TWO-DIMENSIONAL RUIN PROBABILITY FOR SUBEXPONENTIAL CLAIM SIZE[*]

BY

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Abstract. We analyse the asymptotics of ruin probabilities of two insurance companies (or two branches of the same company) that divide between them both claims and premia in some specified proportions when the initial reserves of both companies tend to infinity and generic claim size is subexponential.

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1. INTRODUCTION

We consider a model of two-dimensional risk process with renewal input: claims $\vec{\sigma}_n = (\sigma_{n,1}, \sigma_{n,2})$ arrive in a random input at arrival epochs $\{t_n\}$ with interarrival times $\{\tau_n\}$. There are two insurance companies, company i has initial capital x_i and premium rate p_i, and covers claims $\sigma_{n,i}$, $i = 1$,

2. We assume that the two sequences $\{\vec{\sigma}_n\}$ and $\{\tau_n\}$ are mutually independent and that each of them consists of i.i.d. random variables.

Let $N(t)$ be the number of claims by time $t \geqslant 0$,

$$N(t) = \max\{n \; : \; t_n \leqslant t\},$$

then

$$\vec{S}(t) \; = \; (S_1(t), S_2(t)) \; := \; \sum_{n=1}^{N(t)} \vec{\sigma}_n$$

is the vector of total claim sizes by time t. Further, let

$$\vec{b}(t) \; = \; (b_1(t), b_2(t)) \; := \; \vec{x} + \vec{p}t$$

be the sum of initial capitals and of total premiums by time t, here $\vec{x} = (x_1, x_2)$ and $\vec{p} = (p_1, p_2)$. Introduce ruin probabilities of two types: for any $T \in (0, \infty]$, these are

$$
\begin{aligned}
\psi_\wedge(\vec{x}, T) &= \mathbb{P}\{S_1(t) > b_1(t) \text{ or } S_2(t) > b_2(t) \text{ for some } t \leqslant T\}, \\
\psi_\vee(\vec{x}, T) &= \mathbb{P}\{S_1(t) > b_1(t) \text{ and } S_2(t) > b_2(t) \text{ for some } t \leqslant T\}.
\end{aligned}
$$

Here $\psi_\wedge(x_1, x_2, T)$ describes the ruin probability of at least one insurance company, while $\psi_\vee(x_1, x_2, T)$ corresponds to the ruin of the both insurance companies.

There are several papers—see e.g., Konstantinides and Li [15] and list of references therein—where it is assumed that the pairs $(\sigma_{n,1}, \sigma_{n,2})$ are i.i.d. and have a multivariate regularly varying distribution. We like to consider claim sizes with distributions from a more general subexponential class that includes Pareto and also log-normal and Weibull distributions. We are unaware of any reasonable concept of subexponentiality here. So only two extreme options seem to be doable: either
(i) $\sigma_{n,1}$ and $\sigma_{n,2}$ are independent or
(ii) they are dependent deterministically, say

$$\sigma_{n,1} = l(\sigma_n) \quad \text{and} \quad \sigma_{n,1} = \sigma_n - l(\sigma_n)$$

where $0 \leqslant l(x) \leqslant x$ and σ_n is the cumulative claim of the client n that is covered by the two insurance companies together.

There are several papers that study related problems in direction (i)—see e.g. Li et al. [17], Chen et al. [8], Chen, Wang, Wang [6, 7], Hu and Jiang [14], Lu and Zhang [19], see also references therein. In particular, in the paper by Lu and Zhang [19], a uniform asymptotics over finite intervals have been obtained under some restrictive assumptions on distributions.

Avram et al. [3, 4] studied boundary crossing probabilities of a stochastic process with light-tailed increments. This study was also motivated by ruin probabilities of two insurance companies with proportional claims and the steady state distribution of a tandem queue with two servers (see [18]). Similar considerations were done in Badila et. al. [5] (for mutidimensional risk process) and in [14] (for the two-dimensional risk process with constant interest rate).

The key message of this paper is that in both cases results on the uniform asymptotics may be obtained for a general class of *strong subexponential distributions*, no further restrictions are needed. We only consider the second direction using approach developed in a series of works of Denisov, Foss, Korshunov, Palmowski, and Zachary (see e.g. [9, 11, 12, 13, 16]); similar arguments apply in the first direction.

To make it simple, we assume $l(x)$ to be linear, $l(x) = lx$ for some $l \in (0, 1)$. Then, after some transformations we can reformulate the problem as follows. Let us define

$$(1.1) \qquad S_t = \sum_{i=1}^{N_t} \sigma_i,$$

where N_t is a renewal process with positive i.i.d. inter-arrival times τ_i, and the claim sizes σ_i are positive i.i.d. random variables that do not depend on $N(t)$ and have a common distribution function $F(x)$. Note also that $N_t = \max\{k \geqslant 0 : t_k \leqslant t\}$ for a random walk constructed from the interarrival times:

$$t_n = \sum_{k=1}^{n} \tau_k.$$

Let the boundaries b_1 and b_2 be given by

$$b_1(t) = b_1(t; x_1) = x_1 + p_1 t, \qquad b_2(t) = b_2(t; x_2) = x_2 + p_2 t.$$

We assume that $\mathbb{E}\sigma < \infty$, $\mathbb{E}\tau < \infty$ and that

$$(1.2) \qquad p_1 > p_2, \qquad p_2 > \rho := \mathbb{E}[\sigma]/\mathbb{E}[\tau]$$

for generic τ and σ. In this paper for $T > 0$, we will consider the following boundary crossing probabilities:

$$
\begin{aligned}
\psi_\wedge(x_1, x_2, T) &= \mathbb{P}\{\tau_\wedge(x_1, x_2) \leqslant T\}, \\
\psi_\vee(x_1, x_2, T) &= \mathbb{P}\{\tau_\vee(x_1, x_2) \leqslant T\},
\end{aligned}
$$

for

(1.3) $\tau_\wedge(x_1, x_2) = \inf\{t \geqslant 0 : S_t > b_1(t) \wedge b_2(t)\},$

(1.4) $\tau_\vee(x_1, x_2) = \inf\{t \geqslant 0 : S_t > b_1(t) \vee b_2(t)\},$

where $x \wedge y = \min\{x, y\}$ and $x \vee y = \max\{x, y\}$.

