SPLITTINGS OF EXTENSIONS AND HOMOLOGICAL BIDIMENSION OF THE ALGEBRA OF BOUNDED OPERATORS ON A BANACH SPACE

NIELS JAKOB LAUSTSEN AND RICHARD SKILLICORN

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ABSTRACT. We show that there exists a Banach space E such that:

- the Banach algebra $\mathscr{B}(E)$ of bounded, linear operators on E has a singular extension which splits algebraically, but it does not split strongly;
- the homological bidimension of $\mathscr{B}(E)$ is at least two.

The first of these conclusions solves a natural problem left open by Bade, Dales, and Lykova (*Mem. Amer. Math. Soc.* 1999), while the second answers a question of Helemskii. The Banach space E that we use was originally introduced by Read (*J. London Math. Soc.* 1989).

Nous démontrons qu'il existe un espace de Banach tel que:

- l'algèbre de Banach $\mathscr{B}(E)$ des opérateurs linéaires bornés sur E a une extension singulière qui scinde algébriquement mais qui ne scinde pas fortement;
- la bidimension homologique de $\mathscr{B}(E)$ est au moins deux.

La première de ces conclusions complète les résultats de Bade, Dales et Lykova (*Mem. Amer. Math. Soc.* 1999), et la seconde répond à une question de Helemskii. L'espace de Banach E a été introduit initialement par Read (*J. London Math. Soc.* 1989).

1. Introduction and statement of results

By an *extension* of a Banach algebra $\mathscr{B}$, we understand a short-exact sequence of the form

$$\{0\} \longrightarrow \ker\varphi \longrightarrow \mathscr{A} \xrightarrow{\ \varphi\ } \mathscr{B} \longrightarrow \{0\}, \tag{1.1}$$

where $\mathscr{A}$ is a Banach algebra and $\varphi\colon \mathscr{A} \to \mathscr{B}$ is a continuous, surjective algebra homomorphism. The extension *splits algebraically* (respectively, *splits strongly*) if there is an algebra homomorphism (respectively, a continuous algebra homomorphism) $\rho\colon \mathscr{B} \to \mathscr{A}$ which is a right inverse of φ, in the sense that $\varphi\circ\rho$ is the identity map on $\mathscr{B}$. We say that (1.1) is *admissible* if φ has a right inverse which is bounded and linear, or, equivalently, if $\ker\varphi$ is complemented in $\mathscr{A}$ as a Banach space. Every extension which splits strongly is obviously admissible. The extension (1.1) is *singular* if $\ker\varphi$ has trivial multiplication, in the sense that $ab = 0$ whenever $a, b \in \ker\varphi$.

Bade, Dales, and Lykova [1] carried out a comprehensive study of extensions of Banach algebras, focusing in particular on the following question:

> *For which (classes of) Banach algebras $\mathscr{B}$ is it true that every extension of the form (1.1) which splits algebraically also splits strongly?*

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This question can be viewed as a variation on the theme of automatic continuity. Of course, its answer is positive whenever the Banach algebra $\mathscr{B}$ has the property that *every* algebra homomorphism from $\mathscr{B}$ into a Banach algebra is continuous. A classical theorem of Johnson [5] states that the Banach algebra $\mathscr{B} = \mathscr{B}(E)$ of all bounded operators on a Banach space E has this property whenever E is isomorphic to its Cartesian square $E \oplus E$.

Johnson's result, however, does not extend to all Banach spaces because Read [9] has constructed a Banach space E_{R} such that there exists a discontinuous derivation (and hence a discontinuous algebra homomorphism) from $\mathscr{B}(E_{\mathrm{R}})$. Dales, Loy, and Willis [2] have subsequently given an example of a Banach space E_{DLW} such that all derivations from $\mathscr{B}(E_{\mathrm{DLW}})$ are continuous, but under the assumption of the Continuum Hypothesis, $\mathscr{B}(E_{\mathrm{DLW}})$ admits a discontinuous algebra homomorphism into a Banach algebra.

These results still leave open the above question of Bade, Dales, and Lykova in the case of $\mathscr{B}(E)$ for a general Banach space E: is it true that every extension of $\mathscr{B}(E)$ which splits algebraically also splits strongly? Our first result answers this question in the negative.

