. Since $\mathcal{A}_{ph}$ is a superalgebra of $\mathcal{A}_p$ we have $\operatorname{Lat}\mathcal{A}_{ph} \subseteq \operatorname{Lat}\mathcal{A}_p$. Given a subspace $K \in \operatorname{Lat}\mathcal{A}_p$, as in Eq. (2.3), there are two cases to consider.

Suppose first that $K = M_\lambda M_{\varphi_s} H^2(\mathbb{R})$, where $\varphi_s(x) = e^{-isx^2/2}$, where $s \geq 0, \lambda \in \mathbb{R}$. Then $K \in \operatorname{Lat}\mathcal{A}_{ph}$ if and only if $V_t K \subseteq K$ for $t \geq 0$. Given $f \in H^2(\mathbb{R})$, we have

$$V_t(e^{-isx^2/2}e^{i\lambda x}f(x)) = e^{t/2}e^{-is(e^t x)^2/2}e^{i\lambda(e^t x)}f(e^t x) = e^{t/2}e^{-i(se^{2t})x^2/2}e^{i(\lambda e^t)x}f(e^t x).$$

Thus $V_t K \subseteq K$ if and only if $s = 0$ and $\lambda \geq 0$.

For the second case let $K = L^2[\alpha, +\infty)$, for $\alpha \in \mathbb{R}$. Then $V_t K \subseteq K$ if and only if $\alpha \leq 0$ and so the proof is complete. $\square$

Since $\mathcal{A}_{ph} \subseteq \operatorname{Alg}\mathcal{L}$ is evident, it suffices to prove the converse inclusion. Our strategy is once again to identify the Hilbert-Schmidt operators in these two algebras.

Given a function $k \in L^2(\mathbb{R}^2)$ let k_F, k_{F^*} and $V_t k$ denote the kernel functions of the integral operators $F \operatorname{Int} k F^*$, $F^* \operatorname{Int} k F$ and $V_t \operatorname{Int} k$ respectively. We now note that $k_F = JF_2 k$, where J is the flip operator, with $(Jf)(x,y) = f(x,-y)$, and F_2 is the two-dimensional Fourier transform

$$(F_2 f)(\xi,\omega) = \frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} f(x,y)e^{-i(x\xi+y\omega)}dxdy.$$

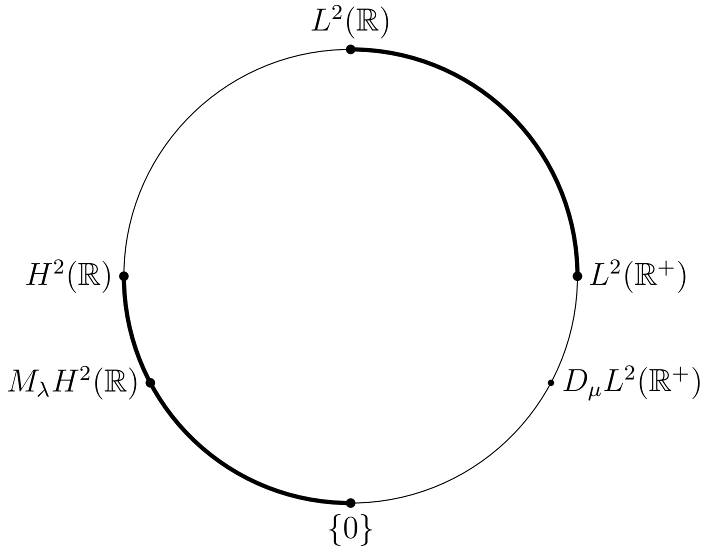

Fig. 3.1 The binest $\mathcal{L}$ shown (in bold lines) as a subset of the Fourier binest.

Indeed

$$\begin{aligned}(F\,\mathrm{Int}\,kF^*f)(x) &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}(\mathrm{Int}\,kF^*f)(y)e^{-ixy}dy \\ &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\left(\int_{\mathbb{R}}k(y,\omega)(F^*f)(\omega)d\omega\right)e^{-ixy}dy \\ &= \frac{1}{2\pi}\int_{\mathbb{R}}\left(\int_{\mathbb{R}}k(y,\omega)\left(\int_{\mathbb{R}}f(\xi)e^{i\omega\xi}d\xi\right)d\omega\right)e^{-ixy}dy \\ &= \frac{1}{2\pi}\int_{\mathbb{R}}\left(\int_{\mathbb{R}}\int_{\mathbb{R}}k(y,\omega)e^{-ixy}e^{i\omega\xi}dyd\omega\right)f(\xi)d\xi \\ &= \int_{\mathbb{R}}(F_2k)(x,-\xi)f(\xi)d\xi.\end{aligned}$$

The significance of the above observation is that we can make use of properties of the 2D Fourier transform, and especially the fact that it commutes with the rotation operators (see Theorem IV.1.1 in[71]). That is

$$F_2R_\theta = R_\theta F_2$$

where R_θ represents the operator of clockwise rotation, for $\theta > 0$, and $\theta \in [-\pi, \pi)$. Considering the rotation operators as acting on the space of the kernel functions we have the following reformulation of the characterization of the Hilbert-Schmidt operators of the parabolic algebra;

$$\mathcal{A}_p \cap \mathcal{C}_2 = \{\operatorname{Int} k \,:\, k \in R_{\pi/4}(L^2(\mathbb{R}^+) \otimes H^2(\mathbb{R}))\} = \{\operatorname{Int} k \,:\, k_F \in R_{\pi/4}(L^2(\mathbb{R}^+) \otimes \overline{H^2(\mathbb{R})})\}.$$

To prove this, one can check that $\operatorname{Int} k \in \mathcal{A}_p \cap \mathcal{C}_2$ if and only if $k(x,y)$ and $k_F(x,y) = 0$, for almost every $x < y$. Hence

$$\mathcal{A}_p \cap \mathcal{C}_2 = \{\operatorname{Int} k \,:\, k \in R_{\pi/4}(L^2(\mathbb{R}^+) \otimes L^2(\mathbb{R}))\} \cap \{\operatorname{Int} k \,:\, k_F \in R_{\pi/4}(L^2(\mathbb{R}^+) \otimes L^2(\mathbb{R}))\}.$$

By our previous observation

$$\begin{aligned} k \in R_{\pi/4}(L^2(\mathbb{R}^+) \otimes L^2(\mathbb{R})) &\Leftrightarrow JF_2 k \in JF_2 R_{\pi/4}(L^2(\mathbb{R}^+) \otimes L^2(\mathbb{R})) \\ &\Leftrightarrow k_F \in R_{-\pi/4}(\overline{H^2(\mathbb{R})} \otimes L^2(\mathbb{R})) \\ &\Leftrightarrow k_F \in R_{\pi/4}(L^2(\mathbb{R}) \otimes \overline{H^2(\mathbb{R})}), \end{aligned}$$

so our claim follows.

The convenience of the above characterization is apparent in the proof of the next lemma. Let $\mathcal{I}_0$ be the closure of the ideal generated by the finite rank operators of $\operatorname{Alg}\mathcal{L}$ with respect to the Hilbert-Schmidt norm.

Lemma 3.4.2. *$\mathcal{I}_0 \subseteq \mathcal{A}_{ph} \cap \mathcal{C}_2$.*

Proof. Let $\operatorname{Int} k$ lie in the ideal $\mathcal{I}_0$. It follows from Proposition 3.3.1 that $k \in L^2(\mathbb{R}^+) \otimes H^2(\mathbb{R})$ and so k_F is an element of $\overline{H^2(\mathbb{R})} \otimes L^2(\mathbb{R}^-)$. Without loss of generality we may assume that $k_F(x,y) = h(x)g(y)$, where $h \in \overline{H^2(\mathbb{R})}$, $g \in L^2(\mathbb{R}^-)$. Define for every

$t \geq 0$ the functions

$$h_t(x) = V_t h(x) = e^{t/2} h(e^t x) \qquad \text{and} \qquad g_t(y) = g(-y).$$

Consequently, each function $k_F^t(x, y) = h_t(x) g_t(x - y)$ lies in $R_{-\pi/4}(\overline{H^2(\mathbb{R})} \otimes L^2(\mathbb{R}^-))$. Since this space can be written as $R_{\pi/4}(L^2(\mathbb{R}^+) \otimes \overline{H^2(\mathbb{R})})$, it follows that $\operatorname{Int} k^t \in A_p \cap \mathcal{C}_2$ where $k^t = (k_F^t)_{F^*}$. Therefore, since $V_t \operatorname{Int} k = F^* V_{-t} \operatorname{Int} k_F F$, it suffices to show that $V_{-t} k_F^t$ converges in norm to k_F as $t \to \infty$.

$$\begin{aligned} V_{-t} k_F^t(x, y) &= e^{-t/2}\, k_F^t(e^{-t}x, y) = e^{-t/2}\, h_t(e^{-t}x)\, g_t(e^{-t}x - y) \\ &= e^{-t/2} e^{t/2} h(e^t e^{-t} x) g(y - e^{-t}x) e^{-t/2} = h(x) g(y - e^{-t}x) \to h(x) g(y), \end{aligned}$$

as $t \to +\infty$. Assume now that g is continuous and let $\epsilon > 0$. Then

$$\begin{aligned} &\int_{\mathbb{R}} \left(\int_{\mathbb{R}} \left| V_{-t} k_F^t(x, y) - k_F(x, y) \right|^2 dy \right) dx = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \left| h(x) \left(g(y - e^{-t}x) - g(y) \right) \right|^2 dy \right) dx = \\ &= \int_{\mathbb{R} \setminus K} |h(x)|^2 \left(\int_{\mathbb{R}} \left| g(y - e^{-t}x) - g(y) \right|^2 dy \right) dx + \int_K |h(x)|^2 \left(\int_{\mathbb{R}} \left| g(y - e^{-t}x) - g(y) \right|^2 dy \right) dx \end{aligned}$$

with K compact subset of $\mathbb{R}$, such that

$$\int_{\mathbb{R} \setminus K} |h(x)|^2 dx \leq \frac{\epsilon}{8\|g\|_2^2}.$$

Hence

$$\int_{\mathbb{R} \setminus K} |h(x)|^2 \left(\int_{\mathbb{R}} \left| g(y - e^{-t}x) - g(y) \right|^2 dy \right) dx \leq 4 \int_{\mathbb{R} \setminus K} |h(x)|^2 \|g\|_2^2 dx \leq \frac{\epsilon}{2}.$$

On the other hand, choose C compact, such that

$$\int_{\mathbb{R} \setminus C} \left| g(y - e^{-t}x) - g(y) \right|^2 dy \leq \frac{\epsilon}{4\|h\|_2^2}$$

for every $x \in K$. In addition, g is uniformly continuous on C, so there exists positive t_0, such that for every $t \geq t_0$ we get

$$|g(y - e^{-t}x) - g(y)|^2 \leq \frac{\epsilon}{4|C| \, \|h\|_2^2},$$

where $|C|$ is the Lebesgue measure of C. Then

$$\begin{aligned}
&\int_K |h(x)|^2 \left(\int_{\mathbb{R}} \left| g(y - e^{-t}x) - g(y) \right|^2 dy \right) dx = \\
&= \int_K |h(x)|^2 \left(\int_{\mathbb{R}\setminus C} \left| g(y - e^{-t}x) - g(y) \right|^2 dy + \int_C \left| g(y - e^{-t}x) - g(y) \right|^2 dy \right) dx \leq \\
&\leq \frac{\epsilon}{4\|h\|_2^2} \int_K |h(x)|^2 dx + \int_K |h(x)|^2 \left(\int_C \frac{\epsilon}{4|C| \, \|h\|_2^2} dy \right) dx = \frac{\epsilon}{4} + \frac{\epsilon}{4} = \frac{\epsilon}{2}.
\end{aligned}$$

Thus

$$\|V_{-t} k_F^t - k_F\|_2 \to 0 \quad \text{as} \quad t \to \infty.$$

The general case is straightforward from the density of continuous functions in L^2. Therefore $V_{-t} \operatorname{Int} k_F^t$ converges to $\operatorname{Int} k_F$ and hence $\operatorname{Int} k \in \mathcal{A}_{ph} \cap \mathcal{C}_2$. □

The next lemma is crucial for the proof of the reflexivity of the triple semigroup algebra and also yields the two-variable variant of the Paley-Wiener theorem given in Corollary 3.4.4.

Given $\theta_0 \in [0, \pi)$, let

$$\begin{aligned}
Q_1^{\theta_0} &= \left\{ (x, y) \in \mathbb{R}^2 : \arctan(y/x) \in \left[-\frac{\pi}{2} - \theta_0, \frac{\pi}{2} \right] \right\} \\
Q_2^{\theta_0} &= \left\{ (x, y) \in \mathbb{R}^2 : \arctan(y/x) \in [-\pi, \theta_0] \right\}.
\end{aligned}$$

Define also the set

$$\mathcal{K}_{\theta_0} = \{k \in L^2(\mathbb{R}^2) : \operatorname{esssupp} k \subseteq Q_1^{\theta_0}\} \cap \{k \in L^2(\mathbb{R}^2) : \operatorname{esssupp} k_F \subseteq Q_2^{\theta_0}\}$$

(see Figure 3.2) and the set

$$\mathbb{S}_{\theta_0} = \overline{span\{R_\theta(L^2(\mathbb{R}^+) \otimes H^2(\mathbb{R})),\, \theta \in \{0, \theta_0\}\}}^{\|\cdot\|}.$$

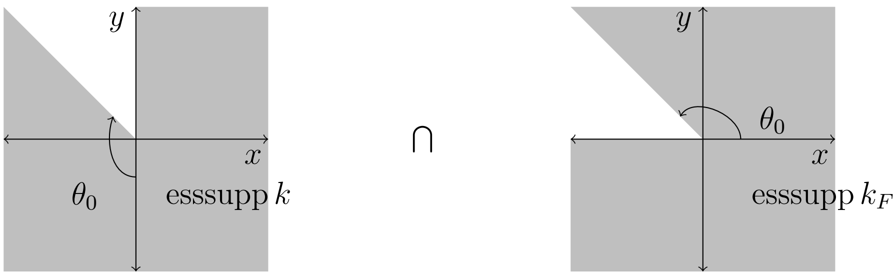

Fig. 3.2 A function $k \in L^2(\mathbb{R}^2)$ is an element of $\mathcal{K}_{\theta_0}$, if and only if both esssupp k and esssupp k_F lie in the respective shaded areas.

Lemma 3.4.3. $\mathcal{K}_{\theta_0} = \mathbb{S}_{\theta_0}$, *for every* $\theta_0 \in [0, \pi)$.

Proof. Let $k \in R_\theta(L^2(\mathbb{R}^+) \otimes H^2(\mathbb{R}))$, with $\theta \in \{0, \theta_0\}$. Expressing the essential support of k in polar coordinates, it is just routine to show that esssupp$k \subseteq Q_1^{\theta_0}$. Also the function k_F lies in the space $JF_2R_\theta(L^2(\mathbb{R}^+) \otimes H^2(\mathbb{R}))$, which can be written as

$$JF_2R_\theta(L^2(\mathbb{R}^+) \otimes H^2(\mathbb{R})) = R_{-\theta}JF_2(L^2(\mathbb{R}^+) \otimes H^2(\mathbb{R})) = R_{-\theta}(\overline{H^2(\mathbb{R})} \otimes L^2(\mathbb{R}^-)).$$

Hence esssupp $k_F \subseteq Q_2^{\theta_0}$, and so it follows that $\mathbb{S}_{\theta_0} \subseteq \mathcal{K}_{\theta_0}$.

To prove the other inclusion, assume that there is a function $k \in \mathcal{K}_{\theta_0} \cap \mathbb{S}_{\theta_0}^{\perp}$. Then the Hilbert space geometry of $L^2(\mathbb{R}^2)$ ensures that

$$\|k + k_S\| > \|k\|, \forall\, k_S \in \mathbb{S}_{\theta_0} \backslash \{0\}. \tag{3.1}$$

Define now the orthogonal projections $P_\theta = proj(R_{-\theta}(L^2(\mathbb{R})\otimes L^2(\mathbb{R}^-))$, $\theta \in \{0, \theta_0, \pi/2\}$. Noting that $P_{\pi/2} = proj(L^2(\mathbb{R}^+) \otimes L^2(\mathbb{R}))$, decompose k as the sum of two orthogonal parts,

$$k = P_{\pi/2}\, k + P_{\pi/2}^{\perp}\, k$$

where $P_{\pi/2}^{\perp} = I - P_{\pi/2}$. Applying to both sides the operator JF_2, we have

$$k_F = (P_{\pi/2}\, k)_F + (P_{\pi/2}^{\perp}\, k)_F.$$

Consider now the representation

$$k_F = P_0(P_{\pi/2}\, k)_F + P_0^{\perp}(P_{\pi/2}\, k)_F + P_{\theta_0}(P_{\pi/2}^{\perp}\, k)_F + P_{\theta_0}^{\perp}(P_{\pi/2}^{\perp}\, k)_F. \tag{3.2}$$

Since $P_0(P_{\pi/2}\, k)_F \in \overline{H^2(\mathbb{R})}\otimes L^2(\mathbb{R}^-)$ which is the space $JF_2(L^2(\mathbb{R}^+)\otimes H^2(\mathbb{R}))$, it follows that $(P_0(P_{\pi/2}\, k)_F)_{F^*}$ lies in $\mathbb{S}_{\theta_0}$. Similarly, taking into account that $k \in L^2(Q_1^{\theta_0})$, we have $P_{\pi/2}^{\perp}\, k \in R_{\theta_0}(L^2(\mathbb{R}^+)\otimes L^2(\mathbb{R}))$, which implies that $(P_{\pi/2}^{\perp}\, k)_F$ lies in $R_{-\theta_0}(\overline{H^2(\mathbb{R})}\otimes L^2(\mathbb{R}))$. Therefore,

$$P_{\theta_0}(P_{\pi/2}^{\perp}\, k)_F \in R_{-\theta_0}(\overline{H^2(\mathbb{R})} \otimes L^2(\mathbb{R}^-))$$

and so $(P_{\theta_0}(P_{\pi/2}^{\perp}\, k)_F)_{F^*}$ lies in $\mathbb{S}_{\theta_0}$.

It follows then, subtracting these operators, that $\operatorname{Int} k'$ is an operator in $\mathcal{K}_{\theta_0}$ where

$$k' = (P_0^{\perp}(P_{\pi/2}\, k)_F)_{F^*} + (P_{\theta_0}^{\perp}(P_{\pi/2}^{\perp}\, k)_F)_{F^*}$$

and

$$k'_F = P_0^\perp (P_{\pi/2}\, k)_F + P_{\theta_0}^\perp (P_{\pi/2}^\perp\, k)_F.$$

The first function in the sum for k'_F is in $\overline{H^2(\mathbb{R})} \otimes L^2(\mathbb{R}^+)$, and is supported in the upper half plane, while the second function is supported in the half plane $y \leq x \tan\theta_0$. However, we also have $k'_F \in L^2(Q_2^{\theta_0})$ and so it follows that the component functions for k'_F have disjoint essential supports, as indicated in Figure 3.3. This figure also depicts the two forms of the semi-infinite lines on which (almost every) restriction of k'_F agrees with the restriction of a function in $\overline{H^2(\mathbb{R})}$. (This local co-analyticity follows from the observations following the identity 3.2.)

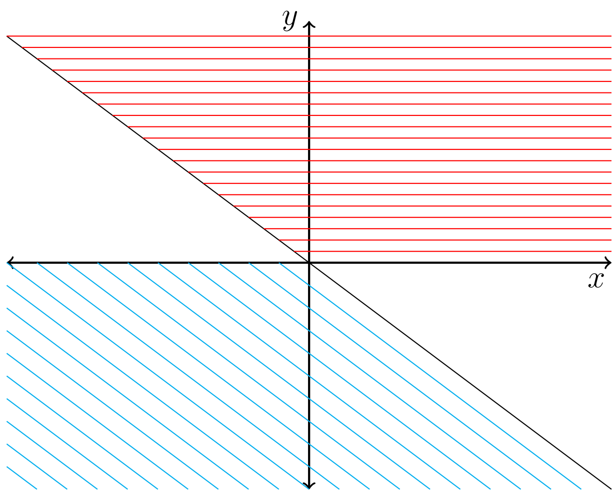

Fig. 3.3 The essential support of k'_F.

It now follows from these essential support observations that

$$k'_F = P_{\theta_0} P_0^\perp (P_{\pi/2}\, k)_F + P_{\theta_0}^\perp P_0 (P_{\pi/2}^\perp\, k)_F$$

and hence that

$$\|k_F'\|^2 = \|P_{\theta_0} P_0^\perp (P_{\pi/2}\, k)_F\|^2 + \|P_{\theta_0}^\perp P_0 (P_{\pi/2}^\perp\, k)_F\|^2.$$

Comparing norms we have

$$\begin{aligned}
\|k\|^2 &= \|P_{\pi/2}\, k\|^2 + \|P_{\pi/2}^\perp\, k\|^2 = \|(P_{\pi/2}\, k)_F\|^2 + \|(P_{\pi/2}^\perp\, k)_F\|^2 \\
&= \|P_{\theta_0} P_0^\perp (P_{\pi/2}\, k)_F\|^2 + \|(P_{\theta_0} P_0^\perp)^\perp (P_{\pi/2}\, k)_F\|^2 + \|P_{\theta_0}^\perp P_0 (P_{\pi/2}^\perp\, k)_F\|^2 + \|(P_{\theta_0}^\perp P_0)^\perp (P_{\pi/2}^\perp\, k)_F\|^2 \\
&= \|k_F'\|^2 + \|(P_{\theta_0} P_0^\perp)^\perp (P_{\pi/2}\, k)_F\|^2 + \|(P_{\theta_0}^\perp P_0)^\perp (P_{\pi/2}^\perp\, k)_F\|^2
\end{aligned}$$

and so $\|k\| \geq \|k_F'\| = \|k'\|$. Since k has been chosen extremally the inequality (3.1) now implies that $\|k\| = \|k'\|$ and so

$$(P_{\theta_0} P_0^\perp)^\perp (P_{\pi/2}\, k)_F = (P_{\theta_0} P_0^\perp)^\perp (P_{\pi/2}^\perp\, k)_F = 0.$$

But $(P_{\pi/2}\, k)_F \in \overline{H^2(\mathbb{R})} \otimes L^2(\mathbb{R})$ and $(P_{\pi/2}^\perp\, k)_F \in R_{-\theta_0}(\overline{H^2(\mathbb{R})} \otimes L^2(\mathbb{R}))$ and so both functions are equal to zero, since every $\overline{H^2(\mathbb{R})}$-slice is zero on a non-null interval. Consequently $k = 0$ and this completes the proof. □

Corollary 3.4.4. *Let $0 < \alpha < \pi/2$ and let C_α be the proper cone of points (x, y) with $x \geq 0$ and $|\arctan y/x| < \alpha$. Then the following conditions are equivalent for a function $k \in L^2(\mathbb{R}^2)$.*

(i) k vanishes on C_α and $F_2 k$ vanishes on $R_{-\pi/2} C_\alpha$.

