Linking Individual and Collective Contests through Noise Level and Sharing Rules

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Abstract

We propose the use of Nitzan's (1991) sharing rule in collective contests as a tractable way of modeling individual contests. This proposal (i) tractably introduces noise in Tullock contests when no closed form solution in pure strategies exists, (ii) satisfies the important property of homogeneity of degree zero, (iii) can be effort or noise equivalent to a standard Tullock contest.

Keywords: Contest; Homogeneity of Degree Zero; Equivalence

JEL classification: C72; D72; D74

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1 Introduction

Suppose N players participate in a contest by exerting costly effort to win a prize of common value V. A crucial modeling element in such setups is the contest success function (CSF), f_i, mapping the vector of non-negative efforts to the probability that player $i \in N$ wins the prize (i.e., $f_i : R_+^N \to [0,1]$ such that $\sum_{i \in N} f_i(.) = 1$). In the CSF proposed by Tullock (1980),

$$f_i^r(e_1, ..., e_N) = \frac{e_i^r}{\sum_{j=1}^N e_j^r} \text{ if } \sum_{j=1}^N e_j > 0 \text{ and } 1/N \text{ otherwise} \qquad (r\text{-function})$$

where $e_i \geq 0$ denotes the effort exerted by player i and $r \geq 0$ determines the level of noise. If $r = 0$ then the noise is maximum and players face a fair lottery. If $r \to \infty$ then there is no noise and the highest effort wins with certainty (an all-pay auction). Different levels of noise can be introduced for intermediate values.

Although the importance of noise when modeling contests is widely accepted, Tullock's otherwise tractable proposal leads to certain modeling challenges: First, when more than two players with asymmetric costs compete, a closed form solution for the equilibrium in pure strategies exists only if $r = 1$. That is, the introduction of noise in asymmetric multiplayer contests becomes intractable. Second, when only two players compete, an equilibrium in pure strategies does not exist for high levels of noise.

Here we propose the allocation of a prize among group members in collective contests as introduced by Nitzan (1991) as a way of addressing these challenges. While guaranteeing tractability, we show that (i) this proposal can be effort or noise equivalent to an r-contest and (ii) several axiomatic and equilibrium properties are similar to Tullock's original proposal. These results differentiate this proposal to a similar approach by Amegashie (2006) as based on Dasgupta and Nti (1998):

$$f_i^\alpha(e_1, ..., e_N) = \frac{e_i + \alpha}{\sum_{j=1}^N e_j + N\alpha} \qquad (\alpha\text{-function})$$

where $\alpha > 0$ is the introduced "tractable" noise parameter.

From an axiomatic perspective, the r-CSF satisfies all desirable properties of imperfect discrimination, anonymity, monotonicity, homogeneity of degree zero (HD0) and Luce's axiom (Skaperdas, 1996). Achieving noise tractability using the α-function requires the sacrifice of HD0. However, HD0 is desirable for contests where the result should be scale invariant, for instance when it should be irrelevant whether effort expenditures are measured in euros or in dollars or whether effort levels are measured in hours or

minutes (see among others Hirshleifer 2000; Malueg and Yates 2006; Alcalde and Dahm 2007; Beviá and Corchón 2015). HD0 is thus viewed as an essential property whenever outlays are in quantifiable units such as money or time. Our proposal satisfies HD0, while sacrificing Luce's axiom. Hence, researchers and contest designers may choose between the current proposal and the α-CSF as alternative ways of introducing "tractable" noise depending on the importance of HD0 versus Luce's axiom. Moreover, and in contrast to the α-function, we show that our proposal can be effort or noise equivalent to Tullock's original proposal.

2   The λ-contest

Following Nitzan (1991) we define

$$f_i^\lambda(e_1, ..., e_N) = \lambda \frac{e_i}{\sum_{j=1}^N e_j} + (1-\lambda)\frac{1}{N} \text{ if } \sum_{j=1}^N e_j > 0 \text{ and } 1/N \text{ otherwise} \qquad (\lambda\text{-function}).$$