The $\psi_\wedge(x_1, x_2, T)$ describes the ruin probability of at least one insurance company before time T (if $T = \infty$ then

$$
\psi_\wedge(x_1, x_2) := \psi_\wedge(x_1, x_2, \infty)
$$

is a ruin probability of at least one company). The $\psi_\vee(x_1, x_2, T)$ and

$$
\psi_\vee(x_1, x_2) := \psi_\vee(x_1, x_2, \infty)
$$

correspond to the ruin of both insurance companies. The first assumption in (1.2) means that the second company has a smaller premium rate than the first company, and the second assumption is the condition under which reserves of both insurance companies tend to infinity. The solutions to the "degenerate two-dimensional" ruin problems strongly depend on the relative position of the vector of premium rates $p = (p_1, p_2)$ with respect to the proportions vector $(1, 1)$. Namely, if the initial reserves satisfy $x_2 \leqslant x_1$, the two lines do not intersect. It follows therefore that $\psi_\wedge$ and $\psi_\vee$ are ruin probabilities of the second and first companies respectively. In this case the asymptotics follows from one-dimensional ruin theory—see e.g. Rolski et al. [20]. Therefore we focus here on the opposite case where

(1.5) $x_1 < x_2.$

In this paper we derive the exact first order asymptotics of these ruin probabilities if x_1, x_2 tend to infinity and when the claims follow a subexponential distribution. We model the claims by subexponential distributions

since many catastrophic events like earthquakes, storms, terrorist attacks etc are used in their description. Insurance companies use e.g. the lognormal distribution (which is subexponential) to model car claims—see Foss et al. [13], Rolski et al. [20] or Embrechts et al. [10] for the further background.

The paper is organized as follows. In next section we present the main results which will be proved in Section 3.

2. MAIN RESULTS

In order to state our results we recall basic notation and notions. Hereinafter we write $f(x, y) \sim g(x, y)$ if $f(x, y)/g(x, y) \to 1$ as $x, y \to \infty$ and $f(x) \sim g(x)$ if $f(x)/g(x) \to 1$ as $x \to \infty$. For a distribution F, $\overline{F}$ denotes the tail distribution function given by $\overline{F}(x) = 1 - F(x)$.

A distribution F on $\mathbb{R}^+$ is called subexponential ($F \in \mathcal{S}$) if and only if $\overline{F}(x) > 0$ for all x and

(2.1) $$\overline{F^{*2}}(x) \sim 2\overline{F}(x) \quad \text{as } x \to \infty,$$

where F^{*2} is the convolution of F with itself.

A distribution F on $\mathbb{R}^+$ is called *strong subexponential* ($F \in \mathcal{S}^*$) if and only if $\overline{F}(x) > 0$ for all x and

(2.2) $$\int\limits_0^x \overline{F}(x - y)\overline{F}(y)\, dy \sim 2m_F \overline{F}(x) \quad \text{as } x \to \infty,$$

where

$$m_F = \int\limits_0^\infty \overline{F}(x)\, dx$$

is the expectation of F. It is known that the property $F \in \mathcal{S}^*$ is a tail property of F, namely, if $F_1 \in \mathcal{S}^*$ and $\overline{F}_1(x) \sim \overline{F}_2(x)$ as $x \to \infty$, then $F_2 \in \mathcal{S}^*$. Further, if $F \in \mathcal{S}^*$ then $F \in \mathcal{S}$ and also $F^s \in \mathcal{S}$ where

$$\overline{F^s}(x) = \min\Big(1, \int\limits_x^\infty \overline{F}(t)\, dt\Big)$$

is the *integrated tail distribution* function determined by F; see [13] for details.

Denote

(2.3) $$m_i = p_i \mathbb{E}\tau - \mathbb{E}\sigma, \qquad i = 1, 2.$$

Since $p_1 > p_2$, we have $m_1 > m_2$.

THEOREM 2.1. *Assume that the distribution F of generic $\sigma \geqslant 0$ is strong subexponential. Then, as x_1, $x_2 \to \infty$,*

$$(2.4) \quad \psi_\wedge(x_1, x_2, T) \sim H_T(x_1, x_2) := \int_0^{\mathbb{E}N_T} \overline{F}(\min\{x_1 + tm_1, x_2 + tm_2\}) \, dt$$

and

$$(2.5) \quad \psi_\vee(x_1, x_2, T) \sim U_T(x_1, x_2) := \int_0^{\mathbb{E}N_T} \overline{F}(\max\{x_1 + tm_1, x_2 + tm_2\}) \, dt$$

holds uniformly for all $T > 0$.

COROLLARY 2.1. *In conditions of Theorem 2.1, as x_1, x_2 and $T \to \infty$,*

$$(2.6) \quad \psi_\wedge(x_1, x_2, T) \sim \int_0^{T/\mathbb{E}\tau} \overline{F}(\min\{x_1 + tm_1, x_2 + tm_2\}) \, dt,$$

and

$$(2.7) \quad \psi_\vee(x_1, x_2, T) \sim \int_0^{T/\mathbb{E}\tau} \overline{F}(\max\{x_1 + tm_1, x_2 + tm_2\}) \, dt.$$

COROLLARY 2.2. *Assume that F is strong subexponential and $a < 1$ is fixed. Then we have the following asymptotics, as $x \to \infty$:*

$$(2.8) \quad \psi_\wedge(ax, x) \sim H(x) := \int_0^\infty \overline{F}(\min\{ax + tm_1, x + tm_2\}) \, dt,$$

and

$$(2.9) \quad \psi_\vee(ax, x) \sim U(x) := \int_0^\infty \overline{F}(\max\{ax + tm_1, x + tm_2\}) \, dt.$$

We can also identify the joint distribution of the ruin times (1.3) and (1.4) and the position at the moment at these ruin times. To do this we will require additional assumptions. Let $e(x)$ be a function tending to infinity as $x \to \infty$. For fixed $y \geqslant 0$ and $v \geqslant 0$ we denote:

$$(2.10) \quad H^{y,v}(x) = \int_{ye(x)}^\infty \overline{F}(\min\{ax + tm_1, x + tm_2\} + ve(x)) \, dt$$

and

$$(2.11) \qquad U^{y,v}(x) = \int\limits_{ye(x)}^{\infty} \overline{F}(\max\{ax + tm_1, x + tm_2\} + ve(x))\, \mathrm{d}t.$$