(ii) k lies in the closed linear span of $R_{\alpha/2}(H^2(\mathbb{R}) \otimes L^2(\mathbb{R}^-))$ and $R_{-\alpha/2}(\overline{H^2(\mathbb{R})} \otimes L^2(\mathbb{R}^+))$.

Our next goal is to make use the previous lemma to show that

$$\operatorname{Alg} \mathcal{L} \cap \mathcal{C}_2 \;=\; \overline{(\mathcal{A}_p \cap \mathcal{C}_2) + \mathcal{I}_0}^{\|\cdot\|_2}.$$

First, we determine the Hilbert-Schmidt operators of $\operatorname{Alg}\mathcal{L}$.

Lemma 3.4.5. $\{k : \operatorname{Int} k \in \operatorname{Alg}\mathcal{L} \cap \mathcal{C}_2\} \subseteq \mathcal{K}_{\pi/4}$.

Proof. Suppose first that $k \in L^2(\mathbb{R}^2)$ is a kernel function such that $\operatorname{Int} k L^2[\lambda, +\infty)$ is a subspace of $L^2[\lambda, +\infty)$, for every $\lambda \leq 0$. Let $x < \lambda < 0$, and take $f \in L^2(\lambda, +\infty)$. Then

$$\int_{\mathbb{R}} k(x,y) f(y) dy = (\operatorname{Int} k f)(x) = 0.$$

Thus $k(x, y) = 0$ for almost for every $y > \lambda$ and $\operatorname{esssupp} k \subseteq Q_1^{\pi/4}$.

Suppose next that $k \in L^2(\mathbb{R}^2)$ and $\operatorname{Int} k M_\lambda H^2(\mathbb{R}) \subseteq M_\lambda H^2(\mathbb{R})$ for every $\lambda \geq 0$. Then the following equivalent inclusions hold for all $\lambda > 0$.

$$\begin{aligned}
&\operatorname{Int} k M_\lambda H^2(\mathbb{R}) \subseteq M_\lambda H^2(\mathbb{R}),\\
&F \operatorname{Int} k F^* F M_\lambda H^2(\mathbb{R}) \subseteq F M_\lambda H^2(\mathbb{R}),\\
&F \operatorname{Int} k F^* D_\lambda L^2(\mathbb{R}^+) \subseteq D_\lambda L^2(\mathbb{R}^+),\\
&F \operatorname{Int} k F^* L^2[\lambda, +\infty) \subseteq L^2[\lambda, +\infty).
\end{aligned}$$

Thus $\operatorname{Int} k_F L^2[\lambda, +\infty) \subseteq L^2[\lambda, +\infty)$, for every $\lambda \geq 0$. Given $x < 0$ and $f \in L^2(\mathbb{R}^+)$ we have

$$\int_{\mathbb{R}} k_F(x,y) f(y) dy = (\operatorname{Int} k_F f)(x) = 0$$

and so it follows that $k_F(x, y) = 0$ for almost for every $y > 0$. Also, for $x \geq 0$ and $f \in L^2[\lambda, +\infty)$ with $\lambda > x$, we again have $(\operatorname{Int} k_F f)(x) = 0$ and so $\operatorname{esssupp} k_F \subseteq Q_2^{\pi/4}$, as required. □

Lemma 3.4.6. *The algebras $\mathcal{A}_{ph} \cap \mathcal{C}_2$ and $\operatorname{Alg}\mathcal{L} \cap \mathcal{C}_2$ coincide.*

Proof. By the previous lemma and Lemma 3.4.3, we have $\{k : \operatorname{Int} k \in \operatorname{Alg} \mathcal{L} \cap \mathcal{C}_2\} \subseteq \mathbb{S}_{\pi/4}$, where

$$\mathbb{S}_{\pi/4} = \overline{R_{\pi/4}(L^2(\mathbb{R}^+) \otimes H^2(\mathbb{R})) + L^2(\mathbb{R}^+) \otimes H^2(\mathbb{R})}^{\|\cdot\|}$$

Thus, noting the form of the kernels for Hilbert-Schmidt operators in $\mathcal{A}_p$ (given prior to Lemma 5.2) and the form for operators in $\mathcal{I}_0$ it follows that

$$\mathbb{S}_{\pi/4} = \{k : \operatorname{Int} k \in \overline{(\mathcal{A}_p \cap \mathcal{C}_2) + \mathcal{I}_0}^{\|\cdot\|_2}\}.$$

Applying Lemma 3.4.2, the desired inclusion follows. □

We have noted in Proposition 1.3.3 that $\mathcal{A}_p \cap \mathcal{C}_2$ contains an operator norm bounded sequence which is an approximate identity for the space of all Hilbert-Schmidt operators. Since this sequence also lies in $\mathcal{A}_{ph}$ it follows from the previous lemma that the weak* closures of $\mathcal{A}_{ph} \cap \mathcal{C}_2$ and $\operatorname{Alg} \mathcal{L} \cap \mathcal{C}_2$ coincide. Thus, the following theorem is proved.

Theorem 3.4.7. *The operator algebra $\mathcal{A}_{ph}$ is reflexive, with $\mathcal{A}_{ph} = \operatorname{Alg} \mathcal{L} = \overline{\mathcal{A}_p + \mathcal{I}}^{w^*}$.*

3.5 The unitary automorphism group of $\mathcal{A}_{ph}$

In the case of the parabolic algebra the group of unitary automorphisms, $X \to \operatorname{Ad} U(X) = UXU^*$, was identified in [38] as the 3-dimensional Lie group of automorphisms $\operatorname{Ad}(M_\lambda D_\mu V_t)$ for λ, μ and t in $\mathbb{R}$. The following theorem shows that the larger algebra $\mathcal{A}_{ph}$ is similarly rigid.

Theorem 3.5.1. *The unitary automorphism group of A_{ph} is isomorphic to $\mathbb{R}$ and equal to $\{\operatorname{Ad}(V_t) : t \in \mathbb{R}\}$.*

Proof. Let $\mathrm{Ad}(U)$ be a unitary automorphism of A_{ph}. Since $\mathcal{A}_{ph} = \mathrm{Alg}\,\mathcal{L}$ it follows from Lemma 3.4.1 and the asymmetric order structure of the binest $\mathcal{L}$ that

$$UH^2(\mathbb{R}) = H^2(\mathbb{R}), \quad UM_\lambda H^2(\mathbb{R}) = M_\mu H^2(\mathbb{R}) \tag{3.3}$$

where $\mu \geq 0$ depends on $\lambda \geq 0$ and $\mu : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous bijection. Also

$$UL^2(\mathbb{R}^-) = L^2(\mathbb{R}^-), \quad UL^2(-\lambda', 0) = L^2(-\mu', 0) \tag{3.4}$$

with $\mu' : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous bijection.

Note that the subspaces $L^2(-\lambda, \infty)$ of $L^2(\mathbb{R}^-)$ form a continuous nest of multiplicity one and so it follows from (3.4) and elementary nest algebra theory (see Davidson [15] for example) that the unitary operator U has the form $U = M_\psi C_f \oplus U_1$, where ψ is a unimodular function in $L^\infty(\mathbb{R}^-)$, $f : \mathbb{R}^- \to \mathbb{R}^-$ is a strictly increasing bijection, and C_f is the unitary composition operator on $L^2(\mathbb{R}^-)$ with

$$(C_f g)(x) = (f'(x))^{1/2} g(f(x)).$$

Let $h \in L^2(\mathbb{R})$. Then for $x \in \mathbb{R}^-$ we have

$$(UM_\lambda h)(x) = (\psi C_f M_\lambda h)(x) = \psi(x) e^{i\lambda f(x)} (f'(x))^{1/2} h(f(x)) = e^{i\lambda f(x)} (Uh)(x).$$

Take $c \in \mathbb{C}^+$ and let $h_c \in H^2(\mathbb{R})$ be the function for which $(Uh_c)(x) = \frac{1}{x+c}$. Then

$$(UM_\lambda h_c)(x) = e^{i\lambda f(x)} \frac{1}{x+c}$$

and so

$$(x+c) g_{\lambda,c}(x) = e^{i\lambda f(x)},$$

where $g_{\lambda,c} = UM_\lambda h_c \in H^2(\mathbb{R})$. By Lemma 3.2.1 the functions $(x+c)g_{\lambda,c}(x)$ are independent of c and agree for all real x. Thus there is a unique extension of $e^{i\lambda f(x)}$ to $\mathbb{R}$, say $\varphi_\lambda(x)$, such that

$$\varphi_\lambda(x) = e^{i\lambda f(x)}, \ \text{for almost every } x \in \mathbb{R}^-$$

and

$$\varphi_\lambda(x) = (x+c)g_{\lambda,c}(x), \ \text{for almost every } x \in \mathbb{R}.$$

It now follows that

$$UM_\lambda h_c = M_{\varphi_\lambda} Uh_c.$$

Since the closed linear span of the functions $h_c = U^* \frac{1}{x+c}$, $c \in \mathbb{C}^+$, is equal to $H^2(\mathbb{R})$, we obtain

$$UM_\lambda h = M_{\varphi_\lambda} Uh. \tag{3.5}$$

for every $h \in H^2(\mathbb{R})$. On the other hand, we have shown that the equation (3.5) also holds for $h \in L^2(\mathbb{R}^-)$. So it follows from the density of $H^2(\mathbb{R}) + L^2(\mathbb{R}^-)$ that $UM_\lambda = M_{\varphi_\lambda} U$. Hence φ_λ is inner. Now (3.3) implies that

$$M_\mu H^2(\mathbb{R}) = M_{\varphi_\lambda} H^2(\mathbb{R}).$$

Therefore, $\varphi_\lambda(x)/e^{i\mu x}$ is equal to a unimodular constant $c_\lambda = e^{i\alpha_\lambda}$ depending on λ. Thus, for every $x \in \mathbb{R}^-$, we have

$$i\lambda f(x) - i\mu x = i\alpha_\lambda$$

or equivalently

$$f(x) = \frac{\mu}{\lambda}x + \frac{\alpha_\lambda}{\lambda}.$$

It follows that $\alpha_\lambda = 0$, since $f(0) = 0$, and that $\mu = \beta\lambda$ for some positive constant β. Thus, for $x < 0$,

$$(C_f h)(x) = \beta^{1/2}h(\beta x) = (V_{\log\beta}h)(x).$$

Writing $t = \log\beta$, we have $Uh = \psi V_t h + U_1 h$, and so with $h(x) = \frac{1}{x+d}$ and $x < 0$ we have $(Uh)(x) = \psi(x)(V_t h)(x)$ and

$$\frac{e^t x + d}{e^{t/2}}(Uh)(x) = \psi(x).$$

By Lemma 3.2.1 again, $\frac{e^t x+d}{e^{t/2}}Uh$ is determined by ψ and there is analytic function φ such that

$$\frac{e^t x + d}{e^{t/2}}Uh = \varphi.$$

We conclude that $Uh = \varphi V_t h$ for all such h and so φ is unimodular. Since $UH^2(\mathbb{R}) = H^2(\mathbb{R})$ it follows that almost everywhere φ is a unimodular constant, η say. Thus $U = \eta V_t$ and the proof is complete. $\square$

Remark 3.5.2. Note that the binest $\mathcal{L}_{\alpha,\beta}$ given by

$$\mathcal{L}_{\alpha,\beta} = \{0\} \cup \{L^2(\alpha', \infty), \alpha' \le \alpha\} \cup \{e^{i\beta' x}H^2(\mathbb{R}), \beta' \ge \beta\} \cup \{L^2(\mathbb{R})\}$$

is equal to $D_\alpha M_\beta \mathcal{L}$. Thus $\mathcal{L}_{\alpha,\beta}$ is unitarily equivalent to $\mathcal{L}$. Also the unitary operator $U = D_\alpha M_\beta$ provides a unitary isomorphism $\operatorname{Ad} U : \operatorname{Alg}\mathcal{L} \to \operatorname{Alg}\mathcal{L}_{\alpha,\beta}$ between their reflexive algebras.

3.6 Further binests

Once again, write $\mathcal{N}_v^-$ and $\mathcal{N}_a^+$ for the subnests of $\mathcal{N}_v$ and $\mathcal{N}_a$ whose union is $\mathcal{L}$. Also let $\mathcal{N}_v^+, \mathcal{N}_a^-$ be the analogous subnests of $\mathcal{N}_v$ and $\mathcal{N}_a$ for which $P_- = (I - P_+)$ is the atomic interval projection for $\mathcal{N}_v^+$ and Q_+ is the atomic interval projection for $\mathcal{N}_a^-$.

By the F. and M. Riesz theorem the orbit of $H^2(\mathbb{R})$ under the Fourier-Plancherel transform F is the subspace $H^2(\mathbb{R})$ together with the three subspaces

$$FH^2(\mathbb{R}) = L^2(\mathbb{R}^+), \quad F^2H^2(\mathbb{R}) = \overline{H^2(\mathbb{R})}, \quad F^3H^2(\mathbb{R}) = L^2(\mathbb{R}^-).$$

More generally, the lattice $\operatorname{Lat}\mathcal{A}_p$, with the weak operator topology for subspace projections, forms one quarter of the Fourier-Plancherel sphere, and the Fourier-Plancherel transform F effects a period 4 rotation of this sphere. (see [45])

We now note that there are 8 binest lattices which are pairwise order isomorphic as lattices and which have a similar status to $\mathcal{L} = \mathcal{N}_a^+ \cup \mathcal{N}_v^-$. These fall naturally into two groupings of 4. Write J for the unitary operator F^2, so that $Jf(x) = f(-x)$. (There will be no conflict here with notation from the previous section.) Writing $\overline{K}$ for $\{\overline{f} : f \in K\}$, these groupings are

$$\mathcal{N}_a^+ \cup \mathcal{N}_v^-, \quad \mathcal{N}_v^+ \cup \overline{\mathcal{N}_a^-}, \quad \overline{\mathcal{N}_a^+} \cup J\mathcal{N}_v^-, \quad J\mathcal{N}_v^+ \cup \mathcal{N}_a^-$$

and

$$\mathcal{N}_a^- \cup \mathcal{N}_v^+, \quad \mathcal{N}_v^- \cup \overline{\mathcal{N}_a^+}, \quad \overline{\mathcal{N}_a^-} \cup J\mathcal{N}_v^+, \quad J\mathcal{N}_v^- \cup \mathcal{N}_a^+$$

forming the orbits of the subspace lattices $\mathcal{N}_a^+ \cup \mathcal{N}_v^-$ and $\mathcal{N}_a^- \cup \mathcal{N}_v^+$ under F. Note that the symbols "+" and "−" indicate the "upper" and "lower" choices for the atomic interval of the nest. Since F induces an order isomorphism of the lattices, F respects these symbols. By Theorem 3.4.7 and the identities

$$FM_\lambda F^* = D_\lambda, \quad FD_\mu F^* = M_{-\mu}, \quad FV_t F^* = V_{-t}$$

it follows that the binest algebras for these 8 binests are (respectively) equal to weak*-closed operator algebras for the following generating semigroup choices for $\{M_\lambda\}, \{D_\mu\}$ and $\{V_t\}$:

$$+++ \quad -+- \quad --+ \quad +--$$

$$++- \quad -++ \quad --- \quad +-+$$

View the lattice $\mathcal{L} = \mathcal{N}_a^+ \cup \mathcal{N}_v^-$ as the right-handed choice in Figure 3.1, write $\mathcal{L}_r$ for $\mathcal{L}$, and view $\mathcal{L}_l = \mathcal{N}_a^- \cup \mathcal{N}_v^+$ as the left-handed choice. From the observations above the 8 binests determine either 1 or 2 unitary equivalence classes of triple semigroup algebras. In fact there are two classes.

Theorem 3.6.1. *The triple semigroup algebra* $\mathcal{A}_{ph} = \operatorname{Alg}\mathcal{L}_r$ *is not unitarily equivalent to triple semigroup algebra* $\mathcal{A}_{ph}^* = \operatorname{Alg}\mathcal{L}_l$

Proof. By Theorem 3.4.7, $\mathcal{A}_{ph}^* = (\operatorname{Alg}(\mathcal{N}_a^+ \cup \mathcal{N}_v^-))^*$ which is the binest algebra for the union of the nests $(\mathcal{N}_a^+)^\perp$ and $(\mathcal{N}_v^-)^\perp$. We have

$$(\mathcal{N}_a^+)^\perp = J\mathcal{N}_a^-, \quad (\mathcal{N}_v^-)^\perp = J\mathcal{N}_v^+$$

and so it suffices to show that the binests

$$\mathcal{N}_a^+ \cup \mathcal{N}_v^-, \quad \mathcal{N}_a^- \cup \mathcal{N}_v^+$$

are not unitarily equivalent.

Suppose, by way of contradiction, that for some unitary U the binest $U(\mathcal{N}_a^+ \cup \mathcal{N}_v^-)$ coincides with $\mathcal{N}_a^- \cup \mathcal{N}_v^+$. Then

$$FU(\mathcal{N}_a^+ \cup \mathcal{N}_v^-) = F(\mathcal{N}_a^- \cup \mathcal{N}_v^+) = \mathcal{N}_v^- \cup \overline{\mathcal{N}_a^+}.$$

We have $\mathcal{N}_v^- = \{L^2(\lambda, \infty), \lambda \leq 0\}$ and so by elementary nest algebra theory, as in the proof of Theorem 3.5.1,

$$FU = M_\psi C_f \oplus X$$

for some unimodular function ψ on $\mathbb{R}^-$ and a composition operator C_f on $L^2(\mathbb{R}^-)$ associated with a continuous bijection f.

We have

$$FU : e^{i\lambda x} H^2(\mathbb{R}) \to e^{-i\mu x}\overline{H^2(\mathbb{R})}$$

with $\mu = \mu(\lambda) : \mathbb{R}^+ \to \mathbb{R}^+$ a bijection.

Take $h_c \in H^2(\mathbb{R})$ such that $FUh_c = \frac{1}{x-c} \in \overline{H^2(\mathbb{R})}$, with $c \in \mathbb{C}^+$. Then, for $x < 0, \lambda > 0$,

$$\begin{aligned}(FUM_\lambda h_c)(x) &= (M_\psi C_f M_\lambda h_c)(x),\\ (FUM_\lambda h_c)(x) &= (e^{i\lambda f(x)} M_\psi C_f h_c)(x),\\ (FUM_\lambda h_c)(x) &= e^{i\lambda f(x)}(FUh_c)(x),\\ g_{\lambda,c}(x) &= e^{i\lambda f(x)}\frac{1}{x-c},\end{aligned}$$

where $g_{\lambda,c} = FUM_\lambda h_c \in \overline{H^2(\mathbb{R})}$. We may apply Lemma 3.2.1 as in the proof of Theorem 3.5.1 (although to $\overline{H^2(\mathbb{R})}$ functions here) to deduce that

$$FUM_\lambda = M_{\varphi_\lambda} FU,$$

where φ_λ is the unique extension of $e^{i\lambda f(x)}$ to the real line. Hence φ_λ is a unimodular function that satisfies

$$M_{\varphi_\lambda}\overline{H^2(\mathbb{R})} = M_{-\mu}\overline{H^2(\mathbb{R})}.$$

This yields that $i\lambda f(x) + i\mu x = 0$ for all $x \in \mathbb{R}^-$, so it follows that $\mu = -\frac{f(x)}{x}\lambda$. This is a contradiction, as desired, since μ is an increasing function. $\square$

The fact that $\mathcal{A}_{ph} = \operatorname{Alg}\mathcal{L}_r$ and $\mathcal{A}_{ph}^* = \operatorname{Alg}\mathcal{L}_l$ fail to be unitarily equivalent expresses the following *chirality* property.

Definition 3.6.2. We say that a reflexive operator algebra $\mathcal{A}$ is **chiral** if

(i) $\mathcal{A}$ and $\mathcal{A}^*$ are not unitarily equivalent, and

(ii) $\operatorname{Lat}\mathcal{A}$ and $\operatorname{Lat}\mathcal{A}^*$ are *spectrally equivalent* in the sense that there is an order isomorphism $\theta : \operatorname{Lat}\mathcal{A} \to \operatorname{Lat}\mathcal{A}^*$ such that for each pair of interval projections $\{P_1 - P_2, Q_1 - Q_2\}$ for $\operatorname{Lat}\mathcal{A}$ the projection pairs

$$\{P_1 - P_2, Q_1 - Q_2\}, \quad \{\theta(P_1) - \theta(P_2), \theta(Q_1) - \theta(Q_2)\}$$

are unitarily equivalent.

While the spectral invariants for a pair of projections are well-known (Halmos [26]) there is presently no analogous classification of binests.

Chapter 4

Norm closed semigroup algebras

4.1 Discrete Crossed Products

Crossed products of C*-algebras were introduced by Murray and von Neumann as a tool for studying groups that act on C*-algebras as automorphisms, since they provide a larger algebra that encodes both the original C*-algebra and the group action. The reader may look for more details in [13, 16, 59, 75]. In this thesis, we will restrict our attention to discrete crossed products, where G is a discrete **abelian** group.

Definition 4.1.1. A **C*-dynamical system** is a triple $(\mathcal{A}, G, \alpha)$ that consists of a unital C*-algebra $\mathcal{A}$, a discrete abelian group G and a homomorphism

$$\alpha : G \to \mathrm{Aut}(\mathcal{A}) : s \mapsto \alpha_s.$$

Given a C*-dynamical system, a **covariant representation** is a pair (π, U), such that π is a representation of $\mathcal{A}$ on some Hilbert space H and $U : s \mapsto U_s$ is a **unitary representation** of G on the same space, that also satisfies the formula

$$U_s \pi(A) U_s^* = \pi(\alpha_s(A)), \, \forall A \in \mathcal{A}, s \in G.$$

We form the complex vector space $\mathcal{A}G$ of finitely supported $\mathcal{A}$-valued functions of G:

$$\mathcal{A}G = \mathrm{span}\{\delta_s \otimes A : s \in G,\ A \in \mathcal{A}\},\ \text{where } \delta_s(t) = \begin{cases} 1, & \text{if } t = s \\ 0, & \text{if } t \neq s \end{cases}$$

and endow it with ring multiplication and involution given by

$$(\delta_s \otimes A) \cdot (\delta_t \otimes B) = (\delta_{s+t} \otimes A\alpha_s(B))$$
$$(\delta_s \otimes A)^* = (\delta_{-s} \otimes \alpha_{-s}(A^*))$$

respectively. The algebra $\mathcal{A}G$ becomes a normed $*$-algebra with the norm:

$$\Big\| \sum_{\substack{s\in F \\ F\subset\subset G}} (\delta_s \otimes A_s) \Big\|_{\ell_1} = \sum_{\substack{s\in F \\ F\subset\subset G}} \|A_s\|,$$

where the notation $F \subset\subset G$ means that F is a finite subset of G. The elements $\sum_{\substack{s\in F \\ F\subset\subset G}} (\delta_s \otimes A_s)$ will be called **(generalized) trigonometric polynomials**.