Theorem 1.1. *There exists a continuous, surjective algebra homomorphism φ from a unital Banach algebra $\mathscr{A}$ onto $\mathscr{B}(E_{\mathrm{R}})$, where E_{R} denotes the above-mentioned Banach space of Read, such that the extension*

$$\{0\} \longrightarrow \ker\varphi \longrightarrow \mathscr{A} \xrightarrow{\varphi} \mathscr{B}(E_{\mathrm{R}}) \longrightarrow \{0\}$$

is singular and splits algebraically, but it is not admissible, and so it does not split strongly.

We do not know whether Read's Banach space E_{R} is essential for this result. Due to the dearth of examples of Banach spaces E for which $\mathscr{B}(E)$ admits a discontinuous homomorphism into a Banach algebra, $E = E_{\mathrm{R}}$ was the most obvious place to start our investigations, and it led to the answer that we were looking for.

Before we can state our second result, we require some more terminology. Let $n \in \mathbb{N}_0$. A Banach algebra $\mathscr{B}$ has *homological bidimension at least n* if there exists a Banach $\mathscr{B}$-bimodule X such that the n^{th} continuous Hochschild cohomology group $\mathscr{H}^n(\mathscr{B}, X)$ of $\mathscr{B}$ with coefficients in X is non-zero. This notion is the topological counterpart of a long established notion in pure algebra. It was introduced by Helemskii, who, together with his students, has studied it for many classes of Banach algebras; see [3] for an overview.

A Banach algebra $\mathscr{B}$ which has homological bidimension zero (so that $\mathscr{H}^1(\mathscr{B}, X) = \{0\}$ for every Banach $\mathscr{B}$-bimodule X) is *contractible*. It is conjectured that a Banach algebra is contractible (if and) only if it is finite-dimensional and semisimple.

If true, this conjecture would imply that the Banach algebra $\mathscr{B}(E)$ has homological bidimension at least one for every infinite-dimensional Banach space E; see [7, Proposition 5.1] for a strong partial result in this direction. It appears to be unknown if a Banach algebra of the form $\mathscr{B}(E)$ for a (necessarily infinite-dimensional) Banach space E can have homological bidimension at least two, a problem that goes back to Helemskii's seminar at Moscow State University. In the case where E is a Hilbert space, this problem is stated explicitly as [4, Problem 21]; see also [1, p. 27]. We shall show that the homological bidimension of $\mathscr{B}(E)$ can be two or greater, again using the above-mentioned Banach space E_{R} of Read.

Theorem 1.2. *There exist a one-dimensional Banach $\mathscr{B}(E_{\mathrm{R}})$-bimodule X and a linear injection from the Banach algebra $\mathscr{B}(\ell_2(\mathbb{N}))$ of bounded operators on the Hilbert space $\ell_2(\mathbb{N})$ into the second continuous Hochschild cohomology group of $\mathscr{B}(E_{\mathrm{R}})$ with coefficients in X. Hence $\mathscr{B}(E_{\mathrm{R}})$ has homological bidimension at least two.*

2. Proofs of Theorems 1.1 and 1.2

The proofs of Theorems 1.1 and 1.2 both rely on a strengthening of the main technical result in Read's paper, as it is stated in [9, Section 4]. This strengthening involves two further pieces of notation. First, we denote by $\mathscr{W}(E_{\mathrm{R}})$ the ideal of weakly compact operators on the Banach space E_{R}. Second, we endow the Hilbert space $\ell_2(\mathbb{N})$ with the trivial multiplication and write $\ell_2(\mathbb{N})^{\sim}$ for its unitization; that is, $\ell_2(\mathbb{N})^{\sim} = \ell_2(\mathbb{N}) \oplus \mathbb{K}1$ as a vector space (where $\mathbb{K}$ denotes the scalar field, either $\mathbb{R}$ or $\mathbb{C}$, and 1 is the formal identity that we adjoin), while the product and the norm on $\ell_2(\mathbb{N})^{\sim}$ are given by

$$(x + \lambda 1)(y + \mu 1) = \lambda y + \mu x + \lambda\mu 1 \quad \text{and} \quad \|x + \lambda 1\| = \|x\| + |\lambda| \qquad (x, y \in \ell_2(\mathbb{N}),\ \lambda, \mu \in \mathbb{K}).$$