Note that $H(x) = H^{0,0}(x)$ and $U(x) = U^{0,0}(x)$. We will assume that there exists some continuous distribution function G such that

$$(2.12) \qquad \lim_{x \to \infty} \frac{H^{y,v}(x)}{H(x)} = \overline{G}(m_1 y + v)$$

and that

$$(2.13) \qquad \lim_{x \to \infty} \frac{U^{y,v}(x)}{U(x)} = \overline{G}(m_2 y + v).$$

THEOREM 2.2. *Assume that there exist a function $e(x) \uparrow \infty$ and a continuous probability distribution G on the positive half-line $(0, \infty)$ such that* (2.12) *and* (2.13) *are satisfied. Then, for $a < 1$, we have:*

(2.14)
$$\lim_{x \to \infty} \mathbb{P}\left\{ \frac{\tau_\wedge(ax, x)}{e(x)} \geqslant y, \quad \frac{S_{\tau_\wedge(ax,x)} - b_\wedge(t)}{e(x)} \geqslant v \Big| \tau_\wedge(ax, x) < \infty \right\} = \overline{G}(m_1 y + v)$$

and

(2.15)
$$\lim_{x \to \infty} \mathbb{P}\left\{ \frac{\tau_\vee(ax, x)}{e(x)} \geqslant y, \quad \frac{S_{\tau_\vee(ax,x)} - b_\vee(t)}{e(x)} \geqslant v \Big| \tau_\vee(ax, x) < \infty \right\} = \overline{G}(m_2 y + v).$$

For similar statements for a linear boundary, see Asmussen and Klüppelberg [2] and Foss et al. [13] in the case of i.i.d. jumps and Asmussen and Foss [1] in the case of Markov modulation.

REMARK 2.1. To provide examples where assumptions (2.12) and (2.13) are satisfied, we choose strong subexponential distribution F for which F^s is self-neglecting. That is, let us assume that there exist a function $e(x) \uparrow \infty$ and a continuous probability distribution G on the positive half-line $(0, \infty)$ such that, for any $y > 0$,

$$(2.16) \qquad \frac{\overline{F}^s(x + ye(x))}{\overline{F}^s(x)} \to \overline{G}(y) \quad \text{as } x \to \infty.$$

From [10, Thm. 3.4.5, p. 158] it follows that F^s is either in the domain of attraction of Frechet distribution or in the the domain of attraction attraction of Gumbel distribution. In the first case

(2.17)

$$F \in \mathrm{RV}(\alpha + 1), \quad F^s \in \mathrm{RV}(\alpha), \quad \overline{G}(y) = (1 + \alpha^{-1}y)^{-\alpha}, \quad e(x) = \alpha^{-1}x,$$

with $\alpha > 0$ (then $\mathbb{E}\sigma < \infty$) for the family $\mathrm{RV}(\alpha + 1)$ of regularly varying distributions. In the second case,

(2.18)
$$\overline{G}(y) = e^{-y}, \qquad e(x) = \frac{m_F \overline{F}^s(x)}{\overline{F}(x)}.$$

To check (2.12) it is convenient to use the following representation

$$
\begin{aligned}
H^{y,v}(x) \;=\;& \frac{1}{m_1}\overline{F}^s(ax + (m_1 y + v)e(x)) \\
&+ \left(\frac{1}{m_2} - \frac{1}{m_1}\right)\overline{F}^s(ax + m_1\widehat{T} + ve(x)) \\
&+ \frac{1}{m_2}\overline{F}^s(x + m_2\widehat{T} + ve(x))
\end{aligned}
$$

(2.19)

for the moment when two lines $ax + m_1 t$ and $x + m_2 t$ cut each other:

$$\widehat{T} \;=\; \frac{(1-a)x}{m_1 - m_2}.$$

Recall also that $H(x) = H^{0,0}(x)$. Now one can check that if (2.17) holds true then indeed (2.12) is satisfied. Similarly, one can prove that in this case assumption (2.12) is also satisfied.

The case (2.18) is much more advanced and it should be analysed case by case using for example [10, Example 3.3.35, p. 149].

In our proof of Theorem 2.1, we use the following simple result that may be of use in other settings.

LEMMA 2.1. *Let $z(t) \geqslant 0$ be an increasing function, F a distribution and N_t a renewal process. Then*

(2.20) $\displaystyle \mathbb{E} \int_0^{N_T} \overline{F}(x + z(t))dt \;\leqslant\; \int_0^{\mathbb{E}N_T} \overline{F}(x + z(t))dt \quad$ *for all x and $T > 0$*

and

(2.21)　$\displaystyle \mathbb{E} \int_0^{N_T} \overline{F}(x + z(t))dt \ \sim \ \int_0^{\mathbb{E}N_T} \overline{F}(x + z(t))dt \quad as\ x,\ T \to \infty.$

If, in addition, the distribution F is a long-tailed, then

(2.22)

$$\mathbb{E} \int_0^{N_T} \overline{F}(x + z(t))dt \ \sim \ \int_0^{\mathbb{E}N_T} \overline{F}(x + z(t))dt \quad as\ x \to \infty\ uniformly\ for\ all\ T > 0.$$

Notice that, in general, the last result is not applicable to an upper limit N_ϱ where ϱ is a stopping time like in (3.16) and (3.17) below.