Each (π, U) covariant representation induces a $*$-homomorphism on $\mathcal{A}G$, since the linear map $\pi \rtimes U$ in $\mathcal{A}G$ with

$$(\pi \rtimes U) \left(\sum_{\substack{s\in F \\ F\subset\subset G}} (\delta_s \otimes A_s) \right) = \sum_{\substack{s\in F \\ F\subset\subset G}} \pi(A_s) U_s$$

is bounded:

$$\left\| (\pi \rtimes U) \left(\sum_{\substack{s\in F \\ F\subset\subset G}} (\delta_s \otimes A_s) \right) \right\| \leq \sum_{\substack{s\in F \\ F\subset\subset G}} \|\pi(A_s)\| \leq \sum_{\substack{s\in F \\ F\subset\subset G}} \|(A_s)\| = \Big\| \sum_{\substack{s\in F \\ F\subset\subset G}} (\delta_s \otimes A_s) \Big\|_{\ell_1}.$$

We define the C^*-algebra $\mathcal{A} \times_\alpha G$ as the completion of $\mathcal{A}G$ with respect to the norm

$$\|F\| := \sup\{\|(\pi \rtimes U)(F)\| \,:\, (\pi, U) \text{ covariant representation of } \mathcal{A}G\}.$$

Observe that $\mathcal{A} \times_\alpha G$ satisfies the universal property :

If (π, U) is a covariant representation of the dynamical system $(\mathcal{A}, G, \alpha)$, then there is a representation $\tilde{\pi}$ of $\mathcal{A} \times_\alpha G$, such that $\tilde{\pi}(\delta_s \otimes A) = \pi(A)U_s$.

To prove that the crossed product norm is a C*-norm and not just a seminorm, we need a covariant representation that admits a faithful representation of $\mathcal{A}G$. By the Gelfand Naimark theorem, let π be a faithful representation of $\mathcal{A}$ on some Hilbert space H. Define the covariant representation $(\tilde{\pi}, \Lambda)$ of $(\mathcal{A}, G, \alpha)$, such that

$$\tilde{\pi} : \mathcal{A} \to B(\ell^2(G, H)) : (\tilde{\pi}(A)x)(s) = \pi(\alpha_{-s}(A))(x(s)) \tag{4.1}$$

and Λ is the **left regular representation** on $\ell^2(G, H)$

$$\Lambda : G \to B(\ell^2(G, H)) : (\Lambda_t x)(s) = x(s - t) \tag{4.2}$$

for all $s, t \in G$, $A \in \mathcal{A}$, $x \in \ell^2(G, H)$. One can easily verify that $\tilde{\pi}$ is a representation of $\mathcal{A}$ and Λ is a unitary representation of G. Also we have the covariance condition;

$$(\Lambda_t \tilde{\pi}(A) \Lambda_t^* x)(s) = (\tilde{\pi}(A)\Lambda_t^* x)(s - t) = \pi(\alpha_{t-s}(A))(\Lambda_t^* x(s - t)) =$$

$$= \pi(\alpha_{-s}\alpha_t(A))(x(s)) = (\tilde{\pi}(\alpha_t(A))x)(s).$$

Note first that the algebra $\mathcal{A}$ is isometrically embedded into the crossed product by the inclusion map

$$\iota : \mathcal{A} \to \mathcal{A} \times_\alpha G : A \mapsto (\delta_0 \otimes A).$$

To prove that ι is an isometry, let $\delta_{s,\xi}$ be the vector in $\ell^2(G, \mathcal{A})$ that is defined by

$$\delta_{s,\xi}(t) := \delta_{s,t}\xi = \begin{cases} \xi, & \text{if } t = s \\ 0, & \text{if } t \neq s \end{cases}.$$

Then

$$\Lambda_t \delta_{s,\xi} = \delta_{s+t,\xi} \quad \text{and} \quad \tilde{\pi}(A)\delta_{s,\xi} = \delta_{s,\pi(\alpha_{-s}(A))\xi},$$

for every $t, s \in G, A \in \mathcal{A}, \xi \in H$. Calculate

$$\|(\delta_0 \otimes A)\|^2 \geq \sup_{\|\xi\|=1} \|(\tilde{\pi} \rtimes \Lambda)(\delta_0 \otimes A)\delta_{0,\xi}\|^2_{\ell^2(G,H)} = \sup_{\|\xi\|=1} \|\tilde{\pi}(A)\delta_{0,\xi}\|^2_{\ell^2(G,H)} =$$
$$= \sup_{\|\xi\|=1} \|\pi(A)\xi\|^2_H = \|A\|^2.$$

The opposite inclusion is straightforward from the fact that $\|A\| = \|(\delta_0 \otimes A)\|_{\ell^1}$.

For every $s \in G$, we denote by V_s the operator

$$V_s : H \to \ell^2(G, H) : \xi \mapsto \delta_{s,\xi}$$

so its adjoint operator has the form $V_s^* : \ell^2(G,H) \to H : x \mapsto x(s)$. Given now any element $\sum\limits_{\substack{s\in F\\ F\subset\subset G}} (\delta_s \otimes A_s) \in \mathcal{A}G$ and $\xi \in H$ we have

$$V_0^*(\tilde{\pi} \rtimes \Lambda)\left(\sum_{\substack{s\in F\\ F\subset\subset G}} (\delta_s \otimes A_s)\right) V_0\xi = \sum_{\substack{s\in F\\ F\subset\subset G}} V_0^*\tilde{\pi}(A_s)\Lambda_s\delta_{0,\xi} = \sum_{\substack{s\in F\\ F\subset\subset G}} V_0^*\tilde{\pi}(A_s)\delta_{s,\xi} =$$

$$= \sum_{\substack{s\in F\\ F\subset\subset G}} V_0^*\delta_{s,\pi(\alpha_{-s}(A_s))\xi} = \sum_{\substack{s\in F\\ F\subset\subset G}} \delta_{s,0}\pi(\alpha_{-s}(A_s))\xi =$$

$$= \pi(A_0)\xi.$$

Hence it follows readily from the equality $\|\pi(A_0)\| = \left\| V_0^*(\tilde{\pi} \rtimes \Lambda)\left(\sum\limits_{\substack{s\in F\\ F\subset\subset G}} (\delta_s \otimes A_s)\right) V_0 \right\|$, that $\|A_0\| \leq \|\sum\limits_{\substack{s\in F\\ F\subset\subset G}} (\delta_s \otimes A_s)\|$. Therefore, one can define the contractive map

$$E_0 : \mathcal{A}G \to \mathcal{A} : \sum_{\substack{s\in F\\ F\subset\subset G}} (\delta_s \otimes A_s) \mapsto A_0. \tag{4.3}$$

Check also that for every $X = \sum\limits_{\substack{s\in F\\ F\subset\subset G}} (\delta_s \otimes A_s) \in \mathcal{A}G$ we get $E_0(XX^*) = \sum\limits_{\substack{s\in F\\ F\subset\subset G}} A_s^*A_s$, so the map E_0 keeps the cone of positive elements of $\mathcal{A}G$ invariant. So we have proved the following

Proposition 4.1.2. *The map E_0 is an expectation* [1] *on $\mathcal{A}G$ and extends by continuity to a map on $\mathcal{A} \times_\alpha G$ with the same properties.*

Define the t-th **Fourier coefficient** of $X \in \mathcal{A} \times_\alpha G$ by

$$E_t(X) = E_0(X(\delta_{-t} \otimes 1)) \in \mathcal{A}.$$

[1] An expectation of a C*-algebra onto a subalgebra is a positive, unital idempotent map.

Note that for every element $X = \sum_{\substack{s\in F\\ F\subset\subset G}} (\delta_s \otimes A_s) \in \mathcal{A}G$ and $t \in G$, we get $E_t(X) = A_t$, and so it follows

$$X = \sum_{\substack{s\in F\\ F\subset\subset G}} (\delta_s \otimes E_s(X)).$$

We can now see that the left regular representation $\tilde{\pi} \rtimes \Lambda$ of $\mathcal{A}G$ is faithful. Given $X = \sum_{\substack{s\in F\\ F\subset\subset G}} (\delta_s \otimes A_s) \in \mathcal{A}G$ such that $\|(\tilde{\pi} \rtimes \Lambda)(X)\| = 0$, then for every $t \in G$ we have

$$\|A_t\| = \|\pi(A_t)\| = \|V_0^*(\tilde{\pi} \rtimes \Lambda)(X(\delta_{-t} \otimes 1))V_0\| \leq \|(\tilde{\pi} \rtimes \Lambda)(X)\| = 0.$$

Therefore $A_t = 0$ for every $t \in G$, but this yields that $X = 0$.

Remark 4.1.3. Since the left regular representation is faithful, we can define the reduced crossed product norm on $\mathcal{A}G$

$$\|\cdot\|_r = \|(\tilde{\pi} \rtimes \Lambda)(\cdot)\|.$$

The norm $\|\cdot\|_r$ does not depend on the choice of the faithful representation π (see [13]). The completion of $\mathcal{A}G$ with respect to the reduced crossed product norm gives rise to the **reduced crossed product**, denoted by $\mathcal{A} \times_\alpha^r G$. Moreover, repeating the proof of Proposition 4.1.2, one can show that the contraction E_0 given by the formula (4.3) extends to an expectation $\tilde{E}_0$ on $\mathcal{A} \times_\alpha^r G$.

Remark 4.1.4. In the general case, the construction via the left regular representation of G is not sufficient to determine the norm of the crossed product. Although in the special case that G is discrete abelian, so amenable [2], the reduced crossed product

[2] A group G is called amenable if there is a left translation invariant state on $L^\infty(G)$

equals the full crossed product. In the following subsection, we will give a proof of this claim in the case where G is the discrete group of real numbers.

4.1.1 Crossed Products by $\mathbb{R}_d$

From now on, the group G is either $\mathbb{Z}$ or $\mathbb{R}_d$; we use $\mathbb{R}_d$ to denote $\mathbb{R}$ equipped with the discrete topology. The theory about crossed products by $\mathbb{Z}$ can be found in [16]. In this section, we develop the theory for $\mathbb{R}_d$.

Proposition 4.1.5. *Let $(\mathcal{A}, \mathbb{R}_d, \alpha)$ be a C^*-dynamical system. Each $X \in \mathcal{A} \times_\alpha \mathbb{R}_d$ has only a countable number of nonzero Fourier coefficients.*

Proof. Let $(Y_n)_n$ be a sequence of generalized trigonometric polynomials in $\mathcal{A} \times_\alpha \mathbb{R}_d$ such that

$$\|X - Y_n\| \leq \frac{1}{n}.$$

We denote by Γ_n the finite set of indices of nonzero Fourier coefficients of Y_n and by Γ the set

$$\cup_{n \in \mathbb{N}} \Gamma_n.$$

The set Γ is countable. Suppose now $k \notin \Gamma$; then

$$\|E_k(X)\| \leq \|E_k(X) - E_k(Y_n)\| + \|E_k(Y_n)\| \leq \|X - Y_n\| \leq \frac{1}{n}$$

for every $n \in \mathbb{N}$. $\square$

Given a C*-dynamical system $(\mathcal{A}, \mathbb{R}_d, \alpha)$, fix $\lambda \in \mathbb{T}$. Then the map $U : s \mapsto (\delta_s \otimes \lambda^s \cdot 1)$ is a unitary representation of $\mathbb{R}_d$, such that

$$U_s \iota(A) U_s^* = (\delta_s \otimes \lambda^s \cdot 1)(\delta_0 \otimes A)(\delta_{-s} \otimes \overline{\lambda^s} \cdot 1) = (\delta_0 \otimes \alpha_s(A)) = \iota(\alpha_s(A)),$$

for every $A \in \mathcal{A}$. Hence the pair (ι, U) is a covariant representation of $(\mathcal{A}, \mathbb{R}_d, \alpha)$, and so the universal property of $\mathcal{A} \times_\alpha \mathbb{R}_d$ induces an automorphism:

$$\varphi_\lambda : \mathcal{A} \times_\alpha \mathbb{R}_d \to \mathcal{A} \times_\alpha \mathbb{R}_d : (\delta_s \otimes A) \mapsto (\delta_s \otimes \lambda^s A).$$

Moreover, given $X \in \mathcal{A} \times_\alpha \mathbb{R}_d$ the map $t \mapsto \varphi_{e^{it}}(X)$ is norm continuous for every $t \in \mathbb{R}$; indeed, one can check it first on the unclosed algebra of trigonometric polynomials and extend it to the closure by a standard approximation argument. So, given $T > 0$, we can define

$$\Phi_T(X) = \frac{1}{2T} \int_{-T}^{T} \varphi_{e^{it}}(X) dt.$$

Check that $\|\Phi_T(X)\| \leq \frac{1}{2T} \int_{-T}^{T} \|\varphi_{e^{it}}(X)\| dt = \|X\|$, so $\|\Phi_T\| \leq 1$. Given a trigonometric polynomial $Y = \sum\limits_{\substack{s \in F \\ F \subset\subset \mathbb{R}}} (\delta_s \otimes A_s)$ in $\mathcal{A} \times_\alpha \mathbb{R}_d$, we have

$$\begin{aligned} \Phi_T(Y) &= \frac{1}{2T} \int_{-T}^{T} \varphi_{e^{it}}(Y) dt = \\ &= \frac{1}{2T} \int_{-T}^{T} \sum_{\substack{s \in F \\ F \subset\subset \mathbb{R}}} (\delta_s \otimes e^{its} A_s) dt = \\ &= \sum_{\substack{s \in F \\ F \subset\subset \mathbb{R}}} (\delta_s \otimes A_s) \frac{1}{2T} \int_{-T}^{T} (\delta_0 \otimes e^{its} \cdot 1) dt. \end{aligned}$$

Compute now the limit $\lim\limits_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} (\delta_0 \otimes e^{its} \cdot 1) dt$.

- $s = 0$. $\lim\limits_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} (\delta_0 \otimes 1) dt = \lim\limits_{T \to \infty} \frac{1}{2T} \cdot 2T (\delta_0 \otimes 1) = (\delta_0 \otimes 1)$;

- $s \neq 0$. $\lim\limits_{T\to\infty} \frac{1}{2T}\int_{-T}^{T}(\delta_0 \otimes e^{its}\cdot 1)dt = \lim\limits_{T\to\infty}\frac{1}{2T}\frac{e^{iTs}-e^{-iTs}}{is}(\delta_0\otimes 1) \to (\delta_0 \otimes 0)$, as $T\to\infty$.

Hence by the linearity of limits, we obtain that

$$\lim_{T\to\infty} \Phi_T(Y) = (\delta_0 \otimes A_0).$$

Define now

$$\Phi_0(Y) = \lim_{T\to\infty} \Phi_T(Y) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \varphi_{e^{it}}(Y)dt.$$

Since $\|\Phi_T(Y)\| \leq \|Y\|$ for all $T > 0$, it follows that $\|\Phi_0(Y)\| \leq \|Y\|$, for every generalized trigonometric polynomial Y. So Φ_0 can be extended to a linear contraction in $\mathcal{A}\times_\alpha \mathbb{R}_d$. In addition, since the family of operators $\{\Phi_T : T > 0\}$ is uniformly bounded, applying a simple approximation argument, it follows that $\Phi_0(X) = \lim\limits_{T\to\infty} \Phi_T(X)$. This proves the following result.

Proposition 4.1.6. *Let E_0 be the expectation defined in Theorem 4.1.2 and $X \in \mathcal{A}\times_\alpha \mathbb{R}_d$. Then $\Phi_0(X) = \lim\limits_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\varphi_{e^{it}}(X)dt = \iota(E_0(X))$.*

Applying standard arguments for kernels of approximating polynomials ([9, 46]), we can obtain the analogue of Bochner - Fejer's theorem.

Given a rationally independent set $\{\beta_1, \ldots, \beta_m\}$ of real numbers and $X \in \mathcal{A}\times_\alpha \mathbb{R}_d$, one can define the **Bochner-Fejer polynomial**

$$\sigma_{(\beta_1,\ldots,\beta_m)}(X) = \sum_{\substack{|\nu_1|<(m!)^2\\ \cdots\cdots\cdots \\ |\nu_m|<(m!)^2}} \left(1-\frac{|\nu_1|}{(m!)^2}\right)\cdots\left(1-\frac{|\nu_m|}{(m!)^2}\right)(\delta_{\frac{\nu_1}{m!}\beta_1+\cdots+\frac{\nu_m}{m!}\beta_m} \otimes E_{\frac{\nu_1}{m!}\beta_1+\cdots+\frac{\nu_m}{m!}\beta_m}(X)). \tag{4.4}$$

Note that a term of $\sigma_{(\beta_1,\ldots,\beta_m)}(X)$ in (4.4) differs from zero if and only if the respective Fourier coefficient of the term is nonzero.

Proposition 4.1.7.

$$\sigma_{(\beta_1,\dots,\beta_m)}(X) = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\varphi_{e^{it}}(X)(\delta_0\otimes K_{(\beta_1,\dots,\beta_m)}(t))dt,$$

where $K_{(\beta_1,\dots,\beta_m)}$ is the Bochner - Fejer kernel for almost periodic functions.

Proof. Fix n and compute

$$\sigma_{(\beta_1,\dots,\beta_m)}(X) = \sum_{\substack{|\nu_1|<(m!)^2\\ \cdots\cdots\\ |\nu_m|<(m!)^2}}\left(1-\frac{|\nu_1|}{(m!)^2}\right)\dots\left(1-\frac{|\nu_m|}{(m!)^2}\right)(\delta_{\frac{\nu_1}{m!}\beta_1+\dots+\frac{\nu_m}{m!}\beta_m}\otimes E_{\frac{\nu_1}{m!}\beta_1+\dots+\frac{\nu_m}{m!}\beta_m}(X)) =$$

$$= \sum_{\substack{|\nu_1|<(m!)^2\\ \cdots\cdots\\ |\nu_m|<(m!)^2}}\left(1-\frac{|\nu_1|}{(m!)^2}\right)\dots\left(1-\frac{|\nu_m|}{(m!)^2}\right)(\delta_0\otimes E_0(X(\delta_{-\frac{\nu_1}{m!}\beta_1-\dots-\frac{\nu_m}{m!}\beta_m}\otimes 1)))(\delta_{\frac{\nu_1}{m!}\beta_1+\dots+\frac{\nu_m}{m!}\beta_m}\otimes 1) =$$

$$= \sum_{\substack{|\nu_1|<(m!)^2\\ \cdots\cdots\\ |\nu_m|<(m!)^2}}\left(1-\frac{|\nu_1|}{(m!)^2}\right)\dots\left(1-\frac{|\nu_m|}{(m!)^2}\right)\Phi_0(X(\delta_{-\frac{\nu_1}{m!}\beta_1-\dots-\frac{\nu_m}{m!}\beta_m}\otimes 1))(\delta_{\frac{\nu_1}{m!}\beta_1+\dots+\frac{\nu_m}{m!}\beta_m}\otimes 1) =$$

$$= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\sum_{\substack{|\nu_1|<(m!)^2\\ \cdots\cdots\\ |\nu_m|<(m!)^2}}\left(1-\frac{|\nu_1|}{(m!)^2}\right)\dots\left(1-\frac{|\nu_m|}{(m!)^2}\right)\varphi_{e^{it}}(X)(\delta_0\otimes e^{-it(\frac{\nu_1}{m!}\beta_1+\dots+\frac{\nu_m}{m!}\beta_m)}\cdot 1)dt =$$

$$= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\varphi_{e^{it}}(X)(\delta_0\otimes K_{(\beta_1,\dots,\beta_m)}(t)\cdot 1)dt.$$

□

Corollary 4.1.8. *For every finite rationally independent set $\{\beta_1,\dots,\beta_m\}$, the map $\sigma_{(\beta_1,\dots,\beta_m)}$ is contractive.*

Let $X\in\mathcal{A}\times_\alpha\mathbb{R}_d$. By the previous lemma we have

$$\begin{aligned}\|\sigma_{(\beta_1,\dots,\beta_m)}(X)\| &\le \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\|\varphi_{e^{it}}(X)\|\|(\delta_0\otimes K_{(\beta_1,\dots,\beta_m)}(t)\cdot 1)\|dt = \\ &= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}K_{(\beta_1,\dots,\beta_m)}(t)dt\|X\| = \|X\|.\end{aligned}$$

Let now $X \in \mathcal{A} \times_\alpha \mathbb{R}_d$ and $(Y_n)_n$ be a sequence of generalized trigonometric polynomials in $\mathcal{A} \times_\alpha \mathbb{R}_d$ that converges to X. Define $\Gamma = \cup_n \Gamma_n$ as in the proof of Proposition 4.1.5 and let $B = (\beta_1, \beta_2, \dots, \beta_m, \dots)$ be a rational basis of Γ.

Theorem 4.1.9. *$\sigma_{(\beta_1,\dots,\beta_m)}(X) \overset{\|\cdot\|}{\to} X$, as $M \to \infty$.*

Proof. We will show first that $\sigma_{(\beta_1,\dots,\beta_m)}(Y_n) \overset{\|\cdot\|}{\to} Y_n$, for every $n \in \mathbb{N}$. Fix some $n \in \mathbb{N}$. Suppose that Y_n is the trigonometric polynomial $\sum\limits_{\substack{s \in F \\ F \subset\subset \mathbb{R}}} (\delta_s \otimes A_s)$. Since B is also a rational basis of the indices of the nonzero Fourier coefficients of Y_n we have

$$\begin{aligned}
&\sigma_{(\beta_1,\dots,\beta_m)}(Y_n) = \\
&= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \varphi_{e^{it}}(Y_n)(\delta_0 \otimes K_{(\beta_1,\dots,\beta_m)}(t) \cdot 1)dt = \\
&= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \sum_{\substack{s \in F \\ F \subset\subset \mathbb{R}}} (\delta_s \otimes e^{its} A_s)(\delta_0 \otimes K_{(\beta_1,\dots,\beta_m)}(t) \cdot 1)dt = \\
&= \sum_{\substack{s \in F \\ F \subset\subset \mathbb{R}}} \left((\delta_s \otimes A_s) \left(\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} e^{its} (\delta_0 \otimes K_{(\beta_1,\dots,\beta_m)}(t) \cdot 1)dt \right) \right) \\
&\to \sum_{\substack{s \in F \\ F \subset\subset \mathbb{R}}} (\delta_s \otimes A_s)(\delta_0 \otimes 1) = \sum_{\substack{s \in F \\ F \subset\subset \mathbb{R}}} (\delta_s \otimes A_s).
\end{aligned}$$

Given now $\epsilon > 0$, choose trigonometric polynomial Y_{n_0} with $\|X - Y_{n_0}\| < \epsilon/3$. Then there exists $m_0 \in \mathbb{N}$, such that $\|Y_{n_0} - \sigma_{(\beta_1,\dots,\beta_m)}(Y_{n_0})\| \leq \epsilon/3$, for every $m > m_0$. Hence, it follows from Corollary 4.1.8 that for all $m > m_0$ we have

$$\begin{aligned}
\|X - \sigma_{(\beta_1,\dots,\beta_m)}(X)\| &\leq \|X - Y_{n_0}\| + \|Y_{n_0} - \sigma_{(\beta_1,\dots,\beta_m)}(Y_{n_0})\| + \|\sigma_{(\beta_1,\dots,\beta_m)}(Y_{n_0} - X)\| \leq \\
&\leq 2\|X - Y_{n_0}\| + \|Y_{n_0} - \sigma_{(\beta_1,\dots,\beta_m)}(Y_{n_0})\| \leq \epsilon.
\end{aligned}$$

□

Corollary 4.1.10. *Let $X \in \mathcal{A} \times_\alpha \mathbb{R}_d$, such that $E_s(X) = 0$, for every $s \in \mathbb{R}$. Then $X = 0$.*

Proof. Since $E_s(X) = 0$, for every $s \in \mathbb{R}$, it follows that $\Phi_s(X) = \iota(E_s(X)) = 0$, for every $t \in \mathbb{R}$. Hence the Bochner-Fejer polynomials of X are trivial, so by Theorem 4.1.9 we have $X = 0$. $\square$

Recall now the left regular representation of the C*-dynamical system, given by the formulas (4.1) and (4.2). As we stated in Remark 4.1.3 the left regular representation gives rise to the reduced crossed product. The following result comes readily from the previous theorem.