As discussed, and compared to the α-CSF, it is easy to show that the λ-function satisfies HD0.[1] Assume linear cost functions with $c_i > 0$ denoting the marginal cost of player i, and without loss of generality assume that $c_1 \leq c_2 \leq ... \leq c_N$, we can define player's i payoff in the λ-contest as:[2]

$$\pi_i^\lambda = f_i^\lambda(e_1, ..., e_N)V - c_i e_i \qquad (1)$$

If $\lambda \in [0, 1]$, then the λ-function satisfies the properties of a CSF and is a convex combination of the most common version of a Tullock CSF ($r = 1$) and of a fair lottery ($r = 0$).[3] Parameter λ is associated to the level of noise in the competition and clearly resembles the effect of r in the r-contest. Low values of λ are associated with high levels of noise. Note however that λ need not be restricted in the $[0, 1]$ interval. When $\lambda > 1$ the proposed function f_i^λ may take values outside $[0, 1]$ and therefore can not be interpreted as a CSF representing probabilities. If $\lambda > 1$, then the proposed function allows for transfers among group members or the presence of a compulsory participation fee (Appelbaum and Katz, 1986; Hillman and Riley, 1989). Since this may imply a negative expected payoff for some contestants our setup may violate voluntary participation and

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[1]The λ-function can be obtained from Beviá and Corchón (2015) by setting $\alpha = \frac{1}{N}$, $s = 1$ and $\beta = \frac{N-1}{N}\lambda$ and the HD0 property can be found from there.

[2]For the reasons of interpretability, players' heterogeneity is introduced through cost asymmetries. This is equivalent to asymmetries in terms of valuations (Gradstein, 1995; Corchón, 2007).

[3]Amegashie (2012) proposes a nested two-player contest that ranges from a Tullock to an all-pay auction. A similar structure can also be found in Grossmann (2014) with a nested α-contest.

hence, as Hillman and Riley (1989) argue, is relevant in situations that involve winners and losers. If one does not want to model such transfers and interpret the λ-function as a CSF then λ needs to be restricted to the [0,1] interval.

2.1 Equilibrium

The λ-contest as presented by its payoffs in (1) has been previously solved in Hillman and Riley (1989).[4]

Remark 1. [*Hillman and Riley (1989)*]

Denote the cost-weighted prize valuations by $V_i = \frac{\lambda V}{c_i}$, there exists a unique equilibrium in pure strategies with player's i effort given by:

$$e_i = \left(1 - \frac{1}{V_i} \frac{(M-1)}{\sum_{j=1}^{M} \frac{1}{V_j}} \right) \frac{(M-1)}{\sum_{j=1}^{M} \frac{1}{V_j}} \tag{2}$$

where M is the number of active players. Player M is the highest marginal cost player for whom the condition $V_M > \frac{(M-2)}{\sum_{i \leq M-1} \frac{1}{V_j}}$ holds.

Using the λ-contest one can solve for the equilibrium efforts in closed form in any asymmetric multiplayer contest. This is not possible in the r-contest. Comparing equilibrium properties across the three (r, α and λ) ways of modeling contests the following results arise:

1. Individual and aggregate effort decreases with the level of noise in both the α and λ-contest. This is different for r-contest where for the two player r-contest comparative statics of aggregate effort with respect to noise levels depend on the degree of asymmetry between the players.

2. Linking the asymmetry with aggregate equilibrium effort in a two-player λ-contest is in line with the standard result of the r-contest (Nti, 1999) and the α-contest since aggregate equilibrium effort decreases in players' asymmetry.

3. In an N-symmetric-players contest adding an additional player increases total effort in the r-contest with an equilibrium in pure strategies and in the λ-contest, while it may decrease total effort in the α-contest.

4. The λ-contest and r-contest can not sustain an equilibrium where all players are inactive while this may occur in the α-contest.

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[4]In their notation, W_i and L_i denote winner and losers' payoffs and the λ-contest is obtained for the particular values $W_i = \frac{\lambda V}{c_i} + (1-\lambda)\frac{V}{c_i N}$ and $L_i = (1-\lambda)\frac{V}{c_i N}$.

2.2 Equivalence

Following the definitions by Chowdhury and Sheremeta (2014):[5]

Definition 1.

- *Contests are **effort equivalent** if they result in the same equilibrium efforts.*

- *Contests are **strategically equivalent** if they result in the same best responses.*

- *Contests are **payoff equivalent** if in equilibrium they result in the same payoffs.*

Following Alcalde and Dahm (2010) we use the effort-elasticity of the probability of winning as a measure of noise in a contest j, i.e., $\nu_i^j(e_1,...e_N) = \frac{\partial f_i^j(e_1,...e_N)}{\partial e_i} \frac{e_i}{f_i^j(e_1,...,e_N)}$ and pay attention to the case in which efforts are equalized.[6]

Definition 2. *Two CSF j and k are **noise equivalent** if and only if $\nu_i^j(e_1^j,...e_N^j) = \nu_i^k(e_1^k,...e_N^k)$ for all $i = 1,...,N$; whenever $e_1^j = e_2^j = ... = e_N^j$ and $e_1^k = e_2^k = ... = e_N^k$.*

It can be shown, by comparing the α-CSF and the r-CSF, that none of the four equivalence properties hold. We now restrict attention to the comparison between the λ and r-contests.