Theorem 2.1. *There exists a continuous, surjective algebra homomorphism ψ from $\mathscr{B}(E_{\mathrm{R}})$ onto $\ell_2(\mathbb{N})^{\sim}$ with $\ker\psi = \mathscr{W}(E_{\mathrm{R}})$ such that the extension*

$$\begin{tikzcd} \{0\} \arrow[r] & \mathscr{W}(E_{\mathrm{R}}) \arrow[r] & \mathscr{B}(E_{\mathrm{R}}) \arrow[r, "\psi"] & \ell_2(\mathbb{N})^{\sim} \arrow[r] & \{0\} \end{tikzcd}$$

splits strongly.

The proof of this result relies on a careful analysis of Read's construction; full details will appear in [8].

In order to prove Theorem 1.1, we require another tool, namely the *pullback* of a diagram of the form

$$\begin{tikzcd} \mathscr{A} \arrow[r, "\alpha"] & \mathscr{C} & \mathscr{B}, \arrow[l, "\beta"'] \end{tikzcd} \tag{2.1}$$

where $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{C}$ are Banach algebras, and $\alpha\colon \mathscr{A} \to \mathscr{C}$ and $\beta\colon \mathscr{B} \to \mathscr{C}$ are continuous algebra homomorphisms. We can define the pullback of this diagram explicitly by the formula

$$\mathscr{D} = \{(a, b) \in \mathscr{A} \oplus \mathscr{B} : \alpha(a) = \beta(b)\}, \tag{2.2}$$

where $\mathscr{A} \oplus \mathscr{B}$ denotes the direct sum of the Banach algebras $\mathscr{A}$ and $\mathscr{B}$. Being a closed subalgebra of $\mathscr{A} \oplus \mathscr{B}$, $\mathscr{D}$ is a Banach algebra in its own right. Let

$$\gamma\colon\ (a, b) \mapsto a, \quad \mathscr{D} \to \mathscr{A}, \qquad \text{and} \qquad \delta\colon\ (a, b) \mapsto b, \quad \mathscr{D} \to \mathscr{B}, \tag{2.3}$$

be the restrictions to $\mathscr{D}$ of the coordinate projections. Then $\alpha \circ \gamma = \beta \circ \delta$, and it can be shown that $\mathscr{D}$, together with the continuous algebra homomorphisms γ and δ, has the following universal property, so that they form a pullback of (2.1) in the categorical sense: for every Banach algebra $\mathscr{E}$ and each pair $\xi\colon \mathscr{E} \to \mathscr{A}$ and $\eta\colon \mathscr{E} \to \mathscr{B}$ of continuous algebra homomorphisms satisfying $\alpha \circ \xi = \beta \circ \eta$, there is a unique continuous algebra homomorphism $\theta\colon \mathscr{E} \to \mathscr{D}$ such that the diagram

$$\begin{tikzcd} \mathscr{E} \arrow[dr, dashed, "\theta"] \arrow[drr, "\xi"] \arrow[ddr, "\eta"'] & & \\ & \mathscr{D} \arrow[r, "\gamma"'] \arrow[d, "\delta"] & \mathscr{A} \arrow[d, "\alpha"] \\ & \mathscr{B} \arrow[r, "\beta"] & \mathscr{C} \end{tikzcd}$$

is commutative.

We now come to our key result, which establishes a connection between extensions and pullbacks.

Proposition 2.2. *Let $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{C}$ be Banach algebras such that there are extensions*

$$\{0\} \longrightarrow \ker\alpha \longrightarrow \mathscr{A} \xrightarrow{\alpha} \mathscr{C} \longrightarrow \{0\} \tag{2.4}$$

and

$$\{0\} \longrightarrow \ker\beta \longrightarrow \mathscr{B} \xrightarrow{\beta} \mathscr{C} \longrightarrow \{0\}, \tag{2.5}$$

and define $\mathscr{D}$, γ, and δ by (2.2) *and* (2.3)*, above. Then δ is surjective, and the following statements concerning the extension*

$$\{0\} \longrightarrow \ker\delta \longrightarrow \mathscr{D} \xrightarrow{\delta} \mathscr{B} \longrightarrow \{0\} \tag{2.6}$$

hold true:

(i) (2.6) *is singular if and only if* (2.4) *is singular.*

(ii) *Suppose that* (2.5) *splits strongly (respectively, splits algebraically, is admissible). Then* (2.6) *splits strongly (respectively, splits algebraically, is admissible) if and only if* (2.4) *splits strongly (respectively, splits algebraically, is admissible).*

Proof. The surjectivity of α implies that δ is surjective, so that (2.6) is indeed an extension.