3. PROOFS

3.1. Proof of Theorem 2.1. As discussed above, it is sufficient to consider the case $x_2 > x_1$. For any $x > 0$ and $\widehat{T} > 0$, let $\widehat{x} = x + p_1\widehat{T}$ and

$$\widehat{b}(t) \ := \ \begin{cases} x + p_1 t & for\ t \leqslant \widehat{T} \\ \widehat{x} + p_2(t - \widehat{T}) & for\ t > \widehat{T}, \end{cases}$$

which is a continuous piece-wise linear function. Then both assertions of the theorem, (2.4) and (2.5), will follow if the following tail asymptotics is proven for all $p_1 > 0$ and $p_2 > 0$, without assumption $p_2 < p_1$:

$$\psi(x, \widehat{T}, T) \ := \ \mathbb{P}\{S_t > \widehat{b}(t) \text{ for some } t \leqslant T\}$$

(3.1)　　　　$\displaystyle \sim \ \int_0^{\mathbb{E}N_T} \overline{F}(\widehat{x}(t))dt \quad as\ x \to \infty \text{ uniformly for all } \widehat{T} \text{ and } T,$

where

$$\widehat{x}(t) \ := \ \begin{cases} x + m_1 t & for\ t \leqslant \widehat{T}/\mathbb{E}\tau \\ x + m_1\widehat{T}/\mathbb{E}\tau + m_2(t - \widehat{T}/\mathbb{E}\tau) & for\ t > \widehat{T}/\mathbb{E}\tau. \end{cases}$$

Since $F \in \mathcal{S}^*$, F is particularly subexponential and long-tailed. For any fixed T, the random variable N_T has a light tail, that is, it possesses a finite exponential moment. In addition, N_T is independent of σ's. Hence, by Theorem 3.37 in [13], for every fixed T,

(3.2)

$$\mathbb{P}\Big\{\sum_{i:t_i \leqslant T} \sigma_i > x\Big\} \ = \ \sum_{j=0}^\infty \mathbb{P}\{N_T = j\}\overline{F^{*j}}(x) \ \sim \ \mathbb{E}N_T\overline{F}(x) \quad as\ x \to \infty,$$

and this equivalence holds uniformly on any T-compact set. This observation together with the following lower and upper bounds

$$\mathbb{P}\Big\{ \sum_{i:t_i\leqslant T} \sigma_i > x + \max\{p_1,p_2\}T \Big\} \ \leqslant \ \psi(x,\widehat{T},T) \ \leqslant \ \mathbb{P}\Big\{ \sum_{i:t_i\leqslant T} \sigma_i > x \Big\}$$

and long-tailedness of F yields that, uniformly on any T-compact set,

$$\psi(x,\widehat{T},T) \ \sim \ \mathbb{E}N_T\overline{F}(x) \quad \text{as } x \to \infty.$$

Taking into account that

$$N_T\overline{F}(x + \max\{m_1,m_2\}T) \ \leqslant \ \int\limits_0^{N_T} \overline{F}(\widehat{x}(t))dt \ \leqslant \ N_T\overline{F}(x)$$

and that F is long-tailed, we prove that (3.1) holds as $x \to \infty$ uniformly on T-compact sets.

Therefore, it remains to prove (3.1) for the case $T \to \infty$. If $T \leqslant \widehat{T}$, then

$$\psi(x,\widehat{T},T) \ = \ \mathbb{P}\Big\{ \sum_{j=1}^n \sigma_j > x + p_1 t_n \text{ for some } n \leqslant N_T \Big\},$$

so, as proven in [16, Theorem 3] for the supremum of a compound renewal process with negative drift $\mathbb{E}\sigma/\mathbb{E}\tau - p_1 = -m_1/\mathbb{E}\tau$, as $x \to \infty$ uniformly for $T \leqslant \widehat{T}$,

$$(3.3) \qquad \psi(x,\widehat{T},T) \ \sim \ \frac{1}{m_1} \int\limits_0^{m_1\mathbb{E}N_T} \overline{F}(x+t)dt$$

$$(3.4) \qquad\qquad = \ \int\limits_0^{\mathbb{E}N_T} \overline{F}(x+m_1 t)dt \ \sim \ \int\limits_0^{\mathbb{E}N_T} \overline{F}(\widehat{x}(t))dt,$$

where the last equivalence follows because $\mathbb{E}N_T \sim T/\mathbb{E}\tau$ as $T \to \infty$ and, for any $\varepsilon > 0$,

$$(1-\varepsilon) \int\limits_0^{T/\mathbb{E}\tau} \overline{F}(x+m_1 t)dt \leqslant \int\limits_0^{(1-\varepsilon)T/\mathbb{E}\tau} \overline{F}(x+m_1 t)dt$$

$$(3.5) \qquad \leqslant \int\limits_0^{(1+\varepsilon)T/\mathbb{E}\tau} \overline{F}(x+m_1 t)dt \ \leqslant \ (1+\varepsilon) \int\limits_0^{T/\mathbb{E}\tau} \overline{F}(x+m_1 t)dt,$$

and similar bounds for the integral of $\overline{F}(\widehat{x}(t))$.

Let us now consider the case $T \to \infty$ and $T > \widehat{T}$. If $\widehat{T}$ is bounded, then we make use of the inequalities

$$\mathbb{P}\{S_t > x + p_1\widehat{T} + p_2t \text{ for some } t \leqslant T\} \leqslant \psi(x, \widehat{T}, T)$$
$$\leqslant \mathbb{P}\{S_t > x - p_1\widehat{T} + p_2t \text{ for some } t \leqslant T\},$$

which reduce—due to long-tailedness of F—the problem to the case $\widehat{T} = T$, a particular case of considered above.

The case where $T \to \infty$, $\widehat{T} \to \infty$ and $T > \widehat{T}$, but $T - \widehat{T}$ is bounded, is very similar. Indeed, in this case it is enough to notice that

$$\mathbb{P}\{S_t > x + p_2(T - \widehat{T}) + p_1t \text{ for some } t \leqslant T\} \leqslant \psi(x, \widehat{T}, T)$$
$$\leqslant \mathbb{P}\{S_t > x - p_2(T - \widehat{T}) + p_1t \text{ for some } t \leqslant T\}.$$

Let us now consider the last remaining case where $T \to \infty$, $\widehat{T} \to \infty$, and $T - \widehat{T} \to \infty$. We demonstrate two approaches, where the first one is based on the uniform equivalences in the case of linear functions obtained in [16] and the second on the discrete time results from [12].