Proposition 1. *For any two-player r-contest with an equilibrium in pure strategies (i.e., r such that $V_1^r + V_2^r > rV_2^r$):*

1. *There exists an effort equivalent λ-contest with $\lambda = \frac{r(V_1 V_2)^r (V_1+V_2)^2}{V_1 V_2 (V_1^r + V_2^r)^2}$.*

2. *There exists no strategically equivalent λ-contest (except for $r = \lambda = 0$ and $r = \lambda = 1$ when the two contests coincide).*

3. *There exists no payoff equivalent λ-contest (except for $r = \lambda = 0$, $r = \lambda = 1$ when the two contests coincide and the symmetric case, $c_1 = c_2$).*

Figure 1 illustrates the result for two asymmetric players. On the left, best responses are different for the λ and r-contest but they intersect at the same effort equivalence point. On the right panel, the value of λ that guarantees payoff equivalence for player 1 only coincides with that providing payoff equivalence for player 2 when the two contests coincide (i.e., $r = \lambda = 1$ and $r = \lambda = 0$).

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[5]In the case of a unique equilibrium (Chowdhury and Sheremeta, 2011), strategic equivalence implies effort equivalence while the opposite need not be true. Moreover, strategic equivalence need not imply payoff equivalence (Chowdhury and Sheremeta, 2014).

[6]The all-pay auction (i.e., $r \to \infty$) is considered a deterministic contest because in case of a tie, an arbitrarily small amount of additional effort is sufficient to secure the prize (Alcalde and Dahm, 2010). In contrast, a marginal increase in effort has no effect on the probability of winning when the two efforts are not equal. This justifies measuring effort-elasticity at equal effort levels.
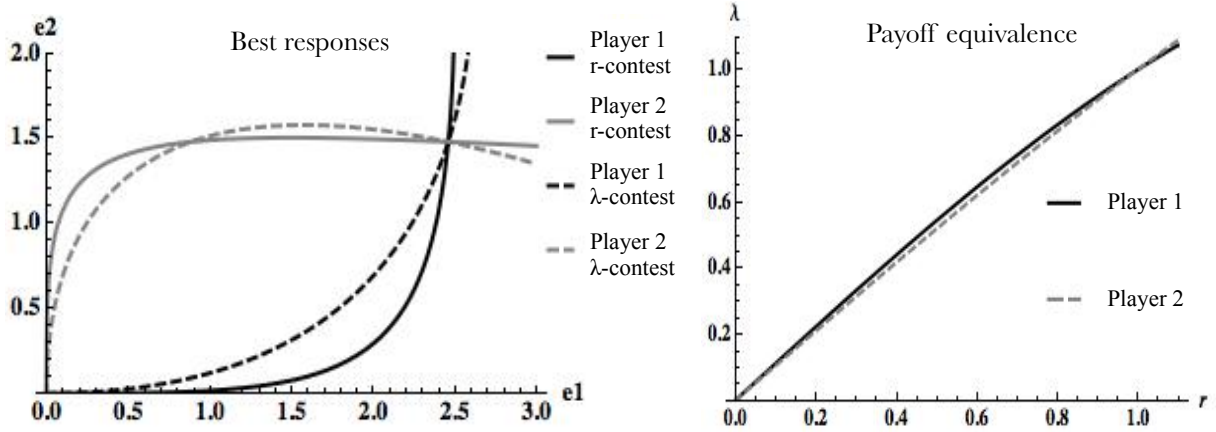
Figure 1: Best response functions and effort equivalence on the left ($r = 0.5$, $\lambda = 0.524729$) and payoff equivalence on the right. For both graphs $V_1 = 20$, $V_2 = 12$.