(i). The restriction of γ to $\ker\delta$ is an isomorphism onto $\ker\alpha$, and the conclusion follows.

(ii). Let $\rho\colon \mathscr{C} \to \mathscr{B}$ be a continuous algebra homomorphism which is a right inverse of β.

$\Rightarrow$. Suppose that $\tau\colon \mathscr{B} \to \mathscr{D}$ is a continuous algebra homomorphism which is a right inverse of δ. Then a direct calculation shows that the continuous algebra homomorphism $\gamma\circ\tau\circ\rho$ is a right inverse of α, so that (2.4) splits strongly.

$\Leftarrow$. Suppose that $\sigma\colon \mathscr{C} \to \mathscr{A}$ is a continuous algebra homomorphism which is a right inverse of α. Then, setting $\tau(b) = (\sigma(\beta(b)), b)$ for each $b \in \mathscr{B}$, we obtain a continuous algebra homomorphism $\tau\colon \mathscr{B} \to \mathscr{D}$. The definition of δ implies that τ is a right inverse of δ, and hence (2.6) splits strongly.

The proof just given applies equally to establish the other two cases. □

Proof of Theorem 1.1. Our aim is to apply Proposition 2.2 with $\mathscr{B} = \mathscr{B}(E_{\mathrm{R}})$, $\mathscr{C} = \ell_2(\mathbb{N})^\sim$, and $\beta = \psi$. Theorem 2.1 shows that, for these choices, we have an extension of the form (2.5) which splits strongly.

Let $q\colon \ell_1(\mathbb{N}) \to \ell_2(\mathbb{N})$ be a bounded, linear surjection. Then $\ker q$ is not complemented in $\ell_1(\mathbb{N})$ because no (complemented) subspace of $\ell_1(\mathbb{N})$ is isomorphic to $\ell_2(\mathbb{N})$. Equip $\ell_1(\mathbb{N})$ with the trivial product, let $\mathscr{A} = \ell_1(\mathbb{N}) \oplus \mathbb{K}1$ be its unitization (defined analogously to the unitization of $\ell_2(\mathbb{N})$, above), and define $\alpha\colon \mathscr{A} \to \mathscr{C}$ by $\alpha(x+\lambda 1) = q(x)+\lambda 1$ for $x \in \ell_1(\mathbb{N})$ and $\lambda \in \mathbb{K}$. Then α is a continuous, surjective algebra homomorphism with kernel $\ker q$, which is uncomplemented in $\ell_1(\mathbb{N})$ and hence in $\mathscr{A}$, so that we have a singular, non-admissible extension of the form (2.4).

Being surjective, q has a linear right inverse $\rho\colon \ell_2(\mathbb{N}) \to \ell_1(\mathbb{N})$, which is multiplicative because $\ell_1(\mathbb{N})$ and $\ell_2(\mathbb{N})$ both have the trivial product. Extend ρ to a linear map between the unitizations $\mathscr{C}$ and $\mathscr{A}$ by making it unital. Then it is an algebra homomorphism which is a right inverse of α, so that the extension (2.4) splits algebraically. Hence Proposition 2.2 produces a singular extension (2.6) of $\mathscr{B} = \mathscr{B}(E_{\mathrm{R}})$ which splits algebraically, but is not admissible. □

Before we proceed to prove Theorem 1.2, let us recall the formal definition of the second continuous Hochschild cohomology group of a Banach algebra $\mathscr{B}$ with coefficients in a Banach $\mathscr{B}$-bimodule X. A 2-*cocycle* is a bilinear map $\Upsilon\colon \mathscr{B}\times\mathscr{B}\to X$ which satisfies

$$a\cdot\Upsilon(b,c)-\Upsilon(ab,c)+\Upsilon(a,bc)-\Upsilon(a,b)\cdot c=0 \qquad (a,b,c\in\mathscr{B}).$$