3.2. Proof based on [16]. Since F is particularly long-tailed, there exists an increasing function $h(x) \to \infty$ such that $h(x) = o(x)$ and

(3.6) $$\overline{F}(x + h(x)) \sim \overline{F}(x) \quad \text{as } x \to \infty,$$

see [13, Lemma 2.19]. For $T > \widehat{T}$,

$$\psi(x, \widehat{T}, T) \leqslant \mathbb{P}\{\sum_{j=1}^{n} \sigma_j > x + p_1 t_n - h(x) \text{ for some } n \leqslant N_{\widehat{T}}\}$$

$$+ \mathbb{P}\{\sum_{j=1}^{n} \sigma_j \leqslant x + p_1 t_n - h(x) \text{ for all } n \leqslant N_{\widehat{T}},$$

$$\sum_{j=1}^{n} \sigma_j > \widehat{x}(t_n) \text{ for some } n \in (N_{\widehat{T}}, N_T]\}$$

(3.7) $$= P_1 + P_2.$$

Again by [16, Theorem 3],

(3.8) $$P_1 \sim \int_0^{\mathbb{E}N_{\widehat{T}}} \overline{F}(x - h(x) + m_1 t)dt \sim \int_0^{\mathbb{E}N_{\widehat{T}}} \overline{F}(x + m_1 t)dt \quad \text{as } x \to \infty,$$

as follows from (3.6).

The second probability on the right hand side of (3.7) equals

$$
\begin{aligned}
P_2 \;=\;& \mathbb{P}\big\{ S_{\widehat{T}} \leqslant x + p_1\widehat{T} - h(x), \\
(3.9)\quad & S_{\widehat{T}} - p_1\widehat{T} + \sum_{j=N_{\widehat{T}}+1}^{n} \sigma_j - p_2(t_n - \widehat{T}) > x \text{ for some } n \in [N_{\widehat{T}}+1, N_T]\big\}.
\end{aligned}
$$

We need to exclude dependence caused by the interval $(\widehat{T}, t_{N_{\widehat{T}}+1}]$. We do it in the following way:

$$
\begin{aligned}
\sup_{n\in[N_{\widehat{T}}+1,N_T]} \Big(\sum_{j=N_{\widehat{T}}+1}^{n} \sigma_j - p_2(t_n - \widehat{T}) \Big) &\leqslant \sigma_{N_{\widehat{T}}+1} \\
&+ \sup_{n\in[N_{\widehat{T}}+2,N_T]} \Big(\sum_{j=N_{\widehat{T}}+2}^{n} \sigma_j - p_2(t_n - t_{N_{\widehat{T}}+1}) \Big),
\end{aligned}
$$

where the two random variables on the right are independent because $t_{N_{\widehat{T}}+1}$ is a stopping time. Further, extending the interval $(t_{N_{\widehat{T}}+1}, T]$ to the interval $(t_{N_{\widehat{T}}+1}, t_{N_{\widehat{T}}+1} + T - \widehat{T}]$ of length $T - \widehat{T}$ we conclude that

$$
\sup_{n\in[N_{\widehat{T}}+2,N_T]} \Big(\sum_{j=N_{\widehat{T}}+2}^{n} \sigma_j - p_2(t_n - t_{N_{\widehat{T}}+1}) \Big),
$$

is stochastically not greater than

$$
\zeta_{T-\widehat{T}} \;:=\; \sup_{t\leqslant T-\widehat{T}} (S_t - p_2 t),
$$

whose tail is equivalent to, again by [16, Theorem 3],

$$
\mathbb{P}\{\zeta > z\} \;\sim\; \int_{0}^{\mathbb{E}N_{T-\widehat{T}}} \overline{F}(z + m_2 y)dy \quad \text{as } z \to \infty.
$$

Fix any $\varepsilon > 0$ such that $m_1 = p_1\mathbb{E}\tau - \mathbb{E}\sigma > \varepsilon$. Then it follows from (3.9) that,

with $\widetilde{\zeta}_{T-\widehat{T}}$ being an independent copy of $\zeta_{T-\widehat{T}}$,

$$
\begin{aligned}
P_2 \;\leqslant\;& \int_{-\infty}^{x-h(x)} \mathbb{P}\{S_{\widehat{T}} - p_1\widehat{T} \in dy\} \mathbb{P}\{\sigma_{N_{\widehat{T}}+1} + \widetilde{\zeta}_{T-\widehat{T}} > x - y\} \\
=\;& \int_{-\infty}^{x-h(x)} \mathbb{P}\{S_{\widehat{T}} - p_1\widehat{T} + (m_1 - \varepsilon)\widehat{T}/\mathbb{E}\tau \in dy + (m_1 - \varepsilon)\widehat{T}/\mathbb{E}\tau\} \\
& \qquad \mathbb{P}\{\sigma_{N_{\widehat{T}}+1} + \widetilde{\zeta}_{T-\widehat{T}} > x - y\} \\
=\;& \int_{-\infty}^{x+(m_1-\varepsilon)\widehat{T}/\mathbb{E}\tau-h(x)} \mathbb{P}\{S_{\widehat{T}} - (\mathbb{E}\sigma + \varepsilon)\widehat{T}/\mathbb{E}\tau \in dy\} \\
& \qquad \mathbb{P}\{\sigma_{N_{\widehat{T}}+1} + \widetilde{\zeta}_{T-\widehat{T}} > x + (m_1 - \varepsilon)\widehat{T}/\mathbb{E}\tau - y\}.
\end{aligned}
$$

(3.10)

We have

$$
\mathbb{P}\{\sigma_{N_{\widehat{T}}+1} + \widetilde{\zeta}_{T-\widehat{T}} > u\} \;\sim\; \mathbb{P}\{\widetilde{\zeta}_{T-\widehat{T}} > u\} + \overline{F}(u) \quad \text{as } u \to \infty,
$$

by [13, Corollary 3.16]. Since the process $S_t - (\mathbb{E}\sigma + \varepsilon)t/\mathbb{E}\tau$ has negative drift $-\varepsilon/\mathbb{E}\tau$ and its value at time $\widehat{T}$ does not exceed the value of its maximum at the same time, we also have

$$
\mathbb{P}\{S_{\widehat{T}} - (\mathbb{E}\sigma + \varepsilon)\widehat{T}/\mathbb{E}\tau > u\} \;\leqslant\; c_1(\varepsilon)\mathbb{P}\{\zeta_{\widehat{T}} > u\},
$$