Figure 2 illustrates how effort equivalence can be obtained through the appropriate choice of value λ^* for any given value r for a given level of asymmetry ($V_1/V_2 = 10$, in this example). Notice that the value of λ^* that guarantees an effort equivalent λ-contest, is not monotonic in r. This occurs because while in the λ-contest individual and aggregate equilibrium efforts are decreasing in the level of noise, this need not be true in the r-contest. Note that, the level of λ that ensures effort equivalence might involve transfers as in Hillman and Riley (1989). This depends on the exact level of asymmetry.
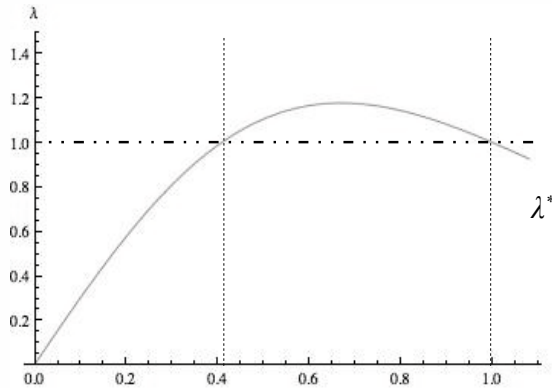
Figure 2: Effort equivalence value of λ, given any r such that an equilibrium in pure strategies exists ($V_1^r + V_2^r > rV_2^r$ with $V_1/V_2 = 10$).

The result on noise equivalence is summarized in Proposition 2. Observe that for asymmetric players the value of λ that guarantees noise equivalence differs from the value that guarantees effort equivalence. Therefore one can obtain both effort and noise equivalence with the same choice of λ only for symmetric players.

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Proposition 2. *For any r-contest with $r \in [0,1]$, the λ-contest and the r-contest are noise equivalent if and only if $\lambda = r$.*

This result implies that for any $r \in [0,1]$, we can find a λ-contest with a comparable level of noise. It is particularly important if we take into account that the r-contest has no closed form solution when $N \geq 3$. That is, for any r-contest with $r \in [0,1)$ and $N \geq 3$, although the r-contest does not have a closed form solution, there exists a noise equivalent λ-contest with $\lambda = r$ and a closed form solution. In contrast to the α-contest that also provides a closed form solution, our proposal can also guarantee noise equivalence with the original r-contest.

2.3 Participation

We have shown that for any r-contest, one can always find an effort equivalent λ-contest. However, in the effort equivalent λ-contest the presence of transfers may be required (i.e., $\lambda > 1$, as in Figure 2 for r belonging to $[0.41, 1]$). These transfers in turn may violate participation constraint (PC) as in Hillman and Riley (1989). The following remark provides the conditions that guarantee voluntary participation.

Remark 2. *In any λ-contest the PC is satisfied if*
$$V_i \sum_{j=1}^{N} \frac{1}{V_j} \frac{\lambda(N-1)+1}{N} \geq \lambda(N-1)(2 - \frac{N-1}{V_i \sum_{j=1}^{N} \frac{1}{V_j}}) \ \forall \ i = 1, ..., N.$$
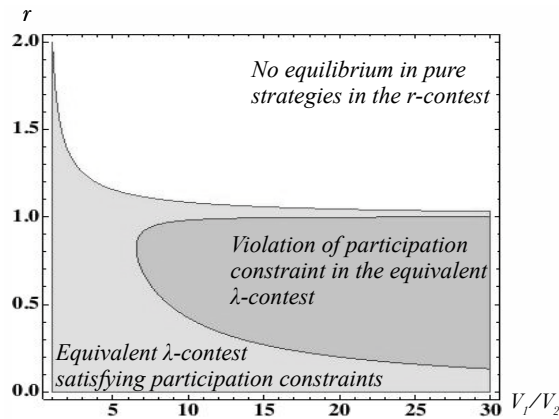
Figure 3: Participation constraint in the effort equivalent λ-contest.

Intuitively, as long as λ is low, that is either no transfers are involved ($\lambda \leq 1$) or transfers are present but are not too punishing the PC is satisfied. Once the required transfers become high enough, then low contributors are severely punished and are better off not participating in the contest.[7] The conditions under which the λ-equivalent contest

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[7]The remark follows from condition (23) in Beviá and Corchón (2015, p. 387).

does not satisfy the PC depend on the specific combination of cost asymmetry and noise level.

In particular, for any two-player r-contest, although an effort-equivalent λ-contest always exists, the latter fails to satisfy PC if

$$\frac{r(V_1 V_2)^r (V_1 + V_2)^2}{V_1 V_2 (V_1^r + V_2^r)^2} > \frac{(V_1 + V_2)^2}{V_1^2 + 2V_1 V_2 - V_2^2}.$$

A graphical representation is given in Figure 3. The darkest area in the lower panel plots the combinations of asymmetry V_1/V_2 and r for which the effort equivalent λ-contest does not satisfy the PC. While the effort equivalent λ-contest satisfies the PC for any level of r when players' asymmetry is low, for higher levels of asymmetry the region of r for which an effort equivalent λ-contest satisfies the PC shrinks.