Proposition 4.1.11. *Let $(\mathcal{A}, \mathbb{R}_d, \alpha)$ be a C*-dynamical system. Then the reduced crossed product $\mathcal{A} \times_\alpha^r \mathbb{R}_d$ coincides with the full crossed product $\mathcal{A} \times_\alpha \mathbb{R}_d$.*

Proof. By the universal property of the full crossed product, there is a representation

$$\varphi : \mathcal{A} \times_\alpha \mathbb{R}_d \to \mathcal{A} \times_\alpha^r \mathbb{R}_d : (\delta_s \otimes A) \mapsto \tilde{\pi}(A)\Lambda_s.$$

It suffices to show that φ is faithful. We need first to point out some observations about these two C*-algebras.

By Remark 4.1.3, one can define on $\mathcal{A} \times_\alpha^r \mathbb{R}_d$ the contractive maps

$$\tilde{E}_t : \mathcal{A} \times_\alpha^r \mathbb{R}_d \to A : \sum_{\substack{s \in F \\ F \subset\subset \mathbb{R}}} \tilde{\pi}(A_s)\Lambda_s \to \tilde{\pi}(A_t).$$

Let now $\{\Phi_t : t > 0\}$ be the family of contractions on $\mathcal{A} \times_\alpha \mathbb{R}_d$, given by the formula

$$\Phi_t(X) = \Phi_0(X(\delta_{-t} \otimes 1)), \tag{4.5}$$

where Φ_0 is the operator defined in Proposition 4.1.6. It follows by routine calculations on the subalgebra of trigonometric polynomials and standard density arguments that $\tilde{E}_t \circ \varphi = \varphi \circ \Phi_t$, for all $t \in \mathbb{R}$.

Let now $X \in \mathcal{A} \times_\alpha \mathbb{R}_d$, such that $\varphi(X) = 0$. Then $(\tilde{E}_t \circ \varphi)(X) = 0$ for every $t \in \mathbb{R}$, which implies that $(\varphi \circ \Phi_t)(X) = 0$. Since the left regular representation is a faithful representation of $\mathcal{A}\mathbb{R}_d$, it follows that $\Phi_t(X) = 0$, for every $t \in \mathbb{R}$. Hence by Corollary 4.1.10 we have $X = 0$. $\square$

As a simple consequence of the above proposition we obtain the following useful inequality. Note that it essentially corresponds to the elementary fact that the norm of an $\mathbb{R}_d \times \mathbb{R}_d$ operator matrix X dominitates the norm of any of its columns.

Proposition 4.1.12. *Let $\mathcal{A}$ be a C^*-algebra acting on a Hilbert space H and ξ be a unit vector in H. For every $X \in \mathcal{A} \times_\alpha \mathbb{R}_d$ and F arbitrarily chosen finite subset of $\mathbb{R}$, we have*

$$\|(\tilde{id} \rtimes \Lambda)(X)\|^2 - \sum_{\substack{s\in F \\ F\subset\subset\mathbb{R}}} \|\alpha_{-s}(E_s(X))\xi\|^2 \geq 0.$$

Proof. Applying Proposition 4.1.11, it suffices to prove the result for the reduced crossed product norm. Let id be the identity representation of $\mathcal{A}$ on H and $Y = \sum_{\substack{s\in F \\ F\subset\subset\mathbb{R}}} (\delta_s \otimes A_s)$ be a generalized trigonometric polynomial in $\mathcal{A} \times_\alpha \mathbb{R}_d$. Note first that

$$\begin{aligned}\Big\|\Big((\tilde{\mathrm{id}} \rtimes \Lambda)(X) - \sum_{\substack{s\in F \\ F\subset\subset\mathbb{R}}} (\tilde{\mathrm{id}} \rtimes \Lambda)(\delta_s \otimes A_s)\Big)\delta_{0,\xi}\Big\|^2 &= \|(\tilde{\mathrm{id}} \rtimes \Lambda)(X)\delta_{0,\xi}\|^2 + \Big\|\sum_{\substack{s\in F \\ F\subset\subset\mathbb{R}}} \tilde{\mathrm{id}}(A_s)\delta_{s,\xi}\Big\|^2 \\ &- \sum_{\substack{s\in F \\ F\subset\subset\mathbb{R}}} \langle(\tilde{\mathrm{id}} \rtimes \Lambda)(X)\delta_{0,\xi}, \tilde{\mathrm{id}}(A_s)\delta_{s,\xi}\rangle - \sum_{\substack{s\in F \\ F\subset\subset\mathbb{R}}} \langle\tilde{\mathrm{id}}(A_s)\delta_{s,\xi}, (\tilde{\mathrm{id}} \rtimes \Lambda)(X)\delta_{0,\xi}\rangle\end{aligned}$$

Since $\tilde{\mathrm{id}}(A_s)\delta_{s,\xi} = \delta_{s,\alpha_{-s}(A_s)\xi} = V_s(\alpha_{-s}(A_s)\xi)$ and $\delta_{s,\xi}$ is orthogonal to $\delta_{t,\eta}$ for $s \neq t$, it follows that

$$\left\|\Big((\tilde{\mathrm{id}} \rtimes \Lambda)(X) - \sum_{\substack{s\in F\\ F\subset\subset\mathbb{R}}} (\tilde{\mathrm{id}} \rtimes \Lambda)(\delta_s \otimes A_s)\Big)\delta_{0,\xi}\right\|^2 = \|(\tilde{\mathrm{id}} \rtimes \Lambda)(X)\delta_{0,\xi}\|^2 + \sum_{\substack{s\in F\\ F\subset\subset\mathbb{R}}} \|\alpha_{-s}(A_s)\xi\|^2$$
$$- \sum_{\substack{s\in F\\ F\subset\subset\mathbb{R}}} \langle V_s^*(\tilde{\mathrm{id}} \rtimes \Lambda)(X)V_0\xi, \alpha_{-s}(A_s)\xi\rangle - \sum_{\substack{s\in F\\ F\subset\subset\mathbb{R}}} \langle \alpha_{-s}(A_s)\xi, V_s^*(\tilde{\mathrm{id}} \rtimes \Lambda)(X)V_0\xi\rangle.$$

One may check that $V_s^*(\tilde{\mathrm{id}} \rtimes \Lambda)(X)V_0 = \alpha_{-s}(E_s(X))$, so adding and subtracting $\sum_{\substack{s\in F\\ F\subset\subset\mathbb{R}}} \|\alpha_{-s}(E_s(X))\xi\|^2$, we obtain that the above expression is equal to

$$\|(\tilde{\mathrm{id}} \rtimes \Lambda)(X)\delta_{0,\xi}\|^2 - \sum_{\substack{s\in F\\ F\subset\subset\mathbb{R}}} \|\alpha_{-s}(E_s(X))\xi\|^2 + \sum_{\substack{s\in F\\ F\subset\subset\mathbb{R}}} \|\alpha_{-s}(E_s(X))\xi - \alpha_{-s}(A_s)\xi\|^2.$$

Note that the last formula takes its lowest value when $\sum_{\substack{s\in F\\ F\subset\subset\mathbb{R}}} \|\alpha_{-s}(E_s(X))\xi - \alpha_{-s}(A_s)\xi\|^2 = 0$, which happens in the case we choose $A_s = E_s(X)$. Since the left hand side is non-negative, we deduce that

$$\|(\tilde{\mathrm{id}} \rtimes \Lambda)(X)\delta_{0,\xi}\|^2 - \sum_{\substack{s\in F\\ F\subset\subset\mathbb{R}}} \|\alpha_{-s}(E_s(X))\xi\|^2 \geq 0.$$

□

4.1.2 Semicrossed products

Definition 4.1.13. Let $(\mathcal{A}, G, \alpha)$ be a C*-dynamical system. If $\mathcal{B}$ is a unital closed subalgebra of $\mathcal{A}$ and G^+ is a unital semigroup of G, we define the **semicrossed product** $\mathcal{B} \times_\alpha G^+$ as the closed subalgebra of the full crossed product, that is generated by the elements $(\delta_0 \otimes b), (\delta_s \otimes 1)$, with $b \in \mathcal{B}$ and $s \in G^+$.

Proposition 4.1.14. *Let $(\mathcal{A}, \mathbb{R}_d, \alpha)$ be a C^*-dynamical system. Then the semicrossed product $\mathcal{A} \times_\alpha \mathbb{R}_d^+$ is equal to the set*

$$\mathcal{A}^{\mathbb{R}^+} = \{X \in \mathcal{A} \times_\alpha \mathbb{R}_d : E_s(X) = 0, \text{ for all } s < 0\}.$$

Proof. If X is a trigonometric polynomial in $\mathcal{A} \times_\alpha \mathbb{R}_d^+$, then it is trivial to see that X lies in $\mathcal{A}^{\mathbb{R}^+}$. The latter set is closed, since it is the intersection of the kernels $\ker E_s$ for all $s > 0$, so the first inclusion is proved. For the converse inclusion, suppose $X \in \mathcal{A}^{\mathbb{R}^+}$. If $X = 0$, there is nothing to prove. If $X \neq 0$, then the only nonzero Fourier coefficients of X have nonnegative indices, so the Fejer-Bochner polynomials of X lie in $\mathcal{A} \times_\alpha \mathbb{R}_d^+$. Hence by Theorem 4.1.9 we have that $X \in \mathcal{A} \times_\alpha \mathbb{R}_d^+$. $\square$

The following corollary follows trivially by routine calculations on the generalized trigonometric polynomials of the semicrossed product algebra.

Corollary 4.1.15. *Let $(\mathcal{A}, \mathbb{R}_d, \alpha)$ be a C^*-dynamical system. The restriction of the expectation E_0 to $\mathcal{A} \times_\alpha \mathbb{R}_d^+$ is a contractive homomorphism onto $\mathcal{A}$.*

4.1.3 The algebra $AP(\mathbb{R})$ revisited

Proposition 4.1.16. *Let G be a discrete abelian group. The crossed product $\mathbb{C} \times G$ (with the trivial action) is isometrically isomorphic to the C^*-algebra $C(\hat{G})$ of continuous functions on the dual group $\hat{G}$.*

Proof. The crossed product $\mathbb{C} \times G$ is a unital commutative algebra, so by the Gelfand transform (see 1.2.9) it is isometrically isomorphic with $\mathcal{C}(\mathfrak{M}(\mathbb{C} \times G))$. Hence it suffices to identify $\mathfrak{M}(\mathbb{C} \times G))$ with $\hat{G}$.

If $\chi \in \mathfrak{M}(\mathbb{C} \times G)$, we may restrict to $G \subset \mathbb{C}G \subseteq \mathbb{C} \times G$. Since χ is multiplicative, $\chi\big|_G$ is a group homomorphism. Define

$$\alpha : \mathfrak{M}(\mathbb{C} \times G)) \to \hat{G} : \chi \mapsto \chi\big|_G.$$

- α is injective; indeed, if $\chi\big|_G = \varphi\big|_G$ then it follows by linearity that $\chi\big|_{\mathbb{C}G} = \varphi\big|_{\mathbb{C}G}$. Hence $\chi = \varphi$, since they are continuous on $\mathfrak{M}(\mathbb{C} \times G))$ and they coincide on a dense subset.

- α is surjective; given $\chi \in \hat{G}$, define the $*$-homomorphism

$$\pi_\chi : \mathbb{C}G \to \mathbb{C} : \sum_{\substack{s \in F \\ F \subset\subset G}} (\delta_g \otimes a_g) \mapsto \sum_{\substack{s \in F \\ F \subset\subset G}} \chi(g) a_g.$$

  Then

$$\left\| \pi_\chi \left(\sum_{\substack{s \in F \\ F \subset\subset G}} (\delta_g \otimes a_g) \right) \right\| = \Big\| \sum_{\substack{s \in F \\ F \subset\subset G}} \chi(g) a_g \Big\| \leq \sum_{\substack{s \in F \\ F \subset\subset G}} \|\chi(g)\| \, \|a_g\| =$$
$$= \sum_{\substack{s \in F \\ F \subset\subset G}} \|a_g\| = \Big\| \sum_{\substack{s \in F \\ F \subset\subset G}} (\delta_g \otimes a_g) \Big\|_{\ell_1}.$$

  So by the universal property of crossed products, we can extend π_χ to a nonzero representation of $\mathbb{C} \times G$ on $\mathbb{C}$.

- α is evidently continuous. Since its domain is a compact space, α is a homeomorphism.

$\square$

Set now G equal to $\mathbb{R}_d$. Applying the above proposition we obtain that $\mathbb{C} \times \mathbb{R}_d$ is isometrically isomorphic with $C(\mathbb{R}_B)$, where $\mathbb{R}_B$ is the Bohr compactification of the real numbers. As we stated in Chapter 1, $C(\mathbb{R}_B)$ can be identified as a C*-algebra with the algebra $AP(\mathbb{R})$ of almost periodic functions ([69]). In the following proposition we provide a proof, using the machinery of crossed products.

Proposition 4.1.17. *The commutative C*-algebras $AP(\mathbb{R})$ and $C(\mathbb{R}_B)$ are isomorphic.*

Proof. By Proposition 4.1.16, we identify $C(\mathbb{R}_B)$ with the crossed product $\mathbb{C} \times \mathbb{R}_d$. Define the covariant representation (π, U) of the C*-dynamical system $(\mathbb{C}, \mathbb{R}_d, \mathrm{id})$ by the formulas

$$\mathbb{C} \to AP(\mathbb{R}) : c \mapsto c \cdot 1$$

and

$$\mathbb{R} \to AP(\mathbb{R}) : \lambda \mapsto e^{i\lambda x}.$$

By the universal property of crossed products, we obtain a representation $\tilde{\pi}$ given by

$$\tilde{\pi} : \mathbb{C} \times \mathbb{R}_d \to AP(\mathbb{R}) : \sum_{\substack{s\in F \\ F \subset\subset \mathbb{R}}} (\delta_s \otimes a_s) \mapsto \sum_{\substack{s\in F \\ F \subset\subset \mathbb{R}}} a_s\, e^{isx}.$$

Let now $X \in \mathbb{C} \times \mathbb{R}_d$, such that $\tilde{\pi}(X) = 0$. One can check that, as in the proof of Proposition 4.1.11 that $(\tilde{\pi} \circ E_\lambda)(X) = (\epsilon_\lambda \circ \tilde{\pi})(X)$, where ϵ_λ is given by the formula (1.6). Hence it follows that $E_\lambda(X) = 0$, for every $\lambda \in \mathbb{R}$, so we have by Theorem 4.1.9 that $X = 0$. Thus, $\tilde{\pi}$ is injective and the proof is complete. □

Consider now the closed subalgebra $AAP(\mathbb{R})$ of analytic almost periodic functions. Applying Proposition 4.1.14 and Corollary 4.1.15, we have the following result.

Proposition 4.1.18.

$$AAP(\mathbb{R}) = \{f \in AP(\mathbb{R}) : \epsilon_\lambda(f) = 0, \ \textit{for every } \lambda < 0\}.$$

Moreover, the compression of the contractive map ϵ_0 to $AAP(\mathbb{R})$ is multiplicative; hence it induces a character x_∞ that satisfies

$$x_\infty\left(\sum_{\substack{\lambda\in F\\ F\subset\subset\mathbb{R}^+}} c_\lambda e^{i\lambda x}\right)\mapsto c_0.$$

As we stated in theorem 1.2.21, the set of continuous automorphisms of $AAP(\mathbb{R})$ is the set $\{\varphi_{c,k} : c \in \mathbb{R}_B, k \in \mathbb{R}^+\}$, where $\varphi_{c,k}$ are the multiplicative linear maps that satisfy

$$\varphi_{c,k}(e^{i\lambda x}) = c(\lambda)e^{ik\lambda x}.$$

Proposition 4.1.19. *Every automorphism $\varphi_{c,k}$ is isometric.*

Proof. Fix some $c \in \mathbb{R}_B$ and $k \in \mathbb{R}^+$. One can check that $(\tilde{\mathrm{id}}, u_{c,k})$, where $\tilde{\mathrm{id}} : \mathbb{C} \to AP(\mathbb{R}) : c \mapsto c \cdot 1_{\mathbb{R}}$ and $u_{c,k} : \mathbb{R}_d \to AP(\mathbb{R}) : \lambda \mapsto c(\lambda)e^{ik\lambda x}$, gives a covariant representation of the C*-dynamical system $(\mathbb{C}, \mathbb{R}_d, \mathrm{id})$. Hence by the universal property, we have a representation of the C^*-algebra $\mathbb{C} \times \mathbb{R}_d \simeq C(\mathbb{R}_B) \simeq AP(\mathbb{R})$ of almost periodic functions, given by

$$\tilde{\mathrm{id}} \rtimes u_{c,k} : AP(\mathbb{R}) \to AP(\mathbb{R}) : e^{i\lambda x} \mapsto c(\lambda)e^{ik\lambda x}. \tag{4.6}$$

The representation $\tilde{\mathrm{id}} \rtimes u_{c,k}$ is evidently faithful, so it is isometric. Moreover, the restriction of $\tilde{\mathrm{id}} \rtimes u_{c,k}$ to the invariant subalgebra $AAP(\mathbb{R})$ is equal to $\varphi_{c,k}$, so the proof is complete. ∎

4.2 The norm closed parabolic algebra A_p

Let $(AP(\mathbb{R}), \mathbb{R}_d, \tau)$ be a C*-dynamical system, where τ induces the group of translation automorphisms:

$$(\tau_s f)(x) = f(x-s),\ \ f \in AP(\mathbb{R}).$$

Our goal in this section, is to prove that the abstract discrete crossed product $AP(\mathbb{R}) \times_\tau \mathbb{R}_d$ is isometrically isomorphic to a concrete C*-algebra acting on $L^2(\mathbb{R})$.

Proposition 4.2.1. *The crossed product $AP(\mathbb{R}) \times_\tau \mathbb{R}_d$ is a simple algebra, i.e. it has no non-trivial two-sided closed ideals.*

Proof. Let J be a non-zero two-sided closed ideal. Hence there exists an element $X \in J$, such that $\Phi_s(X) \neq 0$, for some $s \in \mathbb{R}$. Using the integral formula $\Phi_0(X) = \lim\limits_{T\to+\infty} \frac{1}{2T}\int_{-T}^{T} \varphi_{e^{it}}(X)dt$ that we proved in the previous section, we will prove that $\Phi_s(X)$ belongs to J. Since J is closed, it suffices to prove that $\varphi_{e^{it}}(X) \in J$. Suppose first that X is a generalized trigonometric polynomial $\sum\limits_{\substack{s\in F\\ F\subset\subset\mathbb{R}}} (\delta_s \otimes f_s)$. Compute the product

$$\begin{aligned}
(\delta_0 \otimes e^{itx})X(\delta_0 \otimes e^{-itx}) = &\sum_{\substack{s\in F\\ F\subset\subset\mathbb{R}}} (\delta_s \otimes e^{itx} f_s)(\delta_0 \otimes e^{-itx}) = \\
= &\sum_{\substack{s\in F\\ F\subset\subset\mathbb{R}}} (\delta_s \otimes e^{itx} f_s \tau_s(e^{-itx})) = \\
= &\sum_{\substack{s\in F\\ F\subset\subset\mathbb{R}}} (\delta_s \otimes e^{itx} f_s e^{-it(x-s)}) = \\
= &\sum_{\substack{s\in F\\ F\subset\subset\mathbb{R}}} (\delta_s \otimes e^{its} f_s) = \varphi_{e^{it}}(X).
\end{aligned}$$

Hence, it follows by a standard approximation argument that $(\delta_0 \otimes e^{itx})X(\delta_0 \otimes e^{-itx}) = \varphi_{e^{it}}(X)$ for any $X \in AP(\mathbb{R}) \times_\tau \mathbb{R}_d$.

Similarly, we get $\Phi_s(X) = \Phi(X(\delta_{-s} \otimes 1)) \in J$, so there exists some nonzero $f \in AP(\mathbb{R})$, such that $(\delta_0 \otimes f) \in J$. Since the action of the group can be described by the product of the covariant relation, it follows $(\delta_0 \otimes \tau_s(f)) \in J$ for every $s \in \mathbb{R}$.

Claim: We may assume that $\inf\{|f(x)| : x \in \mathbb{R}\} \geq c > 0$.

Since $f \cdot f^*, nf \in AP(\mathbb{R})$ for every $n \in \mathbb{N}$, we may assume that $f(x) \geq 0$, for every $x \in \mathbb{R}$ and $\|f\| > 2$. Let $\epsilon = \frac{1}{2}$. Then there is $T = T(\epsilon) > 0$, such that for every interval I of length T, there exists $\ell \in I$ that satisfies

$$|f(x+\ell) - f(x)| < \epsilon, \forall x \in \mathbb{R}.$$

On the interval $[0, T]$, we may assume that $f(x) > 1$, for every $x \in [0, \frac{T}{n}]$, for some $n \in \mathbb{N}$ (otherwise, work with $g = \tau_s(f)$, for suitable s). Then, let $f_k = \tau_{k\frac{T}{n}}(f)$, for $k = 0, 1, \ldots, n-1$ and define

$$g(x) = \sum_{k=0}^{n-1} f_k(x), \ x \in \mathbb{R}.$$

Then $g(x) > 1$, for every $x \in [0, T]$. In the general case where $x \in \mathbb{R}$, there exists $\ell \in [x - T, x]$, such that $|f(x-\ell) - f(x)| < \epsilon$. Since ℓ gives that bound uniformly for all $y \in \mathbb{R}$, it yields that $|f_k(x-\ell) - f_k(x)| < \epsilon$, for every $k \in \{0, 1, \ldots, n-1\}$. Therefore, there exists some k, such that $|f_k(x)| > 1 - \epsilon = \frac{1}{2}$. Hence $g(x) > \frac{1}{2}$ and that completes the proof of our claim.