where

$$
\zeta_{\widehat{T}} \;:=\; \sup_{t \leqslant \widehat{T}}(S_t - p_1 t).
$$

Integrating (3.10) by parts, applying the last upper bound and then integrating by parts back we deduce that

$$
\begin{aligned}
\int_{h(x_1)}^{x+(m_1-\varepsilon)\widehat{T}/\mathbb{E}\tau-h(x)} & \mathbb{P}\{S_{\widehat{T}} - (\mathbb{E}\sigma + \varepsilon)\widehat{T}/\mathbb{E}\tau \in dy\} \\
& \mathbb{P}\{\sigma_{N_{\widehat{T}}+1} + \widetilde{\zeta}_{T-\widehat{T}} > x + (m_1 - \varepsilon)\widehat{T}/\mathbb{E}\tau - y\} \\
&= o\big(\mathbb{P}\{\zeta_{T-\widehat{T}} > x + (m_1 - \varepsilon)\widehat{T}/\mathbb{E}\tau\} + \mathbb{P}\{\zeta_{\widehat{T}} > x + (m_1 - \varepsilon)\widehat{T}/\mathbb{E}\tau\}\big),
\end{aligned}
$$

by [13, Theorem 3.28], because $F \in \mathcal{S}^*$. Hence,

$$
\begin{aligned}
P_2 &\leqslant \int_{-\infty}^{h(x)} \mathbb{P}\{S_{\widehat{T}} - (\mathbb{E}\sigma + \varepsilon)\widehat{T}/\mathbb{E}\tau \in dy\} \\
&\qquad \mathbb{P}\{\sigma_{N_{\widehat{T}}+1} + \widetilde{\zeta}_{T-\widehat{T}} > x + (m_1 - \varepsilon)\widehat{T}/\mathbb{E}\tau - y\} \\
&\qquad + o\big(\mathbb{P}\{\zeta_{T-\widehat{T}} > x + (m_1 - \varepsilon)\widehat{T}/\mathbb{E}\tau\} + \mathbb{P}\{\zeta_{\widehat{T}} > x + (m_1 - \varepsilon)\widehat{T}/\mathbb{E}\tau\}\big) \\
&\leqslant \mathbb{P}\{\sigma_{N_{\widehat{T}}+1} + \widetilde{\zeta}_{T-\widehat{T}} > x + (m_1 - \varepsilon)\widehat{T}/\mathbb{E}\tau - h(x)\} + o(...) \\
&\sim \mathbb{P}\{\widetilde{\zeta}_{T-\widehat{T}} > x + (m_1 - \varepsilon)\widehat{T}/\mathbb{E}\tau\} + o(\mathbb{P}\{\zeta_{\widehat{T}} > x + (m_1 - \varepsilon)\widehat{T}/\mathbb{E}\tau\}) \\
\\
&\sim \int_0^{\mathbb{E}N_T - \mathbb{E}N_{\widehat{T}}} \overline{F}(x + (m_1 - \varepsilon)\widehat{T}/\mathbb{E}\tau + m_2 t)dt \\
&\qquad + o(\mathbb{P}\{\zeta_{\widehat{T}} > x + (m_1 - \varepsilon)\widehat{T}/\mathbb{E}\tau\}) \\
&\leqslant \int_{(m_1 - \varepsilon)\widehat{T}/\mathbb{E}\tau}^{\mathbb{E}N_T} \overline{F}(x + m_2 t)dt + o(\mathbb{P}\{\zeta_{\widehat{T}} > x + (m_2 - \varepsilon)\widehat{T}/\mathbb{E}\tau\}).
\end{aligned}
$$

for all sufficiently large $\widehat{T}$. Being substituted together with (3.8) into (3.7) it implies that

$$
\psi(x, \widehat{T}, T) \leqslant (1 + o(1))\Big(\int_0^{\mathbb{E}N_{\widehat{T}}} \overline{F}(x + m_1 t)dt + \int_{(m_1 - \varepsilon)\widehat{T}/\mathbb{E}\tau}^{\mathbb{E}N_T} \overline{F}(x + m_2 t)dt \Big).
$$

Since $\mathbb{E}N_{\widehat{T}} \sim \widehat{T}/\mathbb{E}\tau$ as $\widehat{T} \to \infty$ and due to (3.5) with $T = \widehat{T}$, we get

$$
\psi(x, \widehat{T}, T) \leqslant (1 + o(1))\Big(\int_0^{\widehat{T}/\mathbb{E}\tau} \overline{F}(x + m_1 t)dt + \int_{\widehat{T}/\mathbb{E}\tau}^{\mathbb{E}N_T} \overline{F}(x + m_2 t)dt \Big)
$$

as x, T, $\widehat{T}$, $T - \widehat{T} \to \infty$. So,

$$
(3.11) \quad \psi(x, \widehat{T}, T) \leqslant (1 + o(1)) \int_0^{\mathbb{E}N_T} \overline{F}(\widehat{x}(t))dt \quad \text{as } x,\ T,\ \widehat{T},\ T - \widehat{T} \to \infty.
$$

For the lower bound, let us fix an $\varepsilon > 0$ and follow the standard arguments based on the strong law of large numbers and the single big jump principle. By the strong law of large numbers, there exists an A such that

$$
(3.12) \qquad \mathbb{P}\{|t_n - n\mathbb{E}\tau| < n\varepsilon + A \text{ for all } n \geqslant 1\} \geqslant 1 - \varepsilon.
$$

On the event introduced in (3.12), if $n \leqslant \left[\frac{\widehat{T}-A}{\mathbb{E}\tau+\varepsilon}\right] =: \widehat{n}$, then $t_n \leqslant \widehat{T}$ and hence $n \leqslant N_{\widehat{T}}$. Further, on the event introduced in (3.12), if $n_1 := \left[\frac{\widehat{T}-A}{\mathbb{E}\tau-\varepsilon}\right] < n \leqslant \left[\frac{T-A}{\mathbb{E}\tau+\varepsilon}\right] =: n_2$, then $\widehat{T} < t_n \leqslant T$ and hence $N_{\widehat{T}} < n \leqslant N_T$.