3   Discussion

We propose the use of the λ-function as a tractable way of modeling noise while guaranteeing HD0. Depending on whether HD0 or Luce's axiom is more relevant, we highlight the choice between the λ-contest and the α-CSF proposed by Amegashie (2006). We also show that the λ-contest can be effort or noise equivalent to an r-contest while several equilibrium properties are similar to Tullock's original proposal. The λ-contest can be implemented in applications where the absence of closed form solutions induces focus only on $r = 1$ (e.g., Franke 2012), while it can also be of interest for experiments given the intuitive manner one can introduce noise in the lab.

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4 Appendix

4.1 Proof of Proposition 1

1. When $N = 2$ the condition for a player being active active in the λ-contest is always satisfied. From Remark 1 the equilibrium effort of player i is $e_i = \frac{V_i^2 V_j \lambda}{(V_1+V_2)^2}$ for $i = 1, 2$, $j \neq i$. To prove effort equivalence equalize these equilibrium efforts with the ones of the r-contest (Nti, 1999). Equilibrium efforts of the λ-contest coincide with the ones of the r-contest for $\lambda = \frac{r(V_1 V_2)^r (V_1+V_2)^2}{V_1 V_2 (V_1^r + V_2^r)^2}$.

2. Note that when $r = \lambda = 0$ or $r = \lambda = 1$ the λ-contest and the r-contest coincide, hence strategic equivalence follows immediately in these cases. The best response for player i in the λ-contest is $e_i(e_j) = \max\{0, \sqrt{e_j V_i \lambda} - e_j\}$ while it is not possible to find a closed form solution for the best response of the r-contest. However, as shown in Chowdhury and Sheremeta (2014) effort equivalence is a necessary condition for strategic equivalence. Therefore, strategic equivalence is guaranteed only if the first order conditions of the r-contest are satisfied for any value of e_j after substituting the best responses of the λ-contest with $\lambda = \frac{r(V_1 V_2)^r (V_1+V_2)^2}{V_1 V_2 (V_1^r + V_2^r)^2}$. This is true if and only if $\frac{e_j^r r V_i (A)^{r-1}}{(e_j^r + (A)^r)^2} = 1$, where $A = -e_j + \sqrt{e_j V_i} \sqrt{\frac{r(V_i V_j)^{r-1}(V_i+V_j)^2}{(V_i^r + V_j^r)^2}}$ which is not true for all values of e_j (only for the equilibrium one).

3. By plugin equilibrium efforts in the payoff of player 1 we obtain that the λ-contest induces the same payoff as the one in the r-contest for $\lambda = \frac{(V_1+V_2)^2 (V_1^{2r} - V_2^{2r} - 2r(V_1 V_2)^r)}{(V_1^2 - 2V_1 V_2 - V_2^2)(V_1^r + V_2^r)^2} = \lambda_1$. Similarly, the λ-contest induces payoff equivalence for player 2 if and only if $\lambda = \frac{(V_1+V_2)^2 (V_1^{2r} - V_2^{2r} + 2r(V_1 V_2)^r)}{(V_1^2 + 2V_1 V_2 - V_2^2)(V_1^r + V_2^r)^2} = \lambda_2$. Normalizing $V_2 = 1$ and $V_1/V_2 = v$ we see that $\lambda_1 = \lambda_2$, i.e., payoff equivalence, is only obtained for $V_1 = V_2$ or $r = \{0, 1\}$ (when the two contests coincide).

4.2 Proof of Proposition 2

Player i effort-elasticity in the λ-contest is $\frac{e_1 \sum_{j \neq i} e_j N \lambda}{(e_1 + \sum_{j \neq i} e_j)(\sum_{j \neq i} e_j(1-\lambda) + e_1(1 + (N-1)\lambda))}$ which evaluated at $e_i = e$ $\forall i = 1, ..., N$ is $\lambda \frac{N-1}{N}$. Player i effort-elasticity in the r-contest is $r \frac{\sum_{j \neq i} e_j^r}{e_i^r + \sum_{j \neq i} e_j^r}$ which evaluated at $e_i = e$ $\forall i = 1, ..., N$ is $r \frac{N-1}{N}$. Thus $\lambda = r$ guarantees noise-equivalence.