Now, since the value $\inf\{|f(x)| : x \in \mathbb{R}\}$ is positive, we have by Proposition 1.2.15 that the multiplicative inverse $1/f$ is a bounded almost periodic function. Then

$$(\delta_0 \otimes f)(\delta_0 \otimes 1/f) = (\delta_0 \otimes 1) \in I,$$

so I coincides with the crossed product. $\square$

Remark 4.2.2. The simplicity of crossed product algebras has been studied extensively over the last 50 years (see for example [19, 28]). In particular, Archbold and Spielberg proved in [3] that given a C*-dynamical system $(\mathcal{A}, G, \alpha)$, with $\mathcal{A}$ commutative and G discrete, the crossed product $\mathcal{A} \times_\alpha G$ is simple if and only if the action of the group on $\mathcal{A}$ is minimal[3] and topologically free[4].

Definition 4.2.3. Let B_p be the C*-algebra that is generated by the set of all the multiplication and translation operators M_λ and D_μ acting on $L^2(\mathbb{R})$ respectively. Since the span of the products $M_\lambda D_\mu$ is closed under the operations of ring multiplication and involution, we get that

$$B_p = \overline{\operatorname{span}\{M_\lambda D_\mu \,:\, \lambda, \mu \in \mathbb{R}\}}^{\|\cdot\|}.$$

Theorem 4.2.4. *The C*-algebras $AP(\mathbb{R}) \times_\tau \mathbb{R}_d$ and B_p are isomorphic.*

Proof. Define the covariant representation (π, D), where:

$$\pi : AP(\mathbb{R}) \to B(L^2(\mathbb{R})) : e^{i\lambda x} \mapsto M_\lambda$$

and

$$D : \mathbb{R}_d \to B(L^2(\mathbb{R})) : \mu \mapsto D_\mu.$$

It is trivial to see that π is a representation of $AP(\mathbb{R})$ and D is a unitary representation, so it suffices to prove the covariance relation. Compute

$$D_\mu \pi(e^{i\lambda x}) D_\mu^* = D_\mu M_\lambda D_\mu^*$$

[3]The action of a group G on a C*-algebra $\mathcal{A}$ is called minimal if $\mathcal{A}$ does not contain any non-trivial G-invariant ideals.

[4]An action α on a commutative algebra $\mathcal{A}$ is said to be topologically free if for any finite set $F \subseteq G \backslash \{e_G\}$, the set $\cap_{t \in F} \{\chi \in \mathfrak{M}(\mathcal{A}) | \chi \circ \alpha_t \neq \chi\}$ is dense in $\mathfrak{M}(\mathcal{A})$.

and

$$\pi(\tau_\mu(e^{i\lambda x})) = \pi(e^{i\lambda(x-\mu)}) = e^{-i\lambda\mu}\pi(e^{i\lambda x}) = e^{-i\lambda\mu}M_\lambda,$$

hence the covariant relation holds by the Weyl relations. By the universal property of the crossed product, this yields a representation between two C*-algebras

$$(\pi \rtimes D) : AP(\mathbb{R}) \times_\tau \mathbb{R}_d \to C^*(\pi(AP(\mathbb{R})), D(\mathbb{R}_d)) \,:\, (\pi \rtimes D) \sum_{\substack{s\in F \\ F\subset\subset\mathbb{R}}} (\delta_s \otimes f_s) \mapsto \sum_{\substack{s\in F \\ F\subset\subset\mathbb{R}}} \pi(f_s)D_s.$$

Observe that $C^*(\pi(AP(\mathbb{R})), D(\mathbb{R}_d)) = B_p$. Since $AP(\mathbb{R}) \times_\tau \mathbb{R}_d$ is a simple algebra and $ker(\pi \rtimes D)$ is a two sided ideal, $(\pi \rtimes D)$ is injective, which yields that it is an isometric $*$-isomorphism. □

Remark 4.2.5. By the general theory of crossed products, the mapping

$$E_n : B_p \to B_p : \sum_{\substack{s\in F \\ F\subset\subset\mathbb{R}}} \pi(f_s)D_{\mu_s} \mapsto M_{f_n}$$

is contractive. Moreover, we have a similar expectation for the D_μ operators. By the Weyl relations, we have the covariant relation (ρ, M)

$$\rho : AP(\mathbb{R}) \to B(L^2(\mathbb{R})) : e^{i\lambda x} \mapsto D_\lambda$$

and

$$M : \mathbb{R}_d \to B(L^2(\mathbb{R})) : \mu \mapsto M_{-\mu}.$$

Hence, we have the isomorphism

$$(\rho \rtimes M) : AP(\mathbb{R}) \times_\tau \mathbb{R}_d \to B_p \, : \, (\rho \rtimes M) \sum_{\substack{s \in F \\ F \subset\subset \mathbb{R}}} (\delta_s \otimes f_s) \mapsto \sum_{\substack{s \in F \\ F \subset\subset \mathbb{R}}} \rho(f_s) M_{-s}.$$

Therefore, we have the contractions

$$Z_m : B_p \to B_p : \sum_{\substack{s \in F \\ F \subset\subset \mathbb{R}}} \rho(f_s) M_{-\lambda_s} \mapsto D_{f_m}.$$

Applying the natural isometric isomorphisms $M_f \mapsto f$ and $D_g \mapsto g$, we can identify the range of the maps E_n and Z_m with $AP(\mathbb{R})$.

One may check that $(\rho \rtimes M) \circ (\pi \rtimes D)^{-1} \in \mathrm{Aut}(B_p)$, that sends D_s to M_{-s} and M_λ to D_λ. Since B_p is a concrete operator algebra on $L^2(\mathbb{R})$, by the Stone-von Neumann theorem $(\rho \rtimes M) \circ (\pi \rtimes D)^{-1} = \mathrm{Ad}(F)$, where F is as usual the Fourier-Plancherel transform ([47]).

The closed subalgebra of B_p generated by $\{M_\lambda, D_\mu \, : \, \lambda, \mu \geq 0\}$ is called the **(norm closed) parabolic algebra** and it is denoted by A_p. Evidently,

$$(\pi \rtimes D)^{-1}(A_p) = AAP(\mathbb{R}) \times_\tau \mathbb{R}_d^+,$$

where $AAP(\mathbb{R})$ is the norm closed algebra of analytic almost periodic functions. Applying the contractions E_n, Z_m we obtain by the standard Fejer-Bochner argument that

$$AAP(\mathbb{R}) \times_\tau \mathbb{R}_d^+ = \{X \in AP(\mathbb{R}) \times_\tau \mathbb{R}_d \, : \, E_n(X) = Z_m(X) = 0, \text{ for all } n, m < 0\}.$$

From now on, we identify A_p with the semicrossed product $AAP(\mathbb{R}) \times_\tau \mathbb{R}_d^+$. The first question to wonder for the norm closed algebra is once again the integral domain question, as in the WOT-closed case. The question still seems hard to solve, because of the absence of a first nonzero coefficient. However we can prove that A_p contains no non-trivial idempotents. The following lemma is the key.

Proposition 4.2.6. *The spectrum of every element X in A_p is connected.*

Proof. Let $X \in AAP(\mathbb{R}) \times_\tau \mathbb{R}_d^+$ with spectrum $Sp(X) = U \cup V$, where U, V are non-empty disjoint compact subsets of $\mathbb{C}$. By the density of generalized trigonometric polynomials in A_p, there exists an element $X_0 = \sum_{\substack{s \in F \\ F \subset\subset \mathbb{R}^+}} M_{g_s} D_s$, such that $Sp(X_0)$ is not connected (for this standard Banach algebra fact see for example Theorem 1.1 in [30]). Abusing the notation, we write again that $Sp(X_0) = U \cup V$, for some non-empty disjoint compact sets U and V.

Claim: The norm closed commutative algebra generated by a trigonometric polynomial Z_0, denoted by $A(Z_0)$, is an integral domain.

Let $M > 0$ and let F_n be the finite set of positive indices of the nonzero Fourier coefficients of Z_0^n (so $F_1 = F \backslash \{0\}$). Since Z_0 has only a finite set of nonzero Fourier coefficients, there exists $N > 0$, such that for every $n > N$ we have

$$F_n \cap [0, M] = \emptyset.$$

Define $F_0 = \cup_{n=1}^N F_n \cup \{0\}$. Then for every $t \in [0, M] \backslash F_0$ and $Y = \sum_{n=0}^N c_n Z_0^n$ generalized polynomial we have $E_t(Y) = 0$. Since the subspace of generalized polynomials is dense in $A(Z_0)$ we obtain by continuity of the maps E_t that

$$E_t(Y) = 0, \text{ for all } Y \in A(Z_0).$$

If Y is a nonzero element in $A(Z_0)$, then it has some nonzero Fourier coefficient, say $E_{t_0}(Y)$. Hence the set of indices of nonzero Fourier coefficients of Y in $[0, t_0]$ is finite and nonempty , so it follows that Y has a first nonzero Fourier coefficient.

Let now Y_1, Y_2 be two nonzero elements of $A(Z_0)$ and let m_1 and m_2 be the indices of their respective first nonzero Fourier coefficients. Then $m_1 + m_2$ is the first nonzero Fourier coefficient of the product Y_1Y_2, since

$$E_{m_1+m_2}(Y_1Y_2) = E_{m_1}(Y_1)\tau_{m_1}(E_{m_2}(Y_2))$$

and $E_{m_1}(Y_1), \tau_{m_1}(E_{m_2}(Y_2))$ are two nonzero elements of the integral domain $AAP(\mathbb{R})$. Thus, we proved our claim.

On the other hand, since $Sp(X_0) \subseteq U \cup V$, there are holomorphic functions f, g defined on $U \cup V$, given by $f\Big|_U = g\Big|_V = 1$ and $f\Big|_V = g\Big|_U = 0$. Therefore it follows by Runge's theorem ([14]) and the holomorphic functional calculus ([62]) that $f(X_0), g(X_0) \in A(X_0)$ and

$$f(X_0)g(X_0) = 0,$$

which contradicts the fact that $A(X_0)$ is an integral domain. $\square$

Corollary 4.2.7. *A_p contains no non-trivial idempotents.*

4.2.1 Isometric Automorphisms of A_p

In this section, our goal is to determine the isometric automorphisms of the norm closed parabolic algebra. Interestingly there is a richer diversity than in the WOT-span context. The automorphisms are strongly related to the characters of the discrete real line and the Arens - Singer theory for analytic almost periodic functions [4, 11].

Recall that given a unitary map $U \in B(L^2(\mathbb{R}))$, we can define the automorphism

$$\mathrm{Ad}(U) : B(L^2(\mathbb{R})) \to B(L^2(\mathbb{R})) : T \mapsto UTU^*.$$

For convenience, if $\mathrm{Ad}(U)$ keeps a subspace of $B(L^2(\mathbb{R}))$ invariant, we denote its restriction to the subspace by the same notation. The main theorem of this section is the following.

Theorem 4.2.8. *Let Φ be an isometric automorphism of A_p. Then Φ has the form*

$$\Phi(M_\lambda D_\mu) = c(\mu)d(\lambda)\,\mathrm{Ad}(V_t)(M_\lambda D_\mu),\ \lambda, \mu \in \mathbb{R}^+ \tag{4.7}$$

where $t \in \mathbb{R}$ and c, d are characters of the discrete group of the real numbers. Moreover, the formula (4.7) *gives always a well-defined isometric automorphism of A_p.*

Note that in the special case where the characters c, d are continuous in the standard norm of the reals, then their respective automorphisms are unitarily implemented by M_λ and D_μ, for some $\lambda, \mu \in \mathbb{R}$. The idea of the proof is to work with the induced homeomorphism of the maximal ideal space of the commutative algebra $A_p/\mathfrak{C}_p$, where $\mathfrak{C}_p$ is the commutator ideal of A_p. Similar arguments for the case of crossed products by $\mathbb{Z}^+$ can be found in [60, 73]. The first step is to identify the commutator ideal $\mathfrak{C}_p$. Define the contractions E_n, Z_m as in the previous section and the character x_∞ of $AAP(\mathbb{R})$, as it was defined in 4.1.18.

Lemma 4.2.9. *The commutator ideal $\mathfrak{C}_p$ is equal to the set*

$$\{\alpha \in A_p : E_0(\alpha) = 0, Z_0(\alpha) = 0\}.$$

Proof. If $\alpha = xy - yx \in \mathfrak{C}_p$, then evidently $E_0(\alpha) = Z_0(\alpha) = 0$. On the other hand, for every $\lambda, s > 0$ with λs not equal to $2n\pi$ $(n \in \mathbb{N})$, we have $e^{i\lambda x} = f_s - f_s \circ \tau_s$, where $f_s = e^{i\lambda x}(1 - e^{-i\lambda s})^{-1}$. Hence $e^{i\lambda x} D_s \in \mathfrak{C}_p$, for such λ, s. Since $\mathfrak{C}_p$ is an ideal it follows that $e^{i\lambda x} D_s \in \mathfrak{C}_p$ for every $\lambda, s > 0$. Since these two sets have the same generators (as ideals), the proof is complete. $\square$

Lemma 4.2.10. *$A_p/\mathfrak{C}_p = \{M_f + D_g + \mathfrak{C}_p \,:\, f, g \in AAP(\mathbb{R})\}$.*

Proof. It suffices to prove that the RHS is closed. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence, such that $a_n = M_{f_n} + D_{g_n} + \mathfrak{C}_p$ converging to some $a \in A_p/\mathfrak{C}_p$, that is

$$\inf_{u\in\mathfrak{C}_p} \|a_n - a + u\| \to 0, \text{ as } n \to +\infty.$$

We may assume that $E_0(D_{g_n}) = 0$. Since E_0 is contractive we have that

$$\begin{aligned}
&\|E_0(a_n) - E_0(a)\| \leq \|a_n - a + u\|, \forall u \in \mathfrak{C}_p \\
\Rightarrow &\|M_{f_n} - E_0(a)\| \leq \inf_{u\in\mathfrak{C}_p} \|a_n - a + u\| \to 0, \text{ as } n \to \infty.
\end{aligned}$$

Similarly, we get that $\|D_{g_n} - [Z_0(a) - Z_0(E_0(a))]\| \to 0,$ as $n \to \infty$, so $\|a_n - [E_0(a) + Z_0(a) - Z_0(E_0(a))] + \mathfrak{C}_p\|$ converges to 0, as n goes to infinity. Hence $a = E_0(a) + Z_0(a) - Z_0(E_0(a)) + \mathfrak{C}_p$. □

Let now $\Phi \in \mathrm{Aut}(A_p)$. Then Φ induces an automorphism $\tilde{\Phi} \in \mathrm{Aut}(A_p/\mathfrak{C}_p)$ and a homeomorphism γ_0 between the maximal ideals that contain $\mathfrak{C}_p$, defined by

$$\gamma_0(\zeta)(\alpha + \mathfrak{C}_p) = \zeta(\tilde{\Phi}(\alpha + \mathfrak{C}_p)),\, \zeta \in \mathfrak{M}(A_p/\mathfrak{C}_p).$$

Here, we use the fact that every maximal ideal that contains the commutator ideal is the kernel of a character of the algebra. We want to determine these characters. Write AAP_1 and AAP_2 for the function algebras, both isometrically isomorphic to $AAP(\mathbb{R})$, that are generated by the multiplication and translation unitary semigroups, respectively. Define the mapping

$$\mathfrak{M}(A_p) \to \mathfrak{M}(AAP_1) \times \mathfrak{M}(AAP_2) : \zeta \mapsto (\zeta|_{AAP_1}, \zeta|_{AAP_2}),$$

where the codomain carries the usual product topology.

Lemma 4.2.11. *This map is a homeomorphism onto the subset*

$$(\mathfrak{M}(AAP_1) \times \{x_\infty\}) \cup (\{x_\infty\} \times \mathfrak{M}(AAP_2)).$$

Proof. Let $\zeta \in \mathfrak{M}(A_p)$, such that $\zeta(M_\lambda) \neq 0$, for some $\lambda > 0$. Then it follows by the Weyl relations that $\zeta(D_\mu) = 0$, for every $\mu > 0$. Similarly with the roles of M_λ and D_μ reversed. Hence ζ maps into the set. On the other hand, let (z, x_∞) be a point in the union set. Define on the generalized trigonometric polynomials the multiplicative linear functional ζ by

$$\begin{aligned}\zeta\left(\sum_{\lambda,\mu\in F} c_{\lambda,\mu} M_\lambda D_\mu\right) &= \sum_{\lambda,\mu\in F} c_{\lambda,\mu} z(M_\lambda) x_\infty(D_\mu) = \\ &= \sum_{\lambda\in F} c_{\lambda,0} z(M_\lambda).\end{aligned}$$

But then $\zeta = z \circ E_0$, so it is bounded and extends to a character of A_p. Similarly, we have that for every point (x_∞, z) corresponds the character $z \circ Z_0$. It remains to show that the map is injective and homeomorphic, but this is routine. $\square$

Let χ_∞ be the preimage of (x_∞, x_∞). This the "first coefficient character" on A_p

$$\chi_\infty\left(\sum_{\lambda,\mu\in F} c_{\lambda,\mu} M_\lambda D_\mu\right) = c_{0,0}.$$

Now, Theorem 1.2.22 implies that the maximal ideal space of $AAP(\mathbb{R})$ is the compact topological space $\mathbb{R}_B \times [0,\infty) \cup \{\infty\}$, where $\mathbb{R}_B$ is the Bohr compactification of the real numbers. Write Δ_1, Δ_2 for the maximal ideal spaces of AAP_1 and AAP_2, respectively. Hence, the maximal ideals of A_p that contain $\mathfrak{C}_\mathfrak{p}$ form the connected topological space

$$\Delta_1 \sqcup_{\chi_\infty} \Delta_2.$$

Lemma 4.2.12. *γ_0 fixes χ_∞. Moreover, either $\gamma_0(\Delta_1) = \Delta_1$, or $\gamma_0(\Delta_1) = \Delta_2$.*

Proof. Given $x \in \mathbb{C}^+ \cup \{\infty\}$, let $z_x \in \mathfrak{M}(AAP(\mathbb{R}))$ be the evaluation character at x and ζ_x, η_x be the preimage of the points (z_x, x_∞) and (x_∞, z_x), respectively. Note that the set

$$\mathfrak{M}_{ev}(A_p) = \{\zeta_x, \eta_x : x \in \mathbb{C}^+ \cup \{\infty\}\}$$

is dense in $\Delta_1 \sqcup_{\chi_\infty} \Delta_2$. Also, with the relative product topology, this is homeomorphic to the space

$$(\mathbb{C}^+ \times \{\infty\}) \cup (\{\infty\} \times \mathbb{C}^+) \cap \{(\infty, \infty)\}.$$

Since $\mathfrak{M}_{ev}(A_p)$ is connected, so is the entire character space $\mathfrak{M}(A_p)$ and its homeomorphic space $\Delta_1 \sqcup_{\chi_\infty} \Delta_2$. If we remove the point χ_∞, then the character space, with the relative topology, fails to be connected. We claim that χ_∞ is the only point in the character space with this topological property.

If $\chi \neq \chi_\infty$ is in $\mathfrak{M}_{ev}(A_p)$, then the set of the remaining evaluation characters, with the relative topology, remains connected, and it contains χ in its closure. Hence the space $(\Delta_1 \sqcup_{\chi_\infty} \Delta_2) \backslash \{\chi\}$ remains connected. If χ is a limit character, then once again the space $(\Delta_1 \sqcup_{\chi_\infty} \Delta_2) \backslash \{\chi\}$ contains the dense connected set $\mathfrak{M}_{ev}(A_p)$, so it is connected.

Hence χ_∞ is a fixed point for homeomorphisms.

Consider now the restriction of the homeomorphism γ_0 to $(\Delta_1 \sqcup_{\chi_\infty} \Delta_2) \backslash \{\chi_\infty\}$. Since every homeomorphism maps connected components to connected components, the second assertion of the lemma follows. □

Hence we have two cases.

Case 1 γ_0 keeps Δ_1 and Δ_2 fixed. Let $x \in \mathbb{R}$ and let ζ_x, η_x be the characters defined in the proof of the previous lemma. Since γ_0 keeps Δ_1 invariant, we have

$$0 = \gamma_0(\zeta_x)(D_\mu) = \zeta_x(\tilde{\Phi}(D_\mu)) = E_0(\tilde{\Phi}(D_\mu))(x).$$

Hence $E_0(\tilde{\Phi}(D_\mu)) = 0$ for every $\mu > 0$. Therefore $\tilde{\Phi}(D_\mu + \mathfrak{C}_p) = D_h + \mathfrak{C}_p$, for some $h \in AAP(R)$. Repeating the argument for $\tilde{\Phi}^{-1}$, we have that $\tilde{\Phi}|_{Z_0(A_p/\mathfrak{C}_p)}$ gives an automorphism of $AAP(\mathbb{R})$. Thus, it follows by Theorem 4.1.19 that

$$\tilde{\Phi}(D_\mu + \mathfrak{C}_p) = c(\mu)D_{k_1\mu} + \mathfrak{C}_p, \text{ for some } k_1 > 0,\, c(\mu) \in \mathbb{T}.$$

Applying the same argument on the elements $\tilde{\Phi}(M_\lambda + \mathfrak{C}_p)$, using the η_x characters this time, we get

$$\tilde{\Phi}(M_\lambda + \mathfrak{C}_p) = d(\lambda)M_{k_2\lambda} + \mathfrak{C}_p, \text{ for some } k_2 > 0,\, d(\lambda) \in \mathbb{T}.$$

Hence $\Phi(M_\lambda) = d(\lambda)M_{k_2\lambda} + A$, where A lies in $\mathfrak{C}_p$. The following lemma is the only point of the proof of Theorem 4.2.8 that we will make use of the fact that Φ is isometric.

Lemma 4.2.13. $\Phi(M_\lambda) = d_\lambda M_{k_2\lambda}$.

Proof. First note that

$$\|\Phi(M_\lambda)\| = \|M_\lambda\| = 1 = \|d(\lambda)M_{k_2\lambda}\|.$$

If suffices to prove that every Fourier coefficient of A is zero. We consider the left regular representation $(\tilde{id}, \Lambda)$ of the crossed product. Let F be a finite subset of positive real numbers and ξ a norm one function in $L^2(\mathbb{R})$. By Proposition 4.1.12 we have

$$\begin{aligned}1 = \|\Phi(M_\lambda)\|^2 &\geq \sum_{s\in F\cup\{0\}} \|\tau_{-s}(E_s(\Phi(M_\lambda)))\cdot\xi\|^2_{L^2(\mathbb{R})} = \\ &= \|d(\lambda)e^{ik_2\lambda x}\cdot\xi\|^2_{L^2(\mathbb{R})} + \sum_{s\in F}\|\tau_{-s}(E_s(A))\cdot\xi\|^2_{L^2(\mathbb{R})} = \\ &= 1 + \sum_{s\in F}\|\tau_{-s}(E_s(A))\cdot\xi\|^2_{L^2(\mathbb{R})}.\end{aligned}$$

So $\tau_{-s}(E_s(A)) = 0$, which implies that $E_s(A)$ is the zero function, for every $s \in F$. Since F was arbitrarily chosen, we obtain by Corollary 4.1.10 that $A = 0$. □

Similarly using the left regular representation that corresponds to the $(\rho \rtimes M)$ representation of the crossed product, we obtain $\Phi(D_\mu) = c(\mu)D_{k_1\mu}$.