Then, since $\bar{\sigma}$'s do not depend on the renewal process N_t, we obtain the inequality

$$(3.13) \qquad \psi(x,\widehat{T},T) \geqslant (1-\varepsilon)\mathbb{P}\Big\{\sum_{i=1}^n \bar{\sigma}_i > \widehat{b}(n(\mathbb{E}\tau+\varepsilon)+A) - n\mathbb{E}\sigma$$

$$\text{for some } n \leqslant \widehat{n} \text{ or } n \in (n_1,n_2]\Big\}.$$

Then the standard arguments based on the strong law of numbers—now for σ's—and the single big jump principle finally imply the lower bound

$$\psi(x,\widehat{T},T) \geqslant (1+o(1)) \int_0^{\mathbb{E}N_T} \overline{F}(\widehat{x}(t))dt \quad \text{as } x,\, T,\, \widehat{T},\, T-\widehat{T} \to \infty,$$

which together with (3.11) concludes the proof. $\qquad\qquad\square$

3.3. Proof based on [12]. Let us now give an alternative proof of the asymptotic behaviour of $\psi(x,\widehat{T},T)$ in the case $T \to \infty$ based on the results for discrete time from [12].

Let $S_n = \sum_{k=1}^n \sigma_k$, $\bar{\sigma}_n = \sigma_n - \mathbb{E}\sigma$, and $\bar{S}_n = \sum_{k=1}^n \bar{\sigma}_k \equiv S_n - n\mathbb{E}\sigma$ be a centered random walk. From either Corollary 3 of [11] or Theorem 1 of [9], it follows that, for any random variable γ having a light-tailed distribution (that is, $\mathbb{E}\exp(\delta\gamma) < \infty$ for some $\delta > 0$), we have, for any real C,

$$(3.14) \qquad \mathbb{P}\{\bar{S}_n > x + nC \text{ for some } n \leqslant \gamma\} \sim \mathbb{E}\gamma\overline{F}(x).$$

Take $\varepsilon > 0$ sufficiently small, such that $p_1(\mathbb{E}\tau-\varepsilon) - \mathbb{E}\sigma > 0$ and $p_2(\mathbb{E}\tau-\varepsilon) - \mathbb{E}\sigma > 0$. We take

$$\gamma = \min\{n \geqslant 1 : t_k \geqslant (\mathbb{E}\tau-\varepsilon)k \text{ for all } k \geqslant n\},$$

which is finite almost surely due to the strong law of large numbers. Moreover, this random variable γ is light-tailed because, for $\delta > 0$,

$$\begin{aligned}
\mathbb{P}\{\gamma > n\} &= \mathbb{P}\{t_k < (\mathbb{E}\tau-\varepsilon)k \text{ for some } k \geqslant n\} \\
&= \mathbb{P}\{e^{\delta((\mathbb{E}\tau-\varepsilon)k-t_k)} > 1 \text{ for some } k \geqslant n\} \\
&\leqslant (\mathbb{E}e^{\delta(\mathbb{E}\tau-\varepsilon-\tau)})^n,
\end{aligned}$$

by Doob's inequality provided that $\mathbb{E}e^{\delta(\mathbb{E}\tau - \varepsilon - \tau)} < 1$; it holds for all sufficiently small $\delta > 0$, since the random variable $\mathbb{E}\tau - \varepsilon - \tau$ is bounded above and has negative mean.

Then

$$
\begin{aligned}
\mathbb{P}\{\bar{S}_n > \widehat{b}(t_n) - n\mathbb{E}\sigma &\text{ for some } n \leqslant \gamma\} \\
&\leqslant \mathbb{P}\{\bar{S}_n > x - n\mathbb{E}\sigma \text{ for some } n \leqslant \gamma\} \ \sim \ \mathbb{E}\gamma\overline{F}(x),
\end{aligned}
$$

by (3.14). Therefore, for all $T > 0$,

$$
\begin{aligned}
\psi(x, \widehat{T}, T) \ \leqslant \ & \mathbb{P}\{\bar{S}_n > \widehat{b}(t_n) - n\mathbb{E}\sigma \text{ for some } n \leqslant \gamma\} \\
& + \mathbb{P}\{\bar{S}_n > \widehat{b}(n(\mathbb{E}\tau - \varepsilon)) - n\mathbb{E}\sigma \text{ for some } n \in (\gamma, N_T]\} \\
\leqslant \ & O(\overline{F}(x)) + \mathbb{P}\{\bar{S}_n > \widehat{b}(n(\mathbb{E}\tau - \varepsilon)) - n\mathbb{E}\sigma \text{ for some } n \leqslant N_T\}
\end{aligned}
$$

as $x \to \infty$ uniformly for all $T > 0$.

Now we recall the following result where the class Γ is a class of all counting random variables γ such that, for all n, the event $\{\gamma \leqslant n\}$ does not depend on $\{\sigma_k\}_{k>n}$. For $c > 0$, let $\mathcal{G}_c$ be the class of functions g such that $g(n+1) \geqslant g(n) + c$ for all $n = 1, 2, \ldots$.

THEOREM 3.1. *(see [12, Theorem 2(ii)]). Assume $\bar{S}_n = \sum_1^n \bar{\sigma}_n$ is a centered random walk where the common distribution of $\bar{\sigma}_n$ belongs to the class $\mathcal{S}^*$. Then, for any $c > 0$, uniformly for all $g \in \mathcal{G}_c$ and for all random times $\gamma \in \Gamma$, we have*

$$
\begin{aligned}
\mathbb{P}\{\max_{n \leqslant \gamma}(\bar{S}_n - g(n)) > x\} \ &\sim \ \sum_{n \geqslant 1} \mathbb{P}(\gamma \geqslant n)\mathbb{P}(\bar{\sigma}_1 > x + g(n)) \\
(3.15) \qquad\qquad &\sim \ \sum_{n \geqslant 1} \mathbb{P}(\gamma \geqslant n)\overline{F}(x + g(n)) \quad \text{as } x \to \infty.
\end{aligned}
$$