The set $\mathscr{Z}^2(\mathscr{B},X)$ of continuous 2-cocycles forms a closed subspace of the Banach space of continuous, bilinear maps from $\mathscr{B}\times\mathscr{B}$ into X. Each bounded, linear map $\Omega\colon\mathscr{B}\to X$ induces a continuous 2-cocycle by the definition

$$\delta^1\Omega\colon\ (a,b)\mapsto a\cdot(\Omega b)-\Omega(ab)+(\Omega a)\cdot b,\quad \mathscr{B}\times\mathscr{B}\to X. \tag{2.7}$$

The 2-cocyles of this form are called 2-*coboundaries;* they form a (not necessarily closed) subspace $\mathscr{N}^2(\mathscr{B},X)$ of $\mathscr{Z}^2(\mathscr{B},X)$, and so the quotient

$$\mathscr{H}^2(\mathscr{B},X):=\mathscr{Z}^2(\mathscr{B},X)/\mathscr{N}^2(\mathscr{B},X)$$

is a vector space, which is the *second continuous Hochschild cohomology group* of $\mathscr{B}$ with *coefficients* in X.

Proof of Theorem 1.2. By Theorem 2.1, there are continuous algebra homomorphisms

$$\psi\colon\mathscr{B}(E_{\mathrm{R}})\to\ell_2(\mathbb{N})^\sim \qquad\text{and}\qquad \rho\colon\ell_2(\mathbb{N})^\sim\to\mathscr{B}(E_{\mathrm{R}})$$

such that ρ is a right inverse of ψ. The definition of the unitization implies that we can find maps $\psi_0\colon\mathscr{B}(E_{\mathrm{R}})\to\ell_2(\mathbb{N})$ and $\theta\colon\mathscr{B}(E_{\mathrm{R}})\to\mathbb{K}$ such that

$$\psi(T)=\psi_0(T)+\theta(T)1 \qquad (T\in\mathscr{B}(E_{\mathrm{R}})).$$

We see that θ is a continuous algebra homomorphism, and $X=\mathbb{K}$ is a one-dimensional Banach $\mathscr{B}(E_{\mathrm{R}})$-bimodule with respect to the operations

$$T\cdot\lambda=\theta(T)\lambda \qquad\text{and}\qquad \lambda\cdot T=\theta(T)\lambda \qquad (T\in\mathscr{B}(E_{\mathrm{R}}),\ \lambda\in X). \tag{2.8}$$

Moreover, ψ_0 is bounded and linear. Consequently, for each $U\in\mathscr{B}(\ell_2(\mathbb{N}))$, we can define a continuous, bilinear map $\Upsilon_U\colon\mathscr{B}(E_{\mathrm{R}})\times\mathscr{B}(E_{\mathrm{R}})\to X$ by

$$\Upsilon_U(S,T)=\langle U(\psi_0(S)),\psi_0(T)\rangle \qquad (S,T\in\mathscr{B}(E_{\mathrm{R}})), \tag{2.9}$$

where $\langle\cdot,\cdot\rangle$ denotes the usual Banach-space duality bracket on $\ell_2(\mathbb{N})$, that is, $\langle x,y\rangle=\sum_{n=1}^\infty x_ny_n$ for $x=(x_n)$ and $y=(y_n)$ in $\ell_2(\mathbb{N})$. The map ψ_0 is not multiplicative; more precisely, since $\ell_2(\mathbb{N})$ has trivial multiplication, we have

$$\psi_0(ST)=\theta(S)\psi_0(T)+\theta(T)\psi_0(S) \qquad (S,T\in\mathscr{B}(E_{\mathrm{R}})).$$

A straightforward verification based on this identity shows that Υ_U is a 2-cocycle. Hence we have a map $\Upsilon\colon U\mapsto\Upsilon_U$, $\mathscr{B}(\ell_2(\mathbb{N}))\to\mathscr{Z}^2(\mathscr{B}(E_{\mathrm{R}}),X)$, which is clearly linear.