Now the Weyl relations yield

$$\Phi(M_\lambda D_\mu) = \Phi(e^{i\lambda\mu}D_\mu M_\lambda).$$

The LHS gives

$$\begin{aligned}\Phi(M_\lambda D_\mu) = \Phi(M_\lambda)\varphi(D_\mu) &= d(\lambda)M_{k_2\lambda}c(\mu)D_{k_1\mu} = \\ &= d(\lambda)c(\mu)e^{i\lambda k_1 k_2\mu}D_{k_1\mu}M_{k_2\lambda},\end{aligned}$$

while the RHS is

$$\Phi(e^{i\lambda\mu}D_s M_\lambda) = e^{i\lambda\mu}c(\mu)D_{k_1\mu}d(\lambda)M_{k_2\lambda} = e^{i\lambda\mu}d(\lambda)c(\mu)D_{k_1\mu}M_{k_2\lambda}.$$

Therefore $k_1k_2 = 1$ and so the automorphisms $\Phi(M_\lambda D_\mu) = M_{k_2\lambda}D_{k_1\mu}$ correspond to the automorphisms $\mathrm{Ad}(V_t)$, by taking $t = \log k_1$. Each automorphism of the above form is induced by a covariant respesentation of $(AP(\mathbb{R}), \mathbb{R}_d, \tau)$, so by the universal property of the crossed product it extends to an algebra automorphism of B_p.

Define on $L^2(\mathbb{R})$ the covariant representation $(y_{d,t}, w_{c,t})$ of the C^*-dynamical system $(AP(\mathbb{R}), \mathbb{R}_d, \tau)$, where

$$y_{d,t} : AP(\mathbb{R}) \to B(L^2(\mathbb{R})) : f \mapsto M_{\tilde{\mathrm{id}} \rtimes u_{d,e^t}(f)},$$

where $\tilde{\mathrm{id}} \rtimes u_{d,e^t}$ are given in equation (4.6), and

$$w_{c,t} : \mathbb{R}_d \to B(L^2(\mathbb{R})) : \mu \mapsto c(\mu)D_{\mu e^{-t}}.$$

Indeed, the pair $(y_{d,t}, w_{c,t})$ is a covariant representation, since

$$\begin{aligned} w_{c,t}(\mu)y_{d,t}(e^{i\lambda x})w_{c,t}(-\mu) &= c(\mu)D_{\mu e^{-t}}d(\lambda)M_{\lambda e^t}c(-\mu)D_{-\mu e^{-t}} = \\ &= e^{-i\lambda\mu}d(\lambda)M_{\lambda e^t} = e^{-i\lambda\mu}y_{d,t}(e^{i\lambda x}). = y_{d,t}(\tau_\mu(e^{i\lambda x})) \end{aligned}$$

Hence, by the universal property of the crossed product, we obtain the induced isometric automorphism $y_{d,t} \rtimes w_{c,t}$ of B_p that satisfies

$$M_\lambda D_\mu \mapsto d(\lambda)c(\mu)M_{\lambda e^t}D_{\mu e^{-t}}.$$

It is evident now that the automorphism Φ given in relation (4.7) is of the form $y_{d,t} \rtimes w_{c,t}$ (restricted to A_p), for some $t \in \mathbb{R}$ and $c, d \in \mathbb{R}_B$.

Case 2 γ_0 flips Δ_1 and Δ_2. Repeating the argument of the previous case, we end up with

$$\Phi(M_\lambda) = d(\lambda)D_{k_1\lambda} \quad \text{and} \quad \Phi(D_\mu) = c(\mu)M_{k_2\mu}.$$

Applying again the Weyl commutation relations, we calculate

$$\begin{aligned}\Phi(M_\lambda)\Phi(D_\mu) = e^{i\lambda\mu}\Phi(D_\mu)\Phi(M_\lambda) &\Leftrightarrow d(\lambda)c(\mu)D_{k_1\lambda}M_{k_2\mu} = e^{i\lambda\mu}d(\lambda)c(\mu)M_{k_2\mu}D_{k_1\lambda}\\ &\Leftrightarrow d(\lambda)c(\mu)D_{k_1\lambda}M_{k_2\mu} = e^{i\lambda\mu(1+k_1k_2)}d(\lambda)c(\mu)D_{k_1\lambda}M_{k_2\mu}\end{aligned}$$

which implies that $k_1k_2 = -1$, but this is impossible, since k_1, k_2 are both positive real numbers.

This completes the proof of Theorem 4.2.8. □

4.3 Triple semigroup algebras

As described in the previous section, the dilation operators $\{V_t : t \in \mathbb{R}\}$ implement isometric automorphisms of the C^*-algebra B_p. Let G be the discrete group $\mathbb{R}_d$ or $\mathbb{Z}$ and (B_p, G, v) be the C^*-dynamical system, where v is the group of automorphisms that are unitarily implemented by the operators V_t

$$v : G \to \mathrm{Aut}(B_p) : t \mapsto v_t = \mathrm{Ad}(V_t).$$

Hence, this enables us to define the crossed product, denoted by $B_p \times_v G$. Denote by H_k the contraction from $B_p \times_v G$ onto B_p

$$H_k\Big(\sum_{\substack{\lambda,\mu,t\in F\\ F \text{ finite}}} (\epsilon_t \otimes c_{\lambda,\mu,t}M_\lambda D_\mu)\Big) = \sum_{\substack{\lambda,\mu,t\in F\\ F \text{ finite}}} c_{\lambda,\mu,k}M_\lambda D_\mu.$$

Our next goal is to show that the norm closed algebra

$$B_{ph}^{G} := \|\cdot\|\text{-alg}\{M_\lambda, D_\mu, V_t \,:\, \lambda, \mu \in \mathbb{R}, t \in G\}$$

is isometrically isomorphic to $B_p \times_v G$. By the universal property of the crossed product we have the representation

$$((\pi \rtimes D) \rtimes V) \sum_{\substack{\lambda,\mu,t\in F \\ F \text{ finite}}} (\epsilon_t \otimes c_{\lambda,\mu,t} M_\lambda D_\mu) \mapsto \sum_{\substack{\lambda,\mu,t\in F \\ F \text{ finite}}} c_{\lambda,\mu,t} M_\lambda D_\mu V_t.$$

The following proposition is the key to prove that the above representation is actually an isometric isomorphism.

Proposition 4.3.1. *Given $t_0 \in G$, the mapping*

$$\sum_{\substack{\lambda,\mu,t\in F \\ F \text{ finite}}} c_{\lambda,\mu,t} M_\lambda D_\mu V_t \mapsto \sum_{\substack{\lambda,\mu,t\in F \\ F \text{ finite}}} c_{\lambda,\mu,t_0} M_\lambda D_\mu$$

is contractive, so it extends to a linear contraction $\tilde{H}_{t_0}$ on B_{ph}^{G}.

Proof. It suffices to prove it for $t_0 = 0$. By Poincare's recurrence theorem [24], there exists an increasing unbounded sequence $\{M_n\}_{n\in\mathbb{N}}$ of natural numbers, such that

$$e^{i\lambda M_n} \to 1, \text{ as } n \to \infty \text{ and for all } \lambda \in F.$$

Since $D_{M_n} V_t D_{M_n}^* \overset{\text{WOT}}{\to} 0$ for every $t \neq 0$, one can check that

$$\lim_{n\to\infty} \langle D_{M_n} c_{\lambda,\mu,t} M_\lambda D_\mu V_t D_{M_n}^* f, g\rangle = \begin{cases} \langle c_{\lambda,\mu,0} M_\lambda D_\mu f, g\rangle, & \text{if } t = 0 \\ 0 & \text{if } t \neq 0 \end{cases}$$

Hence

$$\sum_{\substack{\lambda,\mu,t\in F\\ F \text{ finite}}} c_{\lambda,\mu,t} D_{M_n} M_\lambda D_\mu V_t D_{M_n}^* \overset{\text{WOT}}{\to} \sum_{\substack{\lambda,\mu,t\in F\\ F \text{ finite}}} c_{\lambda,\mu,0} M_\lambda D_\mu.$$

Therefore, the proof follows by observing that

$$\left\langle \sum_{\substack{\lambda,\mu,t\in F\\ F \text{ finite}}} c_{\lambda,\mu,t} D_{M_n} M_\lambda D_\mu V_t D_{M_n}^* f, g \right\rangle \leq \| \sum_{\substack{\lambda,\mu,t\in F\\ F \text{ finite}}} c_{\lambda,\mu,t} M_\lambda D_\mu V_t \| \, \|f\| \, \|g\|,$$

for all $f, g \in L^2(\mathbb{R})$. □

Corollary 4.3.2. *The map $(\pi \rtimes D) \rtimes V$ is an isometric $*$-isomorphism of the C^*-algebra $B_p \rtimes_v \mathbb{Z}$.*

Proof. Let $X \in B_p \rtimes_v \mathbb{Z}$, such that $((\pi \rtimes D) \rtimes V)(X) = 0$. Then by the previous proposition we get $\tilde{H}_k(((\pi \rtimes D) \rtimes V)(X)) = 0$, for every $k \in G$. But $\tilde{H}_k \circ ((\pi \rtimes D) \rtimes V) = (\pi \rtimes D) \circ H_k$, since the equality holds for trigonometrical polynomials. Therefore $((\pi \rtimes D) \circ H_k)(X) = 0$, which implies that $H_k(X) = 0$, so $X = 0$. Hence the representation $(\pi \rtimes D) \rtimes V$ is faithful, so isometric. □

We denote by A_{ph}^{G+} the norm closed algebra that is generated by the semigroups of M_λ, D_μ, V_t, where $\lambda, \mu \in \mathbb{R}^+, t \in G^+$. The algebra $A_{ph}^{\mathbb{Z}+}$ is called the **partially discrete triple semigroup algebra**, while the algebra $A_{ph}^{\mathbb{R}+}$ is called the **triple semigroup algebra**.

Let $\mathfrak{C}_{ph}^{G+}$ be the commutator ideal of A_{ph}^{G+}. To describe $\mathfrak{C}_{ph}^{G+}$ we need first the following lemma.

Fix $t > 0$ and let J_t be the closed ideal of $AAP(\mathbb{R})$ generated by the functions of the form

$$e^{i\lambda x} - \varphi_{0,e^t}(e^{i\lambda x}) = e^{i\lambda x} - e^{i\lambda e^t x},$$

for $\lambda > 0$.

Lemma 4.3.3. *The ideal J_t is equal to the ideal $I_0 = \{f \in AAP(\mathbb{R}) : f(0) = x_\infty(f) = 0\}$.*

Proof. It is clear that I_0 contains J_t. To prove the inverse inclusion, note that I_0 has codimension 2, and so it suffices to show that the same holds for J_t. Define the subspace $\tilde{J}_t = span\{a + ce^{ix} : a \in J_t, c \in \mathbb{C}\}$. We claim that $\tilde{J}_t$ is closed.

Let $\{a_n + c_n e^{ix}\}_n$ be a convergent sequence, such that $a_n \in J_t$ and $c_n \in \mathbb{C}$. We claim that the limit of the sequence, say a, lies in $\tilde{J}_t$. Denote by x_1 the character of $AAP(\mathbb{R})$ given by the formula

$$x_1(f) \mapsto f(0). \tag{4.8}$$

Hence $a_n(0) + c_n \to a(0)$. However, since $a_n \in J_t$, it follows that $a_n(0) = 0$, for all $n \in \mathbb{N}$. Therefore

$$c_n \to a(0) \quad \Rightarrow c_n e^{ix} \to a(0)e^{ix}, \quad \text{as } n \to \infty.$$

So $a_n = a_n + c_n e^{ix} - c_n e^{ix} \to a - a(0)e^{ix}$. Since J_t is closed, it contains $a - a(0)e^{ix}$. Hence

$$a = a - a(0)e^{ix} + a(0)e^{ix} \;\in \tilde{J}_t,$$

so $\tilde{J}_t$ is closed.

Hence, it suffices to prove that

$$\tilde{J}_t = \overline{span\{e^{i\lambda x} : \lambda > 0\}}^{\|\cdot\|_\infty}.$$

Since J_t is an ideal in $AAP(\mathbb{R})$, we get

$$e^{i(\kappa+\lambda)x} - e^{i(\kappa+\lambda e^t)x} - e^{ix} \in \tilde{J}_t, \text{ for all } \kappa, \lambda \in (0, \infty).$$

Choose $\kappa + \lambda = 1$, so $e^{i\rho x} \in \tilde{J}_t$ for every $\rho \in [1, e^t)$. Thus, by induction, we have that

$$e^{i\rho e^{(n-1)t}x} - e^{i\rho e^{nt}x} - e^{i\rho e^{(n-1)t}x} = -e^{i\rho e^{nt}x} \in \tilde{J}_t,$$

and

$$e^{i\rho e^{-nt}x} - e^{i\rho e^{-(n-1)t}x} + e^{i\rho e^{-(n-1)t}x} = e^{i\rho e^{-nt}x} \in \tilde{J}_t,$$

for every $\rho \in [1, e^t)$ and $n \in \mathbb{N}$. Hence $e^{i\lambda x} \in \tilde{J}_t$, for all $\lambda \in (0, \infty)$, and hence the proof is complete. $\square$

Proposition 4.3.4. *The commutator ideal $\mathfrak{C}_{ph}^{G+}$ is equal to the set*

$$\ker(E_0 \circ H_0) \cap \ker(Z_0 \circ H_0) \cap \bigcap_{t \in G^+} \left(\ker(\chi_\infty \circ H_t) \cap \ker(x_1 \circ E_0 \circ H_t) \cap \ker(x_1 \circ Z_0 \circ H_t)\right). \tag{4.9}$$

Proof. Let I be the set described in (4.9). Since I is the intersection of kernels of bounded linear operators, it is closed. One can check that if $X = \sum_{\substack{\lambda,\mu,t \in F \\ F \text{ finite}}} c_{\lambda,\mu,t} M_\lambda D_\mu V_t$ is a trigonometric polynomial in I, then it satisfies

1. $c_{\lambda,0,0} = c_{0,\mu,0} = c_{0,0,t} = 0, \text{ for all } \lambda, \mu \in \mathbb{R}^+, t \in G^+;$

2. $\sum_\lambda c_{\lambda,0,t} = \sum_\mu c_{0,\mu,t} = 0, \text{ for all } t \in G^+.$

It is elementary to show that if X, Y trigonometric polynomials in A_{ph}^{G+}, then $XY - YX \in I$. Since multiplication is jointly continuous with respect to the operator norm, it follows by the density of trigonometric polynomials in A_{ph}^{G+} that $XY - YX \in I$, for every $X, Y \in A_{ph}^{G^+}$. Similarly, working first with trigonometric polynomials, we obtain that I is closed under the ideal operations.

For the converse inclusion, let $X \in I$. By Theorem 4.1.9, it suffices to show that $H_t(X)V_t \in \mathfrak{C}_{ph}^{G^+}$, for every $t \in G^+$. As we proved in Lemma 4.2.9, one can check that

$$H_t(X) - E_0(H_t(X)) - Z_0(H_t(X)) + (E_0 \circ Z_0)(H_t(X)) \in \mathfrak{C}_{ph}^{G^+},$$

since it lies in $\mathfrak{C}_p$. Moreover, we obtain by the definition of I that $(E_0 \circ Z_0)(H_t(X)) = (\chi_\infty \circ H_t)(X) = 0$, so it follows that

$$H_t(X)V_t - E_0(H_t(X))V_t - Z_0(H_t(X))V_t \in \mathfrak{C}_{ph}^{G^+}.$$

Hence it suffices to show that $E_0(H_t(X))V_t$ and $Z_0(H_t(X))V_t$ lie in $\mathfrak{C}_{ph}^{G^+}$.

Write $E_0(H_t(X))V_t = M_f V_t$, for some $f \in AAP(\mathbb{R})$. Since $X \in I$, f satisfies the properties $x_\infty(f) = f(0) = 0$. So by the previous lemma there exist $g_n \in AAP(R)$, $n \in \mathbb{N}$, such that $f = \lim_n (g_n - \varphi_{0,e^t}(g_n))$, which implies that $M_f V_t = \lim_n (M_{g_n} V_t - V_t M_{g_n})$, so $M_f V_t$ lies in $\mathfrak{C}_{ph}^{G^+}$. Similarly every element $D_f V_t \in I$ belongs to $\mathfrak{C}_{ph}^{G^+}$, so our proof is complete. $\square$

Before this subsection ends, we prove the existence of two more contractive maps, which will be helpful in the next section.

Proposition 4.3.5. *The maps*

$$\sum_{\substack{\lambda,\mu,t \in F \\ F \text{ finite}}} c_{\lambda,\mu,t} M_\lambda D_\mu V_t \mapsto \sum_{\substack{\lambda,t \in F \\ F \text{ finite}}} c_{\lambda,0,t} V_t$$

$$\sum_{\substack{\lambda,\mu,t \in F \\ F \text{ finite}}} c_{\lambda,\mu,t} M_\lambda D_\mu V_t \mapsto \sum_{\substack{\mu,t \in F \\ F \text{ finite}}} c_{0,\mu,t} V_t$$

are contractive.

Proof. The proof uses similar arguments as in Proposition 4.3.1, working now with the WOT-limits

$$\sum_{\substack{\lambda,\mu,t\in F\\ F \text{ finite}}} c_{\lambda,\mu,t} V_n^* M_\lambda D_\mu V_t V_n \overset{\text{WOT}}{\to} \sum_{\substack{\lambda,t\in F\\ F \text{ finite}}} c_{\lambda,0,t} V_t$$
$$\sum_{\substack{\lambda,\mu,t\in F\\ F \text{ finite}}} c_{\lambda,\mu,t} V_n M_\lambda D_\mu V_t V_n^* \overset{\text{WOT}}{\to} \sum_{\substack{\mu,t\in F\\ F \text{ finite}}} c_{0,\mu,t} V_t,$$

as $n \to \infty$. □

4.3.1 The algebra $A_{ph}^{\mathbb{Z}_+}$

We focus now on the partly discrete triple semigroup algebra $A_{ph}^{\mathbb{Z}_+}$. In order to determine the isometric automorphisms of $A_{ph}^{\mathbb{Z}_+}$, we work again on the induced homeomorphism of the character space onto itself. Define the characters $x_1 \in \mathfrak{M}(AAP(\mathbb{R}))$, such that $x_1(f) = f(0)$, and $\chi_\infty = (x_\infty, x_\infty)$ as before. Let also y_0 be the character in the disc algebra $A(\mathbb{D})$ (see [31]), given by $y_0(f) = f(0)$.

Proposition 4.3.6. *The mapping*

$$\psi : \mathfrak{M}(A_{ph}^{\mathbb{Z}_+}) \to \mathfrak{M}(AAP_1) \times \mathfrak{M}(AAP_2) \times \mathfrak{M}(A(\mathbb{D})) : \chi \mapsto (\chi\big|_{AAP_1}, \chi\big|_{AAP_2}, \chi\big|_{A(\mathbb{D})})$$

is continuous into the subset

$$(\mathfrak{M}(AAP_1) \times \{x_\infty\} \times \{y_0\}) \cup (\{x_\infty\} \times \mathfrak{M}(AAP_2) \times \{y_0\}) \cup$$
$$\cup (\{x_1\} \times \{x_\infty\} \times \mathfrak{M}(A(\mathbb{D}))) \cup (\{x_\infty\} \times \{x_1\} \times \mathfrak{M}(A(\mathbb{D}))) \cup (\{x_\infty\} \times \{x_\infty\} \times \mathfrak{M}(A(\mathbb{D}))).$$

Proof. Let χ be a character in $\mathfrak{M}(A_{ph}^{\mathbb{Z}_+})$. Then

$$\chi\big|_{A_p} \in \mathfrak{M}(A_p) \quad \text{and} \quad \chi\big|_{\|\cdot\|\text{-alg}\{V_t : t\in\mathbb{Z}^+\}} \in \mathfrak{M}(A(\mathbb{D})).$$

One can check that if $\chi\Big|_{A_p}$ does not correspond to a point $\{\chi_\infty, (x_1, x_\infty), (x_\infty, x_1)\}$, then by the commutation relations we get that $\chi(V_t) = 0$, for all positive t.

On the other hand, if $\chi\Big|_{\|\cdot\|\text{-alg}\{V_t : t\in\mathbb{Z}^+\}} \neq 0$, then by the commutation relations we have three cases for $\chi\Big|_{A_p} \in \mathfrak{M}(A_p)$:

1. $\chi(M_\lambda) = 1$ and $\chi(D_\mu) = 0$, which corresponds to the character (x_1, x_∞) in $\mathfrak{M}(A_p)$.

2. $\chi(M_\lambda) = 0$ and $\chi(D_\mu) = 1$, so we get the character (x_∞, x_1).

3. $\chi(M_\lambda) = \chi(D_\mu) = 0$, which gives χ_∞.

Hence the mapping ψ is well defined. Continuity is evident, so the proof is complete. □

Note that every element in the codomain of ψ corresponds to a multiplicative linear functional defined on the non-closed algebra of trigonometric generalized polynomials. Write once again Δ_1, Δ_2 for the sets $\mathfrak{M}(AAP_1) \times \{x_\infty\} \times \{y_0\}$ and $\{x_\infty\} \times \mathfrak{M}(AAP_2) \times \{y_0\}$ respectively. If χ is such a multiplicative functional, then the contraction H_0 yield that χ is bounded and extends to a character of $A_{ph}^{\mathbb{Z}^+}$. Therefore, any maximal ideal of A_p corresponding to a point $(\Delta_1 \sqcup_{\chi_\infty} \Delta_2) \backslash \{\chi_\infty, (x_1, x_\infty), (x_\infty, x_1)\}$ is contained in a unique maximal ideal in $A_{ph}^{\mathbb{Z}^+}$. Similarly, by Lemma 4.3.5 any multiplicative functional of the form $(x_1, x_\infty, y), (x_\infty, x_1, y)$, with $y \in \mathfrak{M}(A(\mathbb{D}))$, is bounded. We denote by Δ_3 the sets of characters that give $\chi(M_\lambda) = 1$, for all λ, and by Δ_4 the characters that satisfy $\chi(D_\mu) = 1$, for all μ.

The pursuit of the continuity of the remaining multiplicative functionals (on the dense subalgebra) that correnspond to the points (x_∞, x_∞, y), we write Δ_0, is more subtle and it remains unclear to the author if this formula can generate a bounded character of $A_{ph}^{\mathbb{Z}^+}$.

Remark 4.3.7. It is trivial to show that given an element u of the commutator ideal of a commutative Banach algebra A, then $\chi(u) = 0$ for every character χ of A. The opposite direction is not true in the case that A contains quasinilpotent elements. A complication with $A_{ph}^{\mathbb{Z}^+}$ is that we cannot determine if the elements of the form $V_t - M_\lambda V_t - D_\mu V_t + \mathfrak{C}_{ph}^{\mathbb{Z}^+}, \lambda, \mu \in \mathbb{R}^+, t \in \mathbb{Z}^+$ are quasinilpotent, a property which turns out to be equivalent to the continuity of specific elements in Δ_0.

We now obtain a partial identification of the character space of $A_{ph}^{\mathbb{Z}^+}$, which is sufficient for our main results in the next section. See Figure 4.1.