Consider function g of the form

$$
g(n) \ = \ \widehat{b}(n(\mathbb{E}\tau - \varepsilon)) - n\mathbb{E}\sigma - x,
$$

which is in the class $\mathcal{G}_{p_1 \wedge p_2}$. By Theorem 3.1, we get that

$$
\begin{aligned}
\mathbb{P}\{\bar{S}_n > \widehat{b}(n(\mathbb{E}\tau - \varepsilon)) - n\mathbb{E}\sigma &\text{ for some } n \leqslant N_T\} \\
&\sim \sum_n \mathbb{P}\{N_T \geqslant n\}\overline{F}(x + g(n)),
\end{aligned}
$$

as $x \to \infty$ and $T \to \infty$. Further, since $\varepsilon > 0$ is arbitrary, we may follow the proof of the upper bound for ψ and let $\varepsilon \to 0$ to obtain an upper bound of the form

$$(1 + o(1)) \sum_n \mathbb{P}(N_T \geqslant n)\overline{F}(\widehat{x}(n)).$$

The last sum may be rewritten as

$$
\begin{aligned}
\mathbb{E} \sum_n \mathbb{I}(N_T \geqslant n)\overline{F}(\widehat{x}(n)) \;\; &= \;\; \mathbb{E} \sum_{n=1}^{N_T} \overline{F}(\widehat{x}(n)) \\
&\sim \;\; \mathbb{E} \int_0^{N_T} \overline{F}(\widehat{x}(t))dt \quad \text{as } x \to \infty.
\end{aligned}
$$

Here the last equivalence follows from the long-tailedness of F. Finally, we may use Lemma 2.1 to conclude with the upper bound (3.11).

Theorem 3.1 is also applicable to (3.13), so hence the correct lower bound for $\psi(x, \widehat{T}, T)$ follows too. $\qquad\square$

3.4. Proof of Lemma 2.1. For any fixed x, the function

$$f(y) \quad := \quad \int_0^y \overline{F}(x + z(t))dt$$

is a concave function in y because $f'(y) = \overline{F}(x + z(y))$ is decreasing in y. Then the upper bound (2.20) follows by Jensen's inequality for concave functions.

Concerning (2.21), notice that, for any fixed $\varepsilon > 0$,

$$\mathbb{E} \int_0^{N_T} \overline{F}(x + z(t))dt \quad \geqslant \quad \mathbb{P}\{N_T > (1 - \varepsilon)\mathbb{E}N_T\} \int_0^{(1-\varepsilon)\mathbb{E}N_T} \overline{F}(x + z(t))dt,$$

where $\mathbb{P}\{N_T > (1 - \varepsilon)\mathbb{E}N_T\} \to 1$ as $T \to \infty$ by the law of large numbers for the renewal process and

$$
\begin{aligned}
\int_0^{(1-\varepsilon)\mathbb{E}N_T} \overline{F}(x + z(t))dt \quad &\geqslant \quad (1 - \varepsilon) \int_0^{\mathbb{E}N_T} \overline{F}(x + z((1 - \varepsilon)t))dt \\
&\geqslant \quad (1 - \varepsilon) \int_0^{\mathbb{E}N_T} \overline{F}(x + z(t))dt,
\end{aligned}
$$

which implies lower bound

$$\mathbb{E} \int_0^{N_T} \overline{F}(x + z(t))dt \quad \geqslant \quad (1 + o(1)) \int_0^{\mathbb{E}N_T} \overline{F}(x + z(t))dt \quad \text{as } x, T \to \infty,$$

which together with upper bound (2.20) justifies (2.21).

Finally, for any fixed T and A, since $z(t) \geqslant 0$ increases,

$$
\begin{aligned}
\mathbb{E} \int_0^{N_T} \overline{F}(x + z(t))dt \;&\geqslant\; \mathbb{E}\Big\{ \int_0^{N_T} \overline{F}(x + z(t))dt; \; N_T \leqslant A \Big\} \\
&\geqslant\; \overline{F}(x + z(A))\mathbb{E}\{N_T; \; N_T \leqslant A\} \\
&\sim\; \overline{F}(x)\mathbb{E}\{N_T; \; N_T \leqslant A\} \quad \text{as } x \to \infty,
\end{aligned}
$$

provided F is long-tailed in which case also

$$
\int_0^{\mathbb{E}N_T} \overline{F}(x + z(t))dt \;\sim\; \overline{F}(x)\mathbb{E}N_T \quad \text{as } x \to \infty,
$$

so hence the asymptotics (2.22) follows uniformly on T-compact sets. Together with (2.21) it completes the proof. □

3.5. Proof of Theorem 2.2. The proof of this result is very similar to the proof of (2.5) of Theorem 2.1. For example to prove (2.14) it suffices to observe that

$$
\begin{aligned}
\mathbb{P}\Big\{ &\frac{\tau_\wedge(ax, x)}{e(x)} \geqslant y, \; \frac{S_{\tau_\wedge(ax,x)} - b_\wedge(t)}{e(x)} \geqslant v; \tau_\wedge(ax, x) < \infty \Big\} \\
&= \; \mathbb{P}\big\{ \bar{S}_n > \min\{ax + p_1 t_n - n\mathbb{E}\sigma, x + p_2 t_n - n\mathbb{E}\sigma\} + ve(x) \\
&\qquad\qquad \text{for some } n \geqslant ye(x)\big\} .
\end{aligned}
$$

□

3.6. Generalisation to n insurance companies. It is more or less clear that a very similar proof works for (3.1) if $\widehat{x}(t)$ is a continuous piece-wise linear increasing function. Then it allows to go beyond two insurance companies and consider a model with an arbitrary number of them. Clearly in high dimension one has to overcome extra combinatorial problems that seem to be doable.

3.7. Open question. We strongly believe that an analogue of Theorem 2.1 holds for stopping times, say ϱ instead of T. We expect the following to be correct:

$$
(3.16) \qquad \psi_\wedge(x_1, x_2, \varrho) \;\sim\; \mathbb{E} \int_0^{N_\varrho} \overline{F}(\min\{x_1 + tm_1, x_2 + tm_2\}) \, dt
$$

and

$$(3.17) \qquad \psi_\vee(x_1, x_2, \varrho) \; \sim \; \mathbb{E} \int\limits_0^{N_\varrho} \overline{F}(\max\{x_1 + tm_1, x_2 + tm_2\}) \, \mathrm{d}t$$

hold as $x \to \infty$ uniformly for all stopping times ϱ with respect to the filtration generated by the renewal process N_t.

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