Suppose that Υ_U is a 2-coboundary for some $U\in\mathscr{B}(\ell_2(\mathbb{N}))$, so that $\Upsilon_U=\delta^1\Omega$ for some bounded, linear map $\Omega\colon\mathscr{B}(E_{\mathrm{R}})\to X$. Since ρ is a right inverse of ψ, we see that $\psi_0(\rho(x))=x$ and $\theta(\rho(x))=0$ for each $x\in\ell_2(\mathbb{N})$. Combining these identities with the definitions (2.7)–(2.9), we obtain

$$\langle Ux,y\rangle=(\delta^1\Omega)(\rho(x),\rho(y))=\theta(\rho(x))\Omega(\rho(y))-\Omega(\rho(x)\rho(y))+\theta(\rho(y))\Omega(\rho(x))=0$$

for each $x,y\in\ell_2(\mathbb{N})$ because $\rho(x)\rho(y)=\rho(xy)=0$. This shows that $U=0$, so that 0 is the only 2-coboundary in the image of Υ. Hence the composition of Υ with the quotient map from $\mathscr{Z}^2(\mathscr{B}(E_{\mathrm{R}}),X)$ onto $\mathscr{H}^2(\mathscr{B}(E_{\mathrm{R}}),X)$ is a linear injection from $\mathscr{B}(\ell_2(\mathbb{N}))$ into $\mathscr{H}^2(\mathscr{B}(E_{\mathrm{R}}),X)$, and the result follows. $\square$

Remark 2.3. There is an underlying connection between Theorems 1.1 and 1.2. To explain it, consider two extensions (2.4) and (2.5) of a Banach algebra $\mathscr{C}$, where the former extension is singular and admissible, but does not split strongly, while the latter splits strongly. Then, by Proposition 2.2, we obtain a singular, admissible extension (2.6) of the Banach algebra $\mathscr{B}$, and this extension does not split strongly. Hence a classical result of Johnson (see [6, Theorem 2.1], or [3, Corollary I.1.11] for an exposition) implies that $\ker\delta$ is a Banach $\mathscr{B}$-bimodule and $\mathscr{H}^2(\mathscr{B},\ker\delta)$ is non-zero, so that $\mathscr{B}$ has homological bidimension at least two.

To apply this result to $\mathscr{B} = \mathscr{B}(E_{\mathrm{R}})$, we take $\mathscr{C} = \ell_2(\mathbb{N})^\sim$ and $\beta = \psi$ as in the proof of Theorem 1.1, so that we have an extension of the form (2.5) which splits strongly by Theorem 2.1. Choose $U \in \mathscr{B}(\ell_2(\mathbb{N}))$ with $\|U\| \leqslant 1$, and turn the vector space $\mathbb{K}\oplus\ell_2(\mathbb{N})$ into a Banach algebra by endowing it with the product and the norm

$$(\lambda,x)(\mu,y) = (\langle Ux,y\rangle,0) \qquad \text{and} \qquad \|(\lambda,x)\| = |\lambda| + \|x\| \qquad (x,y\in\ell_2(\mathbb{N}),\ \lambda,\mu\in\mathbb{K}).$$

Denote by $\mathscr{A}$ the unitization of this Banach algebra, and let $\alpha\colon \mathscr{A}\to\mathscr{C}$ be the natural unital projection. Then α is a continuous, surjective algebra homomorphism, and we have a singular, admissible extension of the form (2.4), which can be shown to split algebraically if and only if it splits strongly, if and only if $U = 0$. Thus, choosing U non-zero, we conclude that $\mathscr{B}(E_{\mathrm{R}})$ has homological bidimension at least two.

A similar argument shows that $\mathscr{B}(E_{\mathrm{DLW}})$ has homological bidimension at least two, where E_{DLW} denotes the Banach space of Dales, Loy, and Willis studied in [2]. To see this, take $\mathscr{B} = \mathscr{B}(E_{\mathrm{DLW}})$ and $\mathscr{C} = \ell_\infty(\mathbb{Z})$, and apply [1, Theorem 3.11(i)] to obtain a singular, admissible extension of $\ell_\infty(\mathbb{Z})$ which does not split strongly.

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Department of Mathematics and Statistics, Fylde College, Lancaster University, Lancaster LA1 4YF, United Kingdom.

E-mail address: `n.laustsen@lancaster.ac.uk, r.skillicorn@lancaster.ac.uk`