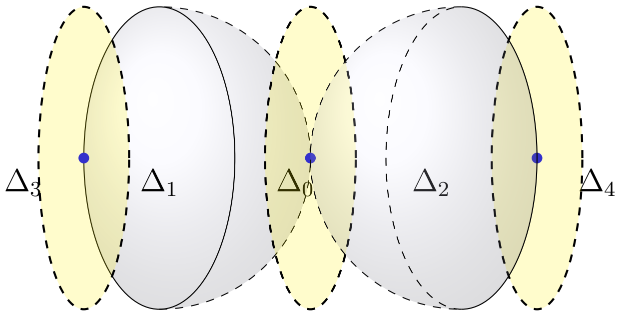

Fig. 4.1 The topological space $\Delta_0 \sqcup \Delta_1 \sqcup \Delta_2 \sqcup \Delta_3 \sqcup \Delta_4$.

Proposition 4.3.8. *The character space $\mathfrak{M}(A_{ph}^{\mathbb{Z}^+})$ has the form $\tilde{\Delta}_0 \sqcup \Delta_1 \sqcup \Delta_2 \sqcup \Delta_3 \sqcup \Delta_4$, where $\tilde{\Delta}_0$ is either the point $\{x_\infty, x_\infty, y_0\}$ or a closed disc in Δ_0.*

Proof. If there is no continuous character of $\mathfrak{M}(A_{ph}^{\mathbb{Z}^+})$ in Δ_0, apart from $\{x_\infty, x_\infty, y_0\}$, then there is nothing to prove. Assume now that χ is a continuous character in Δ_0, so $\chi(V_t) = z^t$ for some $z \neq 0$ in the unit disk. Hence

$$|\sum_t \chi(a_t) z^t| \leq \|\sum_t a_t V_t\|\,,\ a_t \in A_p.$$

Applying the dual automorphisms $\varphi_{e^{i\theta}}$ of $A_p \rtimes_v \mathbb{Z}^+$ for any $\theta \in (0, 2\pi)$, it follows that

$$|\sum_{t\in\mathbb{N}} \chi(a_t)(ze^{i\theta})^t| \leq \|\sum_{t\in\mathbb{N}} e^{i\theta t} a_t V_t\| = \|\sum_{t\in\mathbb{N}} a_t V_t\|.$$

Therefore, by the maximum principle, each multiplicative linear functional of the form $a_t V_t \mapsto \chi(a_t) w^t$, where $|w| \leq |r|$, is continuous. $\square$

Theorem 4.3.9. *The isometric isomorphisms of $A_{ph}^{\mathbb{Z}^+}$ are of the form*

$$\Phi(M_\lambda) = M_{k_1\lambda},\ \Phi(D_\mu) = D_{k_2\mu} \text{ and } \Phi(V_t) = c(t)V_t, \tag{4.10}$$

where $k_1k_2 = 1$ and $c : t \mapsto c(t)$ is multiplicative.

Proof. Let Φ be an isometric isomorphism of $A_{ph}^{\mathbb{Z}^+}$. Once again we consider the induced homeomorphism

$$\gamma : \mathfrak{M}(A_{ph}^{\mathbb{Z}^+}) \to \mathfrak{M}(A_{ph}^{\mathbb{Z}^+}) : \chi \mapsto \chi \circ \Phi^{-1}.$$

It follows by Proposition 4.3.8 that γ fixes the subset of characters $\Delta_p = \Delta_1 \sqcup_{(x_\infty, x_\infty, y_0)} \Delta_2$. Hence the ideal $\mathcal{I} = \cap_{\chi \in \Delta_p} \ker\chi$ is fixed by Φ. By Proposition 4.3.1 it follows that the quotient algebra $A_{ph}^{\mathbb{Z}^+}/\mathcal{I}$ is isomorphic to $A_p/\mathfrak{C}_p$. So the naturally induced automorphism $\tilde{\Phi}$ of the quotient algebra satisfies

$$\tilde{\Phi}(M_\lambda + \mathcal{I}) = d(\lambda) M_{k_1\lambda} + \mathcal{I}$$

$$\tilde{\Phi}(D_\mu + \mathcal{I}) = c(\mu) D_{k_2\mu} + \mathcal{I}$$

where $k_1k_2 = 1$ and c, d are characters of the discrete group of the real numbers. Applying the same argument as in Lemma 4.2.13 we get that $\Phi(M_\lambda) = d(\lambda)M_{k_1\lambda}$ and $\Phi(D_\mu) = c(\mu)D_{k_2\mu}$. Now, since the characters in Δ_3 are continuous, by the commutation relations we get that

$$\Phi(V_t)\Phi(M_\lambda) = \Phi(M_{\lambda e^t})\Phi(V_t) \Rightarrow d(\lambda) = d(\lambda e^t) \Rightarrow d(\lambda) = 1.$$

Similarly, using the continuity of the characters in Δ_4, we get that $c(\mu) = 1$, for every $\mu > 0$. The argument to determine the image of the dilation operators is developed entirely on $L^2(\mathbb{R})$. Since $\Phi(V_t)M_{k_1\lambda} = M_{k_1\lambda e^t}\Phi(V_t)$, if we right multiply both sides by V_t^*, we get

$$\Phi(V_t)V_t^* M_{k_1\lambda e^t} = M_{k_1\lambda e^t}\Phi(V_t)V_t^*.$$

Hence $\Phi(V_t)V_t^*$ commutes with every $M_\lambda, \lambda \in \mathbb{R}$, so it lies in the multiplication algebra $\mathcal{M}_m$, since this algebra is maximal abelian. Mimicking the same argument for the commutation relation with the translation operator, we get that $\Phi(V_t)V_t^*$ is also in the translation algebra $\mathcal{D}_m$. But the intersection of these two algebras is the multiples of the identity operator, so $\Phi(V_t) = c(t)V_t$. We proved that Φ satisfies $\Phi(M_\lambda) = M_{k_1\lambda}$, $\Phi(D_\mu) = D_{k_2\mu}, \Phi(V_t) = c(t)V_t$, where $k_1k_2 = 1$. Moreover since c is multiplicative, we obtain that $c(t) = e^{i\theta t}$, for some $\theta \in [0, 2\pi)$ independent of t. By the universal property of the crossed product, any such mapping can extend to an isometric isomorphism of $A_p \rtimes_v \mathbb{Z}^+$. □

Theorem 4.3.10. *The algebra $A_{ph}^{\mathbb{Z}^+}$ is chiral.*

Proof. It suffices to prove that $A_{ph}^{\mathbb{Z}^+}$ is not isometrically isomorphic to its conjugate algebra $(A_{ph}^{\mathbb{Z}^+})^*$. If Φ was such an isomorphism, then following the same proof as in the previous theorem we get that $\Phi(M_\lambda) = M_{-k_1\lambda}$ and $\Phi(D_\mu) = D_{-k_2\mu}$. But then again, we can prove that $\Phi(V_t)V_t^* = c(t)I$, so $\Phi(V_t) = c(t)V_t \notin (A_{ph}^{\mathbb{Z}^+})^*$. □

4.3.2 The algebra $A_{ph}^{\mathbb{R}^+}$

The approach to the triple semigroup algebra is similar to the case of $A_{ph}^{\mathbb{Z}^+}$. Note that the algebra generated by the unitary semigroup $\{V_t\}_{t\geq 0}$ is isometrically isomorphic to

$AAP(\mathbb{R})$. Writing AAP_3 for this algebra, we obtain that the mapping

$$\mathfrak{M}(A_{ph}^{\mathbb{R}^+}) \to \mathfrak{M}(AAP_1) \times \mathfrak{M}(AAP_2) \times \mathfrak{M}(AAP_3) : \chi \mapsto (\chi\big|_{AAP_1}, \chi\big|_{AAP_2}, \chi\big|_{AAP_3})$$

is continuous into the subset

$$(\mathfrak{M}(AAP_1) \times \{x_\infty\} \times \{x_\infty\}) \cup (\{x_\infty\} \times \mathfrak{M}(AAP_2) \times \{x_\infty\}) \cup$$
$$\cup (\{x_1\} \times \{x_\infty\} \times \mathfrak{M}(AAP_3)) \cup (\{x_\infty\} \times \{x_1\} \times \mathfrak{M}(AAP_3)) \cup (\{x_\infty\} \times \{x_\infty\} \times \mathfrak{M}(AAP_3)).$$

We also keep the notation for $\Delta_1, \Delta_2, \Delta_3, \Delta_4, \Delta_0$ as in the previous section. Note now that each disk is homeomorphic to the topological space $\mathbb{R}_B \times [0, \infty) \cup \infty$. Again, the continuity of the characters in $\Delta_1, \Delta_2, \Delta_3$ and Δ_4 follows from Propositions 4.3.1 and 4.3.5, while it is unknown to the author if the multiplicative linear functionals in Δ_0 are continuous. Moreover, it remains also unclear if Proposition 4.3.8 holds in this case, since we may have continuous limit characters in Δ_0. Nonetheless, let χ_z be the multiplicative functional in Δ_0, that evaluates a function in $AAP3$ to the point z of the upper half plane of $\mathbb{C}$. If χ_z was continuous, then mimicking the proof of 4.3.8, we would get that any multiplicative functional of the form χ_w, where $Im(w) \geq Im(z)$, is continuous. Moreover, any limit character in the closure of the set $\{\chi_w : Im(w) \geq Im(z)\}$ would be continuous.

Given now any isometric isomorphism Φ of $A_{ph}^{\mathbb{R}^+}$, define the induced homeomorphism, say γ, of the character space $\mathfrak{M}(A_{ph}^{\mathbb{R}^+})$ onto itself. Since by Theorem 1.2.22 the set of limit characters has empty interior, it follows that γ permutes the discs. Hence it fixes the set Δ_p of characters that map the family of the dilation operators $\{V_t\}_{t>0}$ to zero. This is the closure of the set of characters of the norm closed parabolic algebra that are extended uniquely in the triple semigroup algebra. Hence using the same arguments we get that the restriction of Φ in A_p is a isometric automorphism of the parabolic algebra. Then, repeating the last argument of the proof of Theorem 4.3.9, we have the corresponding result;

Theorem 4.3.11. *The isometric isomorphisms of $A_{ph}^{\mathbb{R}^+}$ are of the form*

$$\Phi(M_\lambda) = M_{k_1\lambda},\ \Phi(D_\mu) = D_{k_2\mu} \textit{ and } \Phi(V_t) = c(t)V_t,$$

where $k_1k_2 = 1$ and $c : t \mapsto c(t)$ is multiplicative. Furthermore, the algebra $A_{ph}^{\mathbb{R}^+}$ is chiral.

Remark 4.3.12. As we showed in Theorem 3.5.1, the unitary automorphisms of the weak*-closed triple semigroup algebra $\mathcal{A}_{ph}$ are of the form $\mathrm{Ad}(V_t)$. It is still unknown if these are also the isometric isomorphisms of the algebra. In particular, it remains unclear to the author if the dual automorphisms of the norm closed algebra $\mathcal{A}_{ph}^{\mathbb{R}^+}$ can be extended to its weak*-closure.

Chapter 5

Further results/research

5.1 Quasicompact algebras

By the term **quasicompact algebra**, we mean an algebra of the type

$$Q\mathcal{A} = (\mathcal{A} + K(H)) \cap (\mathcal{A}^* + K(H)),$$

where $\mathcal{A}$ is a (usually weak*-closed) operator algebra and $K(H)$ is the ideal of the compact operators on the Hilbert space on which $\mathcal{A}$ acts. We will refer to the algebra $Q\mathcal{A}$ as the quasicompact algebra of $\mathcal{A}$.

In the past, analogous algebras have been studied systematically in the theory of function spaces. The major example is the algebra of **quasicontinuous functions**

$$QC(\mathbb{T}) = (H^\infty(\mathbb{T}) + C(\mathbb{T})) \cap (H^\infty(\mathbb{T})^* + C(\mathbb{T})),$$

a C*-algebra that contains strictly the algebra $C(\mathbb{T})$ of continuous functions on the unit circle (see [18]).

In the case where the operator algebra $\mathcal{A}$ is a selfadjoint algebra, it is trivial to see that $Q\mathcal{A} = \mathcal{A} + K(H)$. In general, the question of whether $Q\mathcal{A}$ is equal to $\mathcal{A} \cap \mathcal{A}^* + K(H)$ seems to be deep.

The first result in this connection for non-selfadjoint operator algebras is related with the quasitriangular algebra $\mathcal{QA}_v$ of the Volterra nest. Let

$$\mathcal{P}_v : \mathcal{C}_2(L^2(\mathbb{R})) \to \mathcal{A}_v \cap \mathcal{C}_2(L^2(\mathbb{R}))$$

be the **triangular truncation operator** with respect to the Volterra nest. Even though $\mathcal{P}_v$ is a contractive projection in the Hilbert-Schmidt norm, it is an unbounded operator with respect to the operator norm (see [15]). This fact leads to the following theorem.

Theorem 5.1.1. *The quasicompact algebra $Q\mathcal{A}_v$ is strictly larger than the algebra $\mathcal{A}_v \cap \mathcal{A}_v^* + K(H)$.*

We omit the proof, because it can be obtained by the same method used to prove Theorem 5.1.5 below, with some simplifications.

In this section, we study the quasicompact algebra of $\mathcal{A}_p$. Theorem 1.2.37 yields that $Q\mathcal{A}_p$ is a C*-algebra. Our goal is to answer the following problem :

Is the quasicompact algebra of $\mathcal{A}_p$ strictly larger than $\mathcal{A}_p \cap \mathcal{A}_p^* + K(H) = \mathbb{C}I + K(H)$?

Lemma 5.1.2. *The restriction of the triangular truncation operator $\mathcal{P}_v\big|_{\mathcal{A}_p + \mathcal{A}_p^*}$ is unbounded.*

Proof. Let p_n be a real coefficient polynomial on $\mathbb{T}$ with supremum norm 1, such that the polynomials $f_n(z) = p_n(z) - \overline{p_n(z)}$ satisfy the property $\|f_n\|_\infty \to 0$. For example, take

$$p_n(z) = c_n \sum_{k=1}^{n} \frac{1}{k} z^k.$$

for appropriate constants c_n. Let Z be a unitary operator in $\mathcal{A}_p$ with full spectrum, such as M_1. Take (F_n) to be a bounded approximate identity of Hilbert-Schmidt operators in the unit ball of A_p. It is trivial to see that F_n^* is a bounded approximate identity on the space of Hilbert-Schmidt operators in $\mathcal{A}_p^*$.

By the functional calculus, we have $\|p_n(Z)\| = \|p_n\| = 1$. Hence there exists a sequence (ξ_n) in the unit sphere of $L^2(\mathbb{R})$, such that $\|p_n(Z)\xi_n\| > 2/3$, for every $n \in \mathbb{N}$. If we fix some $n \in \mathbb{N}$, then we get $p_n(ZF_m)\xi_n \to p_n(Z)\xi_n$. Therefore, we can choose inductively a subsequence (F_{m_n}), which will be denoted by (F_n), such that

$$\|p_n(ZF_n)\xi_n\| > 1/2.$$

Since $p_n(ZF_n)$ is an element of $\mathcal{A}_p$, we have $\langle K, p_n(ZF_n)^*\rangle_{H-S} = 0$, for every Hilbert-Schmidt operator $K \in \mathcal{A}_v$ and $n \in \mathbb{N}$. Thus

$$\begin{aligned}\|\mathcal{P}_v(p_n(ZF_n) - p_n(ZF_n)^*)\| =& \|\mathcal{P}_v(p_n(ZF_n)) - \mathcal{P}_v(p_n(ZF_n)^*)\| = \\ &= \|p_n(ZF_n)\| \geq \|p_n(ZF_n)\xi_n\| > 1/2.\end{aligned}$$

On the other hand, since ZF_n is a contraction for every $n \in \mathbb{N}$, von Neumann's inequality([50]) yields

$$\|p(ZF_n) + q(ZF_n)^*\| \leq \|p + \overline{q}\|$$

for all p, q polynomials in the disc algebra. Taking $p = p_n$ and $q = -p_n$, it follows

$$\|p_n(ZF_n) - p_n(ZF_n)^*\| \leq \|p_n - \overline{p_n}\| \to 0,$$

which completes the proof. □

Lemma 5.1.3. *Let K be a compact operator in $B(L^2(\mathbb{R}))$. Given $\epsilon > 0$, there exists C compact subset of $\mathbb{R}$, such that for all $f \in L^2(\mathbb{R})$ we have*

1. $\|Kf\| \leq \|K\|(\|P_C f\| + \epsilon\|f\|)$;

2. $\|P_{\mathbb{R}\setminus C} K f\| \leq \epsilon\|f\|$.

Proof. It suffices to prove it for K finite rank operator. Let $g_i, h_i \in L^2(\mathbb{R})$, $i \in \{1, \dots, n\}$ such that

$$Kf = \sum_{i=1}^{n} \langle f, g_i \rangle h_i, \text{ for all } f \in L^2(\mathbb{R}).$$

For every compact set C we get

$$\begin{aligned}\|Kf\| &\leq \left\| \sum_{i=1}^{n} \langle P_C f, g_i \rangle h_i \right\| + \left\| \sum_{i=1}^{n} \langle P_{\mathbb{R}\setminus C} f, g_i \rangle h_i \right\| \leq \\ &\leq \|KP_C f\| + \sum_{i=1}^{n} \|P_{\mathbb{R}\setminus C} g_i\| \, \|h_i\| \, \|f\|.\end{aligned}$$

Let $\epsilon > 0$. Choose C_1 such that

$$\|P_{\mathbb{R}\setminus C_1} g_i\| \leq \frac{\epsilon \, \|K\|}{\sum\limits_{i=1}^{n} \|h_i\|}$$

for all g_i. Similarly,

$$\|P_{\mathbb{R}\setminus C} K_f\| = \left\| \sum_{i=1}^{n} \langle f, g_i \rangle P_{\mathbb{R}\setminus C} h_i \right\| \leq \sum_{i=1}^{n} \|g_i\| \, \|P_{\mathbb{R}\setminus C} h_i\| \, \|f\|,$$

so we can choose C_2 that satisfies

$$\max_i \{\|P_{\mathbb{R}\setminus C_2} h_i\|\} \leq \frac{\epsilon \, \|K\|}{\sum\limits_{i=1}^{n} \|g_i\|}.$$

Take $C = C_1 \cup C_2$. $\square$

Lemma 5.1.4. *The algebra $\mathbb{C}I + K(H)$ is norm closed.*

Proof. Let $(c_n I + K_n)_n$ be a Cauchy sequence in $\mathbb{C}I + K(H)$ with $c_n \in \mathbb{C}$ and $K_n \in K(H)$. Then, given $\epsilon > 0$, there exists $N_0 \in \mathbb{N}$ such that for every $n, m \geq N_0$ we have

$$\|(c_n - c_m)I + K_n - K_m\| \leq \epsilon/2.$$

Take $n, m > N_0$. By Lemma 5.1.3, there exists $f \in L^2(\mathbb{R})$ with unit norm such that $\|K_n f\| + \|K_m f\| < \epsilon/4$. Hence we obtain that

$$|c_n - c_m| = \|(c_n - c_m)f\| \leq \|(c_n - c_m)f + (K_n - K_m)f\| + \|(K_n - K_m)f\| \leq \epsilon.$$

Thus $(c_n)_n$ is a Cauchy sequence, which implies that the limit of the sequence $(c_n I + K_n)_n$ lies in $\mathbb{C}I + K(H)$. $\square$

Now, we are in the position to give an affirmative answer to the problem stated above.

Theorem 5.1.5. *The quasicompact algebra of $\mathcal{A}_p$ is strictly larger than the algebra $\mathcal{A}_p \cap \mathcal{A}_p^* + K(H) = \mathbb{C}I + K(H)$.*

Proof. Take operators $p_n(ZF_n)$ as in the proof of Lemma 5.1.2. Since these operators are compact, there exist compact intervals K_n of the real line, such that for any $f \in L^2(\mathbb{R})$ we have

$$\|p_n(ZF_n)f\| \leq \|P_{K_n} f\| + \frac{1}{2^n}\|f\| \tag{5.1}$$

and

$$\|P_{\mathbb{R}\setminus K_n} p_n(ZF_n)f\| \leq \frac{1}{2^n}\|f\| \tag{5.2}$$

where P_{K_n} is the projection on K_n. We also demand

$$\| \left(p_n(ZF_n) - p_n(ZF_n)^*\right) f\| \leq \|p_n(ZF_n) - p_n(ZF_n)^*\| \left(\|P_{K_n} f\| + \frac{1}{2^n}\|f\|\right) \tag{5.3}$$

and

$$\|P_{\mathbb{R}\setminus K_n} \left(p_n(ZF_n) - p_n(ZF_n)^*\right) f\| \leq \frac{1}{2^n}\|p_n(ZF_n) - p_n(ZF_n)^*\| \, \|f\|. \tag{5.4}$$

Choose $t_n >> 0$, in order to force the sets $\Lambda_n = K_n + t_n := \{x_n + t_n \,:\, x_n \in K_n\}$ to be disjoint, and in particular

$$\max \Lambda_n < \min \Lambda_{n+1}.$$

We also write $\Lambda_0 = \mathbb{R} \setminus \cup_{n=1}^{\infty} \Lambda_n$. Since the projection of triangular truncation with respect to the binest commutes with Ad_{D_t}, it follows that the operators

$$A_n = D_{t_n}(p_n(ZF_n))D_{t_n}^*$$

lie in $\mathcal{A}_p$.

<u>Claim 1:</u> Given $f \in L^2(\mathbb{R})$, the sequence $\{\sum_{k=1}^{n} A_k f\}_n$ is convergent.

To prove our claim, it suffices to show that the given sequence is Cauchy. Let $\epsilon > 0$, we need to configure $n_0 = n_0(\epsilon, f) \in \mathbb{N}$ such that for every $n, N > n_0$ we have $\| \sum_{k=n}^{N} A_k f\| \leq \epsilon$. Denote by C the compact set $\cup_{m=n}^{N} \Lambda_m$. Then

$$\left\| \sum_{k=n}^{N} A_k f \right\|^2 = \int_{\mathbb{R}\setminus C} \left| \sum_{k=n}^{N} A_k f \right|^2 + \int_C \left| \sum_{k=n}^{N} A_k f \right|^2.$$

We estimate each integral separately.

- $\displaystyle \int_{\mathbb{R}\setminus C} \left| \sum_{k=n}^{N} A_k f \right|^2 = \left\| P_{\mathbb{R}\setminus C} \sum_{k=n}^{N} A_k f \right\|^2 \leq \left(\sum_{k=n}^{N} \|P_{\mathbb{R}\setminus C} A_k f\| \right)^2 \leq \left(\sum_{k=n}^{N} \frac{1}{2^k}\|f\| \right)^2 = \left(\sum_{k=n}^{N} \frac{1}{2^k} \right)^2 \|f\|^2.$

- $\int_C \left| \sum_{k=n}^{N} A_k f \right|^2 = \int_{\mathbb{R}} \left| \sum_{m=n}^{N} P_{\Lambda_m} \sum_{k=n}^{N} A_k f \right|^2 \leq 2 \int_{\mathbb{R}} \left| \sum_{m=n}^{N} P_{\Lambda_m} A_m f \right|^2 + 2 \int_{\mathbb{R}} \left| \sum_{m=n}^{N} \sum_{\substack{k=n \\ k \neq m}}^{N} P_{\Lambda_m} A_k f \right|^2.$

The first term gives

$$\begin{aligned} \int_{\mathbb{R}} \left| \sum_{m=n}^{N} P_{\Lambda_m} A_m f \right|^2 = \sum_{m=n}^{N} \|P_{\Lambda_m} A_m f\|^2 &\leq \sum_{m=n}^{N} (\|P_{\Lambda_m} f\| + \frac{1}{2^m}\|f\|)^2 \leq \\ &\leq \sum_{m=n}^{N} \left(\|P_{\Lambda_m} f\|^2 + \left(\frac{2}{2^m} + \frac{1}{2^{2m}} \right) \|f\|^2 \right) \leq \\ &\leq \|P_C f\|^2 + \left(\sum_{m=n}^{N} \left(\frac{2}{2^m} + \frac{1}{2^{2m}} \right) \right) \|f\|^2. \end{aligned}$$

Note that for every $\epsilon_1 > 0$, we can choose n_0 big enough such that $\|P_A f\| \leq \epsilon_1 \|f\|$, where $A = \cup_{m=n_0}^{\infty} \Lambda_m$.

For the second term, it follows by relation (5.2) that

$$\begin{aligned} \int_{\mathbb{R}} \left| \sum_{m=n}^{N} \sum_{\substack{k=n \\ k \neq m}}^{N} P_{\Lambda_m} A_k f \right|^2 &= \left\| \sum_{m=n}^{N} \sum_{\substack{k=n \\ k \neq m}}^{N} P_{\Lambda_m} A_k f \right\|^2 = \left\| \sum_{k=n}^{N} \sum_{\substack{m=n \\ m \neq k}}^{N} P_{\Lambda_m} A_k f \right\|^2 = \\ &= \left\| \sum_{k=n}^{N} P_{C \setminus \Lambda_k} A_k f \right\|^2 \leq \left(\sum_{k=n}^{N} \|P_{C \setminus \Lambda_k} A_k f\| \right)^2 \leq \\ &\leq \left(\sum_{k=n}^{N} \frac{1}{2^k} \|f\| \right)^2 = \left(\sum_{k=n}^{N} \frac{1}{2^k} \right)^2 \|f\|^2. \end{aligned}$$

Combining the above estimates we get

$$\left\| \sum_{k=n}^{N} A_k f \right\|^2 \leq \left(3 \left(\sum_{k=n}^{N} \frac{1}{2^k} \right)^2 + 2 \sum_{m=n}^{N} \left(\frac{2}{2^m} + \frac{1}{2^{2m}} \right) + 2\epsilon_1^2 \right) \|f\|^2.$$

Hence, there exists $n_0 \in \mathbb{N}$ such that $\| \sum_{k=n}^{N} A_k f\|^2 \leq \epsilon$, for all $n, N > n_0$, so we proved our claim.

<u>Claim 2:</u> The sequence $\{\sum_{k=1}^{n} A_k\}_n$ is uniformly bounded.

Let $f \in L^2(\mathbb{R})$. Then

$$\left\|\sum_{k=1}^{n} A_k f\right\|^2 = \lim_{M\to\infty}\left\|\sum_{m=0}^{M} P_{\Lambda_m}\sum_{k=1}^{n} A_k f\right\|^2.$$

Then

$$\left\|\sum_{m=0}^{M} P_{\Lambda_m}\sum_{k=1}^{n} A_k f\right\|^2 \leq 2\left\|\sum_{k=1}^{n} P_{\Lambda_k} A_k f\right\|^2 + 2\left\|\sum_{k=1}^{n}\sum_{\substack{m=0\\ m\neq k}}^{M} P_{\Lambda_m} A_k f\right\|^2.$$

Applying now the relation (5.1) we obtain

$$\begin{aligned}\left\|\sum_{k=1}^{n} P_{\Lambda_k} A_k f\right\|^2 &\leq \sum_{k=1}^{n}\left(\|P_{\Lambda_k} f\| + \frac{1}{2^k}\|f\|\right)^2 = \\ &= \sum_{k=1}^{n}\|P_{\Lambda_k} f\|^2 + \sum_{k=1}^{n}\left(\frac{2}{2^k} + \frac{1}{2^{2k}}\right)\|f\|^2 \leq 4\|f\|^2.\end{aligned}$$

Moreover

$$\begin{aligned}\left\|\sum_{k=1}^{n}\sum_{\substack{m=0\\ m\neq k}}^{M} P_{\Lambda_m} A_k f\right\|^2 &\leq \left(\sum_{k=1}^{n}\left\|\sum_{\substack{m=0\\ m\neq k}}^{M} P_{\Lambda_m} A_k f\right\|\right)^2 \leq \left(\sum_{k=1}^{n}\|P_{\mathbb{R}\setminus\Lambda_k} A_k f\|\right)^2 \leq \left(\sum_{k=1}^{n}\frac{1}{2^k}\|f\|\right)^2 = \\ &= \left(\sum_{k=1}^{n}\frac{1}{2^k}\right)^2\|f\|^2 \leq \|f\|^2.\end{aligned}$$

Hence the norms $\|\sum_{k=1}^{n} A_k\|$ are uniformly bounded. Write $K := \sup_n \|\sum_{k=1}^{n} A_k\| < \infty$.

Define the operator A acting on $L^2(\mathbb{R})$ by the formula

$$Af = \lim_n \sum_{k=1}^{n} A_k f,$$

By our first claim the operator A is well defined. One can check that it is also linear by routine calculations. Our second claim tells us that $\|Af\| \leq K\|f\|$, for every $f \in L^2(\mathbb{R})$, so A is bounded. In particular A is by construction the SOT-limit of the sequence

$\{\sum\limits_{k=1}^{n} A_k\}_n$, so it lies in $\mathcal{A}_p$. Define now the Hilbert-Schmidt operators

$$X_n := A_n - A_n^* = D_{t_n}(p_n(ZF_n) - p_n(ZF_n)^*)D_{t_n}^*$$

and note that $\|X_n\| \to 0$. Mimicking similar arguments as above we get that the sequence of the partial sums of $\sum\limits_{n=1}^{\infty} X_n$ is Cauchy with respect to the operator norm, so the norm limit $X = \sum\limits_{n=1}^{\infty} X_n$ lies in the norm closure of the Hilbert-Schmidt operators, which yields that X is a compact operator acting on $L^2(\mathbb{R})$. Since involution is continuous in the WOT-topology, we get that $A - A^* = X$. Therefore,

$$A \in (\mathcal{A}_p + K(H)) \cap (\mathcal{A}_p^* + K(H)),$$

so it remains to show that $A \notin \mathbb{C}I + K(H)$.

Assume that this is not true, so by Lemma 5.1.4 there exists $c \in \mathbb{C}$ and $K \in K(H)$, such that

$$A = cI + K.$$

We left multiply both sides by the projection P_{Λ_0}. Recall that multiplication is separately SOT-continuous, so $P_{\Lambda_0}A$ is the SOT-limit of the operators $\{\sum\limits_{k=1}^{n} P_{\Lambda_0}A_k\}_n$. Moreover we have the estimates

$$\|P_{\Lambda_0}A_k\| \leq \frac{1}{2^k}, \text{ for every } k \in \mathbb{N},$$

so the above sequence converges uniformly to $P_{\Lambda_0}A$. Therefore $P_{\Lambda_0}A$ is a compact operator. Since we created the sets Λ_n by one-sided shifts, Λ_0 is an unbounded set of infinite measure, and this yields that $c = 0$. But this implies that A is compact, which gives the desired contradiction. $\square$

It is natural to consider also the intersection of different quasitriangular algebras. In particular, it remains unclear to the author if the algebra $(\mathcal{A}_v + K(H)) \cap (\mathcal{A}_a + K(H))$ is strictly larger than $\mathcal{A}_p + K(H)$. The question seems to be closely related to the general open problem of a distance formula for the parabolic algebra (see [56] for example).

5.2 Operator Algebras from the discrete Heisenberg semigroup

5.2.1 The algebra $\mathcal{T}_L(\mathbb{H}^+)$

Let $\mathbb{H}$ be the **(discrete) Heisenberg group**, that is the group of all matrices of the form

$$[n, k, m] = \begin{bmatrix} 1 & m & n \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix}$$

where $k, m, n \in \mathbb{Z}$. We write $\mathbb{H}^+$ for the semigroup of $\mathbb{H}$ that consists of the elements $[n, k, m]$ with $k, m \geq 0$ and $n \in \mathbb{Z}$. One can check that $\mathbb{H}^+$ is generated by the elements

$$u = [0, 0, 1], \quad v = [0, 1, 0], \quad w = [1, 0, 0] \quad \textit{and} \quad w^{-1} = [-1, 0, 0].$$

The element w is central and $uv = wvu$.

Recall that the left regular representation of a discrete group G on $\ell^2(G)$ is given by the formula

$$L : G \to B(\ell^2(G)) : (L_g x)(h) = x(g^{-1}h),\ x \in \ell^2(G),\ g, h \in G.$$

We identify the space $\ell^2(\mathbb{H})$ with the Hilbert space $\mathcal{H} = \ell^2(\mathbb{Z}) \otimes \ell^2(\mathbb{Z}) \otimes \ell^2(\mathbb{Z})$ by the map that sends the element of the canonical orthonormal basis of $\mathcal{H}$ corresponding to $w^n v^k u^m \in \mathbb{H}$ to the elementary tensor $e^n \otimes e^k \otimes e^m$. Note that the subspace $H = \ell^2(\mathbb{Z}) \otimes \ell^2(\mathbb{Z}^+) \otimes \ell^2(\mathbb{Z}^+)$ is left invariant by $L(\mathbb{H}^+)$. The weak*-closed algebra generated by the image of the left regular representation of $\mathbb{H}^+$, restricted to the invariant subspace H, will be denoted by $\mathcal{T}_L(\mathbb{H}^+)$. Hence $\mathcal{T}_L(\mathbb{H}^+)$ is generated by the operators L_u, L_v, L_w and L_w^{-1} on H given by:

$$\begin{aligned}
L_u(e^n \otimes e^k \otimes e^m) &= e^{n+k} \otimes e^k \otimes e^{m+1} \\
L_v(e^n \otimes e^k \otimes e^m) &= e^n \otimes e^{k+1} \otimes e^m \\
L_w(e^n \otimes e^k \otimes e^m) &= e^{n+1} \otimes e^k \otimes e^m.
\end{aligned}$$

Since the span of these operators is closed under multiplication, the algebra $\mathcal{T}_L(\mathbb{H}^+)$ coincides with the weak*-closed span of the set

$$\{L_w^n L_v^k L_u^m \,:\, n \in \mathbb{Z},\ k, m \in \mathbb{Z}^+\}.$$

The following result can be found in [2].

Theorem 5.2.1. (Anoussis, Katavolos and Todorov) *The algebra $\mathcal{T}_L(\mathbb{H}^+)$ is reflexive.*

The main tools of the proof in [2] were the bicommutant property of $\mathcal{T}_L(\mathbb{H}^+)$ and the use of a direct integral decomposition for non-selfadjoint algebras ([8]). In the following, we will give a more direct proof of the result by applying a result of Kakariadis [34]. First we need to give a short introduction to the theory of the w*-semicrossed products.

Let $\mathcal{A} \subseteq B(H_0)$ be a unital weak*-closed subalgebra and $\alpha : \mathcal{A} \to \mathcal{A}$ be a contractive weak*-continuous endomorphism of $\mathcal{A}$. Denote by H the Hilbert space $H_0 \otimes \ell^2(\mathbb{Z}^+)$. The w*-semicrossed product $\mathcal{A} \times_\alpha \mathbb{Z}^+$ is a weak*-closed subalgebra of $B(H)$, where we can represent both the algebra $\mathcal{A}$ and the action of the endomorphism α. Define the faithful representation

$$\pi : \mathcal{A} \to B(H) : a \mapsto diag\{\alpha^n(a) \,:\, n \in \mathbb{Z}^+\} = \begin{bmatrix} a & 0 & 0 & \dots \\ 0 & \alpha(a) & 0 & \dots \\ 0 & 0 & \alpha^2(a) & \dots \\ \vdots & \vdots & \ddots & \end{bmatrix}.$$

We also represent $\mathbb{Z}^+$ on H by the isometries $V^n = I_{H_0} \otimes s^n$, where I_{H_0} is the identity operator in $B(H_0)$ and s is the unilateral shift acting on $\ell^2(\mathbb{Z}^+)$. Check that pair (π, V) satisfies the covariant relation

$$\pi(a)V = V\pi(\alpha(a)),$$

hence it will be called **covariant pair**.

Definition 5.2.2. The **w*-semicrossed product** $\mathcal{A} \times_\alpha \mathbb{Z}^+$ is the weak*-closure of the linear space of the "analytic polynomials" $\sum_{n=0}^{N} V^n \pi(a_n),\ a_n \in \mathcal{A}$.

It follows from the covariance relation that the w*-semicrossed product is a unital non-selfadjoint subalgebra of $B(H)$. In particular, $\mathcal{A} \times_\alpha \mathbb{Z}^+$ lies in the w*-tensor product $\mathcal{A} \overline{\otimes} B(\ell^2(\mathbb{Z}^+))$, that is the weak*-closure of the algebra generated by the elementary tensors $a \otimes b$, with $a \in \mathcal{A}$, $b \in B(\ell^2(\mathbb{Z}^+))$.

A standard tool that we use in the theory of w*-semicrossed products is a Fejer-type lemma. Consider for every $s \in \mathbb{T}$ the unitary operator $U_s \in B(H)$, given by $U_s(h \otimes e_n) = e^{ins} h \otimes e_n$. Given $T \in B(H)$ and $n \in \mathbb{Z}$ define the "mth-Fourier coefficient"

$$G_m(T) = \frac{1}{2\pi} \int_{\mathbb{T}} U_s T U_s^* e^{-ims} ds$$

where the integral is considered as the weak*-limit of Riemann sums. One can check that for every $m \in \mathbb{Z}$, the function $G_m(\cdot)$ is weak*-continuous. If we set now

$$\sigma_n(T)(t) = \frac{1}{n+1} \sum_{k=0}^{n} \sum_{m=-k}^{k} G_m(T) e^{imt},$$

then $\sigma_n(T)(0) \xrightarrow{\mathrm{w}^*} T$. The Fourier coefficient $G_m(T)$ can be represented in the "matrix form" of an operator as the mth-diagonal of T. For every $m, n \in \mathbb{Z}^+$, let the "matrix elements" $T_{m,n} \in B(H_0)$ of T be defined by

$$\langle T_{m,n} h, g \rangle = \langle T(h \otimes e_n), g \otimes e_m \rangle,\ h, g \in H_0.$$

Then

$$G_m(T) = \begin{cases} V^m(\sum_{n \geq 0} T_{m+n,n} \otimes p_n), & \text{if } m \geq 0 \\ (\sum_{n \geq 0} T_{n,-m+n} \otimes p_n)(V^*)^{-m}, & \text{if } m < 0, \end{cases}$$

where $p_n \in B(\ell^2(\mathbb{Z}^+))$ is the projection onto $[e_n]$. For the proof of the following theorem we refer the reader to [34].

Proposition 5.2.3. *An operator $T \in B(H)$ belongs to the w^*-semicrossed product if and only if $T_{m,n} \in \mathcal{A}$ for every $m, n \in \mathbb{Z}$ and*

$$G_m(T) = \begin{cases} V^m \pi(T_{m,0}), & \text{if } m \geq 0 \\ 0, & \text{if } m < 0 \end{cases}$$

In the special case where α is implemented by a unitary w acting on the space H_0, so that $\alpha(a) = waw^*$ for all $a \in \mathcal{A}$, we can create a new covariant pair, by transfering the action of the endomorphism α in the representation of $\mathbb{Z}^+$ on H. Indeed, take the pair (ρ, W), where ρ is the representation

$$\rho : B(H_0) \to B(H) : b \mapsto b \otimes 1_{\ell^2(\mathbb{Z}^+)} = \begin{bmatrix} b & 0 & \dots \\ 0 & b & \dots \\ \vdots & \vdots & \ddots \end{bmatrix}$$

and the operator W as follows

$$W = w^* \otimes s = \begin{bmatrix} 0 & 0 & 0 & \dots \\ w^* & 0 & 0 & \dots \\ 0 & w^* & 0 & \dots \\ \vdots & \vdots & \ddots & \ddots \end{bmatrix}.$$

Denote now by $\mathcal{A} \times_w \mathbb{Z}^+$ the weak*-closure of the "analytic polynomials" $\sum_{n=0}^{N} W^n \rho(a_n)$, with $a_n \in \mathcal{A}$. Note that $\mathcal{A} \times_w \mathbb{Z}^+$ is unitarily equivalent to $\mathcal{A} \times_\alpha \mathbb{Z}^+$, via the unitary $Q = \sum_{n \geq 0} w^{-n} \otimes p_n$. Therefore we refer to $\mathcal{A} \times_w \mathbb{Z}^+$ as the w*-semicrossed product, as well. Furthermore, Proposition 5.2.3 yields the following characterization.

Proposition 5.2.4. *An operator $T \in B(H)$ belongs to the w*-semicrossed product if and only if $G_m(T) = W^m \rho(a_m)$ for some $a_m \in \mathcal{A}$, when $m \geq 0$ and $G_m(T) = 0$ when $m < 0$.*

The aforementioned result of Kakariadis is the next theorem ([34], Theorem 2.9). The main argument of the proof is an elaborate adaptation of Sarason's proof for the reflexivity of $H^\infty(\mathbb{T})$ given in [68].

Theorem 5.2.5. (Kakariadis) *If $\mathcal{A}$ is a reflexive algebra, then $\mathcal{A} \times_w \mathbb{Z}^+$ is also reflexive.*

Let now $H_0 = \ell^2(\mathbb{Z}) \otimes \ell^2(\mathbb{Z}^+)$ and $\mathcal{A}$ be the weak*-closed algebra that is generated by the operators ℓ_w, ℓ_w^{-1} and ℓ_v, which act on H_0 as follows

$$\ell_w(e_n \otimes e_k) = e_{n+1} \otimes e_k$$
$$\ell_v(e_n \otimes e_k) = e_n \otimes e_{k+1}.$$

Note that the operators ℓ_w,ℓ_v commute, so $\mathcal{A}$ is unitarily equivalent with the algebra $L^\infty(\mathbb{T}) \times_I \mathbb{Z}^+$, where I is the identity operator acting on $\ell^2(\mathbb{Z})$. Hence it follows from the previous theorem that $\mathcal{A}$ is reflexive. Define now on H_0 the unitary operator

$$\ell_u(e_n \otimes e_k) = e_{n-k} \otimes e_k.$$

Since $\ell_v \ell_u = \ell_w \ell_u \ell_v$, the operator ℓ_u implements an automorphism of $\mathcal{A}$. Hence, if $H = H_0 \otimes \ell^2(\mathbb{Z}^+)$, the semicrossed product $\mathcal{A} \times_u \mathbb{Z}^+$ is a reflexive subalgebra of $B(H)$. We want to determine the generators $\rho(\ell_w), \rho(\ell_v), \ell_u^* \otimes s$ of the w*-semicrossed product. Compute

$$\rho(\ell_w)(e_n \otimes e_k \otimes e_m) = e_{n+1} \otimes e_k \otimes e_m = L_w(e_n \otimes e_k \otimes e_m)$$
$$\rho(\ell_v)(e_n \otimes e_k \otimes e_m) = e_n \otimes e_{k+1} \otimes e_m = L_v(e_n \otimes e_k \otimes e_m)$$

$$(\ell_u^* \otimes s)(e_n \otimes e_k \otimes e_m) = e_{n+k} \otimes e_k \otimes e_{m+1} = L_u(e_n \otimes e_k \otimes e_m).$$

Therefore the algebras $\mathcal{A} \times_u \mathbb{Z}^+$ and $\mathcal{T}_L(\mathbb{H}^+)$ are weak*-closed algebras that share the same generators, so they coincide. Therefore we established again the reflexivity of the latter algebra and thus we obtain a new proof of theorem 5.2.1. $\square$

5.2.2 Some subalgebras of $\mathcal{T}_L(\mathbb{H}^+)$

Let now $\mathcal{T}_L(\mathbb{H}^+)_-$ be the subalgebra of $\mathcal{T}_L(\mathbb{H}^+)$ that is generated by the operators I_H, L_u, L_v, L_w^*. Check that this algebra contains the element $L_uL_v = L_wL_vL_u$, but not the element L_w. Applying the same arguments as above we can identify $\mathcal{T}_L(\mathbb{H}^+)_-$ with a w*-semicrossed product. Indeed, if $\mathcal{A}_-$ is the weak*-closed algebra that is generated by the operators I_{H_0}, ℓ_w^* and ℓ_v, then $\mathcal{A}_-$ can be identified with the algebra $H_\infty(\mathbb{T}) \times_I \mathbb{Z}^+$, with $H_\infty(\mathbb{T})$ viewed as an operator algebra, acting by multiplication on $L^2(\mathbb{T})$. The unitary operator ℓ_u induces an endomorphism of $\mathcal{A}_-$. Since the latter algebra is reflexive, we obtain that $\mathcal{A}_- \times_u \mathbb{Z}^+$ is reflexive. Since $\mathcal{T}_L(\mathbb{H}^+)_-$ is equal to the w*-semicrossed product algebra, we have the following theorem.

Theorem 5.2.6. *The algebra $\mathcal{T}_L(\mathbb{H}^+)_-$ is reflexive.*

Consider now the strictly positive Heisenberg semigroup $\mathbb{H}^{++}$, the subgroup of $\mathbb{H}^+$ that consists of the elements $[k, m, n]$, where $k, m, n \geq 0$. We are interested in the weak*-closed algebra $\mathcal{T}_L(\mathbb{H}^{++})$ generated by the operators L_g, $g \in \mathbb{H}^{++}$ acting on the invariant subspace $\ell^2(\mathbb{H}^{++})$. In particular, using a similar identification of $\ell^2(\mathbb{H}_{++})$ with the Hilbert space $H = \ell^2(\mathbb{Z}^+) \otimes \ell^2(\mathbb{Z}^+) \otimes \ell^2(\mathbb{Z}^+)$, the algebra $\mathcal{T}_L(\mathbb{H}^{++})$ is generated by the operators I_H, L_u, L_v and L_w on H, where I_H is the identity operator in $B(H)$, and the rest are the restriction of the corresponding operators defined in the previous section. Again, by the commutation relations, the algebra $\mathcal{T}_L(\mathbb{H}^{++})$ is equal to the weak*-closed linear span of the products $L_w^n L_v^k L_u^m$, where $n, m, k \in \mathbb{Z}^+$.

It is natural to ask about the reflexivity and possible identifications of the algebra $\mathcal{T}_L(\mathbb{H}^{++})$ with w*-semicrossed products. However, the standard proof that was used above gives no result, so new arguments need to be developed for this case.

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